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EQUIVARIANT COHOMOLOGY OPERATIONS AND THE NON - SIMPLY - CONNECTED SURGERY OBSTRUCTIONS

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EQUIVARIANT COHOMOLOGY OPERATIONS AND THE NON-SIMPLY-CONNECTED SURGERY OBSTRUCTION

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PROEFSCHRIFT

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Aan mijn ouders

Dit proefschrift werd op een voortreffelijke wijze getypt
door Willy van de Sluis-Gerritsen
en van diagrammen voorzien door J.-Baptist Clauwens.

CONTENTS

A large part of geometry is concerned with the study of differential manifolds. Hence one of the most fundamental problems for the geometer is to classify differential manifolds up to isomorphism. To make things more manageable one restricts attention to manifolds with the homotopy type of a given manifold N of dimension n.

The classical approach to this problem is as follows: one considers maps $f : M \to N$ with some extra - so-called "normal" - structure, which is present in the case where f is a homotopy-equivalence. One divides these maps into equivalence-classes under bordism, which is a homotopy problem. Then one tries to do surgery to modify a given map, inside its bordism class, in order to get a homotopy-equivalence.

This process meets an obstruction s(f), which takes its values in the Wall group $L_n(G)$, functorially associated to the fundamental group G of N. The value of s(f) can be read off after doing low-dimensional surgery on M, in order to change f into a highly-connected map, viz. as the stable class of a quadratic form over the group ring, Z[G], defined by taking intersections and self-intersections of spheres in M (in the case that n is even).

A natural question to ask then is: how does this manifestly important quantity s(f) behave under various constructions which can be performed on the map f? One of the most fundamental operations which can be imagined is to take the cartesian product with some fixed manifold V.

In studying this situation one is obstructed by the fact that the map $f \times id : M \times V \to N \times V$ is in general not highly-connected if f is. Hence one has to modify $M \times V$ again in order to be able to read off $s(f \times id)$. In general it is not at all clear how the result of this depends on the original data. For this reason no general formula is known which expresses $s(f \times id)$ in terms of $s(f)$ and (bordism-) invariants of V, except in the simply-connected case $G = 1$.

We take a closer look at $L_{2q}(G)$. An element of it is a class of quadratic forms, and from these one is able to construct some algebraic invariants. Most well-known are the invariants of the type "signature". They in fact only depend on the induced quadratic form over $\mathbb{R}[G]$. In this case one has the advantage that the quadratic form does not in fact depend on the "normal structure". For this reason this case has been extensively studied.

Another type of invariant, which Arf was the first to consider, appear when one reduces coefficients to $\mathbb{F}_2$ instead of $\mathbb{R}$. One is led to the more subtle situation of quadratic forms in characteristic two. Furthermore the ring $\mathbb{F}_2[G]$ is not necessarily semi-simple, as is the case with $\mathbb{R}[G]$; and in applying the algebra to the geometrical situation one no longer has the advantage that the "normal structure" is immaterial.

This is the situation we study in this thesis. We try to solve the problems involved in the "Arf part" of $s(f)$ by giving a definition of it which does not presuppose the map f to be highly-connected. This is a problem raised by L. Shaneson in [12]. We do this for the case G is finite; however it gives information for infinite G by applying it to finite quotients of G.

To accomplish our goal we generalize the technique used by W. Browder [3] in the simply-connected case, which is based on a construction of a quadratic form using algebraic topology instead of the geometric technique of self-intersections. To this end we have to study equivariant algebraic topology.

The overall organization of the material is as follows. Chapter I provides the necessary equivariant algebraic topology. Chapter II contains the construction of our quadratic form and the proof of some of its properties. Chapter III contains the proof that this form determines the "Arf part" of the surgery obstruction s(f).

We now present a detailed description of the contents. In chapter I, §1-5 we recall the construction and elementary properties of the equivariant homology theory and cohomology theory, due to G.E. Bredon [1] and Th. Bröcker [2], and generalize these mildly.

In §6 we generalize Steenrod's cohomology operations to the equivariant case and show that the generalized operations have properties similar to those of the classical operations, and in addition possess some properties which have no classical counterpart (6.4. and 6.5.).

In §7 we consider functional cohomology operations in the equivariant case, prove a general identity (7.2.), and calculate the result in a non-classical example (7.3.).

In §8 we briefly recall the equivariant obstruction theory of G.E. Bredon and apply it to deduce a property of the functional operation associated to the equivariant Steenrod operation.

In §9 we prove a further property of this functional operation, directly, by constructing some intricate homotopies.

In §10 we generalize Cech cohomology to the equivariant case, to complete the picture formed by Bredon's cellular and Bröcker's singular theories; in §11 we use this to state the properties of the Poincaré-duality and Thom-isomorphism in an equivariant setting.

In the first section of chapter II we use among other things, the immersion theory of Hirsch [7] to construct from a "normal map" $f : M \to N$ a G-equivariant map $c : S^k\tilde{N} \to S^k\tilde{M}$ defined on some suspension of the universal covering space of N.

In §2 we make a detailed comparison between the equivariant cohomology of c and the homology of f. In §3 we consider a functional Steenrod operation constructed from c and we show that the properties of this operation proved in chapter I imply that it gives rise to the polar part of some quadratic form. This quadratic form is defined on the equivariant cohomology of the map c with coefficients in the semi-simple ring B, constructed from $\mathbb{F}_2[G]$ by dividing out the Jacobson radical; in so doing we avoid algebraic difficulties.

In §4 we prove that this form behaves in a natural way with respect to inclusions of normal maps and we show that its stable class is an invariant of bordism.

In the second section of chapter III we construct a map with prescribed surgery obstruction in such a careful way that we are able to calculate our quadratic form for it in III.3, only using the naturality and bordism properties. It is shown that in the case at hand, we get precisely the reduction of the surgery quadratic form to B-coefficients.

According to the outline of the proof given in III.1 this implies that that the same is true for any normal map.

CHAPTER I

EQUIVARIANT HOMOLOGY AND COHOMOLOGY THEORY

§1. Introduction of some categories.

1.1

In this section we define some of the basic notions we will work with. The material in the first five sections of this chapter not referring to pairs of groups has been taken from [1] and [2].
In this chapter G will always denote a finite group.

Let $\mathcal{O}_G$ denote the "orbit category", the category with

objects : sets with a transitive action of G and a basepoint, and

morphisms : equivariant maps, not necessarily basepoint-preserving.

This category is equivalent to the category with

objects : subgroups of G, and

morphisms : equivariant maps of the corresponding sets G/K.

Let $\overline{\mathcal{O}}_G$ denote the category with

objects : sets with a transitive action of G, and

morphisms : equivariant maps.

Let $\mathcal{M}_G$ denote the category with

objects : sets with a G-action, and

morphisms : equivariant maps.

Finally denote by $'\mathcal{M}_G$ the category with

objects : sets S with a transitive action of G together with a section s of the projection $S \to S/G$, and

morphisms : equivariant maps, not necessarily preserving this section.

1.2.

Let $\mathcal{A}$ denote the category of abelian groups and homomorphisms of abelian groups. We denote by $\mathcal{C}^G$ resp. $\mathcal{C}_G$ the categroy with

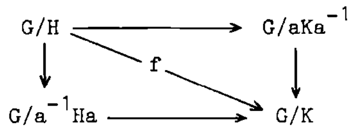
objects : covariant resp. contravariant functors $\mathcal{O}_G \to \mathcal{A}$, and

morphisms : natural transformations of functors.

We will call an element of $\mathcal{C}^G$ resp. $\mathcal{C}_G$ a <u>covariant</u> resp. <u>contravariant coefficient system</u>.

Let $f : G/H \to G/K$ be a morphism in $\mathcal{O}_G$. Say $f(H) = aK$, then $h\ aK = h\ f(H) = f(hH) = f(H) = aK$ hence $a^{-1}h\ a \in K$ for $h \in H$.

This means that f factorizes into a translation and a projection:



In particular $Mor(G/H,G/H) = N(H)/H$, where $N(H)$ denotes the normalizer of H in G. Hence a coefficient system yields $N(H)/H$-modules for every subgroup H of G;

$\mathcal{C}^G$ corresponds to right modules and $\mathcal{C}_G$ corresponds to left modules.

1.3.

A functor $M : \overline{\mathcal{O}}_G \to \mathcal{A}$ of course yields a functor $\mathcal{O}_G \to \mathcal{A}$.

On the other hand let M be a functor $\mathcal{O}_G \to \mathcal{A}$. For each $s,t \in S \in \overline{\mathcal{O}}_G$ there is a $\mathcal{O}_G$ morphism $f_{st} : (S,s) \to (S,t)$ corresponding to the identity mapping of S, hence there is $M(f_{st}) : M(S,s) \xrightarrow{\sim} M(S,t)$.

Now we take the projective limit: $M(S) = \{(x) \in \prod_{s\in S} M(S,s) \mid M(f_{st})x_s = x_t\}$.

That yields a functor $\overline{\mathcal{O}}_G \to \mathcal{A}$.

An element of $\mathcal{M}_G$ resp. $'\mathcal{M}_G$ splits canonically as a sum of orbits lying in $\overline{\mathcal{Q}}_G$ resp. $\mathcal{Q}_G$. Hence a functor $M : \mathcal{Q}_G \to A$ yields a functor $\Sigma M : \mathcal{M}_G \to A$ with the property that $\Sigma M(\coprod_\Lambda X_\lambda) = \bigoplus_\Lambda (\Sigma M(X_\lambda))$, where $\coprod$ denotes disjoint union. In particular we have, for invariant subsets T_1 and T_2 of S:

$M(T_1 \cap T_2) = \Sigma M(T_1) \cap \Sigma M(T_2)$ and

$M(T_1 \cup T_2) = \Sigma M(T_1) + \Sigma M(T_2)$ as subgroups of $\Sigma M(S)$.

1.4.

We can prove things about additive functors N from $\mathcal{M}_G$ to an additive category by proving them for a certain universal category, through which such an N factorizes.

Let $F\mathcal{M}_G$ be the category whose objects are : the free abelian groups FS generated by elements S of $\mathcal{M}_G$, and whose morphisms from FS to FT are: the elements of the subgroup of the group $\mathrm{Hom}_G(FS,FT)$ of equivariant homomorphisms which is generated by $\mathcal{M}_G$-morphisms. $F\mathcal{M}_G$ is an additive category.

There exists an obvious functor F from $\mathcal{M}_G$ to $F\mathcal{M}_G$ and for every additive functor N there is a unique functor $N\otimes$ such that $N = (N\otimes) \circ F$. We define $N\otimes$ on the object FS as N(S); to define $N\otimes$ on morphisms it is sufficient to consider the case that S and T are orbits; it is then easy to check that $N\otimes$ is well-defined by assigning $\sum_{i\in I} N(f_i)$ to $\sum_{i\in I} f_i$, where the f_i are $\mathcal{M}_G$-morphisms from S to T.

1.5.

More generally we can consider the following situation:

Let $\phi : H \to G$ be a fixed homomorphism of groups and define $\underset{\sim}{Q}_\phi$ to be the category consisting of the quotients G/K and H/L and equivariant maps; H-equivariant from H/K to H/L or from H/L to G/K; G-equivariant from G/L to G/K; none from G/K to H/L.

An arbitrary morphism is a composition of a translation, a projection and possibly a map induced by ϕ. Hence a ϕ-coefficient system (i.e. a functor $\underset{\sim}{Q}_\phi \to A$) is determined by
a G-coefficient system N^G,
an H-coefficient system N^H,
and homomorphisms $N^H(H/K) \to N^G(G/\phi K)$ for each $K \subset H$.
For each $S \in M_H$ we get $N^H(S)$ and for each $T \in M_G$ we get $N^G(T)$, and moreover a homomorphism $N^G(T) \to N^H(T)$.

Let M_ϕ be the category with
objects : pairs (S,T) such that $S \in M_H$ and $T \in M_G$ and T is invariant in S.
The morphisms are : equivariant maps of pairs.

Again an arbitrary object can be split as a sum of orbits of H of the form $(S,\emptyset)$ and of orbits of G of the form (T,T); hence an additive functor N on M_ϕ is determined by the $N^H(S) = N(S,\emptyset)$ and $N^G(T) = N(T,T)$ for $S \in M_H$ and $T \in M_G$; in fact:

$$N(S,T) = \frac{N^H(S) \quad \oplus \quad N^G(T)}{\text{diagonal image of } N^H(T)} \quad \text{canonically.}$$

Again there exists a universal additive category FM_ϕ and a functor $F : M \to FM_\phi$, through which all such N factorize. Its objects are pairs (FS,FT).

Applying N to the equations: $(T,\emptyset) \amalg (S-T,\emptyset) = (S,\emptyset)$ and $(T,T) \amalg (S-T,\emptyset) = (S,T)$ we deduce that the following sequence is (split-)exact

$$0 \to N^G(T) \to N(S,T) \to N^H(S)/N^H(T) \to 0.$$

We can also see this by applying N to the exact sequence

$$0 \to (FT,FT) \to (FS,FT) \to (FS,0)/(FT,0) \to 0.$$

In the case that H = G it is interesting to consider only those N for which $N^G = 0$, universal for them is the category with objects the groups FS/FT.

1.6.

Let M be a right G-module, that defines $\overline{M} \in C^G$ by

$\overline{M}(G/H) = M$ mod span $\{mh-m \mid m \in M, h \in H\}$ and

$\overline{M}(i)$ = the canonical projection $\overline{M}(G/H) \to \overline{M}(G/K)$ for i G/H → G/K the map xH → xK, $\overline{M}(i)$ = the map $\overline{M}(G/H) \to \overline{M}(G/g^{-1}Hg)$ mapping x to xg for i the map $G/H \to G/g^{-1}Hg$ mapping xH to $xHg = xg\, g^{-1}Hg$.

Let M be a left G-module, that defines $\underline{M} \in C_G$ by

$\underline{M}(G/H) = \{m \in M \mid hm = m$ every $h \in H\}$ and

$\underline{M}(i)$ = the canonical inclusion $\underline{M}(G/K) \to \underline{M}(G/H)$ for i : G/H → G/K the map xH → xK, $\underline{M}(i)$ = the map $\underline{M}(G/g^{-1}Hg) \to \underline{M}(G/H)$ mapping m to gm for i the map $G/H \to G/g^{-1}Hg$ mapping xH to xHg.

Note that, although left- and right modules over G are in bijection by $gm = mg^{-1}$, there is no such relation between C_G and C^G.

§2. Singular homology theory.

2.1.

Let Top_G be the categroy with <u>objects</u> topological spaces with a continuous action of G, and <u>morphisms</u> equivariant continuous maps.

We will write simply *Top* in case $G = 1$. Similarly, given a homomorphism $\phi : H \to G$ of groups, there is a category Top_ϕ with

<u>objects</u> : pairs (X,Y) such that $X \in Top_H$ and $Y \in Top_G$ an H-invariant subset, and <u>morphisms</u> : equivariant continuous maps of pairs.

There is a functor $S_n^G : Top_G \to M_G$ defined by $S_n^G(X) = \{\text{continuous maps} : \Delta^n \to X\}$, Δ^n being the standard n-simplex. $S_n^G(f)$ maps σ to $f \circ \sigma$. In case $G = 1$ we will write S_n.
Similarly, there is a functor $Top_\phi \to M_\phi$ defined by

$$S_n^\phi(X,Y) = (S_n^H(X), S_n^G(Y)).$$

Given $M \in C^G$ and $X \in Top_G$ define $M_n(X)$ by $M_n(X) = \Sigma M(S_n^G(X)) = M \otimes FS_n^G(X)$. This is called the n^{th} singular chain group of X with coefficients in M. If (X,Y) is a pair in Top_G and $M \in C^G$ we define $M_n(X,Y) = \Sigma MS_n^G(X,Y) = M \otimes FS_n^G(X,Y)$; then the sequence

$$0 \to M_n(Y) \to M_n(X) \to M_n(X,Y) \to 0 \text{ is exact.}$$

Similarly for $(X,Y) \in Top_\phi$ and $N \in C^\phi$, $N_n(X,Y) = N \otimes FS_n^\phi(X,Y)$ is defined and the sequence

$$0 \to N_n^G(Y) \to N_n(X,Y) \to N_n^H(X,Y) \to 0 \text{ is exact.}$$

From now on we will write C_n^G and $\dot{C}_n$ for FS_n^G resp. FS_n; or C_n if $G = 1$.

2.2.

To deduce properties of these chain groups from those of the classical chain groups we use the following

THEOREM [?].

Let $Z \in Top_G$ with trivial G-action, then every natural transformation α $C_n(X) \to C_m(Z \times X)$ yields a natural transformation

α^G $C_n^G(X) \to C_m^G(Z \times X)$ for $X \in Top_G$, and similarly

α^ϕ $C_n^\phi(X,Y) \to C_m^\phi(Z \times X, Z \times Y)$ for $(X,Y) \in Top_\phi$.

P r o o f Let $\jmath \in S_n(\Delta^n)$ be the identity mapping, then $\alpha(\jmath) = \Sigma\, n_i \alpha_i$ for some $\alpha_i \in S_m(Z \times \Delta^n)$ That yields natural transformations $S_n(X) \to S_m(Z \times X)$ by $\sigma \to (id_Z \times \sigma)o\alpha_i$, for $X \in Top_G$ these are equivariant because they are natural, hence they yield natural transformations α_i^G $S_n^G(X) \to S_m^G(Z \times X)$. Then $\alpha^G = \Sigma\, n_i \alpha_i^G$ $FS_n^G(X) \to FS_m^G(Z \times X)$ is the transformation of the theorem.

Furthermore $\alpha_i^\phi = (\alpha_i^H, \alpha_i^G)$ $S_n^\phi(X,Y) = (S_n^H(X), S_n^G(Y)) \to S_m^\phi(Z \times X, Z \times Y)$ is natural hence yields α^ϕ.

Q E D

In particular (take Z = point) the classical boundary operator d_n $C_n(X) \to C_{n-1}(X)$ yields d_n^G $C_n^G(X) \to C_{n-1}^G(X)$ and hence $M \otimes d_n^G$ $M \otimes C_n^G(X) \to M \otimes C_{n-1}^G(X)$, that is $M_n(X) \to M_{n-1}(X)$

DEFINITION. $H_*^G(X,M)$ is the homology of this complex $(M_*(X), M \otimes d_*^G)$.

Similarly one has d_n and hence $N \otimes d_n$ $N_n(X,Y) \to N_{n-1}(X,Y)$ and $H_n^\phi(X,Y,N)$ for $(X,Y) \in Top_\phi$, $N \in C^\phi$

The properties of this homology theory are stated in the following

THEOREM. Let $M \in C^G$.

1) $H_n^G(f,M)$ is a functor on (pairs in) Top_G; from now on we will denote

$H^G(f;M)$ by f_*.

2) If f_0 and f_1 are equivariantly homotopic then $(f_0)_* = (f_1)_*$.

3) For a pair (X,Y) in Top_G there is a long exact sequence:

$$\ldots \to H_n^G(Y;M) \to H_n^G(X;M) \to H_n^G(X,Y;M) \to H_{n-1}^G(Y;M) \to \ldots$$

4) On $\mathcal{Q}_G$ $H_n^G(\ ;M)$ is equal to M for $n=0$ and to 0 for $n\neq 0$.

5) Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an equivariant covering of $X \in Top_G$ such that $X = \bigcup_{i\in I} \operatorname{int} U_i$. Denote by $FS(\mathcal{U})$ the subcomplex of $FS(X)$ generated by the $\sigma : \Delta^n \to X$ such that $\sigma(\Delta^n) \subset U_i$ for some i. Then the inclusion $FS(\mathcal{U}) \to FS(X)$ is a chain-equivalence.

In particular: if (X,Y) is a pair in Top_G and if $W \subset Y$ is G-invariant such that $\overline{W} \subset \operatorname{int} Y$, then the inclusion $(X-W,Y-W) \subset (X,Y)$ induces isomorphisms in $H^G(\ ;M)$.

Similar statements are true for $H^\phi(N \in C^\phi)$:

1) $H_n^\phi(\ ;N)$ is a functor on Top_ϕ.

2) If f_0 and f_1 are homotopic through Top_ϕ morphisms then $(f_0)_* = (f_1)_*$.

3) For $(X,Y) \in Top_\phi$ there is a long exact sequence:

$$\ldots \to H_n^G(Y;N^G) \to H_n^\phi(X,Y;N) \to H_n^H(X,Y;N^H) \to \ldots$$

4) $H_n^\phi(S,T;N) = 0$ for $n\neq 0$ and for discrete S and T,

$H_0^\phi(S,\emptyset;N) = H_0^H(S;N^H) = N^H(S)$.

$H_0^\phi(T,T;N) = H_0^G(T;N^G) = N^G(T)$ and

$H_0^\phi(\ ;N)$ on the inclusion $(T,\emptyset) \subset (T,T)$ is equal to the map $N^H(T) \to N^G(T)$ given by N.

5) If $\mathcal{U} = \{U_i \mid i \in I\}$ is an H-equivariant covering of X such that $\mathcal{U} \cap Y = \{U_i \cap Y \mid i \in I\}$ is a G-equivariant covering of Y then the inclusion $FS(\mathcal{U},\mathcal{U} \cap Y) \to FS(X,Y)$ is a chain-equivalence.

In particular if $(X,Y) \in Top_\phi$ and if $W \subset Y$ is G-invariant such that $\overline{W} \subset \operatorname{int} Y$ then the canonical map $H_n^\phi(X-W,Y-W,N) \to H_n^\phi(X,Y;N)$ is an isomorphism for each n.

For the proof of the first half of this theorem we refer to [2], the proof of the second half is quite similar. In fact it is a direct application of the first theorem in this subsection.

2.3.

We shall have occasion to benefit from the following notational convention. For invariant parts X_1 and X_2 of $X \in Top_G$ we define $S_n^G\{X_1,X_2\}$ to be $S_n^G(X_1) \cup S_n^G(X_2)$, which is contained in, but in general smaller then $S_n^G(X_1 \cup X_2)$. Applying the functors F, $M \otimes$ and the homology functor we find $H_n^G(\{X_1,X_2\},M)$. Similarly for invariant parts X_1 and X_2 of $X \in Top_H$ and for $Y_1,Y_2 \in Top_G$ invariant parts of X_1 resp. X_2 there is defined $H_n^\phi(\{X_1,X_2\},\{Y_1,Y_2\};N)$.

Part 5 of the last theorem states that the canonical map $H_n^G(\{X_1,X_2\};M) \to H_n^G(X_1 \cup X_2,M)$ is an isomorphism, for open X_1 and X_2. The same can be said for relative groups, by using the five-lemma. In particular:

$$H_n^G(X_1,X_1 \cap X_2;M) \cong H_n^G(\{X_1,X_2\},X_2,M) \cong H_n^G(X_1 \cup X_2,X_2;M),$$

the first equality by using the Noether-isomorphism on chain level.

§3. Singular cohomology theory.

3.1.

We define a contravariant functor $L : M_G \to C_G$ by $L(S)(G/H) = F(S^H)$, the free abelian group generated by $S^H = \{s \in S \mid hs = s \text{ every } h \in H\}$, with the obvious values on morphisms.

According to [1] C_G is an abelian category, in particular there is defined an abelian group $\mathrm{Traf}(L(S),M)$ for any $M \in C_G$.
We define the <u>n^{th} singular cochain group of $X \in Top_G$</u> with values in $M \in C_G$ to be $C^n_G(X,M) = \mathrm{Traf}(L \circ S^G_n(X),M)$.

Similarly there is a functor L on $\mathcal{U}_\phi$ such that
$L(S,T)(G/K) = F(T^K)$ and $L(S,T)(H/K) = F(S^K)$ etc.,
hence for $N \in C_\phi$ there is a group $\mathrm{Traf}(L(S,T),N)$, and we can define $C^n_\phi(X,Y,N) = \mathrm{Traf}(L \circ S^\phi_n(X,Y),N)$. For $\phi = id$ or $Y = \emptyset$ or $Y = X$ this reduces to the former definition.

From now on we denote $L \circ S^G_n$ by L^G_n and $L \circ S^\phi_n$ by L^ϕ_n.

3.2.

<u>LEMMA</u>. [1].

$L(S)$ is always a projective object of C_G, and $L(S,T)$ one of C_ϕ.

<u>P r o o f</u>. We note that L is additive, hence it is sufficient to check the statement for an orbit.

i) given the situation

$$\begin{array}{ccc} & & IS \\ & \swarrow & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

where $S = G/K$ it is sufficient to find a lifting of

$$\begin{array}{ccc} & & F(S^K) \\ & \swarrow & \downarrow \\ A(G/K) & \twoheadrightarrow & B(G/K) \end{array}$$

ii) an orbit of (S,T) is of the form $(S,\emptyset)$ or (T,T), hence the problem reduces to situation (i) for H or G.

Q.E.D.

This means that the exactness of

$$0 \to L(T) \to L(S) \to L(S)/L(T) \to 0$$

implies by application of Hom(;M) the exactness of

$$0 \to C^n_G(X,Y;M) \to C^n_G(X;M) \to C^n_G(Y;M) \to 0.$$

Similarly one deduces from the exactness of

$$0 \to L(T,T) \to L(S,T) \to L(S,\emptyset)/L(T,\emptyset) \to 0$$

the exactness of

$$0 \to C^n_H(X,Y;N^H) \to C^n_\phi(X,Y;N) \to C^n_G(Y;N^G) \to 0.$$

Furthermore we remark that the L(S) constitute "sufficiently many" projective objects for the category C_G [1]. This remark will make the construction of Eilenberg-Maclane spaces for Top_G possible.

3.3.

In analogy with the first theorem of I.2.2. we have:

<u>THEOREM</u>. [2].

Let $Z \in Top_G$ with trivial G-action; then every natural transformation $\alpha : C_n(X) \to C_m(Z \times X)$ yields a natural transformation

$\alpha^G : L^G_n(X) \to L^G_m(Z \times X)$ for $X \in Top_G$, and similarly

$\alpha^\phi : L^\phi_n(X,Y) \to L^\phi_m(Z \times X, Z \times Y)$ for $(X,Y) \in Top_\phi$.

<u>P r o o f</u>. Obvious. Note that $S_n(X^K) = S_n(X)^K$.

From this it follows at once that the $C^n_G(X;M)$ form a cochain complex and that the thus defined functor $H^n_G(\ ;M)$ on Top_G has properties similar to those listed in the second theorem of I.2.2. for $H^G_n(\ ;M)$. The same can be said for $H^n_\phi(\ ,\ ;N)$ on Top_ϕ.

§4. The cup-product.

4.1.

Let G_1 and G_2 be groups and let $M_1 : O_{G_1} \to A$ and $M_2 : O_{G_2} \to A$ be co- or contravariant coefficient systems; then we find a coefficient system $M_1 \hat{\otimes} M_2$ for $G_1 \times G_2$ by the map $O_{G_1 \times G_2} \to O_{G_1} \times O_{G_2} \to A$ mapping $(G_1 \times G_2)/H$ through $G_1/p_1H \times G_2/p_2H$ to $M_1(G_1/p_1H) \otimes M_2(G_2/p_2H)$.

From $S_1 \in M_{G_1}$ and $S_2 \in M_{G_2}$ we can form $S_1 \times S_2 \in M_{G_1 \times G_2}$ and $L(S_1 \times S_2)(G/H) = F(S_1 \times S_2)^H = F(S_1^{p_1H} \times S_2^{p_2H}) = FS_1^{p_1H} \otimes FS_2^{p_2H} =$ $= LS_1(G_1/p_1H) \otimes LS_2(G_2/p_2H) = (LS_1 \hat{\otimes} LS_2)(G/H)$ i.e. $L(S_1 \times S_2)$ can be identified with $LS_1 \hat{\otimes} LS_2$.

In particular for $S_1 = S_n^{G_1}(X_1)$ and $S_2 = S_n^{G_2}(X_1)$, where $X_1 \in Top_{G_1}$ and $X_2 \in Top_{G_2}$, so that we can identify $S_1 \times S_2$ with $S_n^{G_1 \times G_2}(X_1 \times X_2)$, we get: $L_n^{G_1 \times G_2}(X_1 \times X_2) = LS_n^{G_1 \times G_2}(X_1 \times X_2) = LS_n^{G_1}(X_1) \hat{\otimes} LS_n^{G_2}(X_2) =$ $L_n^{G_1}(X_1) \hat{\otimes} L_n^{G_2}(X_2)$.

In analogy with 3.3. there is the

<u>THEOREM</u>. A natural transformation of functors on pairs (X_1,X_2) in Top

$\alpha : C_n(X_1) \otimes C_m(X_2) \to C_p(X_1) \otimes C_q(X_2)$

induces a natural transformation of functors on pairs (X_1,X_2) where $X_1 \in Top_{G_1}$ and $X_2 \in Top_{G_2}$:

$\alpha : L_n^{G_1}(X_1) \hat{\otimes} L_m^{G_2}(X_2) \to L_p^{G_1}(X_1) \hat{\otimes} L_q^{G_2}(X_2)$.

<u>P r o o f</u>: as in 3.3.

Firstly this tells us that $L_*^{G_1}(X_1) \hat{\otimes} L_*^{G_2}(X_2)$ has the structure of a chain complex. Secondly we can apply it to the classical Eilenberg-Zilber chain map (then $n = m = p+q$), its homology inverse, and the two homotopies of the two compositions with the identity. Hence there is a generalized Eilenber-Zilber map and it still is a chain-equivalence.

4.2.

We can use the foregoing to construct a crossproduct in cohomology:

$$H^p_{G_1}(X_1;M_1) \otimes H^q_{G_2}(X_2;M_2)$$
$$H(\mathrm{Hom}(L_p^{G_1}(X_1);M_1) \otimes \mathrm{Hom}(L_q^{G_2}(X_2);M_2))$$
$$H(\mathrm{Hom}(L_p^{G_1}(X_1) \hat{\otimes} L_q^{G_2}(X_2);M_1 \hat{\otimes} M_2))$$
$$H(\mathrm{Hom}(L_n^{G_1\times G_2}(X_1\times X_2);M_1 \hat{\otimes} M_2)) = H^n_{G_1\times G_2}(X_1\times X_2;\ M_1 \hat{\otimes} M_2),$$

where the third map is induced from the aforementioned E.Z. chain map.

In case $G_1 = G_2 = G$ say, we can view G as the diagonal subgroup of $G \times G$, and by restricting to G we get a map to $H^n_G(X_1 \times X_2;M_1 \hat{\otimes} M_2)$. Finally in case $X_1 = X_2 = X$ say, we can apply the cohomology of the diagonal map $X \to X \times X$, and we get the cup-product map into $H^n_G(X;M_1 \hat{\otimes} M_2)$.

This cup-product has the classical properties of associativity and commutativity, for instance:

$$y \cup x = (-1)^{pq}T(x \cup y),$$

where T is the coefficient map $M_1 \hat{\otimes} M_2 \to M_2 \hat{\otimes} M_1$ mapping $a \otimes b$ to $b \otimes a$.

We can construct a map as above for the category Top_ϕ; however it has its values in an equivariant cohomology group involving a quadruple of groups. However, applying the group diagonal we find a crossproduct:

$$H^p_\phi(X_1,Y_1;M_1) \otimes H^q_\phi(X_2,Y_2;M_2) \to H^{p+q}_\phi(X_1 \times X_2,\ Y_1 \times Y_2;\ M_1 \hat{\otimes} M_2),$$

and in case $X_1 = X_2 = X$ applying the diagonal of X we get a cup-product:

$$H^p_\phi(X,Y;M_1) \otimes H^q_\phi(X,Y;M_2) \to H^{p+q}_\phi(X,Y;M_1 \hat{\otimes} M_2).$$

§5. The cap-product.

5.1.

Let $M<G_1,G_2>$ be the category with

objects: pairs (S_1,S_2), where $S_1 \in M_{G_1}$ and $S_2 \in M_{G_2}$, and with

morphisms: pairs (f_1,f_2), where f_1 is a M_{G_1} morphism and f_2 is a M_{G_2} morphism.

Let $FM<G_1,G_2>$ be the category with

objects: free abelian groups $F(S_1 \times S_2) = FS_1 \otimes FS_2$ where $(S_1,S_2) \in M<G_1,G_2>$ and

morphisms: homomorphisms lying in the subgroup generated by the homomorphisms induced by $M<G_1,G_2>$ morphisms.

We can view a $M<G_1,G_2>$ morphism as a $M_{G_1\times G_2}$ morphism, and in case $G_1 = G_2 = G$ as a M_G morphism. Similarly for $FM<G_1,G_2>$.

In analogy with the first theorem of 2.2. we have:

THEOREM. [2].

Let $\alpha : C_n(X_1) \otimes C_m(X_2) \to C_p(X_1) \otimes C_q(X_2)$ be a natural transformation of functors on pairs in *Top*; that yields a natural transformation of $FM<G_1,G_2>$ valued functors on pairs (X_1,X_2) with $X_1 \in Top_{G_1}$ and $X_2 \in Top_{G_2}$:

$$FS_n^{G_1}(X_1) \otimes FS_m^{G_2}(X_2) \to FS_p^{G_1}(X_1) \otimes FS_q^{G_2}(X_2).$$

Again one can take the Eilenberg-Zilber map as an example, or rather its homotopy inverse (the case $p = q = n+m$), identifying $FS_p^{G_1}(X_1) \otimes FS_q^{G_2}(X_2)$ with $FS_p^{G_1\times G_2}(X_1 \times X_2)$.

This yields a crossproduct in homology:

$H_n^{G_1}(X_1;M_1) \otimes H_m^{G_2}(X_2;M_2)$

$H_{n+m}(M_1 \otimes FS_*^{G_1}(X_1) \otimes M_2 \otimes FS_*^{G_2}(X_2)) =$

$H_{n+m}((M_1 \hat{\otimes} M_2) \otimes (FS_*^{G_1}(X_1) \otimes FS_*^{G_2}(X_2))$

$H_{n+m}((M_1 \hat{\otimes} M_2) \otimes FS_*^{G_1 \times G_2}(X_1 \times X_2)) = H_{n+m}^{G_1 \times G_2}(X_1 \times X_2;M_1 \hat{\otimes} M_2)$,

given $M_1 \in C^{G_1}$ and $M_2 \in C^{G_2}$.

5.2.

If we would have a natural slant-product

$$H^p_{G_1}(X_1;N) \otimes H_{p+q}^{G_1 \times G_2}(X_1 \times X_2;M^1) \to H_q^{G_2}(X_2;M^2)$$

then, by applying it to the case $p = q = 0$ and X_1 and X_2 of the form G_1/H_1 resp. G_2/H_2, we would find pairings

$$N(G_1/H_1) \otimes M^1(G_1 \times G_2/H_1 \times H_2) \to M^2(G_2/H_2),$$

which are consistent with respect to $\mathcal{O}_{G_1}$ and $\mathcal{O}_{G_2}$ morphisms.

On the other hand: if we are given $N \in C_{G_1}$, $M^1 \in C^{G_1 \times G_2}$, $M^2 \in C^{G_2}$ and consistent maps $N(G_1/H_1) \otimes M^1(G_1 \times G_2/H_1 \times H_2) \to M^2(G_2/H_2)$ we can construct natural maps

$\hat{\phi}$: $\mathrm{Hom}(L(S);N) \to \mathrm{Hom}(\Sigma M^1(S \times T),\Sigma M^2(T))$ according to [2], and using this a slant-product:

$H^p_{G_1}(X_1;N) \otimes H_{p+q}^{G_1 \times G_2}(X_1 \times X_2;M^1)$

$H(\mathrm{Hom}(LS_p^{G_1}(X_1);N) \otimes (M^1 \otimes FS_{p+q}^{G_1 \times G_2}(X_1 \times X_2)))$

$H(\mathrm{Hom}(LS_p^{G_1}(X_1);N) \otimes (M^1 \otimes (FS_p^{G_1}(X_1) \otimes FS_q^{G_2}(X_2))))$

$H(M^2 \otimes FS_q^{G_2}(X_2)) = H_q^{G_2}(X_2;M^2)$,

where the second map is induced by the Eilenberg-Zilber chain map of the last subsection.

If we specialize to the case $G_1=G_2=G$ and consider slant-products $H^p_G(X_1,N) \otimes H^G_{p+q}(X_1 \times X_2,M^1) \to H^G_q(X_2,M^2)$ only, then consistent pairings $N(G/H) \otimes M^1(G/H) \to M^2(G/H)$ are necessary and sufficient. (See [2]). In the case $X_1 = X_2 = X$, composition with the homology of the diagonal $X \to X \times X$ yields a cap-product

$$H^p_G(X,N) \otimes H^G_{p+q}(X,M^1) \to H^G_q(X,M^2).$$

§ 3.

Consider the following situation

$X_1 \in Top_{G_1}$ and Y_1 an invariant part of X_1,

$X_2 \in Top_{G_2}$ and Y_2 an invariant part of X_2.

Since the Eilenberg-Zilber map is natural we get a sum of equivariant maps

$S_r^{G_1}(X_1) \times S_n^{G_2}(X_2) \to S_p^{G_1}(X_1) \times S_q^{G_2}(X_2)$ such that

$S_n^{G_1}(X_1) \times S_n^{G_2}(Y_2) \cup S_n^{G_1}(Y_1) \times S_n^{G_2}(X_2)$ maps to

$S_p^{G_1}(X_1) \times S_q^{G_2}(Y_2) \cup S_p^{G_1}(Y_1) \times S_q^{G_2}(X_2)$,

and we get a natural transformation

$$F(S_r^{G_1\times G_2}(X_1 \times X_2), S_r^{G_1\times G_2}\{X_1 \times Y_2, Y_1 \times X_2\}) \to$$
$$\to F(S_p^{G_1}(X_1) \times S_q^{G_2}(X_2), S_p^{G_1}(X_1) \times S_q^{G_2}(Y_2) \cup S_p^{G_1}(Y_1) \times S_q^{G_2}(X_2)).$$

Since moreover Φ extends to a map

$$\mathrm{Hom}(L(S_1,T_1),N) \to \mathrm{Hom}(M^1 \otimes F(S_1 \times S_2, S_1 \times T_2 \cup T_1 \times S_2), M^2 \otimes F(S_2,T_2))$$

we get a slant-product

$$H^p_{G_1}(X_1,Y_1,N) \otimes H^{G_1\times G_2}_{p+q}(X_1 \times X_2,\{X_1 \times Y_2, Y_1 \times X_2\},M^1) \to H^{G_2}_q(X_2,Y_2,M^2),$$

and for $G_1 = G_2 = G$ and $X_1 = X_2 = X$ a cap-product:

$$H^p_G(X,Y_1;N) \otimes H^G_{p+q}(X,\{Y_1,Y_2\};M^1) \rightarrow H^G_q(X,Y_2;M^2)$$

In case (X,Y_1,Y_2) is an excisive triad we may replace

$$H^G_{p+q}(X,\{Y_1,Y_2\};M^1) \text{ by } H^G_{p+q}(X,Y_1 \cup Y_2;M^1).$$

One can also construct cap-products in the Top_ϕ case; but we will not go into the details.

5.4.

One can easily check that the evaluation map $\hat{\phi}$: $Hom(L(S);N) \otimes \Sigma M^1(S \times T) \rightarrow \Sigma M^2(T)$ is natural with respect to $\mathcal{M}_G$ morphisms $g : S \rightarrow S'$, $h : T \rightarrow T'$. Together with the fact that the Eilenberg-Zilber map is natural and some properties of the homology functor this yields the commutativity of the following diagram and hence the fact that the slant product is natural. Similarly for the cap-product.

$$
\begin{array}{ccccc}
H^p_G(X;N) \otimes H^G_{p+q}(X \times Y;M^1) & \longleftarrow & H^p_G(X';N) \otimes H^G_{p+q}(X \times Y;M^1) & \longrightarrow & H^p_G(X';N) \otimes H^G_{p+q}(X' \times Y';M^1) \\
\downarrow & & \downarrow & & \downarrow \\
H(C^p_G(X;N) \otimes (M^1 \otimes FS^G_{p+q}(X \times Y))) & \longleftarrow & H(C^p_G(X';N) \otimes (M^1 \otimes FS^G_{p+q}(X \times Y))) & \longrightarrow & H(C^p_G(X';N) \otimes (M^1 \otimes FS^G_{p+q}(X' \times Y'))) \\
\downarrow \text{E.Z.} & & \downarrow \text{E.Z.} & \text{naturality of the E.Z. map} & \downarrow \text{E.Z.} \\
H(C^p_G(X;N) \otimes (M^1 \otimes (FS^G_p X \otimes FS^G_p Y))) & \longleftarrow & H(C^p_G(X';N) \otimes (M^1 \otimes (FS^G_p X \otimes FS^G_q Y))) & \longrightarrow & H(C^p_G(X';N) \otimes (M^1 \otimes (FS^G_p X' \times FS^G_q Y'))) \\
\downarrow & & \text{naturality of the evaluation map} & & \downarrow \\
H(M^2 \otimes FS^G_q(Y)) & & \xrightarrow{\hspace{4cm}} & & H(M^2 \otimes FS^G_q(Y'))
\end{array}
$$

5.5.

The slant-product is stable i.e. the following diagram commutes:

$$
\begin{array}{ccc}
H^p_G(X,A;N) \otimes H^G_{p+q}(X \times Y,\ X \times B \cup A \times Y;\ M^1) & & \\
\uparrow \wr & & \\
H^p_G(X,A;N) \otimes H^G_{p+q}(X \times Y, \{X \times B,\ A \times Y\}; M^1) & \xrightarrow{\ \text{slant}\ } & H^G_q(Y,B;M^2) \\
\downarrow 1\otimes\partial & & \downarrow \partial \\
H^p_G(X,A;N) \otimes H^G_{p+q-1}(\{X \times B, A \times Y\}, A \times Y;\ M^1) & & \\
\uparrow \wr & & \\
H^p_G(X,A;N) \otimes H^G_{p+q-1}(X \times B,\ A \times B;\ M^1) & \xrightarrow{\ \text{slant}\ } & H^G_{q-1}(B;M^2)
\end{array}
$$

P r o o f. [5].

Represent $\zeta \in H^G_{p+q}(X \times Y, \{X \times B,\ A \times Y\}; M^1)$ by $z \in M^1 \otimes FS^G_{p+q}(X \times Y, \{X \times B,\ A \times Y\})$, which in turn is the image of $a \in M^1 \otimes FS^G_{p+q}(X \times Y)$.

Represent $\eta \in H^p_G(X,A;N)$ by $y \in \mathrm{Hom}(LS^G_p(X,A);N)$.

Now $\partial\zeta$ is represented by $\partial s = a+b$ say, where $a \in M^1 \otimes FS^G_{p+q}(A \times Y)$ and $b \in M^1 \otimes FS^G_{p+q}(X \times B)$, since z was a cycle. This corresponds under the Noether isomorphism to b, which represents an element of $H^G_{p+q-1}(X \times B,\ A \times B; M^1)$. Finally the slant with η is represented by $\phi(y \otimes E.Z.(b))$, where ϕ denotes the evaluation and E.Z. the Eilenberg-Zilber map.

To calculate $\partial(\eta/\zeta)$ we remark that $\phi(y \otimes E.Z.(s))$ is an element of $M^2 \otimes FS^G_q(Y)$ mapping to the right element $\phi(y \otimes E.Z.(z))$ of $M^2 \otimes FS^G_q(Y)/FS^G_q(B)$ representing η/ζ; hence $\partial(\eta/\zeta)$ is represented by $\partial(y \otimes E.Z.(s))$.

Since the evaluation map is natural it commutes with ∂. By definition of the boundary operator ∂ on a tensorproduct

$\partial(y \otimes E.Z.(s)) = y \otimes \partial E.Z.(s)$ since y was a cocycle. Furthermore

$\partial E.Z.(s) = E.Z.\partial s = E.Z.(a+b) = E.Z.(a) + E.Z.(b)$ since E.Z. is a natural chain map. Finally $\phi(y \otimes E.Z.(a)) = 0$ because

$E.Z.(a) \in M^1 \otimes FS^G_p A \otimes FS^G_q Y$. Q.E.D.

Similarly cap is stable:

$$\begin{array}{ccc}
H^p_G(X,A_2;N) \otimes H^G_{p+q}(X,A_1 \cup A_2;M^1) & & \\
\uparrow & & \\
H^p_G(X,A_2;N) \otimes H^G_{p+q}(X,\{A_1,A_2\};M^1) & \xrightarrow{\text{cap}} & H^G_q(X,A_1;M^2) \\
\downarrow 1\otimes\partial & & \\
H^p_G(X,A_2;N) \otimes H^G_{p+q-1}(\{A_1,A_2\},A_2;M^1) & & \\
\downarrow \text{restriction} & & \downarrow \partial \\
H^p_G(A_1,A_1 \cap A_2;N) \otimes H^G_{p+q-1}(\{A_1,A_2\},A_2;M^1) & & \\
\uparrow \text{noether} & & \\
H^p_G(A_1,A_1 \cap A_2;N) \otimes H^G_{p+q-1}(A_1,A_1 \cap A_2;M^1) & \xrightarrow{\text{cap}} & H^G_{q-1}(A_1;M^2)
\end{array}$$

Also a more complicated stability theorem like 12.20 in [5] is true for equivariant cohomology. In proving it following the lines of loc.cit. it is clarifying to put in some homology-groups involving the { } symbols, as has been done in the above proof.

§6. Steenrod operations.

6.1.

We recall a few facts regarding the construction of the Steenrod squaring operations as can be found in [13,p.271-275.]

The complex $C_*(X) \otimes C_*(X)$ admits an action of $C_2 = \{1,T\}$ by chain maps, given by $T(a \otimes b) = (-1)^{pq}(b \otimes a)$ for $a \in C_p(X)$ and $b \in C_q(X)$. There exists a complex W_* given by:

$W_n = \text{span}\ (e_n, e'_n)$ for $n \geq 0$

$d(e_n) = (-1)^n d(e'_n) = e_{n-1} + (-1)^n e'_{n-1}$ for $n > 0$ and $d(e_0) = d(e'_0) = 0$;

this complex is acyclic and admits a free action of C_2 given by

$Te_n = e'_n$.

There exists a natural C_2-equivariant chain transformation:

$f : W_* \otimes C_*(X) \to C_*(X) \otimes C_*(X)$, uniquely determined up to a natural homotopy. This f can be viewed as a sequence of natural transformations $D_k : C_*(X) \to C_*(X) \otimes C_*(X)$ satisfying the equations

$dD_k = D_{k-1} + (-1)^k\ TD_{k-1} + (-1)^k\ D_k d$ for $k > 0$ and

$dD_0 = D_0 d$;

the correspondence is given by $D_k(x) = f(e_k \otimes x)$.

According to some variant of theorem I.2.2. this yields natural transformation , satisfying the same identities:

$D_k : L^G_n(X) \to L^G_p(X) \mathbin{\hat{\otimes}} L^G_q(X)$ for k, n, p, and q such that $k+n = p+q$.

Consider the map Sq_k:

$$\begin{array}{ccc} \text{Hom}(L^G_n(X);M) \xrightarrow{\text{diag.}} & \text{Hom}(L^G_n(X);M) \otimes \text{Hom}(L^G_n(X);M) & \\ & \downarrow & \\ & \text{Hom}(L^G_n(X) \mathbin{\hat{\otimes}} L^G_n(X), M \mathbin{\hat{\otimes}} M) & \xrightarrow{D_k} \text{Hom}(L^G_{2n-k}(X), M \mathbin{\hat{\otimes}} M). \end{array}$$

Since a cochain $c \in \text{Hom}(L^G_n(X);M)$ is given by the values of the $c(\sigma)$, where $\sigma \in L^G_n(X)(G/H)$, we can say that Sq_k is defined by the formula:

$$\langle Sq_k c, \sigma\rangle = \langle c \otimes c, D_k\sigma\rangle \in M(G/H) \otimes M(G/H) \text{ for}$$

$$c \in \text{Hom}(L^G_n(X);M) \quad \text{and} \quad \sigma \in L^G_{2n-k}(X)\ (G/H).$$

6.2

For $M \in C_G$ define $S^2_\pm M$ to be the coefficient system such that $(S^2_\pm M)(G/H) = M(G/H) \otimes M(G/H)/\text{span}(a \otimes b \pm b \otimes a)$ with the obvious values on morphisms and let p be the canonical projection $M \hat{\otimes} M \to S^2_\pm M$.

PROPOSITION. $p \circ Sq_k$ maps cocycles to cocycles and coboundaries to coboundaries and on cocycles it is additive modulo coboundaries (here $\pm = (-1)^{n-k}$).

P r o o f. We prove only the first statement.

Let $c \in \mathrm{Hom}(L^G_n(X),M)$ such that $\delta c = 0$ and let $\sigma \in L^G_{2n-k}(X)(G/H)$ then $\langle(-1)^k \delta Sq_k c,\sigma\rangle = \langle(-1)^k Sq_k c, d\sigma\rangle = \langle(-1)^k\ c \otimes c,\ D_k d\sigma\rangle =$
$= \langle c \otimes c,\ dD_k\sigma - D_{k-1}\sigma - (-1)^k\ TD_{k-1}\sigma\rangle$,
where the first term $\langle(c \otimes c), D_k\sigma\rangle = 0$ and
the third term $\langle(-1)^k(c \otimes c), TD_{k-1}\sigma\rangle = (-1)^{n+k}T\langle c \otimes c, D_{k-1}\sigma\rangle$ is cancelled by the second term after applying p.

Q.E.D.

Hence $p \circ Sq_{n-k}$ induces a map on cohomology level

$$Sq^k \quad H^n_G(X,M) \to H^{n+k}_G(X,S^2_\pm M), \text{ where } \pm = (-1)^k.$$

These considerations can be extended to the category Top_ϕ without further effort.

6.3.

These Steenrod squaring operations have properties similar to those of the classical Steenrod operations

1) Sq^0 is the coefficient map $M \to S^2_\pm M$ mapping a to $a \otimes a$.

2) $Sq^s c = p(c \cup c)$ for $c \in H^s_G(X,M)$.

3) $Sq^k c = 0$ for $c \in H^s_G(X,M)$ in case $k > s$.

4) Let $u \in H^s_{G_1}(X,A;M)$ and $v \in H^t_{G_2}(Y,B;M)$ and let $(X \times B,\ A \times Y)$ be excisive in $X \times Y$; then $\pi Sq^k(u \times v) = \sum_{i+j=k} Sq^i u \times Sq^j v$ where π is the canonical coefficient map $S^2(M \hat{\otimes} N) \to S^2M \hat{\otimes} S^2N$.

P r o o f.

1) Since for the classical D_k on $C_k(X)$ we have $D_k\sigma = \sigma \otimes \sigma$ according to [13,p.274], the same is true for the generalized D_k.

2) Follows from the fact that D_0 = (E.Z. map)(diagonal map).

3) Is obvious.

4) Is proved by paraphrasing the proof in [13] of the corresponding statement for the classical operations, as we have done already for their construction.

Q.E.D.

Remark: According to the argumentation in [14,p.2-3] it follows from (4) that the Sq^k commute with the coboundary operator in the long exact sequence of a pair, and hence with the suspension isomorphism.

6.4.

We are going to prove a property which has no classical counterpart.

For $M \in C_G$ define $\Lambda^2_{\pm}M$ to be the coefficient system such that $(\Lambda^2_{\pm}M)(G/H) = \text{span}(a \otimes b \pm b \otimes a) \subset M(G/H) \otimes M(G/H) = (M \hat{\otimes} M)(G/H)$.

Hence there is an exact sequence:

$$0 \to \Lambda^2_{\pm}M \to M \hat{\otimes} M \to S^2_{\pm}M \to 0;$$

the Bokstein operator appearing in the corresponding long exact sequence of cohomology will be called B.

THEOREM. The following diagram commutes up to a factor $(-1)^{n+k+1}$:

$$\begin{array}{ccc} H^n_G(X;M) & \xrightarrow{\quad Sq^k \quad} & H^{n+k}_G(X;S^2_\pm M) \\ \Big\downarrow{\scriptstyle Sq^{k+1}} & & \Big\downarrow{\scriptstyle B} \\ H^{n+k+1}_G(X;S^2_\mp M) & \xrightarrow{\quad 1 \pm T \quad} & H^{n+k+1}_G(X;\Lambda^2_\pm M) \end{array}$$

where $1 \pm T$ denotes the obvious coefficient transformation; $\pm = (-1)^k$.

P r o o f. For $x \in H^n_G(X;M)$, represented by the cochain c,

$Sq^k x$ is represented by $p \circ D^*_{n-k}(c \otimes c)$, hence $B \circ Sq^k x$ by $\delta D^*_{n-k}(c \otimes c)$.

On the other hand $(1 \pm T)Sq^{k+1}x$ is represented by $(1+(-1)^k T)D^*_{n-k-1}(c \otimes c)$.

Now we have $<(1+(-1)^k T<D^*_{n-k-1}(c \otimes c),\sigma> =$

$= <c \otimes c, D_{n-k-1}\sigma> + (-1)^k T \;\; D^*_{n-k-1}(c \otimes c),\sigma> =$

$= <c \otimes c, D_{n-k-1}\sigma> + (-1)^{k+n}<c \otimes c, TD_{n-k}\sigma> =$

$= <c \otimes c, dD_{n-k}\sigma> + (-1)^{k+n-1}D_{n-k}d\sigma> = (-1)^{k+n-1}<\delta D^*_{n-k}(c \otimes c),\sigma>$

because the first term vanishes since $\delta(c \otimes c) = 0$. Q.E.D.

6.5.

One can iterate the result of the preceding subsection. For $M \in C_G$ define the coefficient system $A^2_\pm M$ by the formula:

$(A^2_\pm M)(G/H) = \text{kernel of } (1 \pm T) : M(G/H) \otimes M(G/H) \to M(G/H) \otimes M(G/H)$.

We have a commutative diagram:

$$\begin{array}{ccccc} & & H^n_G(X;M) & \xrightarrow{\quad Sq^k \quad} & H^{n+k}_G(X;S^2_\pm M) \\ & & \Big\downarrow{\scriptstyle Sq^{k+1}} & & \Big\downarrow{\scriptstyle B} \\ & {\scriptstyle Sq^{k+2}}\swarrow & H^{n+k+1}_G(X;S^2_\mp M) & \xrightarrow{\quad 1\pm T \quad} & H^{n+k+1}_G(X;\Lambda^2_\pm M) \\ & & \Big\downarrow{\scriptstyle B} & & \Big\downarrow{\scriptstyle C} \\ H^{n+k+2}_G(X,S^2_\pm M) & \xrightarrow{\quad 1\mp T \quad} & H^{n+k+2}_G(X,\Lambda^2_\mp M) & \xrightarrow{\quad j \quad} & H^{n+k+2}_G(X,A^2_\pm M) \end{array}$$

Here C is the Bokstein operation associated with the exact sequence $0 \to A^2_{\pm}M \to M \hat{\otimes} M \to \Lambda^2_{\pm}M \to 0$ and j is the obvious coefficient map.

This means that there is a relation between Sq^k and Sq^{k+2}; $j \circ (1 \mp T) \circ Sq^{k+2} = C \circ B \circ Sq^k$. Hence if we go further, from $A^2_{\pm}M$ to $M \hat{\otimes} M$, then the composition vanishes; so certainly the Sq vanishes after multiplication by 2.

§7. Functional operations.

7.1.

Consider $\phi : H \to G$ as in 1.5.; let $X \in \mathit{Top}_H$, $Y \in \mathit{Top}_G$ and let $f : X \to Y$ be an H-equivariant continuous map. Denoting the cone on X by CX we have $(Y \cup_f CX, Y) \in \mathit{Top}_\phi$. Hence for any $N \in \mathcal{C}_\phi$ there is a long exact sequence:

$$\overset{\delta}{\to} H^n_H(Y \cup_f CX, Y; N^H) \to H^n_\phi(Y \cup_f CX, Y; N) \to H^n_G(Y; N^G) \overset{\delta}{\to} \dots$$

Now we can identify $H^n_H(Y \cup_f CX, Y; H^N)$ with $H^{n-1}_H(X; N^H)$ and using it, δ corresponds to $f^* \circ R$, where R is the transformation $H^n_G(\ ; N^G) \to H^n_H(\ ; N^H)$ given by N, as we see from the diagram:

$$\begin{array}{ccccc}
H^{n-1}_G(Y;N^G) & & \xrightarrow{\quad\delta\quad} & & H^n_H(Y \cup_f CX, Y; N^H) \\
\downarrow{\scriptstyle f^*R} & & & & \downarrow{\scriptstyle f^*} \\
 & \overset{S}{\nearrow} & H^n_H(SX;N^H) & \searrow & \\
H^{n-1}_H(X;N^H) & & \xrightarrow{\quad\delta\quad} & & H^n_H(CX, X; N^H)
\end{array}$$

Hence abbreviating $H^n_\phi(Y \cup_f CX, Y; N)$ to $H^n_\phi(f;N)$ we find a long exact sequence

$$\to H^n_G(Y;N^G) \overset{f^*R}{\to} H^n_H(X;N^H) \overset{\delta}{\to} H^{n+1}_\phi(f;N) \to$$

If ω is a cohomology operation of degree k which is defined for Top_ϕ and which is stable then ω connects two long exact sequences to a commutative ladder. Examples of this situtation are coefficient homomorphisms, Bokstein operations and Steenrod squaring operations.

$$\begin{array}{ccccccccc}
H^{n-1}_G(Y;N^G) & \underset{fR}{\to} & H^{n-1}_H(X;N^H) & \underset{\delta}{\to} & H^n_\phi(f;N) & \underset{j}{\to} & H^n_G(Y;N^G) & \underset{fR}{\to} & H^n_H(X;N^H) \\
\downarrow{\scriptstyle \omega_G} & & \downarrow{\scriptstyle \omega_H} & & \downarrow{\scriptstyle \omega_\phi} & & \downarrow{\scriptstyle \omega_G} & & \downarrow{\scriptstyle \omega_H} \\
H^{n+k-1}_G(Y;M^G) & \to & H^{n+k-1}_H(X;M^H) & \overset{\delta}{\to} & H^{n+k}_\phi(f;M) & \overset{j}{\to} & H^{n+k}_G(Y;M^G) & \to & H^{n+k}_H(X;M^H)
\end{array}$$

Chasing this diagram we see that $\delta^{-1}\omega_\phi j^{-1}$ defines a homomorphism from the subgroup ker $\omega_G \cap$ ker f^*R of $H^n_G(Y;N^G)$ to the quotient coker (ω_H, f^*R) of $H^{n+k-1}_H(X;M^H)$. Henceforward this will be called the functional operation ω_f associated to ω and f.

Of course there is also a functional operation in the case that only one group G is involved.

7.2.

Let ω and Ω be stable Top_ϕ cohomology operations such that $\Omega \circ \omega$ is defined and equal to 0 and consider the following situation:

$X \in Top_H$, $Y \in Top_G$, $Z \in Top_G$, $f : X \to Y$ a Top_H morphism and $g : Y \to Z$ a Top_G morphism such that $g \circ f$ is Top_H homotopic to a constant map.

This homotopy induces Top_H maps $s : Y \cup_f CX \to Z$ and $t : SX \to Z \cup_g CY$.

<u>THEOREM</u>. In this situation one has:

$$\Omega^H \circ \omega^\phi_f \circ g^* = f^* \circ R \circ \Omega^G_g \circ \omega^G.$$

P r o o f.

1) Both sides are defined on the same group:

for the left-hand side that group is $(g^*)^{-1}(\ker\ (\omega_G, f^* \circ R))$,

for the right-hand side that group is $(\omega_G)^{-1}(\ker\ (\Omega_G, g^*))$;

both groups are equal to $\ker\ (g^* \circ \omega_G)$.

2) Both sides have values in the same group:

for the left-hand side that group is Ω^H coker $(\omega_H, f^* \circ R)$,

for the right-hand side that group is $f^* \circ R$ coker (Ω_G, g^*);

both groups are equal to coker $(\Omega^H \circ f^* \circ R)$.

3) The fact that the maps on both sides are equal follows by diagram chasing from diagram (a) on the next page, once we know the diagram to be commutative.

The commutativity of the outer squares follow from the naturality of ω and Ω. The middle square consists of the maps on the outside of diagram (c) on the next page, which is easily seen to be commutative. The lower triangle is similarly exemplified in diagram (b) on the next page. Finally the upper triangle is the result of applying the functor $H^*_\phi(\ ,\ ;N)$ to the triangle of Top_ϕ maps:

$$\begin{array}{ccc} (Y \cup_f CX, Y) & \xrightarrow{(s,g)} & (Z,Z) \\ \nwarrow & & \nearrow {\scriptstyle (g,g)} \\ & (Y,Y) & \end{array}$$

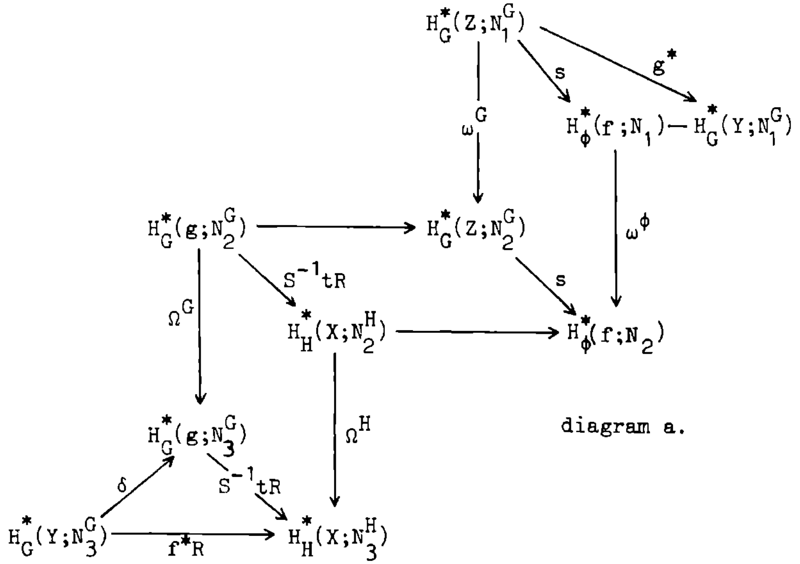


diagram a.

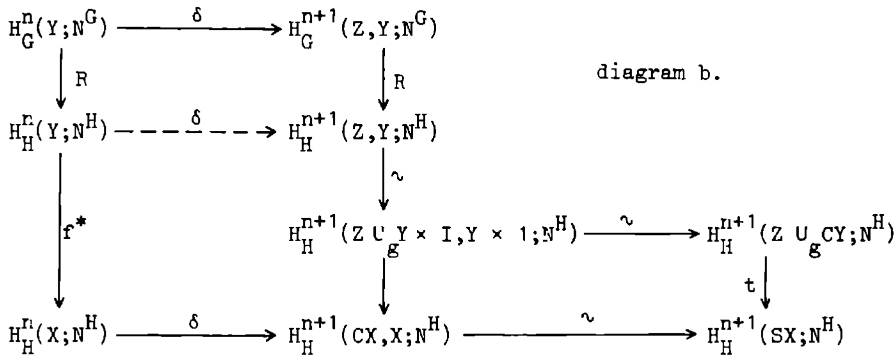


diagram b.

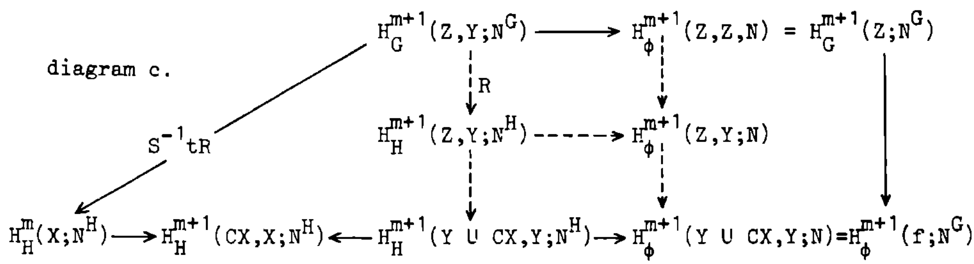


diagram c.

7.3.

In this subsection we calculate the functional operation in an example:

<u>THEOREM</u>. Consider a Top_G morphism $f : SX \to Y$ and let $M \in C_G$ and $y \in H^n_G(Y;M)$. Denote by t the coefficient homomorphism $1 + (-1)^{n-1}T : M \hat{\otimes} M \to M \hat{\otimes} M$. Then we have
$S^{-1} \circ t_f(y \cup y) = (-1)^{n-1}S^{-1} \circ f^*y \cup S^{-1} \circ f^*y$, modulo indeterminacy.

<u>P r o o f</u>. 1) The functional operation t_f is defined on $y \cup y$:
$t(y \cup y) = 0$ because of the commutativity of $\cup$, and
$f^*(y \cup y) = 0$ since cup-products vanish for a suspension.

2) We may assume f to be an inclusion without loss of generality. Let y be the image of $y_1 \in H^n_G(Y,C^+X;M)$, which is represented by the cochain η_1. Since C^-X is contractible, there exists a cochain v_1 on C^-X such that $\delta v_1 = \eta_1|C^-X$, there exists a cochain w on Y extending v_1.

3) Since $\delta(v_1|X) = \eta_1|X = 0$, $v_1|X$ represents an element in $H^{n-1}_G(X;M)$, in fact the element $S^{-1}f^*y$, as is seen from the diagram:

$$\begin{array}{ccccc} \{v_1\} \in H^{n-1}_G(X;M) & \to & y_1|C^-X \in H^n_G(C^-X,X;M) & & \\ & & \uparrow & & \\ & & y_1|SX \in H^n_G(SX,C^+X,M) & \to & y|SX \in H^n_G(SX;M). \end{array}$$

4) Since $\eta_2 = \eta_1 - \delta w$ vanishes on C^-X it represents an element $y_2 \in H^n_G(Y,C^-X;M)$ which maps to $y \in H^n_G(Y;M)$. Then $y_1 \cup y_2 \in H^{2n}_G(Y,SX;M\hat{\otimes}M)$ maps to $y \cup y \in H^{2n}_G(Y,M\hat{\otimes}M)$ hence can be used to calculate $t_f(y \cup y)$. According to §4 $T(y_1 \cup y_2) = (-1)^n y_2 \cup y_1$.

5) We calculate the S of $S^{-1} \circ f^*y \cup S^{-1} \circ f^*y$, which is represented by $v_1 \cup v_1$. To this end we extend it to a C^-X cochain: $w \cup w$.

Then we take the coboundary: $\delta w \cup w + (-1)^{n-1} w \cup \delta w$. Finally we extend to a cocycle on SX which vanishes on C^+X : $\eta_1 \cup w + (-1)^{n-1} w \cup \eta_1$.

6) To show that this cocycle represents $t_f(y \cup y) = \delta^{-1} t(y_1 \cup y_2)$ we have to evaluate δ on it.

To this end we extend it to a cochain on Y : $\eta_1 \cup w + (-1)^{n-1} w \cup \eta_1$. Then we take the coboundary: $(-1)^n \eta_1 \cup \delta w + (-1)^{n-1} \delta w \cup \eta_1$; since $\delta w = \eta_1 - \eta_2$, this equals $(-1)^{n-1}(\eta_1 \cup \eta_2 - \eta_2 \cup \eta_1)$.

On the other hand $t(y_1 \cup y_2) = y_1 \cup y_2 + (-1)^{n-1} T(y_1 \cup y_2) =$
$= y_1 \cup y_2 + (-1)^{n-1}(-1)^n (y_2 \cup y_1)$. Q.E.D.

Remark: For any $x \in H_G^{n-1}(X;M)$ the indeterminacy subgroup of $H_G^{2n-1}(SX;M \hat{\otimes} M)$ contains $St(x \cup x) = 2S(x \cup x)$ hence the sign occuring in the theorem is immaterial. Furthermore there is no need to worry about the sign in the definition of the suspension isomorphism S.

§8. Cellular homology and cohomology; obstruction theory.

8.1.

We define a G-complex X to be a CW complex on which the group G acts by cellular transformations. For any invariant subcomplex A of X (X,A) is called a G-complex pair. According to [1,p.I.1.] such a pair has the equivariant homotopy extension property (denoted HEP) and in particular one has $H_n^G(X,A;M) = \tilde{H}_n^G(X/A;M)$.

For such an X let $w_n(X) \in \mathcal{M}_G$ be the collection of cells of X of dimension n; notice that $w_n(X)^H = w_n(X^H)$. Let X^n denote the n-skeleton of X. Then $H_n(X^n,X^{n-1}) = Fw_n(X) \in F\mathcal{M}_G$ and
$d_{n+1} : H_{n+1}(X^{n+1},X^n) \to H_n(X^n,X^{n-1})$ is an $F\mathcal{M}_G$ morphism; in particular

$M \otimes d_n$ is defined.

8.2.

We now define the <u>cellular homology of X with coefficients in $M \in C^G$</u> $HW_n^G(X;M)$ to be the homology of the complex $(M \otimes Fw_*(X),\ M \otimes d_*)$.

<u>THEOREM</u>. There is a natural isomorphism

$$HW_n^G(X;M) \cong H_n^G(X;M).$$

<u>P r o o f</u>. The statement follows in a purely algebraic fashion from the fact that $M \otimes H_n(X^n,X^{n-1})$ is isomorphic to $H_n^G(X^n,X^{n-1};M)$ in a way respecting the boundary operator d. The algebra can be found in [5, V.1.]

Q.E.D.

8.3.

We define the <u>cellular cohomology of X with coefficients in</u> $M \in C_G$ $HW_G^n(X;M)$ to be the homology of the complex formed by the groups Traf(L o $w_n(X)$;M). Here the boundary operator of $Lw_*(X)$ is defined from the homology sequence of the triple $((X^H)^{n+1},(X^H)^n,(X^H)^{n-1})$ using the identification:

$$L \circ w_n(X)(G/H) = Fw_n(X)^H = Fw_n(X^H) = H_n(X^H)^n,(X^H)^{n-1}).$$

<u>THEOREM</u>. There is a natural isomorphism

$$HW_G^n(X;M) = H_G^n(X;M).$$

<u>P r o o f</u>. According to [1 ,IV.4.] Traf(L o $w_n(X)$;M) is isomorphic to $H_G^n(X^n,X^{n-1};M)$ in a way which respects the boundary operator. The statement is an algebraic consequence of this fact as the following reasoning, modelled on [5, V.1.] shows. One does not need a spectral

sequence as in [1]. Consider the diagram:

$$\begin{array}{ccccccc}
H_G^{n-1}(X^{n-1},X^{n-2};M) & & & & H_G^{n}(X^{n+1},X^{k};M) & & \\
\downarrow j^* & \searrow \partial^n & & & \downarrow & & \\
H_G^{n-1}(X^{n-1},X^{k};M) & \xrightarrow{\partial^*} & H_G^{n}(X^{n},X^{n-1};M) & \xrightarrow{i^*} & H_G^{n}(X^{n},X^{k};M) & \longrightarrow & 0 \\
\downarrow & & & \searrow \partial^{n+1} & \downarrow \partial^* & & \\
0 & & & & H_G^{n+1}(X^{n+1},X^{k};M) & &
\end{array}$$

The diagram is commutative with exact row and columns.

Hence we have: $H_G^n(X,X^k;M) \cong H_G^n(X^{n+1},X^k;M) \cong \ker \partial^* = i^*(\ker \partial^{n+1}) = \mathrm{im}(i^* \ker \partial^{n+1}) \cong \ker \partial^{n+1}/\ker i^* = \ker \partial^{n+1}/\mathrm{im}\, \partial^* = \ker \partial^{n+1}/\mathrm{im}\, \partial^n$ since j^* is onto. The last quotient is $HW_G^n(X;M)$ by definition.

Q.E.D.

Of course there is also a cellular theory for Top_ϕ complex pairs. In fact that is a special case of cohomology with coefficients in a "local coefficient system" as in [1 ,I.5] by using a functor $\theta : K \to \mathcal{Q}_\phi$.

8.4.

We recall some of the facts concerning equivariant obstruction-theory according to [1 , II.1-3.].

Let $Y \in Top_G$ with invariant basepoint y_0 such that Y^H is arcwise connected and simple for each $H \subset G$. Define $\tilde{\omega}_*(Y) \in C_G$ by $\tilde{\omega}_n(Y)(G/H) = \pi_n(Y^H,y_0)$ with the obvious values on morphisms.

<u>THEOREM</u>. Let $f : K^n \cup L \to Y$ be a Top_G morphism, where (K,L) ia a G-complex pair. Then there is defined $\{c_f\} \in HW_G^{n+1}(K,L;\tilde{\omega}_n(Y))$ which only depends on the homotopy-class of $f|K^{n-1} \cup L$ and which vanishes if and

only if $f|K^{n-1} \cup L$ can be extended to $K^{n+1} \cup L$.

By repeated application of this theorem one finds for Y such that $\tilde{\omega}_n(Y) = 0$ if $n \neq r$ and $n < \dim(K-L)$:

THEOREM. Let $f : K \to Y$ be equivariant; then equivariant homotopy-classes (relative L) of maps $g : K \to Y$ (such that $g|L = f|L$) are in bijective correspondence with elements of
$HW^n_G(K,L;\tilde{\omega}_n(Y))$ by the correspondence $g \to \omega^n(g,f)$.
(For the definition of ω in terms of c we refer to [1].)

In particular one can take f to be a constant map in case $L = \emptyset$; in that case we write $\chi_n(g)$ instead of $\omega_n(f,g)$. Furthermore c, and hence χ, is natural with respect to cellular maps.
In particular we have in case Y is a G-complex, and writing

$$\chi_n(Y) \quad \text{for} \quad \chi_n(id_Y) : \chi_n(f) = f^*\chi_n(Y).$$

Now we have the

THEOREM. If $M \in C_G$ there exists a G-complex $K(M,n)$ called the Eilenberg-Maclane G-complex of M such that $\tilde{\omega}_q K(M,n) = M$ for $q = n$ and 0 for $q \neq n$.

Combining these facts with those in the preceding subsection we deduce:

THEOREM. The correspondence $g \to g^*\chi_n(K(M,n))$ induces a bijection between $[[X,K(M,n)]]$ = the set of equivariant maps $X \to K(M,n)$ and $H^n_G(X;M)$.

Henceforth we will abbreviate $\chi_n(K(M,n))$ to χ_n.

8.5.

We need the following theorem, of which Proposition II.7.1. in [1] is the special case X = point:

<u>THEOREM</u>. Let X and K be G-complexes and let $f : X \to K$ be an equivariant map such that $\tilde{\omega}_q(f) : \tilde{\omega}_q(X) \to \tilde{\omega}_q(K)$ is an isomorphism for $q < n-1$ and a surjection for $q = n-1$; then there exists a $K' \supset X$ of the equivariant homotopy-type of K such that $K'-X$ has no cells in dimensions less then n.

<u>P r o o f</u>. Applying a homotopy if necessary we may assume that f is cellular. Then by replacing K by $K \cup_f X \times I$ we may assume that $X \subset K$.

According to [1 , II.5.2.] it follows from $\tilde{\omega}_q(K,X) = 0$ for $q < n-1$ that for $L = X \cup K^{n-1}$ the inclusion $L \subset K$ is equivariantly homotopic (relative to X) to a map $p : L \to X$. Now K is a retract of $K \cup_p L \times I \cup X$.

According to [1, I.1.]: $K \times 0 \cup L \times I \subset K \times I$ is a strong deformation retract, hence $K \times 0 \cup L \times I/L \times 1 \sim_p X \subset K \times I/L \times 1 \sim_p X$ is a strong deformation retract. However $K \times 1/L \times 1 \sim_p X \subset K \times I/L \times 1 \sim_p X$ is also a strong deformation retract. Hence $K \cup L \times I \cup X$ is equivariantly homotopy-equivalent to $K/L \sim_p X$ and there are maps $\phi,\psi : K \xrightarrow{\phi} K/L \sim_p X \xrightarrow{\psi} K$ as in the proof of II.7.1. in
Now one copies that proof starting with the fifth line on page II.18.1. with $K/\sim$ replacing K/L.

Q.E.D.

<u>Remark</u>: Furthermore the diagram:

$$\begin{array}{ccc} X & \longrightarrow & K \\ \cap & \nearrow{\scriptstyle\sim} & \\ K' & & \end{array}$$

commutes up to equivariant homotopy.

<u>COROLLARY</u>. With the same assumptions we have:

$H_q^G(X;M) \to H_q^G(K;M)$ is an isomorphism for $q < n-1$ and a surjection for $q = n-1$, for any $M \in \mathcal{C}^G$.

$H_G^q(X;M) \to H_G^q(K;M)$ is an isomorphism for $q < n-1$ and an injection for $q = n-1$, for any $M \in \mathcal{C}_G$.

8.6.

Consider the specialization at G/H:

$r : \mathrm{Traf}(LS_n^G(X);M) \to \mathrm{Hom}(S_n(X^H);M(G/H))$. This induces a natural and stable transformation $r : H_G^n(X;M) \to H^n(X^H;V)$, where $V = M(G/H)$.

On the other hand an equivariant obstruction-problem $f : X \to Y$ yields by restriction an obstruction-problem $f|H : X^H \to Y^H$. Inspection of the definition of c_f shows that $r(c_f) = c(f|X^H)$; the same is true for χ. Hence if $x \in H_G^n(X,M)$ classifies $f : X \to K(M,n)$ then $r(x) \in H^n(X^H;V)$ classifies $f|H : X^H \to K(M,n)^H$.

In particular: if $f : SK(M.n) \to K(M,n+1)$ is classified by $S\chi(K(M,n))$ then $f|H : SK(V,n) \to K(V,n+1)$ is classified by $rS\chi(K(M,n)) = Sr\chi(K(M,n)) = S\chi(K(V,n))$.

<u>PROPOSITION</u>. If $f : SK(V,n) \to K(V,n+1)$ is classified by $S\chi(K(V,n))$ then f induces in π_q isomorphisms for $q \leq 2n$ and a surjection for $q = 2n+1$.

<u>P r o o f</u>. Since the adjoint map of $f : K(V,n) \to \Omega K(V,n+1)$ induces isomorphisms in the homotopy groups, it does so in the cohomology groups, hence the suspension does so too.

Since the following diagram commutes: if suffices to consider the evaluation map E.

$$\begin{array}{ccc} SK(V,n) & \xrightarrow{\ f\ } & K(V,n+1) \\ \downarrow{\scriptstyle S\ \mathrm{ad}\ f} & \nearrow{\scriptstyle \text{evaluation}} & \\ S\Omega K(V,n+1) & & \end{array}$$

According to [6] and [11], E is homotopy equivalent to a fibration the fibre of which is homotopy-equivalent to $\Omega K(V,n+1) * \Omega K(V,n+1)$; this is a join of (n-1)-connected objects hence 2n-connected.

Hence E induces isomorphisms in π_q for $q \leq 2n$ and a surjection for $q = 2n+1$.

Q.E.D.

<u>COROLLARY</u>.Then the $f : SK(M,n) \to K(M,n+1)$ as above does the same for $\tilde{\omega}_q$ hence according to the last subsection for any $M,N \in \mathcal{C}_G$ the map $H^q_G(K(M,n+1);N) \xrightarrow{f^*} H^q_G(SK(M,n);N)$ is an isomorphism for $q \leq 2n$ and a injection for $q = 2n+1$.

8.7.

Consider the map $H_k : S^kK(M,q) \to K(M,q+k)$ classified by $S^k\chi_q$. Then the following diagram commutes up to equivariant homotopy:

$$\begin{array}{ccc} S^{k-1}H_1 : S^kK(M,q) & \longrightarrow & S^{k-1}K(M,q+1) \\ \Big\downarrow = & & \Big\downarrow H_{k-1} \\ H_k : S^kK(M,q) & \longrightarrow & K(M,q+k) \end{array}$$

Let t be the coefficient map $1 \pm T : S^2_{\mp}M \to M \underset{\cdot}{\hat{\otimes}} M$, where $\pm = (-1)^q$.

It follows from the fact that the functional operation associated to t is natural, that modulo indeterminacy one has:

$$t_{S^{k-1}H_1}(H^*_{k-1}Sq^{q+1}\chi_{q+k}) = t_{H_k}(Sq^{q+1}\chi_{q+k}).$$

The indeterminacy on the left is equal to im t + im $(S^{k-1}H)^*$; the one on the right is im t + im H^*_k. To show that both are equal, one needs to know that the following map is an isomorphism:

$$H^*_{k-1} : H^{2q+k}_G(K(M,q+k);M\hat{\otimes}M) \to H^{2q+k}_G(S^{k-1}K(M,q+1);M\hat{\otimes}M).$$

That this is true follows from repeated application of the corollary in 8.6.

Since $H^*_{k-1}Sq^{q+1}\chi_{q+k} = Sq^{q+1}H^*_{k-1}\chi_{q+k} = Sq^{q+1}S^{k-1}\chi_{q+1} = S^{k-1}Sq^{q+1}\chi_{q+1}$ the left-hand side of the above formula is equal to $S^{k-1}t_{H_1}(Sq^{q+1}\chi_{q+1})$, and since $Sq^{q+1}\chi_{q+1}$ is the reduction of χ^2_{q+1} to $S^2_{\pm}M$ we find this to be equal to $S^{k-1}(S\chi^2_q) = S^k\chi^2_q$, applying I.7.4.

We have proved the following:

<u>THEOREM</u>. $t_{H_k} \circ Sq^{q+1}\chi_{q+k} = S^k(\chi_q \cup \chi_q)$.

§9. <u>Maps to a suspension</u>.

9.1.

Let Y be a G-complex and $X \in Top_G$. We are interested in equivariant maps $f : X \to S^kY$. If $y \in H^q_G(Y;M)$ such that $f^*(S^ky) = 0$ we consider the classifying map $\gamma_y : Y \to K(M,q)$ of y and we denote $S^k\gamma_y \circ f : X \to S^kK(M,q)$ by F.

We define $\psi_f(y)$ to be $Sq^{q+1}_F(S^k\chi_q)$ and we study the behaviour of ψ in this section. For later reference we now state a property of ψ, which is found by combining 8.7. with 7.2.

<u>PROPOSITION</u>. $t\psi(y) = f^*S^k(y \cup y)$.

<u>P r o o f</u>. $t\psi(y) = t\, Sq^{q+1}_F(S^k\chi_q) = F^*S^k(\chi_q \cup \chi_q) = f^*S^k(y \cup y)$.

9.2.

We are going to study $\psi_f(x+y)$. We define $m : K(M,q) \times K(M,q) \to K(M,q)$ by the formula $m^*(\chi_q) = pr^*_1(\chi_q) + pr^*_2(\chi_q)$.

Then if γ_x and $\gamma_y : Y \to K(M,q)$ classify x resp. y, the map

$m \circ (\gamma_x \times \gamma_y) \circ (\text{diagonal map}) : Y \to Y \times Y \to K(M,q) \times K(M,q) \to K(M,q)$

is classifying for x+y since

$$(\text{diagonal})^*(\gamma_x \times \gamma_y)^* m^* \chi_q =$$
$$= (\text{diagonal})^*(\gamma_x \times \gamma_y)^*(pr_1^*(\chi_q) + pr_2^*(\chi_q)) =$$
$$= (pr_1 \circ (\gamma_x \times \gamma_y) \circ \text{diagonal})^*(\chi_q) + (pr_2 \circ (\gamma_x \times \gamma_y) \circ \text{diagonal})^*(\chi_q) =$$
$$= \gamma_x^*(\chi_q) + \gamma_y^*(\chi_q) = x + y.$$

Therefore we study the composition:

$S^k(m \circ (\gamma_x \times \gamma_y) \circ (\text{diag.})) \circ f = S^k m \circ S^k(\gamma_x \times \gamma_y) \circ S^k(\text{diag.}) \circ f.$

We note that for G-complexes A and B with basepoint the natural equivariant map $SA \vee SB \vee A * B \to S(A \times B)$, where * denotes join, is an equivalence. This follows from the well known non-equivariant version using theorem 8.5. or [1 ,II.5.5.].

Furthermore we note that suspension S commutes with one point union $\vee$ (up to a natural equivalence).

Hence we are led to a diagram:

$$\begin{array}{ccccccccc}
X & \to & S^kY & \to & S^k(Y \times Y) & \to & S^k(K(M,q) \times K(M,q)) & \to & S^kK(M,q) \\
 & & & \searrow & \updownarrow & & \updownarrow & \nearrow & \\
 & & & & S^kY \vee S^kY \vee & \to & S^kK(M,q) \vee S^kK(M,q) \vee & & \\
 & & & & S^{k-1}(Y * Y) & & S^{k-1}(K(M,q) * K(M,q)) & &
\end{array}$$

where we have a wedge of three maps in the lower row.

9.3.

<u>PROPOSITION</u>. Consider Top_G morphisms $X \xrightarrow{f} SY \xrightarrow{g} Z$ where $g = g_1 \vee g_2$, then:

i) $(g \circ f)^*(z) = (g_1 \circ f)^*(z) + (g_2 \circ f)^*(z)$.

ii) For a natural operation ω one has:

$\omega_{g \circ f}(z) = \omega_{g_1 f}(z) + \omega_{g_2 f}(z)$, in the sense that the left-hand side is defined whenever the right-hand side is, with smaller indeterminacy.

P r o o f. Obvious.

Applying this to the factorization of the F associated to x+y in 9.2, we conclude that $\psi_f(x+y)$ equals the sum of three terms, two of which are $\psi_f(x)$ and $\psi_f(y)$. The third one can be identified with the functional square associated to the composition:

$$X \to S^k Y \to S^{k-1}(Y * Y) \to S^{k-1}(K(M,q) * K(M,q)) \to S^k K(M,q).$$

The last map here is the $(k-1)^{th}$ suspension of the Hopf map H(m) constructed from m.

In the subsections to come we will prove that already $H(m)^*(S\chi_q) = 0$. Hence we can calculate $\psi_f(x+y)$ by combining naturality of the functional square with the calculation of $Sq^{q+1}_{H(m)}(S\chi_q)$.

9.4.

In chapter II we will use the following version of the foregoing: Let $\phi : H \to G$ be a homomorphism and let $M \in C_G$. Then H acts on K(M,q) by using ϕ. Moreover K(M,q) can be viewed as the Eilenberg-Maclane complex of the $M^H \in C_H$ defined by $M^H(H/K) = M(G/\phi K)$ etc. In fact M even yields an element of C_ϕ.

Now consider an H-complex Y and $X \in Top_H$ and an H-equivariant map $f : X \to S^k Y$. Then given $y \in H^q_H(Y;M^H)$ we can define $\psi(y)$ by using the

Top_ϕ version of the functional square. Everything done so far remains true and we are led to the same problem as in 9.3. since K(M,q) is a G-complex and H(m) is G-equivariant.

9.5.

We define $H : SK(M,q) \to K(M,q+1)$ by the formula $H^*(\chi_{q+1}) = S\chi_q$. We have the adjoint equivariant map $\mathrm{ad}\, H : K(M,q) \to \Omega K(M,q+1)$. We can view H as the composition $E \circ S(\mathrm{ad}\, H)$, where E denotes the natural "evaluation" map $S\,\Omega K(M,q+1) \to K(M,q+1)$.

PROPOSITION. The following diagram commutes up to homotopy:

$$\begin{array}{ccc}
K(M,q) \times K(M,q) & \xrightarrow{\quad m \quad} & K(M,q) \\
\downarrow{\scriptstyle \mathrm{ad}\, H} & & \downarrow{\scriptstyle \mathrm{ad}\, H} \\
\Omega K(M,q+1) \times \Omega K(M,q+1) & \xrightarrow{\text{loop sum } \ell} & \Omega K(M,q+1)
\end{array}$$

Proof. We show that the two compositions have adjoint maps which are equivariantly homotopic. This is done by noting that they classify the same element of $H_G^{q+1}(S(K(M,q) \times K(M,q));M)$.

Q.E.D.

9.6.

We consider the situation where we are given $K,L \in Top_G$ with base-point and morphisms $m : K \times K \to K$ and $j : SK \to L$ such that

$$\begin{array}{ccc}
K \times K & \xrightarrow{\ m\ } & K \\
\downarrow{\scriptstyle \mathrm{ad}\, j} & & \downarrow{\scriptstyle \mathrm{ad}\, j} \\
\Omega L \times \Omega L & \xrightarrow{\ \ell\ } & \Omega L
\end{array}$$

commutes up to equivariant homotopy

where ℓ is the loop multiplication.

PROPOSITION. There exists an equivariant homotopy Λ from $E \circ H(\ell) : \Omega L * \Omega L \to S\Omega L \to L$ to a constant map.

P r o o f. Define $\Lambda : I \times I \times \Omega L \times \Omega L \to L$ by the formula

$$\Lambda(u,v,\sigma,\tau) = \begin{cases} \sigma(2v + u) & \text{for } 2v + u \leq 1, \\ \tau(2v - u - 1) & \text{for } 2v - u \geq 1, \\ \text{basepoint} & \text{for } 1-u \leq 2v \leq 1+u. \end{cases}$$

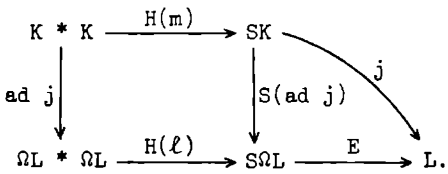
Then $\Lambda(u,v,\sigma,\tau)$ does not depend on τ for $v = 0$ and does not depend on σ for $v = 1$ hence Λ induces a well-defined map $I \times \Omega L * \Omega L \to L$.

Q.E.D.

We conclude that we can extend E to a map

$$E \cup \Lambda : S\Omega L \underset{H(\ell)}{\cup} C(\Omega L * \Omega L) \to L.$$

Since the Hopf construction H( ) is natural, the following diagram commutes up to an equivariant homotopy h:



Hence the map $j \circ H(m) : K * K \to SK \to L$ is homotopic to a constant map; this homotopy, N, is the composition of h and $\Lambda(\text{ad } j * \text{ad } j)$.

In particular we have $H(m)^* H_1^* x_{q+1} = 0$ in the situation of 9.5.; this justifies the statement in 9.3.

9.7.

PROPOSITION. Let $L \in Top_G$ with invariant basepoint p then $d \circ E : S\Omega L \to L \to L \times L$ (where d denotes the diagonal map) is equivariantly homotopic to a map into $L \vee L$.

P r o o f. We define $\Gamma : I \times I \times \Omega L \to L \times L$ by the formula

$$\Gamma(t,v,\sigma) = \begin{cases} (\sigma(2v),p) & \text{for } 2v \leq 1-t, \\ (p,\sigma(2v-1)) & \text{for } 2v \geq 1+t \text{ and} \\ (\sigma(v-\tfrac{1}{2}t+\tfrac{1}{2}),\sigma(v+\tfrac{1}{2}t-\tfrac{1}{2})) & \text{for } 1-t \leq 2v \leq 1+t. \end{cases}$$

This Γ is continuous and $\Gamma(t,v,\sigma) = p$ for $v = 0$ or 1, so induces a well-defined map on $I \times S\Omega L$.

Q.E.D.

A fortiori: denoting the diagonal $L \to L \wedge L$ by Δ, we see that $\Delta \circ E$ is homotopic to the constant map by

$$\Gamma(t,v,\sigma) = \begin{cases} \sigma(v-\tfrac{1}{2}t+\tfrac{1}{2}) \wedge \sigma(v+\tfrac{1}{2}t-\tfrac{1}{2}) & \text{for } 1-t \leq 2v \leq 1+t, \\ p \wedge p & \text{otherwise.} \end{cases}$$

The composition of this homotopy with $H(\ell) : \Omega L * \Omega L \to S\Omega L$, which maps (v,σ,τ) to $(v,\sigma\tau)$, looks as follows:

$$\begin{cases} \sigma(2v-t+1) \wedge \sigma(2v+t-1) & \text{for } 1-t \leq 2v \leq t, \\ \tau(2v-t) \wedge \sigma(2v+t-1) & \text{for } t \leq 2v \leq t+1 \text{ and } 1-t \leq 2v \leq 2-t, \\ \tau(2v-t) \wedge \tau(2v+t-2) & \text{for } 2-t \leq 2v \leq 1+t, \\ p \wedge p & \text{elsewhere.} \end{cases}$$

We extend this as follows:

THEOREM. There exists an $\mathcal{E}$ such that the following diagram commutes up to equivariant homotopy:

$$\begin{array}{ccc} S\Omega L \cup_{H(\ell)} C(\Omega L * \Omega L) & \xrightarrow{E \cup \Lambda} & L \\ \Big\downarrow \text{collapsing map } c. & & \Big\downarrow \Delta \\ S(\Omega L * \Omega L) & \xrightarrow{\mathcal{E}} & L \wedge L \end{array}$$

P r o o f. On $S\Omega L$ the composition $\mathcal{E} \circ c$ will be the constant map and the composition $\Delta \circ (E \cup \Lambda)$ is ΔE; we take the homotopy to be the Γ

constructed in the above proposition. The composition with $H(\ell)$ yields a homotopy for $\Omega L * \Omega L$; we want to extend it to the whole of $C(\Omega L * \Omega L)$.

We define $P : I \times I \times I \times \Omega L \times \Omega L \to L \wedge L$ by the formulas:

$P(t,u,v,\sigma,\tau) =$ (i) $\sigma(u+2v-t+1) \wedge \sigma(u+2v+t-1)$ for the (t,u,v) such that $1-t \leq u+2v \leq t$;

(ii) $\tau(2v-u-t) \wedge \tau(2v-u+t-2)$ for the (t,u,v) such that $2-t \leq 2v-u \leq 1+t$;

(iii) $\tau(2v-u-t) \wedge \sigma(2v+u+t-1)$ for the (t,u,v) such that $2v \geq u+t \geq 2v-1$ and $2-2v \geq u+t \geq 1-2v$;

(iv) $p \wedge p$ elsewhere.

One easily checks that the formulas yield $p \wedge p$ in case one of the inequalities becomes an equality and that the domains defined by the inequalities only intersect in such points. Hence P is well defined by these formulas.

For $v = 1$, which can only happen in the second and the fourth case, P does not depend on σ; likewise for $v = 0$ P does not depend on τ. Hence P yields a well-defined map $I \times I \times \Omega L * \Omega L \to L \wedge L$

For $u = 1$ we find $p \wedge p$. Hence P yields a well-defined map $I \times C(\Omega L * \Omega L) \to L \wedge L$. For $u = 0$ the formulas for P coincide with those for Γ. Hence P yields a homotopy extending Γ.

For $t = 1$ (first, second and fourth case) we find

$\Delta\sigma(u+2v)$ for (u,v) such that $u+2v \leq 1$,

$\Delta\tau(2v-u-1)$ for (u,v) such that $1 \leq 2v-u$,

$p \wedge p$ elsewhere,

which are just the formulas for $\Delta \circ \Lambda$. Hence the homotopy starts right.

For $t = 0$ (third and fourth case) one gets $\tau(2v-u) \wedge \sigma(2v+u-1)$ for the (t,u,v) such that $2v \geq u \geq 2v-1$, $2-2v \geq u \geq 1-2v$, and $p \wedge p$ elsewhere. We define $E(u,(v,\sigma,\tau))$ by this formula; this is possible because it yields $p \wedge p$ for $u = 0$ or 1. Q.E.D.

We may replace the E in the proof by a homotopic one; we shall use the one defined by the less complicated formula:

$$E(u,(v,\sigma,\tau)) = \tau(v) \wedge \sigma(u).$$

From this formula one sees that E can be factorized as follows:

$(E \wedge E) \circ W : S(\Omega L * \Omega L) \to S\Omega L \wedge S\Omega L \to L \wedge L$,

where W maps $(u, (\sigma,v,\tau)) \in S(\Omega L * \Omega L)$ to $(v,\tau) \wedge (u,\sigma) \in S\Omega L \wedge S\Omega L$; in fact W is an equivalence for L a complex.

9.8.

We define a map $J : SK \cup C(K * K) \to S\Omega L \cup C(\Omega L * \Omega L)$ as follows:

a) On $[\frac{1}{2},1] \times K * K$ by the formula

$J(t,(s,x,y)) = (2t-1,(s,\mathrm{ad}\ j(x), \mathrm{ad}\ j(y)))$.

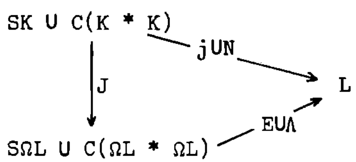
b) On $[0,\frac{1}{2}] \times K * K$ by using the homotopy between

$S(\mathrm{ad}\ j) \circ H(m) : K * K \to SK \to S\Omega L$ and

$H(\ell) \circ (\mathrm{ad}\ j * \mathrm{ad}\ j) : K * K \to \Omega L * \Omega L \to S\Omega L$ from 9.6.

c) On SK by $S(\mathrm{ad}\ j)$.

Then the following diagrams commute:



$$\begin{array}{ccc} SK \cup C(K * K) & \xrightarrow{\text{collaps } c} & S(K * K) \\ \downarrow J & & \downarrow \\ S\Omega L \cup C(\Omega L * \Omega L) & \xrightarrow{c} & S(\Omega L * \Omega L) \end{array}$$

where the map on the right side of the diagram is defined to be the sum (in S-sense) of $S(adj * adj)$ and the constant map, hence is homotopic to $S(adj * adj)$.

We may now draw the following conclusion from theorem 9.7.:

THEOREM. In the situation described in 9.6. there exists a diagram which commutes up to homotopy (everything equivariant):

$$\begin{array}{ccccc} SK \cup C(K * K) & & \xrightarrow{\quad j \cup N \quad} & & L \\ \downarrow \text{collaps } c & & & & \downarrow \text{diagonal} \\ S(K * K) & \xrightarrow{W} & SK \wedge SK & \xrightarrow{j \wedge j} & L \wedge L \end{array}$$

9.9.

Let $A, B \in Top_G$ with invariant basepoint and $M \in C_G$. Then the long exact sequence

$$\ldots\ H^n_G(A \wedge B; M) \rightarrow H^n_G(A \times B; M) \rightarrow H^n_G(A \vee B; M) \ \ldots\ldots$$

splits in a natural way.

Moreover for $a \in H^p_G(A; M^1)$ and $b \in H^q_G(B; M^2)$ such that $p, q > 0$, $a \times b = pr_1^* a \cup pr_2^* b \in H^{p+q}_G(A \times B; M^1 \hat{\otimes} M^2)$ maps to 0 in $H^{p+q}_G(A \vee B; M^1 \hat{\otimes} M^2)$. Hence there is a canonical element, say $a \wedge b$, in $H^{p+q}_G(A \wedge B; M^1 \hat{\otimes} M^2)$ which maps to $a \times b$.

Now we are ready to calculate $Sq^{q+1}_{H(m)}(S\chi_q)$, as announced in 9.3. To this end we remark that in the definition of the functional operation on a pair (X,A) we may interpret the coboundary operator δ $H^n_G(A;M) \to H^{n+1}_G(X,A;M)$ as the cohomology of the natural collapsing map $X \cup CA \to SA$ preceded by the suspension isomorphism. Accordingly we calculate the suspension of $Sq^{q+1}_{H(m)}(S\chi_q)$.

Consider the following diagram, where K stands for $K(M,q)$, L for $K(M,q+1)$, and j for H_1:

$$
\begin{array}{ccc}
 & \overset{j^*}{\dashrightarrow} & S\chi_q \in H^{q+1}_G(SK;\ M) \\
 & & \uparrow \\
\chi_{q+1} \in H^{q+1}_G(L;M) & \xrightarrow{(j\cup N)^*} & H^{q+1}_G(SK \cup C(K * K);\ \ M) \\
\downarrow Sq^{q+1} & & \downarrow Sq^{q+1} \\
\chi_{q+1} \cup \chi_{q+1} \in H^{2q+2}_G(L;S^2M) & \xrightarrow{(j\cup N)^*} & H^{2q+2}_G(SK \cup G(K * K);S^2M) \\
\uparrow \Delta^* & & \uparrow C^* \\
\chi_{q+1} \wedge \chi_{q+1} \in H^{2q+2}_G(L \wedge L;S^2M) & \xrightarrow{E^*} & H^{2q+2}_G(S(K * K);S^2M) \\
\downarrow (j\wedge j)^* & \overset{W^*}{\nearrow} & \\
S\chi_q \wedge S\chi_q \in H^{2q+2}_G(SK \wedge SK;S^2M) & &
\end{array}
$$

The column on the right indicates the construction of $S\ Sq^{q+1}_{H(m)}(S\chi_q)$. The upper square in the diagram is commutative because Sq is natural; the lower square and triangle are precisely the cohomology of the commutative diagram of maps in 9.8.

We can take the image of χ_{q+1} in $H^{q+1}_G(SK \cup C(K * K);M)$ as the lifting of $S\chi_q$ to $H^{q+1}_G(SK \cup C(K * K);M)$ we need, since $H^*_1(\chi_{q+1}) = S\chi_q$. Hence $S\ Sq^{q+1}_{H(m)}(S\chi_q)$ is just the image of $S\chi_q \wedge S\chi_q$ under W.

<u>THEOREM</u>. $S^{q+1}_{H(m)}(S\chi_q) = \chi_q * \chi_q$.

In the situation of 9.3. this implies that

$Sq^{q+1}_{H(m)(\gamma_x * \gamma_y)}(S\chi_q)$ equals the image of $S(x \wedge y)$ in $H^{2q+1}_G(K * K;S^2M)$.

Since the composition $SY \to Y * Y \to S(Y \wedge Y)$ is equal to $S(\Delta)$ we get for the functional Sq^{q+1} of $S\chi_q$ associated to the map $SY \to SK(M,q)$: $S\Delta^*(x \wedge y) = S(x \cup y)$. Hence the third term of $\psi_f(x+y)$ is equal to $f^*S^k(x \cup y)$.

THEOREM. $\psi_f(x+y) = \psi_f(x) + \psi_f(y) + f^*S^k(x \cup y)$.

§10. Equivariant Cech cohomology.

10.1.

Consider a linear action of G on $\mathbb{R}^m$. Then an invariant $X \subset \mathbb{R}^m$ will be called a G-neighborhood retract (GNR) if there exists an invariant neighborhood U of X and an equivariant retraction $r : U \to X$.

PROPOSITION. Let $X \subset \mathbb{R}^m$ be a GNR and let $Y \subset \mathbb{R}^n$ be equivariantly homeomorphic to X; then Y is a GNR.

P r o o f. We have the equivariant maps $i : X \subset U$, $r : U \to X$ and $j : U \subset \mathbb{R}^m$. Since X is a NR it is locally closed hence locally compact; hence Y is locally compact and therefore locally closed. We write $Y = C \cap V$ with C closed and V open; by replacing C by $\bigcap_G gG$ and V by $\bigcap_G gV$ if necessary we may assume that C and V are invariant.

The Tietze Extension Theorem applied to the closed Y in V states that there exists a continuous $f : V \to \mathbb{R}^m$ such that $f|Y = j \circ i \circ h$, where h is the homeomorphism $Y \to X$. Then $\hat{f}$ defined by

$\hat{f}(x) = |G|^{-1} \sum_{g \in G} f(xg^{-1})g$ is equivariant.

Now $\hat{f}^{-1}U$ is invariant and open and $h^{-1}r\hat{f} : \hat{f}^{-1}U \to Y$ is an invariant retraction. Q.E.D.

We will call $Y \in Top_G$ a G-Euclidean neighborhood retract (GENR) if Y is equivariantly homeomorphic to some GNR $X \subset \mathbb{R}^m$.

10.2.

For pairs (X,Y) where $Y \subset X \subset E$ are locally compact invariant parts of some GENR E and for $M \in C_G$ we define:

$$\overline{H}^q_G(X,Y;M) = \lim_{\Lambda} H^q_G(V,W;M),$$

where Λ is the collection of invariant neighborhoods (V,W) of (X,Y) such that $V \supset W$, ordered by inverse inclusion.

PROPOSITION. Let E and E' be GENR's and let $X' \subset E'$ be a locally compact invariant part.

a) If $f : X' \to E$ is continuous and equivariant there exists a continuous and equivariant $F : U' \to E$ such that U' is an invariant open neighborhood of X' and such that $F|X' = f$.

b) If F an $H : E' \to E$ are continuous and equivariant and if $\theta_t : X' \to E$ is an equivariant homotopy between $F|X'$ and $H|X'$ there exists an equivariant homotopy $\Theta_t : U'' \to E$ such that U'' is an invariant open neighborhood of X' and such that $\theta = \Theta|X'$, $\Theta_1 = H|U''$ and $\Theta_0 = F|U''$.

P r o o f. a) We have $i : E \to V$ and $r : V \to E$ where $V \subset \mathbb{R}^n$ is G-invariant and open and where r and i are equivariant and continuous such that $r \circ i = id$. Furthermore one may assume that there is an open $E'' \subset E'$ such that X' is closed in E''; one may assume E'' to be G-invariant.

According to the Tietze theorem $i \circ f : X' \to \mathbb{R}^n$ has an extension $\phi : E'' \to \mathbb{R}^n$, hence (by taking the average) also an equivariant extension Φ. We take $U' = \Phi^{-1}(V)$ and $F = r\Phi$.

b) Just as (a) by paraphrasing [5 ,VIII.6.2.]. Q.E.D.

For an equivariant $f : (X',Y') \to (X,Y)$ where $Y \subset X \subset E$ and $Y' \subset X' \subset E'$ as above there exists an equivariant extension to an open and invariant $U' \supset X'$. Hence for every pair $W \subset V$ of invariant neighborhoods of $Y \subset X$ there is defined

$$F_{VW} : H^q_G(V,W;M) \underset{F}{\to} H^q_G(F^{-1}V,F^{-1}W;M) \to \overline{H}^q_G(X',Y';M).$$

Together they induce a transformation of direct systemshence induce $\overline{F} : \overline{H}^q_G(X,Y;M) \to \overline{H}^q_G(X',Y';M)$.

Similarly it follows from (b) that this $\overline{F}$ does not depend on the extension F of f. Henceforth we will call it $\overline{f}$.

10.3.

Let $Y \subset X$ be locally compact parts of an ENR E and also elements of Top_G. Then $X \subset Map(G,E)$ is invariant and $Map(G,E)$ a GENR hence according to the foregoing $\overline{H}^q_G(X,Y;M)$ is defined and we have:

i) If f and g are equivariantly homotopic then $\overline{f} = \overline{g}$.

ii) $\overline{id} = id$ and $\overline{(f\,g)} = \overline{g}\,\overline{f}$.

iii) In particular $\overline{H}^q_G(X,Y;M)$ only depends on the equivariant homotopy type of (X,Y).

iv) There is a natural exact sequence:

$$\ldots \overline{H}^q_G(X;M) \to \overline{H}^q_G(Y;M) \to \overline{H}^q_G(X,Y;M) \to \overline{H}^{q+1}_G(X;M) \ldots$$

v) There is a natural transformation

$\rho : \overline{H}^q_G(X,Y;M) \to H^q_G(X,Y;M)$ which is an isomorphism if X and Y are GENR's.

vi) For locally compact invariant parts X_1 and X_2 of a GENR E such that X_1-X_2 and X_2-X_1 are open in $X_1 \cup X_2 - X_1 \cap X_2$ there is an excision isomorphism:

$\overline{H}^q_G(X_1 \cup X_2, X_1; M) = \overline{H}^q_G(X_2, X_1 \cap X_2; M)$, hence a Mayer-Vietoris sequence:

$$\ldots \overline{H}^q_G(X_1 \cup X_2; M) \to \overline{H}^q_G(X_1, M) \oplus \overline{H}^q_G(X_2; M) \to \overline{H}^q_G(X_1 \cap X_2; M) \ldots$$

vii) $\overline{H}^q_G(\ \ ;M)$ is "continuous": repeating the limit procedure yields the same groups $\overline{H}$.

10.4.

Let X be a Hausdorff G-space and suppose $X = \bigcup_{i=0}^{m} X_i$, where every X_i is an open invariant GENR. Then X is a GENR.

P r o o f. We may assume X to be a closed part of some $\mathbb{R}^n$ [5, 8.8.], hence we may assume it is an invariant closed part of some $\mathbb{R}^n$ equipped with an action of G. Furthermore it suffices to consider the case m = 1.

Assume that the $r_i : V_i \to X_i$ are equivariant retractions, where the $V_i \subset \mathbb{R}^n$ are open and invariant. Then putting $V_{01} = r_0^{-1}(X_0 \cap X_1) \cap r_1^{-1}(X_0 \cap X_1)$, r_0 and r_1 restrict to neighborhood retractions $V_{01} \to X_0 \cap X_1$.

However $X_0 \cap X_1$ is an open and invariant part of X_0, hence a GENR. Hence there exists an open $U_{01} \supset X_0 \cap X_1$ where r_0 and r_1 are equivariantly homotopic by $r_t : U_{01} \to X_0 \cap X_1$.

Let $U_0 \subset V_0$ and $U_1 \subset V_1$ be open neighborhoods of $X-X_1$ resp. $X-X_2$ such that $\overline{U}_0 \cap \overline{U}_1 = \emptyset$: we may assume that U_0 and U_1 are invariant by replacing by $\bigcap_{g\in G} gU_i$ if necessary.

There exists a continuous $\tau : \mathbb{R}^n \to [0,1]$ such that $\tau|U_0 = 0$ and

$\tau|U_1 = 1$. We may assume τ to be equivariant by replacing it by $\inf \{\tau(g\,x) \mid g \in G\}$ if necessary.

Now $U = U_0 \cup U_1 \cup U_{01}$ is an invariant neighborhood of X and $\rho : U \to X$ is an equivariant retraction where ρ is defined by $\rho|U_0 = r_0$, $\rho|U_1 = r_1$ and $\rho(x) = r_{\tau(x)}(x)$ for $x \in U_{01}$. Q.E.D.

COROLLARY. Let M be a (topological) manifold with a free action of G and let $K \subset M$ be compact and invariant. Then there exists an open and invariant $E \supset K$ in M which is a GENR.

P r o o f. One can cover K by open sets U which are sufficiently small to yield $gU \cap U = \emptyset$ for every $g \neq 1$; hence we can cover K by finitely many such U_i. Take $E = \bigcup_i GU_i$; this a GENR according to the proposition since every GU_i is of the form $G \times U_i$ with U_i an ENR. One easily checks that $G \times U$ is a GENR for U an ENR. Q.E.D.

§11. Poincaré duality.

11.1.

We are interested in the groups $H^n_G(V;M)$ for topological manifolds with a free action of G and for $M \in C_G$. This group is isomorphic to the one we get by replacing M by the module M(G/e) and so is classically known. In this section we give a summary of the facts for later reference.

Let V be a manifold of dimension n then $H_n(V,V-x;\mathbb{F}_2) = \mathbb{F}_2$ for all $x \in V$ and this isomorphism is unique since Aut $\mathbb{F}_2 = \{1\}$. For $A \subset V$ denote by $\Gamma(A)$ the ring of continuous functions $A \to \mathbb{F}_2$. The inclusions $(V,V-A) \subset (V,V-x)$ for $x \in A$ induce a homomorphism

$J_A : H_n(V,V\text{-}A;\mathbb{F}_2) \to \Gamma(A)$.

PROPOSITION. [5 ,VIII.3.3.].

J_A is isomorphic for compact A.

More generally: for neighborhood retracts X and Y in V we have:

$H_n(Y,X;\mathbb{F}_2) = \Gamma_b(V\text{-}X,V\text{-}Y)$, the ring of continuous functions on V-X with compact support which vanish on V-Y.

EXAMPLE. For a compact, connected manifold M of dimension n without boundary we have: $J_M : H_n(M;\mathbb{F}_2) \cong \mathbb{F}_2$.

We denote $J_M^{-1}(1)$ by $\mathcal{O}_M$.

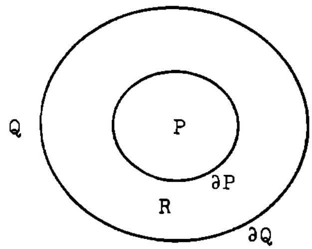
EXAMPLE. For a compact connected manifold L of dimension n with boundary ∂L we consider $M = L \cup \partial L \times [0,1)$.

Then $J_L : H_n(M,M\text{-int } L;\mathbb{F}_2) \cong \mathbb{F}_2$. However this group is by retraction isomorphic to $H_n(L,\partial L;\mathbb{F}_2)$. Hence we get $\mathcal{O}_{L,\partial L} \in H_n(L,\partial L;\mathbb{F}_2)$.

PROPOSITION. [5 ,VII.2.9.]

$\partial : H_n(L,\partial L;\mathbb{F}_2) \to H_{n-1}(\partial L;\mathbb{F}_2)$ maps $\mathcal{O}_{L,\partial L}$ to $\mathcal{O}_{\partial L}$.

PROPOSITION. Let $(P,\partial P)$ and $Q,\partial Q)$ be connected manifolds of dimension n with boundary such that P is a regular domain in Q. Then $\mathcal{O}_{P,\partial P}$ is induced by $\mathcal{O}_{Q,\partial Q}$.

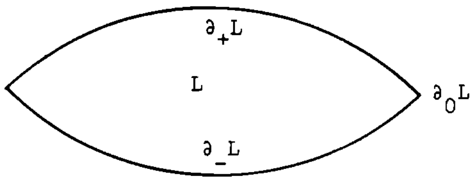


P r o o f and explanation. Consider the diagram:

$$\begin{array}{ccccc}
H_n(P,\partial P;\mathbb{F}_2) & \longrightarrow & H_n(Q,R;\mathbb{F}_2) & \longleftarrow & H_n(Q,\partial Q;\mathbb{F}_2) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
H_n(M,M\text{-int } P;\mathbb{F}_2) & \longrightarrow & H_n(N,N\text{-int } P;\mathbb{F}_2) & \longleftarrow & H_n(N,N\text{-int } Q;\mathbb{F}_2) \\
& \searrow & \swarrow & & \swarrow \\
& & \Gamma_b(\text{int } P) & \xleftarrow{\sim} & \Gamma_b(\text{int } Q)
\end{array}$$

where $R = \overline{Q-P}$, $M = P \cup \partial P \times [0,1) \subset Q$, $N = Q \cup \partial Q \times [0,1)$ hence $N - \text{int}\, P = R \cup \partial Q \times [0,1)$. One easily checks that the diagram commutes.

However $\mathcal{O}_{P,\partial P}$ yields $1 \in \mathbb{F}_2 = \Gamma_b(\text{int}\, P)$ and $\mathcal{O}_{Q,\partial Q}$ yields $1 \in \mathbb{F}_2 = \Gamma_b(\text{int}\, Q)$. Hence $\mathcal{O}_{P,\partial P}$ and $\mathcal{O}_{Q,\partial Q}$ have the same image in $H_n(Q,R;\mathbb{F}_2)$ Q.E.D.



COROLLARY. We consider a bordism L between manifolds with boundary i.e. L is a manifold of dimension n+1 with boundary ∂L which is the union of the manifolds of dimension n $\partial_- L$ and $\partial_+ L$ along their common boundary $\partial_0 L$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}_{L,\partial L} \in H_{n+1}(L,\partial L;\mathbb{F}_2) & \overset{\partial}{\to} & \mathcal{O}_{\partial L} \in H_n(\partial L;\mathbb{F}_2) \\ & & \downarrow \\ & & H_n(\partial L,\partial_+ L;\mathbb{F}_2) \overset{\sim}{\to} \mathcal{O}_{\partial_- L,\partial_0 L} \in H_n(\partial_- L,\partial_0 L;\mathbb{F}_2) \end{array}$$

this follows by combining the second proposition with the last one for the case $(P,\partial P) = (\partial_- L,\partial_0 L)$, $(Q,\partial Q) = (\partial L,\emptyset)$, $R = \partial_+ L$.

11.2.

Consider a k-dimensional real vectorbundle ξ over a sufficiently nice space X (e.g. a manifold) with total space $E(\xi)$ and projection p. Then it can be equipped with an inner product; denote the associated disc bundle by $D(\xi)$ and the sphere bundle by $S(\xi)$.

PROPOSITION. There exists a unique $U_\xi \in H^k(E(\xi),E(\xi)-D(\xi);\mathbb{F}_2)$ such that the restriction to any fibre is precisely $1 \in \mathbb{F}_2 \cong H^k(\text{fibre};\mathbb{F}_2)$, called the Thom class of ξ.

PROPOSITION. In case X is a manifold of dimension n and $A \subset X$, let $\mathcal{O}_A \in H_n(V,V-A;\mathbb{F}_2) \cong \Gamma_b(A)$ correspond to $1 \in \Gamma_b(A)$ and let $\mathcal{O}_{p^{-1}A} \in H_{n+k}(E(\xi),E(\xi)-D(\xi) \cap p^{-1}A;\mathbb{F}_2)$ correspond to 1.
Then we have $p_*(\mathcal{O}_{p^{-1}A} \cap U_\xi) = \mathcal{O}_A$.

P r o o f. The first proposition is well-known.
The second proposition is immediate after writing down the characterizations of U_ξ and $\mathcal{O}$.

COROLLARY. Let $(L,\partial L)$ be a manifold of dimension n with boundary and let ξ be a vectorbundle over L; then ξ can be extended to $M = L \cup \partial L \times [0,1)$. The foregoing applied to X = M and A = int L together with the retraction isomorphisms

$H_n(L,\partial L;\mathbb{F}_2) = H_n(M,M - \text{int } L;\mathbb{F}_2)$ and

$H_{n+k}(D(\xi),D(\xi|\partial L) \cup S(\xi);\mathbb{F}_2) = H_{n+k}(E(\xi),E(\xi)-D(\xi) \cap p^{-1}(\text{int } L);\mathbb{F}_2)$

yields the formula:

$p_*(U_\xi \cap \mathcal{O}_{D(\xi),D(\xi|\partial L) \cup S(\xi)}) = \mathcal{O}_{L,\partial L}$.

We will apply this to X = V/G (notations from 11.1.), noting that $H_n(V/G;\mathbb{F}_2) = H_n^G(V;\mathbb{F}_2)$ canonically. This yields a map $H_G^n(V;M) \to H_G^{n+k}(E(\xi),E(\xi)-D(\xi);M)$ mapping z to $p_*z \cup U_\xi$.

Let $(L,\partial L)$ and ξ be as above. Then there is a commutative ladder of such Thom maps, as a consequence of the naturality and stability of the cup-product:

$$\begin{array}{ccc}
H^q_G(\partial L;M) & \longrightarrow & H^{k+q}_G(D(\xi|\partial L),S(\xi|\partial L);M) \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
H^{q+1}_G(L,\partial L;M) & \longrightarrow & H^{k+q+1}_G(D(\xi),D(\xi|\partial L) \cup S(\xi);M) \\
\downarrow & & \downarrow \\
H^{q+1}_G(L;\ M) & \longrightarrow & H^{k+q+1}_G(D(\xi),\ S(\xi);\ M) \\
\downarrow & & \downarrow \\
H^{q+1}_G(\partial L;\ M) & \longrightarrow & H^{k+q+1}_G(D(\xi|\partial L),\ S(\xi|\partial L);M)
\end{array}$$

11.3.

Consider a manifold V of dimension n with boundary on which G acts freely and let $K,L \subset V$ be invariant and compact, or more generally closed and contained in an open and invariant GENR $E \subset V$. Then for any IF_2G module M, $\overline{H}^q_G(K,L;M)$ is defined and equal to $\varinjlim H^q_G(U,W;M)$ where (U,W) runs through the set of open and invariant parts of V containing (K,L) (since those inside E form a cofinal subsystem).

The fundamental class $\mathcal{O}_K \in H^G_n(V,V-K;IF_2)$ yields through

$j^W : H^G_n(V,V-K;IF_2) \to H^G_n(V,(V-K) \cup W;IF_2) \cong H^G_n(U-L,(U-K) \cup (W-L);IF_2)$

an element $j^W\mathcal{O}_K$. Capping with it yields homomorphisms

$\xi_{UW} : H^q_G(U,W;M) \cong H^q_G(U-L,W-L;M) \to H^G_{n-q}(U-L,U-K;M) \cong H^G_{n-q}(V-L,V-K;M)$

which are consistent hence induce a transformation:

$\overline{H}^q_G(K,L;M) \to H^G_{n-q}(V-L,V-K;M)$.

THEOREM. This map is isomorphic.

P r o o f. We copy the proof of [5,VIII.7.2.] with the following alterations.

Cases (1) to (4): replace $M = IR^n$ by $M = G \times IR^n$; if K is any of the types mentioned, we have $G \times K \subset M$.

Case (5): as in I.10.4. we cover K with "charts" of the form $G \times \mathbb{R}^n$.

Case (6) is diagram chasing, hence does not change. Q.E.D.

The nontrivial facts we used in this proof are the propositions VII.12.22, VIII.7.6. and VIII.7.7. from [5]. These in turn are consequences of VII.12.6. (naturality of cap) and VII.12.20. (stability of cap). We have already seen that these properties of the cap-product remain true in the equivariant case.

11.4.

a) Let V be a compact connected manifold of dimension n without boundary. Then there exists an orientation class $\mathcal{O}_V \in H_n(V;\mathbb{F}_2)$ and the theorem of 11.3. for the case K = V, L = ∅ states that
$\cap \mathcal{O}_V : H^q_G(V;M) \to H^G_{n-q}(V;M)$ is isomorphic for any q.

b) Let L be a compact connected manifold of dimension n with boundary. Then there exists an orientation class $\mathcal{O}_{L,\partial L}$ according to 11.1. and the following diagram commutes:

$$\begin{array}{ccccccc}
H^q_G(L,\partial L;M) & \to & H^q_G(L;M) & \to & H^q_G(\partial L;M) & \overset{\delta}{\to} & H^{q+1}_G(L,\partial L;M) \\
\downarrow \cap \mathcal{O}_{L,\partial L} & & \downarrow \cap \mathcal{O}_{L,\partial L} & & \downarrow \cap \mathcal{O}_{L} & & \downarrow \cap \mathcal{O}_{L,\partial L} \\
H^G_{n-q}(L;M) & \to & H^G_{n-q}(L,\partial L;M) & \to & H^G_{n-q-1}(\partial L;M) & \to & H^G_{n-q-1}(L;M)
\end{array}$$

The vertical arrows in this diagram are isomorphisms.

For the proof of this statement we refer to [5 ,VIII.9.1.]. It relies on the naturality VII.12.6. and the stability VII.12,13, and 14 of the cap-product.

c) Finally for a bordism L between manifolds with boundary $(\partial_+ L, \partial_0 L)$ and $(\partial_- L, \partial_0 L)$ as in the corollary in 11.1. the following diagram commutes:

$$
\begin{array}{ccc}
H_G^q(L,\partial L;M) & \xrightarrow{\cap \mathcal{O}_{L,\partial L}} & H^G_{n+1-q}(L;M) \\
\downarrow & & \downarrow \\
H_G^q(L,\partial_+ L;M) & \xrightarrow{\cap \mathcal{O}_{L,\partial L}} & H^G_{n+1-q}(L,\partial_- L;M) \\
\downarrow \delta & & \downarrow \\
H_G^{q+1}(\partial L,\partial_+ L;M) \cong H_G^{q+1}(\partial_- L,\partial_0 L;M) & \xrightarrow{\cap \mathcal{O}_{\partial_- L,\partial_0 L}} & H^G_{n+1-q}(\partial_- L;M) \\
\downarrow & & \downarrow \partial \\
H_G^{q+1}(L,\partial L;M) & \xrightarrow{\cap \mathcal{O}_{L,\partial L}} & H^G_{n-q}(L;M)
\end{array}
$$

Hence the second horizontal arrow is an isomorphism by the five lemma.

11.5.

The connection made in 11.2. between fundamental classes $\mathcal{O}$ and Thom classes U implies a connection between the Poincaré duality isomorphism and the Thom map.

Consider $(L,\partial L)$ and ξ as in 11.2.

a) The following diagram commutes:

$$
\begin{array}{ccc}
H_G^q(L,\partial L;M) & \xrightarrow{p^* \text{ anc } UE_\xi} & H_G^{q+k}(D(\xi),D(\xi|\partial L) \cup S(\xi);M) \\
\downarrow \cap \mathcal{O}_{L,\partial L} & & \downarrow \cap \mathcal{O}_{D(\xi),D(\xi|\partial L) \cup S(\xi)} \\
H^G_{n-q}(L;M) & \xleftarrow{p_*} & H^G_{n-q}(D(\xi);M)
\end{array}
$$

This follows from the calculation:

$$p_*(p^*x \cup U_\xi) \cap \mathcal{O}_{D(\xi),D(\xi|\partial L) \cup S(\xi)} =$$

$$= p_*(p^*x \cap (U_\xi \cap \mathcal{O}_{D(\xi),D(\xi|\partial L) \cup S(\xi)})) =$$

$$= x \cap p_*(U_\xi \cap \mathcal{O}_{D(\xi),D(\xi|\partial L) \cup S(\xi)}) =$$

$$= x \cap \mathcal{O}_{L,\partial L}.$$

b) The following diagram commutes according to the same formulas:

$$\begin{array}{ccc}
H_G^q(L;M) & \xrightarrow{\quad p^* \text{ and } \cup E_\xi \quad} & H_G^{q+k}(D(\xi),S(\xi);M) \\
\Big\downarrow \cap\mathcal{O}_{L,\partial L} & & \Big\downarrow \cap\mathcal{O}_{D(\xi),D(\xi|\partial L) \cup S(\xi)} \\
H^G_{n-q}(L,\partial L;M) & \xleftarrow{\quad p_* \quad} & H^G_{n-q}(D(\xi),D(\xi|\partial L);M)
\end{array}$$

c) There exists a similar diagram for $H_G^q(\partial L;M)$ but that is a special case of the above.

CHAPTER II

THE GEOMETRY

§1. Normal maps and embeddings.

1.1.

In the following manifold means: a C^∞ differential manifold satisfying the Hausdorff and second countability axiom. In particular it is a metrizable topological space; a metric can be constructed by using a Riemannian structure or an embedding into some Euclidean space.

Let M and N be manifolds; denote by Hom(M,N) the set of continuous maps $M \to N$, and let d be a metric on N.
The fine C^0-topology on Hom(M,N) is defined by prescribing the base; this is formed by the sets

$$W_{f,\delta} = \{g \mid d(g(x),f(x)) < \delta(x) \text{ all } x \in M\},$$

where $\delta : M \to \mathbb{R}$ is strictly positive and continuous.
By restriction we get a topology on $\mathrm{Hom}^\infty(M,N)$, the set of C^∞ maps $M \to N$.

However, a C^∞ map $f : M \to N$ induces a continuous map from M to $J^r(M,N)$, the r^{th} jet bundle of the pair.
The fine C^r-topology on $\mathrm{Hom}^\infty(M,N)$ is the coarsest topology rendering continuous the map $f \to j^r(f)$, $\mathrm{Hom}^\infty(M,N) \to \mathrm{Hom}(M,J^r(M,N))$.

For manifolds embedded in some Euclidean space and with the induced Riemannian structure, the sets $\{g \in \mathrm{Hom}^\infty(M,N) \mid d(g(x),f(x)) < \delta(x)$ and $||Dg(x)-Df(x)|| < \delta(x)$ for every $x \in M\}$ give a base for the fine C^1-neighborhoods of f.

Let $\{C_i \mid i \in I\}$ be a locally finite covering of M by compacta and for each $i \in I$ let (U_i,h_i) resp. (V_i,k_i) be coordinate-systems around C_i resp. $f(C_i)$. Given a set $\delta = \{\delta_i \mid i \in I\}$ of positive numbers, define $X_{f,\delta}$ as the set of $g \in \mathrm{Hom}^\infty(M,N)$ such that

$g(C_i) \subset V_i$, $|k_i fh_i^{-1}(x)-k_i gh_i^{-1}(x)| < \delta_i$ and

$|D(k_i fh_i^{-1})(x)-D(k_i gh_i^{-1})(x)| < \delta_i$, all $i \in I$, $x \in C_i$.

The sets of the form $X_{f,\delta}$ form a base for the fine C^1-topology on Hom(M,N). This is the topology designated by C^1 in the Séminaire Cartan [8].

The <u>coarse</u> C^r-topology is defined in the same way, but in the definition of a base element as above one only demands the inequality on a compact subset of M.

This topology is the one called C^r in the Séminaire Cartan; it can be defined without using a metric, as a compact-open topology. Notice that the fine and the coarse topology coincide only for compact M.

1.2.

Given manifolds M, N without boundary of dimension m, a <u>normal map</u> from M to N consists of:

a vectorbundle ξ over N,

a continuous map $f : M \to N$,

a bundle map lifting $f, \hat{f} : TM+\varepsilon \to TN+\xi$ mapping the fibers isomorphically, where ε is the trivial bundle of the same fibre dimension as ξ, say k.

<u>Remark</u>: This definition is easily seen to be equivalent to C.T.C. Wall's [15] definition.

Let $D^k(2)$ be the open disk of radius 2, $D(\xi)$ the bundle of disks of radius 1 associated to ξ. Then f can be thought of as a map $X = D^k(2) \times M \to Y = D(\xi)$, using projection and zero section, and $\hat{f}$ can be considered as a kind of derivative for f.

Now from the theory of Hirsch and Gromov (see V. Poenaru's talk in A'dam 1970 [10]) we deduce:

PROPOSITION. In these circumstances and assuming M compact, there exists an immersion $F : X \to Y$ such that its derivative DF is homotopic to $\hat{f}$ as bundle isomorphisms. Given r, two choices for F can be connected by a path in the space of all immersions $X \to Y$, continuous with respect to the coarse C^r-topology.

Remark: In the more general case that M is not compact but f is proper one would like to find an F which is proper if restricted to $\overline{D^k(1) \times M}$ and a homotopy which is also proper there.
This however does not follow from the given reference.

Now we apply the "lemme de Thom au but" (Séminaire Cartan no. 6 [8]) using $M \times M \times \Delta(Y)$ as the submanifold of $J^0(X,Y) \times J^0(X,Y) = X \times X \times Y \times Y$ ($\Delta(Y) \subset Y \times Y$ is the diagonal).

The assertion that $F \times F$, restricted to the complement of $\Delta(X)$ in $X \times X$ does not hit the submanifold means that $F|M$ is injective. So the lemma says that every fine C^r-neighborhood of F contains a new immersion F', such that $F'|M$ is injective, provided that

$2 \dim X + 2 \dim M + \dim Y < 2 \dim X + 2 \dim Y$, i.e. $k > \dim M$.

Remark: In the noncompact case with F proper at $M \times \overline{D^k(1)}$ we can choose F' near enough to F as to assure that F' is proper there.

Now F' is a homeomorphism $M \to F'(M)$ and the image $F'(M)$ is closed in Y.

Remark: In the noncompact case that remains true because F' is continuous, proper and injective om M (see Munkres [9] page 20).

According to Munkres [9] lemma 5.7. applied to $A = M$, using the fact that an immersion in codimension 0 is locally a diffeomorphism, there exists a neighborhood of M in X which is mapped homeomorphically onto an open part of Y by F'.

Composing F' with a "scale-transformation in the D^k-direction" we achieve that $M \times \overline{D^k(1)}$ is mapped homeomorphically by the composition F'' onto a closed part of Y whose boundary is $F''(M \times S^{k-1})$.

Now we will show that F' is regularly homotopic to F, and so is also consistent with $\hat{f}$.

LEMMA. Given $f \in \mathrm{Hom}^\infty(X,Y)$, for each r there exists a C^r-fine open $U \ni f$ such that if $g \in U$ there exists a map $h : [0,1] \to \mathrm{Hom}^\infty(X,Y)$, continuous with respect to the coarse topology, such that $h[0,1] \subset U$, $h(0) = f$, $h(1) = g$.

P r o o f following the model of theorem 4.2. in Munkres [9] and using his notations:

For g in a sufficiently small fine neighborhood of f, $f(U_i) \subset O_i$ implies the same is true for g. So it has become a local problem and we prove something analogous to Munkres' theorem 4.1.

So be given $A \subset V \subset \overline{V} \subset U$ as in loc.cit. and make the same Ψ and α. Let $f_1 = f(1-\Psi)+g\Psi$, $F_t = \alpha f_1+(1-\alpha)f$ then f_1 and F are constantly g where $f = g$ and $F_1 = f_1 = g$ on A. Now one can go on as in theorem 4.2.

loc. cit.; one constructs in the same way F_i, F and f_t.

Q.E.D.

Keeping in mind that immersions form a fine open set in $Hom^\infty(X,Y)$ we may choose U small enough as to assure that all h_t are immersions, if f is an immersion.

Summarizing: we have found F'' such that:

F'' is an immersion of $X = M \times D^k(2)$,

DF'' is homotopic to $\hat{f}$,

F'' is a homeomorphism of $V = M \times \overline{D^k(1)}$ to a closed part of $Y = D(\xi)$ with topological boundary $V = M \times S^{k-1}$.

1.3.

According to the theory of Hirsch and Gromov (see V. Poenaru's lecture at A'dam 1970 corr. 2 [10]) the F just constructed is unique up to regular homotopy, hence so is F''.

So let $H : [0,1] \to Hom^\infty(X,Y)$ map into the subset of immersions and be continuous with respect to the coarse topology, and assume that H(0) and H(1) are embeddings of $M \times \overline{D^k(1)}$.

We want to prove that H(0) and H(1) can be joined by a path in the space of embeddings of $M \times \overline{D^k(1)}$.

We may assume - without loss of generality - that H is independent of $t \in [0,1]$ for $t > 1-\varepsilon$ or $t < \varepsilon$ for some $\varepsilon > 0$.

PROPOSITION. In these circumstances there is a different path H' coinciding with H in a neighborhood of 0 and 1 and consisting of immersions, such that

$$ad\ H' : [0,1] \times X \to Y$$
$$(t\ ,\ x) \to H'(t)(x) \quad \text{is } C^\infty.$$

P r o o f. Let $\{K_i \mid i \in I\}$ be a locally finite covering of X by compacta. For fixed $i \in I$ and $t \in [0,1]$ there exists a $\delta_{t,i} \in \mathbb{R}$, $\delta_{t,i} > 0$, such that

$$d_1(H',H(t)) < \delta_{t,i} \text{ on } K_i \Rightarrow H' \text{ is an immersion.}$$

Because $H|K_i$ is continuous there exists a neighborhood U_t of t such that for $s \in U_t$ $d_1(H(s),H(t)) < \frac{1}{2}\delta_{t,i}$ on K_i. So if H' is a path with $d_1(H'(s),H(s)) < \frac{1}{2}\delta_{t,i}$ on K_i for $s \in U_t$ then H'(s) is an immersion on K_i for $s \in U_t$.

We can find a finite subset T of [0,1] such that $\{U_t \mid t \in T\}$ covers [0,1]; let $\delta_i = \min\{\delta_{t,i} \mid t \in T\}$ then if $d_1(H'(s),H(s)) < \delta_i$ on K_i for all $s \in [0,1]$, H'(s) is an immersion on K_i for all $s \in [0,1]$.

Let δ be a strictly positive function on X such that $\delta < \delta_i$ on K_i then if for all s $d_1(H'(s),H(s)) < \delta$ one may conclude that H'(s) is an immersion everywhere. But according to Munkres' theorem 4.6. [9] one can then find an H' such that ad H' is C^∞ (if the $\varepsilon > 0$ used there is sufficiently small nothing changes near t = 0 or 1). Q.E.D.

Extending H' by a constant path one may assume H' to be defined on $\mathbb{R}$.

Because the image of $H' : \mathbb{R} \to \mathrm{Hom}^\infty(X,Y)$ consists of immersions and because $ad\ H' : \mathbb{R} \times X \to Y$ is C^∞, the map

$$G : \mathbb{R} \times X \to \mathbb{R} \times Y$$
$$(t\ ,\ x) \to (t,H'(t)(x)) \text{ is an immersion.}$$

Now we can find in any fine neighborhood of G another immersion G' which

i) coincides with G on $((-\infty,\varepsilon) \cup (1-\varepsilon,\infty)) \times X$

ii) maps $[0,1] \times M$ injectively, provided that $k > \dim M+1$.

Suppose $G'(t,x) = (s(t,x),g'(t,x))$ then the C^∞ map $S : (t,x) \to (s(t,x),x)$ is near enough to the identity map in the fine C^1-sense to be a diffeomorphism $\mathbb{R} \times X \to \mathbb{R} \times Y$, if G' was chosen near enough to G.

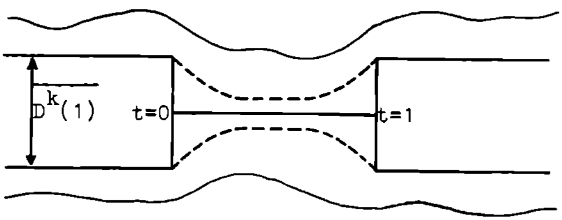
We replace G' by $G'' = G' \circ S^{-1}$ which also satisfies (i) and (ii) and is of the form $(t,x) \to (t,g''(t,x))$.

So there exists a neighborhood of

$(-\infty,0] \times M \times \overline{D^k(1)} \cup [0,1] \times M \times 0 \cup [1,\infty) \times M \times \overline{D^k(1)}$

in $\mathbb{R} \times X$ which is mapped homeomorphically by G''.

By composing G'' with a "t-dependent change of scale in the $\overline{D^k(1)}$-direction" we achieve that a neighborhood of $\mathbb{R} \times M \times \overline{D^k(1)}$ is mapped homeomorphically by the composition G''' (see sketch).



1.4.

In II.1.2. we have seen that F'' maps $M \times \overline{D^k(1)}$ homeomorphically to a closed part V of Y such that ∂V corresponds to $M \times S^{k-1}$. This induces a continuous "collapsing map" $c_F : Y/\partial Y \to V/\partial V = S^k M^+$; here ∂Y is the space of the sphere-bundle of ξ.

However c_F still depends on the choice of the embedding F'. We want to show that c_F is well-determined up to homotopy. So assume two choices have been made and the constructions of II.1.3. have been done.

Let W be a neighborhood of $M \times \overline{D^k(1)}$ in X such that $(-\varepsilon, 1+\varepsilon) \times W$ is mapped homeomorphically by F'' onto an open part of $\mathbb{R} \times Y$. Then

a) because the collapsing map $c : V \to V/\partial V$ is continuous one has a continuous map $(1 \times c) \circ (G''')^{-1}: G'''([0,1] \times W) \to [0,1] \times V/\partial V$.

b) on $[0,1] \times Y - G'''([0,1] \times M \times \overline{D^k(1)})$ the map $(t,y) \to (t,\infty)$ is continuous. (Here ∞ denotes $\partial V/\partial V \in V/\partial V$).

Both maps agree where they are both defined, so one gets a continuous map $c_G : [0,1] \times Y \to [0,1] \times V/\partial V$.

Projecting onto the second factor and factorizing through $[0,1] \times Y/\partial Y$ one finds a homotopy between the two choices for c_F.

Now consider a connected manifold N with basepoint n and a homomorphism from $\pi_1(N,n)$ to a finite group π. Denote the associated covering space of N by $\tilde{N}$; this induces coverings $\tilde{M},\tilde{X},\tilde{Y}$ etc.

Then the map of covering spaces induced by the embedding F''' is a π-equivariant embedding and hence induces a π-equivariant map $c_F : \tilde{Y}/\partial\tilde{Y} \to S^k\tilde{M}^+$, uniquely determined up to π-equivariant homotopy.

From now on we will denote $\tilde{Y}/\partial\tilde{Y}$ by $T(\xi)$.

1.5.

If M and N are compact manifolds with boundary and $(f,\hat{f})$ is a normal map $M \to N$ <u>not</u> satisfying $f(\partial M) \subset \partial N$, we may view M as a regular domain in a manifold without boundary.

The theory of the preceding subsections with the obvious alterations endows us with a homotopy-unique embedding of $M \times \overline{D^k(1)}$ in Y as a closed part with $\partial M \times \overline{D^k(1)} \cup M \times S^{k-1}$ corresponding to the boundary. So collapsing defines a map $T(\xi)/T(\xi|\partial N) \to S^k(M/\partial M)$, and similarly in the equivariant case.

Let $(f,\hat{f})$ be a normal map between compact manifolds with boundary $(M,\partial M)$ and $(N,\partial N)$ such that $f(\partial M) \subset \partial N$.

We introduce the following notation:

$X = M \times D^k(2)$, $Y = D(\xi)$,

$\partial X = \partial M \times D^k(2)$, $\partial Y = D(\xi|\partial N)$.

Because $TM|\partial M = T\partial M + \varepsilon_1$ canonically, a normal map $\partial M \to \partial N$ is induced so there exists a corresponding immersion $F_b : \partial X \times \mathbb{R} \to \partial Y \times \mathbb{R}$ such that $F_b|\partial M \times \overline{D^k(1)} \times \mathbb{R}$ is an embedding. So DF_b is homotopic to $\hat{f}|\partial M$, using these identifications.

<u>Remark</u>: As in II.1.2. we consider f as a map $X \to Y$ hence $f|\partial M$ as a map $\partial X \to \partial Y$ etc.

Because $\partial M \times 0 \subset \partial M \times [0,1]$ is a homotopy-equivalence, $\hat{f}(\partial M \times [0,1]) + id_\varepsilon$ is homotopic to $(\hat{f}|\partial M) \times id_{T[0,1]}$, hence homotopic to $DF_b \times id_{T[0,1]} = D(F_b \times id_{[0,1]})$ rel $(\partial X \times \mathbb{R}) \times 0$.

Extending this homotopy to $X \times \mathbb{R}$ we get $\hat{f}'$, homotopic to $\hat{f}$ rel ∂M and such that $D(F_b \times id_{[0,1]}) = \hat{f}'|(M \times [0,1])$.

According to the relative immersion theorem of Hirsch and Gromov (see V. Poenaru's lecture, theorem 1' [10]) there exists an immersion $F : X \times \mathbb{R} \to Y \times \mathbb{R}$ extending $F_b \times id_{[0,1]}$ on $\partial X \times [0,1] \times \mathbb{R}$ such that DF is homotopic to $\hat{f}'$ rel $\partial X \times [0,1] \times \mathbb{R}$, consequently homotopic to $\hat{f}$ rel $\partial X \times 0 \times \mathbb{R}$.

Using a relative version of the "lemme de Thom au but" (compare

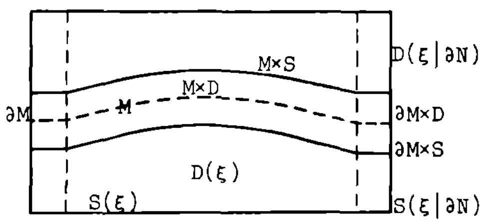
Séminaire Cartan [8 ; exposé 7, corr. 7]) and "scale transformation" one replaces F by a similar F'' embedding $M \times \overline{D^{k+1}(1)}$.

So we get a commutative diagram of embeddings:

$$\begin{array}{ccc} F_b \times id_{[0,1]} : \partial M \times \overline{D^{k+1}(1)} \times [0,1] & \to & D(\xi_{k+1}|\partial N) \times [0,1] \\ \downarrow & & \downarrow \\ F'' : M \times \overline{D^{k+1}(1)} & \longrightarrow & D(\xi_{k+1}) \end{array}$$

Here the vertical arrows are the canonical inclusions, using the isomorphism $D(\xi_{k+1}|\partial N) \times [0,1] \cong D(\xi_{k+1}|\partial N \times [0,1])$.

One proves exactly as in the absolute case that F_b is uniquely determined up to homotopy of embeddings, and for a fixed choice of F_b, F'' is determined up to a homotopy rel $\partial M \times [0,1] \times \overline{D^{k+1}(1)}$. (see sketch)



There exists a collapsing map $D(\xi) \to (\overline{D^{k+1}(1)} \times M)/(S^k \times M) = S^{k+1}M^+$ mapping $S(\xi)$ to the base point; this yields a map

$c : T(\xi) = D(\xi)/S(\xi) \to S^{k+1}M^+$. This maps $D(\xi|\partial N)$ to $S^{k+1}\partial M^+$ hence

$c : T(\xi|\partial N) \to S^{k+1}(\partial M)^+$; there results a map (still called c):

$T(\xi)/T(\xi|\partial N) = D(\xi)/(S(\xi) \cup D(\xi|\partial N)) \to S^{k+1}(M/\partial M)$.

Associated to the embedding of the collar $\partial M \times [0,1] \subset M$ is a collapsing map $M \to C\partial M$ (the cone on ∂M) which maps $\partial M \subset M$ to $\partial M \times 0 \subset C\partial M$ so $M/\partial M$ maps to $S\partial M$.

This map is homotopic to the one which figures in the Puppe sequence of the pair $(M,\partial M)$. Because a commutative diagram of embeddings yields a (strictly!) commutative diagram of collapsing maps, we find a commutative diagram:

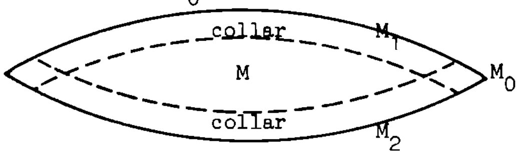
$$\begin{array}{ccccccc}
T(\xi|\partial N) & \longrightarrow & T(\xi) & \longrightarrow & T(\xi)/T(\xi|\partial N) & \longrightarrow & ST(\xi|\partial N) \\
\downarrow c & & \downarrow c & & \downarrow c & & \downarrow c \\
S^{k+1}(\partial M)^{+} & \longrightarrow & S^{k+1}M^{+} & \longrightarrow & S^{k+1}(M/\partial M) & \longrightarrow & S^{k+1}(\partial M)^{+}
\end{array}$$

where c is determined up to homotopy by the normal map. Of course the same is true equivariantly with $\tilde{N},\tilde{M}$ etc. replacing N, M etc.

1.6.

We want to study the situation of a bordism between normal maps of manifolds with boundary. So let

$\partial M_1 = \partial M_2 = M_0$, $\partial M = M_1 \underset{M_0}{\cup} M_2$ (see sketch)



and similarly for N; and for the normal map $(f,\hat{f}) : M \to N$ one has $f(M_i) \subset N_i$, $i = 0,1,2$.

We play it as in the last subsection: first we construct an embedding F_0 of $M \times \overline{D^{k+2}(1)}$ and then an F_1 of $M_1 \times \overline{D^{k+2}(1)}$ which extends $F_0 \times id_{[0,1]}$ and an F_2 of $M_2 \times \overline{D^{k+2}(1)}$ extending $F_0 \times id_{[0,1]}$. So $F_1 \times id$ and $F_2 \times id$ coincide on the collar

$M_0 \times \overline{D^{k+2}(1)} \times [0,1] \times [0,1] \subset M \times \overline{D^{k+2}(1)}$ with $F_0 \times id \times id$ and we can extend to the interior of M.

We get collapsing maps c, uniquely defined up to homotopy:

$$\begin{array}{ccccccc}
T(\xi|N_0) & \longrightarrow & T(\xi|N_1) & \longrightarrow & T(\xi|N_1)/T(\xi|N_0) & \rightarrow & ST(\xi|N_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T(\xi|N_2) & \longrightarrow & T(\xi) & \longrightarrow & T(\xi)/T(\xi|N_2) & \longrightarrow & ST(\xi|N_2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T(\xi|N_2)/T(\xi|N_0) & \rightarrow & T(\xi)/T(\xi|N_1) & \rightarrow & T(\xi)/T(\xi|N_1 \cup N_2) & \rightarrow & S(T(\xi|N_2)/T(\xi|N_0)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
ST(\xi|N_0) & \longrightarrow & ST(\xi|N_1) & \longrightarrow & S(T(\xi|N_1)/(T(\xi|N_0)) & \rightarrow & S^2T(\xi|N_0)
\end{array}$$

$$\Big\Downarrow c$$

$$\begin{array}{ccccccc}
S^{k+2}M_0^+ & \longrightarrow & S^{k+2}M_1^+ & \longrightarrow & S^{k+2}(M_1/M_2) & \longrightarrow & S^{k+3}M_0^+ \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{k+2}M_2^+ & \longrightarrow & S^{k+2}M^+ & \longrightarrow & S^{k+2}(M/M_2) & \longrightarrow & S^{k+3}M_2^+ \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{k+2}(M_2/M_0) & \longrightarrow & S^{k+2}(M/M_1) & \longrightarrow & S^{k+2}(M/M_1 \cup M_2) & \longrightarrow & S^{k+3}(M_2/M_0) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{k+3}M_0^+ & \longrightarrow & S^{k+3}M_1^+ & \longrightarrow & S^{k+3}(M_1/M_0) & \longrightarrow & S^{k+4}M_0^+
\end{array}$$

§2. The cohomology of the collapsing map.

2.1.

As explained in II.1. a normal map of manifolds $f : M \to N$ gives rise to a map $T(\xi) \to S^k\tilde{M}^+$. However, with coefficients in a π-module B one has

$$H_s^\pi(\tilde{N},B) \cong H^s_\pi(\tilde{N},B) \cong H_\pi^{s+k}(D(\xi),S(\xi),B) \cong H_\pi^{s+k}(T(\xi);B)$$

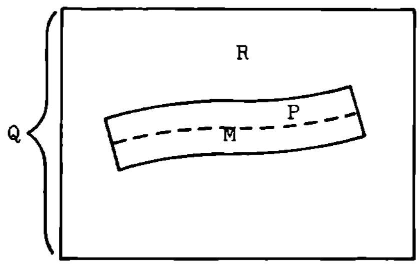
using the Poincaré- and the Thom-isomorphism. We will show that in this way f_* corresponds to c^*.

Furthermore we will show that in the case of manifolds with boundary this correspondence maps the homology ladder of $f : (\tilde{M},\partial\tilde{M}) \to (\tilde{N},\partial\tilde{N})$

into the cohomology ladder of $c : (T(\xi), T(\xi|\partial N)) \to (S^k\tilde{M}^+, S^k\partial\tilde{M}^+)$.

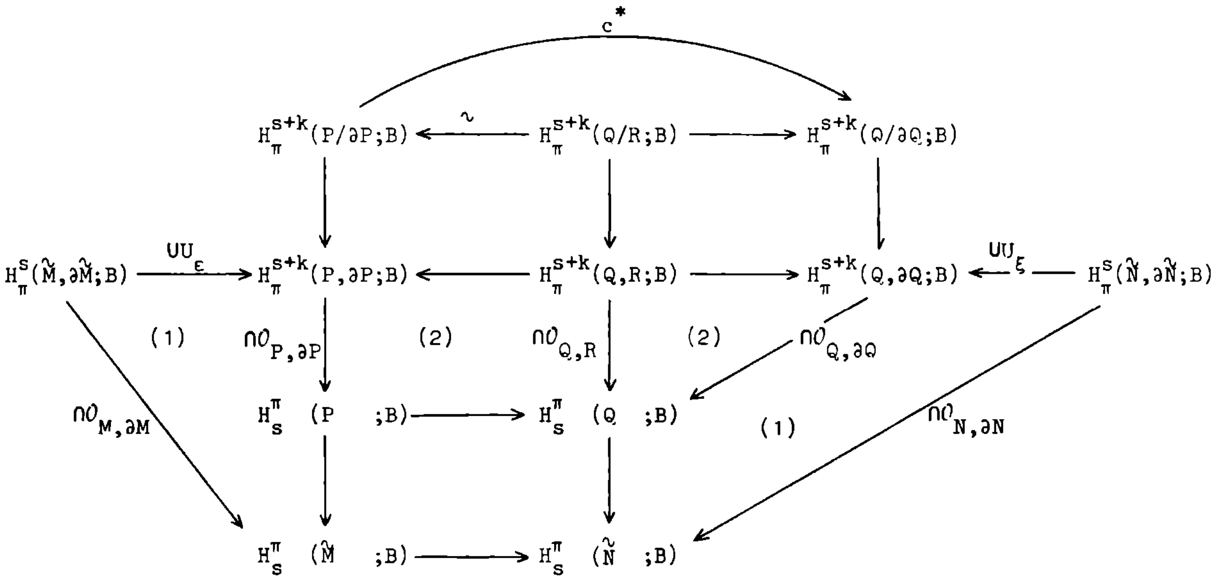
2.2.

First consider the simple case of a normal map f between manifolds with boundary $(M,\partial M)$ and $(N,\partial N)$ not satisfying $f(\partial M) \subset \partial N$. According to 1.5. this induces a collapsing map $c : T(\xi)/T(\xi|\partial N) \to S^k(\tilde{M}/\partial\tilde{M})$ associated to an embedding of $\tilde{M} \times \overline{D^k(1)}$ into $D(\xi)$ (see sketch)



Denote $\tilde{M} \times \overline{D^k(1)}$ by P, $D(\xi)$ by Q, and $\overline{Q-P}$ by R.

We may write down the following commutative diagram:


c*
H_π^{s+k}(P/∂P;B)
H_π^{s+k}(Q/R;B)
H_π^{s+k}(Q/∂Q;B)
H_π^s(M̃,∂M̃;B)
∪U_ε
H_π^{s+k}(P,∂P;B)
H_π^{s+k}(Q,R;B)
H_π^{s+k}(Q,∂Q;B)
∪U_ξ
H_π^s(Ñ,∂Ñ;B)
(1)
∩O_{P,∂P}
(2)
∩O_{Q,R}
(2)
∩O_{Q,∂Q}
∩O_{M,∂M}
H_s^π (P ;B)
H_s^π (Q ;B)
(1)
∩O_{N,∂N}
H_s^π (M̃ ;B)
H_s^π (Ñ ;B)

Here the isomorphisms in the boundary are precisely the relative versions of those in II.1.

The commutativity of this diagram is obvious, except perhaps at (1) and (2); (1) is an application of I.11.5 and (2) is an application of the naturality of the cap-product. Here $\mathcal{O}_{Q,R}$ is the element in $H_{2s+k}(Q,R;\mathbb{F}_2)$ which is the image of both $\mathcal{O}_{P,\partial P}$ and $\mathcal{O}_{Q,\partial Q}$ (see I.11.1. theorem 2).

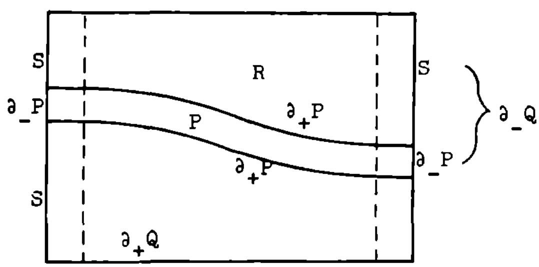
2.3.

According to II.1. a normal map $f : (M,\partial M) \to (N,\partial N)$ yields a system of consistent embeddings and the collapsing maps associated to them. Now we construct three diagrams related to $H(M)$, $H(M,\partial M)$ and $H(\partial M)$. Each of them is commutative for the same reason as the one in II.2.2.

We introduce the notations (see sketch of the situation)

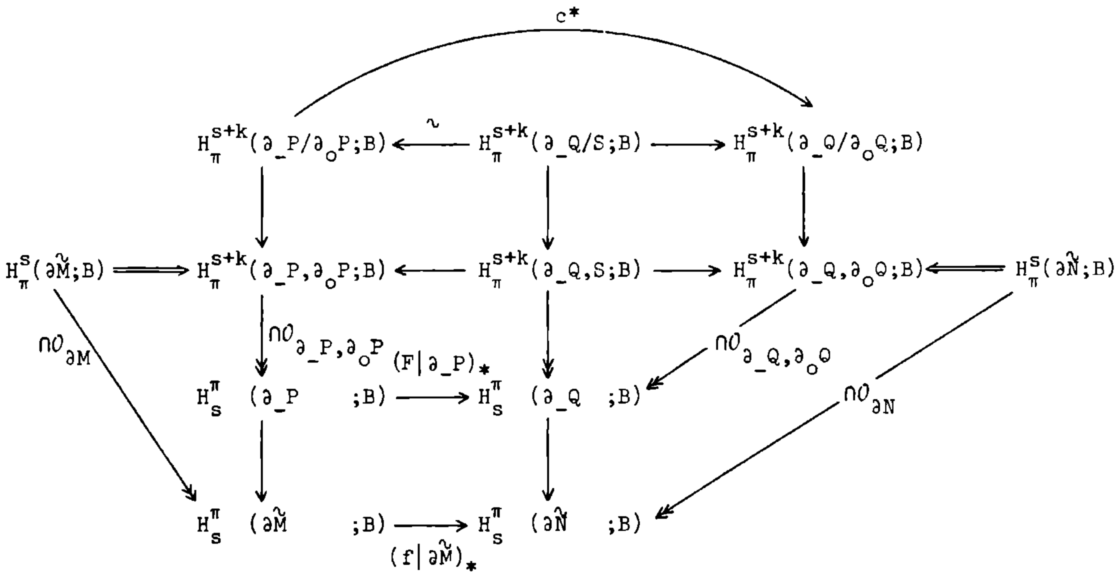
$$P = \overline{D^k(1)} \times \tilde{M}, \quad \partial_+ P = S^{k-1} \times \tilde{M}, \quad \partial_- P = \overline{D^k(1)} \times \partial\tilde{M},$$

$$Q = D(\xi), \quad \partial_+ Q = S(\xi), \quad \partial_- Q = D(\xi|\partial N), \; R=\overline{Q-P}, \; S=\overline{\partial_- Q-\partial_- P}$$



$$
\begin{array}{ccccccccc}
 & & & & \xrightarrow{\quad c^* \quad} & & & & \\
 & & H_\pi^{s+k}(P/\partial_+P;B) & \xleftarrow{\ \sim\ } & H_\pi^{s+k}(Q/R;B) & \longrightarrow & H_\pi^{s+k}(Q/\partial_+Q;B) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H_\pi^{s}(\tilde{M};B) & \Longrightarrow & H_\pi^{s+k}(P,\partial_+P;B) & \longleftarrow & H_\pi^{s+k}(Q,R;B) & \longrightarrow & H_\pi^{s+k}(Q,\partial_+Q;B) & \Longleftarrow & H_\pi^{s}(\tilde{N};B) \\
 & \searrow{\scriptstyle \cap\mathcal{O}_{M,\partial M}} & \downarrow{\scriptstyle \cap\mathcal{O}_{P,\partial P}} & & \downarrow & \swarrow{\scriptstyle \cap\mathcal{O}_{Q,\partial Q}} & & \swarrow{\scriptstyle \cap\mathcal{O}_{N,\partial N}} & \\
 & & H_s^\pi(P,\partial_-P;B) & \longrightarrow & H_s^\pi(Q,\partial_-Q;B) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_s^\pi(\tilde{M},\partial\tilde{M};B) & \xrightarrow{\ f_*\ } & H_s^\pi(\tilde{N},\partial\tilde{N};B) & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & \overset{c^*}{\curvearrowright} & & & & \\
 & & H_\pi^{s+k}(P/\partial P;B) & \overset{\sim}{\longleftarrow} & H_\pi^{s+k}(Q/R\cup\partial\_Q;B) & \longrightarrow & H_\pi^{s+k}(Q/\partial Q;B) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H_\pi^{s}(\tilde{M},\partial\tilde{M};B) & \Longrightarrow & H_\pi^{s+k}(P,\partial P;B) & \longleftarrow & H_\pi^{s+k}(Q,R\cup\partial\_Q;B) & \longrightarrow & H_\pi^{s+k}(Q,\partial Q;B) & \Longleftarrow & H^{s}(\tilde{N},\partial\tilde{N};B) \\
\searrow{\scriptstyle \cap\mathcal{O}_{M,\partial M}} & & \downarrow{\scriptstyle \cap\mathcal{O}_{P,\partial P}} & & \downarrow & & \swarrow{\scriptstyle \cap\mathcal{O}_{Q,\partial Q}} & & \swarrow{\scriptstyle \cap\mathcal{O}_{N,\partial N}} \\
 & & H_s^\pi(P;B) & \xrightarrow{F_*} & H_s^\pi(Q;B) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H_s^\pi(\tilde{M};B) & \xrightarrow{f_*} & H_s^\pi(\tilde{N};B) & & & &
\end{array}
$$


c*
~
H_π^{s+k}(∂_P/∂_oP;B)
H_π^{s+k}(∂_Q/S;B)
H_π^{s+k}(∂_Q/∂_oQ;B)
H_π^s(∂M̃;B)
H_π^{s+k}(∂_P,∂_oP;B)
H_π^{s+k}(∂_Q,S;B)
H_π^{s+k}(∂_Q,∂_oQ;B)
H_π^s(∂Ñ;B)
∩O_∂M
∩O_∂_P,∂_oP
(F|∂_P)_*
∩O_∂_Q,∂_oQ
∩O_∂N
H_s^π (∂_P ;B)
H_s^π (∂_Q ;B)
H_s^π (∂M̃ ;B)
H_s^π (∂Ñ ;B)
(f|∂M̃)_*

The groups at corresponding positions in the three diagrams occur in a long exact sequence. In this way one gets a commutative exact ladder corresponding to each arrow in the diagrams.

a) for the arrows denoted by $\to$ that is the ordinary H_* or H^* ladder of a map of pairs.

b) for the arrows denoted by $\twoheadrightarrow$ that is a Poincaré-duality-ladder as in I.11.4 b or c.

c) for the arrows denoted by $\Rightarrow$ that is a Thom-isomorphism-ladder as in I.11.5.

2.4.

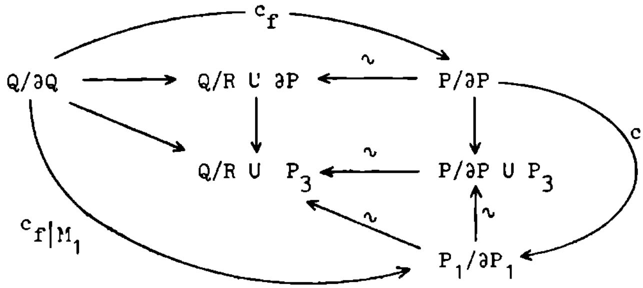
We consider the following situation: $(f,\hat{f})$ a normal map $(M,\partial M) \to (N,\partial N)$, M_1 a regular domain in M, j the inclusion $M_1 \subset M$, such that f maps $M_3 \underset{D}{\equiv} \overline{M-M_1}$ into ∂N.

So we can restrict to a normal map $(M_1,\partial M_1) \to (N_1,\partial N_1)$.

One asks for the relation between $c(f)$ and $c(f|M_1)$.

Again let $P = \tilde{M} \times \overline{D^k(1)}$; analogously $P_1 = \tilde{M}_1 \times \overline{D^k(1)}$, $P_3 = \tilde{M}_3 \times \overline{D^k(1)}$, $Q = D(\xi)$; we construct an embedding $F : P \to Q$ as in the last subsection for manifolds with boundary; then $F|P_1$ is an appropriate embedding for M_1.

Now the inclusion $P_1 \subset P$ gives rise to a collapsing map $c_j : P/\partial P \to P/(\partial P \cup P_3) = P_1/\partial P_1$ which figures in the commutative diagram:



so we get:

PROPOSITION. The correspondence defined in II.2. transforms the diagram:

$$\begin{array}{ccc} H^{\pi}_{s}(\tilde{M}_1;B) & \xrightarrow{j_*} & H^{\pi}_{s}(\tilde{M};B) \\ & {\scriptstyle (f|M_1)_*}\searrow \quad \swarrow{\scriptstyle f_*} & \\ & H^{\pi}_{s}(\tilde{N};B) & \end{array} \quad \text{into} \quad \begin{array}{ccc} H_{\pi}^{s+k}(P_1/\partial P_1;B) & \xrightarrow{c_j^*} & H_{\pi}^{s+k}(P/\partial P;B) \\ & {\scriptstyle (c_{f|M_1})^*}\searrow \quad \swarrow{\scriptstyle c_f^*} & \\ & H_{\pi}^{s+k}(Q/\partial Q;B) & \end{array}$$

2.5.

Let e_j be the collapsing map: $\tilde{M}/\partial\tilde{M} \to \tilde{M}/\tilde{M}_3 \cong \tilde{M}_1/\partial\tilde{M}_1$ then c_j can be viewed as the k-fold suspension of e_j.

From the other hand; $e_j^* : H^s_{\pi}(\tilde{M}_1,\partial\tilde{M}_1;B) = H^s_{\pi}(\tilde{M},\tilde{M}_3;B) \to H^s_{\pi}(\tilde{M},\partial\tilde{M};B)$ corresponds by Poincaré isomorphism to $j_* : H^{\pi}_s(M_1;B) \to H^{\pi}_s(M;B)$.

§3. The construction of the quadratic form.

3.1.

Given a finite group π let A be the group ring $\mathbb{F}_2[\pi]$ of π over the field $\mathbb{F}_2$, endowed with the canonical involution

$- : (\Sigma n_g g) \to \Sigma n_g g^{-1}$.

If I is a two-sided and involution-invariant ideal of A we consider B = A/I. Now π acts from the left on $B \otimes B$; the quotient is isomorphic to B as an abelian group by $b_1 \otimes b_2 \to \overline{b_1} b_2$; here the right action of π on $B \otimes B$ is transformed into the action of π on B by conjugation.

Furthermore the interchange $b_1 \otimes b_2 \to b_2 \otimes b_1$ on $B \otimes B$ corresponds to $-$ on B. In particular the quotient of $S^2B = B \otimes B/\text{span}(b_1 \otimes b_2 - b_2 \otimes b_1$ under the left π-action can be identified with $B/\{b-\bar{b}\}$.

3.2.

Since A is finite, it is certainly Artinian, hence if we take I to be the Jacobsonradical of A (i.e. the set of elements which generate a nilpotent two-sided ideal), the resulting B is semi-simple i.e. every B-module is projective, hence every short exact sequence of B-modules splits.

This implies that for a chain-complex D of B-modules the canonical map $H^n(D;B) \to \text{Hom}_B(H_n(D);B)$ is isomorphic, and similarly the map $H_n(D) \to \text{Hom}_B(H^n(D;B);B)$.

If C is a chain complex equipped with a right π-action then $D = (C \otimes \mathbb{F}_2) \otimes_A B$ is a complex of B-modules.
We then can identify

$$H_n^\pi(C;B) \text{ with } H_n(D) \text{ and}$$
$$H_\pi^n(C;B) \text{ with } H^n(D;B).$$

3.3.

Let $f : (M,\partial M) \to (N,\partial N)$ be a map between manifolds (with boundary) of dimension 2s. Let $\tilde{M}$, $\tilde{N}$ be defined as in II.1.4. Provided that f is of degree one i.e. $f_*\mathcal{O}_{M,\partial M} = \mathcal{O}_{N,\partial N}$, the diagram

$$\begin{array}{ccc} H^s_\pi(\tilde M,\partial\tilde M;B) & \xleftarrow{\quad f^*_{M,\partial M}\quad} & H^s_\pi(\tilde N,\partial\tilde N;B)\\ \Big\downarrow \cap\mathcal{O}_{M,\partial M} & & \Big\downarrow \cap\mathcal{O}_{N,\partial N}\\ H^\pi_s(\tilde M;B) & \xrightarrow{\quad f^M_*\quad} & H^\pi_s(\tilde N;B)\end{array}$$

commutes, hence f^* is injective and f_* is surjective.

We will use the symbol P to denote the Poincaré-isomorphisms $\cap\mathcal{O}_{M,\partial M}$ and $\cap\mathcal{O}_{N,\partial N}$.

This implies a direct-sum splitting of $H^s_\pi(\tilde M,\partial\tilde M;B)$ as $\ker f_*P \oplus \operatorname{im} f^*$ and, related to this, one of $H^\pi_s(\tilde M;B)$ as $\ker f_* \oplus P \operatorname{im} f^*$; similarly for $H^s_\pi(\tilde M;B)$ and $H^\pi_s(\tilde M,\partial\tilde M;B)$.

The canonical map $j : H^\pi_s(\tilde M;B) \to H^\pi_s(\tilde M,\partial\tilde M;B)$ preserves the direct-sum splitting; and if $f|\partial M : \partial M \to \partial N$ <u>is a homotopy-equivalence</u>, the induced map $j : \ker f^M_* \to \ker f^{M,\partial M}_*$ is an isomorphism, as is seen by diagram-chasing.

Furthermore the obvious map $<,> \; : H^\pi_s(\tilde M,\partial\tilde M;B) \to \operatorname{Hom}_B(H^s_\pi(\tilde M,\partial\tilde M;B);B)$ preserves the direct-sum splitting, and is an <u>isomorphism in the case B has been chosen as in II.4.2. to be A/(radical)</u>.

Consider the pairing β associated to the composition:

$$H^s_\pi(\tilde M,\partial\tilde M;B) \underset{P}{\to} H^\pi_s(\tilde M;B) \underset{j}{\to} H^\pi_s(\tilde M,\partial\tilde M;B) \to \operatorname{Hom}_B(H^s_\pi(\tilde M,\partial\tilde M;B))$$

mapping x to $(y \to \langle y, x \cap \mathcal{O}_{M,\partial M}\rangle)$, hence $\beta(y,x) = \langle y, x \cap \mathcal{O}_{M,\partial M}\rangle = (y \cup x) \cap \mathcal{O}_{M,\partial M}$.

<u>PROPOSITION</u>. The restriction of β to $P_*^{-1} \ker f^M$ is a nonsingular pairing.

<u>Remark</u>: to be more precise we should write

$\beta(y,x) = E((y \cup x) \cap \mathcal{O}_{M,\partial M}) = \langle y \cup x, \mathcal{O}_{M,\partial M}\rangle$ where E is the identification of $H^\pi_0(\tilde M;B \otimes B)$ with B.

3.4.

We remark that $H^{\pi}_{s}(\tilde{M},\partial\tilde{M};B)$ inherits a right π-action from B and $H^{s}_{\pi}(\tilde{M},\partial\tilde{M};B)$ a left action. The natural pairing

$< \, , \, > \; : \; H^{\pi}_{s}(\tilde{M},\partial\tilde{M};B) \times H^{s}_{\pi}(\tilde{M},\partial\tilde{M};B) \to B$ has the properties:

$<gx,\sigma> = g<x,\sigma>$

$<x,\sigma g> = <x,\sigma>g.$

Furthermore it follows from the naturality properties of the cap-product with respect to coefficient-homomorphisms that

$gx \cap \mathcal{O} = (x \cap \mathcal{O})g^{-1}$ i.e. the Poincaré-duality-isomorphism $\cap\, \mathcal{O}$ is equivariant. This implies:

<u>PROPOSITION</u>. β is "sesquilinear": $\beta(gy,hx) = g\beta(y,x)h^{-1}$.

3.5.

We have seen in II.1.5. that for manifolds with boundary $(M,\partial M)$, $(N,\partial N)$ of dimension 2s, a normal map $f : M \to N$ defines an equivariant map $T(\xi)/T(\xi|\partial N) \to S^{k}(\tilde{M}/\partial\tilde{M})$ welldefined up to equivariant homotopy.

In the following we consider $x \in H^{s}_{\pi}(\tilde{M},\partial\tilde{M};B)$ satisfying $c^{*}(S^{k}x) = 0$ i.e. $x \cap \mathcal{O}_{M,\partial M} \in \ker f_{*}$. Then the construction of I.9.1. yields a $\psi_{c}(x) \in H^{2s+k}_{\pi}(T(\xi),T(\xi|\partial N);S^{2}B)$ modulo some indeterminacy. By Thom- and Poincaré-isomorphism this cohomology-group is isomorphic to $H^{\pi}_{0}(\tilde{N};S^{2}B) = B/\{b-\bar{b}\}$. The image of $\psi_{c}(x)$ by this isomorphism will be called $q(x)$.

According to I.9. there is a relation between $\psi_{c}(x)$ and the pairing $(y,x) \to c^{*}S^{k}(y \cup x)$, hence between $q(x)$ and the image $b(y,x)$ of $c^{*}S^{k}(y \cup x)$ in B. According to diagram II.2.3. $b(y,x)$ is the image under f_{*} of the Poincaré-dual of $y \cup x$, so $\beta(y,x) = b(y,x)$.

Summarizing we have:

THEOREM. Given a normal map as above, and denoting $\{x \in H^s(\tilde{M},\partial\tilde{M};B) \mid x \cap \mathcal{O}_{M,\partial M} \in \ker f_*^M\}$ by K we have a pair (b,q) such that:

(1) $b : K \times K \to B$ is biadditive.

(2) $b(y,gx) = b(y,x)g^{-1}$.

$b(gy,x) = gb(y,x)$.

(3) $b(y,x) = \overline{b(x,y)}$.

(4) if $f(\partial M) \subset \partial N$, f is of degree one, and $f|\partial M$ is a homotopy equivalence, then b is nonsingular.

(5) $q : K \to B/\{b-\bar{b}\}$ modulo indeterminacy.

(6) $q(x+y) = q(x)+q(y)$ + class of $b(y,x)$.

(7) $q(x) + \overline{q(x)}$ is welldefined and equal to $b(x,x)$.

(8) $q(gx) = g\, q(x)g^{-1}$.

The third property is a consequence of the commutativity of the cup-product: $y \cup x = T(x \cup y)$, where T is the coefficient-homomorphism $B \otimes B \to B \otimes B$ mapping $b \otimes b'$ to $b' \otimes b$; we also use the relation between this T and the involution on B explained in II.3.1.

The last property is a consequence of the naturality of Sq, and hence of ψ_c, with respect to coefficient-homomorphisms.

In other words: if we can prove that the indeterminacy of q vanishes, there will result a quadratic form in the sence of C.T.C. Wall [15, chapter 4]. We will show the vanishing of the indeterminacy in the next few subsections.

3.6.

We consider B as a right module over ρ = Aut B and we apply the theory of I.7. for the canonical homomorphism $\pi \to \rho$.

We modify the definition of ψ_c, using the functional squaring operation equivariant with respect to π and ρ. The considerations of the foregoing subsection remain true since the relevant properties of the operations follow by naturality from the properties proved in I.8. and I.9. for the ρ-equivariant case.

Now the total indeterminacy is, according to I.7.2., equal to:

$$\operatorname{im} Sq^{s+1} + \operatorname{im} (S^k\phi_x \circ c)^* \circ R,$$

where R is the restriction $H^*_\rho \to H^*_\pi$. (as in I.7.)

We consider the two parts of the above sum seperately in the next subsections.

3.7.

We first prove that the first part of the indeterminacy vanishes. To this end we consider the following diagram:

$$\begin{array}{ccc}
H_\pi^{s-1}(\tilde{N},\partial\tilde{N};B) & \longrightarrow & H_\pi^{s+k-1}(T(\xi),T(\xi|\partial N);B) \\
\downarrow Sq^{s+1} & \text{Thom-isomorphisms} & \downarrow Sq^{s+1} \\
H_\pi^{2s}(\tilde{N},\partial\tilde{N};B) & \longrightarrow & H_\pi^{2s+k}(T(\xi),T(\xi|\partial N);B)
\end{array}$$

This diagram commutes because of the Cartan formula I.6.3.: (notice that U is a $\mathbb{F}_2$-cohomology class)

$$Sq^{s+1}(x \cup U_\xi) = Sq^{s+1}x \cup U_\xi + \sum_{i>0} Sq^{s+1-i}x \cup Sq^i U_\xi$$

and because of the next lemma:

<u>LEMMA</u>. $Sq^i U_\xi = 0$ for $i > 0$.

<u>P r o o f</u>. Because $TM \oplus \varepsilon = f^*(TN \oplus \xi)$ for a normal map f, we have for the Stiefel-Whitney classes w_i : $w(M) = f^*(w(N) \cup w(\xi))$.

Denoting the inverse of the total square Sq by $\chi(Sq)$, the relation $w = Sq\, v$ between Stiefel-Whitney classes and Wu-classes implies: $v(M) = f^*(v(N) \cup \chi(Sq)w(\xi))$. Substituting this into

$<v(N) \cup x, \mathcal{O}_{N,\partial N}> = <Sq\, x, \mathcal{O}_{N,\partial N}> = <Sq\, x, f_*\mathcal{O}_{M,\partial M}> = <Sq\, f^*x, \mathcal{O}_{M,\partial M}> =$

$= <v(M) \cup f^*x, \mathcal{O}_{M,\partial M}>$ we find for the latter $<v(N) \cup \chi(Sq)w(\xi) \cup x, \mathcal{O}_{N,\partial N}>$ using $f_*\mathcal{O}_{M,\partial M} = \mathcal{O}_{N,\partial N}$. Hence Poincaré-duality for N implies $v(N) = v(N) \cup \chi(Sq)w(\xi)$, so $\chi(Sq)w(\xi) = 1$ and finally $w(\xi) = 1$.

However $Sq\, U_\xi = w(\xi) \cup U_\xi$, so $Sq\, U_\xi = U_\xi$. Q.E.D.

<u>Remark</u>: One can also conclude that $Sq^i\, U_\xi = 0$ from the fact that ξ is fibre-homotopically trivial, because of Spivak's characterization of the normal spherical fibration [3].

Now the squaring operation at the left side of the diagram vanishes because of dimensional reasons, hence the Sq^{s+1} on the right vanishes, which proves the assertion at the beginning of this subsection.

3.8.

It is more difficult to see that the second contribution to the indeterminacy vanishes.

Consider the commutative diagram:

$$
\begin{array}{ccc}
H_\rho^{2s+k}(K(B,s+k);B') & \xrightarrow[H_k^*]{} & H_\rho^{2s+k}(S^kK(B,s);B') \\
\Big\downarrow R & & \Big\downarrow R \\
H_\pi^{2s+k}(K(B,s+k);B') & \xrightarrow[H_k^*]{} & H_\pi^{2s+k}(S^kK(B,s);B') \\
 & & \Big\downarrow t \\
 & & H_\pi^{2s+k}(T(\xi)/T(\xi|\partial N);B'),
\end{array}
$$

where $t = S^k\phi_x \circ c$.

The fact that $t^* \circ R = 0$ follows from

$(t^* \circ R) \circ H_k^* = t^* \circ H_k^* \circ R = (H_k \circ t)^* \circ R$ since H_k^* is surjective if $B' = S^2B$ and $H_k \circ t$ is homotopic to a constant.

That $H_k \circ t$ is equivariantly homotopic to a constant follows from the fact that it classifies

$(H_k \circ t)^* \chi_{s+k} = t^* S^k\chi_s = c^*(S^k x) = 0$ (because $x \in K$).

That H_k^* is surjective for the special choice of coefficients $B' = S^2B$, will be shown in the next few subsections.

3.9.

As we have seen in I.8.7. the map

$H_{k-1}^* : H_\rho^{2s+k}(K(B,k+s);B') \to H_\rho^{2s+k}(S^{k-1}K(B,s+1);B') = H_\rho^{2s+1}(K(B,s+1);B')$

is isomorphic, hence H_k^* is surjective exactly if its last composition factor is surjective:

$$H_1^* : H_\rho^{2s+1}(K(B,s+1);B') \to H_\rho^{2s+1}(SK(B,s);B').$$

Consider the following commutative diagram; we see by diagram chasing that to prove surjectivity of $H_1^* = \alpha$, it is enough to know that η is surjective and ζ is injective.

$$\begin{array}{ccccc} H_\rho^{2s+1}(K(B,s+1);B') & \xrightarrow{\alpha} & H_\rho^{2s+1}(SK(B,s);B') & \xrightarrow{\gamma} & H_\rho^{2s+1}(H_1;B') \\ \downarrow & & \downarrow \varepsilon & & \downarrow \zeta \\ \mathrm{Traf}(\underline{H}_{2s+1}(K(B,s+1);\mathbb{F}_2);B') & \xrightarrow{\eta} & \mathrm{Traf}(\underline{H}_{2s+1}(SK(B,s);\mathbb{F}_2);B') & \xrightarrow{\delta} & \mathrm{Traf}(\underline{H}_{2s+2}(H_1;\mathbb{F}_2);B') \end{array}$$

(Here the upper row is exact and the lower one has composition 0).

3.10.

We prove the

LEMMA. ζ is isomorphic.

P r o o f. We apply I.8.5. with $n = 2s+2$; this yields a ρ-complex L and a commutative diagram:

$$\begin{array}{ccc} SK(B,s) & \xrightarrow{H_1} & \\ \downarrow & & K(B,s+1) \\ L & \xrightarrow{H_1'} & \end{array}$$

where H' is an equivariant homotopy-equivalence and $L-SK(B,s)$ has only cells in dimensions at least $2s+2$. So we can replace H_1 in the diagram II.3.9. by the pair $(L,SK(B,s))$.

But as G. Bredon remarks after his I.9.5. [1], for a G-complex X without $(n-1)$-cells the pairing $H^n(X;M) \to \mathrm{Traf}(\underline{H}_n(X);M)$ is always isomorphic. This obviously remains true if $\underline{H}_n(X)$ is replaced by $\underline{H}_n(X;\mathbb{F}_2)$, if $2M = 0$.

Applied to $X = L/SK(B,s)$ this yields the desired conclusion. Q.E.D.

3.11.

There remains to be proven that η is surjective. This subsection contains some general considerations needed to do that.

In the following we denote by A^s the $\mathbb{F}_2$-vectorspace of stable $\mathbb{F}_2$-cohomology-operations of degree s; let $\omega_1, \omega_2, \ldots, \omega_m$ be a base of A^s and let A_s be its dual.

Let V be a vector space over $\mathbb{F}_2$, say with base $e_1, e_2, \ldots, e_n$. The Eilenberg-Maclane complex $K(V,s+1)$ is a Aut(V)-complex. Since, by the Hurewicz theorem and the coefficient theorem, $H^{s+1}(K(V,s+1);\mathbb{F}_2) = V^d$ canonically, there is a pairing $A^s \otimes V^d \to H^{2s+1}(K(V,s+1);\mathbb{F}_2)$. It follows from the Künneth formula for $K(V,s+1) = K(\mathbb{F}_2,s+1)^n$, that the $\omega_i(e_j^*)$ constitute a base of $H^{2s+1}(K(V,s+1);\mathbb{F}_2)$, hence the pairing considered above is an isomorphism.

Similarly there exists a pairing (using the cup-product):
$A^s \otimes V^d \oplus V^d \otimes V^d \to H^{2s}(K(V,s);\mathbb{F}_2)$, mapping $Sq^s \otimes f - f \otimes f$ and $f \otimes g - g \otimes f$ to 0 for each f and g in V^d.

Hence there exists an induced map:

$$A^s \otimes V^d \oplus S^2(V^d)/\{Sq^s \otimes f - f \otimes f\} \to H^{2s}(K(V,s);\mathbb{F}_2)$$

and because the $\omega_i(e_j^*)$ together with the $e_{j1}^* \cup e_{j2}^*$ constitute a base of $H^{2s}(K(V,s);\mathbb{F}_2)$ this is also an isomorphism.

Furthermore we may identify the canonical map

$$H^{2s+1}(K(V,s+1);\mathbb{F}_2) \to H^{2s+1}(SK(V,s)\mathbb{F}_2) \cong H^{2s}(K(V,s);\mathbb{F}_2)$$

with the inclusion map:

$$A^s \otimes V^d \to (A^s \otimes V^d \oplus S^2V^d)/\{Sq^s \otimes f - f \otimes f\}.$$

Hence we may identify $H_{2s+1}(K(V,s+1);\mathbb{F}_2)$ with $A_s \otimes V$ and $H_{2s+1}(SK(V,s);\mathbb{F}_2) = H_{2s}(K(V,s);\mathbb{F}_2)$ with

$\{(\sigma,\tau) \in A_s \otimes V \oplus V \otimes V \mid \sigma(Sq^s \otimes f) = \tau(f \otimes f), \tau(f \otimes g) = \tau(g \otimes f)\}$.

The canonical map $H_{2s+1}(SK(V,s);\mathbb{F}_2) \to H_{2s+1}(K(V,s+1);\mathbb{F}_2)$ may be viewed as the projection onto the first summand. So there exists an exact sequence:

$$0 \to \mathrm{im}(1+T) \to H_{2s+1}(SK(V,s);\mathbb{F}_2) \to H_{2s+1}(K(V,s+1);\mathbb{F}_2) \to 0$$

For a subgroup J of Aut(B) we have:

$\underline{H}_i(K(B,s);\mathbb{F}_2)(^{\mathrm{Aut}\ B}/J) = H_i(K(B,s)^J;\mathbb{F}_2) = H_i(K(B^J,s);\mathbb{F}_2)$.

The foregoing considerations lead to an exact sequence of coefficient-systems:

$$0 \to \underline{\mathrm{im}}\ (1+T) \to \underline{H}_{2s+1}(SK(B,s);\mathbb{F}_2) \to \underline{H}_{2s+1}(K(B,s+1);\mathbb{F}_2) \to 0$$

where $\underline{\mathrm{im}}\ (1+T)$ is the coefficient-system whose value at Aut B/ $_J$ is the image of $1+T : B^J \otimes B^J \to B^J \otimes B^J$.

3.12.

We can now prove the statement that η is surjective by applying the half-exact functor Traf(;B') to the exact sequence just derived. For both choices one can make for B':

(a) $B' = \underline{\mathrm{cok}}(1+T)$ i.e. $B'(\mathrm{Aut}\ B/J) = \mathrm{cok}\ (1+T\colon B^J \otimes B^J \to B^J \otimes B^J)$

(b) B' = the coefficient-system constructed from the module

$S^2B = \mathrm{cok}((1+T)\colon B \otimes B \to B \otimes B)$,

one is left to prove that $\mathrm{Hom}_\rho(\mathrm{im}\ (1+T),\ \mathrm{cok}\ (1+T)) = 0$.

<u>SUBLEMMA</u>. Let V be a vectorspace over $\mathbb{F}_2$ of dimension $\neq 2$. Let $T : V \otimes V \to V \otimes V$ map $a \otimes b$ to $b \otimes a$. Then

$\mathrm{Hom}_{\mathrm{Aut}\ V}(\mathrm{im}\ (1+T),\ \mathrm{cok}\ (1+T)) = 0$.

P r o o f. It suffices to show that $f \in \hom(\mathrm{im}\,(1+T), \mathrm{cok}\,(1+T))$ maps $x \otimes y + y \otimes x$ to 0. Suppose $x \otimes y + y \otimes x \neq 0$; this means that x and y are independent hence we can choose a base $e_1, e_2, \ldots, e_n$ such that $e_1 = x$ and $e_2 = y$.

Let $f(e_1 \otimes e_2 + e_2 \otimes e_1) = \alpha e_1 \otimes e_1 + \beta e_2 \otimes e_2 + \gamma e_1 \otimes e_2 + \sum_{i>2} \delta_i e_1 \otimes e_i + \sum_{i>2} \varepsilon_i e_2 \otimes e_i + \sum_{j\geq i>2} \zeta_{ij} e_i \otimes e_j$.

Equivariance of f with respect to the element $A \in \mathrm{Aut}\,V$ defined by $Ae_1 = e_2$, $Ae_2 = e_1$, $Ae_i = e_i$ for $i > 2$, implies that $\alpha = \beta$ and $\delta_i = \varepsilon_i$.

Equivariance of f with respect to the element $A \in \mathrm{Aut}\,V$ defined by $Ae_1 = e_1+e_2$, $Ae_2 = e_2$, $Ae_i = e_i$ for $i > 2$, implies that $\beta = \alpha+\beta+\gamma$ hence $\alpha = \beta = \gamma$, and $\varepsilon_i = \delta_i+\varepsilon_i$ so $= 0$.

Finally, equivariance with respect to the element A defined by $Ae_1 = e_1+e_3$, $Ae_2 = e_2$, $Ae_i = e_i$ for $i > 2$, implies that $f(e_3 \otimes e_2 + e_2 \otimes e_3) = \alpha e_3 \otimes e_3 + \alpha e_2 \otimes e_3$; combined with the equivariance with respect to the element B given by $Be_1 = e_3$, $Be_3 = e_1$ and $Be_i = e_i$ for $i \neq 1,3$, this yields that $f(e_1 \otimes e_2 + e_2 \otimes e_1) = \alpha e_1 \otimes e_1 + \alpha e_1 \otimes e_2$ hence $\alpha = \beta = \gamma = 0$ and $\zeta = 0$.

Q.E.D.

Summarizing: this sublemma yields the surjectivity of η, hence the surjectivity of H_k^* is proved and also the vanishing of the indeterminacy of q. We may thus state the

THEOREM. A normal map between even-dimensional manifolds with boundary, inducing a homotopy-equivalence of the boundaries, gives rise to a nonsingular quadratic form in the sence of C.T.C. Wall.

§4. Properties of the quadratic form.

4.1.

THEOREM. The quadratic form (q,b) is natural for inclusions of normal maps of the type described in II.2.4. with respect to:

$e_j^* : \{x \in H_\pi^s(\tilde{M}_1,\partial\tilde{M}_1;B) \mid c_{f|M_1}^* S^k x = 0\} \to \{x \in H_\pi^s(\tilde{M},\partial\tilde{M};B) \mid c_f^* S^k x = 0\}$

Proof. Consider $x \in H_\pi^s(\tilde{M}_1,\partial\tilde{M}_1;B)$ such that $(c_{f|M_1})^* S^k x = 0$. If $\phi_x : \tilde{M}_1/\partial\tilde{M}_1 \to K(B,s)$ is a map classifying x, then $\phi_x \circ e_j$ is a map classifying $e_j^* x$ and

$$S^k(\phi_x \circ e_j) \circ c_{f|M_1} = S^k\phi_x \circ S^k e_j \circ c_{f|M_1} = S^k\phi_x \circ c_j \circ c_{f|M_1} = S^k\phi_x$$

hence from the definition we see that

$\psi(x)$ constructed from M and

$\psi(e_j^* x)$ constructed from M_1 coincide in $H_\rho^{2s+k}(T(\xi),T(\xi|\partial N);S^2B)$,

hence $q(x) = q(e_j^* x)$ in $B/\{b-\bar{b}\}$.

Furthermore the computation

$$(c_{f|M_1})^* S^k(x \cup y) = c_f^* c_j^* S^k(x \cup y) = c_f^*(S^k(x \cup y)) =$$
$$= c_f^* S^k e_j^*(x \cup y) = c_f^* S^k(e_j^* x \cup e_j^* y)$$

shows that

$$b(x,y) = b(e_j^* x, e_j^* y).$$ Q.E.D.

4.2.

In the remainder of this section we will formulate and prove that the quadratic form (b,q) is, up to some equivalence-relation, invariant under bordism of normal maps.

Consider a bordism of normal maps i.e. the situation of II.1.6. and let $x \in H_\pi^s(\tilde{M},\tilde{M}_2;B)$ such that $c^* S^k x = 0$. Then there is a commutative diagram, according to II.1.6.:

$$\begin{array}{ccccc}
S(T(\xi)/T(\xi|N_2)) & \xrightarrow[c]{} & S^{k+1}(\tilde M/\tilde M_2) & \xrightarrow[S^{k+1}\phi_x]{} & S^{k+1}K(B,s) \\
\uparrow & & \uparrow & & \nearrow \\
S(T(\xi|N_1)/T(\xi|N_0)) & \xrightarrow[c]{} & S^{k+1}(\tilde M_1/\tilde M_0) & & S^{k+1}\phi_{i^*x} \\
\uparrow{\scriptstyle\gamma} & & \uparrow & \nearrow & \\
T(\xi)/T(\xi|N_1 \cup N_2) & \xrightarrow[c]{} & S^k(\tilde M/\tilde M_1 \cup \tilde M_0) & &
\end{array}$$

where i is the inclusion $(\tilde M_1,\tilde M_0) \subset (\tilde M,\tilde M_2)$.

The vertical compositions are equivariantly homotopic to constant maps, hence a functional operation associated to $S^{k+1}\phi_{i^*x} \circ c$ vanishes. In particular the image under γ of $\psi(i^*x)$, which is computed from $S^{k+1}\phi_{i^*x} \circ c$ vanishes.

We will show that the identifications of $H_\pi^{2s+k+1}(S(T(\xi|N_1)/T(\xi|N_0);S^2B)$ and $H_\pi^{2s+k+1}(T(\xi)/T(\xi|N_1 \cup N_2);S^2B)$ with $B/\{b-\bar b\}$ correspond under γ. This shows:

PROPOSITION. In these circumstances is $q(i^*x) = 0$.

4.3.

We notice that γ is a map from the Puppe sequence of the inclusion $T(\xi|N_1 \cup N_2)/T(\xi|N_2) \subset T(\xi)/T(\xi|N_2)$ hence we can identify γ^* with the boundary-operator in the long exact sequence of this pair.

Using the fact that the Thom-isomorphism preserves such a long exact sequence we can identify γ^* with the composition

$$\begin{array}{ccccc}
H_\pi^{2s}(\tilde N_1,\tilde N_0;S^2B) & \underset{g}{\overset{\sim}{\leftarrow}} & H_\pi^{2s}(\tilde N_1 \cup \tilde N_2,\tilde N_2;S^2B) & \underset{h}{\rightarrow} & H_\pi^{2s}(\tilde N_1 \cup \tilde N_2;S^2B) \\
 & & & & \downarrow{\scriptstyle\delta} \\
 & & & & H_\pi^{2s+1}(\tilde N,\tilde N_1 \cup \tilde N_2;S^2B)
\end{array}$$

Finally: $<\delta h^* g^{*-1} x, \mathcal{O}_N> = <h^* g^{*-1} x, \partial \mathcal{O}_N> = <h^* g^{*-1} x, \mathcal{O}_{N_1 \cup N_2}> =$

$= <x, g_*^{-1} h_* \mathcal{O}_{N_1 \cup N_2}> = <x, \mathcal{O}_{N_1, N_0}>$ in $S^2 B \otimes \mathbb{F}_2 = B/\{b - \bar{b}\}$,

which proves the assertion in the last subsection.

4.4.

As in II.4.3. there are direct-sum splittings

$$H^\pi_{s+1}(\tilde{M}, \tilde{M}_1; B) = \ker f_*^{M,M_1} \oplus P \operatorname{im} f^*_{M_1,M_2}$$

$$H^\pi_s(\tilde{M}_1; B) = \ker f_*^{M_1} \oplus P \operatorname{im} f^*_{M_1,M_0}$$

$$H^\pi_s(\tilde{M}; B) = \ker f_*^{M} \oplus \operatorname{im} f^*_{M,M_1 \cup M_2}$$

and corresponding splittings of

$H_\pi^s(\tilde{M}, \tilde{M}_2; B)$, $H_\pi^s(\tilde{M}_1, \tilde{M}_0; B)$ and $H_\pi^{s+1}(\tilde{M}, \tilde{M}_1 \cup \tilde{M}_2; B)$.

Furthermore the maps

$$H^\pi_{s+1}(\tilde{M}, \tilde{M}_1; B) \to H^\pi_s(\tilde{M}_1; B) \to H^\pi_s(\tilde{M}; B)$$

map the kernels into the kernels and hence induce an exact sequence: (the kernel of a surjection of long exact sequences)

$$\ker f_*^{M,M_1} \xrightarrow[i]{} \ker f_*^{M_1} \xrightarrow[p]{} \ker f_*^{M}.$$

Similarly the maps

$$H_\pi^s(\tilde{M}, \tilde{M}_2; B) \to H_\pi^s(\tilde{M}_1 \cup \tilde{M}_2, \tilde{M}_2; B) = H_\pi^s(\tilde{M}_1, \tilde{M}_0; B) \to H_\pi^{s+1}(\tilde{M}, \tilde{M}_1 \cup \tilde{M}_2; B)$$

map the images into the images; hence the direct-sum splitting is preserved.

In II.4. we showed that the cup-product on $H_\pi^s(\tilde{M}_1, \tilde{M}_0)$ induces a nonsingular pairing on $\ker Pf_*$. Notice that instead we can use cup-product and kernel from the isomorphic group $H^s_\pi(\tilde{M}_1 \cup \tilde{M}_2, \tilde{M}_2; B)$.

The three maps

$$\begin{array}{ccccc} H^s_\pi(\tilde{M},\tilde{M}_2;B) & \xrightarrow[\cap \mathcal{O}_{M,M_1 \cup M_2}]{} & H^\pi_{s+1}(\tilde{M},\tilde{M}_1;B) & \longrightarrow & H^\pi_{s+1}(\tilde{M},\tilde{M}_1 \cup \tilde{M}_2;B) \\ & & & & \downarrow \\ & & & & \mathrm{Hom}_B(H^{s+1}_\pi(\tilde{M},\tilde{M}_1 \cup \tilde{M}_2;B);B) \end{array}$$

all preserve the direct-sum splitting and the first and last map are isomorphic (compare II.4.).

The middle map induces an isomorphism: $\ker f_*^{M,M_1} \to \ker f_*^{M,M_1 \cup M_2}$ if $f : (M_2,M_0) \to (N_2,N_0)$ is a homotopy-equivalence, because this implies that $f_* : H^\pi_*(\tilde{M}_1 \cup \tilde{M}_2, \tilde{M}_1;B) \to H^\pi_*(\tilde{N}_1 \cup \tilde{N}_2, \tilde{N}_1;B)$ is isomorphic.

We may conclude that the cup-product on $H^s_\pi(\tilde{M},\tilde{M}_2;B) \times H^{s+1}_\pi(\tilde{M},\tilde{M}_1 \cup \tilde{M}_2;B)$ inducesa nonsingular pairing c on $\ker Pf_*^{M,M_2} \times \ker Pf_*^{M,M_1 \cup M_2}$ in the case considered.

4.5.

PROPOSITION. With the notations already introduced:

$$b(ix,y) = c(x,py).$$

P r o o f. We have:

$b(ix,y) = \langle ix \cup y, \mathcal{O}_{M_1 \cup M_2, M_2} \rangle = \langle ix \cup y, g_* \partial \mathcal{O}_{M,M_1 \cup M_2} \rangle = \langle \delta g^*(ix \cup y), \mathcal{O}_{M,M_1 \cup M_2} \rangle$

where g is the inclusion $(\tilde{M}_1 \cup \tilde{M}_2, \emptyset) \subset (\tilde{M}_1 \cup \tilde{M}_2, \tilde{M}_2)$, and from the other hand: $c(x,py) = \langle x \cup py, \mathcal{O}_{M,M_1 \cup M_2} \rangle$.

Hence the statement follows from the fact that in general, given a triple $X \subset Y \subset Z$ and maps as in the diagram:

$$\begin{array}{ccccc} H^n_\pi(Z,X;B) & \xrightarrow[i]{} & H^n_\pi(Y,X;B) & \xrightarrow[p]{} & H^{n+1}_\pi(Z,Y;B) \\ & \searrow^{h} & & \searrow^{g} & \nearrow_{\delta} \\ & H^n_\pi(Z;B) & \xrightarrow{j} & H^n_\pi(Y;B) & \end{array}$$

one has:

$\delta g(ix \cup y) = \delta(gix \cup gy) = \delta(j(hx) \cup (gy)) = (hx) \cup \delta(gy) = x \cup py.$

Q.E.D.

4.6.

Consider the category with objects the triples (K,b,q), where K is a finitely generated left B-module and (b,q) is a quadratic form on B i.e.

(1) $b : K \times K \to B$ is biadditive.

(2) $b(a_1x,a_2y) = a_1b(x,y)\overline{a_2}$.

(3) $b(y,x) = \overline{b(x,y)}$.

(4) b is nonsingular i.e. induces an isomorphism $K \to K^d = \mathrm{Hom}_B(K,B)$.

(5) $q : K \to B/\{b-\overline{b}\}$.

(6) $q(x+y) = q(x)+q(y) +$ class of $b(x,y)$.

(7) $q(x) + \overline{q(x)} = b(x,x)$.

(8) $q(ax) = aq(x)\overline{a}$ for $x,y \in K$; $a_1,a_2,a \in B$.

The morphisms are the module-homomorphisms preserving b and q.

This category has an obvious concept of "direct sum" (often called "orthogonal sum") hence one can define its Grothendieck-group.

For P a finitely generated left B-module we define $H(P)$ to be the left B-module $P \oplus P^d$, equipped with the q which vanishes on P and P^d and with the b which vanishes on $P \times P$ and $P^d \times P^d$, and which is the canonical pairing on $P \times P^d$.

The quotient of the Grothendieck-group by the subgroup generated by the elements of the form $H(P)$ will be denoted by $L(B)$. The construction of II.3.5. assigns to a normal map of even-dimensional manifolds, inducing a homotopy-equivalence of the boundaries, an element of the group: $L(B)$ (if one is given a homomorphism $\pi_1(N) \to \pi$).

Given a finitely generated left B-module P, consider the set of mappings $\ell : P \times P \to B$ such that:

(i) ℓ is biadditive.

(ii) $\ell(a_1x, a_2y) = a_1\ell(x,y)\overline{a_2}$, $\quad a_1, a_2 \in B$; $x, y \in P$.

An involution T acts on this abelian group by: $(T\ell)(x,y) = \overline{\ell(y,x)}$; given an element $\ell \in \operatorname{coker}(1-T)$ we can define a pair (b_ℓ, q_ℓ) by $b_\ell(x,y) = \ell(x,y) + \overline{\ell(y,x)}$; $q_\ell(x) =$ class of $\ell(x,x)$.

The assignment $\ell \to (b_\ell, q_\ell)$ maps coker(1-T) bijectively to the set of pairs (b,q), satisfying (1)-(3), (5)-(8) above.

For this fact we refer to [16].

4.7.

<u>LEMMA</u>. Let $X \underset{i}{\to} Y \underset{p}{\to} Z$ be an exact sequence of left B-modules and let (b,q) be a nonsingular quadratic form on Y.

Let $c : X \times Z \to B$ also be a nonsingular pairing such that:

$$q(ix) = 0 \quad \text{and} \quad b(ix,y) = c(x,py) \quad \text{for } x \in X,\ y \in Y.$$

Then (Y,b,q) represents 0 in L(B).

<u>P r o o f</u>. Denote im $i \subset Y$ by S. A map $\operatorname{ad}b : S \to (Y/S)^d$ is induced by b because $b(ix_1, ix_2) = 0$ for $x_1, x_2 \in X$. We state that this map is isomorphic.

An element $f \in (Y/S)^d$ can be considered as an element of Y^d, hence, because b is nonsingular, there exists a $y \in Y$ such that $b(a,y) = fp(a)$ for all $a \in Y$. This implies that $c(x,py) = b(ix,y) = 0$ for all $x \in X$, $y \in Y$; hence $py = 0$, so $y \in S$.

If $s \in S$ such that $b(y,s) = 0$ for all $y \in Y/S$ then $s = 0$ because b is nonsingular. This proves the statement.

Because every B-module is projective, we may conclude that $Y = S \oplus Y/S = S \oplus S^d$. Choose P such that $S \oplus P$ is a free module, say P. Then $(Y,b,q) \oplus H(P)$ is a quadratic form with underlying module $Y \oplus B \oplus P^d = F \oplus F^d$ which is free, and b and q vanish on $S \oplus P = F$.

According to lemma 5.3. in [15] (but without mention of bases) the quadratic form is isomorphic to H(F). Because H(F) and H(P) represent 0 in L(B), so does (Y,b,q).

Q.E.D.

Summarizing:

THEOREM. If a normal map of even dimensional manifolds is bordant to a homotopy-equivalence "over π" then the associated quadratic form represents 0 in L(B).

Remark: The condition "bordant ...." means the same as ~ 0 in the notation of [15 ; page 86].

CHAPTER III

THE RELATION BETWEEN THE QUADRATIC FORM AND THE SURGERY OBSTRUCTION

§1. Generalities

1.1.

We recall the following from [15 , chapter 9].

Consider "objects" consisting of the following:

compact manifolds with boundary $(M,\partial M)$ and $(N,\partial N)$ of dimension n,

a map $f : (M,\partial M) \to (N,\partial N)$ of pairs of degree one, inducing a homotopy-equivalence: $\partial M \to \partial N$,

a vectorbundle ν over N and a stable framing F of $TM + f^*\nu$, and finally

a map $w : N \to K$ (K a CW complex) such that $w_1(N)$ factorizes as

$\pi_1(N) \overset{w}{\to} \pi_1(K) \to \{\pm 1\}$.

Introduce the notation $\theta \sim 0$ for an object θ as above to denote that there exist:

compact manifolds $(P,\partial_-P,\partial_+P)$ and $(Q,\partial_-Q,\partial_+Q)$ of dimension n+1 with two boundary parts, such that $(M,\partial M) = (\partial_+P,\partial_+P \cap \partial_-P)$ and

$(N,\partial N) = (\partial_+Q,\partial_+Q \cap \partial_-Q)$,

a map $g : (P,\partial_+P,\partial_-P) \to (Q,\partial_+Q,\partial_-Q)$ of degree one, extending f, and inducing a homotopy-equivalence: $(\partial_-P,\partial_-P \cap \partial_+P) \to (\partial_-Q,\partial_-Q \cap \partial_+Q)$,

a vectorbundle μ over Q extending ν and a stable framing G of $TQ + g^*\mu$, stably extending F, and finally

an extension of w to a map $Q \to K$ such that $w_1(Q)$ factorizes.

The definition: $\theta_1 \sim \theta_2$ if $\theta_1+(-\theta_2) \sim 0$ (where + denotes disjoint union and - denotes change of orientation) defines an equivalence relation on the "objects"; we denote the set of equivalence-classes by $L_n^1(K)$; it has a natural structure of abelian group.

On the other hand there are the Wall groups $L_n(\pi,w)$, which for even n are defined in [15, chapter 5] as equivalence-classes of quadratic forms over $Z[\pi]$. According to the cited reference the surgery obstruction defines an isomorphism $s : L_n^1(K) \to L_n(\pi_1(K),w)$ for $n \geq 4$ if K has finite 2-skeleton. In fact it is stated there for Poincaré-complexes N and Q instead of manifolds, but the fact that we are dealing with manifolds only makes things easier.

Now the constructions of the last chapter endow us with a map $t : L_n^1(K) \to L(B)$, for $\pi = \pi_1(K)$ finite; furthermore the operation of reducing a quadratic form over $Z[\pi]$ to one over B induces a map $r : L_n(\pi,w) \to L(B)$ for n even.

In this chapter we prove the:

THEOREM: $rs = t$

by first constructing an element of $L_n^1(K)$ mapping to an arbitrary given element of $L_n(\pi,w)$ and then showing that our quadratic form for this normal map is just the reduction of the given quadratic form to B coefficients.

§2. The construction of a standard normal map.

2.1.

Assume $n = 2s \geq 6$. Let X^{n-1} be a connected compact manifold with boundary and fundamental group π. Let ϕ be a nonsingular quadratic form on a free module over $R = Z[\pi]$ with base $e_1,e_2,\ldots e_m$. (i.e. ϕ

represents an element of coker (1-T); compare II.4.). We are going to construct a normal map to $X \times [0,1]$ with the class of ϕ as its surgery obstruction, following [15 , theorem 5.8.]; our construction is adapted so as to allow the calculation of our (b,q) for it.

We base our construction on the following algebraic trick: define the quadratic form ψ on a free module over R with base $e_1', e_2', \ldots e_m'$, $e_1'', e_2'', \ldots e_m''$ by $\psi(e_i', e_j') = \psi(e_i'', e_j'') = \psi(e_i', e_j'') = 0$ and $\psi(e_i'', e_j') = \phi(e_i, e_j)$; then the restriction of ψ to the submodule spanned by the $\hat{e}_i = e_i' + e_i''$ is just ϕ : $\psi(\hat{e}_i, \hat{e}_j) = \psi(e_i'+e_i'', e_j'+e_j'') = \phi(e_i, e_j)$. In applying C.T.C. Wall's theorem 5.8. to ψ, one does <u>not</u> introduce self-intersections.

2.2.

We choose 2m disjoint discs in the interior of X and for each one we choose a path connecting it to the basepoint * of X; that is equivalent to choosing liftings to the universal covering $\tilde{X}$; composition with the standard embedding $S^{s-1} \times [-1,1] \times D^{s-1} \to D^{2s-1}$ yields 2m disjoint embeddings $(f_i')^0$ and $(f_i'')^0$.

Let E^{s-1} be the "northern hemisphere" disc in S^{s-1}. Now there are embeddings $\Gamma_i : [0,1] \times D^{s-1} \times D^{s-1} \to \text{int } X$ such that

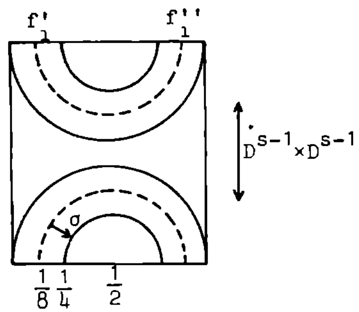
1) $\Gamma_i \mid [0,\frac{1}{4}] \times D^{s-1} \times D^{s-1} = (f_i')^0 \mid E^{s-1} \times [-1,1] \times D^{s-1}$,

2) $\Gamma_i \mid [\frac{3}{4},1] \times D^{s-1} \times D^{s-1} = (f_i'')^0 \mid E^{s-1} \times [-1,1] \times D^{s-1}$,

3) $\Gamma_i \mid [0,1] \times 0 \times 0$ yields together with the chosen paths a zero-homotopic loop. (We identify $[-1,1]$ with $[0,\frac{1}{4}]$ resp. $[\frac{3}{4},1]$.)

We subject the $(f_i')^0$ and $(f_i'')^0$ to simultaneous regular homotopies η_i' resp. η_i'' to new disjoint embeddings $(f_i')^1$ and $(f_i'')^1$ such that the induced "framed immersions" $\eta \times id : S^{s-1} \times [-2,1] \to X \times [-2,-1]$ have intersections and self-intersections as described by ψ. (For details on this, see [15 , proof of theorem 5.8.]).Hence $(f_1')^1$ and $(f_1'')^1$ are also "connected by Γ_i". We may assume that the homotopies, restricted to the complement of E^{s-1}, remain disjoint from the im Γ_1. From now on we denote $\eta_1'(t+2)$ by $(f_1')^t$ and $\eta_1''(t+2)$ by $(f_1'')^t$.

We can use the Γ_i to construct the connected sum $(\hat{f}_i)^t$ of the framed spheres $(f_1')^t$ and $(f_1'')^t$. To be more precise:
Write S^{s-1} as $D^{s-1} \cup [0,1] \times S^{s-2} \cup D^{s-1}$; this yields also a splitting of $S^{s-1} \times D^{s-1} \times [-1,1]$. Define $(\hat{f}_1)^t$ as
$(f_1')^t$ at the "southern hemisphere" on the first D^{s-1} summand,
$(f_1'')^t$ at the "soutnern hemisphere" on the second D^{s-1} summand,
and on the $[0,1] \times S^{s-2}$ summand we define $(\hat{f}_1)^t$ as Γ_1 composed with the embedding $([0,1] \times S^{s-2}) \times ([-1,1] \times D^{s-1}) \to [0,1] \times D^{s-1} \times D^{s-1}$ mapping (τ,v,σ,x) to $(\frac{1}{2} - \frac{3-\sigma}{8} \cos x\tau,\ v(1 - \frac{3-\sigma}{8} \sin x\tau),x)$. (see sketch)

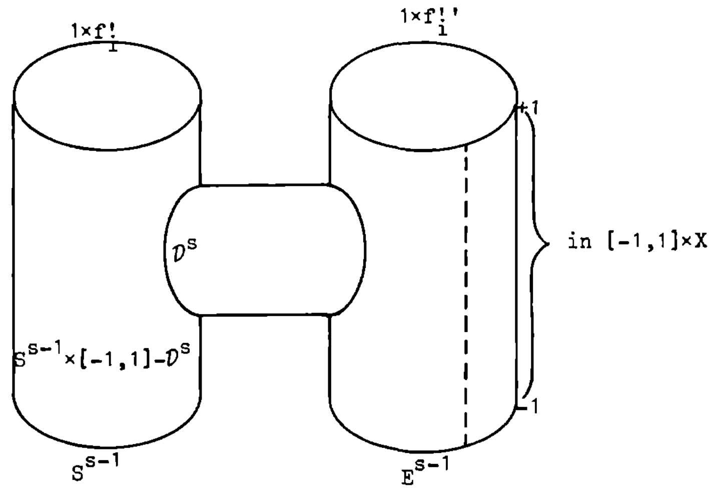


Identifying S^{s-1} with $\{(a,v) \in [-1,1] \times \mathbb{R}^{s-1} \mid a^2+|v|^2 = 1\}$ and replacing in the above formula v by the pair (a,v) we find also an embedding: $([0,1] \times S^{s-1}) \times ([-1,1] \times D^{s-1}) \to [-1,1] \times ([0,1] \times D^{s-1} \times D^{s-1})$.

The image of this map under $id_{[-1,1]} \times \Gamma_i$ in $[-1,1] \times X$ fits together with the image under $id \times (f_i^!)^1$ and $id \times (f_i^{"})^1$ of $([-1,1] \times S^{s-1} - \mathcal{D}^s) \times ([-1,1] \times D^{s-1})$ where $\mathcal{D}^s$ denotes $\{(a,v) \in [-1,1] \times E^{s-1} \mid a^2+|v|^2 \leq 1\}$.

In less precise language: we can form the connected sum of the two cylindres defined by $id \times (f_i^!)$ and $id \times (f_i^{"})$ inside $[-1,1] \times X$. (see sketch.)

We will denote this subset of $[-1,1] \times X$ by T.



Now we can define the manifolds Y_a, Y_b, Y' and Y":

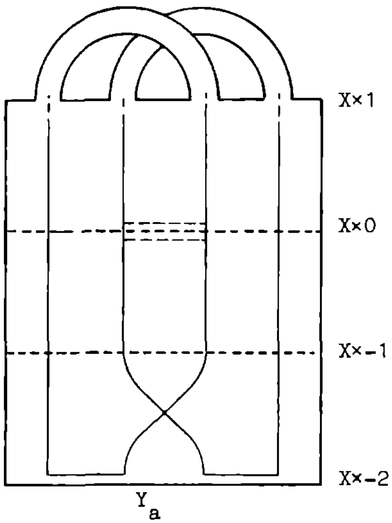
Y_a consists of $[-2,1] \times X$ together with handles $D^s \times D^s$, glued according to the embeddings $(f_i^!)^1$ and $(f_i^{"})^1 : S^{s-1} \times D^s \to X \times 1$.

Y_b consists of $[-2,0] \times X \cup T$ together with the same handles; hence Y_b is a regular domain of Y_a. It can also be considered as $[-2,0] \times X$ together with handles, glued according to the embeddings $\hat{f}_i$. (See the picture.)

Y' is constructed in the same way as Y_a, but only using the $(f_i^!)^1$; similarly

Y" is constructed gluing only the second set of m handles to $[-2,1] \times X$.

In the next subsections we construct a normal from Y_a to $X \times [0,1]$ and consider the induced normal maps on Y_b, Y' and Y".



2.3.

Now that we have defined the manifolds involved we are going to define the normal maps. First we give a more detailed account of the framings in the proof of theorem 5.8. of [15].

Consider X and the regular homotopies η_i of the framed spheres S^{s-1}, which were embedded in the standard way; we can view these homotopies as "framed immersions" $S^{s-1} \times I \to X \times I$. Together with the standardly embedded framed discs D^s that yields a framed immersion of (D^s, S^{s-1}) into $(X \times I, X \times 1)$ i.e. a map
$j : (D^s, S^{s-1}) \times D^s \to (X \times I, X \times 1)$. Denoting by p the projection
$X \times I \to X$ we have a map from
$Y = X \times I \cup$ handles to X, viz. $f = p \cup p \circ j$.

A trivialization F of $T(X \times I) + p^*\nu$ (with $\nu = \nu_X$ there is a canonical one) yields a trivialization of $j^*T(X \times I) \oplus j^*p^*\nu$.
We identify $j^*T(X \times I)$ with the tangent space of the handle by $(id_{TX} \oplus -1) \circ Dj$ which is possible because j is an immersion, hence Dj is isomorphic. The union of these trivializations is a trivialization of $T(X \times I \cup \text{handle}) + (p \cup p \circ j)^*\nu = TY + f^*\nu$.

Our trivialization is the correct one at $X \times I$. In [15, page 10] C.T.C. Wall argues that there exists essentially one trivialization for the handle and that the only thing to be checked is that that one behaves right on the intersection of the handle with $X \times I$. Because the trivialization we constructed for the handle does behave right on the intersection, it is the correct one.

In analogy with this situation we can "fold Y_a along $X \times 0$":
j maps the handles to $X \times [-2,-1]$ using the regular homotopy, and maps $X \times [0,1]$ to $X \times [-1,0]$ by: $(x,t) \to (x,-t)$. With the aid of p: $X \times [-2,0] \to X$ that yields $f : Y_a \to X$. Constructing the framing in

analogy with the above is also straightforward.

By restriction we have j, f and F at our disposal for Y_b, Y' and Y".

2.4.

The f we constructed in the last subsection is a map to X; however we need maps to $X \times I$.

Consider a bordism $(L,\partial_+L,\partial_-L)$ as in I.11.1. and a map $f : L \to Z$ such that $f(\partial_0 L) \subset \partial Z$. By a homotopy we achieve that in a collar $I \times \partial_0 L \subset \partial_+ L$ f maps (t,x) as (1,x); in particular it maps the collar to ∂Z. Now define $f_1 : (L,\partial_+L,\partial_-L) \to (Z\times[0,1],\ Z\times 1 \cup \partial Z\times[0,1],\ Z\times 0)$ by taking the second coordinate equal to the minimum of 1 and $(1/\varepsilon)$ times the distance to ∂_-L, for sufficiently small $\varepsilon > 0$.

A trivialization of $TL+f^*\nu$ yields one of $TL+f_1^*(p^*\nu)$, because $pf_1 = f$, where p is the projection $Z \times I \to Z$.

In this way we get from our construction a normal map of manifolds with boundary of equal dimension n. Notice that $X \times [-1,1] \cup$ handles is mapped to $X \times 1 \subset \partial(X \times I)$ and hence that we have the situation of the "naturality theorem" II.4.1. for the inclusion $Y_b \subset Y_a$.
Similarly for the inclusions of Y' and Y" in Y_a.

§3. The computation of the quadratic form for the model normal map.

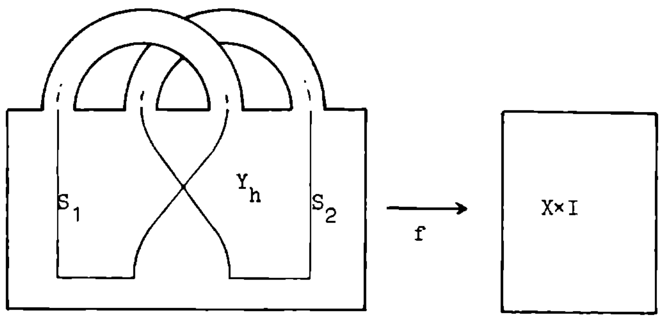
3.1.

The idea of the computation is as follows:
we compute the quadratic form for Y_b using the naturality theorem from the quadratic form for Y_a;
this we compute using naturality from the quadratic forms for Y' and Y";

we compute the quadratic forms for Y' and Y" also by naturality from the one for a certain manifold Y_h, which is a boundary, and we apply II.4.2.

We carry out the construction of C.T.C. Wall [15 , theorem 5.8.] for the quadratic form $H(R^m)$; that yields a normal map $f : Y_h \to X \times I$, inducing a homotopy-equivalence of the boundary because the form is nonsingular.

There are two sets of s-spheres, S_1 and S_2, corresponding with the first resp. second half of the standard base of R^{2m}; each of the spheres consists of the original s-disc, the regular homotopy η of its boundary and the core of the handle (see sketch). These spheres are mapped by f to zero-homotopic spheres.



Now Y_h has the homotopy-type of the one-point union of X and the 2m s-discs, in such a way that f corresponds to the projection on X. Hence with coefficients M equal to R or B the kernel of

$f_* : H_j^\pi(\tilde{Y}_h:M) \to H_j^\pi(\tilde{X} \times I;M)$ vanishes except for j = s,

and in that case it is equal to M^{2m}; the generators are represented by the spheres in S_1 and S_2.

Now we can do surgery on the collection of s-spheres S_1 because these are embedded with trivial normal bundle and without mutual intersections. Because the associated elements of R^{2m} generate a subkernel of $H(R^m)$, the result of this surgery is a homotopy-equivalence (in fact it is a diffeomorphism).

We introduce the following notations:

Z = the trace of the surgery i.e. $Y_h \times I \cup$ the m (s+1)-handles,

$Z_2 = Y_h$, $Z_1 = \overline{\partial Z - Z_2}$, $Z_0 = Z_1 \cap Z_2 = \partial Z_2$.

Then we can apply II.4.2. and we conclude that q and b vanish on the part of $\ker f_* P \subset H^s_\pi(\tilde{Y}_h, \partial\tilde{Y}_h; B)$ which is in the image of $H^s_\pi(\tilde{Z}, \tilde{Z}_1; B)$.

Now according to I.11.4. the following diagram commutes:

$$\begin{array}{ccc} H^s_\pi(\tilde{Z},\tilde{Z}_1;B) & \xrightarrow{\ i\ } & H^s_\pi(\tilde{Z}_2,\tilde{Z}_0;B) \\ \downarrow P & & \downarrow P \\ H^\pi_{s+1}(\tilde{Z},\tilde{Z}_2;B) & \xrightarrow{\ \partial\ } & H^\pi_s(\tilde{Z}_2;B) \end{array}$$

, where P denotes Poincaré duality.

Now $H^\pi_{s+1}(\tilde{Z},\tilde{Z}_2;B) = B^n$ with the elements represented by the (s+1)-handles, which were glued to Y_n during the surgery, as a base; hence the image of it under ∂ is represented by the s-spheres S_1.

We conclude that q and b vanish for the first m base elements of $\ker f_* P$. The same is true for the second m base elements, as is seen by interchanging the roles of S_1 and S_2 in the above discussion.

3.2.

Had we applied the construction of [15 , theorem 5.8.] only for the first m base elements of $H(R^m)$, so with vanishing quadratic form, then we would have got the construction of the normal map $Y' \to X \times I$. We now apply the naturality theorem II.4.1. to the inclusion $Y' \subset Y_h$.

By Poincaré duality e_j^* corresponds to j_* in homology. It is clear that ker $(f|Y')_*$ has a base represented by the collection S_1 of m s-spheres and that j_* maps this to the first half of the base of ker f_*.

We conclude that for $f : Y' \to X \times I$ q and b vanish identically, because they did so on the first half of the base of ker f_*. The same is true for Y''.

3.3.

We apply the naturality theorem II.4.1. to the inclusion $Y' \subset Y_a$. Ker $(f|Y_a)_*$ has a base consisting of elements $e_1', e_2', \ldots e_m', e_1'', e_2'', \ldots e_m''$, which are represented by s-spheres as was the case for Y_h; the first half of those spheres are in Y'.

This tells us that q and b vanish on the corresponding elements of ker $(f|Y_a)_*P$. The same can be said about $Y'' \subset Y_a$ and the second half of the base.

Finally $b(P^{-1}e_i'', P^{-1}e_j')$ can be viewed as the intersection-number of e_i'' and e_j', reduced from R-coefficients to B-coefficients. This number is equal to the intersection-number of the corresponding spheres, hence by construction to $\phi(e_i, e_j)$ (compare III.2.1.). Hence $b(P^{-1}e_i'', P^{-1}e_j') = r\phi(e_i, e_j)$, where r denotes the reduction.

3.4.

Finally we apply the naturality theorem II.4.1. to the inclusion

$k : Y_b \subset Y_a$. Ker $(f|Y_b)_*$ has a base consisting of elements $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m$, which are represented by the spheres formed by im η, T and the cores of the handles. In Y_a these can be viewed as connected sums of corresponding spheres from S_1 and S_2; hence $k_* \hat{e}_i = e_i' + e_i''$ so e_j^* maps $P^{-1}\hat{e}_i$ to $P^{-1}e_i' + P^{-1}e_i''$.

We have

$$q(P^{-1}\hat{e}_i) = q(P^{-1}e_i') + q(P^{-1}e_i'') + b(P^{-1}e_i', P^{-1}e_i'') =$$
$$= 0 + 0 + r\phi(e_i, e_i) = rq_\phi(e_i).$$
$$b(P^{-1}\hat{e}_i, P^{-1}\hat{e}_j) = b(P^{-1}e_i'', P^{-1}e_j'') + b(P^{-1}e_i', P^{-1}e_j') +$$
$$+ b(P^{-1}e_i'', P^{-1}e_j') + b(P^{-1}e_i', P^{-1}e_j'') =$$
$$= 0 + 0 + r\phi(e_i, e_j) + \overline{r\phi(e_j, e_i)} =$$
$$= r\, b_\phi(e_i, e_j),$$

which completes the calculation of this section.

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SAMENVATTING

De techniek van chirurgie wordt gebruikt om te onderzoeken of een gegeven afbeelding van variëteiten bordant is met een homotopie-equivalentie. In dit proefschrift wordt de techniek van W. Browder om de bij een dergelijk probleem optredende obstructie met algebraïsche topologische middelen te vinden uitgebreid van het enkelvoudig samenhangende geval naar het geval van een eindige fundamentaalgroep.
Hierbij wordt gebruik gemaakt van middelen van de equivariante algebraïsche topologie en van de differentiaaltopologie.
In hoofdstuk I wordt hiertoe de equivariante algebraïsche topologie van G. Bredon en Th. Bröcker verder ontwikkeld, met name door de constructie en bestudering van equivariante cohomologie-operaties. Deze blijken naast de vertrouwde, ook eigenschappen te bezitten (stellingen 6.4. en 7.3.) die in het klassieke geval niet aan de dag treden.
In hoofdstuk II §1 wordt het chirurgie-gegeven vertaald in een equivariant topologisch gegeven, waarna in §2 en §3 met de technieken van hoofdstuk I hieruit een niet-singuliere kwadratische vorm in de zin van C.T.C. Wall wordt geconstrueerd. In §4 worden enkele belangrijke eigenschappen van deze constructie afgeleid.
In hoofdstuk III wordt door berekening van deze vorm in een voldoend algemene situatie, aangetoond dat de geconstrueerde grootheid de chirurgie obstructie ten dele vastlegt. Het probleem, gesteld door L. Shaneson, om de chirurgie-obstructie zonder voorbereidende chirurgie te bepalen, is aldus door de resultaten van dit proefschrift in belangrijke mate opgelost.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 17 april 1950 geboren in de gemeente Oss. In december 1966 werd hij winnaar van de vijfde Nederlandse Wiskunde Olympiade. In 1967 behaalde hij het diploma Gymnasium β aan het Titus Brandsma Lyceum te Oss.
Vervolgens studeerde hij van 1967 tot 1970 wis- en natuurkunde aan de Katholieke Universiteit te Nijmegen. Colleges in de wiskunde volgde hij o.a. bij de hoogleraren J.H. de Boer, R.A. Hirschfeld, J.J. de Iongh, H.A.M.J. Oedaijrajsingh Varma, A.C.M. van Rooij en H. de Vries.
Het kandidaatsexamen wiskunde werd in 1969 cum laude afgelegd en het doctoraalexamen in 1970, eveneens cum laude.
In de jaren 1969 en 1970 was hij studentassistent en sinds 1970 is hij als wetenschappelijk medewerker verbonden aan het Mathematisch Instituut van deze Universiteit.
Gedurende het academisch jaar 1971-1972 werd hij door een stipendium van de Niels Stensen Stichting in staat gesteld in Liverpool o.l.v. Prof. C.T.C. Wall onderzoek te verrichten. Een artikel daarover is in voorbereiding.
Vanaf 1972 werd onderzoek verricht voor dit proefschrift. Voorts zijn van de hand van de auteur een artikel in C.R.Acad.Sci Paris en een in Inventiones Mathematicae verschenen. De auteur heeft voordrachten gehouden voor het I.H.E.S. te Bures-sur-Yvette, het Mathematische Forschungsinstitut te Oberwolfach en het Nederlandse Mathematisch Congres 1975 te Utrecht.

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STELLINGEN

1. Het simpliciale complex met hoekpuntenverzameling $a,b,\ldots,\ell$ bestaande uit de simplices

aef, fca, cfk, kfi, fhi, ihe, iej,

eaj, jak, kad, ckd, ℓhg, gjℓ, jgb,

bgd, ged, deh, dhc, hℓc, cℓb, bℓi,

jbi, cba, abd, egf, fgh, kℓj, iℓk en hun zijden

vormt een triangulatie van het orienteerbare oppervlak van geslacht twee op zodanige wijze, dat in elk hoekpunt precies zeven simplices elkaar ontmoeten.

2. Voor een coketencomplex E met coaugmentatie e en met daarmee consistente cup-i-producten $\underset{i}{\cup}$ zij de kegel CE van E gedefinieerd door $(CE)^n = C^n \oplus C^{n-1}$ voor $n > 0$ en $(CE)^0 = C^0 \oplus Z$, met corand-operator δ gedefinieerd door $\delta(a,b) = (\delta a, \delta b - (-1)^n a)$ voor $a \in C^n$ en $\delta(a,b) = (a, e(b) - a)$ voor $a \in C^0$, en met coaugmentatie e gedefinieerd door $e(1) = (e(1),1)$.

Dan wordt een daarmee consistent stelsel van $\underset{i}{\cup}$ - producten gedefinieerd door:

$(a,b) \underset{i}{\cup} (c,d) = (a \underset{i}{\cup} c,\ a \underset{i}{\cup} d + (-1)^{mn+i}\, d \underset{i-1}{\cup} b)$, indien $a \in C^{m+1}$, $c \in C^n$ en $i > 0$;

$(a,b) \underset{0}{\cup} (c,d) = (a \underset{0}{\cup} c,\ a \underset{0}{\cup} d)$ indien $c \in C^n$ en $n > 0$;

$(a,b) \underset{0}{\cup} (c,d) = (a \underset{0}{\cup} c,\ a \underset{0}{\cup} e(d) + d\, b)$ indien $c \in C^0$, dus $d \in Z$.

Met behulp hiervan kan een natuurlijk stelsel van cup-i-producten voor het singuliere coketencomplex van een willekeurige ruimte worden geconstrueerd. De $\cup_0$ is die van Alexander en Whitney en dus associatief. De operatie $a \cup_1$ met vaste a is een derivatie over $\cup_0$.

3. Zij p de partitiefunctie, d.w.z. $p(n)$ is het aantal manieren om n te schrijven als som van natuurlijke getallen.

Dan is het aantal manieren om n te schrijven als som van verschillende oneven getallen op zodanige wijze dat een even (resp. oneven) aantal daarvan congruent is met 3 of 5 modulo 8 gegeven door $\Sigma\ p(\frac{n-k}{4} - \frac{k(k-1)}{2})$, waar gesommeerd wordt over de natuurlijke getallen k die modulo 8 congruent zijn aan n (resp. n+4).

4. Laat V en W vectorruimten zijn van eindige dimensie. Laat voor $i = 1,2,3$ K_i een lineaire deelruimte zijn van V, L_i een lineaire deelruimte zijn van W en P_i een lineaire afbeelding zijn van V naar W. Er bestaat een lineaire afbeelding A van V naar W zodanig dat $(A-P_i)(K_i) \subset L_i$ voor $i = 1,2,3$ precies als aan de volgende vijf onafhankelijke voorwaarden is voldaan:

1) Er bestaat een A met $(A-P_i)(K_i) \subset L_i$ voor $i = 1$ en $i = 2$.

2) Analoog voor $i = 1$ en $i = 3$.

3) Analoog voor $i = 2$ en $i = 3$.

4) Het probleem is oplosbaar als men elke L_i door $L_1+L_2+L_3$ vervangt.

5) Het probleem is oplosbaar als men elke K_i door $K_1 \cap K_2 \cap K_3$ vervangt.

5. Zij k een lichaam en zij $R = k[X]/(X^2)$; k is te beschouwen als een moduul over R waarbij X als 0 werkt.

Dan is een willekeurig ketencomplex C van vrije R-modulen equivalent met het complex gevormd door de R-modulen $H_n(C;k) \otimes_k R$ en de homomorphismen $B_n \otimes X$, waar $B_n : H_n(C;k) \to H_{n-1}(C;k)$ de Bokstein-operator is, geassocieerd met het exacte rijtje van R-modulen:

$0 \to XR \to R \to R/XR \to 0.$

De ketenequivalentieklasse van het ketencomplex C wordt dus vastgelegd door diens homologiegroepen en Boksteinoperaties.

6. Beschouw een inbedding f van de volle torus $S^1 \times D^2$ in de drie-spheer S^3, zodanig dat de hartlijn $f(S^1 \times 0)$ een klaver - bladknoop beschrijft, en zodanig dat het "linking-getal" van $f(S^1 \times 0)$ en $f(S^1 \times p)$ voor $p \in D^2$ gelijk is aan -2.
Het resultaat van het doen van chirurgie op S^3 door middel van f is een varieteit met als fundamentaalgroep de binaire oktaedergroep (van orde 48).

7. Bij het bewijzen van stellingen als de volgende kan met vrucht gebruik worden gemaakt van de meetkunde van twee-dimensionale varieteiten:
Een ondergroep, welke isomorf is met $Z \oplus Z$, van een geamalgameerd product $G_1 *_H G_2$ over een eindige groep H, is geconjugeerd met een ondergroep van G_1 of van G_2.

8. In algebraïsch-topologische bewijzen wordt vaak gebruik gemaakt van ineenstortende spectraalrijen in gevallen, waarin volstaan kan worden met een eenvoudig ad hoc argument.

9. De ruimte van minimale geodeten tussen twee vaste antipodale punten in de Stiefel-varieteit $V_2 IR^n = \{(x,y) \in IR^n \times IR^n; \langle x,x\rangle = \langle y,y\rangle = 1, \langle x,y\rangle = 0\}$ is homeomorf met de suspensie van de Stiefel-varieteit $V_2\, IR^{n-2}$.

10. De voorzieningen, welke door de overheid worden getroffen ten gerieve van de automobilist, gaan vaak ten koste van de voetganger.