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Analysis of a Fractional Order Nonlinear System Based on the Frequency Domain Approximation

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Abstract

The dynamics of nonlinear system is very complicated especially the fractional nonlinear system since they can be found in many areas of engineering and science. The dynamics of the Lorenz system with fractional derivatives is analysed based on the frequency approximation. For a given range of parameters where the non-fractional Lorenz system has periodic orbits, it is found that the fractional Lorenz system exhibits chaos and hyperchaos. A striking finding is that the fractional Lorenz system exhibits hyperchaos, although the total system order is less than 3, which is contrary to the well known conclusion that hyperchaos cannot occur in the integer-order continuous-time autonomous system of order less than 4. Finally, a reasonable explanation is offered for this complicated dynamical phenomenon.

Keywords

Nonlinear System; Fractional Order System; Chaos; Hyperchaos

Introduction

The subject of fractional calculus has gained considerable popularity and attention during the past three decades or so, mainly due to its widespread applications in the fields of science and engineering (Kilbas et al.,2006). Many systems are well known for being able to display fractional-order dynamics, such as viscoelastic systems (Bagley et al.,1991; Koeller, 1984), dielectric polarization (Sun et al.,1984), quantitative finance (Laskin, 2000; Jensen et al.,2003), and the quantum evolution of complex system (Kusnezov et al., 1999). The dynamics of the fractional nonlinear system has also been studied extensively during recent years. According to the Poincare-Bendixson theorem (for a review, see (Wiggin, 2003)), chaos cannot occur in continuous-time autonomous systems of order less than three, which is based on the usual integer order concepts. However, some fractional-order nonlinear systems display chaos when the total order is less than three (Hartley et al., 1995; Ahmad and Sprott, 2003; Grigorenko and Grigorenko, 2003). It is also widely accepted that hyperchaos cannot occur in an integer continuous-time autonomous system of order less than four. In the reference (Li and Chen, 2004), it shows that the fractional-order Rossler equation of an order as low as 3.8 can produce hyperchaos. Most of the researches on fractional chaos or hyperchaotic systems have focussed on the lowest total order of some well known nonlinear systems, which are chaos or hyperchaos when the system orders are integer, and haven't payed much attention to the rich dynamics caused by the fractional order.

In this paper, we investigate the dynamics of the fractional Lorenz system and find that the fractional Lorenz system exhibits chaos even though the normal Lorenz system is not chaotic for the same parameters. Moreover, we find that the fractional Lorenz system of the total order less than 3 displays hyperchaos.

Fractional Order Nonlinear System

The derivative theory of fractional order goes back to a question raised in the year 1695 by L' Hopital to G. W. Leibniz, in which the meaning of derivative of order of 1/2 is discussed. There are several definitions for fractional derivatives, of which the best known is probably the Riemann-Liouville formulation. An alternative is the Caputo definition whose properties are similar to that of the Riemann-Liouville (Kilbas et al.,2006). Caputo's derivative of order α and with the lower limit 0 can be viewed as a sort of regularization of the Riemann-Liouville derivative and is defined as

$$\frac{d^\alpha}{dt^\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, \qquad (1)$$

where $\Gamma(\cdot)$ is the gamma function and $n-1 \le \alpha < n$. The main advantage of the Caputo fractional derivative (1) is that there is a formal generalization of the integer derivative under Laplace transform (Kilbas et al.,2006). Considered all the initial conditions to be zero, the Laplace transform of (1) becomes the more expected and conforming form,

$$L\left(\frac{d^\alpha f(x)}{dt^\alpha}\right) = s^\alpha L(f(t)) \qquad (2)$$

Thus, the fractional integral operator of order α can be represented by the transfer function $F(s) = 1/s^\alpha$ in the frequency domain. The standard definition of the fractional differintegral does not allow direct implementation of the fractional operators in the timedomain. An efficient method to solve this problem is to approximate fractional operators by using standard integer order operators (Charef et al., 1992). In the following simulations, we will mainly use these approximations. The approximation of $1/s^{0.95}$ with an error about 1 dB is given in (Li and Chen, 2004) by

$$\frac{1}{s^{0.95}} \approx \frac{1.2831s^2 + 18.6004s + 2.0833}{s^3 + 18.4738s^2 + 2.6754s + 0.003}. \qquad (3)$$

Analysis of Fractional Order Lorenz System

A famous continuous-time autonomous system is the Lorenz system which shows rich dynamics. Grigorenk et al (Grigorenko and Grigorenko, 2003) has analyzed the fractional Lorenz system and pointed out that the fractional Lorenz system with a total order of less than 3 can exhibit chaos. Here, we analyze a simple fractional Lorenz system with only fractional derivative state given by

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= a(y-x) \\ \frac{dy}{dt} &= cx - xz - y, \\ \frac{dz}{dt} &= xy - bz \end{aligned} \qquad (4)$$

where a, b and c are parameters of the Lorenz system, and α is the fractional order. When α = 1, (4) is equivalent to the classical integer-order Lorenz equation. We assume that the time derivative is in Caputo sense. From the system equation (4), it is seen that the total order is 2.95 if α = 0.95. The phase plot of the chaotic attractor is shown in Fig. 1 with parameters α = 0.95, a = 10, b = 8/3 and c = 100. In this letter, we used the Wolf algorithm to calculate the Lyapunov exponents (Wolf et al., 1985). The largest Lyapunov exponent is $\lambda \approx 3.3$, but for the common Lorenz system $\lambda \approx 0$, which means that the fractional system is chaotic and the integer order Lorenz system is not chaotic. The result differs from that the decrement of the system total dimension induces the damping of the system effectively (Grigorenko and Grigorenko, 2003).

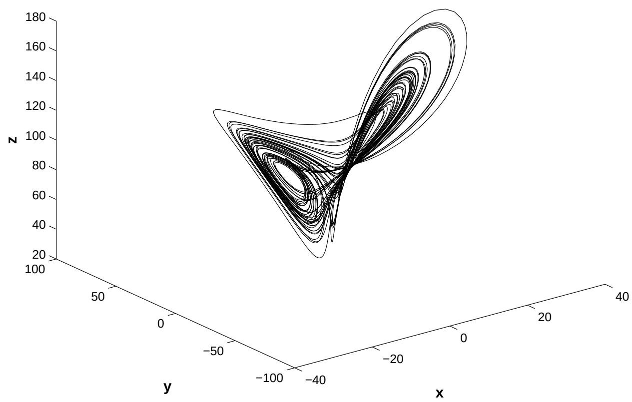

FIG 1. THE CHAOS DEMONSTRATION OF THE FRACTIONAL LORENZ SYSTEM WITH PARAMETERS α= 0.95; a = 10; b = 8/3; c = 100.

When the parameters α = 0.95, a = 10, b = 8/3 and c = 150, the two largest Lyapunov exponents of the fractional system are $\lambda_1 \approx 3.90$ and $\lambda_2 \approx 1.07$, respectively, which means that the system exhibits hyperchaos as shown in Fig. 2. However, the normal Lorenz system orbit with the same parameters is periodic as shown in Fig. 3. It is notable that the fractional nonlinear system exhibits hyperchaos with total order less than 3, which is contrary to the conventional knowledge about integer systems i.e. that the total order should be greater than 3 for hyperchaos.

As can be seen from Fig. 4, the integer order Lorenz system doesn't exhibit chaos, but the fractional order Lorenz system exhibits chaos and hyperchaos in a large range of parameter c. It means that although the system's total order decreases, system dynamics becomes more sensitive.

What is the fundamental reason for these dynamics, and is it really contrary to the conclusion that the order of a continuous autonomous system generating hyperchaos must be more than 4. It may be easily understood by going back to the definition of the fractional derivative operator and the approximation by using standard integer order operators in the frequency domain. According to the approximating method (Charef et al., 1992), the total number of singularities should be infinite if the fractional derivative is exactly represented by the integral transfer function. Both the orders of the numerator and denominator of the system transfer function should be infinite if the fractional system is represented strictly via using standard integer order operators. For example, (3) is the approximation of $1/s^{0.95}$ with a 1 dB error, the total order of the approximation system for (4) is 5, and the approximation system is given by

$$\begin{aligned}
\frac{dx}{dt} &= x_2 \\
\frac{dx_2}{dt} &= x_3 \\
\frac{dx_3}{dt} &= -18.4738x_3 - 2.6574x_2 - 0.003x \\
&\quad +1.2831a(cx_2 - x_2 z - cx + xz + y - x(xy - bz)) \\
&\quad +18.6004a(cx - xz - y) + 2.0833ay. \\
&\quad -a(1.2831x_3 + 18.6004x_2 + 2.0833x) \\
\frac{dy}{dt} &= cx - xz - y \\
\frac{dz}{dt} &= xy - bz
\end{aligned} \tag{5}$$

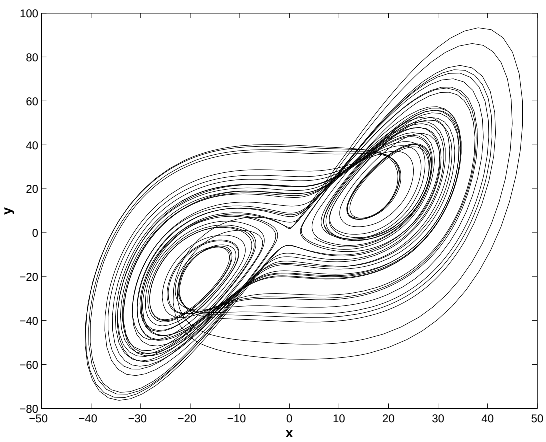

FIG. 2. THE HYPERCHAOS OF THE FRACTIONAL LORENZ SYSTEM WITH PARAMETERS α = 0.95, a = 10, b = 8/3 AND c = 150.

It is easy to find the effect on the nonlinear system if the system orderless than 2. Then, we investigate how the fractional order influences the famous Van der Pol equation given by

$$\begin{aligned}
\frac{d^{\beta}x_1}{dt^{\beta}} &= x_2 \\
\frac{dx_2}{dt} &= -x_1 + 0.2(1 - x_1^2)x_2
\end{aligned} \tag{6}$$

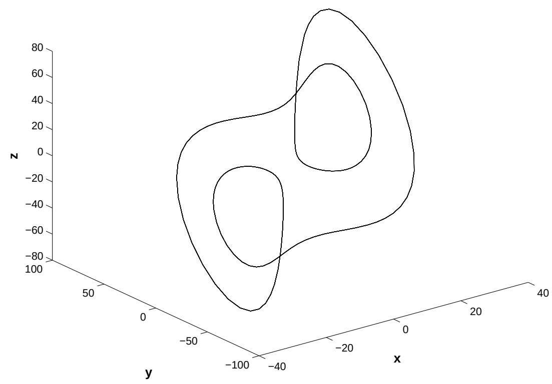

FIG. 3. THE PERIODIC ORBIT OF THE NORMAL LORENZ SYSTEM WITH α = 0.95, a = 10, b = 8/3 AND c = 150.

When β = 1, (6) is the normal Van der Pol equation.

When β= 0.9 and initial condition $(x_1(0), x_x(0)) = (-2, 3)$, the fractional Van der Pol system phase portrait is shown in Fig. 5. It is known that this phase portrait is impossible in the normal Van der Pol equation(that is β = 1) because the trajectory appears to intersect itself in the two coordinate plane. Each point in the plane is supposed to have a unique flow direction according to the Poincare-Bendixson theorem. However, a non-unique flow direction becomes possible for fractional Van der Pol equation. A quite interesting question arised to readers, namely how it can exhibit chaos or hyperchaos in the fractional nonlinear system with total order less than two.

A possible explanation is that some basic characteristics of the integer Lorenz system have been modified. The Lorenz system is symmetrical about the x-y plane when α = 1, but this is not the case for (1) and (4). Moreover, the equilibria of the fractional Lorenz system differs from that of the normal Lorenz system according to (1) and (4). Changes in the symmetry of the system orbit and the equilibria significantly influence on the distribution and linearized characteristics of equilibria as well as the manifolds of the system.

Conclusions

In this work, we analyzed the fractional Lorenz equation. It is found that although both the normal Lorenz system and fractional Lorenz system have a similar two swings shape, the fractional Lorenz system exhibits rich and interesting dynamics which is very different from the result of the standard integer-order dynamics. It is found that the fractional Lorenz system exhibits chaos and hyperchaos with total order less than 3. What is the fundamental reason of the different dynamics? It maybe come from the fractional derivative nonlocal character which is different from the integer dynamics. It is important to systematically develop some methods for the analysis of fractional system. The rich dynamics caused by fractional order are also important topics for future studies in the areas of science and engineering.

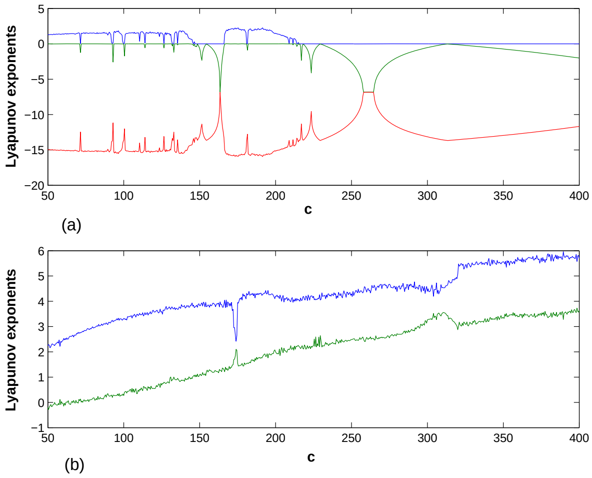

FIG. 4. THE SPECTRUMS OF THE LYAPUNOV EXPONENTS FOR LORENZ SYSTEM WITH a = 10, b = 8/3 (a)THE NORMAL LORENZ SYSTEM (b)THE FRACTIONAL LORENZ SYSTEM WITH $\alpha = 0.95$, WHERE ONLY SHOWS THE TWO LARGEST LYAPUNOVE EXPONENTS

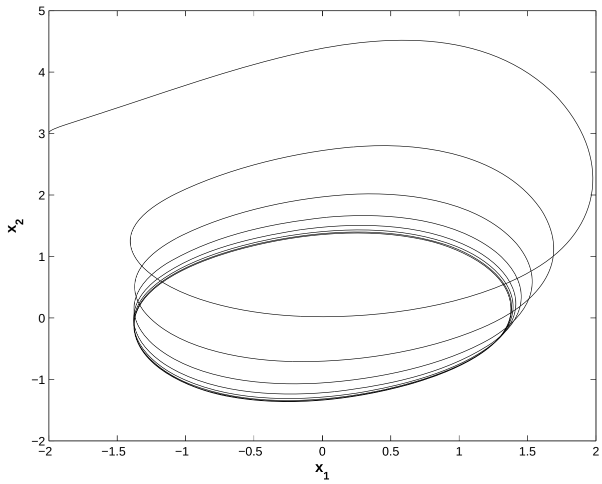

FIG 5. PHASE PLOT WHEN $\beta = 0.9$ FOR VAN DER POL SYSTEM.

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