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C O N T E N T S

I N T R O D U C T I O N

The aim of this work is to give a survey of the space-time symmetry groups of some typical electromagnetic fields in vacuum and in homogeneous media described by constant electric and magnetic permeabilities ε and μ respectively. In the vacuum case, TE and TM modes of rectangular waveguides and resonant cavities are considered [1]. In the case of homogeneous media, isotropic, uniaxial electric and uniaxial magnetic media, media displaying electric and magnetic Faraday effects and naturally optically active isotropic media are dealt with.

In the presence of polarizable matter, the EM-fields considered are solutions of the macroscopic Maxwell equations. Here no attempt is made to derive these macroscopic fields from microscopic ones and a purely phenomenological approach is followed. The transformation properties of these fields depend on the macroscopic description chosen. In this work four such macroscopic descriptions or formulations [2] are considered. In the Chu, Ampère and Boffi formulations the fundamental variables can be split into the polarization variables $\vec{P}$ and $\vec{M}$, and the field variables $\vec{E}$ and $\vec{H}$ or $\vec{B}$ respectively. For the polarization variables the transformation laws with respect to the Lorentz group follow from the microscopic models underlying the Chu, Ampère and Boffi formulations. For the field variables the transformation laws are derived from the postulated covariance of Maxwell's equations (in terms of the fundamental variables) with respect to Lorentz transformations. This in conjunction with the transformation laws for the polarization variables. In the Minkowski formulation, where the polarization variables are not fundamental, the transformation laws of all fundamental variables $\vec{E}_M$, $\vec{B}_M$, $\vec{D}_M$ and $\vec{H}_M$ follow from the covariance postulate. Note that the above covariance requirements are not trivial consequences of the universally accepted Lorentz covariance of the microscopic Maxwell equations. The transition from the microscopic to the macroscopic Maxwell equations is not unambiguous [3]. This consequence of the basic fact, that a macroscopic description is not a complete characterization of a system may explain why macroscopic electrodynamics is still a controversial subject[4][5][6]. For this reason we have compared different formulations from the point of view of space-time symmetries.

After the preliminary chapters I and II, in chapter III the relativistic symmetry groups of transversal electric (TE) and transversal magnetic (TM) modes in empty rectangular waveguides (including evanescent modes) and resonant cavities are determined. Non-propagating modes (the evanescent ones included) have as point group a Shubnikov group, whereas the point group of propagating modes is, in general, of Shubnikov type,only in a frame of reference moving with a relative velocity (with respect to the laboratory system) equal to the propagation velocity of the mode. In the laboratory system the relativistic symmetries, which are not of Shubnikov type,turn out to be mirrors and are called Lorentz mirrors. Chapter IV deals with the space-time symmetries of a monochromatic TEM wave in a homogeneous isotropic medium described by electric and magnetic permeabilities ε and μ respectively. For refractive index $n \gg 1$ the propagation velocity of the wave is small as compared to the speed of light in vacuum and therefore, in this case, it makes sense physically to speak of a non-relativistic approximation and of a non-relativistic limit of the relativistic symmetry group of the wave. The transition to the non-relativistic limit of a relativistic group is made by introducing corresponding Galilei transformations. Formally this can be done for any value $n > 1$ of the refractive index. These formal Galilean symmetries still play an interesting rôle: a relation is found between the relativistic symmetry groups and the Galilei symmetry groups of the field tensors $F_{\mu\nu}$ and $G^{\mu\nu}$ in the Minkowski formulation. In the same chapter the four different formulations are compared from the point of view of relativistic symmetry. It is found that for all formulations the relativistic symmetry groups of the various EM-fields are generated by the same set of symmetry elements. The four formulations differ merely in the way the individual generators are assigned to the relativistic symmetry groups of the various EM-fields. In chapter V the investigation is extended to the homogeneous, anisotropic media mentioned at the beginning. A new type of relativistic symmetry groups is only found in the cases of uniaxial electric and uniaxial magnetic media. Considered in the isotropic limit these groups coincide with the corresponding relativistic symmetry groups of the isotropic case. A more detailed summary of the results is given at the end.

From a more technical point of view let us remark that crystallographic concepts are used to describe the space-time symmetry groups. This was introduced by Janner et. al. [1][7][8] in their first determination of the relativistic symmetry groups of electromagnetic fields and justified later on a more general basis [9].

The work presented here is part of a larger program, the main concern of which is the description of the states of a charged particle in electromagnetic fields with and without the presence of polarizable matter, with the help of the space-time symmetries of such a system. Here this problem is not considered in the general frame work, but the attention is restricted to the case where the equations can be decoupled into the matter-field equations (equations of motion) and the electromagnetic field equations (Maxwell's equations). The problem of determining the space-time symmetries of the system then reduces to the one of finding the invariance operator group of the corresponding equations of motion. This operator group is based on the space-time symmetry group of the (decoupled) electromagnetic fields entering into the equations of motion, in a way which is in principle well known in the vacuum case. In the case where matter is characterized by macroscopic parameters only, the question of how to arrive at the invariance operator group is still a matter of research. There are of course many other problems closely connected with this work. To a number of them some attention has already been given:

(i) The determination of the relativistic symmetry group of the energy-momentum tensor and the possible consequences for a corresponding Lagrangean and thus for the physical system described.

(ii) The determination of the relativistic symmetry groups of covariantly averaged microscopic fields and their relation to the relativistic symmetry groups of the phenomenological fields.

These topics, however, are not presented here, because they have not yet reached the stage generally deemed indispensable for promulgation.

CHAPTER I

SYMMETRIES OF EQUATIONS AND OF SYSTEMS

According to Wigner[10] geometrical and dynamical symmetries are distinguished. Geometrical symmetry is defined in terms of events, whereas dynamical symmetry is a characteristic of the interactions considered. The two are probably best contrasted by means of an example: the inhomogeneous Lorentz group or Poincaré group, IO(3,1), is a geometrical symmetry (or invariance) group of Maxwell's equations, in the sense, that Maxwell's equations are form-invariant under elements of IO(3,1). This is also called covariance of Maxwell's equations. However, this is not saying, that IO(3,1) is the largest group of transformations leaving Maxwell's equations invariant[11][12]. On the other hand there are cases, for which the electromagnetic interaction must be described by potentials. Because the same EM-fields can be derived from different potentials, which are then said to be connected by a gauge transformation, the electromagnetic interaction must be gauge invariant. The gauge group is a dynamical invariance group of the electromagnetic interaction. In this work only the geometrical symmetry comes into the picture, because the symmetries of EM-fields are considered. There is other work, in which invariance operator groups of equations of motion of charged particles in external EM-fields are studied[13]. But then the introduction of the concept of compensating gauge transformation is necessary, because symmetry transformations of EM-fields, in general, do not leave the corresponding potential invariant. Therefore symmetry operators of equations of motion are composed of an element of IO(3,1) and of a gauge transformation. They are examples of symmetries made up of geometrical and dynamical constituent parts.

Considering the symmetries of EM-fields it is natural to start from Maxwell's equations, the fields having to satisfy them. IO(3,1) is a covariance group for these equations and this group furnishes, as it were, the background for the symmetry considerations for EM-fields. Describing physical entities in different, but equivalent coordinate systems establishes the transformation properties of the mathematical entities describing the physical ones. Here equivalent means connected by a (linear) transformation, which is a symmetry of the relevant equations. These mathematical entities can be classified as scalars, vectors or higher rank tensors. This implies the correct transformation properties. Equivalently it is said that the mathematical entities describing the physical ones transform according to (linear) representations of the symmetry group of the relevant equations under an element of that symmetry group. Here the case in point is the EM-field. The symmetry of Maxwell's equations implies that the EM-field transform according to some (linear) representation of IO(3,1).

With the above remarks in mind the concept space-time symmetry of an EM-field is almost self-explanatory: it is an inhomogeneous Lorentz transformation leaving the EM-field in question invariant. Such transformations are subdivided into Shubnikov (or trivial) and other (non-trivial) relativistic transformations. A Shubnikov (or trivial) relativistic transformation is an Euclidean space transformation combined, or not, with time reversal. Taking the example of a uniform magnetic field in the z-direction the inhomogeneous Lorentz transformations leaving the corresponding EM-field tensor

$$F^{\mu\nu} = E \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

invariant form a group, which is the semi-direct product of all space-time translations with the relativistic point group K_H. In terms of a set of generators the latter is:

$$K_H = \{m'_x,\ \bar{1},\ R_z(\phi),\ L_z(\psi) \mid \forall\phi,\ \psi \in \mathbb{R}\} \quad .$$

Here the trivial relativistic generators of K_H are the x-reflection combined with the time reversal, m'_x, the space inversion $\bar{1}$ and the whole one-parameter subgroup of rotations about an angle ϕ around the z-axis $R_z(\phi)$. $L_z(\psi)$, the special Lorentz transformations with velocity in the z-direction are a one-parameter subgroup of non-trivial relativistic transformations. The Shubnikov subgroup of K_H is hence obtained by simply omitting the subgroup $L_z(\psi)$. In the international notation and using a prime to denote time reversal it is $(\infty/m)(2'/m')(2'/m')$ [14].

C H A P T E R II

BACKGROUND IN ELECTRODYNAMICS

II.1 Vacuum Case

Covariant electrodynamics takes place in Minkowski space M. In this four-dimensional space an orthonormal basis $\{e_o=e_t, e_1, e_2, e_3\}$ is chosen, with basis vectors along the t, x, y and z axes, respectively. By writing $x^o = t$ rather than $x^o = ct$, t has been given the dimension of length, in fact t is then the distance travelled by light in vacuum during the corresponding time interval t. The space M is endowed with the relativistic metric given by a diagonal metric tensor: $-g_{00} = g_{11} = g_{22} = g_{33} = 1$. This metric is left invariant by the elements $g = (t, L)$ of IO(3,1), the inhomogeneous Lorentz group or Poincaré group, where t indicates a four-dimensional translation, element of $\mathbb{R}^4$, and L is an element of the homogeneous Lorentz group O(3,1). Under these transformations the relevant mathematical entities transform in a well known manner: scalars are left invariant, vectors transform according to

$$\tilde{x} = gx = (t,L)x = Lx + t \quad \text{or} \quad \tilde{x}^{\nu} = L^{\nu}_{\mu}x^{\mu} + t^{\nu} , \tag{2.1}$$

covariant second rank tensors according to

$$\tilde{A}(x) = LAL^{t}(g^{-1}x) \qquad \text{or} \qquad \tilde{A}^{\mu\nu}(x) = L^{\mu}_{\sigma}L^{\nu}_{\rho}A^{\sigma\rho}(g^{-1}x). \tag{2.2}$$

Covariant and mixed second rank tensors, as well as higher rank tensors, transform correspondingly. For g to be a symmetry of the tensor A the relation $A = \tilde{A}$ must hold componentwise. This symmetry condition can be stated in the following way in matrix language:

$$\tilde{A}(x)L^{\star} = LA(g^{-1}x), \tag{2.3}$$

where $L^{\star} = (L^{-1})^t = (L^t)^{-1}$. In the following conventional Gauss units[15] have been used. Furthermore, the measure of length has been chosen such, that c, the speed of light in vacuum, became the unit of velocity. In these units one has $\varepsilon_o = \mu_o = 1$, and Maxwell's equations are given by

$$\partial_{[\kappa}F_{\lambda\nu]} = 0 \qquad \operatorname{curl}\vec{E} = -\dot{\vec{B}}$$

$$\partial_\nu F^{\lambda\nu} = c^\lambda \quad \text{or in three-dimensional vector notation} \quad \operatorname{div}\vec{B} = 0, \quad \operatorname{curl}\vec{B} = \dot{\vec{E}} + \vec{j} \tag{2.4}$$

$$(\kappa,\lambda,\nu = 0,1,2,3) \qquad \operatorname{div}\vec{E} = \rho\,.$$

In the above formulae [ ] means alternation of indices (see (2.25)). The following identifications have been used:

$$F^{\lambda\nu} = g^{\lambda\sigma}g^{\nu\rho}F_{\sigma\rho} = (\vec{E},\vec{B}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \tag{2.5}$$

and $c^\lambda = (c^o, \vec{c}) = (\rho, \vec{j})$.

The description of the EM-fields by means of a tensor (2.5) implies (up to a sign) the transformation properties of the fields and from these follow the symmetry conditions for $g = (t, L)$:

$$\tilde{F}^{\sigma\rho}(x) = \dot{L}^\sigma_\lambda\, L^\rho_\nu\, F^{\lambda\nu}(g^{-1}x) = F^{\sigma\rho}(x) \tag{2.6}$$

with

$$\dot{L}^\sigma_\lambda = \begin{cases} L^\sigma_\lambda & \text{for } L^o_o > 0 \text{ (L orthochronous)} \\ -L^\sigma_\lambda & \text{for } L^o_o < 0 \text{ (L antichronous).} \end{cases}$$

The slight difference of (2.6) as compared to (2.2) is due to the conventional transformation properties of the electric and magnetic fields under space inversion and time reversal and the combination of the two.

In chapter IV also inhomogeneous Galilei transformations are considered. To this end we take the non-relativistic approximation $|\vec{v}| << c$ and replace Lorentz transformations by the corresponding Galilean ones. Within this framework, the homogeneous Galilei group $G(4)$ is defined as group of real matrices denoted by

$$L(\vartheta,\vec{v},\alpha) = \begin{pmatrix} \vartheta & 0 \\ \vec{v} & \alpha \end{pmatrix} \in G(4) \subset GL(4, \mathbb{R}) \ . \tag{2.7}$$

Here $\vartheta = \pm 1$, $\vec{v} \in \mathbb{R}^3$ is a column vector and $\alpha \in O(3)$. The inhomogeneous Galilei group $IG(4)$ is the semi-direct product of $\mathbb{R}^4$ by $G(4)$, where $G(4)$ operates as group of matrices on $\mathbb{R}^4$. We may write $g = (t, L) \in IG(4)$ for $t \in \mathbb{R}^4$ and $L \in G(4)$. The action of $IG(4)$ on M is then formally the same as in the Poincaré case:

$$gx = (t, L)x = Lx + t \ . \tag{2.8}$$

With $x = (x^o,\vec{x})$, $t = (t^o,\vec{t})$ and $L = L(\vartheta, \vec{v}, \alpha)$ as in (2.7) relation (2.8) is indeed equivalent to the more familiar transformation:

$$\begin{aligned} gx^o &= \vartheta\, x^o + t^o \\ g\vec{x} &= \alpha\vec{x} + \vec{v}x^o + \vec{t} \ . \end{aligned} \tag{2.9}$$

The same formal correspondence between Poincaré case and Galilei case exists for the transformation properties of EM-fields considered as components of the covariant tensor field $F_{\mu\nu}(x)$, which transforms under $g \in IG(4)$ according to

$$g\, F_{\mu\nu}(x) = (L^{-1})^{\mu}_{\sigma}\, (L^{-1})^{\nu}_{\rho} F_{\mu\nu}(g^{-1}x) \ , \tag{2.10}$$

which is formally the same as (2.6) for covariant tensors. Using

$$\dot{L}^{-1}(\vartheta,\vec{v},\alpha) = L(1, -\alpha^{-1}\,\vec{v}, \vartheta\alpha^{-1}) \tag{2.11}$$

the corresponding expressions for the transformation of the $\vec{E}$ and $\vec{B}$ fields are:

$$\begin{aligned} g\,\vec{E}(x) &= \alpha\vec{E}(g^{-1}x) - \vec{v}\times(\alpha\vec{B}(g^{-1}x)) \\ g\,\vec{B}(x) &= \vartheta\alpha\vec{B}(g^{-1}x). \end{aligned} \tag{2.12}$$

From the above follows that in the Galilei case the symmetry condition can be formally formulated in the same way as in the Poincaré case, i.e. (2.6):

$$\tilde{F}_{\mu\nu}(x) = (\dot{L}^{-1})^{\sigma}_{\mu}(L^{-1})^{\rho}_{\nu}F_{\sigma\rho}(g^{-1}x) = F_{\mu\nu}(x)\ . \tag{2.13}$$

Attention has been drawn to the fact that this is not the only possible approach. In particular another transformation law for the EM-fields could be adopted in the Galilei case, differing from (2.12) by elements of the order $\frac{\vec{v}}{c^2}$ [16].

II.2 Material Media

For macroscopic EM-fields in material media Maxwell's equations are:

$$\begin{aligned} &\partial_{[\kappa}F_{\lambda\nu]} = 0 & & \operatorname{curl}\vec{E} = -\dot{\vec{B}} \\ &\partial_{\nu}\,G^{\lambda\nu} = c^{\lambda} & \text{or in three-dimensional vector notation} \quad & \operatorname{div}\vec{B} = 0 \\ & & & \operatorname{curl}\vec{H} = \dot{\vec{D}}+\vec{j} \\ &(\kappa,\lambda,\nu = 0,1,2,3) & & \operatorname{div}\vec{D} = \rho\ . \end{aligned} \tag{2.14}$$

Here the additional identification

$$G^{\lambda\nu} = (\vec{D},\vec{H}) = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & H_z & -H_y \\ -D_y & -H_z & 0 & H_x \\ -D_z & H_y & -H_x & 0 \end{pmatrix} \tag{2.15}$$

has been used. For the tensor fields $F^{\lambda\nu}(x)$ and $G^{\lambda\nu}(x)$ the same transformation properties and therefore the same symmetry conditions hold. The one additional tensor field is not sufficient to describe the EM-fields in material media, because Maxwell's equations cannot be solved, unless the connection between the two pairs of fields $(\vec{E},\vec{B})$ and $(\vec{D},\vec{H})$ is known. This connection is established by the constitutive relations. Restricting to linear constitutive relations they can be expressed by means of the constitutive tensor $\chi^{\lambda\nu\sigma\rho}$:

$$G^{\lambda\nu}(x) = \tfrac{1}{2}\,\chi^{\lambda\nu\sigma\rho}(x)\,F_{\sigma\rho}(x)\,. \tag{2.16}$$

This is the case of instantaneous local coupling. Instantaneous coupling means that the interaction between the fields $(-\vec{E},\vec{B})$ and $(\vec{D},\vec{H})$ at time t_o only depends on the values of these fields at the same instant of time t_o. There are therefore no hysteresis-type of effects. Local coupling means that the interaction between the fields $(-\vec{E},\vec{B})$ and $(\vec{D},\vec{H})$ at point $\vec{x}$ only depends on the value of these fields at point $\vec{x}$. Post [17)] remarks, that these types of behaviour are not independent, because assuming non-instantaneous behaviour space-time covariance will require the acceptance of non-local interaction as an accompanying feature. Dispersion is associated with non-instantaneous and non-local coupling. In more physical terms, the polarization response of a medium, as a rule, is not instantaneous due to the inertia of the microphysical charges in the medium. In addition one expects that the polarization at a particular point in a medium may partly be determined by the dipole fields in the neighbourhood of that point. A real algebraic relation between the fields is not adequate to represent dispersion, even if one makes the components of the constitutive tensor functions of the frequency or wavenumber, because the phase shift between cause and effect cannot be accounted for by a real algebraic relation. A more general linear constitutive relation could be of the form

$$G^{\lambda\nu}(x) = \int \chi^{\lambda\nu\sigma\rho}(x,\xi)\, F_{\sigma\rho}(\xi) d^4\xi, \tag{2.17}$$

with suitable boundaries of integration to ensure causality. The real and the imaginary parts of $\chi^{\lambda\nu\sigma\rho}$ would, for every component, be connected by some kind of Kramers-Kronig relation. Going back to (2.16) it is seen that the constitutive tensor $\chi^{\lambda\nu\sigma\rho}$ "inherits" certain symmetry properties from the EM-field tensors, which are skew-symmetric:

$$\begin{aligned} \chi^{\lambda\nu\sigma\rho} &= -\chi^{\lambda\nu\rho\sigma} \\ \chi^{\lambda\nu\sigma\rho} &= -\chi^{\nu\lambda\sigma\rho}, \quad \text{and in addition one has} \\ \chi^{\lambda\nu\sigma\rho} &= \chi^{\sigma\rho\lambda\nu}. \end{aligned} \tag{2.18}$$

These symmetries of the constitutive tensor are independent of the symmetries of the material media. It follows from (2.18) that the constitutive tensor can be represented by means of a six-by-six matrix, the elements of which depend only on frequency and are constants in the case of monochromatic fields and homogeneous media:

$\chi^{(\lambda\nu)(\sigma\rho)}$	01 $-E_x$	02 $-E_y$	03 $-E_z$	23 B_x	31 B_y	12 B_z
01 D_x	$-\varepsilon_{11}$	$-\varepsilon_{12}$	$-\varepsilon_{13}$	α_{11}	α_{12}	α_{13}
02 D_y	$-\overline{\varepsilon}_{12}$	$-\varepsilon_{22}$	$-\varepsilon_{23}$	α_{21}	α_{22}	α_{23}
03 D_z	$-\overline{\varepsilon}_{13}$	$-\overline{\varepsilon}_{23}$	$-\varepsilon_{33}$	α_{31}	α_{32}	α_{33}
23 H_x	$\overline{\alpha}_{11}$	$\overline{\alpha}_{21}$	$\overline{\alpha}_{31}$	η_{11}	η_{12}	η_{13}
31 H_y	$\overline{\alpha}_{12}$	$\overline{\alpha}_{22}$	$\overline{\alpha}_{32}$	$\overline{\eta}_{12}$	η_{22}	η_{23}
12 H_z	$\overline{\alpha}_{13}$	$\overline{\alpha}_{23}$	$\overline{\alpha}_{33}$	$\overline{\eta}_{13}$	$\overline{\eta}_{23}$	η_{33}

(2.19)

(— means conjugate complex of).

The six-by-six matrix (2.19) splits into four three-by-three submatrices. The diagonal submatrices represent the electric susceptibility ε at fixed $\vec{B}$ (permittivity) and the inverse of the magnetic permeability (at fixed $\vec{E}$). Their real components are associated with birefringence, the imaginary ones with Faraday rotation. As for the off-diagonal matrices, which indicate the presence of magnetoelectric effect, the real part of the components represents physically the Fresnel-Fizeau effect (light waves participate in the motion of the medium in the ratio $(1 - \frac{1}{n^2})$, n being the refractive index) and the imaginary part of the components is associated with natural optical activity of the medium. All this will be treated in more detail in chapter V. By means of matrix (2.19) and by defining the six-vectors

$$\vec{\vec{F}} = \begin{pmatrix} -\vec{E} \\ \vec{B} \end{pmatrix} \quad \text{and} \quad \vec{\vec{G}} = \begin{pmatrix} \vec{D} \\ \vec{H} \end{pmatrix} \tag{2.20}$$

relation (2.16) can now be written in the form

$$\vec{\vec{G}} = \chi^{(\lambda\nu)(\sigma\rho)} \vec{\vec{F}} \; . \tag{2.21}$$

In terms of the field vectors this is

$$\begin{aligned} \vec{D} &= \varepsilon\vec{E} + \alpha\vec{B} \\ \vec{H} &= \alpha_1\vec{E} + \eta\vec{B} \end{aligned} \tag{2.22}$$

with

$$\alpha_1 = -\alpha^\dagger \quad (\dagger \text{ means Hermitean conjugate of}).$$

The matrix (2.19) is called restricted constitutive tensor. From the restricted tensor the full constitutive tensor $\chi^{\lambda\nu\sigma\rho}$ can be built up by means of the symmetry relations (2.18).

In certain cases, take for example the Dirac equation, the electromagnetic interaction must be described by potentials rather than by the EM-fields themselves. The potentials can be regarded as fields which are particular solutions of the inhomogeneous wave equation. The inhomogeneous terms are the sources of the potential field:

$$\Box \text{ potential} = \text{source, where } \Box = \partial_\mu \partial^\mu \quad . \tag{2.23}$$

The most frequently occurring potentials and their corresponding sources are:

Potentials		Sources	
Scalar potential	ϕ	Charge	ρ
Vector potential	$\vec{A}$	Current	$\vec{j}$
Hertz vector	$\vec{\pi}$	Electric dipole field	$\vec{P}$
Fitzgerald vector	$\vec{\mu}$	Magnetic dipole field	$\vec{M}$

With the identification $A_o = -\phi$ and $\vec{A} = (A_x, A_y, A_z)$ the scalar and vector potentials combine into a four-vector A_λ. The field tensor $F_{\sigma\rho} = (-\vec{E}, \vec{B})$ then is related to the four-potential by

$$F_{\sigma\rho} = 2\partial_{[\sigma} A_{\rho]} \quad . \tag{2.24}$$

The factor 2 in (2.24) is a direct consequence of the definition of the alternation symbol which indicates a determinant-like operation:

$$P_{[\kappa\lambda]} = \frac{1}{2!} (P_{\kappa\lambda} - P_{\lambda\kappa})$$
$$P_{[\kappa\lambda\nu]} = \frac{1}{3!} (P_{\kappa\lambda\nu} + P_{\lambda\nu\kappa} + P_{\nu\kappa\lambda} - P_{\lambda\kappa\nu} - P_{\kappa\nu\lambda} - P_{\nu\lambda\kappa}), \text{ etc.} \tag{2.25}$$

The source fields for the Hertz and Fitzgerald vectors are the electric polarization $\vec{P}$ and the magnetic polarization $\vec{M}$ (per unit volume). These are related to the fields by the equations

$$\vec{E} = \vec{D} - \vec{P}$$
$$\vec{B} = \vec{H} + \vec{M} \tag{2.26}$$

Using the following identification $\vec{P}$ and $\vec{M}$ constitute a second rank tensor in Minkowski space M:

$$M^{\lambda\nu} = (-\vec{P},\vec{M}) = \begin{pmatrix} 0 & -P_x & -P_y & -P_z \\ P_x & 0 & M_z & -M_y \\ P_y & -M_z & 0 & M_x \\ P_z & M_y & -M_x & 0 \end{pmatrix} , \tag{2.27}$$

So

$$G^{\lambda\nu} + M^{\lambda\nu} = g^{\lambda\sigma}g^{\nu\rho}F_{\sigma\rho} = F^{\lambda\nu} \tag{2.28}$$

is the covariant formulation of (2.26). Taking the divergence of (2.28) with respect to the index ν and using the second relation of (2.14a) along with (2.24) the result is

$$c^{\lambda} + \partial_{\nu}M^{\lambda\nu} = - \Box A^{\lambda} \quad . \tag{2.29}$$

Here use has been made of the Lorentz condition

$$\partial_{\nu}A^{\nu} = 0 \quad . \tag{2.30}$$

Equation (2.29) indicates how in a macroscopic approach the sources are split into "true" sources and "free" sources, or equivalently into external sources and polarization sources. Separating time and space components and writing things out in three-dimensional vector notation yields:

$\lambda = 0:$	$\rho - \text{div}\,\vec{P} = \text{div}\,\vec{E}$	$\lambda = 1,2,3:$	$\underbrace{\vec{j} + \dot{\vec{P}} + \text{curl}\,\vec{M}} = \text{curl}\,\vec{B} - \dot{\vec{E}}$
	$\rho_{ext} - \rho_{pol} = \text{div}\,\vec{E}$		$\vec{j}_{ext} + \vec{j}_{pol} = \text{curl}\,\vec{B} - \dot{\vec{E}}$
	$\rho = \text{div}(\vec{E}+\vec{P})$		$\vec{j} + \partial_t(\vec{E}+\vec{P}) = \text{curl}(\vec{B}-\vec{M})$
	$\rho = \text{div}\,\vec{D}$		$\vec{j} + \dot{\vec{D}} = \text{curl}\,\vec{H}$.

Assuming zero external sources (2.29) becomes

$$-\partial_{\nu} M^{\lambda\nu} = \Box A^{\lambda} \tag{2.31}$$

This wave equation with polarization sources only, invites the substitution $A^\lambda = \partial_\nu Z^{\lambda\nu}$ with $Z^{\lambda\nu} = -Z^{\nu\lambda}$, so that the structures of the sources $M^{\lambda\nu}$ and of the potentials $Z^{\lambda\nu}$ correspond:

$$-M^{\lambda\nu} = \Box Z^{\lambda\nu} \tag{2.32}$$

(up to a divergence-free part).
The time components of $Z^{\lambda\nu}$ correspond to the Hertz vector $\vec{\pi}$ and the pure space components to the Fitzgerald vector $\vec{\mu}$:

$$Z^{\lambda\nu} = (-\vec{\pi},\vec{\mu}) = \begin{pmatrix} 0 & -\pi_x & -\pi_y & -\pi_z \\ \pi_x & 0 & \mu_z & -\mu_y \\ \pi_y & -\mu_z & 0 & \mu_x \\ \pi_z & \mu_y & -\mu_x & 0 \end{pmatrix} . \tag{2.33}$$

In the presence of non-vanishing external four-current ($c^\lambda \neq 0$) the problem of polarization potentials has been treated by A. Nisbet [18] and W.H. McCrea [19] has given the covariant formulation.

In the terminology of Penfield and Haus [2] the formulation of electrodynamic theory used up to now, would be called the Minkowski formulation. According to these authors a complete formulation of electrodynamics must contain (i) Maxwell's equations and (ii) constitutive relations. To describe EM-fields, six field variables have been defined: $\vec{E}$, $\vec{B}$, $\vec{D}$, $\vec{H}$, $\vec{P}$ and $\vec{M}$. Penfield and Haus [2] adopt the view that only four of these six vector variables are "fundamental", that is, sufficient to build up a complete formulation together with the corresponding constitutive relations and Maxwell's equations. The other two vector variables can be found from the fundamental four by (2.26). Different formulations are then obtained by choosing from the total of six variables four fundamental ones. For every formulation except the Minkowski formulation naturally associated microscopic models are given. With an example let us see in what sense the choice of a basic variable implies such a model. Take the electric field $\vec{E}$ as fundamental variable. The tangential component of $\vec{E}$ is continuous across the surface of a stationary body, whereas the normal component is not. Since the

normal component of $\vec{E}$ changes across the surface, the natural model to associate with this phenomenon is a surface density of electric charge. Such a surface charge could be caused by a conglomeration of miniature electric charge dipoles inside the material. Similarly choosing $\vec{D}$ as fundamental variable means a magnetic current loop model for electric polarization, because in this case the normal component is continuous and the tangential one is not. This situation can be modelled by a conglomeration of miniture magnetic current loops. Since the Minkowski formulation combines the fields $\vec{E}_M$ and $\vec{D}_M$ on one hand, and $\vec{B}_M$ and $\vec{H}_M$ on the other, the above argument also explains why there is no naturally associated model for that formulation: the models implied by the choice of $\vec{E}_M$ and $\vec{D}_M$ as fundamental variables simply are not compatible.
It will be seen that in vacuum (zero polarizations) and in the rest frame of the material medium the EM-fields are equal in all formulations. This is no contradiction since different microscopic models may very well produce the same macroscopic fields: the sources of a magnetic field can be modelled by an array of magnetic charge dipoles, or of electric current loops. The transformation properties of the macroscopic polarization fields are derived from the microscopic model and thus depend on that model. For the other macroscopic fields the transformation properties are derived from the postulated covariance of Maxwell's (phenomenological) equations with respect to IO(3,1). For later reference four different formulations are presented in the following table:

Formulation	Fundamental variables	Redundant variables	Models for electric and magnetic polarizations
Minkowski	$\vec{E}_M, \vec{B}_M, \vec{D}_M, \vec{H}_M$	$\vec{P}_M, \vec{M}_M$	None [Covariance postulate]
Chu	$\vec{E}_C, \vec{H}_C, \vec{P}_C, \vec{M}_C$	$\vec{B}_C, \vec{D}_C$	Electric charge dipole Magnetic charge dipole
Ampère	$\vec{E}_A, \vec{B}_A, \vec{P}_A, \vec{M}_A, \vec{u}$	$\vec{D}_A, \vec{H}_A$	Electric charge dipole Electric current loop
Boffi	$\vec{E}_B, \vec{B}_B, \vec{P}_B, \vec{M}_B$	$\vec{D}_B, \vec{H}_B$	Electric charge dipole Electric current loop

In the column for the fundamental variables the velocity $\vec{u}$ of the moving medium appears only, when it appears in Maxwell's equations, regardless of the fact that it may appear in the constitutive relations or other expressions. The Ampère and Boffi formulations are very similar, in fact for stationary material they are identical. For completeness Maxwell's equations and the transformation properties of the various fields are given in the following for the four formulations listed in the above table. For the Minkowski formulation this information is given in (2.6), (2.14) and indicated in (2.15) and (2.27). In the Chu formulation Maxwell's equations are:

$$\begin{aligned} \operatorname{curl} \vec{H}_C - \partial_t \vec{E}_C &= \partial_t \vec{P}_C + \operatorname{curl} (\vec{P}_C \times \vec{u}) + \vec{j}_{ext} \\ \operatorname{curl} \vec{E}_C + \partial_t \vec{H}_C &= -\partial_t \vec{M}_C - \operatorname{curl} (\vec{M}_C \times \vec{u}) \\ \operatorname{div} \vec{E}_C &= -\operatorname{div} \vec{P}_C + \rho_{ext} \\ \operatorname{div} \vec{H}_C &= -\operatorname{div} \vec{M}_C \, . \end{aligned} \tag{2.34}$$

Here and in the following $\vec{u}$ describes the velocity of the material medium in the original frame of reference, whereas $\vec{v}$ is used for the relative velocity between two frames of reference (boost velocity). The transformation laws for the fundamental fields under a special Lorentz transformation are:

$$\begin{aligned} \tilde{\vec{E}}_C &= \vec{E}_{C_\parallel} + \gamma_{\vec{v}} (\vec{E}_{C_\perp} - \vec{v} \times \vec{H}_C) \\ \tilde{\vec{H}}_C &= \vec{H}_{C_\parallel} + \gamma_{\vec{v}} (\vec{H}_{C_\perp} + \vec{v} \times \vec{E}_C) \\ \tilde{\vec{P}}_C &= \vec{P}_{C_\parallel} + \gamma_{\vec{v}} \vec{P}_{C_\perp} + \gamma_{\vec{v}} (\vec{v} \times (\vec{P}_C \times \vec{u})) \\ \tilde{\vec{M}}_C &= \vec{M}_{C_\parallel} + \gamma_{\vec{v}} \vec{M}_{C_\perp} + \gamma_{\vec{v}} (\vec{v} \times (\vec{M}_C \times \vec{u})) \, . \end{aligned} \tag{2.35}$$

The transformation law for the redundant variables can be found from the transformation law of the fundamental variables:

$$\tilde{\vec{D}}_C = \tilde{\vec{P}}_C + \tilde{\vec{E}}_C = \vec{D}_{C_\parallel} + \gamma_{\vec{v}} \{\vec{D}_{C_\perp} - \vec{v} \times [\vec{H}_C - (\vec{P}_C \times \vec{u})]\}$$

$$\tilde{\vec{B}}_C = \tilde{\vec{M}}_C + \tilde{\vec{H}}_C = \vec{B}_{C_\parallel} + \gamma_{\vec{v}} \{\vec{B}_{C_\perp} + \vec{v} \times [\vec{E}_C + (\vec{M}_C \times \vec{u})]\} \quad , \tag{2.36}$$

with $\gamma_{\vec{v}} = (1-v^2)^{-\frac{1}{2}}$, and parallel ($\parallel$) or orthogonal ($\perp$) is meant with respect to the direction of the boost velocity $\vec{v}$. As for the covariant transformation theory the variables $\vec{E}_C$ and $\vec{H}_C$ transform in the usual tensor manner as given in (2.6) and implied in the notation

$$F_C^{\lambda\nu} = (\vec{E}_C, \vec{H}_C) \quad . \tag{2.37}$$

The transformation laws for the polarization densities $\vec{P}_C$ and $\vec{M}_C$ can also be described by defining the following four-vectors:

$$P_C^\nu = \frac{1}{\gamma_{\vec{u}}} (\gamma_{\vec{u}}^2 (\vec{P}_C . \vec{u}), \vec{P}_C + \gamma_{\vec{u}}^2 (\vec{u}\vec{P}_C) . \vec{u})$$

$$M_C^\nu = \frac{1}{\gamma_{\vec{u}}} (\gamma_{\vec{u}}^2 (\vec{M}_C . \vec{u}), \vec{M}_C + \gamma_{\vec{u}}^2 (\vec{u}\vec{M}_C) . \vec{u}) \quad , \tag{2.38}$$

where $(\vec{u}\,\vec{P}_C)$ and $(\vec{u}\,\vec{M}_C)$ indicate dyadic products. The dyadic product of two vectors is a tensor: $(\overleftrightarrow{ab})_{ij} = a_i b_j$. In these expressions the four-vectors P_C^ν and M_C^ν are referred to the frame in which the material medium moves with velocity $\vec{u}$. The three-vectors $\vec{P}_C$ and $\vec{M}_C$ represent the polarization densities as measured in that frame. Consequently one has in the rest frame of the medium ($\vec{u} = 0$ and $\gamma_{\vec{u}} = 1$):

$${}^oP_C^\nu = (0, {}^o\vec{P}_C) \qquad \text{and} \qquad {}^oM_C^\nu = (0, {}^o\vec{M}_C) \quad ,$$

with ${}^o\vec{P}_C$ and ${}^o\vec{M}_C$ the three-vectors of the polarization densities as referred to the rest frame of the material medium. The four-vectors of the polarization densities (2.38) transform in the following way under $g \in IO(3,1)$:

$$\tilde{P}^{\mu}_{C}(x) = L^{\mu}_{\nu} P^{\nu}_{C}(g^{-1}x)$$
$$\tilde{M}^{\mu}_{C}(x) = \dot{L}^{\mu}_{\nu} M^{\nu}_{C}(g^{-1}x) \quad . \tag{2.39}$$

The expressions (2.38) and (2.39) imply the transformation law for the three-vectors of the polarization densities. First the "old" four-vector is transformed according to (2.39). Then the "new" four-vector is written down in the form of (2.38), that is in terms of the three-vectors of the polarization densities as referred to in the "new" coordinate system. Finally the comparison of the two expressions gives the "new" three-vectors of the polarization densities in terms of the "old" ones, i.e. the transformation law. Take for instance the example of a special Lorentz transformation with velocity $\vec{v}$ in the (negative) z-direction applied to ${}^{o}P^{\nu} = (0, {}^{o}P_x, {}^{o}P_y, {}^{o}P_z)$:

$$\begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & 0 & 0 & \frac{v}{\sqrt{1-v^2}} \\ & 1 & 0 & \\ & 0 & 1 & \\ \frac{v}{\sqrt{1-v^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2}} \end{pmatrix} \begin{pmatrix} 0 \\ {}^{o}P_x \\ {}^{o}P_y \\ {}^{o}P_z \end{pmatrix} = \begin{pmatrix} \frac{v\,{}^{o}P_z}{\sqrt{1-v^2}} \\ {}^{o}P_x \\ {}^{o}P_y \\ \frac{{}^{o}P_z}{\sqrt{1-v^2}} \end{pmatrix}, \quad \frac{1}{\sqrt{1-v^2}} = \gamma_{\vec{v}} \; .$$

The "new" four-vector is

$$\tilde{P}^{\nu} = (\gamma_{\vec{v}}\,(\tilde{\vec{P}}.\vec{v}), \frac{1}{\gamma_{\vec{v}}}\,\tilde{\vec{P}} + \gamma_{\vec{v}}\,(\vec{v}\tilde{\vec{P}}).\vec{v}),$$

with $\tilde{\vec{P}}.\vec{v} = \tilde{P}_z v$ and $(\vec{v}\tilde{\vec{P}}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v\tilde{P}_x & v\tilde{P}_y & v\tilde{P}_z \end{pmatrix}$, therefore,

$$\tilde{P}^{\nu} = \begin{pmatrix} \gamma_{\vec{v}}\, \tilde{P}_z v \\ \frac{1}{\gamma_{\vec{v}}}\, \tilde{P}_x \\ \frac{1}{\gamma_{\vec{v}}}\, \tilde{P}_y \\ \frac{1}{\gamma_{\vec{v}}}\, \tilde{P}_z + \gamma_{\vec{v}}\, v^2 \tilde{P}_z \end{pmatrix} = \begin{pmatrix} \frac{v\,{}^{o}P_z}{\sqrt{1-v^2}} \\ {}^{o}P_x \\ {}^{o}P_y \\ \frac{{}^{o}P_z}{\sqrt{1-v^2}} \end{pmatrix} \quad ,$$

from which one concludes

$$\tilde{P}_x = \gamma_{\vec{v}}\, {}^{o}P_x$$
$$\tilde{P}_y = \gamma_{\vec{v}}\, {}^{o}P_y$$
$$\tilde{P}_z = {}^{o}P_z \quad .$$

This result can also be written as

$$\tilde{\vec{P}} = {}^{o}\vec{P}_{\parallel} + \gamma_{\vec{v}}\, {}^{o}\vec{P}_{\perp} \quad ,$$

which is the usual transformation law for the three-vector of the electric polarization density under a special Lorentz transformation.

The Minkowski fields can be expressed in terms of the Chu fields (and vice versa):

$$\begin{aligned} \vec{E}_M &= \vec{E}_C + \vec{M}_C \times \vec{u} \\ \vec{H}_M &= \vec{H}_C - \vec{P}_C \times \vec{u} \\ \vec{D}_M &= \vec{E}_C + \vec{P}_C \\ \vec{B}_M &= \vec{H}_C + M_C \quad . \end{aligned} \qquad (2.40)$$

One notices that the electric field $\vec{E}_M$ inside moving magnetized material is not the same as the electric field $\vec{E}_C$ and similarly the magnetic fields in the two formulations are not the same either. The fact that these fields are different, however, does not in itself mean that the two formulations are incompatible, because the fields inside the material are not measurable. In free space, where the magnetic and the electric polarizations vanish, the electric and magnetic fields in the two formulations are equal. Furthermore, when the velocity $\vec{u}$ is zero, that is, in the rest frame of the material, they are equal as has already been pointed out. In the Ampère formulation Maxwell's equations are:

$$\begin{aligned}
\text{curl}\,\vec{B}_A - \partial_t\vec{E}_A &= \partial_t\vec{P}_A + \text{curl}(\vec{P}_A \times \vec{u}) + \text{curl}\,\vec{M}_A - \partial_t(\vec{M}_A \times \vec{u}) + \vec{J}_{ext} \\
\text{curl}\,\vec{E}_A + \partial_t\vec{B}_A &= 0 \\
\text{div}\,\vec{E}_A &= -\text{div}\vec{P}_A + \text{div}(\vec{M}_A \times \vec{u}) + \rho_{ext} \\
\text{div}\,\vec{B}_A &= 0.
\end{aligned} \tag{2.41}$$

The transformation laws for the fundamental fields under a special Lorentz transformation are:

$$\begin{aligned}
\tilde{\vec{E}}_A &= \vec{E}_{A_\parallel} + \gamma_{\vec{v}}(\vec{E}_{A_\perp} - \vec{v} \times \vec{B}_A) \\
\tilde{\vec{B}}_A &= \vec{B}_{A_\parallel} + \gamma_{\vec{v}}(\vec{B}_{A_\perp} + \vec{v} \times \vec{E}_A) \\
\tilde{\vec{P}}_A &= \vec{P}_{A_\parallel} + \gamma_{\vec{v}}\,\vec{P}_{A_\perp} + \gamma_{\vec{v}}(\vec{v} \times (\vec{P}_A \times \vec{u})) \\
\tilde{\vec{M}}_A &= \vec{M}_{A_\parallel} + \gamma_{\vec{v}}\,\vec{M}_{A_\perp} + \gamma_{\vec{v}}(\vec{v} \times (\vec{M}_A \times \vec{u}))\ .
\end{aligned} \tag{2.42}$$

For the redundant fields the transformation law can be found from (2.41):

$$\begin{aligned}
\tilde{\vec{D}}_A &= \tilde{\vec{P}}_A + \tilde{\vec{E}}_A = \vec{D}_{A_\parallel} + \gamma_{\vec{v}}\{\vec{D}_{A_\perp} - \vec{v} \times [\vec{B}_A - (\vec{P}_A \times \vec{u})]\} \\
\tilde{\vec{H}}_A &= \tilde{\vec{B}}_A - \tilde{\vec{M}}_A = \vec{H}_{A_\parallel} + \gamma_{\vec{v}}\{\vec{H}_{A_\perp} + \vec{v} \times [\vec{E}_A - (\vec{M}_A \times \vec{u})]\}\ .
\end{aligned}$$

As for the covariant transformation theory the fundamental variables $\vec{E}_A$ and $\vec{B}_A$ transform in the usual tensor manner given in (2.6) and implied in the notation

$$F_A^{\lambda\nu} = (\vec{E}_A, \vec{B}_A) \; . \tag{2.43}$$

The transformation laws for the polarization densities $\vec{P}_A$ and $\vec{M}_A$ can again be described by defining the two four-vectors (see (2.38))

$$\begin{aligned} P_A^{\nu} &= \frac{1}{\gamma_{\vec{u}}} \left(\gamma_{\vec{u}}^2(\vec{P}_A.\vec{u}), \; \vec{P}_A + \gamma_{\vec{u}}^2(\vec{u}\vec{P}_A).\vec{u}\right) \\ M_A^{\nu} &= \frac{1}{\gamma_{\vec{u}}} \left(\gamma_{\vec{u}}^2(\vec{M}_A.\vec{u}) \; , \; \vec{M}_A + \gamma_{\vec{u}}^2(\vec{u}\vec{M}_A).\vec{u}\right) \quad . \end{aligned} \tag{2.44}$$

The translation to the Chu formulation is given by

$$\begin{aligned} \vec{E}_A &= \vec{E}_C + \vec{M}_C \times \vec{u} \\ \vec{B}_A &= \vec{H}_C + \vec{M}_C \\ \vec{P}_A &= \vec{P}_C \\ \vec{M}_A &= \vec{M}_C \quad . \end{aligned} \tag{2.45}$$

In the Boffi formulation Maxwell's equations are:

$$\begin{aligned} \text{curl}\; \vec{B}_B - \partial_t \vec{E}_B &= \partial_t \vec{P}_B + \text{curl}\; \vec{M}_B + \vec{J}_{ext} \\ \text{curl}\; \vec{E}_B + \partial_t \vec{B}_B &= 0 \\ \text{div}\; \vec{E}_B &= -\text{div}\; \vec{P}_B + \rho_{ext} \\ \text{div}\; \vec{B}_B &= 0 \; . \end{aligned} \tag{2.46}$$

The transformation laws for the fundamental fields under a special Lorentz transformation are:

$$\begin{aligned}
\tilde{\vec{E}}_B &= \vec{E}_{B_\parallel} + \gamma_{\vec{v}}(\vec{E}_{B_\perp} - \vec{v} \times \vec{B}_B) \\
\tilde{\vec{B}}_B &= \vec{B}_{B_\parallel} + \gamma_{\vec{v}}(\vec{B}_{B_\perp} + \vec{v} \times \vec{E}_B) \\
\tilde{\vec{P}}_B &= \vec{P}_{B_\parallel} + \gamma_{\vec{v}}(\vec{P}_{B_\perp} + \vec{v} \times \vec{M}_B) \\
\tilde{\vec{M}}_B &= \vec{M}_{B_\parallel} + \gamma_{\vec{v}}(\vec{M}_{B_\perp} - \vec{v} \times \vec{P}_B) \quad .
\end{aligned} \tag{2.47}$$

For the redundant fields the transformation laws are:

$$\begin{aligned}
\tilde{\vec{D}}_B &= \tilde{\vec{E}}_B + \tilde{\vec{P}}_B = \vec{D}_{B_\parallel} + \gamma_{\vec{v}}(\vec{D}_{B_\perp} - \vec{v} \times \vec{H}_B) \\
\tilde{\vec{H}}_B &= \tilde{\vec{B}}_B - \tilde{\vec{M}}_B = \vec{H}_{B_\parallel} + \gamma_{\vec{v}}(\vec{H}_{B_\perp} + \vec{v} \times \vec{D}_B) \quad .
\end{aligned} \tag{2.48}$$

As for the covariant transformation theory the fundamental variables as well as the redundant ones transform in the usual tensor manner given in (2.6) and implied in the notation

$$F_B^{\lambda\nu} = (\vec{E}_B, \vec{B}_B), \; M_B^{\lambda\nu} = (-\vec{P}_B, \vec{M}_B) \text{ and } G_B^{\lambda\nu} = (\vec{D}_B, \vec{H}_B) \quad . \tag{2.49}$$

The Boffi variables are related to the Chu variables by

$$\begin{aligned}
\vec{E}_B &= \vec{E}_C + \vec{M}_C \times \vec{u} \\
\vec{B}_B &= \vec{H}_C + \vec{M}_C \\
\vec{P}_B &= \vec{P}_C - \vec{M}_C \times \vec{u} \\
\vec{M}_B &= \vec{M}_C + \vec{P}_C \times \vec{u} \quad .
\end{aligned} \tag{2.50}$$

C H A P T E R III

BASIC ELECTROMAGNETIC FIELDS IN WAVEGUIDES AND RESONANT CAVITIES

III.1 General Procedure and Illustrating Example

In the expressions for the fields of TE- and TM-modes in waveguides a common propagating factor of the form $\exp(i\omega t - \Gamma z)$ appears, where Γ is the so called propagating constant. According to the value of Γ the following two cases can be distinguished:

(i) $\Gamma = i\beta$, pure imaginary: the mode is propagating,

(ii) $\Gamma = \alpha$, real : the mode is evanescent.

We first consider propagating TE- and TM-modes in ideal rectangular waveguides and resonant cavities as they are given for example in ref. 20). By means of reflections at the waveguide- and cavity-walls these fields are extended to fill the whole three-dimensional Euclidean space; but we still talk - somewhat incorrectly - about the field in a waveguide or in a resonant cavity. Evanescent modes are also considered and in this case the translational symmetry in the axial direction of the waveguide is lost. These fields are formally treated in the same way as the fields with translational symmetry in the axial direction. For our purpose (i.e. the determination of the space-time symmetries of certain EM-fields) the procedure is justified in appendix A. But of course the evaluation, for example, of the states of an electron in such a field requires a careful discussion of the (now suppressed) boundary conditions in the axial direction.

Losses due to imperfectly conducting walls are not considered since they would change the boundary conditions imposed on the EM-field by the waveguide- and cavity-walls and consequently the EM-field could not smoothly be extended to the whole space by reflections of the kind mentioned above.

The explicit expressions for the EM-fields used in the calculations are given in appendix B.

Only EM-fields which have a Fourier expansion of the form

$$F^{\mu\nu}(x) = \sum_{k \in S} \tilde{F}^{\mu\nu}(k) \exp(ik\, x) \tag{3.1}$$

are considered. The set of vectors k **occurring in the expansion (with non-zero** Fourier-coefficients) is called the spectrum S of the EM-field $F^{\mu\nu}(x)$. A finite number of complex vectors k is also admitted (see appendix A). For the rest, the procedure is essentially the same as in **ref.** 7), to which we refer for more details. One first has to determine the spectrum S and the Fourier-coefficients $\tilde{F}^{\mu\nu}(k)$. Next, the primitive translations are obtained from the relation

$$k_\nu\, a^\nu \equiv 0 \pmod{2\pi},\ \forall\, k \in S. \tag{3.2}$$

They form a group U of translations in the Minkowski space M:

$$U = \{a \,|\, k_\nu a^\nu \equiv 0 \pmod{2\pi},\ \forall\, k \in S\}. \tag{3.3}$$

The real vectors of the spectrum S generate a free abelian group of translations U^d in the dual space:

$$U^d = \{k \,|\, \forall\, k \in S\}. \tag{3.4}$$

Since U^d is a discrete group, the set of points obtained by applying U^d to the origin constitute **an n_o-dimensional lattice** Λ^d $(0 \leq n_o \leq 4)$. **A basis** $\{a_*^1, \ldots, a_*^{n_o}\}$ of Λ^d also represents a set of generators of U^d. Denote by Λ the dual lattice of Λ^d and by $\{a_1, \ldots, a_{n_o}\}$ the basis of Λ dual to $\{a_*^1, \ldots, a_*^{n_o}\}$, i.e.:

$$a_*^i \cdot a_j = 2\pi\, \delta_j^i \qquad (i,j = 1, \ldots, n_o). \tag{3.5}$$

If $n_o = 4$, then U consists of discrete translations only, and is generated by a basis of Λ. If $n_o < 4$, all the elements of the orthocomplement V_c of the space generated by Λ^d, also belong to U. Selecting a basis $\{c_{n_o+1}, \ldots, c_4\}$ of V_c, the continuous subgroup U_c of U is generated by the infinitesimal translations $c_i \delta t_i$, $i = n_o+1, \ldots, 4$, along the corresponding basis vectors, so that finally the group U of primitive translations is generated by

$$U = \{a_1, \ldots, a_{n_o}, c_{n_o+1}\delta t_{n_o+1}, \ldots, c_4 \delta t_4\} \cong \mathbb{Z}^{n_o} \oplus \mathbb{R}^{4-n_o} \tag{3.6}$$

$(\delta t_i \in \mathbb{R}$ infinitesimal).

In the case where the spectrum is complex one has to consider the complexification $M \oplus iM$ of the Minkowski space and complex translations also appear (see section 3). The Lorentz transformations are then extended by linearity to the new (real) vector space. The procedure is straight forward, so that in what follows it is no more considered explicitly.

Recall that the spectral group S is the largest subgroup of the Lorentz group O(3,1) that leaves the spectrum invariant:

$$S = \{L \in O(3,1) \,|\, L^{\mu}_{\nu} k^{\nu} \in \mathcal{S}, \; \forall\, k \in \mathcal{S}\}. \qquad (3.7)$$

It turns out, that in all cases here considered the spectral group is either a Shubnikov group or it is the conjugate of a Shubnikov group by a special Lorentz transformation (boost). Therefore, there is a frame of reference, called "Shubnikov frame", in which S appears as Shubnikov group.

The point group K and a set of non-primitive translations u(K) can be found by tracking down all elements L of S which satisfy the matrix relation

$$\dot{L}\, \hat{F}(k) = \hat{F}(Lk) L^{*} \exp\,(i(Lk)u(L)) \qquad (3.8)$$

for suitably chosen non-primitive translations u(L). Writing down relation (3.8) in matrix form, it is assumed that in $F^{\mu\nu}$ and L^{μ}_{ν} the index μ numbers the rows and the index ν the columns. Furthermore L^{*} denotes the inverse transpose of L and $\dot{L}$ equals L for L orthochronous, and -L for L antichronous.

The relativistic symmetry group G of an EM-field is the set of all elements of the Poincaré group IO(3,1) leaving the EM-field invariant. Expressed in terms of the elements of U, K and u(K) the group G is given by

$$G = \{(a + u(L), L) \,|\, a \in U, \; L \in K, \; u(L) \in u(K)\}. \qquad (3.9)$$

It acts as a group of transformations on M according to:

$$(a + u(L), L) \circ x = Lx + a + u(L), \quad \forall\, x \in M \qquad (3.10)$$

and has therefore the multiplication law:

$$(a_1 + u(L_1), L_1)(a_2 + u(L_2), L_2) = (a_1 + u(L_1) + L_1 a_2 + L_1 u(L_2), L_1 L_2). \qquad (3.11)$$

As illustrating example, we have chosen the determination of the symmetry group of the TE_{mn}-fields in rectangular waveguides for the case $\frac{m\pi}{\ell_1} = \frac{n\pi}{\ell_2}$, (u = v). From the tables of appendix B it is seen that these EM-fields are given by:

$$E_x = A \sin Kx \cos Mx \sin Nx$$

$$E_y = B \sin Kx \sin Mx \cos Nx$$

$$E_z = 0$$

$$H_x = C \sin Kx \sin Mx \cos Nx$$

$$H_y = D \sin Kx \cos Mx \sin Nx$$

$$H_z = E \cos Kx \cos Mx \cos Nx$$

where

$$A = \frac{m\pi}{\ell_1} \frac{f}{f_c k_c} E, \quad E = \text{amplitude constant},$$

$$C = -\left(1 - \frac{f_c^2}{f^2}\right)^{\frac{1}{2}} B, \quad B = -A, \quad D = C,$$

$$f_c = \frac{\sqrt{2}}{2} \frac{m}{\ell_1}, \quad k_c = 2\pi f_c, \tag{3.12}$$

$$K^\nu = (\omega, 0, 0, \beta),$$

$$M^\nu = (0, \frac{m\pi}{\ell_1}, 0, 0), \quad N^\nu = (0, 0, \frac{m\pi}{\ell_1}, 0).$$

According to appendix C the spectrum is given by $S = \{\pm k_I, \ldots, \pm k_{IV}\}$, where

$$k_I^\nu = (\omega, \frac{m\pi}{\ell_1}, \frac{m\pi}{\ell_1}, \beta) \overset{\text{def.}}{=} \omega\,(1,u,u,w)$$

$$k_{II}^\nu = (\omega, \frac{m\pi}{\ell_1}, -\frac{m\pi}{\ell_1}, \beta) \overset{\text{def.}}{=} \omega\,(1,u,-u,w)$$

$$k_{III}^\nu = (\omega, -\frac{m\pi}{\ell_1}, \frac{m\pi}{\ell_1}, \beta) \overset{\text{def.}}{=} \omega\,(1,-u,u,w) \tag{3.13}$$

$$k_{IV}^\nu = (\omega, -\frac{m\pi}{\ell_1}, -\frac{m\pi}{\ell_1}, \beta) \overset{\text{def.}}{=} \omega\,(1,-u,-u,w).$$

Thus $\omega u = \omega v = \frac{m\pi}{\ell_1}$ and $\omega w = \beta$. Note that the vectors k_i are isotropic: $-\omega^2 + 2\left(\frac{\pi m}{\ell_1}\right)^2 + \beta^2 = 0$. The Fourier coefficients are real and therefore here:

$$\hat{F}^{\mu\nu}(k_j) = \hat{F}^{\mu\nu}(-k_j) \qquad (j=I, \ldots, IV). \tag{3.14}$$

The Fourier coefficients can be read out of the following table:

Field components	Coefficients	k_I	k_{II}	k_{III}	k_{IV}
E_x	$-\frac{1}{8}A$	+	-	+	-
E_y	$\frac{1}{8}A$	+	+	-	-
E_z	0				
H_x	$-\frac{1}{8}C$	+	+	-	-
H_y	$-\frac{1}{8}C$	+	-	+	-
H_z	$\frac{1}{8}E$	+	+	+	+

Examples:

$$E_x(k_{III}) = \hat{F}^{01}(k_{III}) = -\frac{1}{8}A,$$
$$E_y(k_{III}) = \hat{F}^{02}(k_{III}) = -\frac{1}{8}A. \quad (3.15)$$

From (3.13) one concludes that $n_o=3$, so that $U^d = \{a_*^1, a_*^2, a_*^3\}$.
A possible choice for a basis of S and therefore of Λ^d is:

$$a_*^1 = -(\omega, \frac{m\pi}{\ell_1}, \frac{m\pi}{\ell_1}, \omega w), \; a_*^2 = (0, \frac{2m\pi}{\ell_1}, 0,0), \; a_*^3 = (0,0, \frac{2m\pi}{\ell_1}, 0) \quad (3.16)$$

and the corresponding dual basis is:

$$a_1 = (\lambda, 0, 0, 0), \; a_2 = (\frac{\lambda}{2}, \frac{\ell_1}{m}, 0, 0), \; a_3 = (\frac{\lambda}{2}, 0, \frac{\ell_1}{m}, 0) \quad (3.17)$$

generating the lattice Λ. In (3.16) and (3.17) the components are given with respect to the basis $e_o,\dots,e_3$, after identification of the dual vector space with M by means of the scalar product. The vectors of the subspace V_c are of the form $(r, o, o, \frac{r}{w})$, $r \in \mathbb{R}$. Choose $c_4^\nu = (1, 0, 0, \frac{1}{w})$. Finally the group of primitive translations is given by:

$$U = \{a_1, a_2, a_3, c_4\,\delta t_4\} \cong \mathbb{Z}^3 \oplus \mathbb{R}. \quad (3.18)$$

The spectral group S can be decomposed into cosets with respect to S_o, the subgroup of S leaving each element of S invariant:

$$S = 1S_o + L_2S_o + \dots + L_sS_o. \quad (3.19)$$

It is then sufficient to know S_o and a representative of each coset. Consider an element of S_o. It necessarily leaves the following vectors invariant:
$(0,0,2u,0)$, $(0,2u,0,0)$, $(1,0,0,w)$.
One finds that if this element is not the identity, it is necessarily given by:

$$M_w \overset{\text{def.}}{=} \begin{pmatrix} \frac{1+w^2}{1-w^2} & 0 & 0 & \frac{-2w}{1-w^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2w}{1-w^2} & 0 & 0 & -\frac{1+w^2}{1-w^2} \end{pmatrix}, \tag{3.20}$$

which is of order two, so that

$$S_o = \{M_w\} \cong C_2. \tag{3.21}$$

Recall that the k_i are isotropic, and therefore $w^2 < 1$. Consider the following Lorentz transformation from the laboratory frame K to the inertial frame $\bar{K}$:

$$T(w) : (1, 0, 0, w) \rightarrow (\sqrt{1-w^2}, 0, 0, 0),$$

$$T(w) = \begin{pmatrix} \frac{1}{\sqrt{1-w^2}} & 0 & 0 & \frac{-w}{\sqrt{1-w^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-w}{\sqrt{1-w^2}} & 0 & 0 & \frac{1}{\sqrt{1-w^2}} \end{pmatrix}. \tag{3.22}$$

Noticing that $T(w)\, M_w T(w)^{-1} = m_{\bar{z}}$, where $m_{\bar{z}}$ is the reflection with respect to the hyperplane perpendicular to $\bar{e}_3 = e_{\bar{z}} = T(w)\, e_z$, one gets an easy interpretation of the element M_w as a Lorentz mirror. In the new frame of reference $\bar{K}$ the spectrum $\bar{S}$ is:

$$\begin{aligned} \bar{S} &= T(w)\, S = \{\pm \bar{k}_I, \pm \bar{k}_{II}, \pm \bar{k}_{III}, \pm \bar{k}_{IV}\}, \text{ where} \\ \bar{k}^{\nu}_I &= \omega\, (\sqrt{1-w^2}, u,u,0) \\ \bar{k}^{\nu}_{II} &= \omega\, (\sqrt{1-w^2}, u,-u,0) \\ \bar{k}^{\nu}_{III} &= \omega\, (\sqrt{1-w^2},-u,u,0) \\ \bar{k}^{\nu}_{IV} &= \omega\, (\sqrt{1-w^2},-u,-u,0). \end{aligned} \tag{3.23}$$

Furthermore $\bar{S} = T(w)\; S\; T(w)^{-1}$ and $\bar{S}_o = T(w)\; S_o\; T(w)^{-1} = \{m_{\bar{z}}\}$.
Consider now the subgroup $\bar{D} \subset \bar{S}$ which leaves the set $\bar{k}_j$, $j = I, \ldots, IV$, as a whole invariant. For an element $\bar{L}$ of $\bar{S}$ there are then two possibilities:
(1) $\bar{L} \in \bar{D}$; then $\bar{L}$ leaves $(\sqrt{1-w^2}, 0,0,0)$ invariant and consequently it is of the form:

$$\bar{L} = \begin{pmatrix} 1 & 0 \\ 0 & O_3 \end{pmatrix} ,$$

where O_3 represents an orthogonal transformation in three-dimensional Euclidean subspace.

(2) $\bar{L} \in \bar{S} \setminus \bar{D}$; then $\bar{L}$ is necessarily antichronous, changes therefore the sign of $(\sqrt{1-w^2}, 0,0,0)$ and is of the form:

$$\bar{L} = \begin{pmatrix} -1 & 0 \\ 0 & O_3 \end{pmatrix} .$$

This shows that $\bar{D}$ is a subgroup of $\bar{S}$ of index two and $\bar{S}$ is a magnetic group:

$$\bar{S} = \bar{D} + m_{\bar{t}}\, \bar{D}, \tag{3.24}$$

where $m_{\bar{t}}$ is the mirror with respect to the hyperplane perpendicular to $\bar{e}_o = e_{\bar{t}} = T(w)e_t$. Therefore $\bar{K}$ is a Shubnikov frame. Note that $\bar{S}/\bar{S}_o$ is the symmetry group of the following array of points:

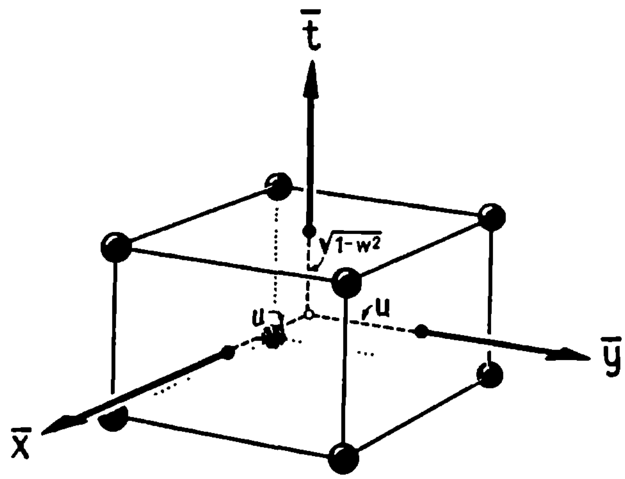

FIG. 1

In the crystallographic notation: $\bar{S}/\bar{S}_o$ = 4 mm1' $=\{4_{\bar{z}}, m_{\bar{x}-\bar{y}}, m_{\bar{t}}\}$, and $\bar{D}/\bar{S}_o$=4 m m is the corresponding non-magnetic subgroup of index two. In the present case this is the group of the square. With $\bar{S}_o = \{m_{\bar{z}}\}$ one finds:

$$\bar{S} = \{4_{\bar{z}}, m_{\bar{x}-\bar{y}}, m_{\bar{z}}, m_{\bar{t}}\} = 4/m\ m\ m\ 1', \tag{3.25}$$

where $4_{\bar{z}}$ is the fourfold rotation about the $\bar{z}$-axis, $m_{\bar{x}}$ the reflection with respect to the hyperplane orthogonal to the $\bar{x}$-axis and similarly for the other elements of $\bar{S}$. Going back to the laboratory frame K, the spectral group becomes:

$$S = \{4_z, m_{x-y}, M_w, M_{\frac{1}{w}}\}, \tag{3.26}$$

where M_w is given by (3.20) and

$$M_{\frac{1}{w}} = T(w)^{-1}\, m_{\bar{t}}\, T(w), \text{ so that } M_w M_{\frac{1}{w}} = m_t m_z = m'_z.$$

Note that w is the velocity of propagation of the TE_{mn}-mode. The Shubnikov frame appears to be the frame in which the corresponding EM-field is so to say at rest, that is it only oscillates in time, but it does not propagate in space. This explains also why for standing waves, as one has for example with the TE_{mnp} and TM_{mnp} cavity modes, as well as for evanescent modes, the laboratory system is already a Shubnikov frame. One also understands why no Shubnikov frame exists for a TEM-wave. Therefore the EM-fields can be divided into three classes: (i) fields whose point group is a Shubnikov group; (ii) fields whose point group is conjugate to a Shubnikov group by a Lorentz boost; (iii) fields for which no Shubnikov frame can be found.

Consider now the point group $\bar{K}$ in the Shubnikov frame $\bar{K}$. The symmetry condition (3.8) then becomes:

$$\dot{\bar{L}}\, \hat{\bar{F}}(\bar{k}) = \hat{\bar{F}}(\bar{L}\,\bar{k})\, \bar{L}^{*} \exp\,(i(\bar{L}\,\bar{k})\, \bar{u}(\bar{L})), \tag{3.27}$$

where $\hat{\bar{F}}(\bar{k}) = \hat{\bar{F}}(T(w)k) = \dot{T}(w)\, \hat{F}(k)\, T(w)^{t}$.

T(w) is given in (3.22). The Fourier coefficients relative to the Shubnikov system are:

$$\hat{\bar{F}}(\bar{k}_j) = \frac{1}{8}\begin{pmatrix} 0 & \epsilon_j A\sqrt{1-w^2} & \tau_j A\sqrt{1-w^2} & 0 \\ -\epsilon_j A\sqrt{1-w^2} & 0 & E & 0 \\ -\tau_j A\sqrt{1-w^2} & -E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.28}$$

where $-\epsilon_I = \tau_I = \epsilon_{II} = \tau_{II} = -\epsilon_{III} = -\tau_{III} = \epsilon_{IV} = -\tau_{IV} = 1$. One verifies that in this particular case $\bar{S} = \bar{K}$ and

$$\bar{K} = \{ 4_{\bar{z}}, m_{\bar{x}-\bar{y}}, m_{\bar{z}}, m_{\bar{t}} \} = 4/m\ m\ m\ 1', \tag{3.29}$$

and a corresponding system of non-primitive translations is given by:

$$\bar{u}(m_{\bar{x}-\bar{y}}) = \bar{u}(m_{\bar{t}}) = \left(\frac{\lambda}{2\sqrt{1-w^2}}, 0,0,0, \right), \quad \bar{u}(4_{\bar{z}}) = \bar{u}(m_{\bar{z}}) = 0.$$

In the laboratory frame K the result can be written as

$$K = \{ 4_z, m_{x-y}, M_w, M_{\frac{1}{\bar{w}}} \} \text{ with} \qquad (3.30)$$

$$u(m_{x-y}) = u(M_{\frac{1}{\bar{w}}}) = (\frac{\lambda}{2}, 0,0,0), \; u(4_z) = u(M_w) = 0.$$

To see that in (3.30) the system of non-primitive translations is correctly given, consider in $\bar{K}$ instead of $\bar{u}(\bar{L}) = (\frac{\lambda}{2\sqrt{1-w^2}}, 0,0,0)$ the equivalent non-primitive translation $\bar{v}(\bar{L}) = (\frac{\lambda}{2\sqrt{1-w^2}}, 0,0, \frac{-\lambda w}{2\sqrt{1-w^2}})$ and verify that indeed $u(L) = T(w)^{-1}\, \bar{v}(\bar{L})$, where $\bar{L}$ stands for $m_{\bar{x}-\bar{y}}$ and for $m_{\bar{t}}$.

III.2 Identification of Generalized Magnetic Space-time Groups

As an example consider the symmetry groups of the TE_{mnp} cavity modes for the case $\frac{m\pi}{\ell_1} = \frac{n\pi}{\ell_2} = \frac{p\pi}{\ell_3}$, $(u = v = w)$. Note that the condition $u = v = w$ does not mean that one has a cubical cavity ($\ell_1 = \ell_2 = \ell_3$). Of course, if this is the case then $m = n = p$. But even so the three coordinate axes are not equivalent, because the restriction to TE-modes implies $E_z = 0$ (see appendix B). Thus the z-axis plays a special rôle. The same is true for the TM-modes, where $H_z = 0$. This is the reason why the TE_{mnp}-modes for $u = v = w$ and for $u = v \neq w$ have the correspondingly same symmetry groups.

According to table 2 of **section** 3 the point group and a set of non-primitive translations are given by

$$K = \{ 4_z, m_{x-y}, m_z, 1' \} = 4/mmm\, 1',$$

$$u(4_z) = 0, \; u(m_{x-y}) = u(m_z) = u(1') = \frac{\lambda}{2} e_o.$$

For this geometrical class table II of **ref.** 21) gives:

System	Geometric class	Isomorphism class	Order	Generators	Arithmetic class	Number
XII	4/mmm1'	$D_4 \times C_2 \times C_2$	32	23 29 26 30	P I E A H	323-327

(3.31)

In ref.21) the groups are given as groups of matrices, so that some Shubnikov groups are identified: e.g. mmm' and 2/m 1'. The symbols P,I,E,A,H represent different centerings of the lattice XII P, whose basis vectors lie along suitable crystallographic axes (see e.g. **ref.**22) for the three-dimensional case).

Table I of **ref.** 3) gives the metric tensor of the XII P-lattice as

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix},$$

so that the corresponding crystallographic axes are orthogonal. A possible basis for the XII P-lattice is

$$\begin{aligned} P_1 &= a_2 - a_3 &&= (0, \tfrac{\ell_1}{m}, -\tfrac{\ell_1}{m}, 0) \\ P_2 &= -a_1 + a_2 + a_3 &&= (0, \tfrac{\ell_1}{m}, \tfrac{\ell_1}{m}, 0) \\ P_3 &= -a_1 + 2a_4 &&= (0, 0, 0, \tfrac{2\ell_1}{m}) \\ P_4 &= a_1 &&= (\lambda, 0, 0, 0), \end{aligned} \tag{3.32}$$

where $\{a_1, \ldots, a_4\}$ is the basis of Λ indicated in section 3. The determinant of the basis transformation (3.32) **being four, the unit cell of XII P contains** four inequivalent points of Λ. Since only the centerings P, I, E, A, H occur, and H is the only one with four points per unit cell (see table IV of **ref.** 21)) the group looked for must belong to the family of groups no. 327 indicated in table II of the same reference. The elements α_i of the point group K are there referred to a basis $\{b_1, \ldots, b_4\}$ obtained from $\{P_1, \ldots, P_4\}$ by the centering H performed according to table IV of **ref.** 21) **by means of the matrix**

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix}^{\bigstar} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

where $\bigstar$ means adjoint, i.e. inverse transposed. Thus

$$\begin{aligned} b_1 &= [\tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}]_P = (\tfrac{\lambda}{2}, \tfrac{\ell_1}{m}, 0, 0) = a_2 \\ b_2 &= [\tfrac{1}{2}, -\tfrac{1}{2}, 0, \tfrac{1}{2}]_P = (\tfrac{\lambda}{2}, 0, -\tfrac{\ell_1}{m}, 0) = a_1 - a_3 \\ b_3 &= [-\tfrac{1}{2}, \tfrac{1}{2}, 0, \tfrac{1}{2}]_P = (\tfrac{\lambda}{2}, 0, \tfrac{\ell_1}{m}, 0) = a_3 \\ b_4 &= [0, 0, -\tfrac{1}{2}, \tfrac{1}{2}]_P = (\tfrac{\lambda}{2}, 0, 0, -\tfrac{\ell_1}{m}) = a_1 - a_4. \end{aligned} \tag{3.33}$$

$[\ldots.]_P$ are the components with respect to the basis (3.32) **and** (....) **those**

with respect to the basis e_o, e_1, e_2, e_3. The relation between the $\{b_1, \ldots, b_4\}$ and the $\{a_1, \ldots, a_4\}$, which is a lattice basis transformation, allows the interpretation of the tabulated matrices as Shubnikov transformations. In our case we find $\alpha_1 = 4_z$, $\alpha_2 = m_y$, $\alpha_3 = m_z$, $\alpha_4 = 1'$, (3.34)
with non-primitive translations:

$$u(\alpha_1) = 0,\ u(\alpha_2) = u(\alpha_3) = u(\alpha_4) = \tfrac{1}{2}(b_2+b_3) = [0,\tfrac{1}{2},\tfrac{1}{2},0]_b. \quad (3.35)$$

Translating the origin by f one gets the equivalent system of non-primitive translations v(K) according to **ref.** 23):

$$v(\alpha) = (1-\alpha)f + u(\alpha). \quad (3.36)$$

In particular for $f = [0, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{2}]_b$ the above system of non-primitive translations (3.35) becomes equivalent to the one of the group defined by the extension no. $327e_1$ as indicated in table II of **ref.** 21:

	α_1	α_2	α_3	α_4
e_1	$-\tfrac{1}{2}(a+b)$	0	0	0

where a, b, c, d correspond to $b_1, \ldots, b_4$ and, of course, e_1 characterizes here a class of inequivalent extensions and has nothing to do with the basis vector along the x-axis. Therefore the symmetry group G of the EM-fields TE_{mnp} (u = v = w) can be identified with the group no. 327 e_1 of the tables of Fast and Janssen [21].

III.3 Survey of the Results

The symmetry groups of the fields considered are indicated in two tables. In table 1, the groups U of primitive translations are given by a set of generators, where discrete or finite translations as well as infinitesimal ones occur. The discrete generators form a basis of the lattice Λ and are noted as $a_i (i = 1, \ldots, n_o)$, whereas the infinitesimal generators are indicated by $c_i\, \delta t_i$, for δt_i infinitesimal $(i = n_o+1, \ldots, 4)$.

Complex translations appear in the case of evanescent modes. Both, the generators for the group of complex primitive translations and those for the subgroup of real primitive translations are given. In a following paper, we hope to be able to show, that it is convenient to consider, as symmetry elements of an EM-field, complex translations also.

In table 2, the point group K is given together with a system u(K) of non-primitive translations relative to an origin displaced by f with respect to that adopted in appendix B, where EM-fields are listed in a standard way[20]. In tables 1 and 2 vectors are given by their components with respect to the basis e_o, ..., e_3.

Note that in all cases where a system u(K) of non-primitive translations different from zero is indicated, this system is also inequivalent to zero, and therefore the corresponding space-time group is non-symmorphic. Furthermore, for generalized magnetic groups, (i.e. four-dimensional crystallographic groups with as point group a Shubnikov group) one also finds the equivalence class of extensions to which thev belong according to the classification of Fast and Janssen[21] (see section 2). In all the cases, and according to the relations (3.9) to (3.11), the space-time groups defined by G ={U, K, u(K)} are the relativistic symmetry groups of the corresponding EM-fields.

Table III.1: Groups of Primitive Translations

EM-fields	Generators of the group U of primitive translations			
	a_1	a_2 or $c_2\delta t_2$	a_3 or $c_3\delta t_3$	a_4 or $c_4\delta t_4$
TE_{mnp} TM_{mnp}	$(\lambda,0,0,0)$	$(\frac{\lambda}{2}, \frac{\ell_1}{m}, 0,0)$	$(\frac{\lambda}{2}, 0, \frac{\ell_2}{n}, 0)$	$(\frac{\lambda}{2}, 0,0, \frac{\ell_3}{p})$
TE_{onp}	$(\lambda,0,0,0)$	$(0,1,0,0)\delta t_2$	$(\frac{\lambda}{2}, 0, \frac{\ell_2}{n}, 0)$	$(\frac{\lambda}{2}, 0,0, \frac{\ell_3}{p})$
TE_{mop}	$(\lambda,0,0,0)$	$(\frac{\lambda}{2}, \frac{\ell_1}{m}, 0,0)$	$(0,0,1,0)\delta t_3$	$(\frac{\lambda}{2}, 0,0, \frac{\ell_3}{p})$
TM_{mno}	$(\lambda,0,0,0)$	$(\frac{\lambda}{2}, \frac{\ell_1}{m}, 0,0)$	$(\frac{\lambda}{2}, 0, \frac{\ell_2}{n}, 0)$	$(0,0,0,1)\delta t_4$
$TE_{mo}\equiv TE_m, TM_m$	$(\lambda,0,0,0)$	$(\frac{\lambda}{2}, \frac{\ell_1}{m}, 0,0)$	$(0,0,1,0)\delta t_3$	$(1,0,0,\frac{1}{w})\delta t_4$
TE_{mn}, TM_{mn}	$(\lambda,0,0,0)$	$(\frac{\lambda}{2}, \frac{\ell_1}{m}, 0,0)$	$(\frac{\lambda}{2}, 0, \frac{\ell_2}{n}, 0)$	$(1,0,0,\frac{1}{w})\delta t_4$
TE_{on}	$(\lambda,0,0,0)$	$(0,1,0,0)\delta t_2$	$(\frac{\lambda}{2}, 0, \frac{\ell_2}{n}, 0)$	$(1,0,0,\frac{1}{w})\delta t_4$
$TE_{mo}^{evan}\equiv TE_m^{evan}$, TM_m^{evan}	$(\frac{\lambda}{2},0,0,\frac{-i\pi}{\alpha})$	$(0,\frac{\ell_1}{m},0,\frac{i\pi}{\alpha})$	$(0,0,1,0)\delta t_3$	$(0,0,0,\frac{2\pi i}{\alpha})$
	$(\frac{\lambda}{2},\frac{\ell_1}{m},0,0)$	$(\frac{\lambda}{2},\frac{-\ell_1}{m},0,0)$	$(0,0,1,0)\delta t_3$	--
$TE_{mn}^{evan}, TM_{mn}^{evan}$	$(\frac{\lambda}{2},0,0,\frac{-i\pi}{\alpha})$	$(0,\frac{\ell_1}{m},0,\frac{i\pi}{\alpha})$	$(0,0,\frac{\ell_2}{n}, \frac{i\pi}{\alpha})$	$(0,0,0,\frac{2\pi i}{\alpha})$
	$(\lambda,0,0,0)$	$(\frac{\lambda}{2}, \frac{\ell_1}{m},0,0)$	$(\frac{\lambda}{2},0, \frac{\ell_2}{n}, 0)$	--
TE_{on}^{evan}	$(\frac{\lambda}{2},0,0,\frac{-i\pi}{\alpha})$	$(0,1,0,0)\delta t_2$	$(0,0, \frac{\ell_2}{n},\frac{i\pi}{\alpha})$	$(0,0,0, \frac{2\pi i}{\alpha})$
	$(\frac{\lambda}{2}, 0, \frac{\ell_2}{n},0)$	$(0,1,0,0)\delta t_2$	$(\frac{\lambda}{2}, 0, \frac{-\ell_2}{n}, 0)$	--

Point groups K (set of generators)	EM-fields	Systems of non-primitive translations u(K)				Change of origin f	Inequivalent extension (acc.to ref.5)
4_z, m_{x-y}, m_z, 1'		$u(4_z)$	$u(m_{x-y})$	$u(m_z)$	u(1')		
	TE_{mnp} u = v	0	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$		327 e_1
	TM_{mnp} u = v	0	0	0	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_1}{m}, \frac{\ell_3}{p})$	327
	TM_{mno} u = v	0	0	$\frac{\lambda}{2} e_o$	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_1}{m}, 0)$	
m_x, m_y, m_z, 1'		$u(m_x)$	$u(m_y)$	$u(m_z)$	u(1')		
	TE_{mnp} u ≠ v	0	0	0	0	$\frac{1}{2}(\frac{\lambda}{2}, \frac{\ell_1}{m}, \frac{\ell_2}{n}, \frac{\ell_3}{p})$	130
	TM_{mnp} u ≠ v	0	0	0	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_2}{n}, \frac{\ell_3}{p})$	130
	TE_{onp}	$\frac{\lambda}{2} e_o$	0	0	0	$\frac{1}{2}(\frac{\lambda}{2}, 0, \frac{\ell_2}{n}, \frac{\ell_3}{p})$	
	TE_{mop}	0	$\frac{\lambda}{2} e_o$	0	0	$\frac{1}{2}(\frac{\lambda}{2}, \frac{\ell_1}{m}, 0, \frac{\ell_3}{p})$	
	TM_{mno} u ≠ v	0	0	$\frac{\lambda}{2} e_o$	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_2}{n}, 0)$	

Point groups K (set of generators)	EM-fields	Systems of non-primitive translations u(K)				Change of origin f	Inequivalent extension (acc. to ref. 5)
4_z, m_{x-y}, M_w, $M_{\frac{1}{w}}$		$u(4_z)$	$u(m_{x-y})$	$u(M_w)$	$u(M_{\frac{1}{w}})$		
	TE_{mn} $u = v$	0	$\frac{\lambda}{2} e_o$	0	$\frac{\lambda}{2} e_o$		
	TM_{mn} $u = v$	$\frac{\lambda}{2} e_o$	0	$\frac{\lambda}{2} e_o$	0		
m_x, m_y, M_w, $M_{\frac{1}{w}}$		$u(m_x)$	$u(m_y)$	$u(M_w)$	$u(M_{\frac{1}{w}})$		
	TE_{mn} $u \neq v$	0	0	0	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_2}{n}, \frac{\lambda}{2w})$	
	TM_{mn} $u \neq v$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	0		
	TE_{on}	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	0	$\frac{\lambda}{2} e_o$		
	$TE_{mo} \equiv TE_m$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	0	$\frac{\lambda}{2} e_o$		
	TM_m	$\frac{\lambda}{2} e_o$	0	$\frac{\lambda}{2} e_o$	0		

Point groups K (set of generators)	EM-fields	Systems of non-primitive translations u(K)			Change of origin f	Inequivalent extension (acc. to ref. 5)
4_z, m_{x-y}, 1'		$u(4_z)$	$u(m_{x-y})$	u(1')		
	TE^{evan}_{mn} u = v	0	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$		
	TM^{evan}_{mn} u = v	0	0	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_1}{m}, 0)$	
m_x, m_y, 1'		$u(m_x)$	$u(m_y)$	u(1')		
	TE^{evan}_{mn} $u \neq v$	0	0	0	$\frac{1}{2}(\frac{\lambda}{2}, \frac{\ell_1}{m}, \frac{\ell_2}{n}, 0)$	
	TM^{evan}_{mn} $u \neq v$	0	0	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, \frac{\ell_2}{n}, 0)$	
	TE^{evan}_{on}	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$		
	$TE^{evan}_{mo} \equiv TE^{evan}_{m}$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$	$\frac{\lambda}{2} e_o$		
	TM^{evan}_{m}	0	0	0	$\frac{1}{2}(0, \frac{\ell_1}{m}, 0, 0)$	

APPENDIX III

III.A Complex Wave Vectors

For the evanescent modes it is necessary to clarify whether the method used for the propagating modes (see section 1) still applies. The evanescent modes can be written as a complex Fourier expansion

$$F^{\mu\nu}(x) = \sum_{k \in S} \hat{F}^{\mu\nu}(k) \exp(ikx), \tag{3.37}$$

where $\hat{F}^{\mu\nu}(k) \in \mathbb{C}$, $k = \ell + im$, $\ell, m, x \in \mathbb{R}^4$. The spectrum S consists of finitely many complex wave vectors k and for this case we shall prove, that the above expansion is unique. This in turn implies that the procedure adopted for real wave vectors can also be used in the complex case.

We have to show that from

$$\tilde{F}^{\mu\nu}(x) = \sum_{\tilde{k} \in \tilde{S}} \hat{\tilde{F}}^{\mu\nu}(\tilde{k}) \exp(i\,\tilde{k}\,x) = \sum_{k \in S} \hat{F}^{\mu\nu}(k) \exp(i\,k\,x) = F^{\mu\nu}(x) \tag{3.38}$$

follows $\quad \tilde{S} = S$ and $\hat{\tilde{F}}^{\mu\nu}(k) = \hat{F}^{\mu\nu}(k), \quad \forall\, k \in S.$ (3.39)

Relation (3.38) can be written in the form:

$$\sum_{k \in S \cup \tilde{S}} a(k) \exp(i\,k\,x) = 0, \quad \forall\, x \in M. \tag{3.40}$$

For $k \in \tilde{S} \cap S$, $a(k) = \hat{F}^{\mu\nu}(k) - \hat{\tilde{F}}^{\mu\nu}(k)$, and for $k \notin \tilde{S} \cap S$, a(k) is either equal to $\hat{F}^{\mu\nu}(k)$ or to $-\hat{\tilde{F}}^{\mu\nu}(k)$. Write now (3.40) as

$$a(k_j) + \sum_{q \neq j} a(k_q) \exp(i[k_q - k_j]x) = 0. \tag{3.41}$$

Since in (3.41) every vector $k_q \neq k_j$ occurs only once and $k_q = \ell_q + i\,m_q$, there is always a vector k_j with a μ-component such that

$$m_j^\mu > m_q^\mu \quad \text{for all } q \neq j\,.$$

Consider now in (3.41) a vector x along the positive μ-axis and let $x^\mu \to \infty$. It then follows that $a(k_j) = 0$. Repeating this same procedure one gets $a(k) = 0$, for all $k \in S$, and this implies (3.39).

III.B Explicit Expressions for the Electromagnetic Fields

In the following list the amplitudes A, B, ..., may have a different meaning for each set of EM-fields; therefore their signification is indicated **separately** for each case. The field expressions incorporate the real part of a factor of the form $\exp(i\omega t - \Gamma z)$, where $\Gamma = \alpha + i\beta$ (see **section 1**). The vectors K, L, M, N, P have components with respect to the basis $e_o, \ldots, e_3$ given by: $K^\nu = (\omega, 0, 0, \beta)$, $L^\nu = (\omega, 0, 0, 0)$, $M^\nu = (0, \frac{m\pi}{\ell_1}, 0, 0)$, $N^\nu = (0, 0, \frac{n\pi}{\ell_2}, 0)$ and $P^\nu = (0, 0, 0, \frac{p\pi}{\ell_3})$, where m, n, p are integers and ℓ_1, ℓ_2, ℓ_3 real numbers indicating the dimensions of the cavity, and correspondingly of the rectangular (ℓ_1, ℓ_2) and of the parallel plane waveguide (ℓ_1). Further, $f_c = \frac{1}{2\pi} k_c$, and E is a constant amplitude. For the other symbols see **ref.20)**. Note that the TE_{mnp}-modes are not defined for m = n = 0 and for p =0; the TM_{mnp}-modes are not defined for m = 0, n = 0 and for m = n = 0; the TE_{mn}-modes are not defined for m = n = 0; the TM_{mn}-modes are not defined for m = 0, n = 0 and m = n = 0; the TE_m- and TM_m-modes are not defined for m = 0.

a. Resonant Cavity Modes: Rectangular Cavities

TE_{mnp}

$$E_x = A \sin Lx \cos Mx \sin Nx \sin Px$$
$$E_y = B \sin Lx \sin Mx \cos Nx \sin Px$$
$$E_z = 0$$
$$H_x = C \cos Lx \sin Mx \cos Nx \cos Px$$
$$H_y = D \cos Lx \cos Mx \sin Nx \cos Px$$
$$H_z = E \cos Lx \cos Mx \cos Nx \sin Px$$

TM_{mnp}

$$E_x = C \cos Lx \cos Mx \sin Nx \sin Px$$
$$E_y = D \cos Lx \sin Mx \cos Nx \sin Px$$
$$E_z = E \cos Lx \sin Mx \sin Nx \cos Px$$
$$H_x = A \sin Lx \sin Mx \cos Nx \cos Px$$
$$H_y = B \sin Lx \cos Mx \sin Nx \cos Px$$
$$H_z = 0$$

$$A = \frac{k}{k_c^2} \frac{n\pi}{\ell_2} E, \quad B = -\frac{k}{k_c^2} \frac{m\pi}{\ell_1} E, \quad C = -\frac{1}{k_c^2} \frac{m\pi}{\ell_1} \frac{p\pi}{\ell_3} E, \quad D = -\frac{1}{k_c^2} \frac{n\pi}{\ell_2} \frac{p\pi}{\ell_3} E,$$

$$k_c^2 = \left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2$$

b. Propagating Modes

Parallel Plane Waveguides

TE_m

$$E_x = 0$$
$$E_y = A \sin Kx \sin Mx$$
$$E_z = 0$$
$$H_x = B \sin Kx \sin Mx$$
$$H_y = 0$$
$$H_z = E \cos Kx \cos Mx$$

where $A = -\frac{m\pi}{\ell_1} \frac{f}{f_c k_c} E$

$$B = -\left(1 - \frac{f_c^2}{f^2}\right)^{\frac{1}{2}} A$$

$$k_c = \frac{m\pi}{\ell_1}$$

TM_m

$$E_x = -B \sin Kx \cos Mx$$
$$E_y = 0$$
$$E_z = E \cos Kx \sin Mx$$
$$H_x = 0$$
$$H_y = A \sin Kx \cos Mx$$
$$H_z = 0$$

<u>Rectangular Waveguides</u>

TE_{mn}

$$E_x = A \sin Kx \cos Mx \sin Nx$$

$$E_y = B \sin Kx \sin Mx \cos Nx$$

$$E_z = 0$$

$$H_x = C \sin Kx \sin Mx \cos Nx$$

$$H_y = D \sin Kx \cos Mx \sin Nx$$

$$H_z = E \cos Kx \cos Mx \cos Nx$$

TM_{mn}

$$E_x = -C \sin Kx \cos Mx \sin Nx$$

$$E_y = -D \sin Kx \sin Mx \cos Nx$$

$$E_z = E \cos Kx \sin Mx \sin Nx$$

$$H_x = A \sin Kx \sin Mx \cos Nx$$

$$H_y = B \sin Kx \cos Mx \sin Nx$$

$$H_z = 0$$

where

$$A = \frac{n\pi}{\ell_2} \frac{f}{f_c k_c} E$$

$$B = -\frac{m\pi}{\ell_1} \frac{f}{f_c k_c} E$$

$$C = -\left(1 - \frac{f_c^2}{f^2}\right)^{\frac{1}{2}} B$$

$$D = \left(1 - \frac{f_c^2}{f^2}\right)^{\frac{1}{2}} A$$

$$k_c^2 = \left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2$$

c. Evanescent Modes

Parallel Plane Waveguides

TE_m^{evan}

$$E_x = 0$$
$$E_y = A\, e^{-\alpha z} \sin Lx \sin Mx$$
$$E_z = 0$$
$$H_x = B\, e^{-\alpha z} \cos Lx \sin Mx$$
$$H_y = 0$$
$$H_z = E\, e^{-\alpha z} \cos Lx \cos Mx$$

TM_m^{evan}

$$E_x = -\, B\, e^{-\alpha z} \cos Lx \cos Mx$$
$$E_y = 0$$
$$E_z = E\, e^{-\alpha z} \cos Lx \sin Mx$$
$$H_x = 0$$
$$H_y = A\, e^{-\alpha z} \sin Lx \cos Mx$$
$$H_z = 0$$

where

$$A = -\frac{m\pi}{\ell_1} \frac{f}{f_c k_c} E$$

$$B = -\frac{m\pi}{\ell_1} \frac{f}{f_c k_c} \frac{\alpha}{k} E$$

$$k_c = \frac{m\pi}{\ell_1}$$

Rectangular Waveguides

TE_{mn}^{evan}

$$E_x = A\, e^{-\alpha z} \sin Lx \cos Mx \sin Nx$$

$$E_y = B\, e^{-\alpha z} \sin Lx \sin Mx \cos Nx$$

$$E_z = 0$$

$$H_x = C\, e^{-\alpha z} \cos Lx \sin Mx \cos Nx$$

$$H_y = D\, e^{-\alpha z} \cos Lx \cos Mx \sin Nx$$

$$H_z = E\, e^{-\alpha z} \cos Lx \cos Mx \cos Nx$$

TM_{mn}^{evan}

$$E_x = -C\, e^{-\alpha z} \cos Lx \cos Mx \sin Nx$$

$$E_y = -D\, e^{-\alpha z} \cos Lx \sin Mx \cos Nx$$

$$E_z = E\, e^{-\alpha z} \cos Lx \sin Mx \sin Nx$$

$$H_x = A\, e^{-\alpha z} \sin Lx \sin Mx \cos Nx$$

$$H_y = B\, e^{-\alpha z} \sin Lx \cos Mx \sin Nx$$

$$H_z = 0$$

where

$$A = \frac{n\pi}{\ell_2} \; \frac{f}{f_c k_c} \; E$$

$$B = -\frac{m\pi}{\ell_1} \; \frac{f}{f_c k_c} \; E$$

$$C = -\frac{m\pi}{\ell_1} \; \frac{f}{f_c k_c} \; \frac{\alpha}{k} \; E$$

$$D = -\frac{n\pi}{\ell_2} \; \frac{f}{f_c k_c} \; \frac{\alpha}{k} \; E$$

$$k_c^2 = \left(\frac{m\pi}{\ell_1}\right)^2 + \left(\frac{n\pi}{\ell_2}\right)^2$$

III.C The Spectra of the EM-Fields

(Components given with respect to the basis $e_o, \ldots, e_3$).

TE_{mnp}	$\pm k_I$	$= \pm (\omega, \frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, \frac{p\pi}{\ell_3})$	$\pm k_V$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, \frac{p\pi}{\ell_3})$
TM_{mnp}	$\pm k_{II}$	$= \pm (\omega, \frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$	$\pm k_{VI}$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$
	$\pm k_{III}$	$= \pm (\omega, \frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, \frac{p\pi}{\ell_3})$	$\pm k_{VII}$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, \frac{p\pi}{\ell_3})$
	$\pm k_{IV}$	$= \pm (\omega, \frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$	$\pm k_{VIII}$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$
TE_m TM_m	$\pm k_I$	$= \pm (\omega, \frac{m\pi}{\ell_1}, 0, \beta)$	$\pm k_{II}$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, 0, \beta)$
TE_{mn}	$\pm k_I$	$= \pm (\omega, \frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, \beta)$	$\pm k_{III}$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, \beta)$
TM_{mn}	$\pm k_{II}$	$= \pm (\omega, \frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, \beta)$	$\pm k_{IV}$	$= \pm (\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, \beta)$
TE_m^{evan}	k_I	$= (\omega, \frac{m\pi}{\ell_1}, 0, i\alpha)$	k_{III}	$= (-\omega, -\frac{m\pi}{\ell_1}, 0, i\alpha)$
TM_m^{evan}	k_{II}	$= (\omega, -\frac{m\pi}{\ell_1}, 0, i\alpha)$	k_{IV}	$= (-\omega, \frac{m\pi}{\ell_1}, 0, i\alpha)$
TE_{mn}^{evan}	k_I	$= (\omega, \frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, i\alpha)$	k_V	$= (-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, i\alpha)$
TM_{mn}^{evan}	k_{II}	$= (\omega, -\frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, i\alpha)$	k_{VI}	$= (-\omega, \frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, i\alpha)$
	k_{III}	$= (\omega, \frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, i\alpha)$	k_{VII}	$= (-\omega, -\frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, i\alpha)$
	k_{IV}	$= (\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, i\alpha)$	k_{VIII}	$= (-\omega, \frac{m\pi}{\ell_1}, \frac{n\pi}{\ell_2}, i\alpha)$

III.D Bases of the Reciprocal Lattices Λ^d

EM-fields	a_*^1	a_*^2	a_*^3	a_*^4
TE_{mnp}, TM_{mnp}	$(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$	$(0,0, \frac{2n\pi}{\ell_2}, 0)$	$(0,0,0\ \frac{2p\pi}{\ell_3})$
TE_{onp}	$(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$	--	$(0,0, \frac{2n\pi}{\ell_2}, 0)$	$(0,0,0, \frac{2p\pi}{\ell_3})$
TE_{mop}	$(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$	--	$(0,0,0, \frac{2p\pi}{\ell_3})$
TM_{mno}	$(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\frac{p\pi}{\ell_3})$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$	$(0,0, \frac{2n\pi}{\ell_2}, 0)$	--
$TE_{mo} \equiv TE_m$, TM_m	$(-\omega, -\frac{m\pi}{\ell_1}, 0, -\beta)$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$	--	$(0,0,0, 2\pi w)$
TE_{mn}, TM_{mn}	$(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -\beta)$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$	$(0,0, \frac{2n\pi}{\ell_2}), 0)$	$(0,0,0, 2\pi w)$
TE_{on}	$(-\omega, 0, -\frac{n\pi}{\ell_2}, -\beta)$	--	$(0,0, \frac{2n\pi}{\ell_2}, 0)$	$(0,0,0, 2\pi w)$
$TE_{mo}^{evan} \equiv TE_m^{evan}$, TM_m^{evan}	$(-2\omega, 0, 0, 0)$ $(-\omega, \frac{m\pi}{\ell_1}, 0, 0)$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$ $(-\omega, -\frac{m\pi}{\ell_1}, 0,0)$	-- --	$(-\omega, -\frac{m\pi}{\ell_1}, 0, -i\alpha)$ --
TE_{mn}^{evan}, TM_{mn}^{evan}	$(-2\omega, 0,0,0)$ $(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, 0)$	$(0, \frac{2m\pi}{\ell_1}, 0,0)$ $(0, \frac{2m\pi}{\ell_1}, 0,0)$	$(0,0, \frac{2n\pi}{\ell_2}, 0)$ $(0,0, \frac{2n\pi}{\ell_2}, 0)$	$(-\omega, -\frac{m\pi}{\ell_1}, -\frac{n\pi}{\ell_2}, -i\alpha)$ --
TE_{on}^{evan}	$(-2\omega, 0, 0, 0)$ $(-\omega, 0, \frac{n\pi}{\ell_2}, 0)$	-- --	$(0,0, \frac{2n\pi}{\ell_2}, 0)$ $(-\omega, 0, -\frac{n\pi}{\ell_2}, 0)$	$(-\omega, 0, -\frac{n\pi}{\ell_2}, -i\alpha)$ --

In the case of evanescent modes both complex and real bases are indicated.

III.E The Spectral Groups

The spectral groups are here indicated as referred to the Shubnikov frame. R_μ denotes the infinitesimal generator of rotations around an axis in the μ-direction.

EM-fields	Conditions	Generators of the spectral group	Shubnikov group
TE_{mnp}, TM_{mnp}	$\frac{m}{\ell_1}, \frac{n}{\ell_2}, \frac{p}{\ell_3}$ all different	$\{m_x, m_y, m_z, 1'\}$	m m m 1'
	$\frac{m}{\ell_1} = \frac{n}{\ell_2} \neq \frac{p}{\ell_3}$	$\{4_z, m_{x-y}, m_z, 1'\}$	4/m m m 1'
	$\frac{m}{\ell_1} \neq \frac{n}{\ell_2} = \frac{p}{\ell_3}$	$\{4_x, m_{y-z}, m_x, 1'\}$	
	$\frac{m}{\ell_1} = \frac{p}{\ell_3} \neq \frac{n}{\ell_2}$	$\{4_y, m_{z-x}, m_y, 1'\}$	
	$\frac{m}{\ell_1} = \frac{n}{\ell_2} = \frac{p}{\ell_3}$	$\{4_z, \bar{3}, 1'\}$	m 3m 1'
TE_{mop}, TE_{onp}, TM_{mno}	$\frac{m}{\ell_1} \neq \frac{n}{\ell_2}$	$\{m_x, m_y, m_z, 1'\}$	m m m 1'
	$\frac{m}{\ell_1} = \frac{n}{\ell_2}$	$\{4_z, m_{x-y}, m_z, 1'\}$	4/m m m 1'
$TE_{mo} \equiv TE_m$, TM_m		$\{R_{\bar{x}}, m_{\bar{x}}, m_{\bar{y}}, m_{\bar{t}}\}$	∞/m m m 1'
TE_{on}		$\{R_{\bar{y}}, m_{\bar{x}}, m_{\bar{y}}, m_{\bar{t}}\}$	
TE_{mn}, TM_{mn}	$\frac{m}{\ell_1} \neq \frac{n}{\ell_2}$	$\{m_{\bar{x}}, m_{\bar{y}}, m_{\bar{z}}, m_{\bar{t}}\}$	m m m 1'
	$\frac{m}{\ell_1} = \frac{n}{\ell_2}$	$\{4_{\bar{z}}, m_{\bar{x}-\bar{y}}, m_{\bar{z}}, m_{\bar{t}}\}$	4/m m m 1'
$TE_{mo}^{evan} \equiv TE_m^{evan}, TM_m^{evan}, TE_{on}^{evan}$		$\{m_x, m_y, 1'\}$	m m 1'
TE_{mn}^{evan}, TM_{mn}^{evan}	$\frac{m}{\ell_1} \neq \frac{n}{\ell_2}$	$\{m_x, m_y, 1'\}$	m m 1'
	$\frac{m}{\ell_1} = \frac{n}{\ell_2}$	$\{m_x, m_{x-y}, 1'\}$	4 m m 1'

CHAPTER IV

TEM PLANE WAVES IN HOMOGENEOUS AND ISOTROPIC SUBSTANCES

IV.1 Relativistic Symmetry Groups

Uniform fields [8], TEM plane waves in vacuum [7] and TE and TM modes in empty wave-guides and resonant cavities [1] having been analyzed, the next step is to try to extend the treatment to fields in material media and the easiest case then to start with is the case of homogeneous, isotropic substances characterized by constant dielectric and magnetic permeabilities ε and μ, which imply the constitutive relations

$$\vec{D} = \varepsilon\vec{E} \quad \text{and} \quad \vec{B} = \mu\vec{H} \ . \tag{4.1}$$

To a given TEM plane wave in vacuum there is one uniquely determined relativistic symmetry group G: the relativistic symmetry group of the corresponding EM-field tensor. In polarizable matter, however, this is no more the case, because one has now to distinguish between the six different field variables $\vec{E}$, $\vec{B}$, $\vec{D}$, $\vec{H}$, $\vec{P}$ and $\vec{M}$ instead of the original two $\vec{E}$ and $\vec{B}$. Of these six, four fundamental ones are chosen to characterize a formulation as already mentioned in chapter II. One can still consider the relativistic symmetry group of at least the fundamental fields, because for a given formulation their transformation properties with respect to elements of the Poincaré group are given (see chapter II). For the non-fundamental or redundant fields, covariant transformation laws do not exist in all formulations. The expressions for the transformed redundant fields have to be derived from the corresponding expressions of fundamental fields by means of the relations (2.26). Two new features appear:

(i) For a given TEM wave there are different relativistic symmetry groups even in homogeneous, isotropic media characterized by ε and μ as above: the symmetry groups of the various fields.

(ii) For a given field the relativistic symmetry group depends on the formulation chosen, i.e. on the macroscopic description adopted.

Therefore several groups are now attached to a TEM wave in a way only determined after the choice of a (macroscopic) formulation. The first feature stems from the fact, that even in homogeneous and isotropic substances ε and μ do not behave as scalars with respect to non-trivial relativistic transformations (as it is the case with respect to the trivial ones). The second feature is due to the different definitions of the various fields in different formulations and accordingly to the different transformation laws, as already discussed in chapter II. There are three main results:

(i) In all the formulations there is a same set of symmetry elements which generate the symmetry groups of the various fields. The choice of a formulation implies how all these elements are to be assigned as generators to the symmetry groups of the various fields (the redundant ones included).

(ii) The symmetry groups depend parametrically on the refractive index only (and not on ε and μ seperately).

(iii) In the limit $n \to 1$, the symmetry group of a particular field (say $(\vec{E},\vec{B})$ e.g.) is not that of a TEM wave in vacuum. Part of the symmetry elements of the TEM wave in the vacuum case are distributed among the symmetry groups (for $n = 1$) of the various fields, in a way, which depends on the formulation. The others do not occur at all.

In this chapter the relativistic symmetry groups of monochromatic TEM waves, linearly, elliptically and circularly polarized, are determined for the four formulations indicated in chapter II, namely the Minkowski, Chu, Ampère and Boffi formulations. Different formulations (having different naturally associated microscopic models) were considered in order to get an indication if, and in what sense, the relativistic symmetries of the EM-fields treated depend on the microscopic structure.

First the EM-fields are found as plane wave solutions of Maxwell's equations, according to the methods given in section V.2 (and specialized to the isotropic case). In all cases the wave vector of the EM-fields is seen to be

$$k_\nu = \omega(-1, 0, 0, n) , \qquad (4.2)$$

with ω the frequency times 2π and $n = \sqrt{\varepsilon\mu}$ the index of refraction. The explicit expressions for the various EM-fields are given in appendix A. Then the relativistic symmetry groups themselves are determined in the way outlined in section III.1. The resulting groups are compared to the relativistic symmetry groups of TEM waves in vacuum by taking $\lim n \to 1$. One remarkable fact is that in this limit a proper subgroup only of the corresponding vacuum case symmetry group is obtained. Take for example the point group of the Minkowski $(\vec{E}_M, \vec{B}_M)$ field tensor. The Shubnikov subgroup is given, for the case of linear polarisation, by (see table IV.2):

$$K_S(F_M^{\mu\nu}) = \{2'_y, m_y, R_z(\pi)\} \quad . \qquad (4.3)$$

Here nothing is changed by taking $\lim n \to 1$. The relativistic or non-trivial generators of $K(F_M^{\mu\nu})$ are M_n and $\tilde{L}_y(\phi,n)$. For these generators things are changed by taking $\lim n \to 1$. First let us consider the Lorentz mirror M_n, which has been introduced in chapter III. In the limit $n = 1$ this Lorentz mirror does not turn into a symmetry element of the corresponding plane wave in vacuum, because it becomes singular and is thus lost. As for $\tilde{L}_y(\phi,n)$ a somewhat closer analysis is necessary:

$$\tilde{L}_y = \begin{pmatrix} \frac{n^2 \mathrm{Ch}\phi - 1}{n^2-1} & 0 & \frac{n\mathrm{Sh}\phi}{\sqrt{n^2-1}} & \frac{n(1-\mathrm{Ch}\phi)}{n^2-1} \\ 0 & 1 & 0 & 0 \\ \frac{n\mathrm{Sh}\phi}{\sqrt{n^2-1}} & 0 & \mathrm{Ch}\phi & \frac{-\mathrm{Sh}\phi}{\sqrt{n^2-1}} \\ \frac{n(\mathrm{Ch}\phi-1)}{n^2-1} & 0 & \frac{\mathrm{Sh}\phi}{\sqrt{n^2-1}} & \frac{n^2-\mathrm{Ch}\phi}{n^2-1} \end{pmatrix} = \tilde{L}_y(\phi,n) \; . \qquad (4.4)$$

Comparing (4.4) with (5.13) and (5.33) in reference 8) one sees that $\tilde{L}_y(\phi,n) = L(-\phi, a)$. Therefore (4.4) can be parametrized differently. By means of the relations

$$\begin{aligned} \mathrm{Ch}\phi &= 1 + \frac{\sigma^2}{2}(n^2-1) \\ \sigma &= \frac{2}{\sqrt{n^2-1}} \mathrm{Sh}\frac{\phi}{2} \end{aligned} \qquad (4.5)$$

$\tilde{L}_y$ can be put in the form

$$L(\sigma,n) = \begin{pmatrix} 1+\frac{n^2\sigma^2}{2} & 0 & +\frac{n\sigma}{2}\sqrt{I} & -\frac{n\sigma^2}{2} \\ 0 & 1 & 0 & 0 \\ +\frac{n\sigma}{2}\sqrt{I} & 0 & 1+\frac{\sigma^2}{2}(n^2-1) & -\frac{\sigma}{2}\sqrt{I} \\ \frac{n\sigma^2}{2} & 0 & \frac{\sigma}{2}\sqrt{I} & 1-\frac{\sigma^2}{2} \end{pmatrix} , \qquad (4.6)$$

with $I = \mathrm{trace}\, L(\sigma,n) = 4 + \sigma^2 (n^2-1)$. Now the limit $n = 1$ can easily be taken:

$$\lim_{n \to 1} L(\sigma,n) = L(\sigma,1) \equiv L(\sigma) \; . \tag{4.7}$$

Therefore (for linear polarization)

$$\lim_{n\to 1} G(F^{\mu\nu}_M) = \{U,(\tfrac{\lambda}{2}\, e_3, 2'_y),(0,m_y),(\tfrac{\lambda}{2}\, e_3, R_z(\pi)),(0,L(\sigma))\,|\,\forall\sigma\in\mathbb{R}\} \; . \tag{4.8}$$

This is easily seen to be a subgroup of the relativistic group of a linearly polarized TEM wave in vacuum:

$$G = \{U,(\tfrac{\lambda}{2}\, e_3, 2'_y),(0,m_y),(\tfrac{\lambda}{2}\, e_3, R_z(\pi)),(0,L(\sigma)),(0,\bar{L}(\rho))\,|\,\forall\sigma,\rho\in\mathbb{R}\} \tag{4.9}$$

(see reference 7). In the above formulae U indicates the subgroup of primitive translations (see section IV.4). For $\tilde{L}_x(\chi,n)$ (see (4.36)) lim $n \to 1$ leads to an analogous result in a similar way:

$$\lim_{n\to 1} \tilde{L}_x(\chi,n) = \bar{L}(\rho) \; . \tag{4.10}$$

IV.2 Determination of the Relativistic Symmetry Elements in the Chu Formulation: a Working Example.

The Chu variables partly transform in the usual form as second rank tensor as in the case of $F^{\mu\nu}_C = (\vec{E}_C,\vec{H}_C)$ and partly as three- or four-vectors as in the case of P^{ν}_C or $\vec{P}_C$. As far as the determination of the relativistic symmetry group of the tensor $F^{\mu\nu}_C$ is concerned, nothing new comes into play as compared with the procedure given in chapter III. However for the three- and four-vectors of electric (and of magnetic) polarization some remarks might be quite useful. Let us first consider the four-vectors P^{ν}_C and M^{ν}_C introduced in chapter II. In the laboratory frame, which is the rest frame of the material medium one has:

$$ {}^{o}P^{\nu}_{C} = (0, {}^{o}\vec{P}_{C}) = (0, {}^{o}P_x, {}^{o}P_y, 0) $$
$$ {}^{o}M^{\nu}_{C} = (0, {}^{o}\vec{M}_{C}) = (0, {}^{o}M_x, {}^{o}M_y, 0) \qquad (4.11) $$

and in the electromagnetic rest frame, which is connected with the laboratory frame by a special Lorentz transformation with velocity $v = \frac{1}{n}$ in z-direction these vectors become:

$$ P^{\nu}_{C} = \frac{1}{\gamma_{\vec{v}}} (-\gamma^2_{\vec{v}} (\vec{P}_C.\vec{v}), \vec{P}_C + \gamma^2_{\vec{v}} (\vec{v}\vec{P}_C).\vec{v}) = (0, {}^{o}P_x, {}^{o}P_y, 0) $$
$$ M^{\nu}_{C} = \frac{1}{\gamma_{\vec{v}}} (-\gamma^2_{\vec{v}} (\vec{M}_C.\vec{v}), \vec{M}_C + \gamma^2_{\vec{v}} (\vec{v}\vec{M}_C).\vec{v}) = (0, {}^{o}M_x, {}^{o}M_y, 0) \; . \qquad (4.12) $$

Here use has been made of the transformation laws

$$ \tilde{P}^{\mu}_{C}(x) = L^{\mu}_{\nu} P^{\nu}_{C}(L^{-1}x) \quad \text{and} \quad \tilde{M}^{\mu}_{C}(x) = (\det L) L^{\mu}_{\nu} M^{\nu}_{C}(L^{-1}x) \; . $$

For Fourier coefficients and in matrix notation the corresponding symmetry conditions can then be written as

$$ L\hat{P}(k) = \hat{P}(Lk)\; e^{i(Lk).u(L)} $$
and
$$ (\det L)\; L\hat{M}(k) = \hat{M}(Lk)\; e^{i(Lk).u(L)} \; . \qquad (4.13) $$

Compare with (3.8). From table IV.5 one sees that the relevant Fourier coefficients in the electromagnetic rest frame are:

$$ \bar{P}(\pm\bar{k}) = \begin{pmatrix} 0 \\ \bar{P}_x(\pm\bar{k}) \\ \bar{P}_y(\pm\bar{k}) \\ 0 \end{pmatrix} \quad \text{with} \quad \begin{aligned} \bar{P}_x(+\bar{k}) &= \bar{P}_x(-\bar{k}) = \frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\, E_o \\ \bar{P}_y(+\bar{k}) &= -\bar{P}_y(-\bar{k}) = -i\, \frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\, E_1 \end{aligned} $$

$$ \bar{M}(\pm\bar{k}) = \begin{pmatrix} 0 \\ \bar{M}_x(\pm\bar{k}) \\ \bar{M}_y(\pm\bar{k}) \\ 0 \end{pmatrix} \quad \text{with} \quad \begin{aligned} \bar{M}_x(+\bar{k}) &= -\bar{M}_x(-\bar{k}) = i\sqrt{\frac{\varepsilon}{\mu}}\, \frac{(\mu-1)n}{\sqrt{n^2-1}} \\ \bar{M}_y(+\bar{k}) &= \bar{M}_y(-\bar{k}) = \sqrt{\frac{\varepsilon}{\mu}}\, \frac{(\mu-1)n}{\sqrt{n^2-1}} \end{aligned} \; . \qquad (4.14) $$

Using the above symmetry conditions in conjunction with the spectral group given in appendix A, the general element of the relativistic point group as referred to the laboratory system is found to be

(i) for the four-vector P^{ν}_{C}:

$$L = \tilde{L}_y(\phi,n)\, M_n^{\eta}\, R_z(\theta)(2'_y)^{\delta}(m_y)^{\varepsilon} \equiv L(\phi,\eta,\theta,\delta,\varepsilon) \tag{4.15}$$

with associated non-primitive translation $u(L)=(\delta \pm \frac{\theta}{\pi})\,\frac{\lambda}{2}\, e_3$.

(ii) for the four vector M^{ν}_{C}:

$$L = \tilde{L}_x(\chi,u)\, M_n^{\eta}\, R_z(\theta)(2'_y)^{\delta}(m_y)^{\varepsilon} \equiv \bar{L}(\chi,\eta,\theta,\delta,\varepsilon) \tag{4.16}$$

with associated non-primitive translation $u(L)=(\delta+\eta \pm \frac{\theta}{\pi})\,\frac{\lambda}{2}\, e_3$.

The point groups for the linearly, elliptically and circularly polarized plane waves are found by taking the following values for the parameters occurring in (4.15) and (4.16):

$$\begin{aligned} &\text{LPW}: \quad \phi,\chi \in \mathcal{R};\ \theta = 0,\pi;\ \eta = 0,\ 1;\ \delta = 0,\ 1;\ \varepsilon = 0,\ 1 \\ &\text{CPW}: \quad \phi,\chi = 0;\ 0 \leq \theta < 2\pi;\ \eta = 0,1;\ \delta = 0,\ 1;\ \varepsilon = 0 \\ &\text{EPW}: \quad \phi,\chi = 0;\ \theta = 0,\ \pi;\ \eta = 0,\ 1;\ \delta = 0,\ 1;\ \varepsilon = 0. \end{aligned} \tag{4.17}$$

As for the three-vectors $\vec{P}_C$ and $\vec{M}_C$ the transformation laws were given already in chapter II (for boosts):

$$\begin{aligned} \tilde{\vec{P}}_C &= \vec{P}_{C_{\parallel}} + \gamma_{\vec{v}}\, \vec{P}_{C_{\perp}} + \gamma_{\vec{v}}\, (\vec{v} \times (\vec{P}_C \times \vec{u})) \\ \tilde{\vec{M}}_C &= \vec{M}_{C_{\parallel}} + \gamma_{\vec{v}}\, \vec{M}_{C_{\perp}} + \gamma_{\vec{v}}\, (\vec{v} \times (\vec{M}_C \times \vec{u})) \ , \end{aligned} \tag{2.35}$$

where $\vec{v}$ is the velocity of the boost and $\vec{u}$ the velocity of the material medium in the original frame of reference. For the determination of the point groups we consider the two cases $\vec{v} = 0$ (Shubnikov) and $\vec{v} \neq 0$ (non-trivial). In the first case $\vec{P}_C$ (and $\vec{M}_C$) transforms the same way in any frame of reference, namely as a polar (axial) vector:

$$\tilde{\vec{P}}_C(x) = \alpha\vec{P}_C(\alpha^{-1}x)$$

and (4.18)

$$\tilde{\vec{M}}_C(x) = \det(\alpha)\ \dot{\alpha}\ \vec{M}_C(\alpha^{-1}x)\ ,$$

where $\alpha \in O(3)$ and $\dot{\alpha} = \begin{cases} -\alpha & \text{if time reversal is applied also} \\ +\alpha & \text{if time reversal is not applied.} \end{cases}$

For the second case one has to examine the term

$$\vec{v} \times (\vec{P}_C \times \vec{u}) = \vec{P}_C(\vec{u}.\vec{v}) - \vec{u}(\vec{v}.\vec{P}_C)\ .$$

One knows that in the rest frame of the wave $\vec{u} = (0,0,-\frac{1}{n})$ and from the spectral group one knows that only $L_x(\chi)$ and $L_y(\phi)$ (see section IV.4 and appendix A) can appear as Lorentz boosts, therefore $\vec{u}.\vec{v} = 0$ in any case. Remains the term $\vec{u}(\vec{v}.\vec{P}_C)$. Since in both frames of reference $\vec{P}_C$ has zero z-component the term $\vec{u}(\vec{v}.\vec{P}_C)$ must vanish for symmetry and therefore $\vec{v} \perp \vec{P}_C$. But in this case $L(\vec{v})$ still cannot be a symmetry, because $L(\vec{v})$ leaves $\vec{P}_{C\parallel}$ invariant and not $\vec{P}_{C\perp}$ (first part of transformation law (2.35)). Therefore one concludes that both $\tilde{L}_x(\chi,n)$ and $\tilde{L}_y(\phi,n)$ do not occur as elements of the point group in the laboratory frame. The same is true for $\vec{M}_C$. The above symmetry conditions in conjunction with the spectral group (appendix A) lead to the following general element of the point group:

$$L = M_n^\eta\ R_z(\theta)(2_y')^\delta(m_y)^\varepsilon \equiv L(\eta,\theta,\delta,\varepsilon)\ . \qquad (4.19)$$

For $\vec{P}_C$ the associated non-primitive translation is

$$u(L) = (\delta \pm \frac{\theta}{\pi}) \frac{\lambda}{2} e_3 , \quad \text{for } \vec{M}_C \text{ it is} \qquad (4.20)$$
$$u(L) = (\delta + \eta \pm \frac{\theta}{\pi}) \frac{\lambda}{2} e_3 .$$

The two groups only differ with respect to the non-primitive translation associated with the generator M_n. A similar argument can be used to discuss the redundant fields $\vec{D}_C$ and $\vec{B}_C$. The transformation laws were given as

$$\tilde{\vec{D}}_C = \tilde{\vec{P}}_C + \tilde{\vec{E}}_C = \vec{D}_{C_{\parallel}} + \gamma_{\vec{v}} \{\vec{D}_{C_{\perp}} - \vec{v} \times [\vec{H}_C - (\vec{P}_C \times \vec{u})]\} \qquad (2.36)$$
$$\tilde{\vec{B}}_C = \tilde{\vec{M}}_C + \tilde{\vec{H}}_C = \vec{B}_{C_{\parallel}} + \gamma_{\vec{v}} \{\vec{B}_{C_{\perp}} + \vec{v} \times [\vec{E}_C + (\vec{M}_C \times \vec{u})]\} \quad .$$

One can show that Lorentz boosts as symmetry elements can appear in the case of linear polarization (for the expressions of the fields see appendix A, table IV.5).
Consider for example $\vec{D}_C$:
here $L_x(\chi)$ is a symmetry, because $\vec{D}_{C_{\parallel}}$ is equal to $\vec{D}_C$ and

$$\vec{v} \times \vec{H}_C = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ H \\ 0 \end{pmatrix} = e_3(vH) = +e_3(vE_o \frac{\varepsilon-1}{\sqrt{n^2-1}}),$$

whereas

$$-\vec{v} \times (\vec{P}_C \times \vec{u}) = -\vec{P}_C\underbrace{(\vec{v}.\vec{u})}_{0} + \vec{u}(\vec{v}.\vec{P}_C) = -e_3(\frac{1}{n} v \frac{n(\varepsilon-1)}{\sqrt{n^2-1}}).$$

$L_y(\phi)$ does not occur as a symmetry. In the same way it can be shown that $L_y(\phi)$ is a symmetry of $\vec{B}_C$ and $L_x(\chi)$ is not. Therefore the relativistic symmetry group of $\vec{D}_C$ is identical with $G(G_M^{\mu\nu})$ and the relativistic symmetry group of $\vec{B}_C$ is identical with $G(F_M^{\mu\nu})$. Correspondingly the same arguments apply for the three- and the four-vectors in the Ampère formulation.

IV.3 The Non-Relativistic Approximation and the Galilean Symmetry Groups.

In section IV.1 the relativistic symmetry groups were also discussed in the limit $n \to 1$ and then compared with the relativistic symmetry groups of the vacuum case. Another very important special case arises for $n \gg 1$, because then the propagation velocity of the TEM wave is much smaller than the speed of light in vacuum,

$$\frac{1}{n} = v \ll 1 \; . \tag{4.21}$$

So indeed the transformation $T(n)$ as well as the relativistic symmetry groups can be considered in their non-relativistic approximation and even in their non-relativistic limit. The transition from the non-relativistic approximation to the non-relativistic limit is made by replacing the Lorentz transformations of the non-relativistic approximation by corresponding Galilean transformations. On the other hand, in the case $n \gg 1$, it is physically meaningful to consider the Galilean symmetry group of TEM waves. Again the symmetry groups depend only on n, but now for any positive value of the refractive index n the Galilean symmetry groups are isomorphic. Therefore, the Galilei symmetry groups of a TEM wave one formally finds in the vacuum case and in a medium for $n = 1$ are respectively the same. This is so, because there is no critical velocity in the Galilei framework. As we know, the relativistic symmetry groups of a TEM wave in a medium for $n = 1$ are not the same as the one found for the corresponding vacuum case, because there, $c = 1$ is a critical velocity. Apart from giving a check on the results of this chapter, because the non-relativistic limit of any relativistic symmetry group for a medium with $n > 1$ must be contained in the corresponding Galilean symmetry group, the considerations of the Galilean symmetry groups are very interesting, because there is a relation between the relativistic symmetry groups and the Galilei symmetry groups of the field tensors $F_{\mu\nu}$ and $G^{\mu\nu}$ in the Minkowski formulation. It was stressed in chapter II that in order to get formally identical symmetry conditions (and also formally identical transformation laws) in the Poincaré and in the Galilei case, the one field tensor had to be used in its covariant form and the other in its contravariant form, i.e. $F^{M}_{\mu\nu} = (-\vec{E}_M, \vec{B}_M)$ and $G^{\mu\nu}_M = (\vec{D}_M, \vec{H}_M)$. This is also the basis of the general linear constitutive relation (2.17). In vacuum $\vec{E}_M = \vec{D}_M = \vec{E}$ and $\vec{B}_M = \vec{H}_M = \vec{B}$, thus $G^{\mu\nu} = F^{\mu\nu} = (\vec{E}, \vec{B})$. The point is now, that the Galilei symmetry group of $(-\vec{E}, \vec{D}) = F_{\mu\nu}$ and of $(\vec{E}, \vec{B}) = F^{\mu\nu}$ are different (even for general fields $\vec{E}$ and $\vec{B}$):

$$G_{Gal}(F^{\mu\nu}) \neq G_{Gal}(F_{\mu\nu}),$$

because the relativistic metric is not invariant under Galilei transformations and therefore $F^{\mu\nu}$ and $F_{\mu\nu}$ are two different fields from the Galilei point of view. The remarkable result is that these groups are precisely those obtained for a medium with $n > 1$ in the non-relativistic limit for the corresponding $F^{M}_{\mu\nu}$ and $G^{\mu\nu}_M$ fields. That this result is non-trivial can be seen from the fact that for $n = 1$ and in the vacuum case the non-relativistic limit of the relativistic groups is, in general, different from the corresponding Galilei symmetry groups. For details see Appendix B.

Let us now treat all this in some more details. First the non-relativistic approximation is considered. In this approximation one has

$$T(n) = \begin{pmatrix} \frac{n}{\sqrt{n^2-1}} & 0 & 0 & \frac{-1}{\sqrt{n^2-1}} \\ & 1 & 0 & \\ & 0 & 1 & \\ \frac{-1}{\sqrt{n^2-1}} & 0 & 0 & \frac{n}{\sqrt{n^2-1}} \end{pmatrix} \xrightarrow{n>>1} T(n) \tilde{=} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{1}{n} & 0 & 0 & 1 \end{pmatrix} \quad (4.22)$$

and

$$M_n = \begin{pmatrix} -\frac{n^2+1}{n^2-1} & 0 & 0 & \frac{2n}{n^2-1} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{2n}{n^2-1} & 0 & 0 & \frac{n^2+1}{n^2-1} \end{pmatrix} \xrightarrow{n>>1} M_n \simeq \begin{pmatrix} -1 & 0 & 0 & \frac{2}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{2}{n} & 0 & 0 & 1 \end{pmatrix} \qquad (4.23)$$

or by means of conjugation

$$M_n = T^{-1}(n) 1' T(n) \simeq \begin{pmatrix} 1 & 0 & 0 & \frac{1}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ \frac{1}{n} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{1}{n} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & \frac{2}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{2}{n} & 0 & 0 & 1 \end{pmatrix},$$

where terms $\sim \frac{1}{n^2} = v^2$ have been neglected. Similarly

$$\begin{aligned} \tilde{L}_x(\chi, n) &= T^{-1}(n)\, L_x(\chi)\, T(n) \\ \tilde{L}_y(\phi, n) &= T^{-1}(n)\, L_y(\phi)\, T(n) \quad . \end{aligned} \qquad (4.24)$$

Now the transition to the non-relativistic limit is made by introducing the corresponding Galilean transformations:

$$T(n) \simeq \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{1}{n} & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\text{limit}]{\text{non-relativistic}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{n} & 0 & 0 & 1 \end{pmatrix} = L_z(-\tfrac{1}{n})$$

(4.25)

with $\vec{v} = (0, 0, -\frac{1}{n})$ (see (2.7)). Further one has

$$M_n \cong \begin{pmatrix} -1 & 0 & 0 & \frac{2}{n} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{2}{n} & 0 & 0 & 1 \end{pmatrix} \xrightarrow[]{\text{non-relativistic limit}} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2}{n} & 0 & 0 & 1 \end{pmatrix} = M(n)$$

and

$$\tilde{L}_x(\chi, n) \xrightarrow{\text{non-relativistic limit}} L_x(v)$$

with $\vec{v} = (\text{Th}\ \chi\ ,\ 0,\ 0),$

$$\tilde{L}_y(\phi, n) \xrightarrow{\text{non-relativistic limit}} L_y(w)$$

with $\vec{w} = (0,\ \text{Th}\ \phi,\ 0),$

because in this last case for example

$$\tilde{L}_y(\phi, n) \to L_z^{-1} L_y(w) L_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{n} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ v & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{n} & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ v & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = L_y(w).$$

The Shubnikov subgroups for the non-relativistic approximation as well as for the non-relativistic limit are the same as in the relativistic case, since these symmetry elements do not depend on n at all.
Next the Galilei symmetry groups of TEM waves in the Minkowski formulation are determined in the usual way. The spectrum is $\mathcal{S} = \{\pm k_\nu\}$, with $k_\nu = \omega(-1,\ 0,\ 0,\ n)$ and the spectral group S is determined from

$$(L^{-1})^\mu_\nu\ k_\mu = \pm\ k_\nu \quad . \tag{4.26}$$

In the wave rest frame the k_ν vector becomes

$$k L_z^{-1}(-\tfrac{1}{n}) = \omega(-1,\ 0,\ 0,\ n) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{n} & 0 & 0 & 1 \end{pmatrix} = \omega(0,\ 0,\ 0,\ n) \tag{4.27}$$

and therefore the spectral group in the wave rest frame is

$$\bar{S} = \{2'_y, m_y, R_z(\theta), 1', L_x(v), L_y(w)\ 0 \leq \theta < 2\pi, v, w \in \mathcal{R}\} \ . \quad (4.28)$$

In the laboratory frame the spectral group is given by

$$S = \{2'_y, m_y, R_z(\theta), M_n, L_x(v), L_y(w) \mid 0 \leq \theta < 2\pi, v, w \in \mathcal{R}\} \ . \quad (4.29)$$

On the basis of the symmetry conditions (compare to (2.6) and (2.13))

$$\dot{L}^{\star} \hat{F}(k) = \hat{F}(Lk)L\, e^{i\varphi} \quad \text{for covariant field tensors} \quad (4.30)$$

and

$$\dot{L}\ \hat{F}(k) = \hat{F}(Lk)L^{\star}\, e^{i\varphi} \quad \text{for contravariant field tensors} \quad (4.31)$$

the Galilei symmetry groups are calculated. By means of the tables IV.2 and IV.4 in section IV.4 one sees that (in the Minkowski formulation)

$$G(F^{M}_{\mu\nu})\big|_{n.r.limit} = G_{Gal}(F_{\mu\nu})$$

$$G(G^{\mu\nu}_{M})\big|_{n.r.limit} = G_{Gal}(G^{\mu\nu}) \quad (4.32)$$

Since for any value of the refractive index n the corresponding Galilei groups are isomorphic, the case n = 1 can formally be considered also, as has already been mentioned. The relation between the relativistic symmetry groups in the lim $n \to 1$, the relativistic symmetry groups in vacuum and the Galilean symmetry groups, is made clear by the following diagram (only non-trivial generators are discussed, since the Shubnikov subgroups G_S are always the same):

Matter Relativistic		Matter Relativistic lim n → 1				Vacuum Relativistic n = 1	
$G(F_{\mu\nu})$	$G(G^{\mu\nu})$ $\Longrightarrow$	$G(F_{\mu\nu})$	$G(G^{\mu\nu})$			$G(F_{\mu\nu})=G(F^{\mu\nu})$	
-	$\tilde{L}_x(\chi,n)$	-	$\bar{L}(\rho)$			$\bar{L}(\rho)$	
$\tilde{L}_y(\phi,n)$	-	$L(\sigma)$	-			$L(\sigma)$	
M_n	M_n	-	-			-	
$\Downarrow$							
Matter Non-Rel.Approx. n>>1		Matter Non-Rel.limit		Matter Galilei		Vacuum Galilei n = 1	
$G(F_{\mu\nu})$	$G(G^{\mu\nu})$ $\Longrightarrow$	$G(F_{\mu\nu})$	$G(G^{\mu\nu})$	$G_{Gal}(F_{\mu\nu})$	$G_{Gal}(G^{\mu\nu})$ $\Rightarrow$	$G_{Gal}(F_{\mu\nu})$	$G_{Gal}(F^{\mu\nu})$
-	$\tilde{L}_x(\chi,n)$	-	$L_x(v)$	-	$L_x(v)$	-	$L_x(v)$
$\tilde{L}_y(\phi,n)$	-	$L_y(w)$	-	$L_y(w)$	-	$L_y(w)$	-
M_n	M_n	$M(n)$	$M(n)$	$M(n)$	$M(n)$	$M(1)$	$M(1)$

The fields are here considered in the Minkowski formulation.

This diagram shows that in matter one has only those relativistic symmetries which have an image in the corresponding Galilei symmetry group. The ones which do not have a corresponding Galilean symmetry transformation do not occur as relativistic symmetries in the presence of matter. One might say, loosely speaking, that the difference in the relativistic symmetries of the $(-E_M, B_M)$ and (D_M, H_M) fields is due to the presence of matter and that for an interpretation of this fact the corresponding Galilean symmetry groups should be kept in mind. Furthermore restricting attention to Shubnikov symmetry elements only (as is usually done) stimulating perspectives are lost.

Note that
$$G(F^M_{\mu\nu})|_{n=1} \cup G(G^{\mu\nu}_M)|_{n=1} = G(F_{\mu\nu}) \text{ and}$$
$$G(F^M_{\mu\nu})|_{n=1} \cap G(G^{\mu\nu}_M)|_{n=1} = G_S(F_{\mu\nu}) \tag{4.33}$$
(in the case of linear polarization).

IV.4 Survey of the Results

As has already been mentioned in section IV.1 the same generating symmetry elements appear in different ways for the four different formulations (see table IV.2). However, the consequences of these symmetries may be different for different formulations. There are now two possibilities:

(i) these symmetries have different physical consequences for different formulations in such a way that experiments could decide which formulation is the best one for a certain class of problems, or

(ii) the different physical consequences for the different formulations are equivalent and therefore the question which formulation is better than the other ones cannot be settled. At least not with the help of such symmetry considerations.

The fundamental question as to which one of the two possibilities is true, is not answered here. Clearly symmetry also depends very much upon how a problem is looked at, in what approximation it is discussed etc. as was already mentioned in the introduction. Furthermore the symmetry considerations presented here are not the only ones possible. An example may make this point more clear: take the vector of electric polarization in the Chu formulation. In table IV.2 the generating symmetry elements are listed for the three-vector of electric polarization and for the corresponding four-vector. One notices that the two corresponding space-time symmetry groups are different, because one of them contains $\tilde{L}_y(\phi,n)$ as an element and the other one does not. Similarly the three-vector and the four-vector of magnetic polarization can be considered. However, these new symmetry groups of the Chu four-vectors are not new in the sense that they contain new generating elements. They are only new in the sense that old symmetry elements are combined differently to generate a new group. Another point is, that covariance is not a natural thing in certain formulations, because for redundant variables such as $\vec{D}$, $\vec{B}$ and $\vec{H}$ in the Chu and Ampère formulation there is no covariant transformation law. The transformation of these variables can of course be derived from the transformation of two fundamental variables, but one of them transforms as part of a tensor, the other one as vector (see (2.38)). In this connection it is interesting to note the remark of Suttorp [6] that for a macroscopic theory of electromagnetism "the principle of form invariance represents an extra requirement, which actually may not be imposed on the theory". Also one might infer that only for the redundant variables there are cases where there is no covariant transformation law, and that according to Penfield and Haus redundant variables can be avoided: in every formulation there are only four variables, the ones chosen to be fundamental for that formulation. But in a sense table IV.2 requires the redundant variables. For example in the Boffi formulation restriction to the fundamental variables would mean the loss of a non-trivial generating element of the relativistic symmetry group, a symmetry element appearing in all the other formulations. Clearly this is not a proof and further research is needed. But by the present investigation new symmetry considerations have been introduced, and it already appears, that they can play an important rôle in the questions presented by macroscopic electrodynamics.

Coming to the results themselves, the symmetry groups of the EM-fields considered are indicated by giving the generating elements. All symmetry groups occurring are non-symmorphic. Since all fields treated in this chapter have the same spectrum, they also all have the same group U of primitive translations.

Table IV.1: Generators of the Group U of Primitive Translations

a_1	$c_2\delta t_2$	$c_3\delta t_3$	$c_4\delta t_4$	
$(0,0,0,\lambda)$	$(0,1,0,0)\delta t_2$	$(0,0,1,0)\delta t_3$	$(1,0,0,\frac{1}{n})\delta t_4$	(4.34)

In the following tables the generating symmetry elements are listed and the EM-fields of which they are symmetries are given. The first table deals with the case of general refractive index $n \neq 1$, the second one with the special case $n = 1$, both for the relativistic symmetry elements. Table IV.4 gives the Galilean symmetry elements (for the Minkowski formulation only). The symmetry groups for the linearly polarized wave, the circularly polarized wave and the elliptically polarized wave are found by choosing the following values for the parameters of the generators:

$$
\begin{aligned}
&\text{LPW:}\quad \chi,\phi,\sigma,\rho,v \in \mathbb{R},\quad \theta=\pi,\quad \eta=1,\quad \delta=1,\quad \varepsilon=1.\\
&\text{CPW:}\quad \chi=\phi=\sigma=\rho=v=0,\quad 0\leq\theta<2\pi,\quad \eta=1,\quad \delta=1,\quad \varepsilon=0.\\
&\text{EPW:}\quad \chi=\phi=\sigma=\rho=v=0,\quad \theta=\pi,\quad \eta=1,\quad \delta=1,\quad \varepsilon=0.
\end{aligned}
\tag{4.35}
$$

As for the notation of the generating elements:

$$\tilde{L}_x(\chi,n) = T^{-1}(n)L_x(\chi)T(n) \quad \text{and} \quad \tilde{L}_y(\phi,n) = T^{-1}(n)L_y(\phi)T(n)$$

with

$$
T(n)=\begin{pmatrix} \frac{n}{\sqrt{n^2-1}} & 0 & 0 & \frac{-1}{\sqrt{n^2-1}} \\ & 1 & 0 & \\ & 0 & 1 & \\ \frac{-1}{\sqrt{n^2-1}} & & & \frac{n}{\sqrt{n^2-1}} \end{pmatrix},\quad L_x(\chi)=\begin{pmatrix} \mathrm{Ch}\chi & \mathrm{Sh}\chi & 0 & 0 \\ \mathrm{Sh}\chi & \mathrm{Ch}\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and}
$$

$$
L_y(\phi) = \begin{pmatrix} \mathrm{Ch}\phi & 0 & \mathrm{Sh}\phi & 0 \\ 0 & 1 & 0 & 0 \\ \mathrm{Sh}\phi & 0 & \mathrm{Ch}\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.36}
$$

further

$$M_n = \begin{pmatrix} -\frac{n^2+1}{n^2-1} & 0 & 0 & \frac{2n}{n^2-1} \\ & 1 & 0 & \\ & 0 & 1 & \\ -\frac{2n}{n^2-1} & & & \frac{n^2+1}{n^2-1} \end{pmatrix}, \quad \bar{L}(\rho) = \begin{pmatrix} 1+\frac{\rho^2}{2} & \rho & 0 & -\frac{\rho^2}{2} \\ \rho & 1 & 0 & -\rho \\ 0 & 0 & 1 & 0 \\ \frac{\rho^2}{2} & \rho & 0 & 1-\frac{\rho^2}{2} \end{pmatrix} \quad \text{and}$$

$$L(\sigma) = \begin{pmatrix} 1+\frac{\sigma^2}{2} & 0 & \sigma & -\frac{\sigma^2}{2} \\ 0 & 1 & 0 & 0 \\ \sigma & 0 & 1 & -\sigma \\ \frac{\sigma^2}{2} & 0 & \sigma & 1-\frac{\sigma^2}{2} \end{pmatrix} . \tag{4.37}$$

Finally the Galilean transformations are

$$L_x(v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L_y(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ w & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{4.38}$$

Note that the above definitions differ from those originally given in ref. 7) by a cyclic permutation of the axes 1,2,3, as a consequence of the different orientation of the wave vector of the TEM wave.

Fields	Shubnikov Transformations			Non-Trivial Transformations				Transl.	Formulation
	$(\frac{\lambda}{2} e_3, 2'_y)^\delta$	$(0, m_y)^\epsilon$	$(\pm\frac{\lambda}{2}\frac{\theta}{\pi} e_3, R_z(\theta))$	$(0, \tilde{L}_x(\chi, n))$	$(0, \tilde{L}_y(\phi, n))$	$(0, M_n)^\eta$	$(\frac{\lambda}{2} e_3, M_n)^\eta$	U	
$(-\vec{E}_M, \vec{B}_M)$	x	x	x		x		x	x	Minkowski
$(\vec{D}_M, \vec{H}_M)$	x	x	x	x		x		x	
$(-\vec{P}_M, \vec{M}_M)$	x	x	x					x	
$(-\vec{E}_C, \vec{H}_C)$	x	x	x					x	Chu
$\vec{P}_C$	x	x	x			x		x	
$\vec{M}_C$	x	x	x				x	x	
$\vec{D}_C$	x	x	x	x		x		x	
$\vec{B}_C$	x	x	x		x		x	x	
P^ν_C	x	x	x		x	x		x	
M^ν_C	x	x	x	x			x	x	
$(-\vec{E}_A, \vec{B}_A)$	x	x	x		x		x	x	Ampère
$\vec{P}_A$	x	x	x			x		x	
$\vec{M}_A$	x	x	x				x	x	
$\vec{D}_A$	x	x	x	x		x		x	
$\vec{H}_A$	x	x	x		x		x	x	
P^ν_A	x	x	x		x	x		x	
M^ν_A	x	x	x	x			x	x	
$(-\vec{E}_B, \vec{B}_B)$	x	x	x		x		x	x	Boffi
$(-\vec{P}_B, \vec{M}_B)$	x	x	x					x	
$(\vec{D}_B, \vec{H}_B)$	x	x	x	x		x		x	

	Shubnikov Transformations			Non-Trivial Transformations		Transl.	
Fields	$(\frac{\lambda}{2}\, e_3, 2'_y)^{\delta}$	$(0, m_y)^{\varepsilon}$	$(\pm\frac{\lambda}{2}\frac{\theta}{}\, e_3, R_z(\theta))$	$(0, \bar{L}(\rho))$	$(0, L(\sigma))$	U	Formulation
$(-\vec{E}_M, \vec{B}_M)$	x	x	x		x	x	Minkowski
$(\vec{D}_M, \vec{H}_M)$	x	x	x	x		x	
$(\vec{P}_M, \vec{M}_M)$	x	x	x			x	
$(-\vec{E}_C, \vec{H}_C)$	x	x	x			x	Chu
$\vec{P}_C$	x	x	x			x	
$\vec{M}_C$	x	x	x			x	
$\vec{D}_C$	x	x	x	x		x	
$\vec{B}_C$	x	x	x		x	x	
P^{ν}_C	x	x	x		x	x	
M^{ν}_C	x	x	x	x		x	
$(-\vec{E}_A, \vec{B}_A)$	x	x	x		x	x	Ampère
$\vec{P}_A$	x	x	x			x	
$\vec{M}_A$	x	x	x			x	
$\vec{D}_A$	x	x	x	x		x	
$\vec{H}_A$	x	x	x		x	x	
P^{ν}_A	x	x	x		x	x	
M^{ν}_A	x	x	x	x		x	
$(-\vec{E}_B, \vec{B}_B)$	x	x	x		x	x	Boffi
$(\vec{P}_B, \vec{M}_B)$	x	x	x			x	
$(\vec{D}_B, \vec{H}_B)$	x	x	x	x		x	

Table IV.4: Generating Elements for Galilean Symmetry Groups ($n \geq 1$)

Fields	Shubnikov Transformations $(\frac{\lambda}{2} e_3, 2'_y)^\delta$	$(0, m_y)^\varepsilon$	$(\pm\frac{\lambda}{2}\frac{\theta}{\pi}, R_z(\theta))$	Non-Trivial Transformations $(0, L_x(v))$	$(0, L_y(w))$	$(0, M(n))^\eta$	$(\frac{\lambda}{2} e_3, M(n))^\eta$	Transl. U	Formulation
$(-\vec{E}_M, \vec{B}_M)$	x	x	x		x		x	x	
$(\vec{D}_M, \vec{H}_M)$	x	x	x	x		x		x	Minkowski
$(\vec{P}_M, \vec{M}_M)$	x	x	x					x	

In the vacuum case the same groups occur as for $n = 1$.

IV.A Explicit Expressions for the EM-Fields

In table IV.5 the Fourier coefficients $\hat{F}(k)$ of the EM-fields for an elliptically polarized TEM plane wave are listed in the various formulations. By means of the relationship $\hat{F}(-k) = \hat{F}(k)$ and by specializing to other polarizations the explicit expressions for all the EM-fields treated can be obtained. The fields are given as referred to the laboratory frame $\mathcal{K}$, which is identical to the rest frame of the (rigid) material medium and in the wave rest frame $\bar{\mathcal{K}}$, which frame moves with the speed $\frac{1}{n}$ in the z-direction of the laboratory frame. In $\bar{\mathcal{K}}$ the wave vector has a particularly simple form: $\bar{k}_\nu = \omega(0,0,0,\sqrt{n^2-1})$. The spectrum is then $\mathcal{S} = \{\pm\bar{k}_\nu\}$ and the spectral group

$$\bar{S} = \{L_x(\chi), L_y(\phi), R_z(\theta), 1', 2'_y, m_y \mid 0 \leq \theta < 2\pi, \chi, \phi \in \mathcal{R}\} \quad . \qquad (4.39)$$

In the laboratory frame the spectral group becomes

$$S = T^{-1}(n)\bar{S}T(n) = \{\tilde{L}_x(\chi,n), \tilde{L}_y(\phi,n), R_z(\theta), M_n, 2'_y, m_y \mid 0 \leq \theta < 2\pi, \chi, \phi \in \mathcal{R}\}. \qquad (4.40)$$

Table IV.5: Fourier Coefficients of EM-Fields

Fields	Laboratory Frame $\mathcal{K}$	Wave Rest Frame $\bar{\mathcal{K}}$			
	All Formulations	Minkowski	Chu	Ampère	Boffi
E_x	E_o	0	$\sqrt{\frac{\varepsilon}{\mu}}\frac{\mu-1}{\sqrt{n^2-1}}E_o$	0	0
E_y	$-iE_1$	0	$-i\sqrt{\frac{\varepsilon}{\mu}}\frac{\mu-1}{\sqrt{n^2-1}}E_1$	0	0
E_z	0	0	0	0	0
B_x	inE_1	$i\sqrt{n^2-1}E_1$	$i\sqrt{n^2-1}\,E_1$	$i\sqrt{n^2-1}\,E_1$	$i\sqrt{n^2-1}\,E_1$
B_y	nE_o	$\sqrt{n^2-1}E_o$	$\sqrt{n^2-1}\,E_o$	$\sqrt{n^2-1}\,E_o$	$\sqrt{n^2-1}\,E_o$
B_z	0	0	0	0	0
D_x	$\sqrt{\frac{\varepsilon}{\mu}}\,nE_o$	$\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}\,E_o$	$\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}\,E_o$	$\frac{n(\varepsilon-1)}{\sqrt{n^2-1}}E_o$	$\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}\,E_o$
D_y	$-i\sqrt{\frac{\varepsilon}{\mu}}\,n\,E_1$	$-i\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}E_1$	$-i\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}\,E_1$	$-i\,\frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\,E_1$	$-i\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}E_1$
D_z	0	0	0	0	0
H_x	$i\sqrt{\frac{\varepsilon}{\mu}}E_1$	0	$i\,\frac{\varepsilon-1}{\sqrt{n^2-1}}\,E_1$	$i\,\frac{\varepsilon-1}{\sqrt{n^2-1}}\,E_1$	0
H_y	$\sqrt{\frac{\varepsilon}{\mu}}\,E_o$	0	$\frac{\varepsilon-1}{\sqrt{n^2-1}}\,E_o$	$\frac{\varepsilon-1}{\sqrt{n^2-1}}\,E_o$	0
H_z	0	0	0	0	0
P_x	$(\varepsilon-1)\,E_o$	$\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}\,E_o$	$\frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\,E_o$	$\frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\,E_o$	$\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}\,E_o$
P_y	$-i(\varepsilon-1)\,E_1$	$-i\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}E_1$	$-i\,\frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\,E_1$	$-i\,\frac{n(\varepsilon-1)}{\sqrt{n^2-1}}\,E_1$	$-i\sqrt{\frac{\varepsilon}{\mu}(n^2-1)}E_1$
P_z	0	0	0	0	0
M_x	$i\sqrt{\frac{\varepsilon}{\mu}}(\mu-1)E_1$	$i\sqrt{n^2-1}\,E_1$	$i\sqrt{\frac{\varepsilon}{\mu}}\,\frac{(\mu-1)n}{\sqrt{n^2-1}}\,E_1$	$i\sqrt{\frac{\varepsilon}{\mu}}\,\frac{(\mu-1)n}{\sqrt{n^2-1}}E_1$	$i\sqrt{n^2-1}\,E_1$
M_y	$\sqrt{\frac{\varepsilon}{\mu}}(\mu-1)E_o$	$\sqrt{n^2-1}\,E_o$	$\sqrt{\frac{\varepsilon}{\mu}}\,\frac{(\mu-1)n}{\sqrt{n^2-1}}\,E_o$	$\sqrt{\frac{\varepsilon}{\mu}}\,\frac{(\mu-1)n}{\sqrt{n^2-1}}E_o$	$\sqrt{n^2-1}\,E_o$
M_z	0	0	0	0	0

Note that in the laboratory frame, i.e. the rest frame of the material medium, the fields are the same in all formulations.

IV.B Non-Relativistic Limit for Parabolic Lorentz Transformations.

The Lorentz transformations, which leave invariant the isotropic vector (1,0,0,1) can be generated by the rotation $R_z(\theta)$, the parabolic Lorentz transformations $L(\sigma)$ and $\bar{L}(\rho)$ and the mirror m_y. Let us consider the relativistic limit for

$$\bar{L}(\rho)=\begin{pmatrix} 1+\frac{1}{2}\rho^2 & \rho & 0 & -\frac{1}{2}\rho^2 \\ \rho & 1 & 0 & -\rho \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}\rho^2 & \rho & 0 & 1-\frac{1}{2}\rho^2 \end{pmatrix} \quad . \tag{4.41}$$

$\bar{L}(\rho)$ can be written as product of two Lorentz boosts $L_x(\chi)$ and $L_z(\psi)$ (with velocities in the positive e_1- and e_3-direction, respectively) and a rotation $R_y(\alpha)$ around the e_2-axis,

$$\bar{L}(\rho) = R_y(\alpha)\; L_z(\psi)\; L_x(\chi), \tag{4.42}$$

where the connection between ρ,α,ψ and χ is given by:

$$\sin\alpha = -\frac{2\rho}{\sqrt{4+\rho^4}}\;;\quad \mathrm{Sh}\psi = -\frac{\rho^2}{2}\,,\quad \mathrm{Sh}\chi = \frac{2\rho}{\sqrt{4+\rho^4}}\,, \tag{4.43}$$

which corresponds to the boost velocities:

$$\mathrm{Th}\psi = -\frac{\rho^2}{\sqrt{4+\rho^4}} \quad \text{and} \quad \mathrm{Th}\chi = \frac{2\rho}{2+\rho^2} \quad . \tag{4.44}$$

Therefore in the non-relativistic limit (n.r.)

$$\bar{L}(\rho)_{|n.r.}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ v_x(\rho) & \frac{2-\rho^2}{\sqrt{4+\rho^4}} & 0 & -\frac{2\rho}{\sqrt{4+\rho^4}} \\ 0 & 0 & 1 & 0 \\ v_z(\rho) & \frac{2\rho}{\sqrt{4+\rho^4}} & 0 & \frac{2-\rho^2}{\sqrt{4+\rho^4}} \end{pmatrix} \tag{4.45}$$

where

$$v_x(\rho) = \frac{2\rho(2-\rho^2)}{(2+\rho^2)\sqrt{4+\rho^4}} + \frac{2\rho^3}{4+\rho^4} \qquad \text{and}$$

$$v_z(\rho) = \frac{4\rho^2}{(2+\rho^2)\sqrt{4+\rho^4}} - \frac{\rho^2(2-\rho^2)}{4+\rho^4} \quad . \tag{4.46}$$

It was shown in section IV.3 that the non-relativistic limit of $\tilde{L}_x(\chi,n)$ for $n > 1$ is

$$\tilde{L}_x(\chi,n)|_{n.r.} = L_x(\chi)|_{n.r.} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ \mathrm{Th}\chi & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \quad . \tag{4.47}$$

The only cases, where $\sin\alpha = 0$, are $\rho = 0$ and $\rho = \infty$. For these values, however, $v_x(\rho) = 0$. There is thus no connection (except the trivial one) between the Galilei transformations obtained as non-relativistic limits of $\tilde{L}_x(\chi,n)$ for $n > 1$ and $n = 1$. The situation is analogous for $\tilde{L}_y(\phi,n)$ (and $L(\sigma)$).

CHAPTER V

TEM WAVES IN ANISOTROPIC HOMOGENEOUS SUBSTANCES

V.1 The EM-Fields

In this chapter treatment will be limited to the Minkowski formulation of electrodynamics in the case of a medium characterized by an anisotropic restricted constitutive tensor. For an exploring type of investigation like the present one,it may be considered sufficient to compare the different formulations in the isotropic case only. To obtain the expressions for the EM-fields which have been used as a basis for calculating the corresponding symmetry groups, several methods can be used, and as has already been pointed out, the results can be specialized in order to get also the EM-fields for the isotropic case as treated in chapter IV. The explicit expressions for the anisotropic case can be found in the appendix.

Probably the most straightforward manner to calculate the EM-fields makes use of Fresnel's equation. Starting point are Maxwell's equations (without external currents and charges) and the (linear) constitutive equations (2.22). For any field $\vec{F}$, plane wave solutions of Maxwell's equations of the form

$$\vec{F} = \mathrm{Re}\ \{\vec{F}_o \exp ikx\}\ ,\ k_\nu = (-\omega,\vec{k}) \tag{5.1}$$

are considered. Substituting these solutions into Maxwell's equations results in

$$\begin{aligned} \vec{k} \times \vec{E} &= \omega\vec{B} \qquad & \vec{k}.\vec{D} &= 0 \\ \vec{k} \times \vec{H} &= -\omega\vec{D} \qquad & \vec{k}.\vec{B} &= 0 \quad . \end{aligned} \tag{5.2}$$

Introduce now

$$\vec{k} = \omega\vec{n} \tag{5.3}$$

and substitute $\vec{H}$ and $\vec{D}$ in $\vec{k} \times \vec{H} = -\omega\vec{D}$ by $\vec{E}$ and $\vec{B}$ with the help of the constitutive equations (2.22) and finally eliminate $\vec{B}$ by means of the relation $\vec{k} \times \vec{E} = \omega\vec{B}$. The result is

$$\sum_{p,q,j,k} (\varepsilon_{il} + \alpha_{ip} n_q e_{pql} + e_{ijk} n_j (\alpha_1)_{kl} + e_{ijk} n_j n_{kp} n_q e_{pql}) E_l = 0, \tag{5.4}$$

with $\alpha_1 = -\alpha^\dagger$ (see (2.22)) and e_{ijk} the totally skew-symmetric Levi-Cività tensor. The above equation (5.4) is an equation for the unknown field $\vec{E}$ and the unknown velocity of propagation or index of refraction $n = |\vec{n}|$. Since (5.4) is a homogeneous system of linear equations the corresponding determinant of the coefficients must vanish in order to guarantee the existence of non-trivial solutions for $\vec{E}$. Thus relation

$$\det(a_{ik}) = 0 \tag{5.5}$$

determines the value of n in terms of the properties of the medium and also the direction of propagation of the plane wave. It is called Fresnel's equation. Then the value of n as determined from Fresnel's equation is introduced in (5.4) and $\vec{E}$ calculated. From (5.2) $\vec{B}$ is found. As an example consider the case of a uniaxial non-magnetic crystal:

$$\varepsilon_{ij} = \delta_{ij}, \qquad \varepsilon_{ik} = \begin{pmatrix} \varepsilon_\perp & & \\ & \varepsilon_\perp & \\ & & \varepsilon_\parallel \end{pmatrix} \qquad \text{and} \qquad \alpha_{ij} = 0 \,. \tag{5.6}$$

With $\vec{l} = \frac{\vec{n}}{n}$ the equation corresponding to (5.4) is

$$\begin{pmatrix} \frac{1}{n^2}\varepsilon_\perp + l_x^2 - 1 & l_x l_y & l_x l_z \\ l_x l_y & \frac{1}{n^2}\varepsilon_\perp + l_y^2 - 1 & l_y l_z \\ l_x l_z & l_y l_z & \frac{1}{n^2}\varepsilon_\parallel + l_z^2 - 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0. \tag{5.7}$$

For a non-trivial solution one must have $\det(a_{ik}) = 0$, this is Fresnel's equation:

$$\varepsilon_{\parallel}\varepsilon_{\perp}^2+n^2[(l_x^2-1)\varepsilon_{\perp}\varepsilon_{\parallel}+(l_y^2-1)\varepsilon_{\perp}\varepsilon_{\parallel}+(l_z^2-1)\varepsilon_{\perp}^2]+n^4[(l_x^2+l_y^2)\varepsilon_{\perp}+l_3^2\varepsilon_{\parallel}] = 0,$$

which in terms of n_i can be written in the form

$$(n^2-\varepsilon_{\perp})(\varepsilon_{\parallel}n_z^2+\varepsilon_{\perp}[n_x^2+n_y^2]-\varepsilon_{\perp}\varepsilon_{\parallel}) = 0 \quad . \tag{5.8}$$

Clearly there are two solutions for n, which means that two different waves can propagate in the medium under the given circumstances (double refraction):

(i) $n^2 = \varepsilon_{\perp}$ corresponding to the ordinary wave

(ii) $\frac{1}{n^2} = \frac{\sin^2\vartheta}{\varepsilon_{\parallel}} + \frac{\cos^2\vartheta}{\varepsilon_{\perp}}$ corresponding to the extraordinary wave with $\vartheta = \sphericalangle(\vec{k}, e_3)$.

With the help of these solutions for n,(5.7) can be solved for the $\vec{E}$-field corresponding to the two cases. Note that solution (ii) displays anisotropic behaviour: as can be expected from the form of ε_{ik} the value of n (or in other words the value of the propagation velocity) depends on the angle ϑ between wave vector and optical axis of the crystal. For the determination of the $\vec{E}$-field a simplifying assumption can be made (without impairing generality): $\vec{l}$ is assumed to lie in the (x,z)-plane or equivalently $l_y = 0$. This assumption makes use of the isotropy with respect to the (x,y)-plane of the crystal. Now $l_y = 0 = n_y$ reduces (5.7) to

$$\begin{pmatrix} \frac{\varepsilon_{\perp}}{\varepsilon_{\parallel}} l_x^2 & 0 & l_x l_z \\ 0 & (\frac{\varepsilon_{\perp}}{\varepsilon_{\parallel}}-1)l_x^2 & 0 \\ l_x l_z & 0 & \frac{\varepsilon_{\parallel}}{\varepsilon_{\perp}} l_z^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \quad . \tag{5.9}$$

One can distinguish the following cases:

(i) $l_x^2 = 0$ $\quad \vec{k} \parallel$ optical axis

(ii) $l_z^2 = 0$ $\quad \vec{k} \perp$ optical axis

(iii) $l_x \neq 0,\ l_z \neq 0$ $\quad \vec{k}$ general in (x,z)-plane.

In the first case ordinary and extraordinary waves are identical. This means that there is no double refraction for $\vec{k} \parallel$ optical axis. In the second case $E_x = E_y = 0$ is implied, so that the solution of (5.9) can only be represented by a linearly polarized plane wave. The same is true in the third case: $E_y = 0$ and therefore $D_y = 0$ is implied. Together with the transversality condition $\vec{k}.\vec{D} = 0$, this illustrates the remark of Landau and Lifschitz[24]: "It should be emphasized that plane waves propagated in anisotropic media are completely linearly polarized in certain planes. In this respect the optical properties of anisotropic media are very different from those of isotropic media. A plane wave propagated in an isotropic medium is in general elliptically polarized, and is linearly polarized only in particular cases". One probably has to remark here that plane waves propagating along the optical axes of the medium form an exception to the above rule. The optical axes are therefore also called axes of isotropy.

The above method for determining the EM-field expressions of a TEM wave propagating in an anisotropic medium has been used by Birss and Shrubsall[25] in conjunction with a perturbation procedure. The basic idea is, that the general solution of Fresnel's equation is a rather tedious affair and that it would be simpler to apply some sort of perturbation technique for the more complicated cases. To this end Fresnel's equation is written in matrix form:

$$(v^2A + vB + C)\vec{E} = 0, \tag{5.10}$$

with $v = \frac{1}{n}$ and

$$A_{ik} = \varepsilon_{ik}$$

$$B_{ik} = \sum_{p,q,j,l} \alpha_{ip} l_q e_{pqk} + e_{ijl} l_j (\alpha_1)_{lk}$$

$$C_{ik} = \sum_{j,l,p,q} e_{ijl} l_j \eta_{lp} l_q e_{pqk} \;.$$

The solution of the special case

$$(v^2 A^o + C^o)\vec{E} = 0 \qquad (5.11)$$

with A^o, $B^o = 0$ and C^o corresponding to (5.6), has just been sketched. What are the effects on these solutions, of small correction terms A', B' and C', so that

$$A = A^o + A', \quad B = B' \text{ and } C = C^o + C' \;? \qquad (5.12)$$

Adopting the language of eigenvalue problems it has already been shown that (5.11) has two eigenvalues v_1^o and v_2^o of physical interest. The corresponding eigenvectors are denoted by $\vec{E}_1^o$ and $\vec{E}_2^o$. The eigenvector corresponding to $v = 0$, non-propagation, is disregarded. Now an attempt is made to build up the solution of the perturbed problem as a linear combination of the solutions of the unperturbed problem:

$$\vec{E} = \lambda_1 \vec{E}_1^o + \lambda_2 \vec{E}_2^o \quad . \qquad (5.13)$$

Substituting this into (5.11) yields:

$$(v^2[A^o + A'] + vB' + [C^o + C'])(\lambda_1 \vec{E}_1^o + \lambda_2 \vec{E}_2^o) = 0 \quad .$$

From the solution of the unperturbed problem one knows:

$$C^o \vec{E}_1^o = -(v_1^o)^2 A^o \vec{E}_1^o$$

$$C^o \vec{E}_2^o = -(v_2^o)^2 A^o \vec{E}_2^o \quad , \text{ so that}$$

$$\lambda_1([v^2-(v_1^o)^2]A^o+v^2A' + vB' + C')\vec{E}_1^o +$$

$$+ \lambda_2([v^2-(v_2^o)^2]A^o+v^2A' + vB' + C')\vec{E}_2^o = 0.$$

Taking the scalar product with $\vec{E}_1^o$ and $\vec{E}_2^o$ gives:

$$\lambda_1([v^2-(v_1^o)^2](\vec{E}_1^o,A^o\vec{E}_1^o)+v^2(\vec{E}_1^o,A'\vec{E}_1^o)+v(\vec{E}_1^o,B'\vec{E}_1^o)+(\vec{E}_1^o,C'\vec{E}_1^o) +$$

$$+\lambda_2([v^2-(v_2^o)^2](\vec{E}_1^o,A^o\vec{E}_2^o)+v^2(\vec{E}_1^o,A'\vec{E}_2^o)+v(\vec{E}_1^o,B'\vec{E}_2^o)+(\vec{E}_1^o,C'\vec{E}_2^o) = 0$$

and

$$\lambda_1([v^2-(v_1^o)^2](\vec{E}_2^o,A^o\vec{E}_1^o)+v^2(\vec{E}_2^o,A'\vec{E}_1^o)+v(\vec{E}_2^o,B'\vec{E}_1^o)+(\vec{E}_2^o,C'\vec{E}_1^o) +$$

$$+\lambda_2([v^2-(v_2^o)^2](\vec{E}_2^o,A^o\vec{E}_2^o)+v^2(\vec{E}_2^o,A'\vec{E}_2^o)+v(\vec{E}_2^o,B'\vec{E}_2^o)+(\vec{E}_2^o,C'\vec{E}_2^o) = 0.$$

The matrix elements $(\vec{E}_i^o,A^o\vec{E}_j^o)$ are then eliminated by showing that, with proper normalization,

$$(\vec{E}_i^o,\ A^o\ \vec{E}_j^o) = \delta_{ij},\ (i,j = 1,2) \quad . \tag{5.14}$$

Physically (5.14) is equivalent with saying that in an anisotropic medium the ordinary and the extraordinary vectors $\vec{D}$ and $\vec{D}'$ are always orthogonal. This fact will also be used for the indicatrix method of determining the expressions for the EM-fields. With the use of (5.14) the equations determining the λ_i of (5.13) are found to be:

$$\begin{aligned}&\{[v^2-(v_1^o)^2]+(\vec{E}_1^o,[v^2A'+vB'+C']\vec{E}_1^o\}\lambda_1+\{(\vec{E}_1^o,[v^2A'+vB'+C']\vec{E}_2^o\}\lambda_2=0\\&\{\vec{E}_2^o,[v^2A'+vB'+C']\vec{E}_1^o\}\lambda_1+\{[v^2-(v_2^o)^2]+(\vec{E}_2^o,[v^2A'+vB'+C]\vec{E}_2^o\}\lambda_2 = 0.\end{aligned} \tag{5.15}$$

To illustrate the procedure, take as an example the case of natural optical activity in isotropic substances. For reasons of simplicity the example of isotropic substances has been chosen. The method, however, works just as well for anisotropic substances. The solution of the unperturbed problem of the propagation of a TEM wave in an isotropic substance is given by:

$$\vec{E}_1^o = \frac{1}{\sqrt{\varepsilon}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{E}_2^o = \frac{1}{\sqrt{\varepsilon}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{5.16}$$

$$v_1^o = v_2^o = v_o = \frac{1}{\sqrt{\varepsilon}}, \quad A_{ik}^o = \varepsilon_{ik} = \varepsilon\delta_{ik} .$$

Note that these solutions satisfy the relation (5.14). For the perturbed problem the following constitutive tensors describing natural optical activity are used:

$$\varepsilon_{ik} = \varepsilon\delta_{ik}, \quad \eta = \delta_{ik} \qquad A_{ik} = A_{ik}^o = \varepsilon\delta_{ik}, \qquad A' = 0$$

$$\alpha_{ik} = (\alpha_1)_{ik} = i\varepsilon'\delta_{ik} \qquad C_{ik} = l_i l_k - \delta_{ik} \qquad C' = 0$$

and

$$B'_{ik} = 2i\varepsilon' \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{5.17}$$

where use has been made of the isotropy of the (unperturbed) medium. In the basis of the unperturbed eigenvectors the matrix elements of B' are given by:

$$(\vec{E}_1^o, B'\vec{E}_1^o) = (\vec{E}_2^o, B'\vec{E}_2^o) = 0$$

$$(\vec{E}_1^o, B'\vec{E}_2^o) = (\vec{E}_2^o, B'\vec{E}_1^o) = -i\alpha, \qquad \alpha = \frac{2\varepsilon'}{\varepsilon} . \tag{5.18}$$

This leads to the following equations for λ_i:

$$(v^2-(v_o)^2)\lambda_1 - i\alpha v\lambda_2 = 0$$

$$i\alpha v \quad \lambda_1 - (v^2-(v_o)^2)\lambda_2 = 0 . \tag{5.19}$$

For a non-trivial solution, the corresponding determinant must vanish: $(v^2-(v_o)^2)^2 - \alpha^2 v^2 = 0$.
Therefore:

$$(v_{1,2})^2 = \tfrac{1}{2}\{\, 2(v_o)^2+\alpha^2 \pm \sqrt{(2(v_o)^2+\alpha^2)^2-4(v_o)^4}\} \quad . \qquad (5.20)$$

Considering ε' small and neglecting therefore terms of the order $(\varepsilon')^2$ and smaller, one has :

$$(v_{1,2})^2 \simeq \frac{1}{\varepsilon^2}(\varepsilon \pm 2\varepsilon'\sqrt{\varepsilon}) \ . \qquad (5.21)$$

In this approximation the result is identical with Born's[26)] result in terms of the refractive index:

$$n_{\pm}^2 = \varepsilon \mp 2n\varepsilon' \ , \qquad n = \sqrt{\varepsilon} \quad . \qquad (5.22)$$

As for considering ε' small and neglecting terms of the order $(\varepsilon')^2$ and smaller Born[26)] says in his book that ε' is of the order of magnitude of the ratio of the lattice constant to the wavelength of the propagating wave. If this order of magnitude is about the same for isotropic media, which are not cubic crystals, then $\varepsilon' \gtrsim 10^{-3}$ for optical frequencies and the above approximation should be quite good. Substituting $v_{1,2}$ into the equations for λ_i one gets:

$$\frac{\lambda_1}{\lambda_2} = \pm i \qquad (\mathcal{O}(\varepsilon')^2), \qquad (5.23)$$

implying that the corresponding $\vec{E}$-field is circularly polarized. Note, that because of $B = B' \neq 0$, there is a term linear in v in equation (5.10). This means that natural optical activity is not a reversible effect in the sense that if v is an eigenvalue then -v is also one, contrary to the Faraday effect (rotating plane of polarization due to an external biasing magnetic field), where the corresponding equation only contains a quadratic term in v. Or, in other words, reflecting the light ray or turning it back and forth, in the case of the Faraday effect, accumulates the rotation angle of the plane of polarization (the problem is not sensitive for the change $v \to -v$). For natural optical activity reflection of the ray means turning back again the plane of polarization.

There is also a more geometrical construction for the polarization of plane waves propagated in an anisotropic medium. This method is called indicatrix method. Here it is only sketched, for details see Landau and Lifshitz[24]. The indicatrix method also illustrates the duality between wave vector and ray vector:

(i) For a given direction of $\vec{n}$ (or $\vec{k}$ for that matter) draw the tensor ellipsoid corresponding to ε_{ik}^{-1}. In main axes coordinates this is an ellipsoid with semi-axes of lengths $\sqrt{\varepsilon_1}$ $\sqrt{\varepsilon_2}$ and $\sqrt{\varepsilon_3}$. Cut this ellipsoid with the plane through the center of the ellipsoid and perpendicular to the given direction of $\vec{n}$. The section is in general an ellipse. The lengths of its axes determine the values of n and their directions indicate the directions of the ordinary and the extraordinary $\vec{D}$ vectors.

(ii) For a given direction of $\vec{s}$, the ray vector draw the tensor ellipsoid corresponding to ε_{ik}. Cut this ellipsoid with the plane through the center of the ellipsoid and orthogonal to the given direction of $\vec{s}$. The section now determines the directions of the ordinary and the extraordinary $\vec{E}$ vectors.

Both procedures give identical results and illustrate again the fact that in general a plane wave propagating in an anisotropic medium is completely linearly polarized in certain planes for both ordinary and extraordinary waves.

Another method still is given by Born[26] for plane waves propagating in crystals. There the resulting fields appear as Fourier series depending on the wave vector and on the lattice vectors of the crystal. The macroscopic fields are represented by the term which is independent of the lattice vectors. This method might prove very useful at a later stage, when microscopic fields will be studied.

V.2 Optical Classification of Non-Magnetic Transparent Media and Cases Treated.

In the preceding section the tensor ellipsoids corresponding to ε_{ik} and ε_{ik}^{-1} were introduced. By considering these ellipsoids of Fresnel, as they are also called, it is easily seen that there are only three optically different kinds of non-magnetic material media, corresponding to the degree of degeneracy the ellipsoids display. The ellipsoid may be of the general type with all three axes of different lengths, or an ellipsoid of rotation with two axes of the same length or, finally, it may be a sphere all directions being equivalent. Accordingly non-magnetic transparent media all fall into one of the following three classes:

(i) Material media with no distinguished crystallographic direction. The corresponding ellipsoids of Fresnel are of the general type with three axes of different lengths, and there are therefore (see Sommerfeld[27]) two central sections, which are circles. These media are called <u>optically biaxial</u>. To this class belong the crystals of the rhombic, monocline and tricline systems. Here $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$ and therefore, in general, $\vec{E}$ and $\vec{D}$ are not parallel.

(ii) Material media with one distinguished crystallographic direction. The corresponding ellipsoids of Fresnel are ellipsoids of rotation and have therefore one central section, which is a circle. These media are called <u>optically uniaxial</u>. One of the dielectrical main axes of the ellipsoids coincides with the distinguished crystallographic direction. To this class belong the crystals of the trigonal, tetragonal and hexagonal systems. Here $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$ and therefore, in general, $\vec{E}$ and $\vec{D}$ are not parallel.

(iii) Material media with three equivalent mutually perpendicular symmetry axes. The corresponding ellipsoids of Fresnel are spheres and consequently any central section is a circle. These media are called optically isotropic. To this class belong the cubic crystals and amorphous matter. Here $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$ and therefore $\vec{E}$ and $\vec{D}$ are parallel.

The nomenclature optically uniaxial and biaxial is derived from the number of circular central sections of Fresnel's ellipsoids. If the central section normal to the direction of the wave vector $\vec{k}$ is a circle, then the propagation velocities of the ordinary and the extraordinary waves are the same (there is actually no extraordinary wave). This kind of direction is called optical axis. Correspondingly the same is true for the central sections normal to the ray vector $\vec{s}$. The optical axes are axes of isotropic behaviour also in the sense that for a plane wave with wave vector $\vec{k}$ parallel to an optical axis all three different types of polarization can occur, not only linear polarization.

In the following table the cases treated in this chapter are listed. The medium is characterized in the first column. A more elaborate characterization in terms of the constitutive tensors of the medium as well as the explicit expressions for the EM-fields are given in the appendix. Case 5, natural optical activity in isotropic media,is treated in this chapter because of its similarity to the Faraday effect. Biaxial media are not considered for reasons of simplicity and, because nothing new is expected to be gained as compared to uniaxial media. The second column gives the relative orientation of the wave vector $\vec{k}$ with respect to the optical axis of the material medium. What happens physically or the corresponding effect is listed in column three.

Table V.1: TEM Plane Waves in Anisotropic Media

	Medium	Direction of $\vec{k}$	Effect
1.1	Uniaxial electric	General	Linear double refraction
1.2		Parallel to optical axis	Isotropic behaviour
1.3		Orthogonal to optical axis	Linear double refraction
2.1	Uniaxial magnetic	General	Linear double refraction
2.2		Parallel to optical axis	Isotropic behaviour
2.3		Orthogonal to optical axis	Linear double refraction
3.1	Electric Faraday	General	Circular double refraction
3.2		Parallel to optical axis	Circular double refraction
3.3		Orthogonal to optical axis	No double refraction. Linear polarization.
4.1	Magnetic Faraday	General	Circular double refraction
4.2		Parallel to optical axis	Circular double refraction
4.3		Orthogonal to optical axis	No double refraction Linear polarization
5.1	Naturally optically active (isotropic)	General	Circular double refraction.

In the case of the electric and magnetic Faraday medium,the optical axis is determined by the exterior biasing magnetic field. For more details see Born[26)].

V.3 Example: Determination of the Relativistic Symmetries in the Case of a Dielectric (Non-Magnetic) Uniaxial Medium

To illustrate certain new features characteristic of anisotropic homogeneous material media and also to show the connection between isotropic and anisotropic such media, case 1.1 of table V.1 is treated in this section. The first step is to determine the EM-fields. Since the medium is anisotropic and because the wave vector

$$k_\nu = \omega(-1, nl_1, 0, nl_3) \qquad (5.24)$$

is not parallel to the optical axis (chosen to be the z-axis), there are two waves propagating in the medium with different velocities. This phenomenon of (linear) double refraction appears as a consequence of the fact that Fresnel's equation, in this case, has two different solutions one for $n_{\sigma}^2 = \varepsilon_\perp$, the ordinary wave, and one for

$$\frac{1}{n_e^2} = \frac{l_1^2}{\varepsilon_\parallel} + \frac{l_3^2}{\varepsilon_\perp} \, , \qquad (5.25)$$

the extraordinary wave,
with $l_1 = \sin\vartheta$, $l_3 = \cos\vartheta$ and $\vartheta = \sphericalangle(\vec{k}$, optical axis). This was shown in section V.1. For the ordinary wave nothing new appears, therefore in the following only the extraordinary wave is treated. In the laboratory frame the Fourier coefficients of the corresponding EM-fields derived by means of Fresnel's equation (see appendix V) are:

$$\hat{F}^{\mu\nu}(\pm k) = \frac{E_o}{2} \begin{pmatrix} 0 & \varepsilon_\perp^{-1} l_3 & 0 & -\varepsilon_\parallel^{-1} l_1 \\ -\varepsilon_\perp^{-1} l_3 & 0 & 0 & -n_e^{-1} \\ 0 & 0 & 0 & 0 \\ \varepsilon_\parallel^{-1} l_1 & n_e^{-1} & 0 & 0 \end{pmatrix} , \quad \hat{G}^{\mu\nu}(\pm k) = \frac{E_o}{2} \begin{pmatrix} 0 & l_3 & 0 & -l_1 \\ -l_3 & 0 & 0 & -n_e^{-1} \\ 0 & 0 & 0 & 0 \\ l_1 & n_e^{-1} & 0 & 0 \end{pmatrix}$$

$$(5.26)$$

and therefore $$\hat{M}_e^{\mu\nu}(\pm k) = \frac{E_o}{2} \begin{pmatrix} 0 & (\varepsilon_\perp^{-1}-1) l_3 & 0 & -(\varepsilon_\parallel^{-1}-1) l_1 \\ -(\varepsilon_\perp^{-1}-1) l_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (\varepsilon_\parallel^{-1}-1) l_1 & 0 & 0 & 0 \end{pmatrix} .$$

The transition to the wave rest frame is now made by means of the transformation

$$T(n)R_y(\vartheta) \equiv T(n,\vartheta) = \begin{pmatrix} \frac{n_e}{\sqrt{n_e^2-1}} & \frac{-l_1}{\sqrt{n_e^2-1}} & 0 & \frac{-l_3}{\sqrt{n_e^2-1}} \\ 0 & l_3 & 0 & -l_1 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{\sqrt{n_e^2-1}} & \frac{n_e l_1}{\sqrt{n_e^2-1}} & 0 & \frac{n_e l_3}{\sqrt{n_e^2-1}} \end{pmatrix}, \tag{5.27}$$

where $T(n)$ is given in (5.53) and $R_y(\vartheta)$ is a rotation over an angle ϑ about the e_2-axis, because the wave vector lies along a general direction in the (x,z)-plane of the laboratory system. In the wave rest frame the Fourier coefficients of the EM-fields become:

$$\bar{\hat{F}}^{\mu\nu}(\pm\bar{k}) = \frac{E_o}{2} \begin{pmatrix} 0 & 0 & 0 & (\epsilon_\perp^{-1}-\epsilon_\parallel^{-1})l_1 l_3 \\ 0 & 0 & 0 & -\frac{\sqrt{n_e^2-1}}{n_e^2} \\ 0 & 0 & 0 & 0 \\ -(\epsilon_\perp^{-1}-\epsilon_\parallel^{-1})l_1 l_3 & \frac{\sqrt{n_e^2-1}}{n_e^2} & 0 & 0 \end{pmatrix} = \frac{E_o\sqrt{n_e^2-1}}{2n_e^2} \begin{pmatrix} 0 & 0 & 0 & -\xi \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 \end{pmatrix}, \tag{5.28}$$

$$\bar{\hat{G}}^{\mu\nu}(\pm\bar{k}) = \frac{E_o}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{\hat{M}}^{\mu\nu}(\pm\bar{k}) = \frac{E_o\sqrt{n_e^2-1}}{2n_e^2} \begin{pmatrix} 0 & -n_e & 0 & -\xi \\ n_e & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \xi & 1 & 0 & 0 \end{pmatrix}.$$

In these expressions

$$\xi = \frac{n_e^2}{\sqrt{n_e^2-1}} (\epsilon_\parallel^{-1} - \epsilon_\perp^{-1}) l_1 l_3 . \tag{5.29}$$

Note that for $\epsilon_\perp = \epsilon_\parallel$ (isotropic medium) one has $\xi = 0$, and the expressions for the EM-fields turn into the corresponding expressions of the isotropic case. One finds now the relativistic symmetries of $\bar{F}^{\mu\nu}$ in the same way as outlined in section III.1 by means of the spectral group

$$\bar{S} = \{L_x(\chi),\ L_y(\phi),\ R_z(\theta),\ 1',\ 2'_y,\ m_y\} \tag{5.30}$$

in the rest frame of the wave and the corresponding symmetry condition

$$\dot{L}\ \bar{\hat{F}}(\bar{k}) = \bar{\hat{F}}(L\bar{k})L^{\star}\ \exp\{i(L\bar{k})u(L)\} \tag{5.31}$$

for L an element of $\bar{S}$, i.e. a Lorentz transformation leaving the wave vector

$$\bar{k}_\nu = \omega(0,0,0,\ \sqrt{n_e^2-1}) \tag{5.32}$$

invariant up to a sign. Therefore this general Lorentz transformation must be of the form

$$L = \begin{pmatrix} L^o_1 & L^o_1 & L^o_2 & 0 \\ L^1_o & L^1_1 & L^1_2 & 0 \\ L^2_o & L^2_1 & L^2_2 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix} \quad \text{and} \quad L^{\star} = \begin{pmatrix} L^o_o & -L^o_1 & -L^o_2 & 0 \\ -L^1_o & L^1_1 & L^1_2 & 0 \\ -L^2_o & L^2_1 & L^2_2 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}. \tag{5.33}$$

These general transformations are substituted into the symmetry condition (5.31) with the result that for symmetry, L must be of the form

$$L = \begin{pmatrix} L^o_o & -\xi(L^o_o - e^{i\alpha}) & L^o_2 & 0 \\ \xi(L^o_o - e^{i\alpha}) & e^{i\alpha} - \xi^2(L^o_o - e^{i\alpha}) & \xi L^o_2 & 0 \\ L^2_o & -\xi L^2_o & L^2_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.34}$$

in the case of $L^3_3 = +1$. However from the form of $\bar{\bar{F}}^{\mu\nu}$ (5.28) and the spectral group (5.30) together with the symmetry condition (5.31), it follows that m_z is an element of the point group and therefore the case $L^3_3 = -1$ need not be considered, because it does not give anything new. Further we remark that the rows and columns of the matrix of a Lorentz transformation are normalized with the consequence that $L^2_o = \pm L^o_2$. We now distinguish the two cases (i) $L^o_2 = 0$ and (ii) $L^o_2 \neq 0$. In the first case, one finds the symmetry element $(0, m_y)$ and the following two elements of the point group of $\bar{\bar{F}}^{\mu\nu}$:

$$M_{\frac{1}{\xi}} \equiv \begin{pmatrix} \frac{\xi^2+1}{\xi^2-1} & -\frac{2\xi}{\xi^2-1} & 0 & 0 \\ \frac{2\xi}{\xi^2-1} & -\frac{\xi^2+1}{\xi^2-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } M_\xi \equiv \begin{pmatrix} -\frac{\xi^2+1}{\xi^2-1} & \frac{2\xi}{\xi^2-1} & 0 & 0 \\ -\frac{2\xi}{\xi^2-1} & \frac{\xi^2+1}{\xi^2-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (5.35)$$

These two Lorentz mirrors furnish the following elements of the relativistic symmetry group of $\bar{\bar{F}}^{\mu\nu}$:

$$\begin{aligned} &\text{for } \xi^2 > 1: \quad (0, M_\xi) \text{ and } (0, M_{\frac{1}{\xi}}) \\ &\text{for } \xi^2 < 1: \quad \left(\frac{\pi}{\omega\sqrt{n_e^2-1}}\, e_3, M_\xi\right) \text{ and } \left(\frac{\pi}{\omega\sqrt{n_e^2-1}}\, e_3, M_{\frac{1}{\xi}}\right). \end{aligned} \qquad (5.36)$$

For $\xi^2 = 1$ the Lorentz mirrors become singular and only their product

$$M_{\frac{1}{\xi}} \; M_\xi = m'_x \qquad (5.37)$$

remains as symmetry element in this case. Another value of ξ^2 is of special interest: that of $\xi^2 = 0$, which represents isotropic medium. In this case the Lorentz mirrors become symmetry elements of order two for isotropic medium:

$$\left(\frac{\pi}{\omega\sqrt{n_e^2-1}}\, e_3,\ M_{\frac{1}{\xi}}\right)\Bigg|_{\xi^2=0} = \left(\frac{\pi}{\omega\sqrt{n_e^2-1}}\, e_3,\ 1'\right)$$
$$\left(\frac{\pi}{\omega\sqrt{n_e^2-1}}\, e_3,\ M_{\xi}\right)\Bigg|_{\xi^2=0} = \left(\frac{\pi}{\omega\sqrt{n_e^2-1}}\, e_3, m_x\right) . \qquad (5.38)$$

Examining the elements of the point group of $\bar{F}^{\mu\nu}$ found so far, one notices that there are representatives in any one of the four cosets of the homogeneous Lorentz group with respect to the proper orthochronous Lorentz group (one-component):

	$0(3,1)_+^\uparrow$		$0(3,1)_-^\downarrow$	$0(3,1)_-^\uparrow$	$0(3,1)_+^\downarrow$
L_o^o	+		-	+	-
det L	+		-	-	+
L	$m_y M_\xi$ $(\xi^2<1)$	$m_y M_{\frac{1}{\xi}}$ $(\xi^2>1)$	$2_y'=m_z m_x'$	m_y	m_x'

This means that the further search for elements of the point group of $\bar{F}^{\mu\nu}$ may be limited to $L \in 0(3,1)_+^\uparrow$. Now we have to consider case (ii) $L_2^o \neq 0$. There are two subcases $L_2^o = \pm L_o^2$ as already pointed out. But from (5.34) we see that by means of $m_y M_\xi$ $(\xi^2 < 1)$ which is an element of the point group, $-L_o^2$ can always be transformed into $+L_o^2$ without changing the coset to which L belongs (because $m_y M_\xi$ for $\xi^2 < 1$ belongs to $0(3,1)_+^\uparrow$). For $\xi^2 > 1$ the same can be done by means of $m_y M_{\frac{1}{\xi}}$. Therefore it is sufficient to consider $L_o^2 = L_2^o$.
For proper orthochronous Lorentz transformations one may apply the method of ref. 8) to determine the one parameter subgroups of the point group. The determinant of (5.34) in the case $e^{i\alpha} = -1$ is -1. The value of the determinant is a continuous function of the components of the corresponding matrix and therefore in our case of ξ. Because for a Lorentz transformation the determinant can only be $\pm$ 1, the determinant of (5.34) for $e^{i\alpha} = -1$ stays -1 for all values of ξ. (5.34) is in this case no element of $0(3,1)_+^\uparrow$ (in contradiction of the assumption made). Note that $\xi^2 = 0$ implies

$L_o^o = L_2^2$ in (5.34). For $e^{i\alpha} = +1$ (5.34) becomes:

$$L = \begin{pmatrix} L_o^o & -\xi(L_o^o-1) & L_2^o & 0 \\ \xi(L_o^o-1) & 1-\xi^2(L_o^o-1) & \xi L_2^o & 0 \\ L_2^o & -\xi L_2^o & L_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (5.39)$$

One now determines L_2^2 from the orthogonality relation between columns with the result

$$L_2^2 = L_o^o - \xi^2(L_o^o-1) \ . \qquad (5.40)$$

Therefore

$$\text{trace } L = 2(L_o^o - \xi^2(L_o^o-1)+1), \quad (L_o^o = 1 \rightarrow \text{trace } L = 4), \qquad (5.41)$$

and with this the invariant axis η for L can be determined with the help of the appendix of ref. 8) :

$$\eta = \rho(-\xi,1,0,0) \ , \qquad \rho \in \mathcal{R} \ . \qquad (5.42)$$

The elements of infinite order of the point group of $\bar{F}^{\mu\nu}$ are then found by means of (5.15), (5.16) and (5.17) in same ref. 8) :

$$\text{trace } L = 4 + \rho^2(1-a^2) \rightarrow L_o^o = 1+\frac{\rho^2}{2} \ , \quad \xi = a \qquad (5.43)$$

and therefore the elements

$$\bar{L}(\rho,\xi) = \begin{pmatrix} 1+\frac{\rho^2}{2} & -\xi\frac{\rho^2}{2} & \frac{\rho}{2} I^{\frac{1}{2}} & 0 \\ \xi\frac{\rho^2}{2} & 1-\xi^2\frac{\rho^2}{2} & \xi\frac{\rho}{2} I^{\frac{1}{2}} & 0 \\ \frac{\rho}{2} I^{\frac{1}{2}} & -\xi\frac{\rho}{2} I^{\frac{1}{2}} & 1+\frac{\rho^2}{2}(1-\xi^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \qquad (5.44)$$

with $I^{\frac{1}{2}} = \sqrt{\text{trace } L(\rho,\xi)}$ belong to the point group of $\bar{F}^{\mu\nu}$.
The corresponding elements of the relativistic symmetry group of $\bar{F}^{\mu\nu}$ are

$$(0,\ L(\rho,\xi))\ , \qquad \forall \rho \in \mathbb{R}\ . \tag{5.45}$$

Again we consider the special case $\xi^2 = 0$, which represents isotropic medium:

$$\bar{L}(\rho,0)= \begin{pmatrix} 1+\frac{\rho^2}{2} & 0 & \frac{\rho}{2} I^{\frac{1}{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\rho}{2} I^{\frac{1}{2}} & 0 & 1+\frac{\rho^2}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.46}$$

with $I^{\frac{1}{2}}\Big|_{\xi=0} = \sqrt{4+\rho^2}$. One choses now a new parametrization:

$$1+\frac{\rho^2}{2} = \mathrm{Ch}\phi \rightarrow \mathrm{Sh}\chi = \sqrt{\mathrm{Ch}^2\phi-1} = \sqrt{\rho^2 + \frac{\rho^4}{4}}$$
$$=\left[\frac{\rho^2}{4}(4+\rho^2)\right]^{\frac{1}{2}} = \frac{\rho}{2} I^{\frac{1}{2}}\ .$$

Therefore we have shown that

$$\bar{L}(\rho,0) = L_y(\phi) \qquad \text{with} \qquad \begin{aligned} \mathrm{Ch}\phi &= 1+\frac{\rho^2}{2} \\ \mathrm{Sh}\phi &= \frac{\rho}{2}\sqrt{4+\rho^2}\ . \end{aligned} \tag{5.47}$$

Again the corresponding symmetry element of the isotropic case has been found for $\xi^2 = 0$. The above calculation exhausts all possibilities and therefore the relativistic symmetry group of $\bar{F}^{\mu\nu}$ is found. As referred to the wave rest frame the general element is given by:

$$L = \bar{L}(\rho,\xi)(M_\xi)^{\eta(\xi)}(2_y')^{\delta}\ (m_y)^{\varepsilon}\ (m_z)^{\zeta} \equiv \bar{L}(\rho,\xi,\eta,\delta,\varepsilon,\zeta) \tag{5.48}$$

with

$$u(L) = \left(\frac{\eta}{2} - [\mathrm{sign}(\xi^2-1)]\ \frac{\eta}{2} + \delta + \zeta\right)\frac{\pi}{\omega\sqrt{n_e^2-1}}\ e_3\ .$$

The parameters can take the following values:

$$\rho,\xi,\phi \in \mathcal{R} \qquad \eta(\xi) = \begin{cases} 0,1 & \text{for } \xi^2 \neq 1 \\ 0 & \text{for } \xi^2 = 1, \end{cases} \quad \theta,\delta,\varepsilon,\zeta = 0,1.$$

For $\xi^2 = 0$ one finds a general element of the corresponding symmetry group for the isotropic case (wave rest frame):

$$L = \bar{L}(\rho,\ 0)(m_x)^{\eta(0)}(2'_y)^{\delta}(m_y)^{\varepsilon}(m_z)^{\zeta} \equiv \bar{L}(\rho,0,\eta,\delta,\varepsilon,\zeta)$$

with (5.49)

$$u(L) = (\eta+\delta+\zeta)\,\frac{\pi}{\omega\sqrt{n_e^2-1}}\,e_3\ .$$

For the $\bar{G}^{\mu\nu}$-field, which in the wave rest frame is identical with the $\bar{G}^{\mu\nu}$-field of the corresponding isotropic case ($\xi^2=0$), the relativistic symmetry group is also the same as in the isotropic case. However the $\bar{M}^{\mu\nu}$-field must be explored along the same lines as $\bar{\bar{F}}^{\mu\nu}$.

V.4 Survey of the Results

The cases treated in this chapter fall into two classes: the naturally anisotropic media and the artificially anisotropic media. In the first class are the uniaxial electric and magnetic media, in the second class the media displaying electric or magnetic Faraday effects. To the second class also we count the naturally optically active medium, because the TEM waves propagating in such a medium do not differ from the TEM waves propagating in media displaying Faraday effects. In the second case of artificially anisotropic media there are no new results as compared to the isotropic case, i.e. no new relativistic symmetry elements are found. The reason is, that the anisotropy is such that TEM waves propagating in these media do not differ essentially from the circularly polarized fields in the isotropic case. The transformation laws are of course also the same and therefore the relativistic symmetry groups are also the same. In the first case of naturally anisotropic media, however, new symmetry elements are found. Here from a technical point of view there are two different kinds of symmetry groups in the wave rest frame: the ones obtained by conjugation of symmetry groups of the isotropic case and the ones, which cannot be obtained in this way. The first kind of groups only seemingly have new symmetry elements as a consequence of the conjugation due to a different orientation of the wave vector k or the EM-fields with respect to the optical axis (assumed to be the z-axis) of the medium. The second kind of groups really contains new symmetry elements. These are the elements $\bar{L}(\rho,\xi)$, $L(\sigma,\xi)$ and M_ξ of the corresponding point groups (see section V.3). For $\xi = 0$ (isotropy) the relativistic symmetry groups of the anisotropic case go over into the corresponding relativistic symmetry groups of the isotropic case. Furthermore it is very interesting to notice that the relation between the field $F^{\mu\nu}$ for electric and the field $G^{\mu\nu}$ for magnetic uniaxial medium with respect to the symmetry elements $(0, \bar{L}(\rho,\xi))$ and $(0, L(\sigma,\xi))$ is exactly the same as in the isotropic case the relation between the fields $F^{\mu\nu}$ and $G^{\mu\nu}$ with respect to the symmetry elements $(0, L_x(\chi))$ and $(0, L_y(\phi))$.

The relativistic symmetry groups, as referred to the wave rest frame, of the EM-fields considered in this chapter are given in table V.3. In the first column the generators occurring for the various relativistic symmetry groups are listed. Each of the following columns corresponds to an EM-field tensor $F = (-\vec{E},\vec{B})$, $G = (\vec{D},\vec{H})$ or $M = (-\vec{P},\vec{M})$ of a particular case or configuration numbered according to table V.1, if only the first two numbers are considered. If all three numbers are considered the numbering coincides with the one in table V.4, where the more elaborate characterization in terms of constitutive tensors, wave vectors and refractive indices is given. The crosses indicate the symmetry elements which generate the relativistic symmetry groups of the corresponding fields. Crosses with a circle around mean that in the laboratory frame the corresponding generator is a non-trivial relativistic symmetry element. Different configurations appearing in the same column have the same relativistic symmetry groups. Cases 1.2 and 2.2 display isotropic behaviour (see chapter IV). In the row u(L) the non-primitive translations of the generators are indicated by numbers:

$$
\begin{aligned}
&1 \text{ corresponds to } && u(L) = \left(0,\ 0,\ 0\ \frac{\pi}{\omega\sqrt{n^2-1}}\right) \\
&2 \text{ corresponds to } && u(L) = \left(0,\ \frac{\pi}{\omega\sqrt{n^2-1}},\ 0,\ 0\right) \qquad (5.50) \\
&3 \text{ corresponds to } && u_{\pm}(L) = \left(0,\ 0,\ 0,\ \frac{\pi}{\omega\sqrt{n_{\pm}^2-1}}\right).
\end{aligned}
$$

Note that in the above expressions for u(L) the appropriate values for n (which depends on the configuration according to table V.4) must be used. The group U of primitive translations is also noted by means of numbers. Because for all the cases treated in this chapter there are three different types of wave vectors. There are also three different translation groups U:

Table V.2: Groups U of Primitive Translations (Laboratory Frame)

Number	Wave vector k_ν	a_1	$c_2\delta t_2$	$c_3\delta t_3$	$c_4\delta t_4$
1	$\omega(-1,nl_1,0,nl_3)$	$(0,\lambda l_1,0,\lambda l_3)$	$(1,\frac{l_1}{n},0,\frac{l_3}{n})\delta t_2$	$(0,0,1,0)\delta t_3$	$(0,-\frac{l_3}{n},0,\frac{l_1}{n})\delta t_4$
2	$\omega(-1,n,0,0)$	$(0,\lambda,0,0)$	$(1,\frac{1}{n},0,0)\delta t_2$	$(0,0,1,0)\delta t_3$	$(0,0,0,1)\delta t_4$
3	$\omega(-1,0,0,n)$	$(0,0,0,\lambda)$	$(0,1,0,0)\delta t_2$	$(0,0,1,0)\delta t_3$	$(1,0,0,\frac{1}{n})\delta t_4$

In the rest frame of the wave, the wave vectors are of the form $\omega(0,0,0,\sqrt{n^2-1})$ (cases 1,3) and $\omega(0,\sqrt{n^2-1},0,0)$ (case 2). Here also the appropriate values for n(according to table V.4) must be substituted into these expressions. Most of the relativistic symmetry groups found are non-symmorphic. There are some exceptions. These are indicated in the row S: s means that the corresponding group is symmorphic and ξ indicates that the corresponding group is symmorphic only for $\xi^2 \geq 1$. All other groups are non-symmorphic. The generators, indicated in table V.3, as has already been pointed out, are referred to the rest frame of the wave. To find the relativistic symmetry groups in the laboratory frame these generators have to be conjugated accordingly. The last row gives the corresponding conjugation matrices C. One then has in the laboratory frame:

$$\tilde{L} = C^{-1} L C . \tag{5.51}$$

These conjugations are again characterized by numbers:

$$\begin{aligned} &1 \text{ corresponds to } C_1 \\ &3 \text{ corresponds to } C_3 \\ &32 \text{ corresponds to } C_3C_2 \end{aligned} \tag{5.52}$$

with

$$C_1 \equiv T_x(n) = \begin{pmatrix} \frac{n}{\sqrt{n^2-1}} & \frac{-1}{\sqrt{n^2-1}} & 0 & 0 \\ \frac{-1}{\sqrt{n^2-1}} & \frac{n}{\sqrt{n^2-1}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$C_3 \equiv T_z(n) = T(n) = \begin{pmatrix} \frac{n}{\sqrt{n^2-1}} & 0 & 0 & \frac{-1}{\sqrt{n^2-1}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-1}{\sqrt{n^2-1}} & 0 & 0 & \frac{n}{\sqrt{n^2-1}} \end{pmatrix} , \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l_3 & 0 & -l_1 \\ 0 & 0 & 1 & 0 \\ 0 & l_1 & 0 & l_3 \end{pmatrix} = R_y\,(\arcsin l_1). \tag{5.53}$$

For the explicit expressions for $\tilde{L}(\rho,\xi)$, $L(\sigma,\xi)$ and M_ξ see section V.3. Furthermore

$$\tau(\xi) \overset{\text{def}}{=} \begin{cases} 1 \text{ for } \xi^2 < 1 \\ 0 \text{ for } \xi^2 \geqslant 1 \end{cases} \quad \text{and} \quad \xi = \frac{\Sigma_{\parallel}^{-1} - \Sigma_{\perp}^{-1}}{2\sqrt{n^2 - 1}} \, l_1 l_3 n^2 \tag{5.54}$$

$$\text{with } \Sigma = \begin{cases} \varepsilon \quad \text{for electric media} \quad \text{(see section V.3)} \\ \mu \quad \text{for magnetic media.} \end{cases}$$

In all these expressions the values of n have to be taken from table V.4.

Table V.3: Relativistic Symmetry Groups for Anisotropic Media

	Configuration (see tables V.1 and V.4)																														
Generators	1.1.1			1.1.2			1.3.1 2.3.2			1.3.2 2.3.1 3.3 4.3			2.1.1			2.1.2			3.1.1 4.1.1			3.1.2 4.1.2			3.2.1 4.2.1 5.1.1			3.2.2 4.2.2 5.1.2			2.2 1.2
	F	G	M	F	G	M	F	G	M	F	G	M	F	G	M	F	G	M	F	G	M	F	G	M	F	G	M	F	G	M	FGM
$(0,L(\sigma,\xi))$																	⊗														ISOTROPIC BEHAVIOUR
$(0,\bar{L}(\rho,\xi)$				⊗																											
$(0,L_x(\chi))$	⊗				⊗									⊗																	
$(0,L_y(\phi))$		⊗						⊗		⊗			⊗			⊗															
$(0,L_z(\psi))$							⊗				⊗																				
$(\tau(\xi)u,M_\xi)^{\tau(\xi)}$				⊗		⊗											⊗	⊗													
$(0,1')$		⊗			⊗			⊗			⊗			⊗						⊗			⊗			⊗			⊗		
$(u,1')$	⊗						⊗			⊗			⊗			⊗			⊗			⊗			⊗			⊗			
$(0,2_x)$	⊗				⊗									⊗		⊗															
$(u,2_x)$		⊗		⊗			x	x	x	x	x	x	⊗				⊗														
$(0,2'_y)$	x	x	x				⊗	⊗		⊗						x		x													
$(u,2'_y)$				x	x	x			⊗		⊗	⊗	x	x	x		x		x	x	x	x	x	x	x	x	x	x	x	x	
$(0,R_z(\pi))$							⊗			⊗																					
$(u,R_z(\pi))$	⊗	⊗	⊗		⊗			⊗					⊗	⊗	⊗	⊗															
$(u_+,R_z(\theta))$																						⊗	⊗	⊗				x	x	x	
$(u_-,R_z(\theta))$																			⊗	⊗	⊗				x	x	x				
$(0,m_y)$				x	x	x	x			x	x	x	x	x	x																
(u,m_y)	x	x	x					x	x							x	x	x													
U		1			1			2			2			1			1			1			1			3			3		
u(L)		1			1			2			2			1			1			1,3			1,3			1,3			1,3		
S				ξ		ξ					s																				
C		32			32			1			1			32			32			32			32			3			3		

In the above table $\sigma,\rho,\chi,\phi,\psi,\xi, \in \mathbb{R}$; $0 \leqslant \theta < 2$;

$$\tau(\xi) = \begin{cases} 1 \text{ for } \xi^2 < 1 \\ 0 \text{ for } \xi^2 \geqslant 1 \end{cases} \quad \text{and} \quad \xi = \frac{\Sigma_\parallel^{-1} - \Sigma_\perp^{-1}}{2\sqrt{n^2-1}} \, l_1 l_3 \, n^2$$

$$\text{with} \quad \Sigma = \begin{cases} \varepsilon \text{ for electric media} \\ \mu \text{ for magnetic media.} \end{cases}$$

⊗,x indicates a symmetry element in the wave rest frame.

⊗ means that the corresponding generator is a non-trivial relativistic symmetry element in the laboratory frame.

A P P E N D I X V

The different cases are listed in table V.4 with the help of numbers. These numbers correspond to the numbers used in table V.1 (up to the last digit, which distinguishes between ordinary and extraordinary waves), where the cases treated were described in a more general way. Along with the constitutive tensor characterizing the medium and the four-dimensional wave vector k^{ν} indicating the relative orientation of the direction of propagation of the TEM wave with respect to the optical axis of the medium, the index of refraction n (which also appears in the wave vector) is given in terms of material properties of the medium for both ordinary and extraordinary waves. In table V.5 the Fourier coefficients of the EM-fields in the laboratory frame are given (for positive k_{ν} vector). The numbering of the cases corresponds to the one of table V.4. For cases 1.2 and 2.2, which display isotropic behaviour, the Fourier coefficients are given for elliptical polarization. The amplitude factor E_0 is suppressed in both tables.

Table V.4: Constitutive Tensors, Wave Vectors and Refractive Indices (Laboratory frame)

Configuration	Constitutive Tensor	Wave Vector k^ν	Reciprocal Refractive Index Squared: $1/n^2$
1.1.1	$\begin{bmatrix} -\varepsilon_\perp &&&&& \\ & -\varepsilon_\perp &&&& \\ && -\varepsilon_\parallel &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}$	$\omega(1, nl_1, 0, nl_3)$	$\varepsilon_\perp^{-1}$
1.1.2		$\omega(1, nl_1, 0, nl_3)$	$l_1^2\varepsilon_\parallel^{-1} + l_3^2\varepsilon_\perp^{-1}$
1.2		$\omega(1, 0, 0, n)$	$\varepsilon_\perp^{-1}$
1.3.1		$\omega(1, n, 0, 0)$	$\varepsilon_\perp^{-1}$
1.3.2		$\omega(1, n, 0, 0)$	$\varepsilon_\parallel^{-1}$
2.1.1	$\begin{bmatrix} -1 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& \frac{1}{\mu_\perp} && \\ &&&& \frac{1}{\mu_\perp} & \\ &&&&& \frac{1}{\mu_\parallel} \end{bmatrix}$	$\omega(1, nl_1, 0, nl_3)$	$\mu_\perp^{-1}$
2.1.2		$\omega(1, nl_1, 0, nl_3)$	$l_1^2\mu_\parallel^{-1} + l_2^2\mu_\perp^{-1}$
2.2		$\omega(1, 0, 0, n)$	$\mu_\perp^{-1}$
2.3.1		$\omega(1, n, 0, 0)$	$\mu_\perp^{-1}$
2.3.2		$\omega(1, n, 0, 0)$	μ^{-1}
3.1.1	$\begin{bmatrix} -\varepsilon & i\varepsilon' &&&& \\ -i\varepsilon' & -\varepsilon &&&& \\ && -\varepsilon &&& \\ &&& 1 && \\ &&&& 1 & \\ &&&&& 1 \end{bmatrix}$	$\omega(1, nl_1, 0, nl_3)$	$(\varepsilon-\varepsilon' l_3)^{-1}$
3.1.2		$\omega(1, nl_1, 0, nl_3)$	$(\varepsilon+\varepsilon' l_3)^{-1}$
3.2.1		$\omega(1, 0, 0, n)$	$(\varepsilon-\varepsilon')^{-1}$
3.2.2		$\omega(1, 0, 0, n)$	$(\varepsilon+\varepsilon')^{-1}$
3.3		$\omega(1, n, 0, 0)$	ε^{-1}
4.1.1	$\begin{bmatrix} -1 &&&&& \\ & -1 &&&& \\ && -1 &&& \\ &&& \eta & -i\eta' & \\ &&& i\eta' & \eta & \\ &&&&& \eta \end{bmatrix}$	$\omega(1, nl_1, 0, nl_3)$	$(\eta-\eta' l_3)$
4.1.2		$\omega(1, nl_1, 0, nl_3)$	$(\eta+\eta' l_3)$
4.2.1		$\omega(1, 0, 0, n)$	$(\eta-\eta')$
4.2.2		$\omega(1, 0, 0, n)$	$(\eta+\eta')$
4.3		$\omega(1, n, 0, 0)$	η
5.1.1	$\begin{bmatrix} -\varepsilon &&& i\varepsilon' && \\ & -\varepsilon &&& i\varepsilon' & \\ && -\varepsilon &&& i\varepsilon' \\ -i\varepsilon' &&& 1 && \\ & -i\varepsilon' &&& 1 & \\ && -i\varepsilon' &&& 1 \end{bmatrix}$	$\omega(1,0,0,n)$	$(\varepsilon-2\varepsilon'\sqrt{\varepsilon})^{-1}$
5.2.1		$\omega(1,0,0,n)$	$(\varepsilon+2\varepsilon'\sqrt{\varepsilon})^{-1}$

Table V.5:

Fourier Coefficients of EM-Fields in the Laboratory Frame (Normalized Amplitude $E_o = 1$).

Fields	Configurations (see Table V.1 and V.4)				
	1.1.1	1.1.2	1.2	1.3.1	1.3.2
E_x	0	$\varepsilon_\perp^{-1} l_3$	$\varepsilon_\perp^{-1} E_o$	0	0
E_y	1	0	$-i\varepsilon_\perp^{-1} E_1$	1	0
E_z	0	$-\varepsilon_\parallel^{-1} l_1$	0	0	$-\frac{1}{n^2}$
B_x	$-nl_3$	0	$i\,\frac{1}{n}\,E_1$	0	0
B_y	0	$\frac{1}{n}$	$\frac{1}{n}\,E_o$	0	$\frac{1}{n}$
B_z	nl_1	0	0	n	0
D_x	0	l_3	E_o	0	0
D_y	n^2	0	$-iE_1$	n^2	0
D_z	0	$-l_1$	0	0	-1
H_x	$-nl_3$	0	$i\,\frac{1}{n}\,E_1$	0	0
H_y	0	$\frac{1}{n}$	$\frac{1}{n}\,E_o$	0	$\frac{1}{n}$
H_z	nl_1	0	0	n	0
P_x	0	$(1-\varepsilon_\perp^{-1})l_3$	$(1-\varepsilon_\perp^{-1})E_o$	0	0
P_y	n^2-1	0	$-i(1-\varepsilon_\perp^{-1})E_1$	n^2-1	0
P_z	0	$-(1-\varepsilon_\parallel^{-1})l_1$	0	0	$-\frac{1}{n^2}(n^2-1)$
M_x	0	0	0	0	0
M_y	0	0	0	0	0
M_z	0	0	0	0	0

Table V.5 (continued):

Fields	Configurations (see Table V.1 and V.4)				
	2.1.1	2.1.2	2.2	2.3.1	2.3.2
E_x	nl_3	0	$-i\frac{1}{n}E_1$	0	0
E_y	0	$-\frac{1}{n}$	$-\frac{1}{n}E_o$	0	$-\frac{1}{n}$
E_z	$-nl_1$	0	0	$-n$	0
B_x	0	l_3	E_o	0	0
B_y	n^2	0	$-iE_1$	n^2	0
B_z	0	$-l_1$	0	0	-1
D_x	nl_3	0	$-i\frac{1}{n}E_1$	0	0
D_y	0	$-\frac{1}{n}$	$-\frac{1}{n}E_o$	0	$-\frac{1}{n}$
D_z	$-nl_1$	0	0	$-n$	0
H_x	0	$\mu_\perp^{-1}l_3$	$\mu_\perp^{-1}E_o$	0	0
H_y	1	0	$-i\mu_\perp^{-1}E_1$	1	0
H_z	0	$-\mu_\parallel^{-1}l_1$	0	0	$-\frac{1}{n^2}$
P_x	0	0	0	0	0
P_y	0	0	0	0	0
P_z	0	0	0	0	0
M_x	0	$(1-\mu_\perp^{-1})l_3$	$(1-\mu_\perp^{-1})E_o$	0	0
M_y	n^2-1	0	$-i(1-\mu_\perp^{-1})E_1$	n^2-1	0
M_z	0	$-(1-\mu_\parallel^{-1})l_1$	0	0	$-\frac{1}{n^2}(n^2-1)$

Table V.5 (continued):

Fields	Configurations (see Table V.1 and V.4)				
	3.1.1	3.1.2	3.2.1 5.1.1	3.2.2 5.1.2	3.3
E_x	l_3	l_3	1	1	0
E_y	i	$-i$	i	$-i$	0
E_z	$-l_1$	$-l_1$	0	0	1
B_x	$-inl_3$	inl_3	$-in$	in	0
B_y	n	n	n	n	n
B_z	inl_1	$-inl_1$	0	0	0
D_x	$\sqrt{\varepsilon}nl_3$	$\sqrt{\varepsilon}nl_3$	$\sqrt{\varepsilon}n$	$\sqrt{\varepsilon}n$	0
D_y	$i\sqrt{\varepsilon}n$	$-i\sqrt{\varepsilon}n$	$i\sqrt{\varepsilon}n$	$-i\sqrt{\varepsilon}n$	0
D_z	$-\sqrt{\varepsilon}nl_1$	$-\sqrt{\varepsilon}nl_1$	0	0	ε
H_x	$-i\sqrt{\varepsilon}l_3$	$i\sqrt{\varepsilon}l_3$	$-i\sqrt{\varepsilon}$	$i\sqrt{\varepsilon}$	0
H_y	$\sqrt{\varepsilon}$	$\sqrt{\varepsilon}$	$\sqrt{\varepsilon}$	$\sqrt{\varepsilon}$	n
H_z	$i\sqrt{\varepsilon}l_1$	$-i\sqrt{\varepsilon}l_1$	0	0	0
P_x	$(n\sqrt{\varepsilon}-1)l_3$	$(n\sqrt{\varepsilon}-1)l_3$	$(n\sqrt{\varepsilon}-1)$	$(n\sqrt{\varepsilon}-1)$	0
P_y	$i(n\sqrt{\varepsilon}-1)$	$-i(n\sqrt{\varepsilon}-1)$	$i(n\sqrt{\varepsilon}-1)$	$-i(n\sqrt{\varepsilon}-1)$	0
P_z	$-(n\sqrt{\varepsilon}-1)l_1$	$-(n\sqrt{\varepsilon}-1)l_1$	0	0	$(\varepsilon-1)$
M_x	0	0	0	0	0
M_y	0	0	0	0	0
M_z	0	0	0	0	0

Table V.5 (continued):

Fields	Configurations (see Table V.1 and V.4)				
	4.1.1	4.1.2	4.2.1	4.2.2	4.3
E_x	l_3	l_3	1	1	0
E_y	i	$-i$	i	$-i$	0
E_z	$-l_1$	$-l_1$	0	0	1
B_x	$-inl_3$	inl_3	$-in$	in	0
B_y	n	n	n	n	n
B_z	inl_1	$-inl_1$	0	0	0
D_x	$\frac{1}{\sqrt{\mu}} nl_3$	$\frac{1}{\sqrt{\mu}} nl_3$	$\frac{1}{\sqrt{\mu}} n$	$\frac{1}{\sqrt{\mu}} n$	0
D_y	$i \frac{1}{\sqrt{\mu}} n$	$-i \frac{1}{\sqrt{\mu}} n$	$i \frac{1}{\sqrt{\mu}} n$	$-i \frac{1}{\sqrt{\mu}} n$	0
D_z	$-\frac{1}{\sqrt{\mu}} nl_1$	$-\frac{1}{\sqrt{\mu}} nl_1$	0	0	1
H_x	$-i \frac{1}{\sqrt{\mu}} l_3$	$i \frac{1}{\sqrt{\mu}} l_3$	$-i \frac{1}{\sqrt{\mu}}$	$i \frac{1}{\sqrt{\mu}}$	0
H_y	$\frac{1}{\sqrt{\mu}}$	$\frac{1}{\sqrt{\mu}}$	$\frac{1}{\sqrt{\mu}}$	$\frac{1}{\sqrt{\mu}}$	$\frac{1}{n}$
H_z	$i \frac{1}{\sqrt{\mu}} l_1$	$-i \frac{1}{\sqrt{\mu}} l_1$	0	0	0
P_x	0	0	0	0	0
P_y	0	0	0	0	0
P_z	0	0	0	0	0
M_x	$-i(n-\frac{1}{\sqrt{\mu}})l_3$	$i(n-\frac{1}{\sqrt{\mu}})l_3$	$-i(n-\frac{1}{\sqrt{\mu}})$	$i(n-\frac{1}{\sqrt{\mu}})$	0
M_y	$(n-\frac{1}{\sqrt{\mu}})$	$(n-\frac{1}{\sqrt{\mu}})$	$(n-\frac{1}{\sqrt{\mu}})$	$(n-\frac{1}{\sqrt{\mu}})$	$\frac{1}{n}(n^2-1)$
M_z	$i(n-\frac{1}{\sqrt{\mu}})l_1$	$-i(n-\frac{1}{\sqrt{\mu}})l_1$	0	0	0

S U M M A R Y

In the first chapter some preliminary remarks on the symmetry of systems and the symmetry of equations are made. Chapter II gives the background in electromagnetic theory necessary for the following chapters.

In chapter III the space-time symmetry is investigated of transverse electric (TE) and transverse magnetic (TM) modes in empty rectangular waveguides and resonant cavities [3]. It is shown that in some of these cases relativistic symmetries occur in the form of Lorentz mirrors, i.e. reflections with respect to a hyperplane perpendicular to a space or time direction conjugated by a special Lorentz transformation (boost). In section III.1 the general procedure applied to determine the symmetry groups of EM-fields is illustrated by means of an example. This procedure is essentially the same for the vacuum case and for EM-fields in material media. Some of the symmetry groups turn out to be generalized magnetic groups, i.e. crystallographic space groups in four dimensions with a Shubnikov point group. By Shubnikov transformation we mean Euclidean space transformations, combined or not, with time reversal. The identification of certain relativistic symmetry groups with generalized magnetic groups is discussed in section III.2. In the last section the results of this chapter are summed up.

The relativistic and Galilean symmetry groups of TEM waves in isotropic, and homogeneous media are determined in chapter IV[28]. The first section treats the relativistic symmetries of a TEM wave in the Minkowski, Chu, Ampère and Boffi formulations (see ref. 2)) and for linear, circular and elliptical polarization of the wave. The special limit of refractive index n = 1 is discussed and compared with the vacuum case. Section IV.2 gives an example: the determination of the relativistic symmetry elements of a TEM wave

in the Chu formulation. The Galilean symmetry groups and the non-relativistic approximation of the relativistic symmetry groups form the subject discussed in section IV.3. An interesting relation is found between the relativistic symmetry groups and the Galilei symmetry groups of the field tensors $F_{\mu\nu}$ and $G^{\mu\nu}$ in the Minkowski formulation. The last section sums up the results of this chapter and in an appendix some more details are given on the non-relativistic approximation of parabolic Lorentz transformations and on the corresponding Galilei transformations.

The last chapter deals with TEM waves in anisotropic and homogeneous media. In the first section some methods for deriving the explicit expressions for the EM-fields of a TEM wave in an anisotropic homogeneous medium are sketched. In section V.2 the classification of transparent media from an optical point of view into anaxial, uniaxial and biaxial media is discussed. An example is given in section V.3 illustrating in some detail how the new symmetry elements (new with respect to the isotropic case) are found. In the last section the results of this chapter are discussed.

The most important results of the present work can be summarized in the following way:

(i) Only in the vacuum case one single relativistic symmetry group can be associated with a TEM wave. In a medium characterized by constant electric and magnetic permeabilities ε and μ there are several different relativistic symmetry groups associated with a TEM wave: the symmetry groups of the various fields.

(ii) In the cases where a wave rest frame can be defined (i.e. for $n > 1$ and for the propagating modes in waveguides) conjugated boosts and Lorentz mirrors are found as non-trivial relativistic symmetry elements in the laboratory frame. The conjugation is given by the transformation relating the laboratory frame (rest frame of material medium if present) and the wave rest frame. This is of course no more so for $n = 1$ or for a single TEM wave in vacuum, in which cases the Lorentz mirrors disappear and the boosts become parabolic Lorentz transformations (see appendix IV.B).

(iii) For a homogeneous medium characterized by electric and magnetic permeabilities ε and μ the relativistic symmetry groups depend on the refractive index n only (and not on ε and μ seperately).

(iv) For a given field of a TEM wave in a material medium the relativistic symmetry group depends on the formulation chosen, i.e. on the macroscopic description adopted. It is found that for all formulations the relativistic symmetry groups of the various EM-fields are generated by the same set of symmetry elements. The four formulations differ merely in the way the individual generators are assigned to the relativistic symmetry groups of the various EM-fields.

(v) In the limit $n \to 1$, the relativistic symmetry group of a TEM wave in a medium of refractive index n does not coincide with the relativistic symmetry group of a TEM wave in vacuum.

(vi) For refractive index $n > 1$ and isotropic, homogeneous medium, the non-relativistic limit of the relativistic symmetry groups of a TEM wave is precisely the corresponding Galilei symmetry group. For $n = 1$ and for a TEM wave in vacuum this is not true any more.

(vii) Considered in the isotropic limit the relativistic symmetry groups for a TEM wave propagating in a uniaxial, homogeneous medium coincide with the corresponding relativistic symmetry groups of the isotropic case.

(viii) Of the anisotropic media treated, only the uniaxial electric and the uniaxial magnetic ones lead to new results as compared with the isotropic case.

S A M E N V A T T I N G

Het eerste hoofdstuk bevat voorbereidende opmerkingen over de symmetrie van systemen en de symmetrie van vergelijkingen. In hoofdstuk II wordt een samenvatting gegeven van de electrodynamica die gebruikt wordt in de volgende hoofdstukken.

In hoofdstuk III wordt de symmetrie in ruimte en tijd onderzocht van transversale elektrische (TE) en transversale magnetische (TM) modes in rechthoekige golfgeleiders en trilholten [3]. Het wordt aangetoond, dat in sommige gevallen relativistische symmetrieën voorkomen. In het laboratorium-systeem hebben de niet-triviale relativistische symmetrieën de gedaante van Lorentz-spiegels, d.w.z. Lorentz-transformaties geconjugeerd met spiegelingen aan een hypervlak loodrecht op de tijdas of op een zuiver ruimtelijke vector, door een speciale Lorentz-transformatie. In paragraaf III.2 wordt de algemene procedure, die wordt gebruikt om de symmetrie van EM-velden te bepalen, geillustreerd met een voorbeeld. Deze procedure is in wezen dezelfde voor het geval van vacuum als voor dat van EM-velden in materie. Enige van deze symmetrie-groepen blijken gegeneraliseerde magnetische groepen te zijn, d.w.z. kristallografische ruimtegroepen in vier dimensies met een Shubnikov-puntgroep. Onder Shubnikov-transformatie verstaan we transformaties van de Euklidische ruimte al of niet gekombineerd met tijdsomkeer. De identificatie van bepaalde relativistische symmetrie-groepen met gegeneraliseerde magnetische groepen wordt besproken in paragraaf III.2. In de laatste paragraaf worden de resultaten van dit hoofdstuk opgesomd.

De relativistische en Galilei-symmetriegroepen van TEM golven in isotrope, homogene media worden bepaald in hoofdstuk IV[28]. De eerste paragraaf behandelt de relativistische symmetrie van een TEM golf in de formulering van Minkowski, Chu, Ampère en Boffi (zie ref. 2)) en zowel voor lineaire, circulaire en elliptische polarisatie van de golf. De speciale limiet van brekingsindex n = 1 wordt besproken en vergeleken met het geval van vacuum. In paragraaf IV.2 wordt een voorbeeld gegeven: de bepaling van de relativistische symmetrie-elementen van een TEM golf in de Chu formulering. De Galilei-symmetriegroepen en de niet-relativistische benadering van de relativistische symmetriegroepen vormen het onderwerp van paragraaf IV.3. Men vindt een interessante relatie tussen de relativistische symmetriegroepen en de Galilei-symmetriegroepen van de veldtensoren $F_{\mu\nu}$ en $G^{\mu\nu}$ in de formulering van Minkowski. In de laatste paragraaf worden de resultaten van dit hoofdstuk opgesomd. In een appendix worden details gegeven van de niet-relativistische benadering van parabolische Lorentz-transformaties en hun corresponderende Galilei-transformaties.

Het laatste hoofdstuk behandelt TEM golven in anisotrope, homogene media. In de eerste paragraaf worden enige methoden geschetst om expliciete uitdrukkingen af te leiden voor de EM-velden van een TEM golf in een anisotroop, homogeen medium. In paragraaf V.2 wordt de classificatie van transparante media besproken in het raam der optica. Men heeft dan anaxiale, uniaxiale en biaxiale media. Een voorbeeld wordt gegeven in paragraaf V.3, waar in detail wordt aangetoond hoe de nieuwe symmetrie-elementen (nieuw vergeleken met het isotrope geval) worden gevonden. In de laatste paragraaf worden de resultaten van dit hoofdstuk besproken.

De belangrijkste resultaten van dit onderzoek kunnen als volgt worden samengevat:

(i) Alleen in het geval van vacuum kan een enkele relativistische symmetriegroep worden toegekend aan een TEM golf. In een medium, dat wordt gekenmerkt door konstante elektrische en magnetische permeabiliteiten ϵ en μ, zijn er verschillende relativistische symmetriegroepen toe te kennen aan een TEM golf: de symmetriegroepen van de verschillende velden.

(ii) In de gevallen, waar een rustsysteem van een golf kan worden gedefinieerd (d.w.z. voor n > 1 een voor zich voortplantende modes in golfgeleiders) worden gekonjugeerde speciale Lorentz-transformaties en Lorentz-spiegels als niet-triviaal-relativistische symmetrie-elementen in het laboratoriumsysteem gevonden. De konjugatie wordt bepaald door de transformatie, die het laboratoriumsysteem (het rustsysteem van de eventueel aanwezige materie) en het rustsysteem van de golf verbindt. Dit is natuurlijk niet langer zo voor $n = 1$ of voor een enkele TEM golf in vacuum, in welke gevallen de Lorentz-spiegels verdwijnen en de speciale Lorentz-transformaties overgaan in parabolische Lorentz-transformaties (zie appendix IV.B).

(iii) Voor een homogeen medium, gekarakteriseerd door elektrische en magnetische permeabiliteiten ε en μ, hangen de relativistische symmetriegroepen alleen af van de brekingsindex n (en niet van ε en μ afzonderlijk).

(iv) Voor een gegeven veld van een TEM golf in materie hangt de relativistische symmetrie groep af van de gekozen formulering, d.w.z. van de gebruikte macroscopische beschrijving. Men vindt, dat voor alle formuleringen de relativistische symmetriegroepen van de verschillende EM-velden voortgebracht worden door dezelfde verzameling symmetrie-elementen. Het verschil tussen de vier formuleringen zit in de manier waarop de afzonderlijke generatoren bij de symmetriegroepen van de verschillende EM-velden horen.

(v) De relativistische symmetriegroepen van een TEM golf in een medium gekarakteriseerd door de brekingsindex n is in de limiet $n = 1$ niet dezelfde als de relativistische symmetriegroep van een TEM golf in vacuum.

(vi) In de formulering van Minkowski is de niet-relativistische limiet van de relativistische symmetriegroepen van een TEM golf in een isotroop homogeen medium met brekingsindex $n > 1$ identiek met de corresponderende Galilei-symmetriegroepen. In het geval van $n = 1$ en in het geval van een TEM golf in vacuum is dit niet meer waar.

(vii) In de isotrope limiet gaan de relativistische symmetriegroepen van een TEM golf in een uniaxiaal, homogeen medium over in de corresponderende relativistische symmetriegroepen van een TEM golf in een isotroop medium.

(viii) Van alle anisotrope media, die in dit proefschrift zijn behandeld,leveren, vergeleken met het isotrope medium, alleen de uniaxiaal elektrische en de uniaxiaal magnetische media nieuwe resultaten op.

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STELLINGEN

I

Bepaalt men de electromagnetische velden met behulp van de methode van Birss en Shrubsall, dan moet de susceptibiliteitstensor, in het geval van magneto-electrisch effect, voldoen aan zekere condities, die niet worden genoemd door Birss en Shrubsall.

Birss, R.R. and Shrubsall, R.G., Phil.Mag. 15(1967)687.

II

In de klassieke mechanika kan ieder autodiffeomorfisme van de configuratie-ruimte worden uitgebreid tot een kanonieke transformatie van de fase-ruimte.

Abraham, R., Foundations of Mechanics, Benjamin (Amsterdam,1967).

III

De tijdsomkeer-operatie, $(q'_i,p'_k)=(q_i,-p_k)$, is alleen dan een kanonieke transformatie van de fase-ruimte, als de dimensie daarvan gelijk is aan 4n, n een natuurlijk getal.

IV

Noch de groep van kanonieke transformaties, noch de groep gedefinieerd door van Hove, bevat alle symmetrie-transformaties van een willekeurige Hamilton-funktie.

Van Hove, L., Sciences.-T.XXVI,fasc.6,(1951).

VI

In de meeste boeken over electrodynamica worden de transformatie wetten van electromagnetische velden en van de daarbij horende potentialen onder Lorentz-transformaties onvolledig behandeld.

VII

De transformatie eigenschappen van klassieke velden zo als gegeven door Roman zijn onjuist.

Roman, P., Introduction to Quantum Field Theory, Wiley (New York, 1969)p.16.

VIII

Op den duur zal de wereld om ons heen alleen leefbaar kunnen worden gehouden, als de groei van de bevolking en de expansie van de industrie onder kontrole worden gebracht.

IX

In zijn school "Summerhill" heeft A.S. Neill gedurende meer dan dertig jaren, in de praktijk, een positieve bijdrage gegeven tot een meer menswaardige opvoeding van kinderen.

Neill, A.S., Theorie und Praxis der antiautoritären Erziehung, Rowohlt (Reinbek bei Hamburg, 1969); Summerhill: Pro und Contra, Rowohlt (Reinbek bei Hamburg, 1970).

A. Bieri

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