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Coexistence of Multiple Strange Attractors Governed by Different Initial Conditions in a Deterministic System

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Abstract

This paper presents a new four-dimension autonomous system which shows extraordinary dynamical properties . Chaotic attractor and periodic attractor or hyper-chaotic attractor and quasi-periodic attractor, which are governed by different initial conditions instead of the system parameters, can coexist in the deterministic system. These interesting phenomena are verified through numerical simulations and analyses including time series, phase portraits, Poincaré maps, bifurcation diagrams, and Lyapunov exponents.

1. Introduction

In general, many deterministic systems always display one of these forms such as static state, periodic or quasi-periodic motion, chaos or hyper-chaos, and so on. If a system is said to be chaotic, its behavior is so unpredictable as to show random dynamics owing to great sensitivity to the initial conditions. This property implies that two trajectories emerging from two different closely initial conditions separate exponentially with the passage of time. The necessary requirements for a deterministic smooth nonlinear system to be chaotic are that the system must be nonlinear, and be at least three dimensional. The fact that some dynamical systems showing the above necessary conditions possess such a critical dependence on the initial conditions has been known since the end of the last century. However, only in the last thirty years, experimental observations have pointed out that.

Lorenz found the atmosphere dynamical model in 1963 [1]. Since then, the Lorenz system and other chaotic systems, such as the Rössler system [2] and the Chua circuit equation [3, 4], have been investigated profoundly and comprehensively. The theory of chaos, when being applied to the fields of control, synchronization and security communication, can help us to realize important value of chaos in practical engineering [5-8]. Moreover, chaotic systems are common in nature. Many natural phenomena can be characterized as being chaotic. They can be found in meteorology, solar system, heart and brain of living organisms and so on. Therefore, chaos has become an important research topic in the fields of nonlinearity in the past thirty years, and some new characteristics of chaos were also found.

It is common that a nonlinear system with changeable system parameters shows different dynamical behaviors such as periodic state, quasi-periodic state, chaos or hyper-chaos, and so on. From previous researches for example in references [9-13], it can be found that the bifurcation phenomena can be obtained from the chaotic systems by varying one parameter and the bifurcation diagrams are not sensitive to the conditions. In other words, the nonlinear systems show the same chaotic attractors under different initial conditions, which means that changing initial conditions can not affect the chaotic properties of the system.

The logical question is whether the initial conditions can affect the nonlinear dynamics especially about strange attractors. In this paper, a new nonlinear system is proposed to study the coexistent phenomena of several kinds of attractors governed by the different initial conditions. There are several kinds of coexistence phenomena in this system including the coexistence of periodic attractor and chaotic attractor, the coexistence of quasi-periodic attractor and hyper-chaotic attractor, and so on. The phase diagrams, Lyapunov exponents, time series, Poincaré maps and bifurcation diagrams are used to illustrate and judge different dynamics.

2. The proposed model

Consider the following four-dimensional (4-D) nonlinear dynamical system with quadratic coupling terms:

$$\begin{cases} \dot{x} = -ax - e\omega + yz \\ \dot{y} = by + xz \\ \dot{z} = cz + f\omega - xy \\ \dot{\omega} = d\omega - gz \end{cases} \tag{1}$$

where x, y, z, ω are state variables and a, b, c, d, e, f, g are real constant parameters. Let $a = 50,\ c = 10,\ d = 0.2,\ e = 10,\ f = 16,\ g = 0.5$ and keep parameter b negative and variable. When $b = -16$, this system shows a beautiful four-wing chaotic attractor [14, 15], as shown in Fig. 1.

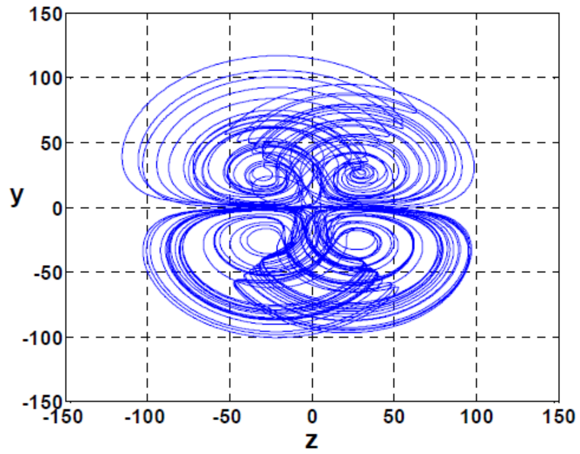


Fig. 1: The 4-wing hyper-chaotic attractor of system (1) projected on $y-z$ plane

For a nonlinear system, one positive Lyapunov exponent means chaos, and two positive Lyapunov exponents imply hyper-chaos. Here, the four Lyapunov exponents of system (1) are denoted as $\lambda_1, \lambda_2, \lambda_3$ and λ_4. When $b = -16$, the Lyapunov exponents of system (1) are $\lambda_1 = 2.4736$, $\lambda_2 = 0.5175$, $\lambda_3 = 0$, and $\lambda_4 = -58.7248$ which mean system (1) is in hyper-chaotic [16, 17].

For system (1), some other attractors with different b also can be obtained. It is well known that for many classical chaotic systems [1-4], the bifurcation phenomena only depend on parameters instead of initial conditions, which means that the basic dynamics of system are not changed if the system parameters are fixed.

The orbits of a system with fixed parameters and variable initial conditions usually show one kind of dynamics such as the periodic or quasi-periodic orbits, chaos and hyper-chaos, and so on. However, system (1) exhibits the coexistence phenomena of two kinds of attractors which depend not only on the system parameters but also the initial conditions. In the following two sections, we will investigate the surprising coexistent dynamics caused by different initial conditions not the changeable system parameters.

3. Coexistence of hyper-chaotic attractor and quasi-periodic attractor

System (1) is symmetric with respect to y-axis which can be easily proven via the transformation $(x,y,z,\omega)\to(-x,y,-z,-\omega)$. In the following parts of this paper, $I_{00}=[1,\ 1,\ 1,\ 1]$ and $I_{01}=[1,\ \text{-}1,\ 1,\ 1]$ are chosen as two different initial conditions, and the red solid line denotes the orbit from initial condition I_{00} and the black dashed line represents the orbit from I_{01} in the following figures.

To analyze the dynamics of the orbits starting from different initials, the general methods are time series curve, phase portrait, bifurcation diagram, Poincaré map, Lyaponov exponent, and so on. For system (1), fix parameters $a=50,\ c=10,\ d=0.2,$ $e=10,\ f=16,\ g=0.5$ and $b=-4.5$, the time series curves and the phase portraits are shown in Fig. 2 and Fig. 3, respectively.

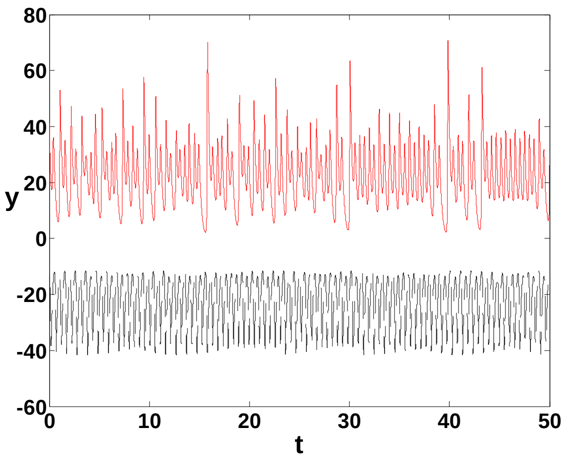


Fig. 2: Time series curves of system (1) with initial conditions I_{00} and I_{01} : $b=-4.5$

From Fig. 2, it is easy to find that the upper part of time series is more disorder than the lower curve. However, the lower one is almost regular and periodic. Fig. 2 illustrates that system (1) has various types of orbits under two different initial conditions. To further investigate the dynamics of this system, other techniques should be used.

Figs. 3(a) and 3(b) are the phase portraits of system (1). As can be seen from Fig. 3(a) that the orbit from I_{00} shows chaotic dynamics. While the orbit from I_{01} which is shown in Fig. 3(b) is periodic and moves in a two dimensional torus which will be further proved by the Lyapunov exponents which can also be used to distinguish the different attractors under different conditions. Lyapunov exponents in Table 1 mean that system (1) has different kinds of dynamics under the initial conditions of I_{00} and I_{01}. Therefore, it is possible that there exist several types of attractors in one nonlinear system with fixed system parameters.

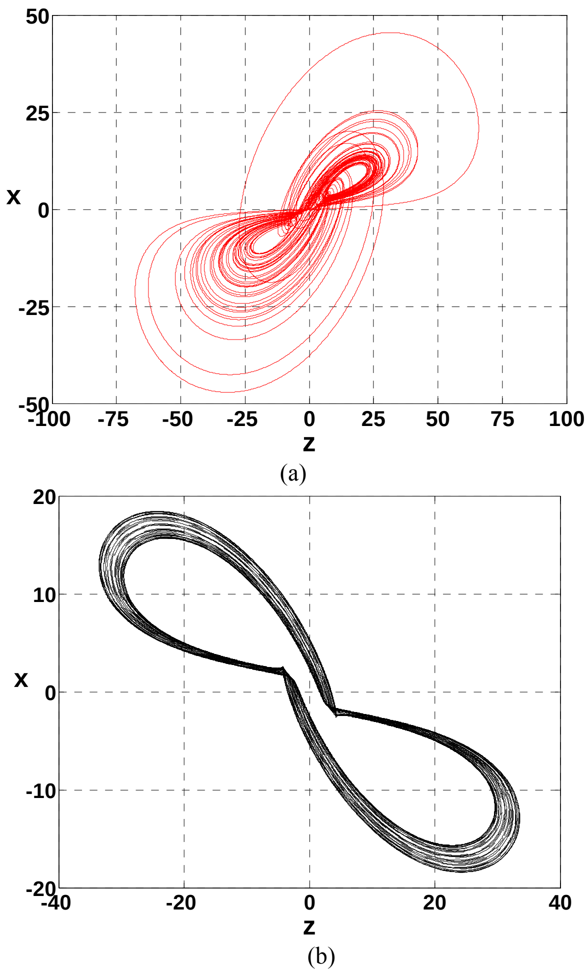
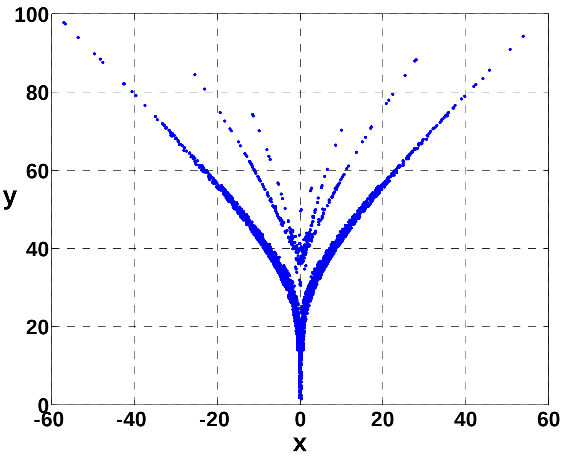


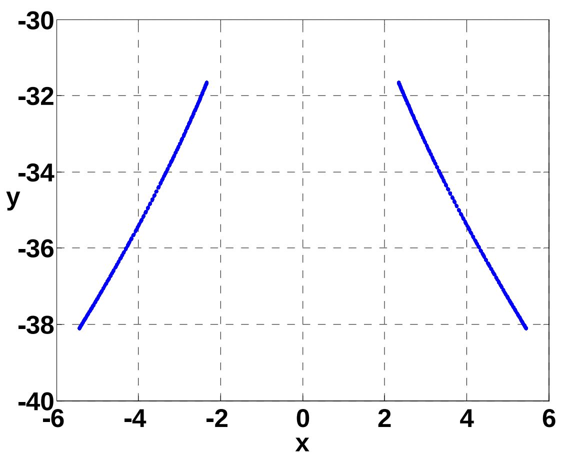
Fig. 3: Phase portraits of system (1) with $b=-4.5$: (a) and (b) projection on the $x-z$ plane

Table 1
Lyapunov exponents corresponding to different initial conditions I_{00} and I_{01} with $b=-4.5$

Initial Value	λ_1	λ_2	λ_3	λ_4	Dynamics
I_{00}	0.6445	0.1443	0	-45.0877	Hyper-chaos
I_{01}	0	0	-0.2975	-44.0031	Quasi-periodic orbit



(a)



(b)

Fig. 4: Poincaré section with $z=0$ and $b=-4.5$: (a) I_{00} , (b) I_{01}

As an important analysis technique, Poincaré map can reflect bifurcation and folding properties of nonlinear dynamics. Fig. 4 shows the returned Poincaré maps of system (1) with parameter $b=-4.5$ projected on $x-y$ plane under initial conditions I_{00} and I_{01}. Clearly, the returned map in Fig. 4(a) is symmetrically disordered and folded, which is an important feature to distinguish chaotic attractor from other types of attractors. While the one in Fig. 4 (b) shows two symmetrical straight lines. For the returned Poincaré map of a quasi-periodic attractor, it always exhibits an almost closed circle or a bounded straight line. Therefore, it can be seen from Fig. 4(b) that the returned map starting from I_{01} represents the dynamics of a quasi-periodic orbit.

Furthermore, in order to investigate the dynamics of system (1) in the neighborhood of $b=-4.5$ under different initial conditions, it is better to use the bifurcation diagram. Here, I_{00} and I_{01} are still used as the initial conditions. The bifurcation diagrams of state variable y are shown in Fig. 5 with respect to b in the interval [-4.55,-4.45].

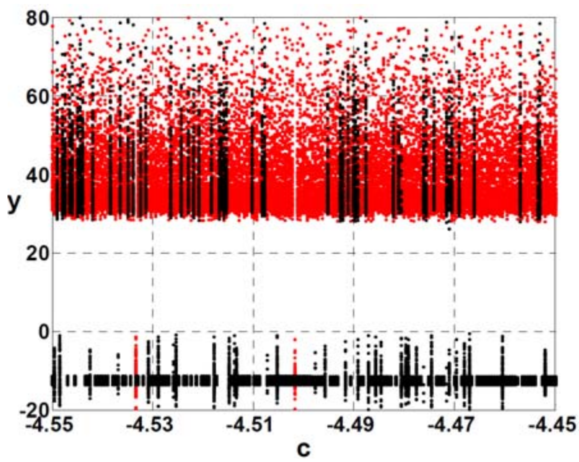


Fig. 5: Bifurcation diagram versus b in interval [-4.55, -4.45] with initial conditions I_{00} and I_{01}

The upper red dots and the lower black dots in Fig. 5 show the bifurcation data starting from different initial conditions I_{00} and I_{01} , respectively. It can be seen that the upper dots are irregular in a wide range and the lower dotes distribute regularly in some narrow intervals. The bifurcation diagrams also show that these different behaviors of system (1) are caused by different initial conditions I_{00} and I_{01}. Moreover, the occurrence of quasi-periodic in the lower part is not continuous but intermittent between the quasi-periodic and chaos with respect to b.

Hence, it can be concluded that the coexistence of hyper-chaotic attractor and quasi-periodic attractor subject to the different initial conditions really exists in system (1).

4. Coexistence of chaotic attractor and periodic attractor

Except the coexistence phenomenon of chaotic attractor and quasi-periodic attractor, there is another type of coexistence phenomenon, i.e. chaotic attractor and periodic attractor in this deterministic system when system parameter b is in the interval [-1.65, -1.64].

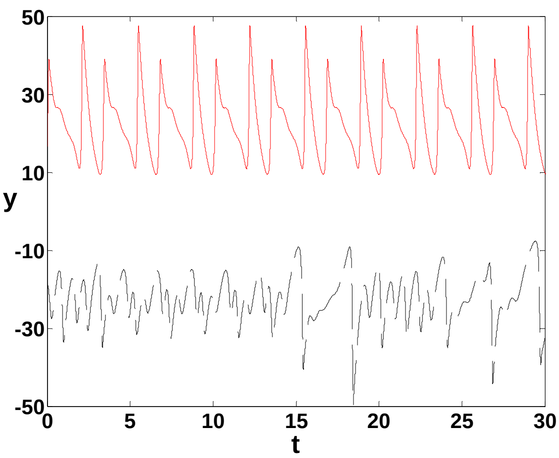


Fig. 6: Time series of system (1) with initial conditions I_{00} and I_{01} : $b = -1.647$

The methods mentioned in section 3 are also used, and the same initial conditions I_{00} and I_{01} are still used to analyze this phenomenon.

Fig. 6 shows the time series of system (1) with parameter $b = -1.647$. It shows that the red line denotes a periodic signal, while the dashed line shows a disorder signal. So it means that there are different dynamical characteristics corresponding to two different initial conditions I_{00} and I_{01}, respectively.

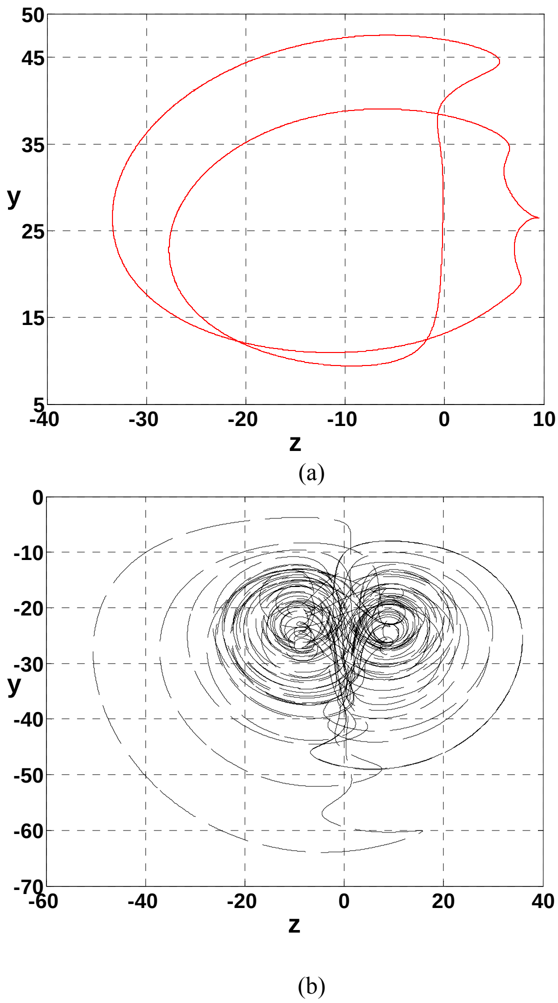
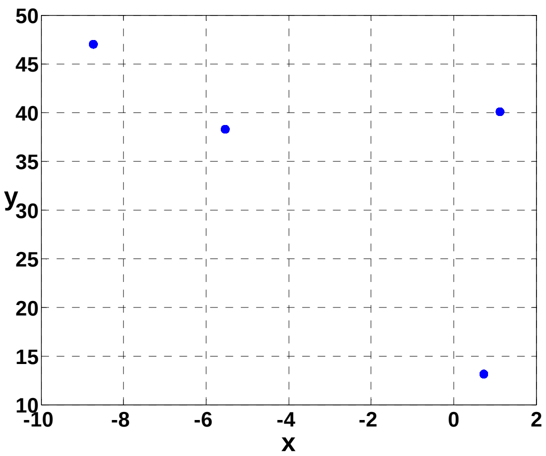


Fig. 7: Phase portrait of system (1) with $b = -1.647$: (a) and (b) projected on the $x - z$ plane

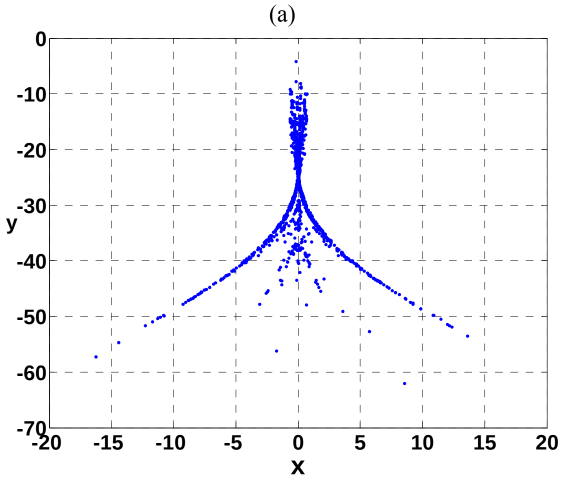
Two types of attractors, which are generated under two different initial conditions I_{00} and I_{01}, are shown in Figs. 7(a) and 7(b), respectively. As can be seen from Fig. 7, one is a periodic attractor and another is a chaotic attractor. Lyapunov exponents given in Table 2 can mathematically prove that the orbit starting from I_{00} is periodic orbit and the orbit starting from I_{01} is chaos.

Table 2
Lyapunov exponents corresponding to different initial conditions I_{00} and I_{01} with $b=-1.647$

Initial Value	λ_1	λ_2	λ_3	λ_4	Dynamics
I_{00}	0	-0.0925	-0.4532	-40.9027	Periodic orbit
I_{01}	0.2837	0	-0.3303	-41.3996	Chaos



(a)



(b)

Fig. 8: Poincaré maps on the crossing section $z=0$ with $b=-1.647$: (a) I_{00} , (b) I_{01}

The returned Poincaré maps also illustrate that one is a periodic attractor and another is a chaotic attractor using different initial conditions I_{00} and I_{01}, as shown in Fig. 8. The four spots in the returned Poincaré map of Fig. 8(a) implies the orbit from I_{00} is a period-2 orbit. However, Fig. 8 (b) shows a series of disordered dots, which confirm that this attractor is chaotic attractor.

Fig. 9 is the bifurcation diagram of variable y with respect to parameter b in the interval [-1.65, -1.64]. Similarly, it can be concluded that the dynamics of system (1) from I_{00} are the periodic, multi-periodic or chaotic with different b. While the dynamics from I_{01} is always chaotic. Therefore, it can be found that there is also coexistence of several kinds of attractors caused by different initial conditions.

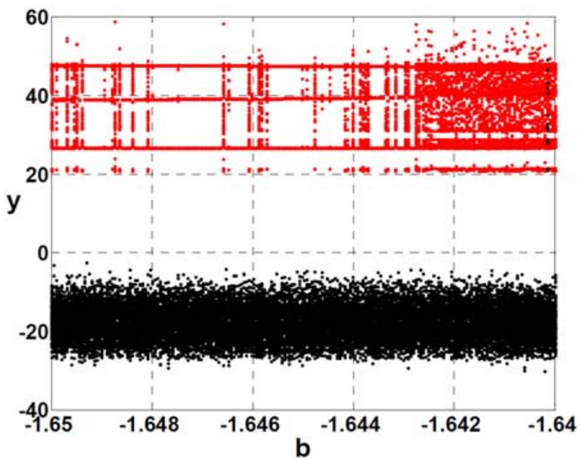


Fig. 9: Bifurcation diagram with respect to b in the interval [-1.65, -1.64] with I_{00} and I_{01}

5. Conclusion

In this paper, two kinds of initial-originated coexistent phenomena of attractor have been investigated for a new four-dimensional autonomous dynamical system. One is the coexistence of hyper-chaotic attractor and quasi-periodic attractor, and another one is the coexistence of chaotic attractor and periodic attractor. The time series curve, the phase portrait, the Poincaré map, the bifurcation diagram and the Lyaponov exponent techniques were conducted to convince the positive answer of the new phenomenon. As can be seen from the results of the proposed nonlinear system, it is incomplete for the bifurcation of nonlinear systems which only depends on the system parameters. The bifurcation may subject to the initial conditions for some systems. From this point of view, the initial-originated study provides a new and effective method to enrich dynamic behavior in some nonlinear dynamical systems.

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