EFFECTIVE RANGE FORMALISM AND THE ELECTROMAGNETIC INTERACTION WITH APPLICATIONS TO PROTON-PROTON SCATTERING

G. J. M. AUSTEN

EFFECTIVE RANGE FORMALISM AND THE ELECTROMAGNETIC INTERACTION WITH APPLICATIONS TO PROTON-PROTON SCATTERING

PROMOTOR:
PROF. DR. IR. J. J. DE SWART

EFFECTIVE RANGE FORMALISM AND THE ELECTROMAGNETIC INTERACTION WITH APPLICATIONS TO PROTON-PROTON SCATTERING

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE
WISKUNDE EN NATUURWETENSCHAPPEN
AAN DE KATHOLIEKE UNIVERSITEIT TE NIJMEGEN, OP GEZAG VAN
DE RECTOR MAGNIFICUS, PROF. DR. P. G. A. B. WIJDEVELD,
VOLGENS BESLUIT VAN HET COLLEGE VAN DEKANEN
IN HET OPENBAAR TE VERDEDIGEN
OP DONDERDAG 18 MAART 1982
DES NAMIDDAGS TE 4 UUR

door

GERARDUS JOZEF MATHIAS AUSTEN

geboren te Heerlen

krips repro meppel

Dankwoord

Op deze plaats wil ik graag iedereen bedanken die het totstandkomen van dit proefschrift mogelijk heeft gemaakt.

In de eerste plaats zijn dit de huidige en voormalige medewerkers van het Instituut voor Theoretische Fysica te Nijmegen die door middel van opbouwende discussies en ideeen steeds voor mij klaar stonden. In het bijzonder zou ik hier willen noemen Dr. M.M. Nagels en Dr. T.A. Rijken op wiens werk ik heb voortgebouwd. Zonder de inspanningen van Drs. W. van der Sanden zou het niet mogelijk zijn geweest om de ideeen in de praktijk te toetsen en de programmatuur in korte tijd klaar te krijgen.

Verder wil ik Mevrouw W. Vink danken, die op bekwame wijze ervoor gezorgd heeft dat het manuscript uitgetikt en drukklaar gemaakt werd.

Bovendien wil ik noemen de medewerkers van de diverse dienstverlenende afdelingen van de Faculteit der Wiskunde en Natuurwetenschappen, in het bijzonder de afdeling Illustratie en de medewerkers van het Universitair Reken Centrum.

CONTENTS

In this thesis on effective range theory and the electromagnetic interaction, we study two aspects which are of importance in the analysis of proton-proton scattering data and the description of this process with the help of a potential model.

The first part of this thesis describes a generalized effective range formalism which at present is used to analyse experimental proton-proton scattering data for laboratory energies below 30 MeV. In principle it could be used up to 300 MeV; in this way the original energy region for which effective range theory was believed to work fine (0 - 10 MeV) is extended up to the first inelastic channel.

The main goal of the first part is to provide a thorough basis for the modified effective range theory. We will consider only single channel scattering and omit generalizations to coupled channels (which are rather straightforward). As an application we will show this theory can be used to determine the pion-nucleon coupling constant.

The effective range theory was originally developed fourty years ago [La 44, Sch 47]. In its most simple form it gives a prescription of the total phase shift as a function of the energy in terms of a few phenomenological parameters. However, in proton-proton scattering this total phase shift is very difficult to parametrize since it contains also a part which is generated by the electric interaction. Therefore this part is subtracted from the total phase shift and the remainder, the so-called nuclear phase shift is parametrized. In this way Coulomb- and vacuum polarization corrected effective range formulas have been derived in the past [Be 49, He 60].

Also one has tried to incorporate in the effective range theory some information of the most well-known part of the nuclear interaction the one pion exchange mechanism. However, this was done only in an approximated form or with the help of techniques which are difficult to handle in practice [CFS 59, No 64]. In the most frequently used method - the Cini, Fubini, Stanghellini approximation - two approximations were made which tend to cancel each other in the 0 - 10 MeV region. the one pion cut is replaced by one pole - weakening the total attraction - and the remainder of the interaction is parametrized such that it cannot support the short range repulsion, which has the reverse effect. The first approximation appeared to be critical when a few years ago the second approximation was removed [Em 79, Sa 79] and a pion-nucleon constant $g^2/4\pi = 20$ was obtained from the 0 - 30 MeV proton-proton data.

In this thesis we will show how to incorporate in the effective range theory every interaction which can be solved in closed form or computed with numerical methods. Also we will pay attention to the phenomenological representation of that part of the interaction that is unknown (or less well-known)

The basic idea behind the generalized effective range theory is that the interaction between two particles can be considered as to consist of two parts. one part for which one has a good theoretical description (for instance a potential model) and another part for which this description does not exist or is too complicated to handle in first instance. In proton-proton scattering one could consider the electric interaction and the long range pion-exchange mechanism as the well-known part, the medium- and short-range nuclear interaction

as the second and less well-known part.

The generalized effective range theory will combine these two aspects of the interaction in a simple way and predict phase shifts as a function of the energy once one has choosen a suitable parametrization of the second part of the interaction; independent of this phenomenological parametrization the first part of the interaction, included in the definition of the effective range function, is always represented correctly in the total phase shift.

The success of effective range theory strongly depends on the amount of the interaction which can be taken into account exactly. Every part of the interaction which is not or incorrectly incorporated in the basic effective range functions must be represented in the phenomenological part. This is especially the case for the long range electric interaction since it involves an infinite number of terms in a partial wave series of the amplitude.

Therefore this electromagnetic interaction is considered in detail in the second part of this thesis. A method is developed to obtain a potential in configuration space which describes the electromagnetic interaction not only in lowest (nonrelativistic) order, but which also contains relativistic corrections. It is possible to apply this potential to higher scattering energies or to situations where, due to other interactions, the particles obtain relativistic velocities in the potential region. Effective range functions for this "relativistic" Coulomb potential are derived in the last chapter.

The results of the second part of this thesis are not only of importance for the actual application of the effective range theory. In fact the same method can also be applied to the strong interactions.

Starting from a field theoretical description of the interaction, we will review and modify the standard techniques which are used to obtain effective potentials which must describe this interaction. Since this reduction can only be performed by successive approximations, we will use a method based on an expansion in terms of the nucleon mass. In this way it is possible to pursue a consistent course in the book-keeping of terms which must be kept and neglected.

Part One:

Effective Range Theory

CHAPTER I

Effective range theory: introduction and historical review

1. Basic effective range theory

Effective range theory has been used in physics for more than thirty years to characterize and analyse two particle scattering in the low energy (near threshold) region. In this thesis we will deal mainly with two nucleon scattering. However, the results can often be applied to any other two particle system.

Effective range theory borrows its importance from the fact that for low energies, scattering is mainly determined by the s-wave phase shift. This can be verified experimentally by observing the spherical symmetry of the scattering in the center of mass system at low energies*. Experimentally we can measure the cross section at different energies of the incident particle which gives the behaviour of the s-wave phase shift δ_0 as a function of the energy. It appears that δ_0 (and therefore the cross section), up to first order, is determined by just two numbers as a function of the energy near threshold in the effective range approximation:

$$k \cot \delta_0 = -\frac{1}{a} + \frac{1}{2} r k^2 \qquad (1)$$

with k the center of mass momentum of a nucleon**.

---

* For proton-proton scattering however, little is seen of this symmetry due to the Coulomb interaction, but apart from this complication the discussion will hold for a suitable definition of the phase shift. We will return to this point later.

** The usual convention $\hbar = c = 1$ is used.

The quantity "a" is called the scattering length and is related to the total cross section at zero energy by (neglecting Coulomb complications):

$$\sigma_{tot}(E = 0) = 4\pi\, a^2 \quad * \qquad (2)$$

which shows that a has the dimension of a length. The other quantity "r" is called the effective range and has also the dimension of a length. The name "scattering length" comes from the observation that the scattering wave function (radial, s-wave), outside the range of the potential appears to be:

$$\sim \sin(kr + \delta_0) \underset{\delta \to 0}{\approx} \sin(k(r-a))$$

and seems to be shifted over a length "a" with respect to the wave function without scattering. The value of the "effective range" is a rough indication of the spatial size of the interaction as we will see later.

The fact that the function $k \cot \delta_0$ (and in proton-proton scattering a combination like. $k\, C_0^2(\eta) \cot \delta_0 + 2\eta k\, h(\eta)$) has been taken to parametrize, is explained in effective range theory which was developed by Landau [La 44] and by Schwinger [Sch 47] using the variational method and further exploited by Blatt and Jackson [Bl 49]. A simpler deduction was proposed by Bethe [Be 49] who also modified the theory for proton-proton scattering (Coulomb interaction). In this chapter we will

---

*For neutron-proton scattering there are in fact two s-wave phases, one dealing with total spin triplet, the other with total spin singlet scattering and the formula becomes:

$$\sigma_{tot} = 4\pi \left(\frac{3}{4} a_t^2 + \frac{1}{4} a_s^2 \right) .$$

follow Bethe's method since it is simple and intuitively clear.

Consider the scattering of a particle in the $\ell = 0$ state by a (central) potential (or two particle scattering in the "relative" system).

The radial part of the wave function satisfies the radial Schrodinger equation; for energy E_1 and wave function u_1:

$$u_1'' + (k_1^2 - V(r))\ u_1 = 0 \tag{3a}$$

with k_1 the center of mass momentum and $V(r)$* the potential multiplied with two times the reduced mass. For another energy E_2 and wave function u_2, we have:

$$u_2'' + (k_2^2 - V(r))\ u_2 = 0 \tag{3b}$$

Equations (3a) and (3b) are second order differential equations and each has two independent solutions. The "physical" wave function has the boundary condition:

$$u_1(0) = u_2(0) = 0 \tag{4}$$

Phase shifts δ_1 and δ_2 are determined by the asymptotic ($r \to \infty$) limit of these solutions; for potentials which fall off faster than $\frac{1}{r}$ for $r \to \infty$ one has:

$$\begin{aligned} u_1(r) &\underset{r\to\infty}{\sim} \sin(k_1 r + \delta_1) \\ u_2(r) &\underset{r\to\infty}{\sim} \sin(k_2 r + \delta_2) \end{aligned} \tag{5}$$

Let us normalize the wave functions by:

---

*For the moment the potential is assumed to be energy independent and local.

$$u_1(r) \xrightarrow[r\to\infty]{} \frac{\sin(k_1 r + \delta_1)}{\sin \delta_1} \equiv \bar{u}_1(r)$$
$$u_2(r) \xrightarrow[r\to\infty]{} \frac{\sin(k_2 r + \delta_2)}{\sin \delta_2} \equiv \bar{u}_2(r) \qquad (6)$$

which implies:

$$\bar{u}_1(0) = \bar{u}_2(0) = 1$$

and

$$\bar{u}_1'' + k_1^2 \bar{u}_1 = \bar{u}_2'' + k_2^2 \bar{u}_2 = 0$$

Multiplying eq. (3a) with u_2 and eq. (3b) with u_1 and subtracting both equations, we obtain after integrating the result over r from 0 to infinity:

$$(u_2 u_1' - u_1 u_2')\Big|_0^\infty = (k_2^2 - k_1^2) \int_0^\infty u_1 u_2 \, dr \qquad (7a)$$

The same procedure gives for the barred functions:

$$(\bar{u}_2 \bar{u}_1' - \bar{u}_1 \bar{u}_2')\Big|_0^\infty = (k_2^2 - k_1^2) \int_0^\infty \bar{u}_1 \bar{u}_2 \, dr \qquad (7b)$$

Subtracting the last two equations the values at infinity at the left hand side do not contribute since barred and unbarred functions coincide by definition. For r = 0 there is only a contribution from the barred ones due to boundary condition (4) and therefore, using eq. (6) we get:

$$k_2 \cot \delta_2 - k_1 \cot \delta_1 = (k_2^2 - k_1^2) \int_0^\infty (\bar{u}_1 \bar{u}_2 - u_1 u_2) \, dr \qquad (8)$$

This equation is exact and forms the basis of the classic effective range theory. As it stands, however, it gives no immediate information since the barred functions in the integrand at the right hand side contain the phase shifts (appearing also in the left hand side) and the unbarred physical wave functions are not measurable. Equation (8) can be derived

in another way when one realizes that the physical wave functions u_1 and u_2 are orthogonal for $E_1 \neq E_2$ and therefore:

$$(k_2^2 - k_1^2) \int_0^\infty (\bar{u}_1\bar{u}_2 - u_1u_2)\, dr = (k_2^2 - k_1^2) \int_0^\infty \bar{u}_1\bar{u}_2\, dr \quad .$$

Since $\bar{u}_1$ and $\bar{u}_2$ are explicitly known (eq. (6)), the remaining integral can be calculated, resulting into the left hand side of eq. (8) (see [Au 58]). Defining the "mixed effective range":

$$\rho(k_1^2, k_2^2) = 2 \int_0^\infty (\bar{u}_1\bar{u}_2 - u_1u_2)\, dr \tag{9}$$

eq. (8) becomes:

$$k_2 \cot \delta_2 - k_1 \cot \delta_1 = \frac{1}{2} (k_2^2 - k_1^2)\, \rho(k_1^2, k_2^2) \tag{8b}$$

We observe that the main contribution to the integral, defining ρ, comes from <u>inside</u> the potential range. In this region the physical wave function will depend only slightly on the momentum (if it is not too big). Also the comparison functions $\bar{u}_1$ and $\bar{u}_2$ are in first order independent on the energy for small r since they are defined to be 1 in the origin. Therefore the <u>approximation</u>:

$$\rho(k_1^2, k_2^2) \simeq \rho(0,0) = r \tag{10a}$$

is probably not too crude for small momenta. Using the result of standard scattering theory that the s-wave phase shift goes to zero linear in k one defines:

$$-\frac{1}{a} = \lim_{k_1 \to 0} k_1 \cot \delta_1 \tag{10b}$$

In this way we obtain the effective range <u>approximation</u> ($k \equiv k_2$):

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} r k^2 \tag{11}$$

which is the lowest order Taylor expansion of the <u>exact</u> formula:

$$k \cot \delta = -\frac{1}{a} + \frac{1}{2} k^2 \rho(k^2, 0) \tag{12}$$

It is difficult to make an estimate of the accuracy of the approximation eq. (11) without explicit knowledge of the potential. The arguments based on the energy independence of the physical and comparison wave functions lead to the conditions:

$$\left.\begin{aligned} k^2 &\ll V(r) \\ kr &\ll 1 \end{aligned}\right\} \quad \text{inside the potential region}$$

For the neutron-proton scattering system in the $^1S_0$ state, which potential in the outer regions approximates the one pion potential:

$$0.16\ M_{red}\ \frac{e^{-m_\pi r}}{r}$$

these conditions (for $r = \frac{1}{m_\pi}$) lead to:

$$E_{lab} \ll 20 \text{ MeV}.$$

Another estimate is based on the analytic properties in k^2 of eq. (12). The s-wave scattering amplitude can be written as:

$$\frac{e^{2i\delta} - 1}{2ik} = \frac{1}{k \cot \delta - ik} = \frac{1}{-\frac{1}{a} + \frac{1}{2}\rho(k^2,0)k^2 - ik} \qquad (13)$$

It follows from standard scattering theory that this amplitude (and therefore $\rho(k^2,0)$) will possess dynamical singularities in the complex k plane starting at:

$$k = \pm i \frac{m_\pi}{2}$$

with m_π the lowest exchanged mass in the potential. Therefore an expansion of $\rho(k^2,0)$ will diverge for momenta: $|k| > \frac{m_\pi}{2}$ which corresponds to the condition:

$$E_{lab} > 10 \text{ MeV}.$$

In the neutron-proton system the effective range approximation appears to be good within a few per cent up to 40 MeV.

2. Effective range theory in the proton-proton system

The proton-proton system differs from the neutron-proton system (considered in section 1) in several aspects:

1. The interaction is complicated by long range electric interactions (Coulomb + vacuum polarization);
2. Experimental data are very accurate.

Due to the electric interaction the theory as developed in section 1 cannot be used. Besides the complication that the asymptotic form of the wave function as used in eq. (5) ff is not correct, it appears that effective range functions of the form: k cot δ will have an essential singularity at $E_{lab} = 0$ MeV due to the Coulomb interaction, and a cut starting at $- 10^{-3}$ MeV due to the vacuum polarization. These problems were solved by Bethe and Heller [Be 49, He 60].

The Coulomb singularities are removed in the effective range function (s-wave) $P^C(k^2)$:

$$P^C(k^2) \equiv k\, C_0^2(\eta)\, \cot \delta^C + 2\eta k\, h(\eta) \tag{14}$$

with δ^C the phase shift defined by the asymptotic limit of the physical wave function:

$$\begin{aligned} &u(0) = 0 \\ &u(r) \underset{r\to\infty}{\longrightarrow} \sim \sin\,(kr - \eta \log 2kr + \sigma_0 + \delta^C) \end{aligned} \tag{15}$$

η = Coulomb parameter = $\frac{M\alpha}{2k}$, σ_0 = Coulomb phase = arg $\Gamma(1 + i\eta)$.

In eq. (14) $C_0(\eta)$ and $h(\eta)$ are complicated functions which depend only on the Coulomb interaction. For their definition we refer to chapter III. A similar procedure can be applied to the vacuum polarization. Defining: $\delta^E = \delta^C - \tau_0$, τ_0 being the vacuum polarization phase shift, Heller derived the effective range function $P^{C,V}(k^2)$:

$$P^{c,v}(k^2) \equiv \frac{k\, C_0^2(\eta)}{1-\phi_0} [(1+\chi_0) \cot \delta^E - \tan \tau_0] + 2\eta k\, (h(\eta) + \ell_0(\eta)) \quad (16)$$

with χ_0, ϕ_0 and ℓ_0 functions obtained from the vacuum polarization interaction only. The function $P^{c,v}(k^2)$ removes also the nearest vacuum polarization singularities at - 10^{-3} MeV. It appears that the nearest singularity is now determined by the pion interaction at E_{lab} = - 10 MeV.

The behaviour of $P^{c,v}(k^2)$ as a function of the energy, is roughly the same as the one derived for the $^1S_0$ neutron-proton system eq. (12) and is discussed in chapter VI.

Due to the very accurate proton-proton sacttering data in the low energy region, the effective range approximation of the effective range function:

$$P^{c,v}(k^2) = - \frac{1}{a^{c,v}} + \frac{1}{2} r^{c,v} k^2 \quad (17)$$

is too crude; one adds another expansion parameter p - the shape - and obtains what is called the shape dependent approximation:

$$P^{c,v}(k^2) = - \frac{1}{a^{c,v}} + \frac{1}{2} r^{c,v} k^2 - p^{c,v} r^3 k^4 \quad (18)$$

This approximation is believed to work fine up to 10 MeV; for larger energies deviations become bigger (partially caused by the cut at - 10 MeV from the pion interaction). In the shape dependent approximation of $P^{c,v}(k^2)$ the parameters $a^{c,v}$, $r^{c,v}$ and $p^{c,v}$ are purely phenomenological parameters, determined by experiment only. To get rid of one parameter and also to extend approximately the radius of convergence of the series, the Cini-Fubini-Stanghellini approximation was derived [CFS 59, Wo 62, No 64]:

$$p^{c,v}(k^2) = -\frac{1}{a^{c,v}} + \frac{1}{2} r^{c,v} k^2 - r^3 \frac{p^{c,v} k^4}{1+q^{c,v} k^2} \qquad (19)$$

$p^{c,v}$, $q^{c,v}$ = functions of $a^{c,v}$, $r^{c,v}$ and pion coupling constant

We will comment later on the accuracy of this approximation.

From the discussion up to now it will be clear that effective range theory has two different aspects:

(i) the calculation of a suitable effective range function which must be as smooth as possible as a function of energy;

(ii) the choice of a suitable parametrization of the effective range function with a minimum number of phenomenological parameters.

These two points are closely connected with each other and with the analytic properties of the effective range function. We will not use Bethe's method, as described in section 1, for several reasons:

(i) For higher partial waves and/or Coulomb interactions it looses its simplicity and involves much computational work;

(ii) In Bethe's method the analytic properties of the effective range function do not clearly emerge;

(iii) It is difficult to find out which parametrization must be used and how many parameters will be needed.

General effective range theory for finite range potentials

1. Introduction

Consider single channel scattering, described by the radial Schrodinger equation:

$$\left(\frac{d^2}{dr^2} + (k^2 - \frac{\ell(\ell+1)}{r^2} - V)\right) u = 0 \qquad (1)$$

with V a local, energy independent potential (multiplied with two times the reduced mass), satisfying the following properties:

$V(r) \xrightarrow[r\to\infty]{}$ drops exponentially

$V(r)$ is analytic in r for Re $r \geq 0$.

Generalizations for potentials which have a $1/r$ tail or a simple pole at the origin will be treated in the following chapters.

The phase shift δ_ℓ of V, is determined from the asymptotic behaviour of the regular solution of (1), defined by:

$$u(r) \xrightarrow[r\to 0]{} \sim \frac{r^{\ell+1}}{(2\ell+1)!!}$$
$$u(r) \xrightarrow[r\to\infty]{} \text{constant} \times (\cos\delta_\ell\, j_\ell(kr) + \sin\delta_\ell\, n_\ell(kr))kr \qquad (2)$$

where j_ℓ and n_ℓ * are the spherical Bessel and Neumann functions.

The collection of all δ_ℓ's contains the scattering information of the potential V. For instance the scattering amplitude is given by:

---

*We take the Messiah convention for n_ℓ, i.e. an opposite sign for n_ℓ compared with most authors [Me 61]:

$$n_\ell(z) \xrightarrow[z\to 0]{} z^{-\ell-1}\,(2\ell-1)!!$$

$$f_V(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{e^{2i\delta_\ell} - 1}{2ik} P_\ell(\cos\theta) \qquad (3)$$

The phase shifts δ_ℓ will be fitted in such a way that they can reproduce the experimental information. To connect data at different energies one often uses the effective range expansion:

$$P(k^2) \equiv k^{2\ell+1} \cot\delta_\ell = -\frac{1}{a_\ell} + \frac{1}{2} r_\ell k^2 + \ldots \qquad (4)$$

However, the series expansion(4) has a radius of convergence in k^2 which is determined by the behaviour of the potential for $r \to \infty$.

Assuming:

$$V(r) \xrightarrow[r\to\infty]{} \sim e^{-\mu_L r} \qquad (5)$$

then we will prove that (4) is convergent for $|k^2| < \mu_L^2/4$.

Suppose the potential is a superposition of two parts:

$$V(r) = V_L(r) + V_S(r) \qquad (6)$$

with: $V_L(r) \xrightarrow[r\to\infty]{} \sim e^{-\mu_L r}$

$V_S(r) \xrightarrow[r\to\infty]{} \sim e^{-\mu_S r}$

and: $\mu_S > \mu_L$, V_L and V_S both analytic in r (Re $r \geq 0$).

The potential V_L will be supposed to be known in closed form and with a range which is <u>longer</u> than the potential V_S which is unknown (or not well known). The expansion of the effective range function (4) is limited by the potential V_L with the longest range.

The purpose of general effective range theory is to disentangle the information from the potentials V_L and V_S in such a way that finally we have a formalism in which only the V_S potential is represented by phenomenological parameters; this in contrast with eq. (4) where a_ℓ and r_ℓ parametrize V_S as well as V_L.

As applications of the decomposition (6) one could imagine:

np scattering with: $V_L = 0$

V_S = nuclear potential

which is in fact the trivial decomposition and will ultimately lead to the old effective range formalism, eq. (4);

or:

np scattering with: V_L = pion potential

V_S = nuclear minus the pion potential

After generalization to potentials with an r^{-1} tail and/or a simple pole in the origin:

pp scattering with: V_L = Coulomb potential

V_S = vacuum polarization + nuclear potential

which is obtained by using the effective range function $P^C(k^2)$, eq. (1.14);

or:

pp scattering with: V_L = Coulomb + vacuum polarization potential

V_S = nuclear potential

leading to the effective range function $P^{C,V}(k^2)$, eq. (16);

or:

pp scattering with: V_L = Coulomb + vacuum polarization + pion potential

V_S = nuclear minus the pion potential.

The effective range function $P^L(k^2)$ for a long range potential V_L, is determined by the properties of the solutions of the radial Schrödinger equation in which only V_L is present. This will be studied in the next section.

2. The long range potential V_L

Again we start with the radial Schrodinger equation, now for potential V_L only:

$$\left(\frac{d^2}{dr^2} + (k^2 - \frac{\ell(\ell+1)}{r^2} V_L) \right) v = 0 \qquad * \tag{7}$$

This second order linear differential equation has two independent solutions, which we will define by boundary conditions at the origin: the regular solution v_R with:

$$v_R(r) \underset{r\to 0}{\longrightarrow} \frac{1}{(2\ell+1)!!} r^{\ell+1} + O(r^{\ell+2}) \tag{8}$$

the irregular solution v_I with:

$$v_I(r) \underset{r\to 0}{\longrightarrow} (2\ell-1)!! \, r^{-\ell} + O(r^{-\ell+1}) \tag{9}$$

They satisfy the Wronskian relation:

$$W(v_R, v_I) = v_R' v_I - v_I' v_R = 1$$

The regular solution is uniquely determined by the boundary condition (8). The irregular solution, however, is not completely fixed by (9): one can add a multiple of the regular solution; we will return on this point in the following (see also appendix I).

Defining $\bar{j}_\ell$ and $\bar{n}_\ell$ by:

$$\begin{aligned} \bar{j}_\ell(r) &= \frac{1}{k^{\ell+1}} j_\ell(kr)\, kr \\ \bar{n}_\ell(r) &= k^\ell n_\ell(kr)\, kr \end{aligned} \tag{10}$$

one can write the integral equations:

---

*We will omit the subscript indicating the angular momentum in the following.

$$v_R(r) = \bar{j}_\ell(r) + \int_0^\infty G(r,r')\, V_L(r')\, v_R(r')\, dr'$$
$$v_I(r) = a\,\bar{n}_\ell(r) + b\,\bar{j}_\ell(r) + \int_\varepsilon^\infty G(r,r')\, V_L(r')\, v_I(r')\, dr' \qquad (r > \varepsilon) \tag{11}$$

with:

$$a = W(\bar{j}, v_I)\Big|_{r=\varepsilon}$$
$$b = W(v_I, \bar{n})\Big|_{r=\varepsilon} \tag{12}$$

and
$$G(r,r') = 0 \qquad r < r'$$
$$= \bar{j}_\ell(r)\bar{n}_\ell(r') - \bar{n}_\ell(r)\bar{j}_\ell(r') \qquad r \geq r' \tag{13}$$

The number ε is finite, not equal to zero. The integral equation for v_I is not defined for $\varepsilon = 0$ due to the singularity of the integrand at $r = 0$ (like $r^{-2\ell}$). In appendix I we give more details about the regular and irregular solution and a short proof of the following analytic properties.

- v_R, v_I, $\bar{j}_\ell$ and $\bar{n}_\ell$ are for finite r entire functions of k^2. This is a consequence of the fact that the boundary conditions (8), (9) are analytic in k^2;
- v_R and $\bar{j}_\ell$ are entire functions in r (Re $r \geq 0$);
- v_I is analytic in r (Re $r \geq 0$) except for a singularity at the origin and a cut starting at r = 0 along the negative axis. The modified Neumann function $\bar{n}_\ell$ has only a pole in the origin.

We can rewrite eq. (11) into:

$$v_R(r) = (1 + \int_0^r \bar{n}_\ell\, V_L\, v_R\, dr')\bar{j}_\ell(r) - (\int_0^r \bar{j}_\ell\, V_L\, v_R\, dr')\,\bar{n}_\ell(r) \tag{14}$$

and:

$$v_I(r) = (a - \int_\varepsilon^r \bar{j}_\ell V_L\, v_I\, dr')\bar{n}_\ell(r) + (b + \int_\varepsilon^r \bar{n}_\ell V_L\, v_I\, dr')\,\bar{j}_\ell(r) \tag{15}$$

Since V_L has a finite range we can write v_R and v_I for $r \to \infty$ as a linear combination of the Bessel and Neumann functions (x kr). We will do this in a special way and define three functions: N_L, H_L and δ_L connected with the asymptotic limit:

$$\begin{aligned} v_R &\xrightarrow[r\to\infty]{} N_L \,(\cos\delta_L\, j_\ell + \sin\delta_L\, n_\ell)\, kr \\ v_I &\xrightarrow[r\to\infty]{} \frac{1}{N_L k}\,(\cos\delta_L\, n_\ell - \sin\delta_L\, j_\ell)\, kr \\ &\quad - H_L N_L\,(\cos\delta_L\, j_\ell + \sin\delta_L\, n_\ell)\, kr \end{aligned} \tag{16}$$

In eq. (16) we have used: $W(v_R, v_I) = 1$.

Using eq. (10), (14), (15) and (16) we obtain:

$$\tan\delta_L = -\,k^{2\ell+1}\;\frac{\int_0^\infty \bar{j}_\ell\, V_L\, v_R}{1 + \int_0^\infty \bar{n}_\ell\, V_L\, v_R}$$

$$N_L \cos\delta_L = \frac{1}{k^{\ell+1}}\;\left(1 + \int_0^\infty \bar{n}_\ell\, V_L\, v_R\right) \tag{17}$$

$$N_L \sin\delta_L = -\,k^{\ell} \int_0^\infty \bar{j}_\ell\, V_L\, v_R$$

$$\frac{\cos\delta_L}{N_L k} - H_L N_L \sin\delta_L = k^{\ell}\,\left(a - \int_\varepsilon^\infty \bar{j}_\ell\, V_L\, v_I\right)$$

$$-\frac{\sin\delta_L}{N_L k} - H_L N_L \cos\delta_L = k^{-\ell-1}\,\left(b + \int_\varepsilon^\infty \bar{n}_\ell\, V_L\, v_I\right)$$

Analytic properties of N_L, H_L and δ_L are derived by first defining the functions:

$$R^{\pm} = \int_0^\infty (\bar{n}_\ell + i\,\bar{j}_\ell\, k^{2\ell+1})\; V_L\, v_R\, dr \tag{18}$$

which are related to the well-known Jost-functions*:

---

*[Ne 66] eq. 12.144 after rewriting, using our conventions.

$$\mathcal{L}_{\ell\pm} = 1 + R^{\pm}$$

Furthermore we define

$$S^{+} = b \mp i\, k^{2\ell+1}\, a + \int_{\varepsilon}^{\infty} (\bar{n}_{\ell} \pm i\, \bar{j}_{\ell} k^{2\ell+1}) V_L\, v_I\, dr \qquad (19)$$

In appendix I we prove:

- S^{+}, R^{+} are analytic in the whole k-plane except on the negative imaginary axis: $k \leq -i\,\frac{\mu_L}{2}$;
- S^{-}, R^{-} are analytic in the whole k-plane except on the positive imaginary axis: $k \geq +i\,\frac{\mu_L}{2}$;
- $\lim_{k\to\infty} R^{\pm}(k) = O(\frac{1}{k})$ except along the negative/positive imaginary axis.
- $\lim_{k\to\infty} \frac{S^{\mp}}{k^{2\ell+1}} = \mp i + O(\frac{1}{k})$ except for the negative/positive imaginary axis.
- $S^{+*}(-k^*) = S^{\pm}(k)$
  $S^{+*}(k^*) = S^{\mp}(k)$
  $R^{\pm *}(-k^*) = R^{\mp}(k)$
  $R^{\pm *}(k^*) = R^{\mp}(k)$ (20)

The properties of $S^{\pm}$ are only valid when there is a restriction on the amount of regular solution contained in the definition of the irregular solution (see appendix I). Using these properties and eqs (17), (18), (19) one can prove:

- $k^{-2\ell-1} \tan \delta_L = \frac{1}{i} \frac{R^{-}-R^{+}}{2+R^{+}+R^{-}} k^{-2\ell-1}$ is analytic in the complex k-plane except for singularities along the imaginary axis for $k \geq i\,\frac{\mu_L}{2}$ and $k \leq -i\,\frac{\mu_L}{2}$, and except for (isolated) poles coming from zero's of $2+R^{+}+R^{-}$.

The symmetry properties eq. (20) give: $k^{-2\ell-1} \tan \delta_L$ is a real function

(on the real axis) and is even in k.

Since $\int_0^\infty \bar{J}_\ell \, V_L \, V_R$ is, in general, at $k^2 = 0$ unequal to zero (note that $\bar{J}_\ell$ and V_R have boundary conditions which do not depend on k^2), we can use the inverse of $k^{-2\ell-1} \tan \delta_L$ and obtain the effective range function and expansion eq. (4). Furthermore:

$\lim_{k \to \infty} k^{-2\ell-1} \tan \delta_L = O\,(k^{-2\ell-2})$ from which we observe, among others, that the s-wave effective range function:

$$P_{\ell=0}(k^2) = k \cot \delta_L$$

is linear in k^2 for $k^2 \to \infty$ and that $\lim_{k^2 \to \infty} \delta_L = O\left(\frac{1}{k}\right)$.

$$k^{\ell+1} N_L(k) \, e^{-i\delta_L} = 1 + R^+ = \mathcal{L}_{\ell^+} \qquad (21a)$$

is analytic except for singularities on the negative imaginary axis: $k \leq -i \frac{\mu_L}{2}$.

Furthermore $k^{\ell+1} N_L(k)$ is real on the real axis and even in k.

Since $\lim_{k\to\infty} \delta_L = O\left(\frac{1}{k}\right)$ and $\lim_{k\to\infty} (1+R^+) = 1 + O\left(\frac{1}{k}\right)$ one obtains:

$$\lim_{k \to \infty} k^{\ell+1} N_L(k) = 1 + O\left(\frac{1}{k}\right)$$

From the expression for $N_L \cos \delta_L$, eq. (17) one observes that:

$\lim_{k\to 0} k^{\ell+1} N_L(k) =$ finite (in general).

Note that $e^{-i\delta_L}$ has also singularities in the upper half plane, starting at $k = +i \frac{\mu_L}{2}$. Therefore $N_L(k)$ must have a singularity structure at the same place in the upper half plane, in such a way that it cancels the singularities of $e^{-i\delta_L}$ in the product: $N_L \, e^{-i\delta_L}$. This is an important property which allows calculation of $N_L(k)$ with dispersion integrals. Similar properties hold for:

$$k^{\ell+1} N_L(k) \, e^{+i\delta_L} = 1 + R^- = \mathcal{L}_{\ell^-} \qquad (21b)$$

except that upper- and lower-plane are interchanged.

- The expression.

$$-k^{\ell+1}\, e^{-i\delta_L}\, N_L(k)\ \left(\frac{1}{N_L^2 k} + H_L\right) = S^+(k)$$

has the same analytic properties as the Jost functions, eq. (21a). Dividing through $f_{\ell+} = 1 + R^+$ we get:

$$\frac{1}{N_L^2 k} + H_L = -\frac{S^+}{1 + R^+} \tag{22a}$$

which is analytic except for singularities along the negative imaginary axis: $k \leq -i\frac{\mu_L}{2}$. Moreover this function has <u>poles</u> due to the <u>zero's</u> of the Jost function, which are usually interpreted as bound state or resonance poles. With upper and lower plane interchanged, similar properties hold for:

$$\frac{1}{N_L^2 k} - H_L = \frac{S^-}{1 + R^-} \tag{22b}$$

Furthermore one proves: $H_L(k)$ is <u>real</u> on the real k-axis and is <u>even</u> in k.

Using the asymptotic limits for S^+, R^+ and $N_L(k)$ one obtains: $\lim_{k\to\infty} H_L(k) = O(k^{2\ell})$ and for the zero energy limit: $\lim_{k\to 0} H_L(k) =$ finite. Once N_L is calculated, this allows calculation of $H_L(k)$ with dispersion integrals. Subtraction is needed due to the behaviour for $k \to \infty$; therefore H(k) can be determined up to a polynomial in k^2 of degree $k^{2\ell}$ from its singularity structure in the k^2 plane. This ambiguity corresponds with the freedom one has in the definition of the irregular solution. Changing the irregular solution by adding a multiple of the regular solution:

$$v_I \to v_I + F(k^2)\ v_R \tag{23}$$

will give the same analytic properties for eq. (22) (and H_L), provided

$F(k^2)$ is an analytic function in k^2 with $\lim_{|k|\to\infty} F(k^2) = O(k^{2\ell})$ (see appendix I). One proves that $S^{\pm}$ similarly changes:

$$S^{+}(k) \to S^{+}(k) + (1 + R^{\pm}(k))\, F(k^2)$$

Therefore:

$$H_L(k^2) \to H_L(k^2) - F(k^2) \tag{24}$$

The functions N_L, H_L and δ_L, determined by the long range potential V_L, are used in the definition of the effective range function of the total potential: $V = V_L + V_S$, as we will show in the following.

3. The total potential: $V = V_L + V_S$

Let us return to the total potential: $V = V_L + V_S$, and its physical (regular) solution: u. The Schrodinger equation reads:

$$\left(\frac{d^2}{dr^2} + \left(k^2 - \frac{\ell(\ell+1)}{r^2} - V\right)\right) u = 0 \tag{25}$$

with the boundary conditions for u(r) at r = 0:

$$u(r) \underset{r\to 0}{\longrightarrow} \frac{r^{\ell+1}}{(2\ell+1)!!} + O(r^{\ell+2}) \tag{26}$$

The ℓ^{th} partial wave phase shift δ is determined by the asymptotic behaviour of u(r):

$$u(r) \underset{r\to\infty}{\longrightarrow} \text{constant} \quad (\cos\delta\, j_\ell + \sin\delta\, n_\ell)\, kr \tag{27}$$

Just as the regular solution v_R, defined in section 2, the wave function u(r) is analytic in k^2 for finite r since it has boundary conditions at r = 0 which are analytic in k^2. It satisfies the integral equation:

$$u(r) = v_R(r) + \int_0^\infty dr'\, G^L(r,r')\, V_S(r')\, u(r') \tag{28}$$

with:

$$G^L(r,r') = 0 \qquad r < r'$$
$$G^L(r,r') = v_I(r')\, v_R(r) - v_R(r')\, v_I(r) \qquad r \geq r' \tag{29}$$

G^L is the complete Greens function of the long range potential.

We rewrite eq. (28) into:

$$u(r) = (1 + \int_0^r v_I\, V_S\, u\, dr')\, v_R(r) - (\int_0^r v_R\, V_S\, u\, dr')\, v_I(r) \tag{30}$$

The integrals: $\int_0^r dr'\, v_I\, V_S\, u$ and $\int_0^r dr'\, v_R\, V_S\, u$ are analytic in k^2 in the whole complex k^2 plane since v_I, v_R, u and V_S are analytic, for finite r. This analyticity is affected in the limit $r \to \infty$. Defining:

$$A \equiv 1 + \int_0^\infty dr'\, v_I\, V_S\, u$$
$$B \equiv -\int_0^\infty dr'\, v_R\, V_S\, u \tag{31}$$

and using the inequalities of appendix I, eq. (5) and eq. (14), which also hold for the regular solution u(r), one proves:

A and B are even and real in k and analytic in the region:

$|\mathrm{Im}\, k| \leq \frac{\mu_S}{2}$ if: $|V_S(r)| \underset{r\to\infty}{\longrightarrow} \simeq e^{-\mu_S r}$.

Moreover, when $V_S(r)$ is an analytic* potential, one proves that:

A and B are analytic in the whole k-plane except for singularities along the imaginary axis for $k \geq i\frac{\mu_S}{2}$ and $k \leq -i\frac{\mu_S}{2}$.

Therefore, except for simple poles coming from zeros of B, $\frac{A}{B}$ is even and analytic in the same region. Due to the boundary conditions

---

*Meant is analyticity in r. See [Ne 66] for a more careful definition and discussion; we will always use potentials analytic in the $\frac{1}{2}\pi$ class, and in this chapter they are also restricted to:

V(0) = finite.

at $r = 0$ also the limit $k \to 0$ is, in general, well defined for $\frac{A}{B}$ and unequal zero. Combining eqs (27), (30), (31) and (16) we obtain:

$$u(r) \underset{r\to\infty}{\longleftrightarrow} A\, v_R(r) + B\, v_I(r)$$

$$\underset{r\to\infty}{\to} (A\, N_L \cos\delta_L - \frac{B}{N_L k} \sin\delta_L - B\, H_L\, N_L \cos\delta_L) j_\ell(kr)\, kr$$

$$+ (A\, N_L \sin\delta_L + \frac{B}{N_L k} \cos\delta_L - B\, H_L\, N_L \sin\delta_L) n_\ell(kr)\, kr$$

So:

$$\cot\delta = \frac{A\, N_L \cos\delta_L - \frac{B}{N_L k} \sin\delta_L - B\, H_L\, N_L \cos\delta_L}{A\, N_L \sin\delta_L + \frac{B}{N_L k} \cos\delta_L - B\, H_L\, N_L \sin\delta_L} \tag{32}$$

Defining:

$$\boxed{\delta \equiv \delta_L + \delta_S^L} \tag{33}$$

one rewrites eq. (32) into the simple form:

$$\boxed{\frac{\cot\delta_S^L}{N_L^2 k} + H_L = \frac{A}{B}} \tag{34}$$

The <u>effective range function</u> of the total potential V with respect to its long range component V_L is <u>defined</u> as:

$$\boxed{P^L(k^2) \equiv \frac{\cot\delta_S^L}{N_L^2 k} - H_L} \tag{35}$$

These definitions need some comments:

- the phase shift δ_S^L, defined by eq. (33), can be considered as the phase shift of the <u>total</u> potential: $V = V_L + V_S$ <u>with respect</u> to solutions of the potential V_L. Defining:

$$v_R^\infty = \frac{1}{N_L} v_R$$
$$v_I^\infty = N_L k\, v_I + H_L\, N_L\, k\, v_R \tag{36}$$

one proves:

$$v_R^{\infty} \xrightarrow[r\to\infty]{} (\cos\delta_L\, j_\ell + \sin\delta_L\, n_\ell)\, kr$$

$$v_I^{\infty} \xrightarrow[r\to\infty]{} (-\sin\delta_L\, j_\ell + \cos\delta_L\, n_\ell)\, kr \qquad (37)$$

and:

$$u \xrightarrow[r\to\infty]{} \simeq (\cos\delta_S^L\, v_R^{\infty} + \sin\delta_S^L\, v_I^{\infty})$$

$$\xrightarrow[r\to\infty]{} (\cos(\delta_L + \delta_S^L)\, j_\ell + \sin(\delta_L + \delta_S^L)\, n_\ell)\, kr$$

The analyticity of δ_S^L is limited by singularities starting at $k = \pm i\frac{\mu_L}{2}$ coming from the long range potential V_L.

- The functions N_L and H_L have singularities along the imaginary axis starting at $k = \pm i\frac{\mu_L}{2}$.

- However the singularities from δ_S^L, N_L and H_L, starting at $k = \pm i\frac{\mu_L}{2}$ cancel out in the effective range function $P^L(k^2)$, analytic in the complex plane, except for simple poles and singularities along the imaginary axis starting at $k = \pm i\frac{\mu_S}{2}$; if $\mu_S > \mu_L$ then an expansion in k^2 of $P^L(k^2)$ will converge better than a similar expansion of the effective range function of the total phase shift: $P(k^2) = k^{2\ell+1}\cot\delta$.

- As stated before, $P^L(k^2)$ is an even and real function in k, and one can prove:

$$\lim_{k\to\infty} P^L(k^2) = O\,(k^{2\ell+2})$$

$$\lim_{k\to 0} P^L(k^2) \neq 0 \qquad \text{, in general.}$$

When we take as V_L the trivial potential $V_L \equiv 0$, we obtain: $\delta_L = 0$, $N_L = k^{-\ell-1}$, and H_L can be chosen to be: $H_L = 0$ ($V_R = \bar{j}_\ell$, $V_I = \bar{n}_\ell$). In this case P^L becomes:

$$P^L \xrightarrow[V_L=0]{} k^{2\ell+1}\cot\delta_S^L = k^{2\ell+1}\cot\delta$$

which is the ordinary effective range function. Considering all this

for the neutron-proton interaction as described by the one-boson-exchange models with lightest mesons the π and η (neglecting for the moment 2π uncorrelated exchange) we observe that: $P(k^2)$ has a cut in the complex energy plane starting at - 10 MeV (laboratory energy). Taking as V_L the pion exchange potential, it will lead to a larger analyticity region for $P^L(k^2)$: $P^L(k^2)$ has a cut starting at about - 150 MeV. Suitable parametrizations of $P^L(k^2)$ will be discussed later.

In the next section we will discuss the connection of the effective range function with the partial wave amplitude and derive some more properties.

4. Effective range functions and the partial wave amplitude

We define outgoing and incoming solutions of the long range potential V_L:

$$v_+ = \left(\frac{1}{N_L^2 k} + H_L \right) v_R + v_I$$
$$v_- = \left(\frac{-1}{N_L^2 k} + H_L \right) v_R + v_I \qquad (38)$$

In the asymptotic limit we have:

$$v_\pm(r) \xrightarrow[r\to\infty]{} \frac{e^{\mp i\delta_L}}{N_L k} \; h_\ell^{(\pm)}(kr) \; kr \qquad * \qquad (39)$$

The solutions $v_\pm(r)$ are analytic in the whole complex k-plane for finite r, except for singularities along the negative- (v^+) and positive-(v^-) imaginary axis for $|k| \geq \frac{\mu_L}{2}$ (conform eqs (22)). Moreover

---

*$h_\ell^{(\pm)}(kr) \equiv n_\ell(kr) \pm i\, j_\ell(kr)$

$\lim_{r\to\infty} h_\ell^{(\pm)}(kr) = e^{i(kr-(\ell\pi/2))}/kr$

they have poles at the zeros of the Jost functions $\mathcal{L}_{\ell^{\pm}}$. With the usual methods one proves:

$$|v_+(r)| \leq \text{constant} \quad \left|\frac{e^{\mp i\delta_L}}{N_L k^{\ell+1}}\right| \quad \left(\frac{1+|kr|}{|r|}\right)^{\ell} e^{\mp(\mathrm{Im}\, k)r} \tag{40}$$

The integral equation (30) for the solution of the total potential V can be written as:

$$u(r) = +\frac{N_L^2 k}{2i}\left(v_+(r)\ (1 + \int_0^r v_- V_S\, u\, dr')\right.$$
$$\left. - v_-(r)\ (1 + \int_0^r v_+ V_S\, u\, dr')\right) \tag{41}$$

For real k one has:

$$u(r) \underset{r\to\infty}{\longrightarrow} \simeq e^{i\delta_S^L} v_+(r) - e^{-i\delta_S^L} v_-(r)$$
$$\underset{r\to\infty}{\longrightarrow} \frac{1}{N_L k}\left(e^{i(\delta_L+\delta_S^L)}\ h_\ell^{(+)} - e^{-i(\delta_L+\delta_S^L)}\ h_\ell^{(-)}\right) kr \tag{42}$$

Therefore:

$$e^{2i\delta_S^L} = \frac{1 + \int_0^\infty v_- V_S\, u\, dr'}{1 + \int_0^\infty v_+ V_S\, u\, dr'} \tag{43}$$

and of course: $\delta = \delta_L + \delta_S^L$

Due to the limited analyticity of v_+ and v_- in the integrands of eq. (43), $e^{2i\delta_S^L}$ (and also δ_S^L) is only analytic for $|\mathrm{Im}\, k| \leq \frac{\mu_L}{2}$ (except for possible poles from zeros of the denominator): Analyticity is restricted in first instance by the long range potential V_L. Furthermore at the bound state or resonance energies of the long range potential V_L we have:

$$e^{2i\delta_S^L} = 0 \tag{44}$$

since v_+, $\left(\frac{1}{N_L^2 k} + H_L\right)$, has a pole at these energies. This corresponds with the fact that bound states (resonances) of the long range potential V_L are (in general) different from bound states and

resonance energies of the total potential: $V_L + V_S$:

if: $e^{2i\delta_L} = \infty$

then, in general: $e^{2i\delta} = e^{2i\delta_L} e^{2i\delta_S^L} \neq \infty$

The effective range expression (34) automatically takes care of this since at the bound state poles of the potential V_L:

$$N_L^2 k = 0 \quad , \quad H_L N_L^2 k = 1 \quad \text{and} \quad \frac{A}{B} \neq \infty$$

(the proof of this is straightforward).

The analyticity of eq. (43) is limited in the upper half plane (physical sheet) to the region: $\text{Im } k \leq \frac{\mu_L}{2}$. Therefore we define a quantity which is analytic in the upper half plane for: $\text{Im } k \leq \frac{\mu_S}{2}$:

$$-\frac{\int_0^\infty v_R V_S u}{1 + \int_0^\infty v^+ V_S u} = \frac{N_L^2 k}{2i} (e^{2i\delta_S^L} - 1) \equiv N_L^2 k^2 f_{\delta_S^L} \tag{45}$$

It is related with the asymptotic limit:

$$u(r) \xrightarrow[r\to\infty]{} \simeq (v_R + N_L^2 k^2 f_{\delta_S^L} v_+) \tag{46}$$

which is proven by rewriting eq. (30) into:

$$u(r) = (1 + \int_0^r dr' v_+ V_S u) v_R(r) - (\int_0^r dr' v_R V_S u) v^+(r) \tag{47}$$

We change the normalization of u(r) defining:

$$u^\infty(r) = \frac{u(r)}{1 + \int_0^\infty dr' v_+ V_S u} \tag{48}$$

The wave function of the total potential: $u^\infty(r)$ is analytic in the whole complex k-plane except for singularities in the lower half plane, along the imaginary axis for $\text{Im } k \leq -\frac{\mu_L}{2}$ and except for poles due to the vanishing of the denominator of eq. (48) (bound states or resonances of

the total potential: $V_L + V_S$, compare with eq. (43) and prove: $e^{2i\delta} = \infty$). Furthermore $u^\infty(r)$ satisfies:

$$u^\infty(r) = v_R(r) - v_R(r)\left(\int_r^\infty dr' v_+ V_S u^\infty\right) - v_+(r)\left(\int_0^r dr' v_R V_S u^\infty\right)$$

$$= v_R(r) - \int_0^\infty G^{L,+}(r,r') V_S(r') u^\infty(r') dr' \qquad (49)$$

with:

$$G^{L,+}(r,r') = v_R(r_<)\, v_+(r_>) \qquad (50)$$

where $r_<$ indicates the smaller of r and r', and $r_>$ the larger.

We obtain:

$$N_L^2 k^2 f_{\delta_S^L} = - \int_0^\infty v_R\, V_S\, u^\infty\, dr' \qquad (51)$$

Using the inequalities for v_R and v_+ one obtains:

$$\left|G^{L,+}(r,r')\right| \leq \frac{\text{constant}}{\left|1+R^+\right|} \left(\frac{|r_<|}{1+|kr_<|}\right)^{\ell+1} \left(\frac{1+|kr_>|}{|r_>|}\right)^{\ell} e^{|\text{Im } k|r_< - \text{Im } k\ r_>} \qquad (52)$$

and for Im $k \geq 0$:

$$\left|G^{L,+}(r,r')\right| \xrightarrow[r,r'\to\infty]{} \simeq e^{-\text{Im } k(r_>-r_<)} \qquad (53)$$

Except for those points in the upper half plane where a long range bound state is present, this gives an additional exponential damping in r and r' of the integrand of eq. (49) in the upper half plane. Neglecting these bound states for the moment one proves the following properties with the usual methods using eqs (49), (51), (52):

$N_L^2 k^2 f_{\delta_S^L}$ is analytic in the upper half of the complex momentum plane, except for singularities along the imaginary axis starting at $+i\frac{\mu_S}{2}$ and poles corresponding with bound states of the total potential V (note: N_L and $f_{\delta_S^L}$ are only analytic up to $k = i\frac{\mu_L}{2}$!).

Moreover: the successive Born approximations of u^∞, from eq. (49) (in symbolic notation):

$$u^{\infty} = v_R - G V_S v_R + G V_S G V_S v_R - \ldots$$

lead to successive Born approximations of $N_L^2 k^2 f_{\delta_S^L}$:

$$N_L^2 k^2 f_{\delta_S^L} = - v_R V_S v_R + v_R V_S G V_S v_R - \ldots \qquad (54)$$

which has an <u>additive</u> singularity structure, which means:

the n^{th} Born approximation of $N_L^2 k^2 f_{\delta_S^L}$, defined by:

$$+- v_R \underbrace{V_S G V_S G \ldots V_S}_{n \text{ times } V_S} v_R$$

is analytic up to $k = + i \, n \frac{\mu_S}{2}$ in the upper half plane.

Otherwise stated:

the dynamical singularities in the upper half plane (not the bound states!) along the imaginary axis for $N_L^2 k^2 f_{\delta_S^L}$ from $k = i \frac{\mu_S}{2}$ up to $k = i \, n \frac{\mu_S}{2}$ are completely given by the first (n-1) Born approximations.

For these reasons it is $N_L^2 k^2 f_{\delta_S^L}$ which is most useful in dispersion techniques (however, an explicit solution of the wave functions corresponding with the long range potential V_L is required). In the <u>lower</u> half plane these properties do not hold: $N_L^2 k^2 f_{\delta_S^L}$ is only analytic up to $k = - i \frac{\mu_L}{2}$!! Returning to the bound states of V_L, we note that at these energies: $N_L^2 k^2 = 0$; therefore with eq. (44) we get:

$$f_{\delta_S^L} = - \frac{1}{2ik_{\beta_L}} \quad ; \quad N_L^2 k_{\beta_L}^2 f_{\delta_S^L} = 0 \quad ; \quad k_{\beta_L} = \text{momentum of } V_L \text{ bound state} \qquad (55)$$

Now we are ready to make the connection with the total partial wave amplitude; using $\delta = \delta_L + \delta_S^L$ one obtains:

$$f_\delta \equiv \frac{e^{2i\delta} - 1}{2ik} = \frac{e^{2i\delta_L} - 1}{2ik} + e^{2i\delta_L} \frac{e^{2i\delta_S^L} - 1}{2ik}$$

$$\equiv f_{\delta_L} + e^{2i\delta_L} f_{\delta_S^L} \qquad (56)$$

Using the properties of $N_L^2 k^2 \, f_{\delta_S^L}$ and those of $\frac{e^{2i\delta_L}}{N_L^2 k^2}$ (eq. (21)) one obtains the well-known result:

$e^{2i\delta_L} f_{\delta_S^L}$ is analytic in the upper half momentum plane up to $k = +i\frac{\mu_S}{2}$ (except bound state poles).

The effective range function (35) can now be obtained in another way:

$$P^L = \frac{1}{N_L^2 k} \cot \delta_s^L + H_L = \frac{i}{N_L^2 k} + H_L + \frac{1}{f_{\delta_S^L} N_L^2 k^2} \tag{57}$$

From the properties of $N_L^2 k^2 \, f_{\delta_S^L}$ and $i/(N_L^2 k) + H_L$ it is clear that P^L is analytic in the upper half momentum plane up to $k = i\frac{\mu_S}{2}$ (except for isolated poles if $f_{\delta_S^L} = 0$). Since the left hand side is even in k (δ_S^L is odd, N_L^2 and H_L are even in k!) the same property also holds for the lower half plane. That P^L is real on the real axis, follows from the fact that δ_S^L, N_L and H_L are real for physical k.

In the next section we will see how we can use all this to define effective range functions without the restrictions of local, non-relativistic Schrödinger theory.

5. Definition of the effective range function using its analytic properties

Elastic single channel effective range theory can be derived if the total partial wave amplitude f_δ satisfies the decomposition theorem:

Take.

$$f_\delta = f_{\delta_L} + \text{remainder} \tag{58}$$

with f_{δ_L} *the partial wave amplitude for a long range interaction which*

forms part of the total interaction.

Total interaction = long range + short(er) range interaction

$$(\text{range} = 1/\mu_L) \qquad (\text{range} = 1/\mu_S) \tag{59}$$

Then the singularity structure in the upper half momentum plane must be split up according to:

f_δ : *analytic up to* $k = +i\frac{\mu_L}{2}$

f_{δ_L} : *analytic up to* $k = +i\frac{\mu_L}{2}$

remainder. analytic up to $k = +i\frac{\mu_S}{2}$

provided: $\mu_L < \mu_S$, *and except for poles corresponding with bound states (either from the total interaction or from its long range part).*

This decomposition theorem has a much wider range than local, non-relativistic Schrodinger scattering theory alone and can be applied to most types of interactions whether nonlocal, energy dependent or derived from field theory. It is also valid for coupled channels, however, we will use it only for single channel partial waves for the moment.

Due to unitarity we can write eq. (58) on the physical axis as:

$$f_\delta = f_{\delta_L} + e^{2i\delta_L} f_{\delta_S^L} \tag{60}$$

with:

$$f_{\delta_S^L} = \frac{e^{2i\delta_S^L} - 1}{2ik} \quad ; \quad \delta = \delta_L + \delta_S^L \quad ; \quad \delta_S^L,\ \delta_L \text{ real} \quad .$$

This is of course also true for unphysical momenta, however, in that case δ_S^L is not necessarily real. Suppose we could find two functions N_L and H_L with the analytic properties:

A {
- $k^{\ell+1} N_L e^{-i\delta_L}$ *is analytic in the upper half plane and has (simple) zeros at the bound states of the long range interaction.*
- $k^{\ell+1} N_L$ *is even and real in* k.
- $k^{\ell+1} N_L \xrightarrow[|k|\to\infty]{} 1$ and $\lim_{k\to 0} k^{\ell+1} N_L$ = *finite number.*

B {
- $\frac{1}{N_L^2 k} + H_L$ *is analytic in the upper half plane except for (simple) poles at the bound states. At the bound states the residue of* $H_L + \frac{1}{N_L^2 k}$ *must be <u>twice</u> the residue of* $1/(N_L^2 k)$
- H_L *is even and real in* k.
- $\left|\frac{H(k)}{k^{2\ell+1}}\right| \xrightarrow[k\to\infty]{} 0$ and $H(k) \xrightarrow[k\to 0]{}$ *finite number*

Then we can define: $P^L = \frac{\cot \delta_S^L}{N_L^2 k} + H_L$ (61)

and prove:

$$P^L = \frac{i}{N_L^2 k} + H_L + \frac{1}{f_{\delta_S^L}\, e^{2i\delta_L}} \frac{e^{2i\delta_L}}{N_L^2 k^2} \tag{62}$$

is even and real in k, analytic for momenta $|k| < \frac{\mu_S}{2}$ except for possible poles corresponding with: $f_{\delta_S^L} = 0$ (i.e. the short range interaction becomes "transparent"). If we have a complete knowledge of the long range interaction (i.e. phase shifts and bound states) the functions N_L and H_L can be solved in terms of δ_L and β_L (bound state momenta); assuming for the moment that there are no bound states, requirement A leads to the dispersion integral, valid for k in the upper half plane:

$$\ln\,(k^{\ell+1} N_L(k)) - i\,\delta_L(k) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\ln\,(k'^{\ell+1} N_L) - i\delta_L}{k'-k}\, dk' \tag{63}$$

For k in the upper half plane, infinitely close to the positive real

axis, eq. (63) leads to:

$$k^{\ell+1} N_L(k) = e^{-\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\delta_L}{k'-k} dk'} = e^{-\frac{2}{\pi} P \int_0^{\infty} k' dk' \frac{\delta_L(k')}{k'^2-k^2}} \quad (64)$$

where we have used the fact that δ_L is odd in k. Furthermore requirement B leads to the subtracted dispersion relation:

$$\frac{1}{N_L^2 k} + H_L - ik^{2\ell+1} = \sum_{j=0}^{2\ell} \frac{1}{j!} H^{(j)}(0) k^j + \frac{k^{2\ell+1}}{2\pi i} \int_{-\infty}^{+\infty} \frac{\frac{1}{N_L^2 k'} + H_L - ik'^{2\ell+1}}{k'^{2\ell+1}(k'-k)} \quad (65)$$

Taking the real part of the left and right hand side we obtain for k close to the real axis*:

$$H_L(k) = \sum_{j=0}^{2\ell} \frac{1}{j!} H^{(j)}(0) k^j + \frac{2}{\pi} k^{2\ell+2} P \int_0^{\infty} \frac{\frac{1}{N_L^2 k'^{2\ell+2}} - 1}{k'^2-k^2} dk' \quad (66)$$

Using eqs (64) and (66) N_L and H_L can be calculated once the phase shift δ_L is known for all energies. This could be of importance if there is

---

*Use:

$$\frac{1}{k'^{2\ell+1}(k'-k-i\varepsilon)} = \frac{1}{k^{2\ell+1}(k'-k-i\varepsilon)} - \frac{1}{k^{2\ell+1}(k'-i\varepsilon)} - \frac{1}{k^{2\ell}(k'-i\varepsilon)^2} \cdots \frac{1}{k(k'-i\varepsilon)^{2\ell+1}}$$

and:

$$\int_{-\infty}^{+\infty} dk' \frac{F(k')}{(k'-i\varepsilon)^n} = P \int_{-\infty}^{+\infty} dk' \frac{F(k')-F(0)\ldots-\frac{k'^{n-2}}{(n-2)!} F^{(n-2)}(0)}{k'^n} + \frac{i\pi}{(n-1)!} F^{(n-1)}(0)$$

no Schrodinger equation which could be used to solve N_L, H_L and δ_L: for instance δ_L is given by a Born approximation or by an effective range approximation. We will return on this in chapter IV.

Formula (64) can be generalized for the case that there are bound state poles in the upper half plane. Suppose there are poles at $k = i\beta_n$, then consider:

$$\prod_n \frac{k+i\beta_n}{k-i\beta_n} k^{\ell+1} N_L e^{-i\delta_L} \tag{67}$$

which has no zeros in the upper half plane since the zeros of the Jost function are simple.

Now we can take the logarithm of (67) and follow the same steps as before. One gets:

$$k^{\ell+1} N_L(k) = \prod_n \left(1 + \frac{\beta_n^2}{k^2}\right) e^{-\frac{2}{\pi} P \int_0^\infty k' \, dk' \frac{\delta_L(k')}{k'^2-k^2}} \tag{68a}$$

for real physical k, or:

$$k^{\ell+1} N_L(k) = \prod_n \left(1 + \frac{\beta_n^2}{k^2}\right) e^{-\frac{1}{\pi} \int_{-\infty}^{+\infty} dk' \frac{\delta_L(k')}{k'-k} + i\,\delta_L(k)} \tag{68b}$$

for k in the upper half plane, or:

$$k^{\ell+1} N_L(k) = \prod_n \left(1 + \frac{\beta_n^2}{k^2}\right) e^{-\frac{1}{\pi} \int_{-\infty}^{+\infty} dk' \frac{\delta_L(k')}{k'-k} - i\,\delta_L(k)} \tag{68c}$$

in the lower half plane.

The singularity in eqs (68a,b,c) at k = 0 is only apparent since $\lim_{k\to 0} \delta_L(k) = n\pi$ * which causes the principal value integral in eq. (68a) to be singular too, giving a finite value for $k^{\ell+1} N_L$ at k = 0.

---

*Absence of CDD poles and property "A" are assumed.

CHAPTER III

Effective range theory for potentials which contain the Coulomb interaction

1. Introduction

In this chapter we will consider the problems which arise when the potential contains the Coulomb interaction with infinite range. For proton-proton scattering the total potential is written as the sum of the (finite range) nuclear potential and the (infinite range) electric potential:

$$V = V_{nuc} + V_{elec} \tag{1}$$

In the real physical world V_{elec} will contain the Coulomb, magnetic moment and vacuum polarization potential.

For the moment we will consider only the Coulomb potential in its nonrelativistic form:

$$V_{elec} = \frac{2M_{red}\,\alpha_F}{r} \tag{2}$$

with M_{red} the reduced mass and α_F the fine structure constant.

In chapter VI of part II of this thesis we will study the case of a "relativistic" Coulomb potential (containing in fact also an r^{-2} part). For the vacuum polarization, which can be evaluated only numerically, we refer to chapter IV.

When one tries to include the potential of eq. (2) in the calculations of chapter II, one meets two problems:
First of all the potential is not analytic in r anymore but contains a simple pole at the origin. This is not a serious problem since all of the results can easily be generalized for potentials being analytical

except for a simple pole at the origin.

More serious is the fact that the potential falls off like r^{-1} for $r \to \infty$, which implies that the limiting wavefunction for $r \to \infty$ cannot be written as a fixed (independent of r) linear combination of Bessel and Neumann functions as in eq. (II.2).

However, considering single channel partial wave scattering, we could write the radial wave function u_ℓ, for potential V and the ℓ'th partial wave, in the asymptotic limit as:

$$u_\ell(r) \xrightarrow[r\to\infty]{} (\cos \delta_N^C F_\ell + \sin \delta_N^C G_\ell) \tag{3}$$

with F_ℓ and G_ℓ the standard regular and irregular Coulomb wave functions, and δ_N^C the so-called nuclear phase shift (with respect to Coulomb wave functions). The total wave function $\psi(\vec{r})$ is a linear combination of $u_\ell(r)$ with the spherical harmonic $Y_\ell^0(\theta)$:

$$\psi(\vec{r}) = \sum_{\ell=0}^{\infty} a_\ell \frac{u_\ell(r)}{kr} Y_\ell^0(\theta) \quad . \tag{4}$$

Choosing:

$$a_\ell = (i)^\ell \sqrt{4\pi(2\ell+1)}\, e^{i(\delta_N^C+\sigma_\ell)} \tag{5}$$

with:

$$\sigma_\ell = \arg \Gamma(\ell+1+i\eta) \quad \text{and} \quad \eta = \frac{M_{red}\,\alpha_F}{k}$$

one obtains:

$$\psi(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{\ell=0}^{\infty} e^{i\sigma_\ell} \frac{\sqrt{4\pi(2\ell+1)}}{kr} (i)^\ell F_\ell(\eta,kr)\, Y_\ell^0(\theta)$$

$$+ \sum_{\ell=0}^{\infty} e^{i\sigma_\ell} \frac{e^{2i\delta_{N-1}^C}}{2ik} \sqrt{4\pi(2\ell+1)}\, (i)^\ell \frac{G_\ell + iF_\ell}{r}\, Y_\ell^0(\theta) \tag{6}$$

The first summation of eq. (6) is the solution of pure Coulomb scattering ($V_{nuc} = 0$, $\delta_N^C = 0$) and can be rewritten as ($r \to \infty$):

$$e^{i(kz+\eta \ln k(r-z))} - \frac{\eta}{2k \sin^2 \frac{\theta}{2}} e^{-i\eta \ln (\sin^2 \frac{\theta}{2}) + 2i\sigma_0}$$

$$\times \frac{e^{i(kr-\eta \ln 2kr)}}{r} \qquad (7)$$

which represents an incoming "plane wave" along the z-direction and an outgoing spherical wave multiplied with the Coulomb amplitude:

$$f_c(\theta) = -\frac{\eta}{2k \sin^2 \frac{\theta}{2}} e^{-i\eta \ln (\sin^2 \frac{\theta}{2}) + 2i\sigma_0} \qquad (8)$$

The decomposition in partial wave amplitudes reads:

$$f_c(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{e^{2i\sigma_\ell} - 1}{2ik} P_\ell(\cos \theta) \qquad (9)$$

The summation on the right hand side, however, does not converge in the ordinary sense, i.e. point-like. The partial sums diverge oscillatory and have no limit.

This problem can be solved by considering this partial wave sum as an Abel-summation [Ge 79] or considering eq. (9) in the sense of distributions [Ta 74].

The second summation of eq. (6) gives for $r \to \infty$:

$$\sum_{\ell=0}^{\infty} e^{2i\sigma_\ell} \frac{e^{2i\delta^C_{N,\ell}} - 1}{2ik} \sqrt{4\pi(2\ell+1)}\, Y^0_\ell(\theta) \quad \frac{e^{i(kr-\eta \ln 2kr)}}{r} \qquad (10)$$

and modifies the outgoing spherical wave of eq. (7), leading to the total amplitude:

$$f(\theta) = f_c(\theta) + \sum_{\ell=0}^{\infty} e^{2i\sigma_\ell} \frac{e^{2i\delta^C_{N,\ell}} - 1}{2ik} \sqrt{4\pi(2\ell+1)}\, Y^0_\ell(\theta) \qquad (11)$$

On the partial wave level this can be written as (with the restriction concerning the summation of the Coulomb part):

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell\, P_\ell(\cos \theta) \qquad (12a)$$

$$f_\ell = f_{\sigma_\ell} + e^{2i\sigma_\ell} f_{\delta^C_{N,\ell}}$$

$$= \frac{e^{2i\sigma_\ell} - 1}{2ik} + e^{2i\sigma_\ell} \frac{e^{2i\delta^C_{N,\ell}} - 1}{2ik} \qquad (12b)$$

Eq. (12b) resembles eqs (II.56) and (II.58). Therefore σ_ℓ can be considered in a way as the phase shift caused by the Coulomb potential and:

$$\delta = \sigma_\ell + \delta^C_{N,\ell} \qquad (13)$$

as the <u>total</u> phase shift for the <u>total</u> potential: $V = V_{nuc} + V_{elec}$.

In the next section we will review some properties of the Coulomb functions F_ℓ and G_ℓ and related quantities which we will need in the following.

2. The Coulomb wave functions

Consider the radial Schrödinger equation for the Coulomb potential:

$$v''_\ell + \left(k^2 - \frac{2M_{red}\,\alpha_F}{r} - \frac{\ell(\ell+1)}{r^2}\right) v_\ell = 0 \qquad (14)$$

which, using $\rho = kr$, can be rewritten into:

$$\frac{\partial^2}{\partial\rho^2} v_\ell + \left(1 - \frac{2\eta}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right) v_\ell = 0 \qquad (15)$$

with:

$$\eta = \frac{M_{red}\,\alpha_F}{k} \quad .$$

As two independent solutions of eq. (15) one usually takes the so-called Coulomb wave functions $F_\ell(\eta,\rho)$ and $G_\ell(\eta,\rho)$. The first one is regular at the origin. Their asymptotic behaviour for $\rho \to \infty$ is:

$$F_\ell(\eta,\rho) \underset{\rho\to\infty}{\longrightarrow} \sin\left(\rho - \eta \log 2\rho - \frac{1}{2}\ell\pi + \sigma_\ell\right)$$

$$G_\ell(\eta,\rho) \underset{\rho\to\infty}{\longrightarrow} \cos\left(\rho - \eta \log 2\rho - \frac{1}{2}\ell\pi + \sigma_\ell\right) \qquad (16)$$

where the Coulomb phase shift is defined by:

$$\sigma_\ell = \arg \Gamma\ (\ell + 1 + i\eta) \tag{17}$$

For the regular Coulomb wave function one can derive the series expansion:

$$F_\ell(\eta,\rho) = C_\ell(\eta) \sum_{n=\ell+1}^{\infty} A_n^\ell(\eta)\ \rho^n \tag{18}$$

with:

$$C_\ell(\eta) = \left(\frac{2\pi\eta}{e^{2\pi\eta}-1}\right)^{1/2} \frac{1}{(2\ell+1)!!} \prod_{s=1}^{\ell} \left(1 + \frac{\eta^2}{s^2}\right)^{1/2} \quad , \ \ell > 0$$

$$C_0(\eta) = \left(\frac{2\pi\eta}{e^{2\pi\eta}-1}\right)^{1/2} \tag{19}$$

and:

$$A_{\ell+1}^\ell(\eta) = 1 \quad , \quad A_{\ell+2}^\ell(\eta) = \frac{\eta}{\ell+1}$$

$$(n(n-1)-\ell(\ell+1))\ A_n^\ell(\eta) = 2\eta\ A_{n-1}^\ell(\eta) - A_{n-2}^\ell(\eta) \tag{20}$$

From eqs (18) and (20) one derives:

$$F_\ell(\eta,0) = 0 \quad , \quad \frac{d}{dr} F_\ell(\eta,kr) \xrightarrow[r\to 0]{} (\ell+1)\ C_\ell(\eta)\ k^{\ell+1}\ r^\ell$$

Therefore the boundary conditions at r = 0 are <u>not</u> analytic in k^2; hence $F_\ell(\eta,\rho)$ is not analytic for finite r and will have poles at the same places as $C_\ell(\eta)$.

We define:

$$\bar{F}_\ell(\eta,\rho) = \frac{1}{(2\ell+1)!!\ k^{\ell+1}\ C_\ell(\eta)} F_\ell(\eta,\rho) \tag{21}$$

which has the boundary conditions (similar to eq. (II.8)):

$$\bar{F}_\ell(\eta,\rho) \xrightarrow[r\to 0]{} \frac{1}{(2\ell+1)!!}\ r^{\ell+1} + O(r^{\ell+2}) \quad .$$

Therefore $\bar{F}_\ell(\eta,kr)$ will be an entire function of k^2 and r. The irregular

Coulomb function $G_\ell(\eta,\rho)$ has the expansion:

$$G_\ell(\eta,\rho) = \frac{2\eta}{C_0^2(\eta)} F_\ell(\eta,\rho) \quad \{\ln 2\rho + \frac{q_\ell(\eta)}{p_\ell(\eta)}\} + D_\ell(\eta) \sum_{n=-\ell}^{\infty} a_n^\ell(\eta)\ \rho^n \tag{22}$$

with the definitions:

$$\frac{q_\ell(\eta)}{p_\ell(\eta)} = \sum_{s=0}^{\ell} \frac{s}{s^2+\eta} - \sum_{s=1}^{2\ell+1} \frac{1}{s} + \mathrm{Re}\ \frac{\Gamma'(1+i\eta)}{\Gamma(1+i\eta)} + 2\gamma + \frac{r_\ell(\eta)}{p_\ell(\eta)}$$

$$r_\ell(\eta) = \frac{(-1)^{\ell+1}}{(2\ell)!}\ \mathrm{Im} \sum_{n=0}^{2\ell} \{ \frac{2^n}{n!(2\ell+1-n)} \frac{\Gamma(i\eta-\ell+n)}{\Gamma(i\eta-\ell)} \}$$

$$p_\ell(\eta) = \frac{2^{2\ell+1}\eta}{(2\ell+1)\{(2\ell)!\}^2} \prod_{n=1}^{\ell} (n^2+\eta^2) \qquad , \qquad \ell > 0$$

$$p_0(\eta) = 2\eta$$

$$D_\ell(\eta) = \frac{1}{(2\ell+1)\ C_\ell(\eta)} \tag{23}$$

and γ Euler's constant.

The coefficients $a_n^\ell(\eta)$ satisfy:

$$a_{-\ell}^\ell(\eta) = 1 \quad , \quad a_{\ell+1}^\ell(\eta) = 0 \quad , \quad A_n^\ell(\eta) = 0 \quad \text{for } n > \ell + 1$$

$$(n-\ell-1)(n+\ell) a_n^\ell(\eta) = 2\eta\ a_{n-1}^\ell(\eta) - a_{n-2}^\ell(\eta) - (2n-1)\ p_\ell(\eta)\ A_n^\ell(\eta) \tag{24}$$

Also $G_\ell(\eta,\rho)$ satisfies non-analytic boundary conditions at $r = 0$:

$$G_\ell(\eta,\rho) \underset{r\to 0}{\longrightarrow} \{D_\ell(\eta)\ k^{-\ell}\ r^{-\ell} + \ldots\} + \frac{2\eta}{C_0^2(\eta)} F_\ell(\eta,\rho)\ \ln r$$

$$+ \{\frac{q_\ell(\eta)}{p_\ell(\eta)} + \ln 2k\} \frac{2\eta}{C_0^2(\eta)} F_\ell(\eta,\rho) \tag{25}$$

Defining:

$$\bar{G}_\ell(\eta,\rho) = (2\ell-1)!!\ \{(2\ell+1)\ k^\ell\ C_\ell(\eta)\ G_\ell(\eta,\rho) - (q_\ell(\eta) + p_\ell(\eta)\ \ln 2k)\ \frac{k^\ell\ F_\ell(\eta,\rho)}{C_\ell(\eta)}\ \} \tag{26}$$

One can prove:

$$\bar{G}_\ell(\eta,\rho) \underset{r\to 0}{\to} (2\ell-1)!!\ r^{-\ell}\ \{1 + \ldots\} + ((2\ell-1)!!)^2\ (2\ell+1)\ k^{2\ell+1} \times p_\ell(\eta)\ \bar{F}_\ell(\eta,\rho)\ \ln r \tag{27}$$

with the term of order $r^{2\ell+1}$, between the brackets in the first term of eq. (27), vanishing.

The boundary condition at r = 0 of $\bar{G}_\ell(\eta,\rho)$ is analytic in k^2 and therefore $\bar{G}_\ell(\eta,\rho)$ is an entire function of k^2 and also in r, except for the branch cut due to the ln r term and the pole at r = 0.

In this way we have constructed a regular and irregular solution, $\bar{F}_\ell$ and $\bar{G}_\ell$, similar to the solutions V_R and V_I used in section II.2. Comparing eqs (21), (26) with eq. (II.16), using eq. (16), it seems reasonable to define:

$$N_C(k) = \frac{1}{(2\ell+1)!!\ C_\ell(\eta)\ k^{\ell+1}}$$
$$H_C(k) = (2\ell+1)!!\ (2\ell-1)!!\ k^{2\ell+1}\ (q_\ell(\eta) + p_\ell(\eta)\ \ln 2k) \tag{28}$$

Indeed, when we use the definition of the effective range function $P^C \equiv (\cot \delta_N^C)/(N_C^2 k) + H_C$ again, we obtain:

$$P^C = ((2\ell+1)!!\ C_\ell(\eta))^2\ k^{2\ell+1}\ \cot \delta_N^C + (2\ell+1)!!\ (2\ell-1)!!\ k^{2\ell+1}\ (q_\ell(\eta) + p_\ell(\eta)\ \ln 2k) \tag{29}$$

which is, up to a polynomial of degree $k^{2\ell+1}$ which is conventionally put into the parametrized right hand side of the effective range

identity, indeed the correct form of the Coulomb effective range function (cf. eq. (14) or references [Sw 62], [Ja 50]). Note that we did not prove yet that this function will also have analytic properties as in chapter II.

Consider the Coulomb counterpart of eq. (II.21a):

$$k^{\ell+1} N_C(k)\, e^{-i\sigma_\ell(k)} = \frac{1}{(2\ell+1)!!\, C_\ell(\eta)}\, e^{-i\sigma_\ell(\eta)} \tag{30}$$

Using:

$$C_\ell(\eta) = \frac{2^\ell\, e^{-\pi\eta/2}\, |\Gamma(\ell+1+i\eta)|}{\Gamma(2\ell+2)} \qquad \text{for real } \eta \quad ([Ab\ 65], 14.1.7)$$

one obtains:

$$k^{\ell+1} N_C(k)\, e^{-i\sigma_\ell} = \frac{e^{\pi\eta/2}\, \Gamma(\ell+1)}{\Gamma(\ell+1+i\eta)} \tag{31}$$

which can be continued analytically in the complex k-plane and indeed corresponds with the Coulomb Jost function ([Ne 66], eq. 14.41a).

This function is highly singular for k = 0 and has zeros when:

$$\ell+1+i\eta = n \qquad , \qquad n = 0, -1, -2, \ldots \tag{32}$$

corresponding with the Coulomb bound states.

Furthermore:

$$\lim_{k\to\infty} k^{\ell+1} N_C(k)\, e^{-i\sigma_\ell} = 1$$

but in contrast with the result in section II.2 $k^{\ell+1} N_C(k)$ is <u>not</u> even in k (we will return on this).

The counterpart of eq. (II.22a) is:

$$\frac{ik}{N_C^2 k^2} + H_C = i\, k^{2\ell+1} \left((2\ell+1)!!\right)^2 C_\ell^2(\eta) + (2\ell+1)!!\,(2\ell-1)!! \times k^{2\ell+1} (q_\ell + p_\ell \ln 2k) \tag{33}$$

Using: $C_0^2(\eta) = 2\pi\eta\, (e^{2\pi\eta} - 1)^{-1}$ and $C_\ell^2(\eta) = \dfrac{p_\ell(\eta)\, C_0^2(\eta)}{2\eta\,(2\ell+1)}$,

one writes eq. (33) into:

(polynomial of degree $k^{2\ell}$) + (polynomial of degree $k^{2\ell+1}$ with zeros at $2\eta = \pm 1, \pm 2, \ldots, \pm\ell$) $\times$ $(ie^{-\pi\eta}\Gamma(1+i\eta)\Gamma(1-i\eta)$

$+ \eta(\psi(1+i\eta) + \psi(1-i\eta) - 2 \ln \eta)$ (33b)

The poles from the Γ and ψ terms cancel out partially; due to the zeros in the front factor there are only simple poles left at the positions of the Coulomb bound states, eq. (32).

Furthermore, there is a cut starting at $k = 0$ due to the $\ln \eta$ term and an essential singularity at $k = 0$. This cut is conventionally put along the negative k-axis and therefore:

$\frac{ik}{N_C^2 k^2} + H_C$ is analytic in the upper half plane except for poles at the Coulomb bound states*.

From the fact that N_C, H_C and σ_ℓ are <u>real</u> on the positive real axis follows:

$$\begin{aligned} N_C^*(k^*) &= N_C(k) \\ H_C^*(k^*) &= H_C(k) \\ \sigma_\ell^*(k^*) &= \sigma_\ell(k) \end{aligned} \tag{34}$$

One proves also:

$$\begin{aligned} N_C(+k) &= (-1)^{\ell+1} e^{\pi\eta} N_C(-k) \\ \sigma_\ell(+k) &= -\sigma_\ell(-k) \\ H_C(-k) &= H_C(k) \pm \frac{ik(e^{2\pi\eta}-1)}{(kN_C(k))^2} \quad , \quad \begin{matrix} - \text{ for } \mathrm{Im}\, k > 0 \\ + \text{ for } \mathrm{Im}\, k < 0 \end{matrix} \end{aligned} \tag{35}$$

---

*Corresponding properties, with upper- and lower-half plane interchanged are valid for: $k^{\ell+1} N_C e^{+i\sigma_\ell}$ and $-(ik)/(N_C^2 k^2) + H_C$. The proof is left to the reader.

Using eqs (34), (35) and the analytic properties one derives for k in the upper half plane and a repulsive Coulomb (no bound states):

$$k^{\ell+1} N_C(k) e^{-i\sigma_\ell} = \exp\{-\frac{2}{\pi}\int_0^\infty dk' \frac{k'}{k'^2-k^2}\sigma_\ell(k') + \frac{\pi\eta}{2}\}$$

$$\frac{ik}{(k N_C(k))^2} + H_C(k) = ik + \frac{2k^2}{\pi}\int_0^\infty dk' \frac{(\frac{1}{N_C k'})^2 - 1}{k'^2-k^2} \quad (\ell=0) \tag{36}$$

(Compare with eqs (II.64 - II.68).)

In the next section we will not use the usual "Coulomb functions" c_ℓ, q_ℓ, p_ℓ, ... anymore, only N_C, H_C, σ_ℓ. We quote once more their definition:

$$\bar{F}(r) = N_C F(r)$$

$$\bar{G}(r) = \frac{1}{N_C k} G(r) - H_C N_C F(r) \tag{37}$$

$$W(\bar{F},\bar{G}) \equiv \bar{F}'\bar{G} - \bar{F}\bar{G}' = 1$$

3. Long range potentials containing the Coulomb interaction

We assume that the total potential V can be decomposed according to:

$$V = V_S + V_L + \frac{2 M_{red} \alpha_F}{r} \tag{38}$$

with:

$$V_S(r) \xrightarrow[r\to\infty]{} \sim e^{-\mu_S r}$$

$$V_L(r) \xrightarrow[r\to\infty]{} \sim e^{-\mu_L r}$$

$$\mu_L < \mu_S \quad .$$

As far as the behaviour for $r \to 0$ is concerned, we will assume that V_L and V_S are analytic in r except for possibly a simple pole in the origin. This is a restriction with respect to the usual assumption

in scattering theory which allows a $r^{-2+\varepsilon}$ behaviour at the origin. Since field theoretical potentials have been shown to make sense in the external region only and since there is nothing which guarantees that the potential picture holds at small distances we consider the assumption of analyticity (except for a simple pole) for the potential not too restrictive (it includes also Yukawian potentials). As stated before, the treatment of the potentials with a simple pole at r = 0, does not differ essentially from the finite case.

Following the same steps as in section II.2 we first consider the long range part of eq. (38)*:

$$V_L + \frac{2 M_{red} \alpha_F}{r}$$

The regular and irregular solutions are defined with the boundary conditions at r = 0.

$$v_R(r) \xrightarrow[r\to 0]{} \frac{1}{(2\ell+1)!!} r^{\ell+1} + O(r^{\ell+2})$$

$$v_I(r) \xrightarrow[r\to 0]{} (2\ell-1)!! \, r^{-\ell} + O(r^{-\ell+1}) \qquad ** \qquad (39)$$

They satisfy the integral equations:

---

*Sometimes we will refer to this long range part with: V_L, i.e. putting the Coulomb part into V_L: "$V_L \equiv V_L + \frac{2M_r \alpha_F}{r}$". When explicit formulas are used, containing V_L, one must always use the "finite range" part as V_L; the Coulomb part is represented by the use of Coulomb wave functions instead of Bessel or Neumann functions.

**This is of course not sufficient; as before we will fix the "amount of regular solutions contained in V_I" by requiring that the "$r^{\ell+1}$" term in the expansion must be zero.

$$v_R(r) = \bar{F}_\ell(r) + \int_0^r (\bar{F}_\ell(r)\ \bar{G}_\ell(r') - \bar{G}_\ell(r)\ \bar{F}_\ell(r'))\ V_L(r')\ v_R(r')\ dr'$$

$$v_I(r) = a\ \bar{G}_\ell(r) + b\ \bar{F}_\ell(r) + \int_\varepsilon^r (\bar{F}_\ell(r)\ \bar{G}_\ell(r') - \bar{G}_\ell(r)\ \bar{F}_\ell(r'))\ V_L(r')\ v_I(r')\ dr' \qquad (40)$$

with:

$$a = W(\bar{F}, v_I)\Big|_{r=\varepsilon}$$
$$b = W(v_I, \bar{G})\Big|_{r=\varepsilon} \qquad (41)$$

Since the boundary conditions (39) are analytic in k^2, the solutions v_R and v_I are also analytic in k^2 (for finite r). The functions N_L, H_L and the phase shift δ_L^C of V_L in the presence of the Coulomb field are defined by:

$$v_R \xrightarrow[r\to\infty]{} N_L\ (\cos\delta_L^C\ F_\ell + \sin\delta_L^C\ G_\ell)$$
$$v_I \xrightarrow[r\to\infty]{} \frac{1}{N_L k}\ (\cos\delta_L^C\ G_\ell - \sin\delta_L^C\ G_\ell) \qquad (42)$$
$$- H_L\ N_L\ (\cos\delta_L^C\ F_\ell + \sin\delta_L^C\ G_\ell)$$

From eqs (37), (40), (42) follows:

$$N_L \cos\delta_L^C = N_C\ (1 + \int_0^\infty dr'\ \bar{G}_\ell\ V_L v_R) + H_C\ N_C \int_0^\infty dr'\ \bar{F}_\ell\ V_L v_R$$
$$N_L \sin\delta_L^C = -\frac{1}{N_C k} \int_0^\infty dr'\ \bar{F}_\ell\ V_L\ v_R \qquad (43)$$

and:

$$-\frac{\sin\delta_L^C}{N_L k} - H_L N_L \cos\delta_L^C = H_C N_C\ \{-a + \int_\varepsilon^\infty \bar{F}_\ell\ V_L v_I\ dr'\}$$
$$+ N_C\ \{b + \int_\varepsilon^\infty \bar{G}_\ell V_L v_I\ dr'\}$$
$$\frac{\cos\delta_L^C}{N_L k} - H_L N_L \sin\delta_L^C = \frac{1}{N_C k}\ \{a - \int_\varepsilon^\infty \bar{F}_\ell\ V_L v_I\ dr'\} \qquad (44)$$

Equations (43) can be used to form the linear combinations:

$$N_L\, e^{-+i\delta_L^C} = N_C \{1 + \int_0^\infty (\bar{G}_\ell + (H_C \pm \frac{1}{N_C^2 k}) \bar{F}_\ell)\, V_L\, v_R\, dr'\} \qquad (45)$$

The integral in the right hand side of eq. (45), with ∞ replaced by a finite upper bound has the following analytic properties:

- $\bar{G}_\ell$, $\bar{F}_\ell$ and v_R are analytic in k^2 for finite r. Since $(H_C +\!- \frac{1}{N_C^2 k})$ is analytic in the upper/lower momentum plane, except for poles at $k = +i\beta_C, -i\beta_C$ - with $i\beta_C$ a Coulomb bound state - and the cut along the negative k-axis, this is also the case for the integral with finite upper bound.

One proves:

$$\bar{G}_\ell + (H_C +\!- \frac{1}{N_C^2 k})\, \bar{F}_\ell = \frac{1}{N_C k} (G_\ell +\!- i F_\ell)$$

This combination has an exponential damping for $r \to \infty$ if k lies in the upper/lower momentum half plane. Now one can prove with the usual methods that the integral for $r \to \infty$ is uniformly converging in this region and therefore*:

$\frac{N_L}{N_C} e^{-+i\delta_L^C}$ *is analytic in the upper/lower complex k half plane except for singularities at the Coulomb bound states* $(+\!-i\beta_C)$.

Also $N_L\, e^{-+i\delta_L^C}$ has singularities in the upper/lower half plane (although N_C is zero at the Coulomb bound states, this zero is not simple and does not cancel the pole from the integral). This is not the result we expected from the properties of eq. (II.21). However, δ_L^C is not the

---

*In fact all of this is not trivial due to the r^{-1} behaviour for $r \to \infty$ in the potential. Bounds on the wave functions for $\ell = 0$ are given e.g. by Cornille and Martin [Co 62].

"total phase shift" of the long range potential + Coulomb, but rather.

$$\delta_L^C + \sigma_\ell$$

Multiplying eq. (45) with $k^{\ell+1}\, e^{-+i\sigma_\ell}$ we obtain:

$$k^{\ell+1}\, N_L(k)\, e^{-+i(\delta_L^C+\sigma_\ell)} = k^{\ell+1}\, N_C(k)\, e^{-+i\sigma_\ell} \times \{1 + \int_0^\infty (\bar{G}_\ell + (H_C +- \frac{1}{N_C^2 k})\bar{F}_\ell) V_L v_R dr'\} \quad (46)$$

The factor: $k^{\ell+1}\, N_C\, e^{-+i\sigma_\ell}$ is analytic in the upper/lower half plane and contains simple zeros at $k = +i\beta_C/-i\beta_C$ which cancel the poles from the integral at the same places.

One obtains: $k^{\ell+1}\, N_L\, e^{-+i(\sigma_\ell+\delta_L^C)}$ is analytic in the complex k plane except for: singularities due to V_L starting at $k = -+ i\frac{\mu_L}{2}$ along the imaginary axis, the branch cut along the negative k-axis and the essential singularity at k = 0 from the Coulomb interaction.

Furthermore one proves that:

$$k^{\ell+1}\, N_L\, e^{-i(\delta_L^C+\sigma_\ell)} = 0 \leftrightarrow 1 + (\int_0^\infty (\bar{G}_\ell + (H_C + \frac{1}{N_C^2 k})\, \bar{F}_\ell V_L v_R dr' = 0$$

$$\leftrightarrow\ v_R \xrightarrow[r\to\infty]{} \sim (G_\ell + i\, F_\ell) \xrightarrow[r\to\infty]{} \sim e^{-(\mathrm{Im}\, k)r}$$

Therefore, a zero of $k^{\ell+1}\, N_L\, e^{-i(\delta_L^C+\sigma_\ell)}$ corresponds with an exponential damping of the regular wave function in the upper half momentum plane, i.e. bound states of the potential: $V_L + \frac{2M_{red}\,\alpha_F}{r}$. The procedure, leading to eq. (46), can also be applied to eqs (44). One obtains:

$$N_L\, e^{-+i(\delta_L^C+\sigma_\ell)}(H_L +- \frac{1}{N_C^2 k}) = -\, N_C\, e^{-+i\sigma_\ell} \{b - a(H_C +- \frac{1}{N_C^2 k}) + \int_\varepsilon^\infty (\bar{G}_\ell + (H_C +- \frac{1}{N_C^2 k})\, \bar{F}_\ell)\, V_L v_I dr'\} \quad (47)$$

With the same arguments one proves:

Expression (47) is analytic in the upper/lower half plane. Since $N_L\, e^{-i(\delta_L^C+\sigma_\ell)}$ has the same properties and is zero at the bound state momenta of the total long range potential one obtains:

$H_L + \frac{1}{N_L^2 k}$ is analytic in the upper/lower half plane except for poles at $k = +i\beta_L, -i\beta_L$ where $i\beta_L$ is the momentum of a bound state of: $V_L + \frac{2M_{red}\,\alpha_F}{r}$. In the other half plane singularities start at $k = -+ i\frac{\mu_L}{2}$ and there could be poles at $k = \gamma_L/-\gamma_L$ with γ_L the momentum of a resonance (or virtual bound state). Also the cut along the negative k-axis and the essential singularity is present.

Considering the symmetry properties in the momentum plane: the fact that N_L, H_L, δ_L^C are real on the positive k-axis gives:

$$\begin{aligned} N_L^*(k^*) &= N_L(k) \\ H_L^*(k^*) &= H_L(k) \\ \delta_L^{C*}(k^*) &= \delta_L^C(k) \end{aligned} \tag{48}$$

The transformations for $k \to -k$ are not so simple as in chapter II. One proves:

$$N_L(k)\, e^{-i(\delta_L^C(k)+\sigma_\ell(k))} = (-1)^{\ell+1}\, e^{\pi\eta}\, N_L(-k)\, e^{+i(\delta_L^C(-k)+\sigma_\ell(-k))}$$

$$\left(\frac{1}{N_L^2 k} + H_L\right)_{+k} = \left(\frac{-1}{N_L^2 k} + H_L\right)_{-k} \tag{49}$$

for $\mathrm{Im}\, k > 0$.

With the analytic properties, eqs (48) and (49) lead to:

$$e^{-i\delta_L^C(k)}\, k^{\ell+1}\, N_L(k) = k^{\ell+1}\, N_C(k)\, \exp\{-\frac{2}{\pi}\int_0^\infty dk'\, \frac{k'}{k'^2-k^2}\, \delta_L^C(k')\}$$

$$\frac{1}{N_L^2 k} + H_L(k) = \text{constant} + ik + \frac{2k^2}{\pi}\int_0^\infty dk'\, \frac{\left(\frac{1}{N_L k'}\right)^2 - 1}{k'^2-k^2} \tag{50}$$

for $\ell = 0$, $\mathrm{Im}\, k > 0$ and the case that the potential $V_L + \frac{2M_{red}\,\alpha_F}{r}$ has

no bound states.

Equation (50) can be used to compute N_L and H_L if the phase shift δ_L^C is known from other sources.

4. The effective range function

In this section we will include the short range part into the potential and derive its effective range function with respect to the long range potential with Coulomb interaction.

First we will follow the method used in section II.3, then the analytic properties of the amplitude will be considered according to section II.4.

The regular solution of the total potential V, defined in eq. (38) satisfies the integral equation:

$$u(r) = (1 + \int_0^r dr'\ v_I(r')\ V_S(r')\ u(r'))\ v_R(r) - (\int_0^r dr'\ v_R(r')\ V_S(r')\ u(r'))\ v_I(r) \qquad (51)$$

It has the boundary condition:

$$u(r) \underset{r\to 0}{\longrightarrow} \frac{1}{(2\ell+1)!!}\ r^{\ell+1} + O(r^{\ell+2})$$

and is an analytic function of k^2.

In the asymptotic limit we write it as:

$$u(r) \underset{r\to\infty}{\longrightarrow} A\ v_R(r) + B\ v_I(r) \qquad (52)$$

with:

$$A = 1 + \int_0^\infty dr'\ v_I\ V_S\ u$$
$$B = -\int_0^\infty dr'\ v_R\ V_S\ u \qquad (53)$$

Since: $V_S(r) \underset{r\to\infty}{\longrightarrow} \sim e^{-\mu_S r}$, A and B will be analytic functions in k^2

up to $k = \pm i \frac{\mu_S}{2}$, where singularities start along the imaginary axis.

Using eqs (42) and the definition of the phase shift δ_N^C for potential V, according to eq. (3), one obtains:

$$\frac{1}{N_L^2 k} \cot \delta_S^L + H_L = \frac{A}{B} \tag{54}$$

with:

$$\delta_N^C = \delta_L^C + \delta_S^L$$

Therefore the effective range function:

$$P^L = \frac{1}{N_L^2 k} \cot \delta_S^L + H_L \tag{55}$$

will be analytic in k^2 except for singularities along the negative k^2 axis for $k^2 \leq -\frac{\mu_S^2}{4}$ and except for poles due to the zeros of B.

Also at the threshold, $k = 0$, P^L is well defined in general due to the k-independent boundary conditions (except when B accidentally is zero at this point). Furthermore, taking $V_L = 0$ (only Coulomb interaction as long range potential), one obtains: $v_R = \bar{F}_\ell$, $v_I = \bar{G}_\ell$, and $N_L = N_C$, $H = H_C$, $\delta_S^L = 0$, leading to: $P^L(k^2) = P^C(k^2)$ (the ordinary Coulomb effective range function, eq. (29)).

Now we will turn to the partial wave amplitude. According to eq. (12b) the total amplitude can be written as the sum of the Coulomb partial wave amplitude and a correction term for the nuclear part:

$$f_\ell = f_{\sigma_\ell} + e^{2i\sigma_\ell} f_{\delta_N^C} \tag{56}$$

with:

$$f_{\sigma_\ell} = \frac{e^{2i\sigma_\ell} - 1}{2ik} \quad , \quad f_{\delta_N^C} = \frac{e^{2i\delta_N^C} - 1}{2ik}$$

Using: $\delta_N^C = \delta_L^C + \delta_S^L$ one gets:

$$f^C_{\delta_N} = f^C_{\delta_L} + e^{2i\delta^C_L} f^L_{\delta_S}$$

$$= \frac{e^{2i\delta^C_L}-1}{2ik} + e^{2i\delta^C_L} \frac{e^{2i\delta^L_S}-1}{2ik} \tag{57}$$

which splits the nuclear part of the amplitude into a long range (V_L) and a short range (V_S) part. Note that:

$$f_\ell = f_{\sigma_\ell} + e^{2i\sigma_\ell} f^C_{\delta_L} + e^{2i(\sigma_\ell+\delta^C_L)} f^L_{\delta_S} \tag{58}$$

The first two terms on the right hand side represent the partial wave amplitude for the potential: $V_L + \frac{2 M_{red} \alpha_F}{r}$.

We define outgoing and incoming (irregular) solutions of the long range potential:

$$v_{+-} = (H_L +- \frac{1}{N_L^2 k}) v_R + v_I \tag{59}$$

with:

$$v_{+-} \xrightarrow[r\to\infty]{} \frac{e^{+-i\delta^C_L}}{N_L k} (G_\ell +- i F_\ell) \xrightarrow[r\to\infty]{} \sim e^{+-ikr}$$

These wave functions are analytic in k except for the singularities caused by the $(H_L +- \frac{1}{N_L^2 k})$ factor. One obtains:

$$e^{2i\delta^L_S} = \frac{1 + \int_0^\infty v_- V_S u}{1 + \int_0^\infty v_+ V_S u} \tag{60}$$

with singularities along the imaginary axis starting at $k = +- i \frac{\mu_L}{2}$ and poles, coming from v_- and from zeros of the denominator. Also the Coulomb cut starting at k = 0 along the negative k-axis is present.

A better behaviour in the (physical) upper half momentum plane has:

$$N_L^2 k^2 \frac{e^{2i\delta_S^L}-1}{2ik} = -\frac{\int_0^\infty v_R\, V_S\, u}{1 + \int_0^\infty v_+\, V_S\, u} \qquad (61)$$

which is analytic up to $k = +i\frac{\mu_S}{2}$ except for poles from zeros of the denominator (corresponding with bound states of V). At the threshold it still contains the essential Coulomb singularity and of course the cut along the negative axis.

These singularities can be made explicit, using eqs (59) and (53) - (55):

$$N_L^2 k^2 \frac{e^{2i\delta_S^L}-1}{2ik} = \frac{1}{P^L(k^2) - (\frac{1}{N_L^2 k} + H_L)} \qquad (62)$$

The factor $(1/N_L^2 k + H_L)$ is responsible for the Coulomb singularities at $k = 0$ and singularities starting at $k = -i\frac{\mu_L}{2}$ in the lower half plane. The effective range function $P^L(k^2)$ does not contain singularities in upper or lower half plane up to $k = +- i\frac{\mu_S}{2}$ (neglecting accidental poles).

The function defined by eq. (61) with nice analytic properties in the physical plane, has (for $V_L = 0$!!) been derived by Hamilton [Ha 73], calling it the effective function, and is the starting point in dispersive calculations for the Coulomb corrections on the hadronic amplitude. We define:

$$u^\infty = \frac{u}{(1 + \int_0^\infty v_+\, V_S\, u\, dr')} \qquad (63)$$

which satisfies:

$$u^\infty(r) = v_R(r) - \int_0^\infty dr'\, G^{L,+}(r,r')\, V_S\, u^\infty \qquad (64)$$

with:

$$G^{L,+}(r,r') = v_R(r_<)\ v_+(r_>)$$

Now:

$$N_L^2 k^2\ f_{\delta_S^L} = -\int_0^\infty v_R\ V_S\ u^\infty\ dr' \qquad (65)$$

As in the case without Coulomb interaction one can prove that the singularity structure of $N_L^2 k^2\ f_{\delta_S^L}$ is "additive" in the successive Born approximations of the integral equation (64) (the proof of this, which needs bounds on $G^{L,+}$, is not trivial; see [Co 62]). Note that this additivity is not present in the effective range function P^L: the discontinuity along the first part of the cut is also influenced by the higher Born approximation. One obtains:

$$\text{disc}\ \frac{A}{B} = \text{disc}\ \left(\frac{1}{N_L^2 k^2 f_{\delta_S^L}} + \left(\frac{1}{N_L^2 k} + H_L\right)\right)$$

$$= \frac{-\ \text{disc}\ (N_L^2 k^2\ f_{\delta_S^L})}{|N_L^2 k^2\ f_{\delta_S^L}|} \qquad \text{in the upper half plane.}$$

The numerator is linear in the potential V_S for momenta: $i\frac{\mu_S}{2} < k \leq \mu_S$. However, the denominator contains all orders.

Finally we observe that at the bound state momenta of the long range potential, $i\beta_L$:

$$N_L^2 k^2\ f_{\delta_S^L} = 0$$

and:

$$e^{2i(\sigma_\ell+\delta_L)}\ f_{\delta_S^L} \xrightarrow[k\to i\beta_L]{} -\frac{1}{2ik}\, e^{2i(\sigma_\ell+\delta_L)} \qquad (\xrightarrow[k\to i\beta_L]{} \infty)$$

Therefore the poles at $k = i\beta_L$ in the different terms of the total amplitude:

$$f_{\ell} = f_{\sigma_{\ell}} + e^{2i\sigma_{\ell}} f_{\delta_L} + e^{2i(\sigma_{\ell}+\delta_L)} f_{\delta_S^L}$$

will cancel each other giving a finite total sum.

CHAPTER IV

The effective range identity: parametrizing the effective range function

1. Introduction

Once the long range potential is known the functions N_L, H_L and δ_L (δ_L^C) can be computed. The phase shift δ_N^C of the total potential is given by:

$$\delta_N^C = \delta_L^C + \delta_S^L$$
$$\frac{1}{N_L^2 k} \cot \delta_S^L + H_L = \frac{A}{B} \qquad (1)$$

If the short range potential V_S is known explicitly one could compute A and B from eqs (3.53) and use them in (1). However, it will be clear that it is much simpler in this case to calculate the total phase shift δ_N^C directly by solving the Schrödinger equation. In that case the short range potential must be parametrized (coupling constants, masses) and these parameters are adjusted such as to give an accurate description of the experimental data (cross sections, polarizations, binding energies, quadrupole moment and so on).

This is the standard method which Hamada-Johnston [Ha 62], Bressel-Kerman [Br 69], Reid [Re 68] and many others use for the analysis of proton-proton scattering data up to 300 MeV lab energy. However, this method has several disadvantages:

- it is rather computer time consuming since for every experimental energy the Schrödinger equation must be solved several times.
- it requires an a priori knowledge of the form of the potential, long range as well as intermediate and short range.

However, only the long range part (one pion exchange and electric interactions) is believed to be well-known. The medium and short-range behaviour of the nucleon-nucleon interaction is more difficult to derive but it seems to be more or less well described in terms of ρ-, ω-, and ε-exchange together with some additional repulsion.

At really short distances, less than 0.4 or 0.5 fm, the interactions obtained naively from meson exchange must be cut off in some (phenomenological) way. Also the origin of the short range repulsion ("core") remains unclear yet.

All this implies that the potential will contain pure phenomenological functions which could lead to the following troubles:

- since the predicted cross sections, polarizations, etc, are very sensitive for relatively small changes in the parameters, fitting to the experimental data directly is difficult. Moreover, if the potential contains wrong or pathological sub-potentials, fitting the parameters will lead to a potential with a very delicate balance between the several parts in order to compensate for these unphysical features.
- although different potentials, which agree with each other in the long range part, result into (roughly) the same phase shifts, the behaviour for smaller distances could be substantially different; however, they agree about the following features:

The nucleon-nucleon interaction seems to have:

- a long range one pion exchange (OPE) part (+ possible electric interactions)
- a medium range strong attractive force
- a short range repulsion.

For these reasons we propose the following procedure to analyse

nucleon-nucleon scattering data:

1. include in the long range potential V_L all potentials which are explicitly known and calculate the functions N_L, H_L and δ_L (δ_L^C) (for instance for p-p scattering: OPE + electric potentials).

2. parametrize the right hand side of eq. (1) in such a way that it is able to describe some general characteristics of the short range part (for p-p: medium range attraction + short range repulsion).

Disadvantages of this procedure are:

- the parametrization must be done for every partial wave separately, therefore increasing the number of parameters. Perhaps one must find a method to predict parameters for higher partial waves from the lower ones.
- the parameters are not "physical", in the sense that they merely represent in another way experimental data; they only connect data at different energies with each other. It is not possible to extract from them in a simple way "physical" quantities like coupling constants or to use them in another physical situation (for instance crossing from $NN \to NN$ to $N\bar{N} \to N\bar{N}$).

On can see already on beforehand how many parameters are at <u>least</u> necessary to describe the nucleon-nucleon interaction; namely 6, corresponding with:

- strength and range of the OPE potential,
- strength and range of the medium range interaction,
- strength and range of the short range repulsion.

Whether and how these characteristics are really observable will be considered in chapter VI.

In the next section we will give a special parametrization of the

right hand side of eq. (1): the pole approximation.

2. The effective range function and the pole approximation

The most simple parametrization of the effective range function P_L is the Taylor expansion in k^2, written in the well-known form:

$$P_L(k^2) = -\frac{1}{a_S^L} + \frac{1}{2} r_S^L k^2 - r_S^{L\,3} p_S^L k^4 + \ldots \qquad (2)$$

The notation a_S^L, r_S^L, ... is used in order to indicate the fact that we are dealing with the parametrization of δ_S^L (phase shift of the short range potential with respect to the long range potential).

The series eq. (2) is converging for momenta:

$$|k^2| < \frac{\mu_S^2}{4} \qquad (3)$$

(or less when B = 0 at shorter distance from the origin, causing a simple pole in P_L).

The bound eq. (3) is caused by dynamical singularities along the negative k^2 axis: $k^2 < -\frac{\mu_S^2}{4}$. In order to improve convergence for momenta at a finite distance ($\neq 0$) of the singularities one often uses Padé approximants. A Padé approximant $Q^{[N,M]}(k^2)$ of the function $P_L(k^2)$, is defined as the ratio of two polynomials:

$$Q^{[N,M]}(k^2) = \frac{R_M(k^2)}{S_N(k^2)} \qquad (4)$$

with R_M and S_N polynomials of degree M and N in k^2 respectively, with coefficients determined by the requirement*:

---

*For a review of the properties of Padé approximants we refer to: G.A. Baker jr.: Essentials of Padé approximants; Academic Press Inc. (1975).

$$\lim_{k^2 \to 0} \left(Q^{[N,M]}(k^2) - P_L(k^2)\right) \Rightarrow O(k^{2(N+M+1)}) \qquad (5)$$

Under certain conditions one can prove that:

$\lim_{N,M\to\infty} Q^{[N,M]}(k^2) = P_L(k^2)$ (meant is not especially the effective range function, but any function with singularities in the complex k^2 plane, providing it satisfies certain properties). This is in principle valid for k^2 in the entire complex plane.

Eqs (4) and (5) are a pure mathematical definition of the parametrization:

$$P_L(k^2) = Q^{[N,M]}(k^2) \qquad (6)$$

In the following we will demonstrate the physical content of such an approximation.

We recall the properties of $N_L^2 k^2 f_{\delta_S^L}$ as defined in eq. (3.61): In the physical k^2 plane (corrsponding with the upper half momentum plane) it has singularities along the negative k^2 axis for: $k^2 \leq -\frac{\mu_S^2}{4}$. Typical examples are cuts if the potential is a superposition of Yukawa's, poles if the total potential consists of exponentials in r. Due to the mapping from $k \to k^2$ it has also a cut along the positive real axis, with discontinuity:

$$\text{disc}\,(N_L^2 k^2 f_{\delta_S^L}) = 2i\,\text{Im}\,(N_L^2 k^2 f_{\delta_S^L})_{k^2+i\varepsilon} = \frac{2i}{N_L^2 k} \left|N_L^2 k^2 f_{\delta_S^L}\right|^2 \qquad (7)$$

Therefore for physical energies:

$$\text{Im}\,(N_L^2 k^2 f_{\delta_S^L}) = \frac{1}{N_L^2 k} \left|N_L^2 k^2 f_{\delta_S^L}\right|^2 \qquad (8)$$

which will be called the modified optical theorem. For momenta: $-\frac{\mu_S^2}{4} < k^2 < 0$ one proves: $N_L^2 k^2 f_{\delta_S^L}$ is real and from eq. (3.50) one derives. $\lim_{k\to\infty} k^{\ell+1} N(k) = 1$. Furthermore it appears that $N_L^2 k^2 f_{\delta_S^L}$ is

bounded for $k^2 \to 0$ in the physical plane, except when a (bound state) pole is lying on top of the threshold.

Following the "N/D" method [Che 60] we write:

$$N_L^2 k^{2\ell} f_{\delta_S^L} = \frac{N}{D} \tag{9}$$

where $N(k^2)$ contains only (dynamical) left hand singularities in the complex k^2 plane and is analytic everywhere else; $D(k^2)$ contains only the unitarity cut and is analytic everywhere else.

Furthermore we require:

$$\lim_{|k^2| \to \infty} D(k^2) = 1 \tag{10}$$

and from $\lim_{k \to \infty} \delta_S^L = O\left(\frac{1}{k}\right)$ and $\lim_{k \to \infty} N_L^2 k^{2\ell} = O(k^{-2\ell})$ it follows:

$$\lim_{|k^2| \to \infty} N(k^2) \sim k^{-2\ell-2} \tag{11}$$

In principle one could calculate the singularities in the left hand plane from eqs (3.64) and (3.65). One believes that for smaller energies only the nearby singularity structure is of importance so that it is sufficient to calculate only the first and second Born approximation [Lu 64]. A problem, however, is that even the first Born approximation can only be calculated (if V_S is known) if the long range potential V_L is rather trivial: $V_L = 0$, $V_L = \frac{2M_{red}\,\alpha_F}{r}$ or any potential which can be solved in closed form. Therefore, instead of parametrizing the short range potential V_S, we choose to parametrize the singularity structure of $N_L^2 k^{2\ell} f_{\delta_S^L}$. Of all the possible methods, we will consider only the approximation of the left hand singularities by a sequence of poles, merely because the equations can then be solved in closed form. In principle it is possible to simulate in this way almost every singularity structure; in practice, however, this would

require a large number of poles and the simplicity would be lost.

We will take one pole for each characteristic feature of the short range potential. For s-wave proton-proton scattering this would mean:

- for V_L = electric interaction: 3 poles representing: V_S = OPE + medium range attraction + short range repulsion.

- for V_L = OPE potential + electric interaction: 2 poles representing: medium + short range interaction.

Approximating the left hand singularities by n poles, $N(k^2)$ must be of the form:

$$N(k^2) = \sum_{i=1}^{n} \frac{\lambda_i}{k^2+\mu_i^2} \tag{12}$$

with 2n free (real) parameters: $\{\lambda_i, \mu_i\}$. However, the condition eq. (11) gives ℓ restrictions:

$$\sum_{i=1}^{n} \lambda_i \mu_i^{2k} = 0 \qquad ; \qquad k = 0, \ldots, \ell-1 \tag{13}$$

and therefore the total number of free parameters is only $(2n-\ell)$. From eqs (13) it follows that only a nontrivial ($N \neq 0$) solution is possible if: $n > \ell$; therefore for the ℓ^{th} partial wave a minimum of $\ell+1$ poles is required (with $\ell+2$ free parameters).

$N(k^2)$ can be written in several forms:

$$N(k^2) = \sum_{i=1}^{n} \frac{\lambda_i}{k^2+\mu_i^2} = \frac{1}{k^{2\ell+2}} \left(\nu_0 + \sum_{i=1}^{n} \frac{\nu_i}{k^2+\mu_i^2}\right)$$

$$= \frac{U_{n-\ell-1}(k^2)}{V_n(k^2)} \tag{14}$$

with V_n and $U_{n-\ell-1}$ polynomials in k^2 of degree n and $n-\ell-1$ respectively; of course the coefficients $\{\nu_i\}$ are related to $\{\lambda_i, \mu_i\}$.

From the analytic properties of D, (eq. (7)) one derives the integral equation:

$$D(k^2) = 1 - \frac{1}{\pi} \int_0^\infty dk'^2 \frac{1}{k'^2 - k^2} \frac{1}{N_L^2 k^2} N(k'^2) \qquad (15)$$

Using eqs (14) and (2.65), (2.66) (also valid if the Coulomb interaction is included, eq. (3.50)) one can perform the integration resulting into:

$$D(k^2) = 1 - N(k^2) \left(\frac{1}{N_L^2 k} + H_L - W_\ell(k^2) \right) + \sum_{i=1}^{n} \frac{\tau_i}{k^2 + \mu_i^2} \qquad (16)$$

with $W_\ell(k^2)$ a polynomial of degree ℓ (the first term in the right hand side of eq. (2.66)) and τ_i a real number:

$$\tau_i = \frac{2\nu_i}{\pi} \int_0^\infty dk' \frac{1}{N_L^2 k'^{2\ell+2}} \frac{1}{k'^2 + \mu_i^2} \qquad *$$

Therefore:

$$N_L^2 k^2 f_{\delta_S^L} = \frac{N}{D} = \frac{1}{1 + \frac{\sum_{i=1}^{n} \frac{\tau_i}{k^2+\mu_i^2}}{N(k^2)} + W_\ell(k^2) - \frac{1}{N_L^2 k} - H_L}$$

$$= \frac{1}{\frac{R_n(k^2)}{S_{n-\ell-1}(k^2)} - \frac{1}{N_L^2 k} - H_L} \qquad (17)$$

with R_n and $S_{n-\ell-1}$ real polynomials in k^2 of degree n and $n-\ell-1$ respectively.

Comparing with eq. (3.62) gives:

$$P_L(k^2) = \frac{R_n(k^2)}{S_{n-\ell-1}(k^2)} \qquad (18)$$

---

*This integral can be rewritten with help of the integral (2.66), (3.50) (generalized for $\ell \neq 0$).

An n-pole approximation of $N_L^2 k^2 f_{\delta_S^L}$ therefore is equivalent with a Padé approximation of order $[n-\ell-1, n]$ of the effective range function:

$$P_L(k^2) = Q_L^{[n-\ell-1,n]}(k^2) \qquad (19)$$

The number of coefficients in the Padé $Q_L^{[n-\ell-1,n]}(k^2)$ is $(2n-\ell)$; therefore instead of fitting $\{\lambda_1, \mu_1\}$ (with restrictions of eq. (13)) it is more convenient to consider the Padé coefficients as free parameters*. In the past mixed forms have been derived with (a few) Padé coefficients as well as pole parameters as independent parameters. We will consider them in the next section for the cases $V_L = 0$ and $V_L = \frac{2M_{red}\,\alpha_F}{r}$; therefore they are parametrizations for the "classical" effective range functions.

3. CFS approximations

CFS approximations have been used to include in nucleon-nucleon scattering (np and pp) some information about the OPF interaction. First we will consider the np case with $V_L = 0$. The short range interaction - identical with the total potential - contains one pion exchange + medium range attraction + short range repulsion.

In the original CFS approximation, derived for s-waves, the OPE cut is replaced by one pole with:

$$\text{position} = -\frac{\mu_\pi^2}{2}$$
$$\text{residue} = \frac{f^2 M}{2} \quad (M = \text{nucleon mass}) \qquad (20)$$

*It is not true, however, that every choice of Padé coefficients leads to "physical" poles in $N_L^2 k^2 f_{\delta_S^L}$, i.e. along the negative k^2 axis, with real residues.

This choice can be justified in several ways. The argument of Cini-Fubini-Stanghellini [CFS 59] was based on the observation that the total (partial wave summed) amplitude will have a pole at: $\vec{k}^2 = -\mu_\pi^2$, with residue $f^2 M$. Since the total amplitude at $\theta = 90^\circ$ can be approximated for small energies by the s-wave partial amplitude (p-waves do not contribute at 90° and $\ell \geq 2$ waves are small) one has:

$$\frac{e^{2i\delta_0}-1}{2ik} \simeq f_{tot}(\theta = 90^\circ) = \frac{f^2 M}{2k^2+\mu_\pi^2} + \ldots \tag{21}$$

which corresponds with (20).

Another argument is that eq. (21) is the [1,0] Padé of the first Born approximation (giving the OPE cut) of the s partial wave amplitude:

$$f^{BA,\pi}_{\ell=0} = \frac{f^2 M}{4k^2} \ln\left(1 + \frac{4k^2}{\mu_\pi^2}\right) \longrightarrow_{k \to 0} \frac{f^2 M}{2k^2+\mu_\pi^2} \tag{22}$$

The medium- and short ranged interactions are also replaced by poles; using one pole for both interactions together one can solve the equations in section 2 * ($N_L k = 1$, $H_L = 0$, $\delta_L = 0$) and arrive at:

$$P(k^2) = k \cot \delta_N = -\frac{1}{a} + \frac{1}{2} r k^2 - \frac{c\mu_\pi^{-3} k^4}{1 + d\,\mu_\pi^{-2} k^2}$$

with:

$$d = \left(2 - \frac{f^2 M}{\mu_\pi}\left(\frac{3}{2}\sqrt{2} - \frac{4}{\mu_\pi a} - \frac{1}{2}\mu_\pi r\right)\right) \Big/ \left(1 - \frac{f^2 M}{\mu_\pi}\left(\frac{1}{4}\sqrt{2} - \frac{1}{\mu_\pi a}\right)\right)$$

$$c = \left(1 - \frac{1}{2} d\right)\left(2\sqrt{2} - \mu_\pi r - \frac{4}{\mu_\pi a}\right) \tag{23}$$

Using one pole for the pion according to eq. (20), and one pole for the medium- and short range interaction each, one obtains:

---

*Of course this is not the method used by CFS or the most simple method for this trivial potential. See also [Wo 62], [No 64].

$$P(k^2) = k \cot \delta_N = -\frac{1}{a} + \frac{1}{2} r k^2 - \frac{p k^2 (1+tk^2)}{(1+qk^2)(1+sk^2)} \qquad (24)$$

where p and q are complicated functions of a, r, t, s, f^2 and μ_π.

The approximation of eq. (23), with a total of two poles in the amplitude will be called the CFS (2) approximation. Eq. (24), which will result in a total of three poles, will be called the CFS (3) approximation.

The CFS approximations can be generalized in order to account for the Coulomb interaction [Wo 62], [No 64]. One takes:

$$V_L = \frac{M \alpha_F}{r} \quad ;$$

$$N_L^2 k^2 = \frac{1}{C_0^2} \quad ; \qquad H_L = 2\eta k\, h(\eta) \qquad (25)$$

and uses the same method as before, replacing the OPE cut by one pole and using one or more poles for the remaining interactions. One takes for the pion pole:

$$\text{position} = -\frac{\mu_\pi^2}{2} \times \gamma_C$$

$$\text{residue} = \frac{f^2 M}{2} \times \delta_C \qquad (26)$$

with γ_C and δ_C Coulomb modification factors (we will return on this in the following).

Representing the medium- and short-range interactions by <u>one</u> pole, one arrives at the Coulomb corrected CFS (2) approximation:

$$P_C(k^2) = C_0^2 k \cot \delta_N^C + 2\eta k\, h(\eta)$$

$$= -\frac{1}{a_N^C} + \frac{1}{2} r_N^C - \frac{pk^4}{1+qk^2} \qquad (27)$$

p,q = very complicated functions of a_N^C, r_N^C, μ_π, f^2, α_F.

Emmen [Em 79] calculated the case where the vacuum polarization

is included in the long range potential, and also the generalization of CFS (3). We refer to chapter V for an alternative form of the CFS approximations, which in many aspects is more convenient to use.

In chapter VI we will consider the merits of the CFS (2) and CFS (3) approximations applied in nucleon-nucleon scattering.

To determine the Coulomb modification factors in eq. (26) we will follow Wong and Noyes [Wo 62].

Consider the OPE part of the total interaction, for the $^1S_0$ wave represented by the Yukawa potential:

$$V_\pi = - M f^2 \frac{e^{-\mu_\pi r}}{r} \tag{28}$$

The OPE cut in $N_C^2 k^2 f_{\delta_N^C}$ is determined by the first Born approximation of eqs (64 and (65) in V_π (the singularity structure is additive!):

$$N_C^2 k^2 f_{\delta_N^C} = \frac{}{1\pi} - \int_0^\infty v_R(r') V_\pi(r') v_R(r') dr' \tag{29}$$

Since V_L = Coulomb potential we have:

$$v_R(r) = \bar{F}_0(r) = \frac{1}{C_0 k} F_0$$

and:

$$N_C^2 k^2 f_{\delta_N^C} = \frac{Mf^2}{1\pi\, k^2 C_0^2} \int_0^\infty dr' \frac{e^{-\mu_\pi r'}}{r'} F_0^2(r') \tag{30}$$

This can be written in closed form:

$$N_C^2 k^2 f_{\delta_N^C} = - \frac{Mf^2}{1\pi\, 4k^2} \left(\frac{\mu_\pi 1}{2k} \right)^{-2-i\eta} \left(1 + \frac{\mu_\pi 1}{2k}\right)^{2i\eta} \times F\left(1+i\eta, 1+i\eta; 2; - \frac{4k^2}{\mu_\pi^2}\right) \tag{31}$$

with F the ordinary hypergeometric function.

F(a;b;c;2) is analytic in the z plane, cut along the real axis from z = 1 to ∞, corresponding with a cut from $k^2 = - \frac{\mu_\pi^2}{4}$ to $- \infty$ in the k^2 plane. This cut can be made explicit by using the identity ([Ab 65],

eq. 15.3.6, [Wo 62]):

$$F(1+i\eta,1+i\eta;2;-\frac{4k^2}{\mu_\pi^2}) = \frac{\Gamma(-2i\eta)}{\Gamma^2(1-i\eta)} F(1+i\eta,1+i\eta,1+2i\eta;1+\frac{4k^2}{\mu_\pi^2})$$

$$+ (1+\frac{4k^2}{\mu_\pi^2})^{-2i\eta} \frac{\Gamma(2i\eta)}{\Gamma^2(1+i\eta)} F(1-i\eta,1-i\eta;1-2i\eta;1+\frac{4k^2}{\mu_\pi^2})$$

giving:

$$N_C^2 k^2 f_{\delta_N^C 1\pi} = \frac{Mf^2}{\mu_\pi^2} \left\{ \frac{\Gamma(-2i\eta)}{\Gamma^2(1-i\eta)} (1+\frac{2k}{i\mu_\pi})^{2i\eta} F(1+i\eta,1+i\eta;1+2i\eta;1+\frac{4k^2}{\mu_\pi^2}) \right.$$

$$\left. + \frac{\Gamma(2i\eta)}{\Gamma^2(1+i\eta)} (1-\frac{2k}{i\mu_\pi})^{-2i\eta} F(1-i\eta,1-i\eta;1-2i\eta;1+\frac{4k^2}{\mu^2}) \right\} \quad (32)$$

We can now clearly see that neither F nor the Γ functions are singular in the upper half plane for $k > \frac{i\mu_\pi}{2}$ and that the discontinuity along the OPE cut is given only by the $(1-\frac{2k}{i\mu_\pi})^{-2i\eta}$ factor.

Expanding F and the Γ functions in orders of η, leaving terms of order η^2 and higher one finds:

$$N_C^2 k^2 f_{\delta_N^C 1\pi} \Big|_{\substack{k=ix+\epsilon \\ x > \frac{\mu_\pi}{2} \\ k=ix-\epsilon}} = i\pi \frac{Mf^2}{2k^2} \left|1-\frac{2k}{i\mu_\pi}\right|^{-2i\eta} \quad (33)$$

One sees that eq. (33) reduces to the familiar expression for the one meson cut in the zero charge limit.

The multiplicative factor $\left|1-\frac{2k}{i\mu_\pi}\right|^{-2i\eta}$ is the Coulomb modification of this cut.

The Noyes approximation of this cut is to replace it by a pole at $k = i\frac{\mu_\pi}{\sqrt{2}}$ with the residue of eq. (20) multiplied with this factor at $k = i\frac{\mu_\pi}{\sqrt{2}}$. Therefore in eq. (26):

$$\gamma_C = 1 - \frac{M\alpha_F}{\mu_\pi}\sqrt{2} \quad (34)$$
$$\delta_C = |\sqrt{2}-1|$$

The OPE-cut is represented by a pole at the same position as in

the non-Coulomb case. However, the Coulomb modification factor is greater than 1 and increases rapidly when k approaches the branching point at $i\frac{\mu_\pi}{2}$. Therefore the position of the effective pole should perhaps be smaller than $i\frac{\mu_\pi}{\sqrt{2}}$; in this way the points closer to the branching point would obtain more weight.

This can be verified explicitly by constructing the [1,0] Padé in k^2 of the Born approximation eq. (31); expanding eq. (31) in k^2 (keeping η fixed), one obtains:

$$N_C^2 k^2 f_{\delta_N^C} \overset{k\to 0}{\underset{1\pi}{\simeq}} \frac{Mf^2}{\mu_\pi^2}\left(1+\frac{2k}{\mu_\pi i}\right)^{2i\eta}\left\{1-(1+i\eta)^2\frac{2k^2}{\mu_\pi^2}+\ldots\right\} \qquad (35a)$$

Neglecting powers of k^4, η^2 and higher one has.

$$N_C^2 k^2 f_{\delta_N^C} \overset{k\to 0}{\underset{1\pi}{\simeq}} \frac{\frac{Mf^2}{2}\left(1+\frac{4}{3}\frac{\eta k}{\mu_\pi}\right)}{k^2+\frac{\mu_\pi^2}{2}\left(1-\frac{8}{3}\frac{\eta k}{\mu_\pi}\right)} \qquad (35b)$$

This corresponds with:

$$\begin{aligned}\gamma_C &= \left(1-\frac{8}{3}\frac{\eta k}{\mu_\pi}\right)\\ \delta_C &= \left(1+\frac{4}{3}\frac{\eta k}{\mu_\pi}\right)\end{aligned} \qquad (35c)$$

Compared with the Wong-Noyes prescription eq. (34) this results into a pole which lies 7 % (1.4 MeV lab energy for NN) closer to the threshold and a residue which is 3 % lower. Although this is rather small, it could result into larger differences in the shape corrections. For $^1S_0$ np * a reduction of the pole position and residue with 7 % and 3 % respectively, would lead to a shape which is roughly 20 % larger in the low energy region compared with eq. (23).

---

*Only used as an example since the results can be calculated in a simple way.

At the end of this chapter we will anticipate some results of chapter VI. It appears that the approximations CFS (2) and CFS (3) are not accurate enough when applied to low energy proton-proton scattering (0 - 30 MeV) (in view of the very accurate data nowadays). This was already noted by Naisse [Na 77] who compared the CFS (2) shape prediction with the shape from a potential model.

In our opinion the replacement of the OPE cut by one simple pole is too crude, especially for energies close to threshold.

CHAPTER V

Approximations and numerical calculations of N_L, H_L and δ_L

1. Introduction

The effective range function P_L can only be used in a practical situation when numerical values can be obtained for N_L, H_L and δ_L.

However, only for a few cases ($V_L = 0$, V_L = Coulomb-potential, V_L = Borgmann potentials) these functions can be computed exactly in closed form. In the other cases we must calculate them numerically or use approximations.

The approximations can be divided into three classes:

1. Pole approximations;
2. Born approximations;
3. Approximations using as input a simplified analytical structure.

Sometimes combinations are possible.

In the Born approximations all functions are expanded in the coupling constant and only the lowest term is kept. Unitarity in the amplitude is violated and also the nearest singularities are not exact (additivety of the singularity structure is only valid the amplitude). Unitarity and the nearest singularities can be taken exactly into account by using approximations of type 3. One could calculate the phase shift δ_L using for instance the N/D method and after that N_L and H_L with formulas like eq. (3.50). As input one could take the discontinuity of the most nearby cuts. One believes that the behaviour of the amplitude at lower energies is influenced largely by the characteristics of the "nearby" cuts, while the effects of the more

"distant" cuts may be neglected [Lu 64].

Born approximations will work fine for weak potentials or larger energies.

We have considered pole approximations already in chapter IV: the singularities in the complex plane are approximated by one or more poles. Their advantage is that everything can be solved in closed form.

In the next sections we will consider only pole and Born approximations. Furthermore some aspects of (exact) numerical calculations in pp scattering with V_L = Coulomb + vacuum polarization + OPE potential will be considered.

2. Pole approximations

2.1. Without Coulomb interaction

Suppose the long range potential V_L contains no long range Coulomb interaction. We will restrict ourselves to the s-wave and replace the (dynamical) singularity structure of:

$$f_{\delta_L} = \frac{e^{2i\delta_L} - 1}{2ik} \qquad (1)$$

by a simple pole at $k = i\frac{\mu}{\sqrt{2}}$ with residue $\frac{f^2_M}{2}$ (cf. eq. (4.20)). Solving the N/D equations [Che 60] will give:

$$k \cot \delta_L = \frac{\mu^2}{2}\left(\frac{2}{f^2_M} - \frac{1}{\mu\sqrt{2}}\right) + \left(\frac{2}{f^2_M} + \frac{1}{\mu\sqrt{2}}\right) k^2 \qquad (2)$$

N_L and H_L can be obtained with the help of eqs (2.64) and (2.66), which is straightforward but rather tedious.

It is more convenient to use the analytic properties of $k_L^2 N_L^2 e^{-2i\delta_L}$ and $ik/N_L^2 k^2 + H_L$ (eqs (2.21) and (2.22)). From eq. (2) one obtains:

$$e^{2i\delta_L} = \frac{(k+\alpha)(k+\beta)}{(k-\alpha)(k-\beta)}$$

with: $\alpha = \imath \frac{\mu}{\sqrt{2}}$, the position of the dynamical singularity;

and: $\beta = \imath \frac{\mu}{\sqrt{2}} \frac{f^2 M\sqrt{2} - 4\mu}{f^2 M\sqrt{2} + 4\mu}$, the real or virtual bound state of V_L.

We require $k^2 N_L^2 e^{-2\imath\delta_L}$ to be analytic in the upper half plane with a (double) zero at the bound state position; moreover $k^2 N_L^2$ must be real and even in k. One sees easily that:

$$k^2 N_L^2 = \frac{k^2 - \beta^2}{k^2 - \alpha^2} \tag{4}$$

does the job (also: $\lim_{k\to\infty} k^2 N_L^2 = 1$).

Furthermore, we require that $(\imath k/k^2 N_L^2 + H_L)$ may not have singularities in the upper half plane except for the bound states, where it must have a simple pole. Furthermore, H_L must be real and even in k, and we will make use of the ambiguity in H_L by requiring $H_L(0) = 0$. One obtains*:

$$H_L(k^2) = \frac{\imath}{\beta} (\beta^2 - \alpha^2) \left(1 + \frac{\beta^2}{k^2 - \beta^2}\right) \tag{5}$$

Eqs (4) and (5) determine the form of the effective range function P_L, which is parametrized in order to give the phase shift δ_S^L.

---

*One must distinguish between two cases: virtual and real bound state. For a virtual bound state, $-\beta$ lies in the upper half plane and therefore the pole in $\imath k/N_L^2 k^2$ at this point must be cancelled by H_L.

For a real bound state, $-\beta$ lies in the lower half plane and gives no troubles. However $-\imath k/N_L^2 k^2 + H_L$ must be analytic in the lower half plane and gives the same result. It corresponds with the condition that the residue of H_L and $\imath k/N_L^2 k^2$ must be the same at a bound state pole.

Using the n-pole approximation of section IV.2, where the dynamical singularities of the short range potential V_S are also represented by poles, one obtains the following equations which must be used to obtain the <u>total</u> phase shift δ:

$$k \cot \delta_L = -\frac{1}{a_L} + \frac{1}{2} r_L k^2 \qquad \text{(eq. (2))}$$

$$\equiv Q_0^{[0,1]}(k^2)$$

$$\frac{1}{N_L^2 k} \cot \delta_S^L + H_L = Q_L^{[n-1,n]}(k^2) \qquad \text{(eq. (4.6))}$$

$$\delta = \delta_L + \delta_S^L \qquad (6)$$

The long range potential V_L is represented by a one pole approximation, the short range V_S by an n pole approximation. It can be expected therefore that the <u>total</u> potential $V = V_L + V_S$ is in fact approximated by (n+1) poles. Indeed, one proves:

$$k \cot \delta = Q_0^{[0,1]}(k^2) + \frac{1}{(\alpha+\beta)^2} \frac{(k^2 - \alpha^2)^2}{Q_L^{[n-1,n]} + Q_0^{[0,1]} + \text{const.}} \qquad (7)$$

with: $\text{const} = 1 \frac{\alpha}{\beta} (\alpha-\beta)$.

Therefore (cf. eq. (4.19)):

$$k \cot \delta = \frac{\text{polynomial in } k^2 \text{ of degree (n+1)}}{\text{polynomial in } k^2 \text{ of degree (n)}} \qquad (8)$$

For n = 1 the short range potential is represented by one pole and one can prove that eq. (7) is indeed equivalent with the CFS (2) approximation: eq. (4.23). For n = 2 eq. (7) leads to the CFS (3) approximation of eq. (4.24). However, eqs (6) and (7) are quite general and allow us to use other than <u>pole</u> approximations for the short range potential. For every parametrization of P_L, eq. (6) will guarantee that there is a pole, representing V_L, at the same place (according to eq. (4.20)) in the total partial wave amplitude:

$$f_\delta = \frac{e^{2i\delta} - 1}{2ik}$$

2.2 With Coulomb interaction

Suppose the long range interaction consists out of the Coulomb potential and a finite range potential (e.g. the OPE potential). In that case the expression:

$$N_C^2 k^2 \, f_{\delta_L^C} = N_C^2 k^2 \, \frac{e^{2i\delta_L^C} - 1}{2ik} \tag{9}$$

will have in the upper half momentum plane only dynamical singularities corresponding with the finite range potential (modified by the Coulomb interaction).

Using a one pole approximation for these singularities, one proves, following section IV.2, for $\ell = 0$:

$$N_C^2 k^2 \, f_{\delta_L^C} = \frac{1}{-\frac{1}{a_L^C} + \frac{1}{2} r_L^C k^2 - \frac{ik}{N_C^2 k^2} - H_C} \tag{10}$$

with a_L^C and r_L^C functions of the pole position and residue, which can be calculated explicitly. We define:

$$\begin{aligned} F^+(k) &= -\frac{1}{a_L^C} + \frac{1}{2} r_L^C k^2 - \frac{ik}{N_C^2 k^2} - H_C \\ F^-(k) &= -\frac{1}{a_L^C} + \frac{1}{2} r_L^C k^2 + \frac{ik}{N_C^2 k^2} - H_C \end{aligned} \tag{11}$$

Denoting the position of (pure) Coulomb bound states with β_C, the position of the dynamical singularity of eq. (9) by α and the bound state of V_L by β (assuming the Coulomb potential is repulsive)*, one

*Therefore, Im $\beta_C < 0$, Im $\alpha > 0$ and Im $\beta > 0$ for a real bound state; Im $\beta < 0$ for a virtual bound state.

proves (cf. eqs (3.33)):

F^+ and F^- are analytic in the complex momentum plane except for the cut along the negative axis, the essential singularity at the origin and except for poles at β_C (F^+) and $-\beta_C$ (F^-).

Furthermore F^+ will be zero in the upper half plane for $k = \alpha$ and possibly $k = \beta$. In the lower half plane F^- is zero at $k = -\alpha$ and $k = -\beta$.

According to section III.3 the functions N_L and H_L must be chosen such that:

$$k^2 N_L^2 e^{-2i(\delta_L + \sigma_0)} \tag{12}$$

is analytic in the upper half plane, with a (double) zero at $k = \beta$.

Also:

$$\frac{ik}{(N_L k)^2} + H_L \tag{13}$$

must be analytic in the upper half plane, except for a pole at $k = \beta$.

Choose:

$$k^2 N_L^2 = k^2 N_C^2 \frac{4F^+(k)\, F^-(k)}{(r_L^C)^2 (k^2 - \alpha^2)^2} \tag{14}$$

Now one obtains:

$$k^2 N_L^2 e^{-2i(\delta_L+\sigma_0)} = k^2 N_C^2 e^{-2i\sigma_0} \frac{4(F^+(k))^2}{(r_L^C)^2 (k^2-\alpha^2)^2} \tag{15}$$

The double zero of the denominator at $k = \alpha$ is cancelled by a corresponding one of the numerator. In the lower half plane the double zeros at $k = \beta_C$ (from $k^2 N_C^2 e^{-2i\sigma_0}$) are cancelled by the pole of F^+ at this position. The only zero in the upper half plane is at $k = \beta$, where F^+ vanishes.

Furthermore, one sees that $k^2 N_L^2$ is real on the positive real

axis and the symmetry property of eq. (3.49) can be verified with help of eqs (3.35). For $k \to \infty$ one obtains: $\lim_{k\to\infty} k^2 N_L^2 = 1$.

Of course, one can obtain eq. (14) from the integral expression eq. (3.50) using:

$$\frac{1}{N_C^2 k} \cot \delta_L^C + H_C = -\frac{1}{a_L^C} + \frac{1}{2} r_L^C k^2 \tag{16}$$

The derivation of H_L, using the properties of eq. (13), is similar. After manipulating the several expressions one finds that:

$$H_L = \frac{1}{8} (r_L^C)^2 (k^2-\alpha^2)^2 \frac{F^+ + F^-}{F^+ F^-} + \frac{1}{a_L^C} - \frac{1}{2} r_L^C k^2 \tag{17}$$

gives:

$$\frac{ik}{k^2 N_L^2} + H_L = \frac{1}{4} (r_L^C)^2 \frac{(k^2-\alpha^2)^2}{F^+} + \frac{1}{a_L^C} - \frac{1}{2} r_L^C k^2 \tag{18}$$

and has the correct properties.

The term: $-\frac{1}{a_L^C} + \frac{1}{2} r_L^C k^2$, analytic in the whole momentum plane, is subtracted in order to obtain the correct asymptotic limit:

$$\lim_{k\to\infty} \left(\frac{ik}{k^2 N_L^2} + H_L \right) \to ik + \text{const.}$$

The function H_L is real for physical k and it is convenient to gauge it such that: $H(0) = 0$. One obtains:

$$H_L(k^2) = \frac{(r_L^C)^2}{8} (k^2-\alpha^2)^2 \frac{F^+ + F^-}{F^+ F^-} - \frac{1}{2} r_L^C k^2 + \frac{a_L^C (r_L^C)^2}{4} \alpha^4 \tag{19}$$

The effective range function P_L is now determined. Using the n-pole approximation for the short range potential V_S again, the <u>total</u> phase shift δ_N^C of $V = V_L + V_S$ with respect to Coulomb functions is parametrized according to:

$$\frac{1}{kN_C^2} \cot \delta_L^C + H_C = -\frac{1}{a_L^C} + \frac{1}{2} r_L^C k^2 \qquad \text{(eq. (16))}$$

$$\equiv Q_C^{[0,1]}(k^2)$$

$$\frac{1}{kN_L^2}\cot\delta_S^L + H_L = Q_L^{[n-1,n]}(k^2)$$

$$\delta_N^C = \delta_L^C + \delta_S^L \tag{20}$$

It is not trivial to prove that eqs (20) are equivalent with:

$$\frac{1}{kN_C^2}\cot\delta_N^C + H_C = Q_C^{[0,1]}(k^2) - \frac{(r_L^C)^2(k^2-\alpha^2)^2}{4(Q_L^{[n-1,n]} + Q_C^{[0,1]} + \text{const.})} \tag{21}$$

with:

$$\text{const.} = -\frac{(r_L^C)^2\, a_L^C\, \alpha^4}{4} + \frac{1}{a_L^C}$$

For n = 1, eq. (21) is equivalent with the Coulomb corrected CFS (2) approximation (the position and residue of the V_L pole must be chosen according to eqs (4.26) and (4.34)). Taking n = 2 gives the Coulomb corrected CFS (3) approximation.

3. Born approximations

In this section we will consider Born approximations for N_L, H_L, δ_L for the long range potential:

$$V_L(r) = -Mf^2\frac{e^{-\mu r}}{r} \tag{22}$$

We will treat only the $\ell = 0$ case; generalizations for higher partial waves and for a V_L which contains also a Coulomb interaction, will be left to the reader and are straightforward.

The first Born approximation we will consider, is derived by expanding formulas (2.17) in f^2, keeping only the lowest orders; using: $v_R(r) \simeq \bar{j}_0(kr) = \frac{\sin kr}{k}$ and $v_I(r) \simeq \bar{n}_0 = \cos kr$, and replacing ε by 0 since everything should be independent on it, one obtains:

$$\delta_L = -\frac{1}{k}\int_0^\infty dr'\, V_L(r')\, \sin^2(kr')$$

$$N_L k = 1 + \frac{1}{k}\int_0^\infty dr'\, V_L(r')\, \sin kr' \cos kr' \qquad (23)$$

$$H_L = \int_0^\infty dr'\, (\sin^2 kr' - \cos^2 kr')\, V_L(r')$$

However, as it stands H_L is only properly defined if $V_L(0)$ = finite (caused by replacing ε by 0 too hasty). One could add a cut-off factor to the definition in eq. (22). However, one obtains also a finite expression for H_L if one subtracts the value at $k^2 = 0$:

$$H_L(k^2) - H_L(0) = \int_0^\infty dr'\, (\sin^2 kr' - \cos^2 kr' + 1)\, V_L(r')$$

$$= 2\int_0^\infty dr'\, \sin^2 kr'\, V_L(r') \qquad (24)$$

The integrals can be solved explicitly, giving:

$$\delta_L(k) = \frac{Mf^2}{4k}\ln\left(1 + \frac{4k^2}{\mu^2}\right)$$

$$kN_L(k) = 1 - \frac{Mf^2}{2k}\arctan\left(\frac{2k}{\mu}\right) \qquad (25)$$

$$H_L(k^2) - H_L(0) = -\frac{Mf^2}{2}\ln\left(1 + \frac{4k^2}{\mu^2}\right)$$

We will call the approximations of eq. (25), the Born I approximations.

The Born II approximations, which will appear to be slightly better than Born I in practice, start with the same approximation for the phase shift δ_L, however, used in combinations with eqs (2.64) and (2.66) in order to calculate N_L and H_L.

N_L can be computed exactly, H_L is expanded in f^2; neglecting terms of order f^6 and higher one obtains:

$$\delta_L(k) = \frac{Mf^2}{4k}\ln\left(1 + \frac{4k^2}{\mu^2}\right) \qquad \text{(by definition)}$$

$$kN_L(k) = e^{-(f^2M/2k)\arctan(2k/\mu)} \qquad (26)$$

$$H_L(k^2) - H_L(0) = -\frac{Mf^2}{2}\left(1 - \frac{f^2 M\mu}{k^2}\right) \ln\left(1 + \frac{4k^2}{\mu^2}\right) - \frac{2f^4M^2}{\mu} + O(f^6)$$

One could consider the Born II approximations, eq. (26), as a consistent set of functions N_L, H_L, δ_L which are <u>exact</u> for a long range potential V_L which would give the phase shift:

$$\delta_L = \frac{Mf^2}{4k} \ln\left(1 + \frac{4k^2}{\mu^2}\right)$$

Therefore, the Born II approximation represents in a certain sense a real "physical" situation (which is an approximation of eq. (22)), while the Born I can be considered as an "approximation of an approximation".

In section 5 we will compare the pole and the Born approximations with the functions N_L, H_L and δ_L calculated exactly with the help of numerical methods. The numerical aspects will be treated in the next section.

4. Numerical calculation of the effective range functions in proton-proton scattering

In this section we will consider the numerical procedures for the calculation of the "OPE-Coulomb-vacuum polarization" effective range function in the S-wave ($^1S_0$).

As long range potential V_L we take:

$$V_L(r) = -f^2 M \frac{e^{-\mu r}}{r} + \frac{M\alpha_F}{r} + M\, V_{vac}(r) \qquad (27)$$

with M the proton mass, μ the neutral pion mass, $f^2 = \frac{g^2}{4\pi}\frac{\mu^2}{4M^2}$, g^2 the π^0pp coupling constant squared, and V_{vac} the vacuum polarization potential [Du 57]. In principle the calculation of N_L, H_L and δ_L is simple and is based on their definitions in section III.3:

(i) Solve the radial Schrödinger equation:

$$v'' + [k^2 - V_L(r)]\, v = 0 \quad ,$$

twice, with boundary conditions:

$$v_R(r) \underset{r\to 0}{\longrightarrow} r + O(r^2)$$

$$v_I(r) \underset{r\to 0}{\longrightarrow} 1 + O(r)$$

(ii) Using a suitable method, solve the differential equation up to a certain point where the OPE and vacuum polarization part can be neglected with respect to the Coulomb potential.

(iii) Calculate δ_L^C, N_L^C and H_L^C using the definitions of eq. (3.42) by matching the calculated wave functions to Coulomb wave functions.

To solve the differential equation we use the Numerov method [Nu 33] (sometimes called the Cowell method) which solves the second order differential equation:

$$v'' = Av \tag{28}$$

with the relations:

$$\begin{aligned} \bar{v}(r) &= \left(1 - \frac{h^2}{12} A(r)\right) v(r) \\ \bar{v}(r+h) &= \frac{2 + \frac{5}{6} h^2 A(r)}{1 - \frac{h^2}{12} A(r)} \bar{v}(r) - \bar{v}(r-h) + O\left(\frac{h^6 v^{(vi)}(r)}{240}\right) \end{aligned} \tag{29}$$

This method gives rather accurate phase shifts, however, the accuracy on N_L and H_L is much less when the potential has a singular point at r = 0*. Therefore we solve the differential equation from

---

*see next page.

r = 0 up to a certain point with a series expansion and use the Numerov method as soon as the bad influence of the singular point r = 0 on the accuracy can be neglected.

The potential eq. (27) is expanded in r:

$$k^2 - V_L(r) = a + \frac{b}{r} + c\,\frac{\ln r}{r} + d\,r + O(r^2) \ldots$$

with:

$$a = k^2 + Mf^2\mu - \frac{\alpha_F^2 M}{2}\,m$$

$$b = -M\alpha_F - Mf^2 + \frac{2\alpha_F^2 M}{3\pi}\left(\gamma + \frac{5}{6} + \ln m\right)$$

$$c = \frac{2\alpha_F^2 M}{3\pi}$$

$$d = -\frac{Mf^2\mu^2}{2} + \frac{2\alpha_F^2 M}{2\pi}\,m^2 \qquad (30)$$

with γ Eulers constant, $\gamma = 0.5772\ldots$, and m the electron mass. We refer to Durand [Du 57] for details about the vacuumpolarization potential.

The solution of the radial Schrödinger equation in the neighbourhood of r = 0 can be written as:

$$v(r) = \sum_{n,m} \alpha_{nm}\, r^n \ln^m r \qquad (31)$$

and one obtains recurrence relations for the coefficients α_{nm} in the

---

*This can be seen in the following way: take for instance a potential which is very repulsive near the origin; in that case the regular solution will decrease exponentially to zero for $r \to 0$. This means that the wave function is almost zero in that region (compared with the value for $r \to \infty$). The phase shift is more sensitive for the radius of the repulsion than the actual value of the potential; this is however not the case for N_L and H_L. Try it with a repulsive square well.

usual way.

The result for the regular solution v_R, with boundary conditions: $v_R(0) = 0$, $v_R'(0) = 1$ appears to be:

$$v_R(r) = r - \frac{1}{2} br^2 - \frac{c}{2} r^2 (\ln r - \frac{3}{2})$$
$$+ (-\frac{1}{6} a + \frac{1}{12} b^2 - \frac{19}{72} bc + \frac{83}{432} c^2) r^3$$
$$+ (\frac{bc}{6} - \frac{19}{72} c^2) r^3 \ln r + \frac{1}{12} c^2 r^3 \ln^2 r + O(r^4) \qquad (32)$$

The irregular solution v_I has boundary condition $v_I(0) = 1$, and furthermore we require that the amount of <u>regular</u> solution contained in v_I (i.e. terms proportional with $r + \ldots$) is zero.

We get:

$$v_I(r) = 1 + (c-b) r \ln r - \frac{c}{2} r \ln^2 r + (-\frac{1}{2} a + \frac{27}{8} bc - \frac{3}{4} b^2 - \frac{73}{16} c^2)$$
$$\times r^2 + (-\frac{11}{4} bc + \frac{1}{2} b^2 + \frac{33}{8} c^2) r^2 \ln r + (\frac{3}{4} bc - \frac{13}{8} c^2) r^2$$
$$\times \ln^2 r + \frac{c^2}{4} r^2 \ln^3 r + (-\frac{1}{18} ab - \frac{137}{108} b^2 c + \frac{7}{36} b^3 + \frac{1241}{432} bc^2$$
$$- \frac{1}{6} d + \frac{17}{108} ac - \frac{5213}{2592} c^3) r^3 + (-\frac{2}{9} ac - \frac{431}{144} bc^2 + \frac{67}{72} b^2 c$$
$$- \frac{1}{12} b^3 + \frac{331}{108} c^3 + \frac{1}{6} ab) r^3 \ln r + (\frac{55}{48} bc^2 - \frac{5}{24} b^2 c$$
$$- \frac{469}{288} c^3 + \frac{ac}{12}) r^3 \ln^2 r + (-\frac{1}{6} bc^2 + \frac{59}{144} c^3) r^3 \ln^3 r$$
$$- \frac{c^3}{24} r^3 \ln^4 r + O(r^4) \qquad (33)$$

In practice the series expansions are used from $r = 0$ up to $r = 0.005\ \lambda_\pi$ ($\lambda_\pi \simeq 1.4$ fm); then the Numerov method is used with initial step length: $h = 0.001\ \lambda_\pi$. The step size is doubled at 0.2, 0.6, 1.4, 3.0 and 6.2 λ_π and we integrate out up to 15 fm, where the pion potential can be neglected. This takes about 1000 steps and an

accuracy of more than 6 digits is reached in the interval $0 < E_{lab} <$ 300 MeV. For $r > 15$ fm the vacuum polarization potential still exists with a range of 200 fm, which means we still have more than 1000 fm to go before it can be neglected. Instead of using the Numerov method to perform this integration (which would take something like 10000 steps more), we corrected the results obtained at 15 fm in another way. We quote some results from the variable phase approach to potential scattering [Ca 67] modified for the long range Coulomb potential:

The solution of the radial Schrodinger equation can be written as:

$$\begin{aligned} V(r) &= \alpha(r)\ (\cos\delta(r)\ F(r) + \sin\delta(r)\ G(r)) \\ V'(r) &= k\alpha(r)\ (\cos\delta(r)\ F'(r) + \sin\delta(r)\ G'(r)) \end{aligned} \tag{34}$$

(in fact this is the definition of $\alpha(r)$ and $\delta(r)$). $F(r)$ and $G(r)$ are the standard Coulomb regular and irregular (s-wave) wave functions. One proves*:

$$\begin{aligned} \delta'(r) &= -\frac{1}{k} V_L(r)\ \{\cos\delta(r)\ F(r) + \sin\delta(r)\ G(r)\}^2 \\ \alpha'(r) &= -\frac{1}{k} \alpha(r)\ V_L(r)\ \{\cos\delta(r)\ F(r) + \sin\delta(r)\ G(r)\} \\ &\qquad \times \{\sin\delta(r)\ F(r) - \cos\delta(r)\ G(r)\} \end{aligned} \tag{35}$$

Note that for the <u>regular</u> solution:

$$\begin{aligned} \lim_{r\to\infty} \delta(r) &= \delta_L^C \\ \lim_{r\to\infty} \alpha(r) &= N_L \end{aligned} \tag{36}$$

The regular solution is solved with the help of the Numerov method up

---

*See note on page 45 : V_L = long range potential eq. (27) - Coulomb part.

to 15 fm; therefore $\delta(15)$ and $\alpha(15)$ are known. The extrapolation to infinity is made by integrating eqs (35):

$$\delta(\infty) = \delta(15) - \frac{1}{k} \int_{15}^{\infty} dr\, V_L(r)\, (\cos\delta(r)\, F(r) + \sin\delta(r)\, G(r))^2$$

$$\alpha(\infty) = \alpha(15)\, \exp\{-\frac{1}{k} \int_{15}^{\infty} dr\, V_L(r)\, (\cos\delta(r)\, F(r) + \sin\delta(r)\, G(r)) \times (\sin\delta(r)\, F(r) - \cos\delta(r)\, G(r))\} \tag{37}$$

The functions $\delta(r)$ and $\alpha(r)$, which occur in the integrands, are, however, unknown. We will approximate them by a constant: the value at 15 fm. This must be a good approximation since V_L between 15 fm and ∞ is just the vacuum polarization which is rather small, and so $\delta(r)$ and $\alpha(r)$ will remain close the values they had at 15 fm. In fact this procedure is a kind of first Born approximation for the vacuum polarization in the region: $15 < r < \infty$. (Effective range functions corrected for vacuum polarization only, have been calculated in the past always in Born approximation: [He 60].)

One obtains:

$$\delta_L^C = \bar{\delta} - \frac{1}{k}\cos^2\bar{\delta}\, FVF - \frac{2}{k}\sin\bar{\delta}\cos\bar{\delta}\, FVG - \frac{1}{k}\sin^2\bar{\delta}\, GVG$$

$$N_L = \bar{N}\exp\{-\frac{1}{k}\sin\bar{\delta}\cos\bar{\delta}\,(FVF - GVG) - \frac{1}{k}(\sin^2\bar{\delta} - \cos^2\bar{\delta})FVG\} \tag{38}$$

with: $\bar{\delta} = \delta(15)$, $\bar{N} = \alpha(15)$

and:
$$\begin{pmatrix} FVF \\ FVG \\ GVG \end{pmatrix} \equiv \int_{15}^{\infty} dr\, V_{vac}(r) \begin{pmatrix} F^2(r) \\ F(r)\, G(r) \\ G^2(r) \end{pmatrix}$$

These integrals are calculated once as a function of the energy. The corrections can then be made for every long range potential (containing vacuum polarization) without solving the equations every time up to 1000 fm. For the irregular solution v_I, defining H_L, a similar procedure

is applied.

For v_I it is convenient to define:

$$\beta(r) = \cot \delta(r) \tag{39}$$

with $\delta(r)$ defined in eq. (34) ($\delta(r)$ is not the same function as in the regular case!). One proves:

$$\beta'(r) = \frac{1}{k} V_L(r) \; (\beta(r) \; F(r) + G(r))^2 \tag{40}$$

and:

$$\lim_{r\to\infty} \beta(r) = \frac{- N_L^2 k \; H_L + \tan \delta_L^C}{1 - N_L^2 k \; H_L \tan \delta_L^C}$$

The value of $\beta(15)$ can be calculated from the values of v_I and v_I' at this point. One extrapolates to ∞ with:

$$\beta(\infty) = \bar{\beta} + \frac{1}{k} \int_{15}^{\infty} V_L(r) \; (\beta(r)F(r) + G(r))^2$$

$$\simeq \bar{\beta} + \frac{1}{k} \bar{\beta}^2 \; FVF + \frac{2}{k} \bar{\beta} \; FVG + \frac{1}{k} GVG \tag{41}$$

where $\bar{\beta} = \beta(15)$.

Finally we mention two trivial checks which are made after a calculation of N_L, H_L and δ_L and which give an indication of the numerical error.

The first one takes place at 15 fm and checks whether the Wronskian relation: $v_R' \; v_I - v_R \; v_I' = 1$ is still satisfied. Deviations give an indication of the error caused by the series expansion and the Numerov method.

The second check gives an indication of the error caused by using the above-mentioned extrapolation technique from 15 fm to ∞. After the calculation of δ_L^C, N_L and $\beta(\infty)$, we use <u>these</u> functions instead of $\bar{\delta}$, $\bar{N}$ and $\bar{\beta}$ in the right hand side of eqs (38) and (41), and calculate

δ_L^C, N_L and H_L again. The difference between the new and old values gives an impression of the neglected higher Born terms. For energies between 0.3 and 0.5 MeV an accuracy of five digits is reached, above 0.5 MeV more than six.

One more problem are calculations at zero energy which are necessary when one tries to gauge H_L with the condition: $H_L(0) = 0$. For $k = 0$ the wave functions F and G are not well-defined and also N_L will diverge (repulsive Coulomb). Although one can get around these singularities we will avoid it, since the gauging at zero energies is not essential. We choose energy 0.35003 MeV as subtraction point; this influences only the actual value of a_S^L, however, not the physical observable total phase shift: $\delta_N^C = \delta_S^L + \delta_L^C$.

5. Approximations versus exact calculations

In this section we will compare the pole- and Born-approximations with each other and with the results obtained by solving the radial Schrodinger equation exactly. We will neglect Coulomb- and vacuum-polarization since they only make calculations more complicated than necessary for our purpose: a qualitative insight in the performance of the several methods.

We use the long range potential:

$$V_L(r) = - Mf^2 \frac{e^{-\mu r}}{r} \qquad (42)$$

to represent the OPE interaction (for instance in the $^1S_0$ np channel).

In figures V.1 and V.2 we have plotted the functions: $1/N_L^2k^2$, H_L and δ_L, in the one-pole (CFS) approximation, the Born I and II approximation, together with the Schrodinger (S) solution as a function

of k^2 ($k^2 = 1 \Leftrightarrow E_{lab} \simeq 40$ MeV). Figure V.1 contains the result for the "physical" coupling constant $f^2 = 0.08$, figure V.2 for $f^2 = 0.01$.

The results speak for themselves, one could conclude that $f^2 = 0.08$ is certainly too big to get reliable results from the Born approximations. For $f^2 = 0.01$ the Born approximations (in particular Born II) correspond within a few per cent with the Schrodinger solution. Always the Born II approximation appears to be the best, while mostly CFS is the worst approximation of the exact solution.

It would therefore be obvious to conclude that it is better to use the Born approximations than the CFS formulas. This is, however, not always true.

One must realize that the long range potential V_L (and therefore N_L, H_L, δ_L) is not directly observable, only the total phase shift: $\delta_N = \delta_L + \delta_S^L$.

The "short range" phase shift δ_S^L is parametrized in the one pole approximation of P_L as:

$$\frac{1}{N_L^2 k} \cot \delta_S^L + H_L = -\frac{1}{a_S^L} + \frac{1}{2} r_S^L k^2 \qquad (43)$$

In practice, however, a_S^L and r_S^L are fixed by the total phase shift $\delta_N = \delta_L + \delta_S^L$ via its effective range relation:

$$k \cot \delta_N = -\frac{1}{a_N} + \frac{1}{2} r_N k^2 + \ldots \qquad (44)$$

a_N and r_N, the total scattering length and effective range, are experimentally known and therefore a_S^L, r_S^L must be refitted for every approximation of the long range potential V_L in order to give the same a_N and r_N. Once this is done eq. (43) will predict only the deviation of $k \cot \delta_N$ from the shape independent approximation: $-\frac{1}{a_N} + \frac{1}{2} r_N k^2$.

For this purpose we define the shape function p

$$p(k^2) = \frac{k \cot \delta_N + \frac{1}{a_N} - \frac{1}{2} r_N k^2}{k^4} \qquad (45)$$

for $a_N = -10\ \lambda_\pi$, $r_N = 2\ \lambda_\pi$ ($\lambda_\pi \simeq 1.4$ fm) and $f^2 = 0.08$ the shape functions for the different approximations are given in figure V.3a.

Now the Born approximations appear to be totally wrong, giving even the wrong sign for the shape function!! In contrast with this, the CFS approximation has only a deviation of 20 % compared with the Schrodinger results. For a smaller coupling constant, $f^2 = 0.01$, the Born approximations improve, which is shown in figure V.3b. Note that the CFS approximation now deviates 30 % from the Schrodinger solution (however, the absolute value of P has decreased). These unexpected good results of the one pole (CFS) approximation are partially explained by figure V.4. We have plotted the ratio of the CFS-, Born-approximations and the Schrödinger functions. It appears that these ratios for the CFS approximations are about the same and almost a constant as a function of the energy (0.6 - 0.5). Therefore multiplying eq. (43), calculated for the CFS approximation, with a factor 1/0.6 (both left- and righthand side) will result into an equation which is in fact the corresponding Schrodinger effective range identity (within 15 %).

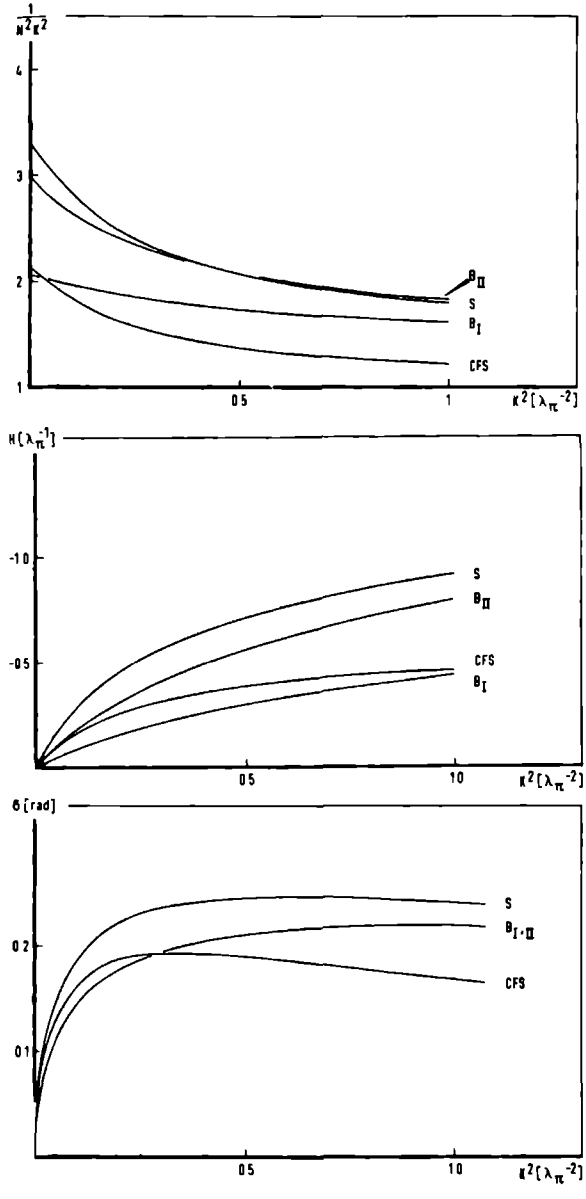


Figure V.1: The functions $1/N_L^2k^2$, H_L and δ_L calculated with the help of the Schrödinger equation (S), Born (BI, BII) and one pole (CFS) approximations; coupling constant $f^2 = 0.08$.

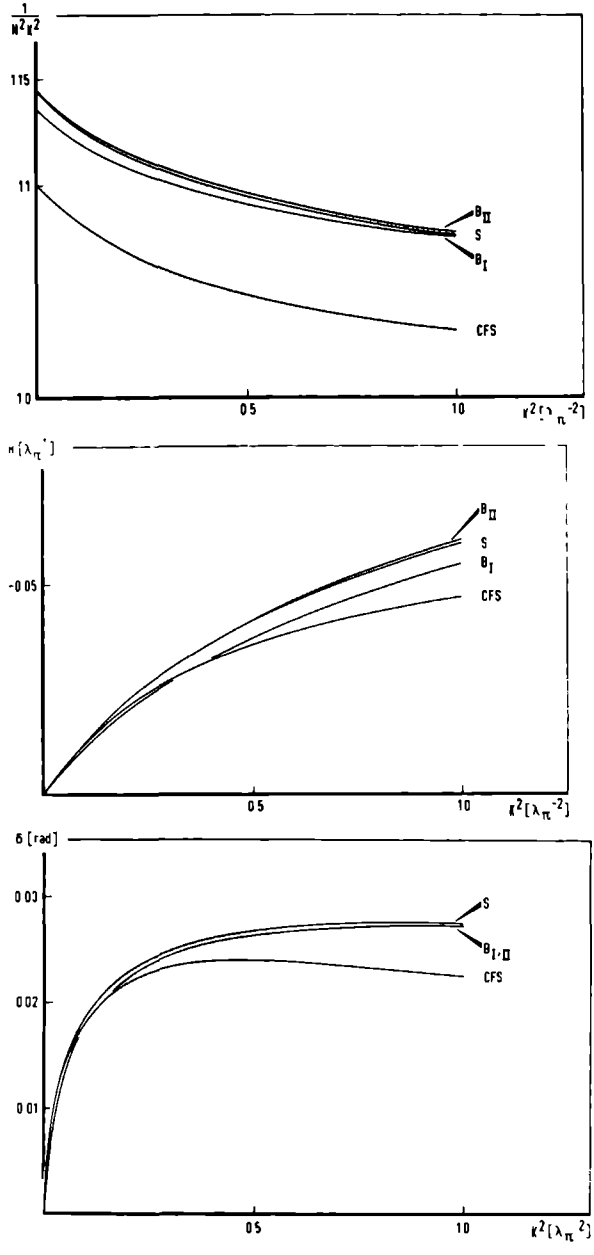


Figure V.2: The same as figure V.1; coupling constant $f^2 = 0.01$.

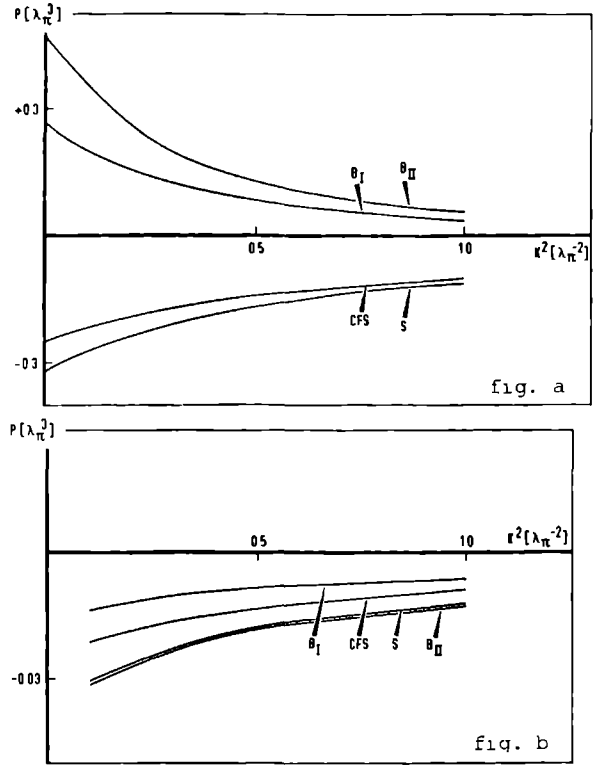


Figure V.3: Shape functions for k cot δ_l obtained from a one pole approximation of the short range potential; notation as in fig. V.1. a: $\gamma^2 = 0.08$; b: $\gamma^2 = 0.01$.

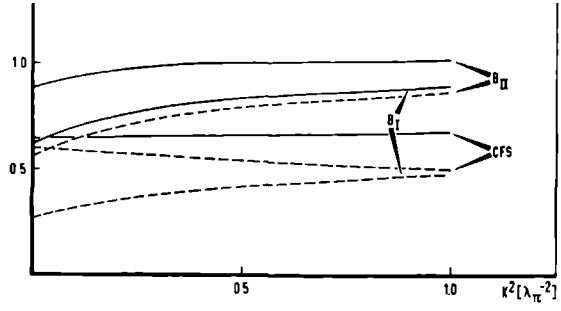


Figure V.4: Ratio for $1/N_L^2 k^2$ (continuous curve) and d_L (dashed curve) of the different approximations and the Schrödinger solutions.

CHAPTER VI

Application to the proton-proton system: determination of the π^0pp coupling constant from low energy scattering data

1. General characteristics of the NN system in the $^1S_0$ partial wave

To study the behaviour of the effective range function in the $^1S_0$ nucleon-nucleon partial wave, we will leave out the Coulomb- and vacuumpolarization potential. Using Coulomb- and vacuumpolarization corrected effective range functions will give the same general behaviour differing only in minor points (e.g. the scattering length which is - 7.8 fm instead of - 17 or - 24 fm).

As discussed already in section 4.1 the s-wave nuclear interaction can be divided into three parts:

1. A long range attraction with a range of 1.4 fm due to pion exchange.
2. A medium and short range interaction (overall attractive) with a range of 0.5 - 0.7 fm caused by the exchange of heavier mesons (ρ, ω, ε, 2π and so on).
3. A short range repulsion not due to the exchange of simple mesons and often represented by a phenomenological hard or soft core or by a more physical Pomeron "exchange" potential.

These three features can be recognized in the behaviour of the effective range function k cot δ_N as a function of the energy. In figure VI.1 we have used the Nijmegen OBE potential [Na 78]* to calculate the

*The pomeron coupling constant and pion coupling constant are slightly changed to reproduce a $^1S_0$ np scattering length of - 23.7 fm.

effective range function from E_{lab} = 0 to 250 MeV. Other models would give plots with the same characteristics.

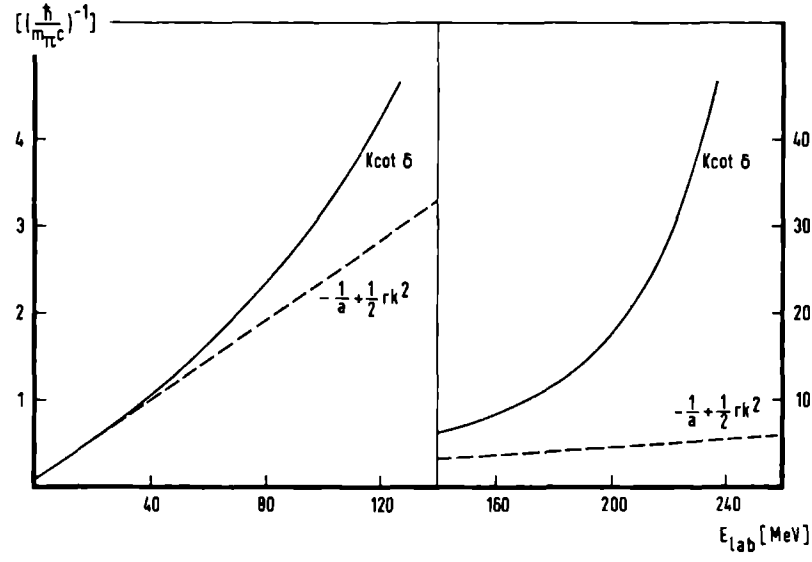


Figure VI.1: Effective range function k cot δ_N and shape independent approximation in the $^1S_0$ np wave.

These characteristics are: the effective range function is positive from 0 MeV up to about 250 MeV; at this point it has a pole and it becomes negative for higher energies. The pole in this region is caused by the zero of the phase shift near 250 MeV and is a consequence of the repulsive core which starts dominating at higher energies. At lower energies the total potential is overall attractive. However, the influence of the repulsive part is felt also at lower energies. This is seen in figure VI.2 where we have plotted the shape function:

$$P_{OBE}(k^2) = \frac{k \cot \delta_N + \frac{1}{a_N} - \frac{1}{2} r_N k^2}{k^4} \qquad (1)$$

Note that the shape independent approximation of the effective range function: $p(k^2) = 0$, corresponds with the replacement of the complicated left hand singularity structure of the NN interactions by only one pole (which is attractive). It will be clear that this is a poor way to represent the three-fold nature of the NN interaction.

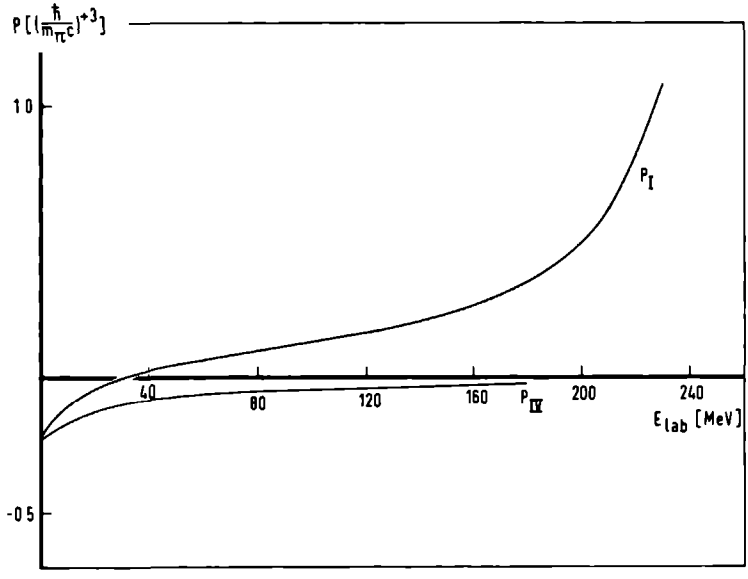


Figure VI.2: Shape function of the OBE model (P_I) and the CFS (2) approximation (P_{IV}).

In fact the shape function of the NN interaction is negative below 30 MeV, becomes positive then and has the above-mentioned pole at 250 MeV, caused by the repulsion. Furthermore we note that the approximations in which the medium- and short range interaction are replaced by just one pole (e.g. the CFS (2) approximation with $V_L = 0$ and $V_S \equiv$ 2 poles*, or the approximation $V_L = -f^2 M \frac{e^{-\mu r}}{r}$ and $V_S \equiv$ 1 pole;

*equivalent with $V_L \equiv$ 1 pole, $V_S \equiv$ 1 pole.

see also figure V.3), will give a negative shape for every energy.

For the CFS (2) approximation this is observed in eq. (4.23):

$$p_{CFS(2)}(k^2) = - \frac{c\ \mu_\pi^{-3}}{1 + d\ \mu_\pi^{-2}\ k^4} \qquad (2)$$

$(c,d > 0)$

It appears that the pole representing the short range interaction must be "attractive" to produce the same a_N and r_N as the NN interaction. Only when the medium-range attraction and the short-range repulsion are each represented by <u>one pole</u> the "cross-over" at 30 MeV can be explained.

The CFS (3) approximation gives (eq. (4.24)):

$$p_{CFS(3)}(k^2) = - \frac{p(1+tk^2)}{(1+qk^2)(1+sk^2)} \qquad (3)$$

with: $p > 0$, $q > 0$, $t,s < 0$.

The cross-over is at $k^2 = - \frac{1}{t}$, the pole position at $k^2 = - \frac{1}{s}$.

In the CFS approximation the OPE interaction is approximated by one pole. However, we could also represent it by a Yukawa potential:

$$V_L = - Mf^2 \frac{e^{-\mu r}}{r}$$

and calculate its effective range function exactly. Approximating V_S by two effective poles results into a shape function $p_{Y(2)}$ defined by:

$$P_L(k^2) = - \frac{1}{a_S^L} + \frac{1}{2} r_S^L k^2 - \frac{p_S^L k^4}{1+q_S^L k^2}$$

$$\delta_N = \delta_L + \delta_S^L \qquad (4)$$

$$p_{Y(2)}(k^2) = \frac{k \cot \delta_N + \frac{1}{a_N} - \frac{1}{2} r_N k^2}{k^4}$$

with a_S^L, r_S^L functions of a_N and r_N. Choosing $p_S^L < 0$ (not too big) and $q_S^L < 0$, will also result into a cross-over and pole for $p_{Y(2)}(k^2)$, as we will see in the next section.

In figure VI.3 we have plotted P_{OBE} again, on an enlarged scale for the energy region 0 - 30 MeV. The shaded area and the dotted curve give an indication of the magnitude of the errors on the experimental data in terms of the shape function. Note that these errors correspond with pp scattering single energy analysis and are shifted to the horizontal axis (p = 0) for a clearer view.

We arrive at the important conclusion:

Already at energies below 30 MeV one must account for the three-fold nature of the NN interaction in the parametrization of the phase shift δ_V, in proton-proton scattering.

In the next section we will compare the two possible candidates: CFS (3) and "Yukawa + 2 poles" with the shape predicted by the OBE model.

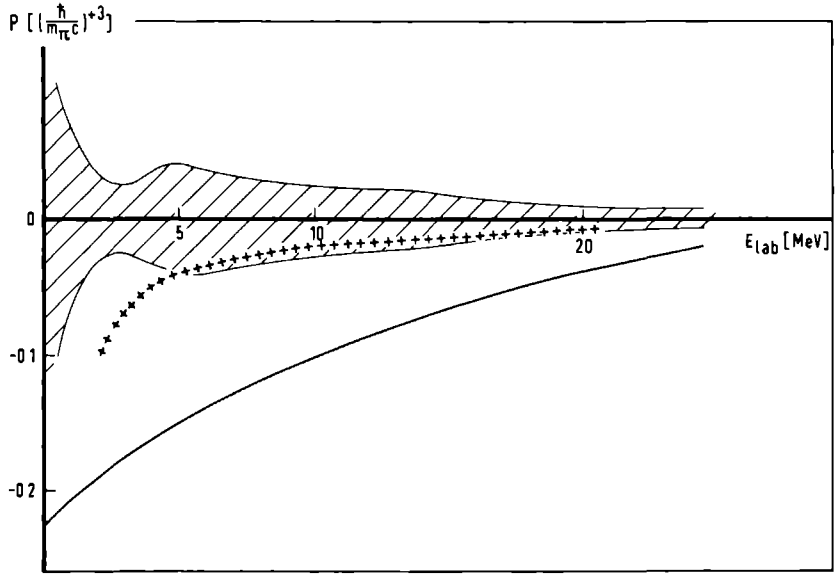


Figure VI.3: OBE potential shape (P_I in fig. VI.2) for the energy region 0 - 30 MeV. Shaded area gives indication of the experimental error due to the error in the phase shift, dotted curve variation caused by an error of 0.015 fm in the effective range r_N. Both are shifted to the p = 0 axis.

2. Determination of the pion coupling constant from s-wave phase shifts; test with the help of a potential model

In section 1 we saw that only the CFS three pole or the "Yukawa effective range function with 2 poles" are possible candidates for a suitable parametrization of the NN phase shift in an energy region from 0 - 300 MeV.

These parametrizations each depend on 6 independent parameters, for which we will take:

a_N: scattering length

r_N: effective range

$\frac{g^2}{4\pi}, f^2$: pion coupling constant

μ: pion mass

E_{co}: the cross-over point, i.e. the energy where the shape function = 0 ($E_{co} \approx 30$ MeV)

E_{po}: the pole position, i.e. the energy where the phase shift passes zero ($E_{po} \approx 250$ MeV).

Only the CFS (3) approximation can be expressed in closed form in these parameters (eq. (3)). For the "Yukawa + 2 poles" approximation (short hand: Y(2)) these numbers must be calculated numerically using eqs (4) and the method described in section V.4.

From these 6 parameters, the pion mass is known from other sources. Also the pion-nucleon coupling constant can be considered as known from other experiments. However, one must realize that in pp scattering it is the π^0 which causes the nearest branch cut, while in the other experiments (mainly πN scattering or threshold pion photoproduction) it is the charged pion coupling constant which is determined. Therefore pp scattering is almost the only source from which we

can determine the π^0 coupling constant and we will consider it as a free parameter.

From the other four parameters E_{po} is only of importance when one analyses the 0 - 300 MeV region. For analyses in the 0 - 40 MeV energy band it is sufficient to fix E_{po} at 250 MeV (or ∞) and consider only a_N, r_N, f^2 and E_{co} as being free; in this way one includes already in the low energy analysis some information from scattering at higher energies.

Before turning to the actual proton-proton data in section 3, we will first test the different methods (CFS (3), Y(2)) in a non-trivial way with the help of the Nijmegen OBE potential model [Na 78]. It is important to realize that in the ideal case that CFS (3) and Y(2) were exact (i.e. no approximations in the theory; in our case the poles); they should reproduce the potential model predictions (whether the OBE model represents physics, is another question).

In figure VI.4 we have plotted, for the 0 - 30 MeV region, the shape functions defined in eqs (1), (2), (3) and (4), each calculated with the same parameters:

$a_N = -23.7$ fm , $r_N = 2.8$ fm , $\frac{g^2}{4\pi} = 14.213$, $\mu = 138$ MeV ,

$E_{co} \approx 30$ MeV , $E_{po} \approx 250$ MeV (roughly)*

---

*This is of course not true for $p_{CFS(2)}$, eq. (2) which cannot produce a cross-over and pole; it is fixed by a_N, r_N, f^2 and μ only (eq. 4.23). For technical reasons we did not use E_{co} and E_{po}, but points at 35 and 160 MeV to fix the parametrization in CFS (3) and Y(2).

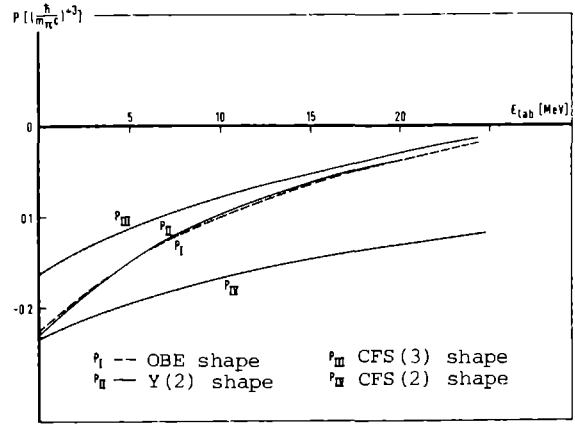


Fig. VI.4: The OBE shape (P_I) with the predictions of: - the two- and three-pole CFS approximations (P_{III}, P_{IV}); - the Yukawa potential for the pion with two poles for the short range interaction (P_{II}).

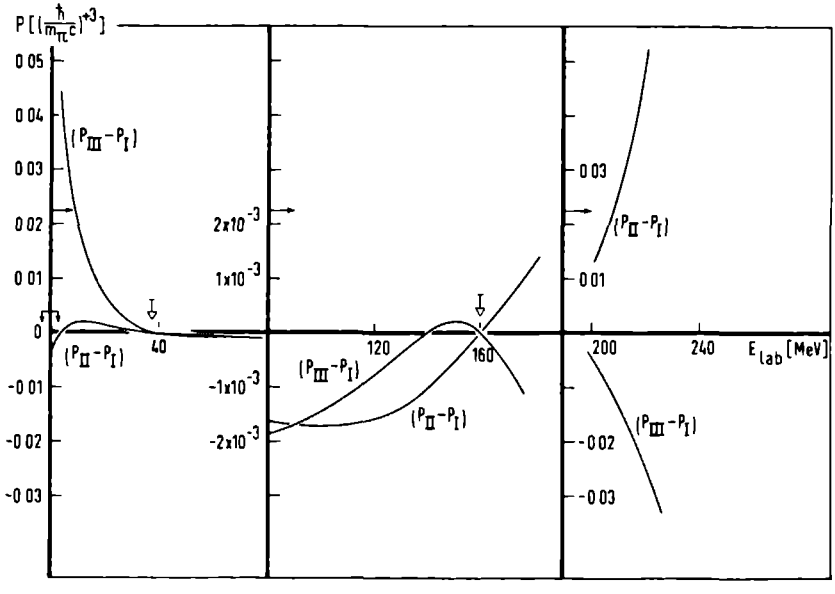


Fig. VI.5: Differences between the approximated shapes and the OBE shape; between 80 and 190 MeV the vertical scale is enlarged.

Comparing with the OBE model one observes:

- CFS(2) has too much shape (of course, it contains no repulsion and therefore no cross-over),
- CFS(3) predicts a shape which is too small;
- Y(2) agrees remarkably well with the "exact" shape;
- Considering the "error-band" in figure VI.3, the difference between CFS (3) and Y(2) is certainly statistical relevant in a multi-energy analysis of low energy pp data.

In figure VI.5 we have plotted the difference between the exact (OBE) shape and the CFS (3), respectively Y(2) shape for energies between 0 and 240 MeV. We see that for energies between 40 and 180 MeV both CFS (3) and Y(2) differ only little with the OBE shape; in terms of the s-wave phase shift they correspond with deviations in the order of 0.01 - 0.05 degrees (experimental errors in pp scattering are in the order of 0.5 degree). Only in the neighbourhood of the 250 MeV pole tne differences become larger. However, in the phase shift this corresponds only with an error of approximately 0.2 degree.

We conclude:

- the effective range model, Y(2), in which OPE is represented by a Yukawa potential and the remaining interaction by two poles, is able to represent a complicated potential model very well (certainly far within experimental errors);
- representing the OPE interaction also by a pole will result into deviations in the 0 - 30 MeV region comparable with the errors in s-wave single energy phase shift-analysis.

This does not mean that it is not possible to obtain a good fit using the CFS (3) approximation. This is illustrated in figure VI.6 where we

plotted the same OBE shape with $\frac{g^2}{4\pi} = 14.21$, however, used $\frac{g^2}{4\pi} = 16$ and 18 in the calculation of CFS(3) and Y(2). It appears that the shape is very sensitive to the pion coupling constant.

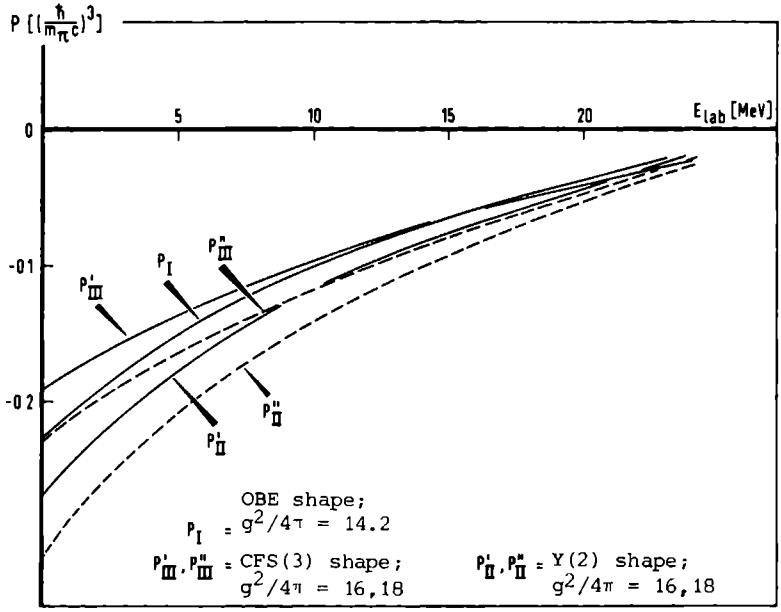


Figure VI.6: Dependence of the shape function on the pion-nucleon coupling constant.

One observes that CFS(3) with a coupling constant of 17 (20 % too high) will reproduce the shape of the OBE model with a coupling constant of 14.2 reasonable. That the OPE part of the interaction seems to be too weak in the CFS(3) approximation is clear, since the left hand pion cut starting at - 10 MeV is replaced by a pole at - 20 MeV with a weight which is an average of the total discontinuity from - 10 MeV up to - ∞.

To obtain an insight into what can be expected if one tries to determine the π^0pp coupling constant from pp scattering data, we first

tried to reproduce the coupling constant used in the OBE model with the help of the effective range formalism with OBE phase shifts as input data. To be as realistic as possible we modified the OPE potential used in [Na 78] with some additional nonlocal terms and refitted slightly. The coupling constant used in the potential model was: $\frac{g^2_\pi}{4\pi} = 13.61$. After that, proton-proton s-wave phase shifts* were calculated at energies between 0 and 30 MeV representative for the experimental cross-section measurements nowadays. Also phase shifts at 50, 100, 150, 225 and 325 MeV were calculated corresponding with the energies of the recent VPI analysis [Ar 80]. Errors were assigned to the phase shifts in three different ways:

(a) Using the numerical errors (ranging from 3.10^{-3} degrees at 0.5 MeV to 3.10^{-2} degree at 150 MeV);

(b) Using an experimental error resulting from single energy analysis at that point [Sa 79, Ar 80];

(c) Using the same error as (b), we varied also the phase shift around the calculated value to simulate a statistical distribution.

In table VI.1 we compare the pion coupling constant predicted by the Y(2) and CFS(3) effective range functions from OBE phase shifts in two different energy regions: 0 - 30 MeV and 0 - 150 MeV, using errors (a):

*For technical reasons vacuum polarization was omitted.

$\frac{g^2_{OBE}}{4\pi}$ = 13.61	Y(2)	CFS(3)
0 - 30 MeV	13.9 ± 0.4	16.44
	χ^2/data = 0.02	χ^2/data = 1.34
0 - 150 MeV	13.89 ± 0.13	14.48
	χ^2/data = 0.07	χ^2/data = 9.13

Table VI.1: π coupling constant and χ^2/data determined from potential phase shifts with numerical error.

Since numerical errors are not of statistical nature the χ^2/data gives only an impression how well the potential phases can be reproduced by an effective range model. The error quoted is not the statistical error but the maximum variation possible with the phase shifts within the error-band. The Y(2) effective range formalism is able to reproduce the potential phase shifts within the numerical error and also predicts the pion coupling constant within 2 %. For the CFS (3) this is not the case. As expected the resulting pion coupling is too large (~ 20 %) in the 0 - 30 MeV region. Adding data at higher energies improves the prediction since CFS (3) becomes better in the region above 40 MeV. Table VI.2 is the result of the calculations with the experimental errors (b) and (c). One observes the error which can be expected in the analysis of experimental data and the influence which a statistical distribution of the data may have. Results for the CFS (3) method are similar with exception of the central value which is too high (16 - 17 in the 0 - 30 MeV region).

$\frac{g^2_{OBE}}{4\pi} = 13.61$	Y(2), errors (b)	Y(2), errors (c)
0 - 30 MeV	13.9 ± 1.4	13.6 + 1.4
		χ^2/data = 0.98
0 - 150 MeV	13.9 ± 0.6	13.0 ± 0.6
	χ^2/data = 0.0005	χ^2/data = 0.88
0 - 325 MeV	13.8 ± 0.5	
	χ^2/data = 0.001	

Table VI.2: π coupling constant determined by Y(2) effective range function with experimental errors (see text).

We conclude: if an OBE potential model represents physics reasonable, one can determine the pion-nucleon coupling constant very well with the help of the Y(2) effective range formalism. The CFS(3) method will result into coupling constants which are too big: 7 % - 25 % depending on the energies which are used: data at lower energies will give a larger deviation.

Before turning to the actual data we want to make a few remarks on these numerical checks. It seems to be possible now to determine $g^2/4\pi$ with a "model" error of 0.3 and an experimental error of 1.4 in the low energy region (0.6 if one includes data up to 150 MeV). However, we have "proven" this only if nature is represented reasonable by a Nijmegen OBE potential. Besides the pion, this model contains as lowest meson mass 549 MeV from the η. Therefore the quantity $N_L^2 k^2 f_{\delta_S}^L$ (with $V_L \equiv$ OPE interaction) has singularities starting at - 160 MeV in the complex (laboratory) energy plane. It seems to be reasonable

that a cut structure at such a distance from the origin can be represented by two poles as is done in Y(2).

However, a phenomenon like uncorrelated two-pion exchange (two pion exchange diagrams minus the contribution already taken into account by the iteration of the one pion exchange potential in the Schrodinger equation) will produce an additional cut starting as low as - 40 MeV, and could influence the fit. Whether this is indeed the case, can only be seen after calculations with the help of an explicit representation of the uncorrelated two pion exchange mechanism [Pa 70, Ta 52, Br 53]. If the assumption and approximations made in the effective range theory are clearly in conflict with experimental data we expect that some or all of the following things will happen:

I. The pion coupling constant strongly depends on the energy range which is used. This is for instance observed for CFS (3) in table VI.1.

II. In this section we have dealt only with s-waves. In the analysis of experimental data, however, one must also parametrize the peripheral waves which are less dependent on the inner parts of the potential and which will give also a value for the pion coupling constant. In the case of a clear conflict between theory and experiment the value obtained for $g^2/4\pi$ from the s-wave will differ from the value obtained from the peripheral waves.

3. Determination of the π^0pp coupling constant from experimental data

In this section we will not consider the rather complicated and technical process of data collection, calculation of observables from phase shifts, search procedures and so on, which are connected with the actual analysis of proton-proton scattering.

We refer to the internal reports of Emmen, van Dongen and van der Sanden [Em 79, Do 79, Sa 79], and to the work of Sher, Signell and Heller [Sh 70], for details about low energy analyses. Data up to 500 MeV are analysed by the Livermore and VPI group [MG 68, MG 69, Ar 74, Ar 80]. Also the paper of Naisse [Na 77] is of interest since he pays special attention to the CFS (2) approximation in the 0 - 12 MeV energy region.

The data consisted out of more than fifty groups of cross-section and polarization measurements between 0.3376 and 30 MeV. To check the consistency of the s-wave effective range function with higher energies we added as data also the most recent single energy s-wave phase shifts at 50, 100, 150, 200 and 325 MeV from [Ar 80].

The parametrization of the phase shifts was done with the help of old and new techniques:

- the f and higher partial waves are represented by the Coulomb corrected one pion exchange mechanism (Born approximation);
- the p- and d-waves are parametrized according to Sher's [Sh 70] method with the only difference that the central, tensor and spin-orbit p-wave combination are used as independent waves rather than the $^3P_0$, $^3P_1$, and $^3P_2$ waves.
- the s-wave was approximated in two ways:
  a. the Coulomb- and vacuumpolarization corrected CFS (3) approximation;
  b. the two pole approximation of the effective range function $P_L(k^2)$ with: $V_L \equiv$ Coulomb- + vacuumpolarization + OPE-interaction*. We referred to this method already with the notation: Y(2).

*OPE-interaction is represented by a Yukawa potential.

A least chi-squared fit was performed for the data sets:

I 0 - 30 MeV experimental data

II 0 - 30 MeV experimental data +

50, 100, 150 MeV s-wave phase shifts [Ar 80]

III 0 - 30 MeV experimental data +

50, 100, 150, 200 and 325 MeV s-wave phases [Ar 80].

As far as the s-wave is concerned we searched for a_N, r_N, $g^2/4\pi$, E_{co} in data sets I and II, keeping E_{po} fixed at 253 MeV. For dataset III we also searched for the pole position E_{po}.

The results for the pion-nucleon coupling constant are given in table VI.3.

s-wave parametrization / dataset	Y(2)	CFS (3)
I	15.9 ± 1.0	17.8 ± 1.0
II	15.89 ± 0.56	17.59 + 0.55
III	15.21 ± 0.47	16.42 ± 0.47

Table VI.3: pion coupling constant determined with the help of both s-wave parametrizations for the energy regions I, II and III.

In table VI.3 the same coupling constant is used in the s-wave parametrization and the peripheral waves. Fitting the coupling constant in the s-wave ($g_S^2/4\pi$) and the peripheral waves ($g_>^2/4\pi$) separately we obtain the results of table VI.4.

	Y(2)	CFS (3)
I	$g_S^2/4\pi = 16.4 \pm 1.3$	$g_S^2/4\pi = 20.8 \pm 1.3$
	$g_>^2/4\pi = 15.4 \pm 1.3$	$g_>^2/4\pi = 15.4 \pm 1.3$
II	$g_S^2/4\pi = 15.9 \pm 0.6$	$g_S^2/4\pi = 17.8 \pm 0.6$
III	$g_S^2/4\pi = 15.2 \pm 0.5$	$g_S^2/4\pi = 16.5 \pm 0.5$
	$g_>^2/4\pi = 15.5 \pm 1.4$	$g_>^2/4\pi = 15.9 \pm 1.4$

Table VI.4: "s-wave" and "peripheral wave" pion coupling constants.

In table VI.4 one observes the phenomena already expected from the previous section:

- the "s-wave" coupling constant determined by CFS (3) is much larger than the value obtained by using Y(2);
- the CFS (3) value is strongly dependent on the analysed energy region and drops more than 25 % when data at higher energies are added;
- the low energy "s-wave" result for CFS (3) is clearly in conflict with the coupling constant determined from the peripheral waves;
- the Y(2) coupling constants are (within the error) not dependent on the energy region, and agree with the value from the peripheral waves.

Therefore the CFS(3) method can be rejected as a suitable parametrization. This is not because CFS (3) will not fit the experimental data*, but because the fitted value of the coupling constant is not stable and not in agreement with the value from the peripheral waves.

*In fact the chi-squared is somewhat higher than Y(2) but this does not exclude the phase shifts as predicted by CFS (3).

The Y(2) parametrization seems to be a reliable method within the current understanding of the nucleon-nucleon interaction. However, the predicted value of the coupling constant could be influenced by singularities which are rather close to the origin (for instance uncorrelated two pion exchange). Since the predicted coupling constant is not significantly dependent on the analysed energy region and also in agreement with the one from the peripheral waves there is no real indication that this is also the case.

Another cause which could influence the fitted value of $g^2/4\pi$ is a systematic deviation in the s-wave phase shift. This will be clear since the pion coupling is roughly proportional with the deviation from the straight line in the low energy region in the effective range plot (the shape).

This phenomenon is demonstrated in a fit we performed using s-wave phases as input data also in the 0 - 30 MeV region. We used phase shifts from the single energy analysis from van der Sanden and Emmen [Sa 79, Em 79] (where necessary we corrected them for vacuumpolarization approximately). To these data we added the results from the Livermore group [MG 69] up to 325 MeV.

To our surprise we obtained:

$$g_\pi^2/4\pi = 13.5 \pm 0.7 \quad .$$

This is rather strange when one realizes that the single energy phase shifts in the low energy region were obtained from (almost) the same set of experimental data*, which in an energy dependent fit gave:

*There are added some new data at 9.85 MeV and recent data from the VPI group [Ar 80] since that time, which resulted into a lower error; the fitted value of $g_\pi^2/4\pi$ was not affected significantly.

15.2 ± 0.5.

In the single energy phase shift analysis one is, however, not able to fit all of the higher partial waves, and therefore (potential) model values for the tensor and spin orbit p-wave combination were used in [Em 79] and [Sa 79]. These model values (for δ_T) appeared to be slightly and systematically too negative compared with the results of the multi-energy analysis where δ_T was fitted (deviation ~ 8 %).

The pp differential cross section can be written as [Sh 70]:

$$\sigma_{pp} = \sigma_{pp}(\text{nuc}) + \sigma_{pp}(\text{int}) + \sigma_{pp}(\text{Coul+vp}) \qquad (5)$$

with σ_{pp}(nuc), σ_{pp}(int) and σ_{pp}(Coul+vp) the so-called nuclear, interference and Coulomb + vacuumpolarization terms. For higher energies the $^1S_0$ nuclear phase shift is mainly determined by the nuclear part*:

$$k^2\,\sigma_{pp}(\text{nuc}) = \sin^2\delta_{^1S_0} + 18\,\delta_T^2\left(\frac{36}{25} + \left(\frac{\delta_{LS}}{\delta_T}\right)^2\right)$$

$$+ 9\cos^2\theta\ (\text{p-wave phases}) \qquad (6)$$

Therefore it is the quantity:

$$\sin^2\delta_{^1S_0} + 18\,\delta_T^2\left(\frac{36}{25} + \left(\frac{\delta_{LS}}{\delta_T}\right)^2\right) \approx \sin^2\delta_{^1S_0} + 26\,\delta_T^2$$

$$\left(\frac{\delta_{LS}}{\delta_T}\ \text{is small}\right) \qquad (7)$$

which is determined in a single energy analysis, rather than $\delta_{^1S_0}$.

*At 90° the interference term is zero and the argument holds exactly. For smaller angles this interference term influences the total cross section. However, this intereference term depends both on $\delta_{^1S_0}$ and on δ_C; therefore a variation in $\delta_{^1S_0}$ can be corrected by fitting δ_C. The interference part of the single energy analysis will therefore probably be correct since δ_C is fitted (δ_T and δ_{LS} not).

Therefore we corrected the δ_{1S_0} from the single energy analysis with the formula:

$$\sin^2 \delta_{1S_0}(\text{corrected}) + 26\ \delta_T^2(\text{multi-energy}) = \sin^2 \delta_{1S_0}(\text{single-energy}) + 26\ \delta_T^2(\text{potential model}) \quad (8)$$

These corrections appeared to be small, ranging from + 0.014° at 5 MeV, + 0.09° at 10 MeV up to 0.3° at 26 MeV. The extra "shape" caused by these corrections will influence the pion coupling constant obtained from a fit to these phase shifts. We obtained after these corrections:

$$g_\pi^2/4\pi = 15.0 \pm 0.7 \quad ,$$

which agrees with the fit directly to the data. Therefore it appears that the determination of an "s-wave" pion coupling constant is strongly correlated with the p-waves, especially the tensor combination δ_T; therefore polarization measurements and cross sections at 90° are important.

Summarizing our result: the π^0pp coupling constant as determined by the data below 30 MeV is:

$$15.9 \pm 1.0 \quad .$$

Taking also s-wave information up to 325 MeV into account:

$$15.21 \pm 0.47 \quad .$$

These values seem to be rather high compared with the "conventional" value: 14.43 ± 0.41 [MG 69] obtained by a multi-energy phase shift analysis. However, Viollier [Vi 74], using the phase shifts of [MG 69] in dispersion relations and correcting for the Coulomb interaction, arrived at: 15.3 ± 0.3.

As a final remark: it must be possible to decrease the error on the coupling constant by including also the peripheral waves for $E_{lab} > 30$ MeV in the fit. Since the Livermore analysis gives an error of 0.4 (from the peripheral waves) it is likely that an accuracy of 0.3 could be reached in a multi-energy fit of the data between 0 and 300 MeV.

APPENDIX

In this appendix we will sketch the proof of the different analytic properties for the solutions of the radial Schrödinger equation:

$$v'' + (k^2 - \frac{\ell(\ell+1)}{r^2} - V)\ v = 0 \qquad (1)$$

We strongly recommend the reader first to study chapter 12 of the book of Newton [Ne 66], or chapters 11 and 12 of the book of Taylor [Ta 72]. For the moment we will assume that V is a potential analytic in r, independent of k^2, in some region which contains the origin r = 0. Modifications for potentials which have a simple pole at r = 0 will be given between: < >.

Equation (1) has a regular solution of the form:

$$v_R(r) = \frac{1}{(2\ell+1)!!}\, r^{\ell+1} \sum_{i=0}^{\infty} r_i\, r^i \quad , \quad r_0 = 1,\ r_1 = 0$$
$$<r_0 = 1,\ r_1 \neq 0> \qquad (2)$$

where $v_R(r)$ is real analytic in k^2 and analytic in r in the same region as V(r).

This can be proven by deriving and manipulating the integral equation*:

$$v_R(r) = \bar{j}(r) + \int_0^r (\bar{j}(r)\,\bar{n}(r') - \bar{n}(r)\,\bar{j}(r'))\, V(r')\, v_R(r')\, dr' \qquad (3)$$

as is done by Newton and Taylor [Ne 66, Ta 72], or by studying the recursion relation for the expansion coefficients r_i:

$$(n+2)(n+2\ell+3) r_{n+2} + k^2 r_n - \sum_{i+j=n} v_i r_j = 0 \qquad (4)$$

with v_i defined by the expansion for V(r):

---

*$\bar{n}$ and $\bar{j}$ as defined in chapter II.

$$V(r) = \sum_{0}^{\infty} v_\iota \, r^\iota \qquad \langle V(r) = \sum_{-1}^{\infty} v_\iota \, r^\iota \rangle \tag{5}$$

The series (5) converges within a certain distance from the origin.

One proves: the coefficients r_n are real polynomials in k^2 with highest terms k^n for even n and k^{n-3} for odd n <k^n even n, k^{n-1} odd n>. Therefore a finite summation in (2) is always analytic in k^2. One can prove that the summation (2) is uniformly converging (for bounded k^2 and r) and therefore $v_R(r)$ is also analytic in k^2.

Furthermore, with the help of the integral equation (3) one proves:

$$|v_R(r)| \leq \text{constant} \frac{|r|^{\ell+1}}{(1 + |kr|)^{\ell+1}} \, e^{|\mathrm{Im}\, kr|} \tag{6}$$

So far, as the regular solution v_R, is concerned these properties are well-known. However, eq. (1) has another, irregular solution, of the form:

$$v_I(r) = C(k^2) \, v_R(r) \ln r + (2\ell-1)!! \, r^{-\ell} \sum_{\iota=0}^{\infty} s_\iota \, r^\iota \tag{7}$$

with $s_0 = 1$, $s_1 = 0$ <$s_0 = 1$, $s_1 \neq 0$>.

That v_I is indeed a solution can be proven by putting it into the Schrodinger equation (1) and deriving the relation:

$$(n+2)(n+1-2\ell) \, s_{n+2} + k^2 s_n + C(k^2)(2n-2\ell+3) r_{n-2\ell+1} - \sum_{\iota+\jmath=n} v_\iota s_\jmath = 0 \tag{8}$$

From (8) the coefficients s_n (polynomials in k^2 of the same degree as r_n) can be calculated and also the function $C(k^2)$:

$$C(k^2) = \frac{1}{2\ell+1} \left(\sum_{\iota+\jmath=2\ell-1} v_\iota s_\jmath - k^2 s_{2\ell-1} \right) \tag{9}$$

which is a polynomial in k^2 of degree $k^{2\ell-2}$ <$k^{2\ell}$>.

The coefficient $s_{2\ell+1}$ is not defined from the recursion relation (8) and can be choosen freely it corresponds with adding a multiple of the regular solution to the irregular solution. We will define our standard irregular solution with the choice:

$$s_{2\ell+1} = 0 \tag{10}$$

It is quite possible to make another choice; it will not affect the properties in the following as long as:

$s_{2\ell+1}(k^2)$ is a polynomial of k^2 of degree (2ℓ).

With the help of equation (7) and(8) one can prove:

v_I is a real analytic function of k^2 and except for the ln r singularity also analytic in r in the same region as V(r).

For the irregular solution one proves the integral equation:

$$v_I(r) = a\bar{n}(r) + b\bar{j}(r) + \int_{\varepsilon}^{r} dr' \, (\bar{j}(r)\bar{n}(r') - \bar{n}(r)\bar{j}(r'))V(r')v_I(r') \tag{11a}$$

with: $r > \varepsilon > 0$

$$\text{and: } \left.\begin{aligned} a\,\bar{n}(\varepsilon) + b\,\bar{j}(\varepsilon) &= v_I(\varepsilon) \\ a\bar{n}'(\varepsilon) + b\bar{j}'(\varepsilon) &= v_I'(\varepsilon) \end{aligned}\right\} \equiv \left\{\begin{aligned} a &= -\,W(v_I, \bar{j})\big|_{\varepsilon} \\ b &= W(v_I, \bar{n})\big|_{\varepsilon} \end{aligned}\right. \tag{11b}$$

Note that the integral in (11a) is not defined for $\varepsilon = 0$ since the integrand is singular for $r \to 0$ $(\ell \neq 0)$ <all ℓ>. Equation (11a) can be used to solve $v_I(r)$ in iterated form; with the help of this method one proves:

$$|v_I(r)| \leq \text{constant} \left\{ |a| \left(\frac{1+|kr|}{|r|}\right)^{\ell} + |b| \left(\frac{|r|}{1+|kr|}\right)^{\ell+1} + |a| \left(\frac{|r|}{1+|kr|}\right)^{\ell+1} \left(\frac{1+|k\varepsilon|}{|\varepsilon|}\right)^{2\ell} \right\} e^{|\mathrm{Im}\, kr|} \tag{12}$$

In the following we will need some more restrictions on the analyticity of the potential V. We will assume that V(r) is analytic in the region Re $r > 0$, finite in the origin or atmost having a simple

pole here, and falls off exponentially like $e^{-\mu r}$ (or $e^{-\mu r}/r$, etc.) for $r \to \infty$, Re $r > 0$.

When one is interested in the analytic properties of effective range functions, it appears to be convenient to define the integrals:

$$
\begin{aligned}
R^{+} &= \int_{0}^{\infty} (\bar{n} + i \bar{j} k^{2\ell+1}) \, V \, v_R \, dr' \\
R^{-} &= \int_{0}^{\infty} (\bar{n} - i \bar{j} k^{2\ell+1}) \, V \, v_R \, dr' \\
S^{+} &= b - i k^{2\ell+1} a + \int_{\varepsilon}^{\infty} (\bar{n} + i \bar{j} k^{2\ell+1}) \, V \, v_I \, dr' \\
S^{-} &= b + i k^{2\ell+1} a + \int_{\varepsilon}^{\infty} (\bar{n} - i \bar{j} k^{2\ell+1}) \, V \, v_I \, dr'
\end{aligned}
\tag{13}
$$

Although S^{+-} seem to depend on the value of ε this is in fact <u>not</u> the case, as can be proven with the help of eqs. (11).

With the usual methods, using the analyticity in k of the integrand, the analyticity of a and b in k^2, the analyticity and exponential decreasing of the potential and the bounds of eqs (6) and (12) one proves:

- S^{+}, R^{+} are analytic in the complex k-plane except for the negative imaginary axis where singularities are possible for
  $-\infty < \text{Im } k \leq -\mu/2$.
- S^{-}, R^{-} are analytic in the complex k-plane except for the positive imaginary axis where singulartities are possible for
  $+\mu/2 \leq \text{Im } k < \infty$.

Furthermore, one obtains:

$$
\begin{aligned}
S^{+-*}(-k^*) &= S^{+-}(k) \\
S^{+-*}(k^*) &= S^{-+}(k) \\
R^{+-*}(-k^*) &= R^{+-}(k) \\
R^{+-*}(k^*) &= R^{-+}(k)
\end{aligned}
\tag{14}
$$

Using eq. (6) one proves:

$$\lim_{|k|\to\infty} |R^{+-}(k)| = O\left(\frac{1}{|k|}\right) \quad \text{(except negative/positive imaginary axis)}$$

$$< < O\left(\frac{1}{|k|^{1-\delta}}\right) \quad \text{with } \delta \text{ any number } > 0 > \qquad (15)$$

From the recursion relations (4) and (8) one derives for $\varepsilon = \frac{\text{constant}}{k}$:

$$\lim_{|k|\to\infty} a = 1 + O\left(\frac{1}{k^2}\right)$$

$$\lim_{|k|\to\infty} b = O(k^{2\ell}) \qquad (16)$$

Using the bound eq. (12) and the fact that S^{+-} does not depend on ε one gets:

$$\lim_{|k|\to\infty} \frac{S^{+-}}{k^{2\ell+1}} = {}^{-}_{+}\, 1 + O\left(\frac{1}{|k|}\right) \quad \text{(except for the negative/positive imaginary axis)} \qquad (17)$$

Note 1:

Sometimes it is convenient to have a bound on $v_I(r)$, like eq. (12), which is not dependent on ε. Using eq. (12) and eq. (16) one proves immediately:

$$|v_I(r)| \leq \text{constant} \left(\frac{1+|kr|}{|r|}\right)^{\ell} e^{|\text{Im } kr|} \qquad (18)$$

for $|r| > \frac{\text{constant}}{|k|}$ and $|k| > |k_{min}| \neq 0$.

With the help of the expansion eq. (7) the validity of eq. (18) can be extended also to $|r| < \frac{\text{constant}}{|k|}$.

Furthermore, using eq. (16) and eqs (7) and (8) again in the region $|k| < |k_{min}|$, one proves:

$$|v_I(r)| \leq \left\{ \text{constant} \left(\frac{1+|kr|}{|r|}\right)^{\ell} + \text{constant} \left(\frac{|r|}{1+|kr|}\right)^{\ell+1} \right\} e^{|\text{Im } kr|} \qquad (19)$$

which is valid for all r and k (Re r > 0).

Note 2:

In chapter II we derived for potentials which are finite in the origin:

$$\delta(k) \xrightarrow[|k|\to\infty]{} O\left(\frac{1}{|k|}\right)$$

$$k^{\ell+1} N(k) \xrightarrow[k\to\infty]{} 1 + O\left(\frac{1}{k}\right)$$

$$\frac{H(k^2)}{k^{2\ell}} \xrightarrow[k\to\infty]{} \text{constant}$$

For potentials which have a simple pole in the origin this becomes:

$$\lim_{k\to\infty} \delta(k) < O\left(\frac{1}{|k|^{1-\delta}}\right)$$

$$\lim_{k\to\infty} (k^{\ell+1} N(k) - 1) < O\left(\frac{1}{|k|^{1-\delta}}\right)$$

$$\lim_{k\to\infty} \frac{H(k^2)}{k^{2\ell}} < \text{constant } |k|^{\delta}$$

} with δ any number > 0

Part Two:

The Electromagnetic Interaction

CHAPTER I

Introduction and motivation

In the second part of this thesis we will study some aspects of the electromagnetic interaction. It is impossible for us to give at this place a survey of the many different techniques used to describe this type of interaction. In the following we will only try to outline the historical background of one of the situations where an accurate description of the electromagnetic interaction is necessary: low energy proton-proton scattering.

Field theoretical methods are extensively used in the last four decades to describe the electromagnetic interaction between charged particles. The problems concerning the infrared catastrophe, the self energy of the charged particle, mass- and charge renormalization are solved within quantum electrodynamics. The predictions for quantities like the hydrogen bound state levels, the anomalous magnetic moment of the electron and muon are in excellent agreement with experiment.

The success of quantum electrodynamics encouraged the application of the same kind of techniques to other types of interactions. In the following we will refer especially to the interaction between two nucleons.

The electromagnetic force is mediated by the photon which is transmitted between the source (the electromagnetic current) and the charged particle. This electromagnetic force is present in the two nucleon system. However, besides that, another much stronger type of interaction is present.

Yukawa proposed that the nucleon is the source of a force field,

called the meson field, in the same way as an electrically charged object is the source of an electromagnetic field. The quantum associated with this field was called the pion, discovered in 1947. Until about 1960 many people have tried to construct field theoretical nuclear potentials due to one pion exchange (OPE) or two pion exchange. However, these methods failed to describe the experimental data.

After the discovery of the ρ and ω meson, fields associated with these particles were added, leading to the so-called One Boson Exchange models (OBE). Extensive work in the past two decades shows that a more or less satisfactory description is reached (for references we refer to the theses of Nagels [Na 75] and Rijken [Ry 75]). However, this in principle field theoretical description of the nuclear interaction differs in at least two aspects from standard quantum electrodynamics.

First of all the nucleons and mesons are not "fundamental", but consist out of quarks. This in contrast with the "fundamental" electrons and electromagnetic field (photon). This shows that a simple description like OBE models is at most a good approximation of the physical world. One can doubt whether all of the concepts of standard field theory can be applied.

Secondly, when one assumes that field theory is applicable to these interactions, one meets the difficulty that the intrinsic strength of the nuclear interaction is much bigger than in the electromagnetic case. The coupling between charge and photon is small enough to allow for a perturbative procedure which "converges" rather fast*. This is not the case for the strong interactions: truncation of the series

---

*In fact one believes it is a so-called asymptotic series.

after a finite number of terms will lead to serious troubles. First of all one looses "physics" which is contained in higher order exchanges and which cannot be neglected as in electrodynamics due to the smallness of the coupling constant. Secondly one comes in conflict with unitarity.

Unitary troubles can be solved by using the so-called Bethe-Salpeter equation [BS 51]; however, the loss of some "physics" contained in higher order terms is inevitable. As an additional complication, the Bethe-Salpeter equation, even in approximated form, cannot be solved exactly and must be treated numerically which is very (computer) time consuming and not trivial.

For these reasons several approximation schemes were developed for the Bethe-Salpeter equation, from which we mention the Blankenbecler-Sugar-Logunov-Tavkhelidze (BSLT) "three dimensional" scattering equations. The BSLT-equations are equations in momentum space and give difficulties when one tries to include the electromagnetic interaction (with a zero mass photon). Therefore they are often translated to configuration space which, however, leads to other approximations. At this point one has an equation which resembles the ordinary Schrödinger equation in coordinate space in many aspects and the electromagnetic interaction is often represented by a simple Coulomb potential.

This situation, from physics up to physical model is depicted in figures I.1 and I.2.

With the help of the last type of approximations a part of the Nijmegen High Energy Physics group has studied the nucleon-nucleon interaction in detail during the last decade, and found no clear

physics = interacting quarks → local renormalizable field theory with nucleon and mesons →

approximate renormalization using "physical" coupling constants and masses; neglect "self energy" diagrams → consider only one boson exchange diagrams: unitarize w.h.o. the Bethe-Salpeter equation

→ approximate BS equation by more convenient ones: BSLT eqs → Schröd. eqs

Figure I.1.

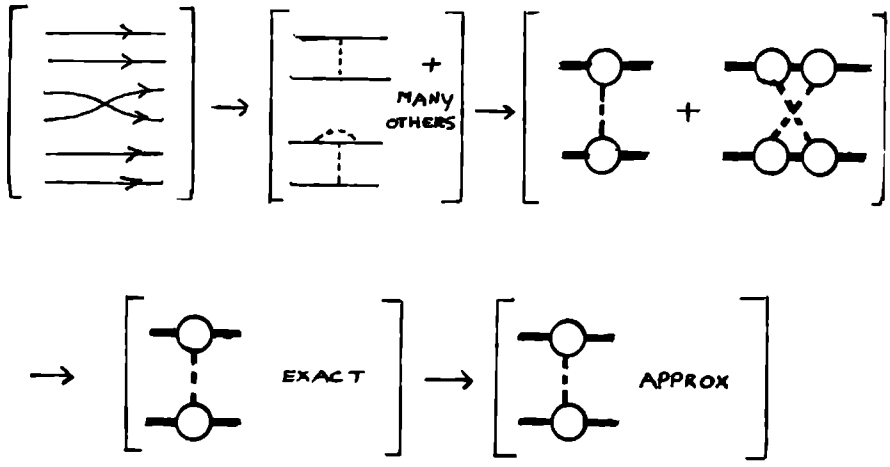


Figure I.2.

indication that one of the approximations was clearly bad, considering the experimental data*.

Much work was done, especially in the last step of the approxima-

*At this point we must mention the work of Rijken [Ry 75] who arrived ultimately at essentially the same equations using another starting point: analytic S-matrix approach using the "New Strip Approximation".

tion, to make the Schrödinger equations as equivalent as possible to the BSLT-equations. A clear advantage of the (configuration space) Schrodinger equation is that the electric interaction can be added in a rather simple way to the strong interaction. Due to the zero mass of the photon it is extremely difficult to handle this in the (momentum space) BSLT-equations. Up to a few years ago, the electric interaction was represented by a simple Coulomb potential of the form $\frac{\alpha}{r}$; one arrives at this potential by simple using the approximation scheme, described above, using the photon exchange mechanism, and it is fully consistent with the local configuration space potentials as derived by Nagels and Rijken [Na 75, Ry 75] between 1970 and 1975. However, since that time the Nijmegen group has improved its description of the strong nuclear interaction by including also nonlocal types of interaction in its configuration space potentials (the step BSLT → Schrodinger equation forces one to neglect some nonlocal terms; the most crude approximation leads to local potentials). Also the electric part of the interaction needed therefore revision and we used the same approximations as for the nuclear interaction in first instance.

However, to our (unpleasant) surprise, this led to a nonunique electric potential which was in general not able to describe a very fundamental problem in physics: the bound state levels of the hydrogen atom. Unless we made some special assumptions it predicted only the Balmer part of the energy levels, but failed in the description of the fine structure splitting.

This was a serious short-coming of our conventional method since the fine structure splitting in the bound state levels corresponded with the newly calculated (nonlocal) corrections in the potentials.

A critical revision of all the approximations was necessary which led ultimately to a different approach of the last two steps of the approximations: from boson exchange diagrams via one boson exchange to three dimensional approximations and the configuration space potential.

In the second part of this thesis we will consider the last two steps of approximations in figures I.1 and I.2. Our starting point will be boson exchange diagrams where the renormalization has been taken into account by using physical coupling constants, masses and form factors. Typical renormalization features like the anomalous magnetic moment, the Lamb shift, or the vacuum polarization are therefore not contained in our starting model and must be put in by hand as effective potentials.

Although the title suggests that the following is only a calculation of the electric potential in nucleon-nucleon scattering, it will appear that the methods can also be applied, in principle, to the strong nuclear interaction. In fact, the electric potential derived in this thesis, can only be applied in combination with a nuclear interaction derived with the same methods.

In chapter II we will give a short review of the Bethe-Salpeter equation, the three dimensional BSLT equations and we will show that the last approximation in general leads (in lowest order) to an incorrect bound state structure for the hydrogen atom.

In chapter III we will use a "φ^3" like model to outline the basic concept of the derivation of a configuration space potential from boson exchange diagrams. Since an exact calculation is not possible this is done in a perturbative way, using the inverse of the mass of

the scattering particles as expansion parameter.

In chapters IV and V we will apply this method to more "physical" cases: spin 0 - spin 0 and spin 1/2 - spin 1/2 scattering and show that the derived potential describes the hydrogen bound state levels well, including terms of order α_F^4 ($\alpha_F \equiv$ fine structure constant).

In chapter VI we will consider the electric potential in low energy proton-proton scattering and derive modified effective range functions.

The Bethe-Salpeter equation, BSLT three dimensional approximation and the electromagnetic interaction

1. Introduction and the Bethe-Salpeter equation

In this section we will list some of our conventions, review some well known properties of the summation of Feynman diagrams with the help of the Bethe-Salpeter (B.S.) equation and discuss the approximations in the Blankenbecler-Sugar (BSLT) pseudopotential approach.

We are interested in the elastic scattering of two spin-1/2 particles a and b:

$$a + b \to a + b \tag{1}$$

with initial four-momenta p_a and p_b and final momenta p'_a and p'_b. One-particle states are normalized according to:

$$\langle p,s|p',s'\rangle = (2\pi)^3 2E(p)\delta^3(\vec{p}-\vec{p}')\delta_{ss'} \tag{2}$$

with s and s' denoting the component of the spin along the z-direction and:

$$E(p) = (\vec{p}^2 + m^2)^{1/2} \tag{3}$$

m being the mass of the particle; in the case of unequal mass scattering we will denote the mass of particle a with m and its on shell energy with E(p), for particle b with M and $D(\vec{p}) = (\vec{p}^2 + m^2)^{1/2}$ respectively.

The Dirac spinors we use are normalized according to:

$$\bar{u}(\vec{p},s')u(\vec{p},s) = 2m\,\delta_{s's} \tag{4a}$$

for positive energy spinors, and:

$$\bar{v}(\vec{p},s')v(\vec{p},s) = -2m\,\delta_{s's} \tag{4b}$$

for negative energy spinors. Fourvectors will be denoted: $p = (p^0,\vec{p})$, and:

$$p^2 \equiv p_\mu p^\mu = \vec{p}^2 - p^{0\,2} \tag{5}$$

For the Dirac spinors we will use the Pauli-Dirac representation:

$$u(\vec{p},s) = \sqrt{E(\vec{p}) + m}\begin{pmatrix} \chi_s \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E(\vec{p}) + m}\,\chi_s \end{pmatrix} \tag{6}$$

with $\vec{\sigma}$ the Pauli spin matrices and χ_s a Pauli spinor. The γ-matrices have the following form:

$$\vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix} \quad , \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{7}$$

and: $\bar{u}(\vec{p},s) = u^+(\vec{p},s)\gamma_4$.

The scattering process (1) is described with the help of the scattering operator T with matrix elements:

$$\langle f|T|i\rangle \equiv \langle p_a',s_a';p_b',s_b'|T|p_a,s_a;p_b,s_b\rangle \tag{8}$$

It is customary to define an M matrix, which is a 16x16 matrix in spinor space, and which, sandwiched between Dirac spinors gives the corresponding T-matrix elements:

$$\langle f|T|i\rangle = \bar{u}(\vec{p}_a',s_a')\bar{u}(\vec{p}_b',s_b')M_{fi}(p',p;P)u(\vec{p}_a,s_a)u(\vec{p}_b,s_b) \tag{9}$$

with P the total four-momentum of the system:

$$P = p_a + p_b = p_a' + p_b' \tag{10}$$

and p the relative momentum defined by

$$\begin{aligned} p_a &= \mu_1 P + p \\ p_b &= \mu_2 P - p \end{aligned} \qquad \text{with } \mu_1 + \mu_2 = 1 \tag{11}$$

and similarly for p'. In the center-of-mass frame one has:

$$P = (\sqrt{s},0) \quad ; \quad p_a = (E(\vec{p}),\vec{p}) \quad ; \quad p_b = (D(\vec{p}),-\vec{p}) \; ;$$
$$p'_a = (E(\vec{p}'),\vec{p}') \quad , \quad p'_b = (D(\vec{p}'),-\vec{p}') \; ;$$

and therefore:

$$\begin{aligned} p &= (\mu_2 E(\vec{p}) - \mu_1 D(\vec{p}),\vec{p}) \\ p' &= (\mu_2 E(\vec{p}') - \mu_1 D(\vec{p}'),\vec{p}') \end{aligned} \tag{12}$$

The M matrix is defined for all four-momenta p, p', P, however represents only the physical scattering process (1) when p and p' satisfy equation (12) and $P = (E(\vec{p})+D(\vec{p}),0) = (\sqrt{s},0)$, which puts both the initial and final particles on the energy- and the mass-shell.

In field theory the M-matrix can be obtained when one sums the infinite set of Feynman diagrams of fig. II.1.

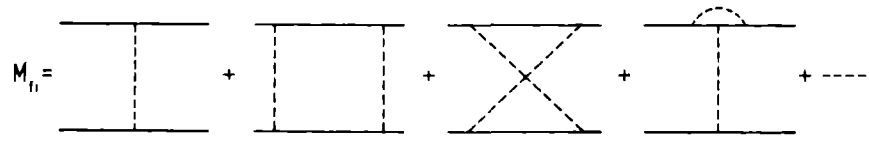


fig. II.1

This infinite summation is clearly an impossible task, and one even does not know whether this series converges. Truncation of this series will give problems with the unitarity of the scattering matrix. By rearranging the diagrams of fig. II.1 one can obtain a matrix equation

of the form:

$$M_{fi}(p',p;P) = M^{irr}_{fi}(p',p;P) + \int \frac{d^4k}{(2\pi)^4} M^{irr}_{fi}(p',k;P)G(k;P)M(k,p;P) \tag{13}$$

which is the so-called Bethe-Salpeter equation [BS 51; Ge 51; Sch 51] without 2-particle unitarity troubles when M^{irr} is truncated.

M^{irr} stands for the irreducible kernel and consists of all "irreducible" Feynman diagrams (fig. II.2).

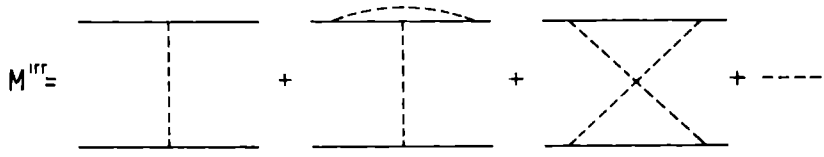


fig. II.2

G is the Greens function, which for two spin-1/2 particles consists of the two-particle propagator:

$$G(k,P) = -i \left[\frac{(\mu_1 P+k)_\mu \gamma^\mu + im}{(\mu_1 P+k)^2+m^2-i\varepsilon}\right]\left[\frac{(\mu_2 P-k)_\mu \gamma^\mu + iM}{(\mu_2 P-k)^2+M^2-i\varepsilon}\right] \tag{14}$$

The BS equation can be depicted as:

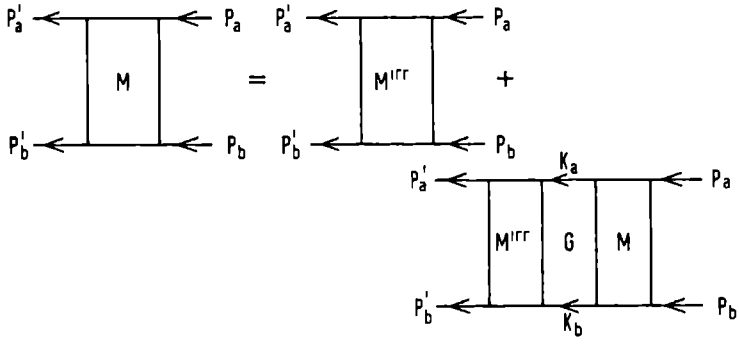


fig. II.3

The BS-equation is still an equation which (except for some simple cases) is not exactly solvable. Moreover one can doubt whether it is valid to describe a scattering process like NN → NN with the help of a local renormalizable field theory. The nucleons are not considered as confined three quark states but as elementary particles (e.g. the bare nucleon) which are "dressed" with meson clouds and renormalized to give physical coupling constants and masses (we refer to [Sw 77] for an extensive discussion on this point).

For these reasons one takes only a limited number of diagrams in the irreducible kernel into account and uses the BS-equation merely as a method to incorporate some general features of a relativistic scattering process: relativistic unitarity, retardation, covariance, and several conservation laws.

Most of the time, the first step is to remove all self-energy graphs from the irreducible kernel and to consider them as being represented by the use of physical masses and coupling constants (form factors) in the propagators and vertices. Then we are left with the M^{irr} depicted in fig. II.4.

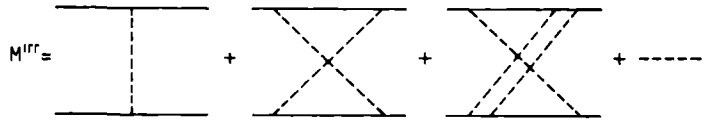


fig. II.4

Also this M^{irr} must be truncated. A commonly used approximation is to keep only the first (one-meson-exchange) term. This gives, once the BS-equation is solved, the so-called ladder approximation M_L to M:

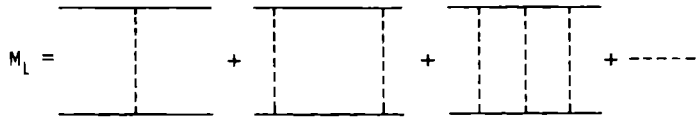


fig. II.5

The crossed (fourth order) diagram is also sometimes taken into account, however, one has rarely gone beyond the fourth order because of the great computational work involved.

The BS-equation with the approximated irreducible kernel is still a very complicated equation and hard to solve. Its main problem is the four-dimensional integration over the momenta of the intermediate particles and the complicated singularity structure of the kernel due to the Lorentz metric of the particle propagators. To simplify the numerical treatment one often applies a "Wick rotation" [Wi 54] to the BS-equation. The basic idea is to remove the singularities due to the Lorentz metric via a rotation of the integration path $k_0 \to ik_0$ and a simultaneous transformation of the external variable $p_0 \to ip_0$. However, the validity of this procedure cannot be rigorously proven. Also the integral equation involving two independent variables $|\vec{k}|$ and $|k_0|$ (the integration over the angle can be avoided by making a partial wave decomposition) requires much computer space and time. Instead of explicitly solving the equation with matrix inversion, one often iterates the BS-equation, starting with M^{irr}, by putting the approximate solution in the righthandside of equation (13) again; in this way one generates the coefficients of an expansion in the coupling constant

(fig. II.5) which can be used to construct a Padé approximant of the exact solution. For detailed information we refer to the literature [Le 67, Le 66; Fl 75], a review of the BS-equation is given by Nakanishi [Na 69].

Because of these difficulties in the BS-equation some alternative equations have been proposed which may be considered as certain approximations to the BS-equation such that relativistic covariance and unitarity are preserved but the integration over the relative energy variable is removed. These equations, called the Blankenbecler-Sugar-Logunov-Tavkhelidze equations (BSLT-eqs), will be discussed next.

2. BSLT-equations or relativistic three-dimensional scattering equations

The unitarity of the S matrix implies for the transition matrix T_{fi} as produced by the BS-equation via eq. (9)*:

$$T^{+}_{fi} - T_{fi} = (2\pi)^4 i \sum_{\substack{\text{spins} \\ n}} \int \frac{d^3p_{a_n} d^3p_{b_n}}{(2\pi)^6} \frac{1}{4E(\vec{p}_{a_n}) D(\vec{p}_{b_n})} \times T^{+}_{fn} \delta^4(P_n - P_i) T_{ni} \qquad (15)$$

Using the total and relative four-momenta as defined by eqs (10) and (11) it is not difficult to rewrite this as:

$$T^{+}_{fi} - T_{fi} = (2\pi)^2 i \sum_{\substack{\text{spins} \\ n}} \int \frac{d^4k_n}{(2\pi)^4} \delta(P^0 - E(\vec{k}) - D(\vec{k})) \times T^{+}_{fn} \frac{\delta(k^0_n - \mu_2 E(\vec{k}_n) + \mu_1 D(\vec{k}_n))}{4E(\vec{k}_n) D(\vec{k}_n)} T_{ni} \qquad (16)$$

*We restrict ourselves to the energy region where we have only (physical) two-particle intermediate states: i.e. two-particle unitarity.

Define the operation "‡":

$$M_{fi}^{\ddagger} = \gamma_4^{(a)}\gamma_4^{(b)}\, M_{fi}^{+}\, \gamma_4^{(a)}\gamma_4^{(b)}$$

Since in eq. (16) only physical two-particle states contribute, i.e. positive energy states on mass- and energy-shell, we can write this as:

$$\bar{u}(\vec{p}_a')\bar{u}(\vec{p}_b')\{M_{fi}^{\ddagger} - M_{fi}\}\, u(\vec{p}_a)u(\vec{p}_b) = (2\pi)^2 i \sum_{\substack{\text{spins}\\ n}} \int \frac{d^4k_n}{(2\pi)^4}$$

$$\bar{u}(\vec{p}_a')\bar{u}(\vec{p}_b')\, M_{fn}^{\ddagger}\, u(\vec{k}_a)u(\vec{k}_b)\, \frac{\delta(P^0 - E(\vec{k}_n) - D(\vec{k}_n))\,\delta(k^0 - \mu_2 E(\vec{k}_n) + \mu_1 D(\vec{k}_n))}{4E(\vec{k}_n)D(\vec{k}_n)}$$

$$\times\, \bar{u}(\vec{k}_a)\bar{u}(\vec{k}_b) M_{ni}\, u(\vec{p}_a)u(\vec{p}_b) \qquad (17)$$

Following Blankenbecler and Sugar and others [Bl 66; Lo 63], the BS two-particle propagator is written as a sum of two terms:

$$G = g + (G - g) \qquad (18)$$

where g is to be an appropriate nonrelativistic propagator corresponding to G, and is constructed in such a way that it possesses the singularity structure of G in the physical region. The BS-equation (13) can now be written in symbolic operator form as two equations:

$$\begin{aligned} M &= W + W\, g\, M \\ W &= M^{irr} + M^{irr}(G - g)W \end{aligned} \qquad (19)$$

Here W is called the pseudopotential which must be "hermitean" in the physical two-particle region where only elastic scattering is possible:

$$W^{\ddagger} = W$$

This requirement coincides with the unitarity condition as required by (17) for M and which must be generated by g in the first equation of (19):

$$M^{\ddagger}_{fi} - M_{fi} = \int \frac{d^4k_n}{(2\pi)^4} M^{\ddagger}_{fn}(g^{\ddagger}_{(k_n,P^0)} - g_{(k_n,P^0)}) M_{ni} \tag{20}$$

resulting into:

$$g^{\ddagger}(k,P^0) - g(k,P^0) = \pi^2 \delta(P^0 - E(\vec{k}) - D(\vec{k}))\delta(k^0 - \mu_2 E(\vec{k}) + \mu_1 D(\vec{k}))$$

$$\times \frac{1}{E(\vec{k})D(\vec{k})} \sum_{\text{spins}} u(\vec{k}_a)\bar{u}(\vec{k}_a)u(\vec{k}_b)\bar{u}(\vec{k}_b) \tag{21}$$

Of course, eq. (21) does not determine totally the exact form of g. It only guarantees that the singularity structure of g in the physical region gives a "hermitean" pseudopotential W and a relativistic two-particle unitary transition matrix T (in fact also G satisfies eq. (21) but it has also a singularity structure in the unphysical region due to its ability to propagate negative energy states).

The most common and useful form of g is:

$$g(\vec{k},P^0) = h(\vec{k},P^0)2\pi\delta(k^0 - \mu_2 E(\vec{k}) + \mu_1 D(\vec{k}))$$

$$\times \sum_{\text{spins}} u(\vec{k}_a)\bar{u}(\vec{k}_a)u(\vec{k}_b)\bar{u}(\vec{k}_b) \tag{22}$$

and eq. (21) is equivalent with:

$$\text{Im } h(\vec{k},P^0) = -\frac{\pi}{4E(\vec{k})D(\vec{k})} \delta(P^0 - E(\vec{k}) - D(\vec{k})) \tag{21'}$$

In this case the integration over k^0 is trivial. Defining W_{fi} as the matrix elements of W_{fi}, obtained by sandwiching W_{fi} between Dirac spinors, the BSLT-equation for the T matrix between positive energy states reads:

$$T_{fi}(\vec{p}',\vec{p};\sqrt{s}) = W_{fi}(\vec{p}',\vec{p};\sqrt{s}) + \sum_{\substack{\text{spins}\\ n}} \int \frac{d^3\vec{k}}{(2\pi)^3} W_{fn}(\vec{p}',\vec{k};\sqrt{s})$$

$$\times h(\vec{k},\sqrt{s})\ T(\vec{k},\vec{p},\sqrt{s}) \tag{23}$$

Note that the fourth component of the momentum k, needed in the

calculation of W, is given by:

$$k^0 = \mu_2 E(\vec{k}) - \mu_1 D(\vec{k}) \tag{24}$$

The pseudopotential W is in general dependent on the energy $\sqrt{s}$. This has two reasons. First of all in the definition of the pseudopotential, $W = M^{irr} + M^{irr}(G - g)W$, the energy dependence enters via the iteration due to the (G - g) term. Secondly M^{irr} could contain terms like:

$$(p_a + p_a')_\mu (p_b + p_b')^\mu = \text{(use (11) and (24))} =$$
$$(\vec{p} + \vec{p}')^2 - (2\mu_1\sqrt{s} + \mu_2(E(\vec{p})+E(\vec{p}')) - \mu_1(D(\vec{p})+D(\vec{p}')))$$
$$\times (2\mu_2\sqrt{s} - \mu_2(E(\vec{p})+E(\vec{p}')) + \mu_1(D(\vec{p})+D(\vec{p}'))) \tag{25}$$

This energy dependence is often considered as undesirable, and causes some authors to make special choices for the weights with the purpose to get rid of this feature. However, in fact it is an automatic consequence of the BSLT reduction and it violates or contradicts nothing as long as:

The BSLT pseudo equation is considered as an effective equation with the only purpose to produce the correct scattering matrix.

Hence. it is an error to consider the pseudopotential as a quantum mechanical potential and to impose on it all requirements which a (nonrelativistic) potential must satisfy (e.g. energy independent). Note that the transition matrix T now satisfies a three dimensional equation which appears to be easily solvable once h is determined (we will return to this point). In fact all remaining problems due to the relativistic nature of the BS equation have moved now into the equation for the pseudopotential:

$$W = M^{irr} + M^{irr}(G - g)W \tag{26}$$

which still is a fourdimensional integral equation. The hope is that one is able to make such a good choice for g that in the low energy region the second term on the righthandside of eq. (26) is a small correction and that the approximation:

$$W = M^{irr} \tag{27}$$

is not too crude.

In making the approximation (27) however one runs the risk to introduce unphysical features in the scattering problem. For instance Swart [Sw 77] showed that for equal mass NN scattering the introduction of a scalar exchange via a derivative coupling leads to a spin-flip term in the pseudopotential, which is forbidden by charge independence. This term is zero when one makes the symmetric choice for μ_1 and μ_2: $\mu_1 = \mu_2 = \frac{1}{2}$.

A similar phenomenon can be seen in eq. (25). Taking $\mu_1 = 1$, $\mu_2 = 0$ one has:

$$(p_a+p_a')_\mu (p_b+p_b')^\mu = (\vec{p}+\vec{p}')^2 - (D(\vec{p})+D(\vec{p}'))(2\sqrt{s}-D(\vec{p})-D(\vec{p}')) \tag{28}$$

The lefthandside of eq. (28) is clearly symmetric under particle exchange: a $\leftrightarrow$ b. However, the righthandside not since it transforms into:

$$(\vec{p}+\vec{p}')^2 - (E(\vec{p})+E(\vec{p}'))(2\sqrt{s}-E(\vec{p})-E(\vec{p}'))$$

which is only equal to the righthandside of (28) for on shell momenta, e.g. if:

$$\sqrt{s} = E(\vec{p}) + D(\vec{p}) = E(\vec{p}') + D(\vec{p}')$$

Of course, all these phenomena will disappear when one takes more terms of the pseudopotential into account; the next step would be:

$$W = M^{irr} + M^{irr}(G-g)M^{irr} \tag{29}$$

However, unless one solves for the pseudopotential exactly (which is equivalent with solving the B.S. equation) one can always run into troubles.

We will now turn to the definition of the function h occurring in the propagator g, eq. (22). The usual procedure is to assume real analyticity of h in some variable, except for singularities given by eq. (21'). We mention three usual forms:

Thompson (analyticity in $\sqrt{s}$), [Th 70]:

$$h_T = \frac{1}{4ED} \frac{1}{\sqrt{s}-E(\vec{p})-D(\vec{p})+i\varepsilon} \tag{30}$$

Partovi and Lomon (analyticity in s), [Pa 70]:

$$h_P = \frac{E+D}{2ED} \frac{1}{s-(E(\vec{p})+D(\vec{p}))^2+i\varepsilon} \tag{31}$$

Nagels (analyticity in q_s^2; $\sqrt{s} \equiv E(q_s)+D(q_s)$), [Na 75]:

$$h_N = \frac{1}{2(E+D)} \frac{1}{q_s^2-p^2+i\varepsilon} \tag{32}$$

As has been pointed our by Yaes [Ya 71], there is an infinite set of Green functions satisfying (21). They differ only a multiplicative factor which is a function of $\sqrt{s}$ and momentum $\vec{p}^2$, and which becomes unity on the energy shell.

It will be convenient to write h_T, h_P and h_N in terms of the nonrelativistic Lippmann-Schwinger propagator h_L defined by:

$$h_L = \frac{2M_r}{q_s^2-p^2+i\varepsilon} \quad ; \quad M_r = \frac{mM}{m+M} \tag{33}$$

One obtains $h = N^2(\vec{p},\sqrt{s}) * h_L$ (34)

with:

$$N_T^2 = \frac{1}{4ED}\,\frac{1}{2M_r}\,\frac{q_s^2-p^2}{\sqrt{s}-E-D}$$

$$N_P^2 = \frac{E+D}{2ED}\,\frac{1}{2M_r}\,\frac{q_s^2-p^2}{s-(E+D)^2} \qquad (35)$$

$$N_N^2 = \frac{1}{2(E+D)}\,\frac{1}{2M_r}$$

Note that for on shell momentum $\vec{p}$:

$$N_T^2 = N_P^2 = N_N^2 = \frac{1}{2\sqrt{s}}\,\frac{1}{2M_r} \qquad (36)$$

The BSLT-equation (23) reads (omitting the particle state labels):

$$T(\vec{p}',\vec{p}\,\sqrt{s}) = W(\vec{p}',\vec{p};\sqrt{s}) + \int \frac{d^3k}{(2\pi)^3}\, W(\vec{p}',\vec{k};\sqrt{s})\; N(\vec{k})$$

$$\times \frac{2M_r}{q_s^2-k^2+i\varepsilon}\; N(\vec{k})\; T(\vec{k},\vec{p};\sqrt{s}) \qquad (37)$$

Defining the non-relativistic $\mathcal{T}$ operator and potential V by:

$$\mathcal{T}(\vec{p}',\vec{p};\sqrt{s}) = N(\vec{p}')\; T(\vec{p}',\vec{p};\sqrt{s})\; N(\vec{p})$$

$$V(\vec{p}',\vec{p};\sqrt{s}) = N(\vec{p}')\; W(\vec{p}',\vec{p};\sqrt{s})\; N(\vec{p}) \qquad (38)$$

We see that $\mathcal{T}$ satisfies the Lippmann-Schwinger equation with the potential V:

$$\mathcal{T}(\vec{p}',\vec{p};\sqrt{s}) = V(\vec{p}',\vec{p};\sqrt{s}) + \int \frac{d^3k}{(2\pi)^3}\; V(\vec{p}',\vec{k};\sqrt{s})$$

$$\times \frac{2M_r}{q_s^2-p^2+i\varepsilon}\; \mathcal{T}(\vec{k},\vec{p};\sqrt{s}) \qquad (39)$$

Moreover, one proves that non-relativistic expressions go over in their relativistic counterpart by replacing $\mathcal{T}$ by T. For instance: The unpolarized differential cross section $\frac{d\sigma}{d\Omega}$ from:

$$\frac{p'}{p}\,\frac{|\mathcal{T}(\vec{p}',\vec{p})|^2}{4\pi^2}\,M_r^2 \qquad \text{non-relativistically}$$

into

$$\frac{p'}{p}\,\frac{1}{64\pi^2 s}\,|T(\vec{p}',\vec{p})|^2 \qquad \text{relativistically;}$$

and the unitarity equation, from

$$\mathcal{T}^+ - \mathcal{T} = 2\pi i \int \frac{d^3k}{(2\pi)^3}\,\delta\left(\frac{q_s^2-k^2}{2M_r}\right)\mathcal{T}^+\mathcal{T}$$

into

$$T^+ - T = (2\pi)^2 i \int \frac{d^4k}{(2\pi)^4}\,\delta(k^0-\mu_2 E(\vec{k})+\mu_1 D(\vec{k}))$$

$$\times\,\delta(\sqrt{s}-E(\vec{k})-D(\vec{k}))\,\frac{1}{4ED}\,T^+T \qquad .$$

Note that the differences between the BSLT propagators appear as differences in the off shell behaviour of the Lippmann-Schwinger potentials .

Also the $\mathcal{T}$ operators will differ, but only in their off shell behaviour. This is also a reason to be carefull with calculating for instance quantities as a Quadrupole moment with wavefunctions obtained from V. The $\mathcal{T}$ operators will only give correct scattering phases and bound state energies. Even this will only be true if the pseudo-potential W is solved exactly. Approximations like eq. (27) will lead to differences in the results calculated with N_T, N_L or N_N. Which choice is best (which means the smallest "G-g") cannot be said at beforehand. Therefore we will consider them in a situation where the (exact) solution is known: the electron bound to an infinite heavy atomic nucleus by a Coulomb potential.

3. The electric potential as derived from the BSLT equations

For a hydrogen-like atom with an infinitely heavy nucleus the Dirac equation with the Coulomb potential:

$$V = -\frac{Ze^2}{4\pi r} = -\frac{Z\alpha_f}{r} \qquad (40)$$

gives bound-state energies (we put $\hbar = c = 1$) [BS 57].

$$B_{n,j} = -\frac{m}{2}\left(\frac{(Z\alpha_f)^2}{n^2} + \frac{(Z\alpha_f)^4}{n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) + \ldots\right) \qquad (41)$$

The first term of eq. (41) corresponds with the non-relativistic Balmer formula, the second term is the first relativistic correction and is called the fine-structure splitting.

For the actual hydrogen atom there are several corrections for eq. (41) which we will list according to their order in α_f.

α_f^2: m must be replaced by the reduced mass,

α_f^4: - other corrections due to the finite mass of the proton (recoil corrections),

- interaction between the magnetic moment of the proton and the magnetic moment of the electron (hyperfine splitting);

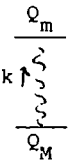
α_f^5: - mass and charge renormalization; vacuum polatization (Lamb shift);

- remaining corrections from two photon exchange diagrams;

and so on.

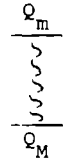
Taking the limit $M \to \infty$ and expanding $B_{n,j}$ up to order α^4 must always give eq. (41) (proton magnetic moment $\sim \frac{1}{M}$), when one uses a theory which claims to describe the electromagnetic interaction between two spin $\frac{1}{2}$ particles. A theory which does not stand this test is more or less useless. Of course, the Bethe-Salpeter equation with an appropriate irreducible kernel is correct. However, the number of terms in the irreducible kernel which are needed depends on the gauge of the electromagnetic interaction. In the following we will use two gauges explicitly:

- The manifestly Lorentz invariant Feynman gauge where the one photon exchange contribution is given by:



$$= Q_m Q_M \frac{\gamma_\mu \gamma^\mu}{k^2 - i\varepsilon} \tag{42}$$

- The Coulomb gauge with:



$$= Q_m Q_M \left\{ \frac{\gamma_0 \gamma^0}{\vec{k}^2 - i\varepsilon} + \frac{\vec{\gamma}\cdot\vec{\gamma} - (\vec{\gamma}\cdot\hat{k})(\vec{\gamma}\cdot\hat{k})}{k^2 - i\varepsilon} \right\} \tag{43}$$

with $\hat{k} = \vec{k}/|\vec{k}|$ and Q_m, Q_M the charges of the particles.

The Coulomb gauge is not manifestly Lorentz invariant. However, it has the advantage that the dominating term for small energies, the $\gamma_0\gamma^0$ part, is instantaneous and therefore does not depend on k^0. All the effects of the relative time are moved into the second term (exchange of "transverse photons") which is smaller in magnitude. Therefore we expect that eq. (43) is suitable for a theory in which the retardation is not taken exactly into account and is approximated (for instance the BSLT equations with approximated pseudopotential). This in contrast with the prescription (42) where the retardation modifies the dominating term ($\gamma_0\gamma^0$ part) and is cancelled by the "longitudinal" photon in higher order diagrams. In the BS-equation, using eq. (42), the irreducible kernel can be approximated by the one-photon-exchange graph to give the bound state energies of eq. (41). However, in the Feynman gauge one has to take also the crossed box into account (just to obtain the cancellation of the time like and longitudinal photons).

Now we will continue with the BSLT-approximations of the BS-equation. For the pseudo-potential we will take the lowest order

approximation given by eq. (42) or eq. (43). To arrive at the bound state energies we will follow the program:

1. calculate the pseudo-potential matrix elements between positive energy spinors using the prescription (24) for the zero components of the momenta.
2. transform the pseudo-potential to a Lippmann-Schwinger potential using eq. (38).
3. make an expansion in terms of the momenta and keep only terms of order 0 and $(\text{momentum})^2/(\text{mass})^2$.
4. transform the potential in momentum space to a configuration space potential.
5. calculate the bound states.

Step 3 is necessary to perform the transformation to a configuration space potential. Instead of making the expansion <u>after</u> step 1 and 2 we will give the results of step 1 and 2 immediately in expanded form:

1. Using the standard procedure (see [Na 75]) we arrive at:

$$
\begin{aligned}
W(\vec{p},\vec{p}') = Q_m Q_M \frac{4Mm}{\underline{k}^2} \Bigg\{ & (1 + \frac{\vec{q}^2}{2M^2} + \frac{\vec{q}^2}{2m^2} + \frac{\vec{q}^2}{Mm} + X) \\
& - i(\vec{\sigma}_1+\vec{\sigma}_2)\cdot\frac{(\vec{q}\times\vec{k})}{2Mm} - i(\frac{\vec{\sigma}_1}{4m^2} + \frac{\vec{\sigma}_2}{4M^2})\cdot(\vec{q}\times\vec{k}) \\
& + \frac{(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})}{4Mm} - (\vec{\sigma}_1\cdot\vec{\sigma}_2)\frac{\vec{k}^2}{4Mm} \\
& - \frac{\vec{\sigma}_1\cdot(\vec{k}\times\vec{q})\,\sigma_2\cdot(\vec{k}\times\vec{q})}{16M^2m^2} + \text{terms}\left(\frac{\text{momentum}}{\text{mass}}\right)^4 \Bigg\}
\end{aligned}
\tag{44}
$$

with: $\vec{k} = \vec{p}' - \vec{p}$; $\vec{q} = \frac{\vec{p}+\vec{p}'}{2}$

and: $X = 0$ Feynman gauge (45)

$$X = - \frac{(\vec{k}\cdot\vec{q})^2}{Mm\vec{k}^2} \qquad \text{Coulomb gauge} \qquad (45)$$

Furthermore:

$$\underline{k}^2 = \vec{k}^2 - k_0^2 = \vec{k}^2 \left(1 - \frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2} \left(\frac{\mu_2}{m} - \frac{\mu_1}{M}\right)^2\right) \qquad (46)$$

in the Feynman gauge, using (24)

and:

$$\underline{k}^2 = \vec{k}^2$$

in the Coulomb gauge.

Note that the potential in the Feynman gauge depends on the choice of the weights, in contrast with the potential in the Coulomb gauge (in fact the dependence on μ_1 and μ_2 occurs only in higher order momentum terms). In the limit $M \to \infty$ only a few terms survive:

$$\underset{M\to\infty}{W(p,p')} \cong Q_m Q_M \frac{4Mm}{\vec{k}^2} * \left(1 + \frac{\vec{q}^2}{?m^2} + \lim_{M\to\infty} \left(\frac{\mu_2}{m} - \frac{\mu_1}{M}\right)^2 \frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2} - 1 \frac{\vec{\sigma}_1\cdot(\vec{q}\times\vec{k})}{4m^2}\right) \qquad (47)$$

for the Feynman gauge. The Coulomb gauge gives the same expression without the term which depends on the weights and corresponds with the choice:

$$\mu_1 = \frac{M}{M+m} \quad , \quad \mu_2 = \frac{m}{M+m} \qquad \text{in the Feynman gauge.}$$

2. For the transformation to the LS potential it is convenient to rewrite eq. (35), once expanded, into the form:

$$N(\vec{p})N(\vec{p}') = \frac{1}{4Mm}\left(1 - \frac{q_s^2}{2Mm} + \lambda\left(q^2 + \frac{k^2}{4} - q_s^2\right) + \ldots\right) \qquad (48)$$

with:

$$\lambda_P = - \frac{1}{4M^2} - \frac{1}{4m^2}$$

$$\lambda_T = - \frac{1}{4M^2} - \frac{1}{4m^2} - \frac{1}{4Mm} \qquad (49)$$

$$\lambda_N = - \frac{1}{2Mm}$$

Note that the λ dependent term ($\equiv$ pseudo-propagator dependent) vanishes

for on-shell momenta

$$q^2 + \frac{k^2}{4} = q_s^2 \tag{50}$$

this is also true for the retardation terms in eq. (46):

$$\vec{q}\cdot\vec{k} = 0 \qquad \text{if} \qquad \vec{p}^{\,2} = \vec{p}\,'^{2} = q_s^2 \tag{50'}$$

3. The LS-potential in momentum space now reads (for $M \to \infty$, $Q_M = -Q_m = e$).

$$V = -\frac{e^2}{k^2}\left(1 + \alpha\frac{q^2}{m^2} - \beta\frac{k^2}{m^2} + \gamma\frac{q_s^2}{m^2} + \frac{\delta}{m^2}\frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2}\right) + \frac{e^2}{\vec{k}^2}\,i\,\frac{\vec{\sigma}\cdot(\vec{q}\times\vec{k})}{4m^2} \tag{51}$$

with:
$$\alpha = \frac{1}{2} + \lambda m^2$$
$$\beta = -\frac{1}{4}\lambda m^2$$
$$\gamma = -\lambda m^2$$
$$\delta = \lim_{M\to\infty} m^2\left(\frac{\mu}{m} - \frac{1}{M}\right)^2 \quad .$$

This potential is dependent on the gauge (Coulomb gauge $\delta \equiv 0$), on the weights and also on the choice of the pseudo-propagator. On energy shell (eq. (50) and (50')) nowever, this dependence disappears and the potential is:

$$V_{\text{on shell}} = -\frac{e^2}{\vec{k}^2}\left(1 + (\alpha+\gamma)\frac{q_s^2}{m^2} - (\beta+\frac{\alpha}{4})\frac{k^2}{m^2}\right) + \frac{e^2}{\vec{k}^2}\,i\,\frac{\vec{\sigma}\cdot(\vec{q}\times\vec{k})}{4m^2}$$
$$= -\frac{e^2}{\vec{k}^2}\left(1 + \frac{q_s^2}{2m^2} - \frac{\vec{k}^2}{8m^2}\right) + \frac{e^2}{\vec{k}^2}\,i\,\frac{\vec{\sigma}\cdot(\vec{q}\times\vec{k})}{4m^2} \tag{52}$$

for all gauges, weights and pseudo-propagators.

4. The momentum space potential can be transformed to configuration space using the formula's of Appendix C.

5. The calculation of the bound state energies is now completely straightforward by first solving the Schrödinger equation for the

Coulomb potential: $-\frac{\alpha_f}{r}$, and then treating the remaining terms in first order perturbation theory. We comment only on two points:

a. The Lippmann-Schwinger equation (39) is an eigenvalue problem in the center-of-mass momentum squared q_s^2, and not in the center-of-mass energy. Therefore in the hydrogen atom one solves the bound state momentum and calculates the bound state energies via:

$$B = \lim_{M\to\infty} (\sqrt{M^2 + q_s^2} + \sqrt{m^2 + q_s^2} - M - m) \qquad (53)$$
$$= \sqrt{m^2 + q_s^2} - m$$

b. Once transformed to configuration space, the potential eq. (51) will contain some $\delta^{(3)}(\vec{r})$ functions which influence only s-waves. On the other hand there is also a $\vec{L}\cdot\vec{S}$ potential present which acts only in $\ell \geq 1$ states. These two potentials cooperate in such a way that the resulting bound state energies do not depend on $\delta_{\ell,0}$ (caused by the $\delta^{(3)}(\vec{r})$ function) anymore but only on ℓ and j (total angular momentum) as continuous variables.

The bound state energies (up to order α^4) for the potential of eq. (51) are:

$$B_{n,\ell,j} = -\frac{m}{2}\left\{\frac{\alpha_f^2}{n^2} + \frac{\alpha_f^4}{n^3}\left(\frac{1}{j+1/2} - \frac{(2\alpha+2\gamma-1/4)}{n}\right.\right.$$
$$\left.\left. - (2\alpha+8\beta-1)\,\delta_{\ell,0} + \frac{8\alpha+2\delta-2}{2\ell+1}\right)\right\} \qquad (54)$$

Using the values of α,β,γ and δ of eq. (51) one obtains:

$$B_{n,\ell,j} = -m\left\{\frac{1}{2}\frac{\alpha_f^2}{n^2} + \frac{\alpha_f^4}{2n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n} + \frac{2+8\lambda m^2+2\delta}{2\ell+1}\right)\right\} \qquad (55)$$

Note that the correct bound-states are obtained (eq. (41)) only if:

$$2 + 8\lambda m^2 + 2\delta = 0 \qquad (56)$$

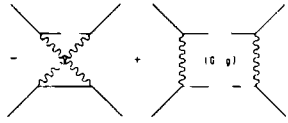
In table II.1 one sees whether the BSLT with the most simple pseudo-potential satisfies this requirement for different pseudo-propagators, weights and gauges:

	Thompson	Partavi-Lomon	Nagels
Coulomb gauge or: Feynman gauge $\mu_1 = \frac{M}{M+m}$; $\mu_2 = \frac{m}{M+m}$	yes	yes	no
Feynman gauge $\mu_1 \neq \frac{M}{M+m}$; $\mu_2 \neq \frac{m}{M+m}$	no	no	no

Table II.1: the cases for which the BSLT equation with one photon exchange gives the correct hydrogen atom ($M \to \infty$).

For those cases which do not yield the correct hydrogen atom, the next term in the pseudo-potential must be taken into account to obtain the correct answer:

$$W^{(4)} = M^{(4)}_{irr} + M^{(2)}_{irr} (G - g) M^{(2)}_{irr} \tag{57}$$



For those cases which give the correct result the next term in the pseudo-potential must therefore be small: $W^{(4)} \approx 0$ (at least up to α^4 in B). This in fact is the advantage of the BSLT equation: using a first order diagram as potential with a special pseudo-propagator (and weights) causes higher order corrections to be small. However, one does not know in advance which is the most optimal choice and must therefore explicitly calculate $W^{(4)}$.

In the next sections we will give a method to construct explicitly the correct first order potential and propagator in such a way that always the one photon (particle) exchange diagram and the planar box and crossed box diagrams are represented correctly (up to a certain accuracy) by using this potential and (pseudo)propagator in a BSLT-like equation.

Three dimensional equations in "φ^3" theory

1. Choice of the pseudopropagator

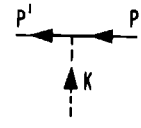
In this section we will present a method to construct a three dimensional equation which will give (up to a certain order) the same on-shell scattering matrix elements as the BS-equation.

We will use a φ^3-like theory merely because it is the most simple theory to outline the basic ideas behind our method. Applications to spin-1/2 - spin-1/2 and spin-0 - spin-0 scattering will be presented in the next sections in a more concise form.

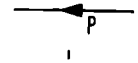
In this "φ^3" theory, applied to the scattering of scalar nucleons with mass M-exchanging scalar bosons with mass μ, the following set of Feynman rules is used:

a. Vertex function: $ig\,\delta^{(4)}(p-p'+k)$:

(g is the coupling constant)



b. Nucleon propagator: $\dfrac{1}{i(p^2+M^2-i\varepsilon)}$:



c. Boson propagator: $\dfrac{1}{i(k^2+\mu^2-i\varepsilon)}$:

d. $\int \dfrac{d^4k}{(2\pi)^4}$ for a loop integration.

e. An overall factor $+i$.

The Bethe-Salpeter equation in the center-of-mass system reads*:

*For spinless external particles there is of course no difference between M and T or N and W as used in the previous section.

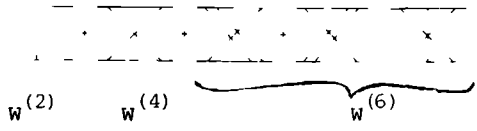
$$M(p',p,\sqrt{s}) = M^{1rr}(p',p,\sqrt{s}) + \int \frac{d^4k}{(2\pi)^4} M^{1rr}(p',k;\sqrt{s})\, G(k,\sqrt{s})\, M(k,p,\sqrt{s}) \tag{1}$$

with:

$$G(p;\sqrt{s}) = \frac{+1}{(p^2+M^2-i\varepsilon)\,((P-p)^2+M^2-i\varepsilon)} \tag{2}$$

with $P = (\sqrt{s},\vec{0})$ the total momentum.

Furthermore we do not consider renormalization corrections so that M^{1rr} is represented by:

$M^{1rr} =$ 

$+ \ldots$ (3)

The external legs are on-shell e.g.:

$$\begin{aligned} p_a &= \left(\frac{\sqrt{s}}{2}, \vec{p}\right) \quad , \quad p_b = \left(\frac{\sqrt{s}}{2}, -\vec{p}\right) \\ p'_a &= \left(\frac{\sqrt{s}}{2}, \vec{p}'\right) \quad , \quad p'_b = \left(\frac{\sqrt{s}}{2}, -\vec{p}'\right) \end{aligned} \tag{4}$$

Just like the BSLT-equations the three dimensional equation to be constructed will have a pseudopotential: $w(\vec{p}',\vec{p},\sqrt{s})$, dependent only on the three momenta and the total energy, and a pseudopropagator: $g(\vec{p},\sqrt{s})$ which, used in the three dimensional equation:

$$M = W + WgM \tag{5}$$

gives the correct <u>on-shell</u> M matrix element; stated otherwise: in the formal solution of eq. (5):

$$M = W + WgW + WgWgW + \ldots \tag{6}$$

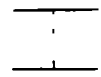
the pseudopropagator g serves to generate diagrams (or part of diagrams) which do not occur in W, but <u>are</u> contributing to M:

$$M = M^{1rr} + M^{1rr}GM^{1rr} + \ldots \tag{7}$$

In practice it will go just the other way around: first we choose a g and an initial W and compare the iteration of eq. (6) with those of eq. (7). Differences between the solutions will be added to the pseudopotential W and the procedure is iterated with this corrected W. In fact this is what happens in the pseudopotential equation (2.26).

Unlike the BSLT-equations we will not use a general principle like the unitarity condition to construct a pseudopropagator, but use the original BS-equation as a guide to choose g.

Therefore let us look more closely at the diagrams which are contained in the BS-equation.

a. The one-boson-exchange (OBE) diagram:  contained in M^{irr}:

its on-shell value is given by:

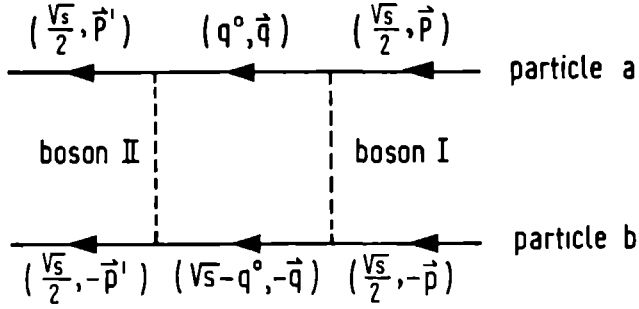
$$M^{irr(2)}(\vec{p}',\vec{p};\sqrt{s}) = \frac{-g^2}{(\vec{p}'-\vec{p})^2+\mu^2-i\varepsilon} \tag{8}$$

This term must be included in the pseudopotential; for reasons which will be clear later we will take:

$$W^{(2)}(\vec{p}',\vec{p};\sqrt{s}) = \frac{-g^2}{(\vec{p}'-\vec{p})^2+\mu^2-(E_p-E_{p'})^2-i\varepsilon} \tag{9}$$

Note that on-shell: $E_p = E_{p'} = \frac{1}{2}\sqrt{s}$. (10)

b. The OBE-diagram is iterated by the BS-equation and gives the first reduceble diagram (the so-called planar box diagram):



given by:

$$\int \frac{d^4q}{(2\pi)^4} \frac{-g^2}{\varepsilon^2-(q^0-\sqrt{s}/2)^2-i\delta} \frac{1}{(E_q^2-q^{0^2}-i\delta)\,(E_q^2-(\sqrt{s}-q^0)^2-i\delta)} \frac{-g^2}{\varepsilon'^2-(q^0-\sqrt{s}/2)^2-i\delta} \tag{11}$$

with $\varepsilon^2 = (\vec{p}-\vec{q})^2 + \mu^2$, $\varepsilon'^2 = (\vec{p}'-\vec{q})^2 + \mu^2$.

To construct a three dimensional counterpart we will perform the q^0 integration. The integrand of eq. (11) has poles in the complex q^0 plane located at:

$q^0 =$ $+ E_{\vec{q}} - i\delta$: positive energy pole particle a.

$- E_{\vec{q}} + i\delta$: negative energy pole particle a.

$\sqrt{s} - E_{\vec{q}} + i\delta$: positive energy pole particle b.

$\sqrt{s} + E_{\vec{q}} - i\delta$: negative energy pole particle b.

$\frac{\sqrt{s}}{2} \pm \varepsilon \mp i\delta$: boson I on mass shell.

$\frac{\sqrt{s}}{2} \pm \varepsilon' \mp i\delta$: boson II on mass shell.

Their positions in the complex plane are displayed in figure III.1:

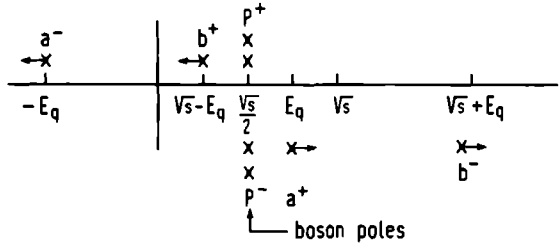


Figure III.1: Singularities of the box diagram; arrows indicate movement of poles for increasing $|\vec{q}|$.

When the intermediate particles are both on their mass shell, i.e. when

$$q^0 = E_{\vec{q}} \qquad \text{and} \qquad \sqrt{s} - q^0 = E_{\vec{q}}$$

the poles a^+ and b^+ coincide. It is this pinching which is responsible

for the unitarity cut and happens if: $q^0 = E_q = \frac{\sqrt{s}}{2}$.

Closing the integration in q^0 in the lower half plane we will pick up the contributions of four poles: a^+, b^- and $P^-_{I,\ II}$ which we denote by writing a "▶" on the corresponding line in the box diagram:

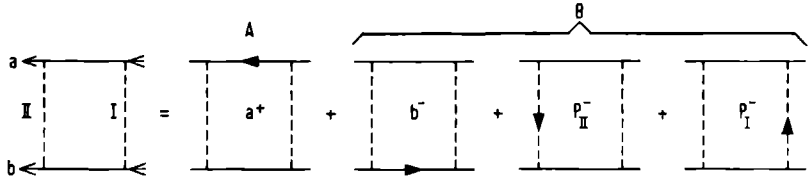


Figure III.2: closing q^0 integration in lower half plane.

Note that the direction of the arrow "▶" indicates whether it is a positive or negative energy pole: when it is parallel with the direction of time (from right to left) it is a positive energy pole, otherwise a negative energy.

Closing the contour in the upper half plane would give of course the same result for the box diagram, however, it would be the sum of four other diagrams:

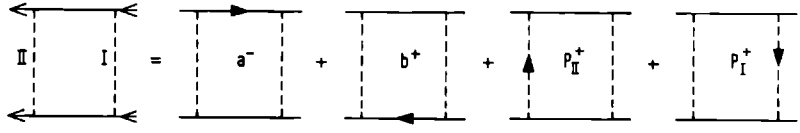


Figure III.3. closing q^0 integration in upper half plane.

In our case where particle a and b have the same mass the diagrams of figs. III.2 and III.3 are identical one by one. However, for unequal mass scattering this is not the case: only the sum is equal. We will

meet a similar situation in the crossed box.

Furthermore note that the diagrams on the right in figs. III.2 and III.3 represent three dimensional integrals. To calculate the integrand one can use the rules:

(i) put the particle with the "▶" on its mass shell with the zero component of the momentum flowing in the direction of the arrow.

(ii) calculate the four momenta of the other particles assuming energy-momentum conservation at each vertex.

(iii) replace the —▶— , —◀— lines by $\frac{1}{2E_q}$ and ↑ , ↓ by $\frac{1}{2\varepsilon}$ (or $\frac{1}{2\varepsilon'}$).

(iv) use Feynman rules for the other lines and leave the integration in the zero component of the intermediate momentum.

The same procedure can be applied to the next term in M^{irr}: the crossed box graph. Closing integration in the lower half plane gives the decomposition of fig. III.4, closing in the upper half plane fig. III.5:

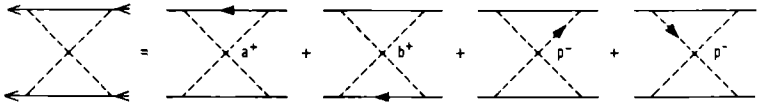


Figure III.4: crossed box decomposition.

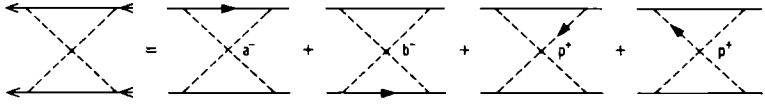


Figure III.5: crossed box decomposition.

Note that in the first decomposition two positive energy nucleon poles are contributing and in the second decomposition two negative energy nucleon poles. Calculating the diagrams explicitly shows that the two positive energy poles of fig. III.4 cancel each other almost giving the contribution of the two negative energy poles of fig. III.5. It is therefore more convenient to use the decomposition of fig. III.5 rather then fig. III.4.

The decomposition of a two loop diagram, the sixth order ladder-graph, is given in fig. III.6.

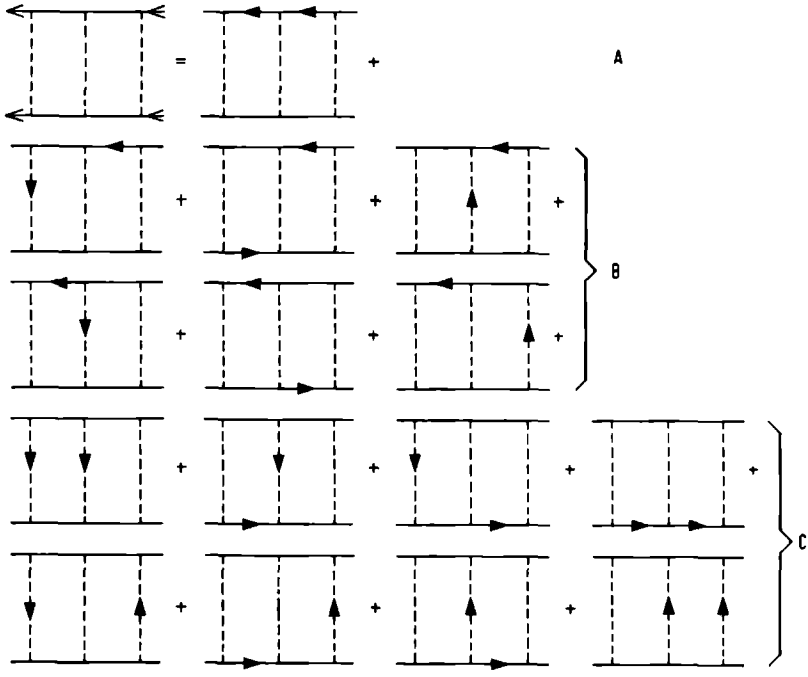
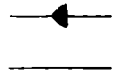


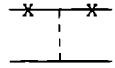
Figure III.6: decomposition of sixth order laddergraph.

The diagrams are ordered such that it invites to the following construction of a three dimensional equation:

- choose as pseudopropagator in symbolic notation:

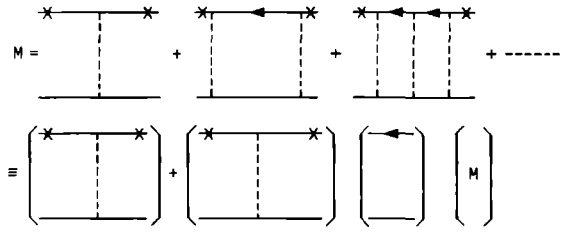
$g =$ 

- and as first order pseudopotential:

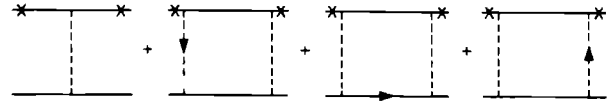
$W^{(2)} =$  , corresponding with eq. (13.9),

where X denotes that these particles must be put on their positive energy mass shells.

Then the equation $M = W^{(2)} + W^{(2)} g M$ will generate the terms of type A in figs III.2 and III.6:



Terms of type B are not generated by the iteration, and will be put in the pseudopotential as fourth order contribution:

$W^{(2)} + W^{(4)} =$ 

In 6th order the iteration will now generate the terms of type B, but terms of type C are still missing. Moreover, we have neglected the fourth order crossed box contribution. Including the graphs of fig. III.5 in the pseudopotential the total potential correct up to fourth order reads:

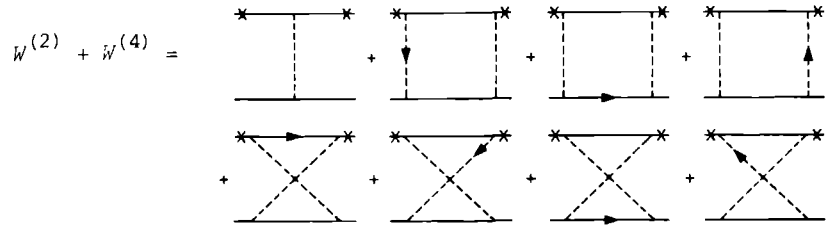
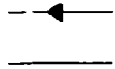


Figure III.7: second and fourth order pseudopotentials.

Calculating the pseudopropagator g =  we obtain the three dimensional equation:

$$M(p_a',p_b';p_a,p_b) = W(p_a',p_b';p_a,p_b) + \int \frac{d^3q}{(2\pi)^3} W(p_a',p_b';p_a,p_b)$$

$$\frac{1}{2E_q\sqrt{s}(\sqrt{s}-2E_q+i\varepsilon)} M(q_a,q_b;p_a,p_b) \qquad (12)$$

with

$$p_a = (E_{\vec{p}},\vec{p}) \quad , \quad p_b = (\sqrt{s}-E_{\vec{p}},-\vec{p}) \quad , \quad p_a' = (E_{\vec{p}'},\vec{p}')$$

$$p_b' = (\sqrt{s}-E_{\vec{p}'},-\vec{p}') \quad , \quad q_a = (E_{\vec{q}},\vec{q}) \quad , \quad q_b = (\sqrt{s}-E_{\vec{q}},-\vec{q}) .$$

This propagator g was first proposed by Gross [Gro 69] and corresponds with the choice:

$$N_{Gr}^2 = \frac{\sqrt{s}+2E_p}{\partial E_p \sqrt{s} M} \qquad (13)$$

or

$$\lambda_{Gr} = -\frac{1}{4M^2} \qquad (14)$$

(Compare with eqs (2.35), (2.49), putting E = D and M = m.)

So far everything is equivalent with the BSLT-equations except for the arguments used for the choice of the pseudopropagator and the rather complicated notation for the pseudopotential.

In the next step we will use the fact that we are not interested in a three dimensional equation which represents the BS-equation for

all momenta and masses: our main goal is to construct a three dimensional equation which is equivalent to the BS-equation in the non-relativistic region and includes some terms which are able to account for the lowest order relativistic effects in the BS-equation. For this purpose we have to define what a non-relativistic limit and a relativistic correction is.

2. The 1/M expansion

The BS-equation is an integral equation in momentum space and it appears to be convenient to use the following procedure:

- expand the BS-equation in powers of 1/M (scattering mass) with respect to the point $M = \infty$ for fixed momenta, exchanged masses and coupling constants. In the limit $M \to \infty$, keeping all momenta fixed, the velocities (p/E) of the "external" and "internal" scattering particles will go to zero. However, in doing so one meets a difficulty which we try to explain with the help of the non-relativistic Schrödinger equation:

$$\{\vec{p}^{\,2} + 2M_r g_s^2 V(r)\}\ \psi = q_s^2\ \psi \qquad (15)$$

with g_s^2 a dimensionless coupling constant.

It is clear that taking the limit $M \to \infty$ and keeping g_s^2 fixed will affect the effective "strength" of the potential, which seems to be proportional with M (at fixed momenta!). The same phenomenon is observed in the expression of the cross-section in terms of the scattering matrix:

$$\frac{d\sigma}{d\Omega} = \frac{p'}{p}\ \frac{|\mathcal{T}_{(\vec{p}',\vec{p})}|^2}{4\pi^2}\ M_r^2 \quad .$$

Therefore we do not fix the dimensionless coupling constant in the limit $M \to \infty$ but fix the related quantity f_s^2 with the dimension of mass:

$$f_s^2 = M\, g_s^2 \quad .$$

For "φ^3" the coupling constant g^2 used in the vertex has the dimension MeV^2. We introduce the related quantity f^2 with the dimension of a mass:

$$f^2 = g^2/M^2$$

and then keep f^2 fixed in taking the limit $M \to \infty$. We will denote the limit $M \to \infty$, while keeping the momenta and f^2 fixed, with LIM.

Expanding the M matrix in powers of f^2:

$$M = \sum_{n=0}^{\infty} M^{(2n)} \quad , \quad M^{(2n)} \sim f^{2n}$$

one can prove:

$$\text{LIM } M^{(2n)} \sim f^{2n} \times \text{constant} \times M$$

and therefore:

$$\text{LIM } M \sim \text{constant} \times M$$

Since:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \left| M(p',p) \right|^2$$

also $\text{LIM } \frac{d\sigma}{d\Omega} \sim$ constant.

We write

$$M(p',p) \underset{\substack{f^2,\text{momenta} \\ \text{fixed}}}{=} M_0(p',p) + \frac{M_1(p',p)}{M} + \frac{M_2(p',p)}{M^2} + \ldots \qquad (16)$$

M_0 will be called the non-relativistic limit, and M_1, M_2, .. the (first) relativistic corrections, with $M_1 \sim$ constant$_1$ M. Following the arguments of section 2 the "non-relativistic" $\mathcal{T}$ operator was defined by:

$$\mathcal{T}(\vec{p}',\vec{p}) = N_{Gr}(\vec{p}')\, M(p',p)\, N_{Gr}(\vec{p}) \qquad . \qquad (17)$$

Since $\text{LIM } N_{Gr} \sim 1/M$ we have:

$$\text{LIM } \mathcal{T} \sim \text{constant}/M$$

The corresponding expansion for $\mathcal{T}$ is:

$$\mathcal{T}(\vec{p}',\vec{p}) = \mathcal{T}_0(\vec{p}',\vec{p}) + \frac{\mathcal{T}_1(\vec{p}',\vec{p})}{M} + \frac{\mathcal{T}_2(\vec{p}',\vec{p})}{M^2} + \ldots \qquad (18)$$

with $\mathcal{T}_i \sim \text{constant}_i/M$.

satisfies by construction:

$$\mathcal{T}(\vec{p}',\vec{p}) = V(\vec{p}',\vec{p}) + \int \frac{d^3\vec{k}}{(2\pi)^3} V(\vec{p}',\vec{k}) \frac{M}{q_s^2-k^2+i\epsilon} \mathcal{T}(\vec{k},\vec{p}) \qquad (19)$$

or

$$\mathcal{T} = V + V\hat{g}M\mathcal{T}$$

with

$$\hat{g}(\vec{p}) = \frac{1}{q_s^2-p^2+i\epsilon} \quad \text{and} \quad V(\vec{p}',\vec{p}) = N_{Gr}(\vec{p}')W(\vec{p}',\vec{p})N_{Gr}(\vec{p})$$

In order to obtain $\mathcal{T}(\vec{p}',\vec{p})$ correctly up to the first three terms of eq. (18), we must calculate the first three terms of the potential $V(\vec{p}',\vec{p})$:

$$V(\vec{p}',\vec{p}) = V_0(\vec{p}',\vec{p}) + \frac{V_1(\vec{p}',\vec{p})}{M} + \frac{V_2(\vec{p}',\vec{p})}{M^2} + \ldots \qquad (20)$$

with $V_i \sim \text{constant}_i/M$.

This potential used in eq. (19) will then, after the transformation of eq. (17) which also can be expanded in powers of 1/M, give the first three terms of the M matrix in eq. (16). In first instance the calculation of V_0, V_1 and V_2 seems to be a formidable task since we must first expand V in the coupling constant (corresponding with fig. III.6):

$$V = V^{(2)} + V^{(4)} + V^{(6)} + \ldots$$

and then perform the $\frac{1}{M}$ expansion for each term; rearranging them gives:

$$V_0 = V_0^{(2)} + V_0^{(4)} + V_0^{(6)} + \ldots$$

$$V_1 = V_1^{(2)} + V_1^{(4)} + V_1^{(6)} + \ldots$$

and so on.

However, calculating these terms gives an interesting pattern. We can show that

$$
\begin{aligned}
V^{(2)} &= V_0^{(2)} + V_2^{(2)}/M^2 + \ldots \\
V^{(4)} &= 0 + V_2^{(4)}/M^2 + \ldots \\
V^{(6)} &= 0 + 0 + \ldots
\end{aligned}
\tag{21}
$$

The fact that in $V^{(4)}$ the $1/M^2$ term is leading and in $V^{(6)}$ the $1/M^3$ term, is caused by strong cancellations between the several diagrams and it is unlikely to be an accident. We conjecture (although we cannot prove it in general) that the same cancellations also happen at higher orders. So:

In order to calculate the first three terms of eq. (16) it is sufficient to consider only the second and fourth order (in coupling constant) diagrams for the pseudopotential.

The same statement is not true for an expansion of $\mathcal{T}$ in the coupling constant: $\mathcal{T}^{(6)}$ has terms of order $\mathcal{T}_0^{(6)}$ and $\mathcal{T}_2^{(6)}$ but these are generated completely by the pseudopotential up to fourth order by solving the LS-equation (eq. (19)).

In the next section we will calculate the expansion of eq. (21) for the first three terms and obtain the non-vanishing terms up to second order in 1/M.

3. Calculation of the first three terms in the 1/M expansion for the potential.

The calculation of the second order (in coupling constant) expansion is trivial.

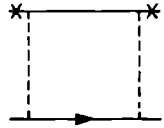
Using eqs (9), (14), (19) and (2.48) we obtain:

$$V^{(2)}(\vec{p}',\vec{p}) = -\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2} * \left| 1 - \frac{q_s^2}{2M^2} - \frac{1}{4M^2}(q^2 + \frac{k^2}{4} - q_s^2) \right.$$

$$\left. + \frac{(\vec{q}\cdot\vec{k})^2}{k^2+\mu^2}\frac{1}{M^2} + \ldots \right| \qquad (22)$$

with $\vec{q}$ and $\vec{k}$ defined as in eq. (2.44).

For the fourth order expansion we use the terms of fig. III.7 and obtain:



$$= \sim \frac{f^4}{M^5} \int \text{(momentum dep terms)}$$

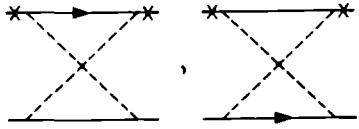
Using:

$$MV^{(4)}(\vec{p}',\vec{p}) = MN(\vec{p}')\, W^{(4)}(\vec{p}',\vec{p})N(\vec{p})$$

$$\overset{\rightarrow}{M\to\infty} \frac{1}{M} \lim_{M\to\infty} W^{(4)} \sim \frac{1}{M^6} \quad .$$

We see that it contributes to $V_6^{(4)}$.

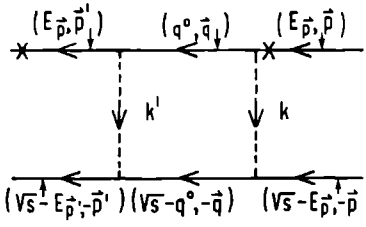
Similar one shows:



$$= \sim \frac{f^4}{M^3} * \int \text{(mom.dep.terms)}$$

contributing to $V_4^{(4)}$.

For the remaining terms of the box diagram the following notation is used:



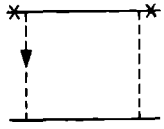
$$k = (E_{\vec{p}}-q^0, \vec{p}-\vec{q})$$

$$k' = (q^0-E_{\vec{p}'}, \vec{q}-\vec{p}')$$

and the abbreviations.

$$\varepsilon^2 = \varepsilon^2_{\vec{k}} = \vec{k}^2+\mu^2 \quad , \quad \varepsilon'^2 \equiv \varepsilon^2_{\vec{k}'} = \vec{k}'^2+\mu^2 \quad .$$

Then we obtain (the integration $\int \frac{d^3q}{(2\pi)^3}$ is always implicitly understood):



$$= \frac{1}{E^2_{\vec{q}} - (E_{\vec{p}'}+\varepsilon')^2} \frac{1}{E^2_{\vec{q}} - (\sqrt{s}-E_{\vec{p}'}-\varepsilon')^2}$$

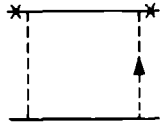
$$* \frac{f^2 M}{\varepsilon^2 - (E_{\vec{p}'}-E_{\vec{p}}+\varepsilon')^2} * \frac{-f^2 M}{2\varepsilon'}$$

Taking the limit $M \to \infty$ this diagram becomes:

$$\frac{f^4}{8\varepsilon'^3(\varepsilon^2-\varepsilon'^2)} * \left(1 + \frac{q_s^2-\vec{p}'^2}{M\varepsilon'} + \frac{2\varepsilon'}{\varepsilon^2-\varepsilon'^2} * \frac{\vec{p}'^2-\vec{p}^2}{2M} + \ldots\right) \qquad (23)$$

This diagram contributes to $V_1^{(4)}$ and $V_2^{(4)}$.

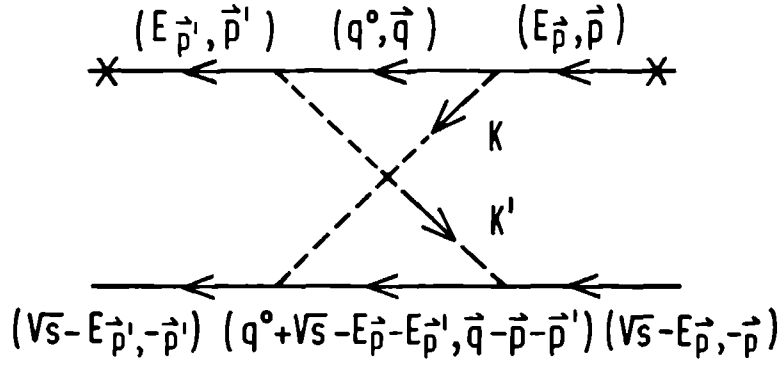
Similarly one obtains:



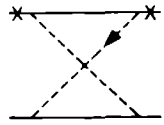
$$\xrightarrow[M\to\infty]{} \frac{f^4}{8\varepsilon^3(\varepsilon'^2-\varepsilon^2)} * \left(1 + \frac{q_s^2-\vec{p}^2}{M\varepsilon} + \frac{2\varepsilon}{\varepsilon'^2-\varepsilon^2}\right.$$

$$\left. * \frac{\vec{p}^2-\vec{p}'^2}{2M} + \ldots\right) \qquad (24)$$

In the crossed box the momenta are defined according to:

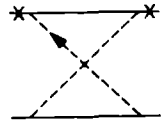


defining $\vec{q}' = \vec{q} - \vec{p} - \vec{p}'$ one obtains



$$\underset{M\to\infty}{\to} \frac{-f^4}{8\varepsilon^3(\varepsilon'^2-\varepsilon^2)} * \left(1 + \frac{\varepsilon}{M} - \frac{\vec{q}^2+\vec{q}'^2+\vec{p}'^2-\vec{p}^2-2q_s^2}{2\varepsilon M} + \frac{2\varepsilon}{\varepsilon'^2-\varepsilon^2}\frac{\vec{p}'^2-\vec{p}^2}{2M} + \ldots\right) \qquad (25)$$

and



$$\underset{M\to\infty}{\to} \frac{-f^4}{8\varepsilon'^3(\varepsilon^2-\varepsilon'^2)} * \left(1 + \frac{\varepsilon'}{M} - \frac{\vec{q}^2+\vec{q}'^2+\vec{p}^2-\vec{p}'^2-2q_s^2}{2\varepsilon' M} + \frac{2\varepsilon'}{\varepsilon^2-\varepsilon'^2}\frac{\vec{p}^2-\vec{p}'^2}{2M} + \ldots\right) \qquad (26)$$

These diagrams give contributions to $V_1^{(4)}$ and $V_2^{(4)}$.

However, one observes the cancellation of the terms contributing to $V_1^{(4)}$ if one adds the expressions (23) to (26). For the calculation of the remaining terms, contributing to $V_2^{(4)}$, we change variables:

$$\begin{aligned} \vec{p} &= \vec{\pi} + \vec{\Delta} \quad ; \quad \vec{p}' = \vec{\pi} - \vec{\Delta} \\ \vec{q} &= \vec{\pi} - \vec{\delta} \quad ; \quad \vec{q}' = -\vec{\pi} - \vec{\delta} \\ \vec{k} &= \vec{\delta} + \vec{\Delta} \quad ; \quad \vec{k}' = \vec{\Delta} - \vec{\delta} \end{aligned} \qquad (27)$$

and $\int \frac{d^3\vec{q}}{(2\pi)^3} \to \int \frac{d^3\vec{\delta}}{(2\pi)^3}$.

Considering the terms of the form: $\frac{f^4}{8\varepsilon^3(\varepsilon'^2-\varepsilon^2)^2} * 2\varepsilon \frac{(\vec{p}'^2-\vec{p}^2)}{2M}$ and $\frac{f^4}{8\varepsilon'^3(\varepsilon^2-\varepsilon'^2)^2} * 2\varepsilon' \frac{(\vec{p}^2-\vec{p}'^2)}{2M}$ it is not difficult to prove that they sum up to an expression:

$$\sim \frac{\text{even function in } \vec{\delta}}{\vec{\Delta}\cdot\vec{\delta}}$$

which vanishes after the integration $\int \frac{d^3\vec{\delta}}{(2\pi)^3}$.

The remaining terms sum up to:

$$- \frac{f^4}{8\epsilon^2\epsilon'^2 M} + \frac{f^4}{16M(\epsilon^2-\epsilon'^2)} \left\{ \frac{2\vec{\delta}^2+8\vec{\pi}\cdot\vec{\Delta}-2\vec{\Delta}^2}{\epsilon'^4} - \frac{2\vec{\delta}^2-8\vec{\tau}\cdot\vec{\Delta}-2\vec{\Delta}^2}{\epsilon^4} \right\} \tag{28}$$

Use the identities:

$$\vec{\nabla}_\delta \frac{\vec{\delta}+\vec{\Delta}}{\epsilon^2\epsilon'^2} = \frac{1}{\epsilon^2\epsilon'^2} - 2\frac{\delta^2-\Delta^2}{\epsilon^2\epsilon'^4} + 2\frac{\mu^2}{\epsilon^4\epsilon'^2}$$
$$\vec{\nabla}_\delta \frac{\vec{\delta}-\vec{\Delta}}{\epsilon^2\epsilon'^2} = \frac{1}{\epsilon^2\epsilon'^2} - 2\frac{\delta^2-\Delta^2}{\epsilon^4\epsilon'^2} + 2\frac{\mu^2}{\epsilon^2\epsilon'^4} \tag{29}$$

and $\int d^3\vec{\delta}\ \vec{\nabla}_\delta \left(\frac{\vec{\delta}\pm\vec{\Delta}}{\epsilon^2\epsilon'^2}\right) = \int d^3\vec{\delta}\ \frac{\epsilon^4+\epsilon'^4}{(\epsilon^2-\epsilon'^2)} \frac{\vec{\pi}\cdot\vec{\Delta}}{\epsilon^4\epsilon'^4} = 0$

The contribution of the fourth order diagrams to $V_2^{(4)}$ is:

$$V_2^{(4)} = \frac{1}{4M^2} W_2^{(4)} = \frac{f^4\mu^2}{32M^3} \int \frac{d^3\vec{\delta}}{(2\pi)^3} \left(\frac{1}{\epsilon^2\epsilon'^4} + \frac{1}{\epsilon^4\epsilon'^2}\right) \tag{30}$$

Assuming that the contributions to the $1/M$ and $1/M^2$ term from the higher order diagrams are zero, the L.S.-potential correct up to second order in the $1/M$ expansion is:

$$V_0 + V_1 + V_2 = -\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2} \left| 1 - \frac{q_s^2}{2M^2} - \frac{1}{4M^2}(\vec{q}^2 + \frac{k^2}{4} - q_s^2) + \frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2+\mu^2}\frac{1}{M^2} \right| + \frac{f^4\mu^2}{32M^3} \int \frac{d^3\vec{q}}{(2\pi)^3} \left(\frac{1}{\epsilon^2\epsilon'^4} + \frac{1}{\epsilon^4\epsilon'^2} \right) \tag{31}$$

However, this potential has the unpleasant feature that it is not linear in the coupling constant f^2 (also the BS irreducibel kernel contains non-linear diagrams!). Although in principle one could work with a potential of this form, it leads immediately to <u>non-additive</u> potentials. This means that if mesons of different masses are exchanged, e.g. μ_1 and μ_2, the <u>total</u> potential is <u>not</u> the <u>sum</u> of the potentials derived for the meson with mass μ_1 and the meson with mass μ_2 separately, but has additional contributions which contain μ_1 <u>and</u> μ_2:

$$V(\mu_1\&\mu_2) = V(\mu_1) + V(\mu_2) + W(\mu_1,\mu_2) \tag{32}$$

Furthermore we note that eq. (30) can be rewritten as:

$$\frac{f^4\mu^2}{32M^3}\int\frac{d^3\vec{q}}{(2\pi)^3}\left\{\frac{1}{\vec{k}^2+\mu^2}\,\frac{M}{q_s^2-\vec{q}^2+i\varepsilon}\,\frac{q_s^2-\vec{q}^{\,2}}{M(\vec{k}'^2+\mu^2)^2}\right.$$
$$\left.+\frac{q_s^2-\vec{q}^2}{M(\vec{k}^2+\mu^2)^2}\,\frac{M}{q_s^2-\vec{q}^2+i\varepsilon}\,\frac{1}{\vec{k}'^2+\mu^2}\right\} \qquad (33)$$

and that the potential:

$$U = -\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2}\left(1+\frac{\mu^2}{\vec{k}^2+\mu^2}\,\frac{q_s^2-\vec{p}^2}{2M^2}+\frac{\mu^2}{\vec{k}'^2+\mu^2}\,\frac{q_s^2-\vec{p}'^2}{2M^2}\right) \qquad (34)$$

in the Lippmann-Schwinger equation (eq. (19)) with external momenta on energy shell will produce terms:

$$U^{(2)} = U_0^{(2)} = -\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2}$$

$$U^{(4)} = U_0^{(4)} + U_2^{(4)} + U_4^{(4)} = \frac{f^4}{16M^2}\int\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{1}{\vec{k}^2+\mu^2}\,\frac{M}{q_s^2-\vec{q}^2+i\varepsilon}\,\frac{1}{\vec{k}'^2+\mu^2}$$
$$+\frac{f^4\mu^2}{32M^4}\int\frac{d^3\vec{q}}{(2\pi)^3}\left\{\frac{1}{\vec{k}^2+\mu^2}\,\frac{M}{q_s^2-\vec{q}^2+i\varepsilon}\,\frac{q_s^2-\vec{q}^2}{(\vec{k}'^2+\mu^2)^2}\right.$$
$$\left.+\frac{q_s^2-\vec{q}^2}{(\vec{k}^2+\mu^2)^2}\,\frac{M}{q_s^2-\vec{q}^2+i\varepsilon}\,\frac{1}{\vec{k}'^2+\mu^2}\right\}$$
$$+\frac{f^4\mu^4}{64M^6}\int\frac{d^3\vec{q}}{(2\pi)^3}\left\{\frac{q_s^2-\vec{q}^2}{(\vec{k}^2+\mu^2)^2}\,\frac{M}{q_s^2-\vec{q}^2+i\varepsilon}\,\frac{q_s^2-\vec{q}^2}{(\vec{k}'^2+\mu^2)^2}\right\} \qquad (35)$$

Therefore: with external momenta on the energy shell the non-linear term $V_2^{(4)}$ is identical with the term $U_2^{(4)}$ of the once iterated non-local potential U which is linear in the coupling constant. Therefore the potential of eq. (31) is (up to $1/M^4$ terms) equivalent with:

$$V = -\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2}\left[1-\frac{q_s^2}{2M^2}-\frac{1}{4M^2}(\vec{q}^2+\frac{k^2}{4}-q_s^2)+\frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2+\mu^2}\,\frac{1}{M^2}\right.$$
$$\left.+\frac{\mu^2}{\vec{k}^2+\mu^2}\,\frac{q_s^2-\vec{q}^2-\frac{k^2}{4}}{M^2}\right] \qquad (36)$$

The last term in eq. (36) will give no contribution in $V^{(2)}$ on energy shell but generates exactly the term $V_2^{(4)}$ of eq. (30) in the iteration: VgV. The potential is now linear in the coupling constant at the cost of an extra momentum dependent term. Moreover one can prove, that these potentials are <u>additive</u> (at least in the framework of "φ^3" theory).

Leaving details to the interested reader one shows:

(i) in eq. (32):

$$W(\mu_1,\mu_2) = \frac{f_1^2 f_2^2}{32M^3} \int \frac{d^3\vec{q}}{(2\pi)^3} \left\{ \frac{\mu_1^2}{(\vec{k}^2+\mu_1^2)^2 (\vec{k}'^2+\mu_2^2)} + \frac{\mu_2^2}{(\vec{k}^2+\mu_1^2)(\vec{k}'^2+\mu_2^2)^2} \right\}$$

$$+ (1 \leftrightarrow 2)$$

(ii) these terms are generated automatically by:

$$V(\mu_1)gV(\mu_2) + V(\mu_2)gV(\mu_1)$$

with V the potential of eq. (36).

(iii) therefore:

$$V(\mu_1 \& \mu_2) = V(\mu_1) + V(\mu_2)$$

In this way we arrive at a potential with relativistic corrections which is linear in the coupling constant and also additive, and which reproduces the first three terms of the 1/M expansion in the BS M-matrix although the BS irreducible kernel is <u>not</u> linear and <u>not</u> additive.

For the proof that the contributions of the sixth order diagram to $V_0^{(6)}$, $V_1^{(6)}$ and $V_2^{(6)}$ are zero we refer to appendix A.

Finally we wish to remark that it is essential to take the crossed box into account. Neglecting the crossed box but calculating the planar box will lead to a non-vanishing term $V_1^{(4)}$ which, in general, is larger in magnitude then the next contribution: $V_2^{(4)}$. Since BSLT-

equations with the most simple pseudopotential: W = one boson exchange diagram, will not contain a $V_1^{(4)}$ contribution (it enters via the M^{irr}(G-g)W term) we arrive at the conclusion that it is better to use a BSLT-equation than the BS-equation in the ladder approximation.

4. Comparison between different pseudopotentials

The correct pseudopotential (up to order $1/M^2$) which must be used in a Lippmann-Schwinger equation is given by (36). It contains the following contributions:

$$-\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2}\left(1-\frac{q_s^2}{2M^2}\right) = \text{on shell value.}$$

$$\left.\begin{aligned}
&+\frac{f^2}{16M^3}\frac{1}{\vec{k}^2+\mu^2}\left(\vec{q}^2+\frac{k^2}{4}-q_s^2\right)\\
&-\frac{f^2}{4M^3}\frac{1}{\vec{k}^2+\mu^2}\frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2+\mu^2}\\
&-\frac{f^2}{4M^3}\frac{1}{\vec{k}^2+\mu^2}\frac{\mu^2}{\vec{k}^2+\mu^2}\left(q_s^2-\vec{q}^2-\frac{k^2}{4}\right)
\end{aligned}\right\}\ \text{zero on shell, contributing only for off-shell momenta.}$$

However, in the context of the 1/M expansion the last three terms are not completely independent. In appendix B it is shown that:

$$\frac{\mu^2}{M^2(\vec{k}^2+\mu^2)^2}\left(\vec{q}^2+\frac{k^2}{4}-q_s^2\right) = \frac{2(\vec{q}\cdot\vec{k})^2}{M^2(\vec{k}^2+\mu^2)^2} - \frac{\vec{q}^2+\frac{k^2}{4}-q_s^2}{2M^2(\vec{k}^2+\mu^2)} + O\left(\frac{1}{M^4}\right) \quad (37)$$

when used in a Lippmann-Schwinger equation with the external legs on shell. This means that an equivalent form of eq. (36) is:

$$V = -\frac{f^2}{4M}\frac{1}{\vec{k}^2+\mu^2}\left(1-\frac{q_s^2}{2M^2}+\frac{1}{4M^2}\left(\vec{q}^2+\frac{k^2}{4}-q_s^2\right)-\frac{1}{M^2}\frac{(\vec{q}\cdot\vec{k})^2}{\vec{k}^2+\mu^2}\right) \quad (38)$$

It is incorrect to use for one potential eq. (36) and for the other eq. (38) in a superposition of different potentials: in that case additivety is lost. Explicitly:

$$V(\mu_1 \& \mu_2) = V(\mu_1; \text{eq. (36)}) + V(\mu_2; \text{eq. (36)})$$

and also

$$V(\mu_1 \& \mu_2) = V(\mu_1; \text{eq. (38)}) + V(\mu_2; \text{eq. (38)})$$

but <u>not</u>:

$$V(\mu_1 \& \mu_2) = V(\mu_1; \text{eq. (36)}) + V(\mu_2; \text{eq. (38)}) \quad \text{(wrong)}$$

but

$$V(\mu_1 \& \mu_2) = V(\mu_1; \text{eq. (36)}) + V(\mu_2; \text{eq. (38)}) + W(\mu_1, \mu_2) \tag{39}$$

The BSLT-equations with the pseudopotential $W = M^{(2)}_{irr}$ will give

$$V_{BSLT(2)} = -\frac{f^2}{4M} \frac{1}{\vec{k}^2+\mu^2} \left(1 - \frac{q_s^2}{2M^2} + \lambda(\vec{q}^2 + \frac{\vec{k}^2}{4} - q_s^2) + (\mu_2-\mu_1)^2 \times \frac{(\vec{q}\cdot\vec{k})^2}{M(\vec{k}^2+\mu^2)}\right) \tag{40}$$

with μ_1 and μ_2 the weights and:

$$\lambda_P = -\frac{1}{2M^2} \quad \text{(for spin 0 particles also called } \lambda_{BBS})$$

$$\lambda_T = -\frac{3}{4M^2}$$

$$\lambda_N = -\frac{1}{2M^2}$$

These BSLT-potentials differ already in the first non-local terms with the correct potential eq. (38). In the limit $\mu \to 0$ the two non-local terms will cancel partially and we obtain, using eq. (37) once more:

$$V = -\frac{f^2}{4M} \frac{1}{\vec{k}^2} \left(1 - \frac{q_s^2}{2M^2}\right) \tag{41}$$

$$V_{BSLT(2)} = -\frac{f^2}{4M} \frac{1}{\vec{k}^2} \left(1 - \frac{q_s^2}{2M^2} + \left(\lambda + \frac{(\mu_2-\mu_1)^2}{4M^2}\right)(\vec{q}^2 + \frac{\vec{k}^2}{4} - q_s^2)\right) \tag{42}$$

For $\mu = 0$ the term $V_2^{(4)}$ (eq. (30)) vanishes, representing the contributions of the planar box and crossed box not taken into account by

the iteration of the one boson exchange diagram with the Gross propagator. Therefore the difference between (41) and (42) is not so much the neglect of the crossed box diagram but rather the non-optimal choice of the pseudopropagator. For the Gross pseudopropagator:

$$\mu_1 = 1 \quad ; \quad \mu_2 = 0 \quad ; \quad \lambda_{Gr} = -\frac{1}{4M^2} \quad \text{(cf eq. (14))}$$

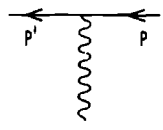
and indeed: eq. (42) gives eq. (41).

The electromagnetic potential: spin-0 - spin-0 scattering

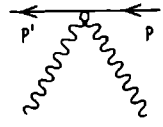
In this chapter we will derive a potential which describes the electromagnetic interaction between two particles with spin zero. Again we will not give the complete potential but only the lowest three terms in the 1/M expansion, representing the "non-relativistic" part of the electromagnetic interaction and the first two relativistic corrections. We will consider unequal mass scattering and assign masses m and M to particles a and b, according to the convention used in section 2.

The Feynman rules change with respect to chapter III into:

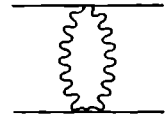
a. Vertices:



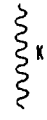
$= ie(p+p')_\mu * \delta$ function



$= -2ie^2 g_{\mu\nu} * \delta$ function

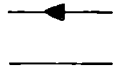
b. factor $\frac{1}{2}$ for graphs like  containing closed photon loop.

c. photon propagator:



$= \frac{g_{\mu\nu}}{ik^2}$ *)

As in section 3 we define the pseudopropagator in symbolic notation:

g = 

*in fact we will use: $(g_{\mu\nu})/(i(k^2+\mu^2))$, so that one is able to generalize results to massive vector meson exchange.

one derives:

$$g(q,\sqrt{s}) = N_{Gr}^2(q,\sqrt{s}) * \frac{2M_r}{q_s^2-\vec{q}^2+i\varepsilon} \tag{1}$$

with

$$N_{Gr}^2 = \frac{q_s^2-\vec{q}^2}{2E_q((\sqrt{s}-E_q)^2-D_q^2)} * \frac{1}{2M_r} \tag{2}$$

and

$$M_r = \frac{mM}{m+M}$$

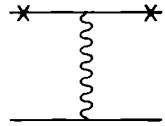
In the $\frac{1}{M}$ ($\frac{1}{m}$) expansion:

$$N_{Gr}^2 = \frac{1}{4Mm}\left(1 + \frac{q_s^2}{4m^2} - \frac{q_s^2}{2Mm} - \frac{\vec{q}^2}{4m^2}\right) \tag{3}$$

Note that this choice of g, which puts particle "a" on its mass shell, will in the first instance lead to a non-relativistic potential ($V = N_{Gr}WN_{Gr}$) which is <u>not</u> symmetric in m and M.

Also the lowest order pseudopotential is not symmetric:

$W^{(2)}$ =



or:

$$W^{(2)} = \frac{e^2\{(E_{\vec{p}}+E_{\vec{p}'})(2\sqrt{s}-E_{\vec{p}}-E_{\vec{p}'})+(\vec{p}+\vec{p}')^2\}}{\vec{k}^2+\mu^2-(E_{\vec{p}}-E_{\vec{p}'})^2} \tag{4}$$

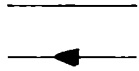
expanded:

$$W^{(2)} = 4Mme^2\left(1+\left(\frac{1}{4m^2}-\frac{1}{4Mm}\right)(\vec{p}^2+\vec{p}'^2) + \left(\frac{1}{2M^2}+\frac{1}{2Mm}\right)q_s^2 + \frac{(\vec{p}+\vec{p}')^2}{4Mm} + \frac{(\vec{p}^2-\vec{p}'^2)^2}{4m^2(\vec{k}^2+\mu^2)}\right)\frac{1}{\vec{k}^2+\mu^2} \tag{5}$$

One obtains the L.S. potential:

$$V_0^{(2)} + V_1^{(2)} + V_2^{(2)} = \frac{e^2}{\vec{k}^2+\mu^2}\left(1 + \left(\frac{1}{4m^2}+\frac{1}{2Mm}\right)\vec{q}^2 + \left(\frac{1}{16m^2}-\frac{1}{8Mm}\right)\vec{k}^2 + \left(\frac{1}{4m^2}+\frac{1}{2M^2}\right)q_s^2 + \frac{(\vec{k}\cdot\vec{q})^2}{m^2(\vec{k}^2+\mu^2)}\right) \tag{6}$$

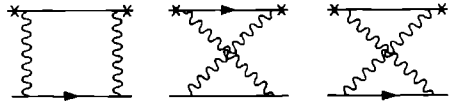
This potential is derived by closing the intermediate momenta of the ladderdiagrams in the lower half plane (compare figure III.2). Closing the contour in the upper half plane (compare figure III.3) would lead to:

$g =$  and: $W^{(2)} =$ 

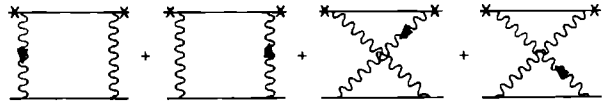
and the LS-potential of eq. (6) with m and M interchanged. Note that these potentials are the same on energy shell: $q_s^2 = \vec{q}^2 + \vec{k}^2/4$ and $(\vec{k}\cdot\vec{q})^2 = 0$. However, the off-shell behaviour is different and they would ultimately lead to different scattering matrix elements (even up to the first three terms in the $\frac{1}{M}$, $\frac{1}{m}$ expansion).

It is essential that one adds the remaining contributions of the planar box and crossed box diagram, because it will restore the symmetry between the $\frac{1}{m}$ and $\frac{1}{M}$ terms in the expansion.

As in "φ^3" theory the diagrams:



do not contribute to $V_0^{(4)}$, $V_1^{(4)}$ and $V_2^{(4)}$. For the remaining diagrams of the fourth order:



one obtains in expanded form, using the tricks of chapter 3 (the calculations are straightforward but rather tedious)*:

$$W_0^{(4)} + W_1^{(4)} + W_2^{(4)} = 0 + 0 + 2e^4 M \times \int \frac{d^3\vec{\delta}}{(2\pi)^3} \left[\frac{\mu^2}{\varepsilon^2 \varepsilon'^4} + \frac{\mu^2}{\varepsilon^4 \varepsilon'^2} + \frac{2}{\varepsilon^2 \varepsilon'^2} \right] \qquad (7)$$

*Again each diagram contributes in leading order to $W_1^{(4)}$, however, in the summation they cancel each other.

The LS-potential $V^{(4)}$ is obtained by multiplying with $\frac{1}{4Mm}$; just as in chapter 3 this contribution is quadratically in the coupling constant and it will be included in $V^{(2)}$ through an additional term which contributes only off-shell (in the second Born term):

$$\frac{e^2}{\vec{k}^2+\mu^2}\left(\frac{M+m}{2Mm^2}\left(q_s^2-\vec{q}^2-\frac{\vec{k}^2}{4}\right)+\frac{M+m}{2Mm^2}\left(q_s^2-\vec{q}^2-\frac{\vec{k}^2}{4}\right)\frac{\mu^2}{\vec{k}^2+\mu^2}\right) \qquad (8)$$

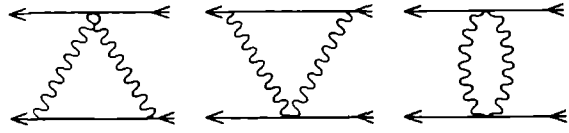
Adding the contribution of eq. (6) and (8) gives a potential which is still not symmetric in M and m, however, using the identity of eq. (3.37).

$$\frac{\mu^2}{m^2(\vec{k}^2+\mu^2)^2}\left(\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2\right)=\frac{2(\vec{q}\cdot\vec{k})^2}{m^2(\vec{k}^2+\mu^2)^2}-\frac{\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2}{2m^2(\vec{k}^2+\mu^2)}+O\left(\frac{1}{m^4}\right)$$

one can rewrite the $(\vec{k}\cdot\vec{q})^2$ term in eq. (6) (representing the expanded retardation term) and obtain the symmetric result:

$$V=\frac{e^2}{\vec{k}^2+\mu^2}\left(1+\frac{1}{2}\left(\frac{1}{M^2}+\frac{1}{Mm}+\frac{1}{m^2}\right)q_s^2-\frac{1}{4Mm}\vec{k}^2\right.$$
$$\left.-\frac{1}{2Mm}\left(\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2\right)\frac{\mu^2}{\vec{k}^2+\mu^2}\right) \qquad (9)$$

To obtain the complete fourth order potential one must include the seagull graphs:



Performing the integration of the zero component of the loop momentum one obtains the decomposition of picture (4.1) (we closed the integration such that only negative energy particle poles contribute; however, closing in the other half plane would give the same result):

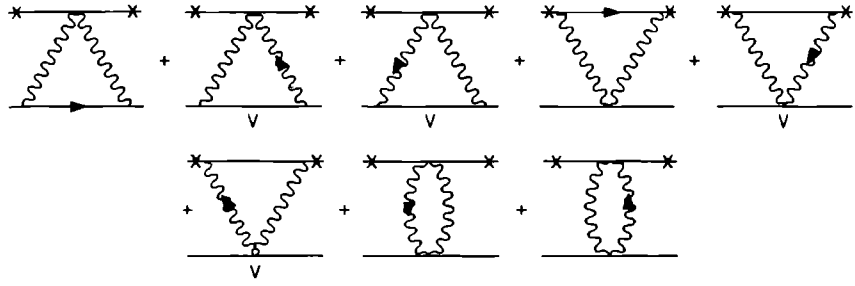


Figure 4.1: three dimensional representation seagull graphs; V indicating diagrams contributing to $W_2^{(4)}$.

Only four diagrams contribute to $W_2^{(4)}$:

$$W^{(4)}_{\text{seagull}} = W_2^{(4)} + \ldots$$

$$= -\,2(M+m)e^4 \int \frac{d^3\vec{\delta}}{(2\pi)^3}\ \frac{1}{\epsilon^2\epsilon'^2} + \ldots \qquad (10)$$

Also this term will be represented in $V_2^{(2)}$ through an additional nonlocal term:

$$\frac{e^2}{\vec{k}^2+\mu^2} * \frac{\vec{q}^2 + \frac{\vec{k}^2}{4} - q_s^2}{4M_r^2} \qquad (11)$$

The total LS-potential, derived from a second and fourth order expansion in e, containing the first three terms in the $\frac{1}{M}$, $\frac{1}{m}$ expansion is therefore:

$$V_0 + V_1 + V_2 = \frac{e^2}{\vec{k}^2+\mu^2}\left(1 + \left(\frac{1}{4M_r^2} - \frac{1}{2Mm}\right) q_s^2 + \frac{\vec{q}^2}{4M_r^2}\right.$$

$$\left. + \left(\frac{1}{16M_r^2} - \frac{1}{4Mm}\right)\vec{k}^2 - \frac{\mu^2}{\vec{k}^2+\mu^2}\,\frac{\vec{q}^2 + \frac{\vec{k}^2}{4} - q_s^2}{2Mm}\right) \qquad (12)$$

Used in the Lippmann-Schwinger equation with reduced mass $\frac{Mm}{M+m}$ at center of mass momentum q_s^2, this potential will give scattering matrix elements which are related to the BS scattering matrix M by:

$$\mathcal{V}(\vec{p}',\vec{p},\sqrt{s}) = N_{Gr}(\vec{p}',\sqrt{s})M(p',p;\sqrt{s})N_{Gr}(\vec{p},\sqrt{s}) \qquad (13)$$

We emphasize once more that this is only valid when the external particle-momenta are <u>on-shell</u> (energy shell for the LS-equation, mass shell for the BS-equation). For photon exchange the potential is:

$$V = \frac{e^2}{\vec{k}^2}\left(1 + \frac{\vec{q}^2}{4M_r^2} - \left(\frac{1}{4Mm} - \frac{1}{16M_r^2}\right)\vec{k}^2 + \left(\frac{1}{4M_r^2} - \frac{1}{2Mm}\right)q_s^2\right) \qquad (14)$$

and in the limit $M \to \infty$:

$$V = \frac{e^2}{\vec{k}^2}\left(1 + \alpha\frac{\vec{q}^2}{m^2} - \beta\frac{\vec{k}^2}{m^2} + \gamma\frac{q_s^2}{m^2}\right) \qquad (15)$$

with:

$$\alpha = \frac{1}{4} \quad , \quad \beta = -\frac{1}{16} \quad , \quad \gamma = \frac{1}{4} \quad .$$

Following the method in section 2, one can prove that for attraction ($e^2 < 0$) this potential will give bound states at energies:

$$B_{n,\ell} = m\left\{-\frac{\alpha_f^2}{2n^2} + \frac{\alpha_f^4}{2n^3}\left(\frac{2\alpha+2\gamma-\frac{1}{2}}{n} + (2\alpha+8\beta)\delta_{\ell,0} - \frac{8\alpha}{2\ell+1}\right) + \ldots\right\} \qquad (16)$$

(Note the $\delta_{\ell,0}$ term, caused by $\delta^{(3)}(\vec{r})$ functions in the configuration space potential.)

Using the values for α, β and γ:

$$B_{n,\ell} = m\left\{-\frac{1}{2}\frac{\alpha_f^2}{n^2} + \frac{\alpha_f^4}{2n^3}\left(\frac{3}{4n} - \frac{2}{2\ell+1}\right) + \ldots\right\} \qquad (17)$$

The bound states for a particle satisfying the Klein-Gordon equation in the presence of a static Coulomb potential can be solved exactly, giving [Me 61]:

$$B_{n,\ell} + m = m\left(1 + \frac{\alpha_f^2}{(n-\varepsilon_\ell)^2}\right)^{-1/2} \qquad (18)$$

with

$$\varepsilon_\ell = \ell + \frac{1}{2} - \{(\ell + \frac{1}{2})^2 - \alpha_f^2\}^{1/2}$$

Expanded in the fine structure constant α_f, eq. (18) will give indeed eq. (17).

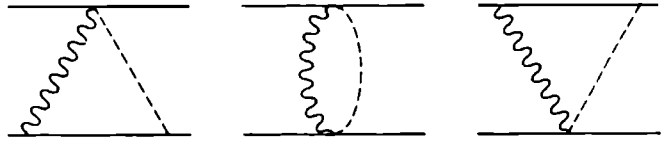
A BSLT-equation with a pseudopotential approximated by the one photon exchange contribution, would give a potential with the same on shell value, which means:

$$\begin{aligned} \alpha + \gamma &= \frac{1}{2} \\ \text{always:} \qquad -\beta + \frac{\gamma}{4} &= \frac{1}{8} \\ 2\alpha + 8\beta &= 0 \end{aligned} \tag{19}$$

$$B_{n,\ell} = m\{-\frac{1}{2}\frac{\alpha_f^2}{n^2} + \frac{\alpha_f^4}{2n^3}(\frac{3}{4n} - \frac{8\alpha}{2\ell+1}) + \ldots\}$$

One obtains the correct bound state levels if the pseudopropagator is chosen such that: $\alpha = \frac{1}{4}$.

Finally we wish to remark that the potential of eq. (12) is in general not additive; using this potential in combination with a potential of the φ^3 type by just adding them, will result into terms corresponding with non existing diagrams like:



This can be avoided by not making the step from eq. (10) to eq. (11), which will lead to the non-linear potential:

$$V = \frac{e^2}{\vec{k}^2+\mu^2}\left(1 + \frac{1}{2}\left(\frac{1}{M_r^2} - \frac{1}{Mm}\right)q_s^2 - \frac{1}{4Mm}\vec{k}^2 - \frac{1}{2Mm}\left(\vec{q}^2 + \frac{\vec{k}^2}{4} - q_s^2\right)\frac{\mu^2}{\vec{k}^2+\mu^2}\right)$$

$$- \frac{M+m}{2Mm}e^4 \int \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{(\vec{p}-\vec{q})^2+\mu^2}\frac{1}{(\vec{p}'-\vec{q})^2+\mu^2} \tag{20}$$

The potential for spin-1/2 - spin-1/2 scattering

The derivation of the electromagnetic potential between two spin-1/2 particles is more complicated than the treatment of the spin-0 case because of the fact that the propagators and vertices become matrices which involve positive and negative energy states.

The propagator of a spin-1/2 particle, which is off its mass shell, can be written in terms of Dirac spinors in the following way:

$$\frac{m-i\not{p}}{i(p^2+m^2-i\varepsilon)} = \frac{1}{2E_{\vec{p}}} \left[\frac{\sum_s u(\vec{p},s)\bar{u}(\vec{p},s)}{p^0-E(\vec{p})+i\varepsilon} + \frac{\sum_s v(\vec{p},s)\bar{v}(\vec{p},s)}{p^0+E(\vec{p})-i\varepsilon} \right] \qquad (1)$$

The factors in the numerator, $\Lambda \sim u\bar{u}$ and $\Sigma\, v\bar{v}$, act as projection operators on the positive and negative energy states (except for a normalization factor).

For the vertex, describing the coupling of a spin-1/2 particle, with anomalous magnetic moment κ, with a vector particle, we take:

$$i\Gamma_\mu = i\gamma_\mu + \frac{\kappa}{2m} i\, \sigma_{\nu\mu}\, k^\nu \qquad (2)$$

with $k^\nu = (p'-p)^\nu$.

For the photon propagator we will use the Feynman gauge (Coulomb gauge will be treated at the end of this chapter).

The Bethe-Salpeter equation is a matrix equation in spinor space which couples positive and negative energy states. We are interested in the scattering from positive energy states into positive energy states, which is described by sandwiching the M matrix between Dirac spinors:

$$T_{fi}^{++,++}(p_f,p_i;P) = \bar{u}(\vec{p}_f)\bar{u}(-\vec{p}_f)M_{fi}(p_f,p_i;P)u(\vec{p}_i)u(-\vec{p}_i) \qquad (3)$$

Due to the fact that: $\Gamma_\mu^{+-} \neq 0$, the intermediate states will contain in general a mixture of positive and negative energy states.

As before we will decompose a Feynman diagram by performing the integration of the zero component of the intermediate momentum. However, now we will also indicate the positive (+) and negative (-) energy states; the diagram of figure III.2 now becomes (for the $T^{++,++}$ matrix element):

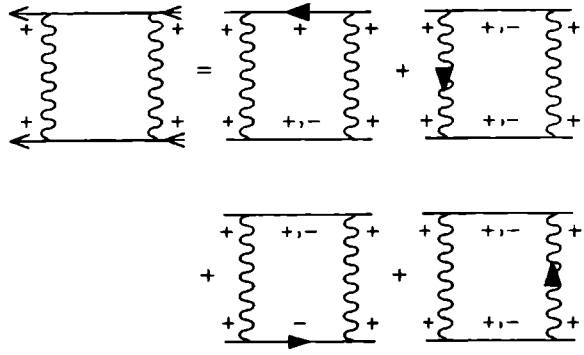


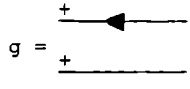
Figure V.1: decomposition box diagram in 3 dimensional integrals.

If a spin-1/2 intermediate particle lies on its positive (negative) energy shell, it will not contain negative (positive) components. However, the accompanying particle remains in general off shell and contains therefore positive - as well as negative energy states, except when the three momentum $\vec{P}$ is on-shell:

$$\sqrt{s} = E(\vec{p}) + D(\vec{p}) \quad .$$

In this way the box diagram of fig. V.1 will contain 12 subdiagrams. However, in the $\frac{1}{M}$ ($\frac{1}{m}$) expansion it will appear that in the first three terms only those diagrams contribute which contain no negative Dirac spinors.

As pseudopropagator we define in symbolic notation:



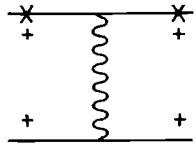
one proves:

$$g(\vec{q},\sqrt{s}) = \frac{1}{4E_q D_q} \frac{\sum_s u_1(\vec{q})\bar{u}_1(\vec{q}) \sum_s u_2(-\vec{q})\bar{u}_2(-\vec{q})}{(\sqrt{s}-E_q-D_q+i\varepsilon)} \tag{4}$$

This pseudopropagator corresponds with the choice of Thompson [Th 70] and is, in contrast with the spin-0 - spin-0 case, symmetric in the "upper" and "lower" particle.

The projection operator in the numerator is included in the matrix elements. The lowest order pseudopotential then reads:

$W^{(2)}$ =



corresponding with:

$$W^{(2)}(p',p;\sqrt{s}) = \frac{\bar{u}_1(\vec{p}')\bar{u}_2(-\vec{p}')\Gamma_{1\mu}\Gamma_2{}^{\mu}u_1(\vec{p})u_2(-\vec{p})}{(\vec{p}'-\vec{p})^2+\mu^2-(E_{p'}-E_p)^2-i\varepsilon} \tag{5}$$

We recall that this pseudopropagator and pseudopotential will generate all the diagrams of the form:

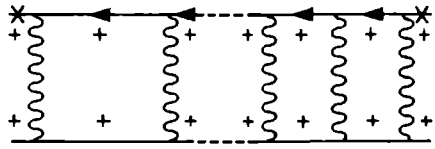


Figure V.2

in the three dimensional equation:

$$T^{++,++}(p',p;P) = W^{(2)}(p',p;\sqrt{s}) + \int \frac{d^3\vec{q}}{(2\pi)^3} W^{(2)}(p',q;\sqrt{s}) \times g(\vec{q},\sqrt{s})T^{++,++}(q,p;\sqrt{s}) \tag{6}$$

The pseudopotential (5) is symmetric with respect to particles 1 and 2 except for the retardation term in the denominator.

Expanding the pseudopotential eq. (5) in $\frac{1}{m}$, $\frac{1}{M}$, one obtains, neglecting $W_3^{(2)}$, ... terms:

$$W^{(2)}(\vec{p}',\vec{p};\sqrt{s}) = \frac{4Mme^2}{\vec{k}^2+\mu^2} * \{(1 + (\frac{1}{2M^2} + \frac{1}{2m^2} + \frac{1}{Mm})\vec{q}^2 - (\frac{\kappa_1}{4m^2} + \frac{\kappa_2}{4M^2})\vec{k}^2 + \frac{(\vec{q}\cdot\vec{k})^2}{(\vec{k}^2+\mu^2)m^2})$$
$$+ (-\frac{1}{4m^2} + \frac{(1+\kappa_1)}{2m^2} + \frac{(1+\kappa_1)}{2Mm})\, i\,\vec{\sigma}_1\cdot(\vec{k}\times\vec{q})$$
$$+ (-\frac{1}{4M^2} + \frac{(1+\kappa_2)}{2M^2} + \frac{(1+\kappa_2)}{2Mm})\, i\,\vec{\sigma}_2\cdot(\vec{k}\times\vec{q})$$
$$- \frac{1}{4Mm}(1+\kappa_1)(1+\kappa_2)(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 - (\vec{k}\cdot\vec{\sigma}_1)(\vec{k}\cdot\vec{\sigma}_2))\} \qquad (7)$$

with: $\vec{k} = (\vec{p}'-\vec{p})$ and $\vec{q} = \frac{\vec{p}+\vec{p}'}{2}$.

To obtain the corresponding LS potential one writes:

$$g(\vec{q},\sqrt{s}) = N_T^2(\vec{q},\sqrt{s}) * \frac{2M_r}{q_s^2-\vec{q}^2+i\varepsilon} \qquad (8)$$

and

$$V = N_T(\vec{p}';\sqrt{s})\; W(\vec{p}',\vec{p};\sqrt{s})\; N_T(\vec{p};\sqrt{s}) \qquad (9)$$

Giving:

$$V_0^{(2)} + V_1^{(2)} + V_2^{(2)} = \frac{e^2}{\vec{k}^2+\mu^2} * \{(1 + (\frac{1}{4M^2} + \frac{1}{4m^2} + \frac{3}{4Mm})\vec{q}^2$$
$$+ (\frac{1}{4M^2} + \frac{1}{4m^2} - \frac{1}{4Mm})q_s^2 - (\frac{1}{16M^2} + \frac{1}{16m^2} + \frac{1}{16Mm} + \frac{\kappa_1}{4m^2} + \frac{\kappa_2}{4M^2})\vec{k}^2$$
$$+ \frac{(\vec{q}\cdot\vec{k})^2}{(\vec{k}^2+\mu^2)m^2}) + (\text{spin dependent terms of eq. (7)})\} \qquad (10)$$

The first fourth order correction is the diagram:

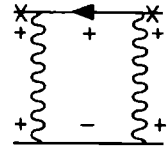


Figure V.3

To estimate its contribution in the $\frac{1}{M}$, $\frac{1}{m}$ expansion, table V.1 can be helpful:

matrix element	γ_0	$\vec{\gamma}_1$	$\frac{\sigma_{v0} k^v}{m}$	$\frac{\vec{\sigma}_{v1} k^v}{m}$
++ --	m	mom	$\frac{mom^2}{m}$	$mom + mom \frac{k^0}{m}$
+- -+	mom	m	mom	$k^0 + \frac{mom^2}{m}$

Table V.1: leading term in $\frac{1}{M}, \frac{1}{m}$ expansion for matrix elements of vertices between different spin-1/2 states with mass m, "mom" indicates terms which are proportional with some momentum.

The "contribution of a propagator" is dependent on the amount of four momentum flowing through it. A particle line (mass m) of type ⟶ or ⟵ always contributes $\frac{1}{m}$. The diagrams of fig. V.2, with n photon lines, are proportional with:

$$e^{2n} \frac{(Mm)^n}{(M+m)^{n-1}} \quad , \quad M, m \to \infty \tag{11}$$

Therefore the coupling constant will be written as (compare with chapter 3):

$$f^2 = \frac{Mm}{M+m} e^2 \tag{12}$$

and one proves:

$$\lim_{\substack{M,m\to\infty \\ \vec{p}',\vec{p},f^2 \text{ fixed}}} \frac{T^{++,++(2n)}}{M+m} \to f^{2n} \text{ constant} .$$

In the same way, the diagram of fig. V.3 gives:

$$\frac{T^{(4)}\ (\text{figure V.3})}{M+m} \underset{M,m\to\infty}{\to} \frac{(M\&m)^3}{M^4 m^3}$$

and contributes to $M_4^{(4)}$ and higher terms in the $\frac{1}{M}$, $\frac{1}{m}$ expansion.

We will not calculate the remaining fourth order contributions explicitly, but rather give the total result*:

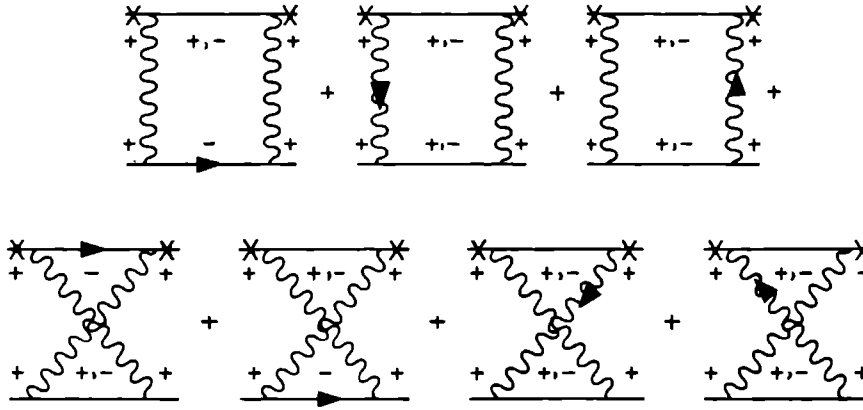


Figure V.4.

$$\to \int \frac{d^3q}{(2\pi)^3} \{2e^4 m\mu^2 \left(\frac{1}{\varepsilon^2\varepsilon'^4} + \frac{1}{\varepsilon^4\varepsilon'^2} \right) + \frac{2Me^4}{\varepsilon^2\varepsilon'^2} \} \qquad (13)$$

Again several terms are contributing to $W_1^{(4)}$, however, in the sum they cancel each other, leaving only the term of eq. (13) which contributes to $W_2^{(4)}$.

The LS potential $V_2^{(4)}$ is obtained by multiplying with $\frac{1}{4Mm}$ ($\simeq N_T(\vec{p}',\sqrt{s})N_T(\vec{p},\sqrt{s})$); once more we will include this contribution in

---

*For these diagrams one can neglect those with a negative energy state and approximate the vertices by:

$$(\Gamma_\mu\Gamma^\mu)^{++,++} \simeq (\Gamma_0\Gamma^0)^{++,++} \simeq 4Mm \quad .$$

$V^{(2)}$ by an additional $V_2^{(2)}$ term which is zero on shell:

$$\frac{e^2}{\vec{k}^2+\mu^2}\left(\frac{M+m}{4Mm^2}\left(q_s^2-\vec{q}^2-\frac{\vec{k}^2}{4}\right)+\frac{M+m}{2M^2m}\left(q_s^2-\vec{q}^2-\frac{\vec{k}^2}{4}\right)\frac{\mu^2}{\vec{k}^2+\mu^2}\right) \qquad (14)$$

This correction contributes to the central part of the potential and is not symmetric in M and m; however, adding the (asymmetric) retardation term of eq. (10) and using the LS-equivalence:

$$\frac{(\vec{q}\cdot\vec{k})^2}{m^2(\vec{k}^2+\mu^2)^2} \equiv \frac{\mu^2}{2m^2(\vec{k}^2+\mu^2)^2}\left(\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2\right)+\frac{\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2}{4m^2(\vec{k}^2+\mu^2)}$$

one obtains:

retardation + fourth order corrections →
(eq. 10) (eq. 14)

$$\frac{e^2}{\vec{k}^2+\mu^2}\left(\frac{1}{2Mm}\left(q_s^2-\vec{q}^2-\frac{\vec{k}^2}{4}\right)\frac{\mu^2}{\vec{k}^2+\mu^2}+\frac{1}{4Mm}\left(q_s^2-\vec{q}^2-\frac{\vec{k}^2}{4}\right)\right) \qquad (15)$$

which is symmetric in m and M.

When 6th and higher order Feynman diagrams give no additional contributions, the first three terms in the $\frac{1}{m}$, $\frac{1}{M}$ expansion of the LS potential for spin-1/2 - spin-1/2 scattering are:

$$\begin{aligned}V_0+V_1+V_2 = \frac{e^2}{\vec{k}^2+\mu^2}\Bigg(&1+\left(\frac{1}{4M^2}+\frac{1}{4m^2}+\frac{1}{2Mm}\right)\vec{q}^2+\left(\frac{1}{4m^2}+\frac{1}{4M^2}\right)q_s^2\\&-\left(\frac{1}{8Mm}+\frac{\kappa_1}{4m^2}+\frac{1}{16m^2}+\frac{\kappa_2}{4M^2}+\frac{1}{16M^2}\right)\vec{k}^2-\frac{1}{2Mm}\left(\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2\right)\frac{\mu^2}{\vec{k}^2+\mu^2}\Bigg)\\&+\frac{e^2}{\vec{k}^2+\mu^2}\Bigg\{\left(-\frac{1}{4m^2}+\frac{(1+\kappa_1)}{2m^2}+\frac{(1+\kappa_1)}{2Mm}\right)i\vec{\sigma}_1\cdot(\vec{k}\times\vec{q})\\&\quad+\left(-\frac{1}{4M^2}+\frac{(1+\kappa_2)}{2M^2}+\frac{(1+\kappa_2)}{2Mm}\right)i\vec{\sigma}_2\cdot(\vec{k}\times\vec{q})\\&\quad-\frac{1}{4Mm}(1+\kappa_1)(1+\kappa_2)\left(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2-(\vec{k}\cdot\vec{\sigma}_1)(\vec{k}\cdot\vec{\sigma}_2)\right)\Bigg\}\end{aligned} \qquad (16)$$

Before discussing some characteristics of this potential, we note that one can obtain the same potential in the Coulomb gauge, when $\frac{\Gamma_\mu\Gamma^\mu}{\vec{k}^2-i\varepsilon}$ is replaced by: $\frac{\Gamma_0\Gamma^0}{\vec{k}^2-i\varepsilon}+\frac{\vec{\Gamma}\cdot\vec{\Gamma}-(\vec{\Gamma}\cdot\hat{k})(\vec{\Gamma}\cdot\hat{k})}{k^2-i\varepsilon}$

In that case the second order potential of eq. (10) would give the same result, except for the retardation term (instantaneous Coulomb) and the additional $\frac{(\vec{\Gamma}\cdot\hat{k})(\vec{\Gamma}\cdot\hat{k})}{k^2}$ term (longitudinal photon):

$$(V_0^{(2)} + V_1^{(2)} + V_2^{(2)})_{\text{Coulomb gauge},\mu=0} =$$

$$\frac{e^2}{\vec{k}^2} * \left\{ 1 + \left(\frac{1}{4M^2} + \frac{1}{4m^2} + \frac{3}{4Mm} \right)\vec{q}^2 + \left(\frac{1}{4M^2} + \frac{1}{4m^2} - \frac{1}{4Mm} \right)q_s^2 \right.$$

$$\left. - \left(\frac{1}{16m^2} + \frac{1}{16M^2} + \frac{1}{16Mm} + \frac{\kappa_1}{4m^2} + \frac{\kappa_2}{4M^2} \right)\vec{k}^2 - \frac{(\vec{q}\cdot\vec{k})^2}{Mmk^2} \right\}$$

$$+ (\text{spin dependent terms of eq. (7)}) \qquad (17)$$

As for the corrections of fig. V.4, eq. (17) is the total potential since there are no photon poles for the $\Gamma_0\Gamma^0$ part of this vertex and the other terms do not contribute to $W_1^{(4)}$ and $W_2^{(4)}$ (see note on page 186). Substituting into eq. (17):

$$\frac{(\vec{q}\cdot\vec{k})^2}{Mmk^4} \equiv \frac{\vec{q}^2 + \frac{\vec{k}^2}{4} - q_s^2}{4Mmk^2}$$

will give again eq. (16) ($\mu = 0$).

In the spin dependent terms of eq. (16) one recognizes the different interaction mechanisms from a charged moving particle with mass m and magnetic moment $\frac{1+\kappa_1}{2m}$ with the fields produced by a moving charged particle with mass M and magnetic moment $\frac{1+\kappa_2}{2M}$:

$$-\frac{e^2}{4Mm}\frac{(1+\kappa_1)(1+\kappa_2)}{\vec{k}^2}\left(\vec{k}^2\vec{\sigma}_1\cdot\vec{\sigma}_2 - (\vec{k}\cdot\vec{\sigma}_1)(\vec{k}\cdot\vec{\sigma}_2)\right)$$

$\Leftrightarrow$ the interaction energy of the magnetic moment of particle 1 with the B field produced by the magnetic moment of particle 2.

$$\frac{e^2}{\vec{k}^2}\frac{(1+\kappa_1)}{2m^2} i\vec{\sigma}_1\cdot(\vec{k}\times\vec{q})$$

$\Leftrightarrow$ the interaction energy of the moving magnetic moment of particle 1

with the electric field produced by particle 2.

$$\frac{e^2}{\vec{k}^2}\,\frac{(1+\kappa_1)}{2Mm}\,i\vec{\sigma}_1\cdot(\vec{k}\times\vec{q})$$

⇔ the interaction energy of the magnetic moment of particle 1 with the magnetic field produced by the moving charge of particle 2.

$$\frac{e^2}{\vec{k}^2}\,\frac{(1+\kappa_2)}{2Mm}\,i\vec{\sigma}_2\cdot(\vec{k}\times\vec{q})$$

⇔ the interaction energy of the charge of particle 1 with the magnetic field produced by the magnetic moment of particle 2.

$$\frac{e^2}{\vec{k}^2}\,\frac{(1+\kappa_2)}{2M^2}\,i\vec{\sigma}_2\cdot(\vec{k}\times\vec{q})$$

⇔ the interaction energy of the charge of particle 1 in the electric field produced by the moving magnetic moment of particle 2.

$$-\frac{1}{4m^2}\,\frac{e^2}{\vec{k}^2}\,i\vec{\sigma}_1\cdot(\vec{k}\times\vec{q}) \quad , \quad -\frac{1}{4M^2}\,\frac{e^2}{\vec{k}^2}\,i\vec{\sigma}_2\cdot(\vec{k}\times\vec{q})$$

⇔ Thomas terms which describe the precession of the spins of particles 1 and 2.

The central part of the potential contains several contributions. It is convenient to rewrite it into the form:

$$\frac{e^2}{\vec{k}^2}\left(1+\left(\frac{1}{4M^2}+\frac{1}{4m^2}+\frac{1}{2Mm}\right)\left(\vec{q}^2+\frac{\vec{k}^2}{4}\right)+\left(\frac{1}{4m^2}+\frac{1}{4M^2}\right)q_s^2\right.$$
$$\left.-\left(\frac{1}{4Mm}+\frac{(1+\kappa_1)}{4m^2}-\frac{1}{8m^2}+\frac{(1+\kappa_2)}{4M^2}-\frac{1}{8M^2}\right)\vec{k}^2\right) \qquad (18)$$

After transformation to configuration space the $\left(\vec{q}^2+\frac{\vec{k}^2}{4}\right)$ term will contain no $\delta^{(3)}(\vec{r})$ function and therefore:

$$V_c(\vec{r}) = (\text{r-dependent and momentum dependent terms})$$
$$-e^2\left(\frac{1}{4Mm}+\frac{(1+\kappa_1)}{4m^2}-\frac{1}{8m^2}+\frac{(1+\kappa_2)}{4M^2}-\frac{1}{8M^2}\right)\delta^{(3)}(\vec{r}) \qquad (19)$$

The terms:

$$\left(\frac{(1+k_1)}{4m^2} - \frac{1}{8m^2}\right)\delta^{(3)}(\vec{r}) \quad \text{and} \quad \left(\frac{(1+k_2)}{4M^2} - \frac{1}{8M^2}\right)\delta^{(3)}(\vec{r})$$

are the Darwin terms for the two spin-1/2 particles caused by the Zitterbewegung.

The remainder:

$$\frac{e^2}{\vec{k}^2}\left(1 + \left(\frac{1}{4M^2} + \frac{1}{4m^2} + \frac{1}{2Mm}\right)\left(\vec{q}^2 + \frac{\vec{k}^2}{4}\right) + \left(\frac{1}{4M^2} + \frac{1}{4m^2}\right) q_s^2 - \frac{1}{4Mm}\vec{k}^2\right) \tag{20}$$

is exactly equal to the central potential found in the spin-0 - spin-0 case: eq. (4.14). (The Darwin term for spin-0 - spin-0 is of order $\left(\frac{1}{m^4}, \frac{1}{M^4}\right)$ and therefore does not appear in this order; [BD 65] page 203.) The central potential contains the static Coulomb-potential - interaction between the charge of particle 1 and the electric field produced by particle 2 - and also terms which represent the interaction between the moving charge and the magnetic field. Also relativistic corrections are present (for instance the well-known replacement $\sqrt{p^2+m^2} \to m + \frac{p^2}{2m} - \frac{p^3}{8m^3} + \ldots$). In the limit $M \to \infty$, for electron-proton scattering with $\kappa_1 = 0$, eq. (16) can be rewritten into the form of eq. (2.51) with:

$$\alpha = \frac{1}{4} \quad , \quad \beta = \frac{1}{16} \quad , \quad \gamma = \frac{1}{4} \quad , \quad \delta = 0 \tag{21}$$

which, using eq. (2.54), will give the correct hydrogen atom with an infinitely heavy proton: eq. (2.41). However, we can do more. Neglecting terms proportional with $\frac{1}{M^2}$ and proportional with the magnetic moment of the proton one obtains for a Dirac electron ($\kappa_1 = 0$):

$$V = \frac{e^2}{\vec{k}^2}\left(1 + \left(\frac{1}{4m^2} + \frac{1}{2Mm}\right)\vec{q}^2 + \frac{1}{4m^2} q_s^2 - \left(\frac{1}{8Mm} + \frac{1}{16m^2}\right)\vec{k}^2\right)$$
$$+ \frac{e^2}{\vec{k}^2}\left(\frac{1}{2Mm} + \frac{1}{4m^2}\right) i\sigma_{el} \cdot (\vec{k}\times\vec{q}) \tag{22}$$

This interaction must reproduce the first order reduced mass corrections to the fine structure levels. A particle with reduced mass m_r ($= \frac{mM}{m+M}$) in the central potential:

$$V = -\frac{e^2}{\vec{k}^2}\,(1 + \alpha\vec{q}^{\,2} - \beta\vec{k}^{\,2} + \gamma q_s^2)$$

will give bound states:

$$B_{n,\ell} = m_{red}\,\{-\frac{1}{2}\frac{\alpha_f^2}{n^2} + \frac{\alpha_f^4}{2n^3}\,(m_{red}^2\,\frac{(2\alpha+2\gamma) - \frac{1}{4}\frac{M^2+m^2-Mm}{M^2m^2}}{n}$$

$$+ m_{red}^2(2\alpha+8\beta)\delta_{\ell,0} - m_{red}^2\,\frac{8\alpha}{2\ell+1}\,) + \ldots\,\} \qquad (23)$$

The spin-orbit term will give an additional correction for $\ell \neq 0$:

$$\begin{aligned} & m_{red}^3\,\frac{\alpha_f^4}{n^3}\,(\frac{1}{2m^2} + \frac{1}{Mm})\,(\frac{1}{\ell+\frac{1}{2}} - \frac{1}{j+\frac{1}{2}}) \qquad (\ell \neq 0) \\ & = m_{red}^3\,\frac{\alpha_f^4}{n^3}\,(\frac{1}{2m^2} + \frac{1}{Mm})\,(\frac{1}{\ell+\frac{1}{2}} - \frac{1}{j+\frac{1}{2}} - \delta_{\ell,0}) \qquad (\text{all } \ell) \end{aligned} \qquad (24)$$

Adding eqs (23), (24) and expanding, keeping only terms up to α^4 and $\frac{m}{M}$, gives:

$$B_{n,j} = m_{red}\,\{-\frac{1}{2}\frac{\alpha_f^2}{n^2} + \frac{\alpha_f^4}{2n^3}\,(\frac{\frac{3}{4} - \frac{1}{4}\frac{m}{M}}{n} - \frac{1}{j+\frac{1}{2}})$$

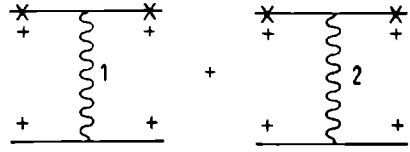
$$+ O(\alpha^5) + O(\alpha^4\,\frac{m^2}{M^2}) + \ldots\,\} \qquad (25)$$

This formula is in agreement with results obtained by others (see e.g. [Gr 69], eq. (4.4)).

Including the terms proportional with the magnetic moment of the proton in eq. (16) will give rise to the hyperfine splitting. For a complete calculation on this point we refer to [Gr 69].

Finally we wish to mention that the vector meson exchange potential of eq. (16) is additive. The proof is rather straightforward: the lowest order pseudopotential $W^{(2)}$ (eq. (5)) is additive almost by

definition:

$W^{(2)}(\mu_1\&\mu_2)$ = 

$$= W^{(2)}(\mu_1) + W^{(2)}(\mu_2) \quad .$$

The fourth order potential, calculated from fig. V.4, appears to be:

$$W^{(4)}(\mu_1\&\mu_2) = \int \frac{d^3\vec{q}}{(2\pi)^3} 2Me^4 \left(\frac{1}{\varepsilon_1^2\varepsilon_1'^2} + \frac{1}{\varepsilon_2^2\varepsilon_2'^2} + \frac{1}{\varepsilon_1^2\varepsilon_2'^2} + \frac{1}{\varepsilon_2^2\varepsilon_1'^2} \right)$$

$$+ 2me^4 \left(\frac{\mu_1^2}{\varepsilon_1^2\varepsilon_1'^4} + \frac{\mu_1^2}{\varepsilon_1^4\varepsilon_1'^2} + \frac{\mu_2^2}{\varepsilon_2^2\varepsilon_2'^4} + \frac{\mu_2^2}{\varepsilon_2^4\varepsilon_2'^2} + \frac{\mu_1^2}{\varepsilon_1^4\varepsilon_2'^2} + \frac{\mu_1^2}{\varepsilon_2^2\varepsilon_1'^4} \right.$$

$$\left. + \frac{\mu_2^2}{\varepsilon_2^4\varepsilon_1'^2} + \frac{\mu_2^2}{\varepsilon_1^2\varepsilon_2'^4} \right) \quad ; \quad \text{with } \varepsilon_1^2 = k^2 + \mu_1^2 \quad ; \quad \varepsilon_1'^2 = k'^2 + \mu_1^2$$

$$\varepsilon_2^2 = k^2 + \mu_2^2 \quad ; \quad \varepsilon_2'^2 = k'^2 + \mu_2^2$$

$$= W^{(4)}(\mu_1) + W^{(4)}(\mu_2) + (\text{mixed terms}) \qquad (26)$$

However, it is not difficult to show that this non-additive contribution is generated completely by two second order potentials which are zero on shell (and additive!):

$$\frac{4Mme^2}{\vec{k}^2+\mu_1^2} \left(\frac{M+m}{4Mm^2} (q_s^2 - \vec{q}^2 - \frac{\vec{k}^2}{4}) + \frac{M+m}{2M^2m} (q_s^2 - \vec{q}^2 - \frac{\vec{k}^2}{4}) \frac{\mu_1^2}{\vec{k}^2+\mu_1^2} \right)$$

$$+ \frac{4Mme^2}{\vec{k}^2+\mu_2^2} \left(\frac{M+m}{4Mm^2} (q_s^2 - \vec{q}^2 - \frac{\vec{k}^2}{4}) + \frac{M+m}{2M^2m} (q_s^2 - \vec{q}^2 - \frac{\vec{k}^2}{4}) \frac{\mu_2^2}{\vec{k}^2+\mu_2^2} \right)$$

In this way the LS potential $V_0 + V_1 + V_2$ becomes additive.

The electric potential as derived in the Coulomb gauge, eq. (17), is <u>not</u> additive to the vector-meson exchange potentials of eq. (16); one can verify this by explicit calculation.

Potentials for scalar meson exchange can be derived in the same

way. They appear to be additive to each other and to the vector-meson exchange potentials if one follows the procedure:

(i) calculate $V^{(2)}$ by putting <u>one</u> of the two particles on mass shell;

(ii) calculate $V_1^{(4)} + V_2^{(4)}$ from the remaining meson pole diagrams and include this contribution in $V_2^{(2)}$ as a term which is zero on energy shell and linear in the coupling constant squared;

(iii) finally the contribution from the retardation in the meson propagator can be replaced by two other terms (appendix B). It is not necessary to do this; however, applying this trick to one sub-potential and not to the others will destroy additivity. Note that it will restore the symmetry in the potential which was broken due to step (i) and (ii).

The electric potential in low energy nucleon-nucleon scattering

1. Corrections for the Coulomb amplitude

In the preceding chapter we derived the electric potential between two spin-1/2 particles and applied it to the hydrogen atom. Another field for which an accurate description of the electric interaction is necessary is low energy proton-proton scattering. Due to its long range nature the electric interaction is dominating in the forward direction. This "Coulomb peak" will grow in amplitude and width with respect to the "nuclear" part of the scattering, if the energy decreases.

In this section we will consider the modifications in the usual pp-scattering theory due to the first "relativistic" corrections in the electric potential. We will restrict the discussion to the terms which are independent of the magnetic moment because we believe that these terms are more important for lower energies than the corrections due to the electric spin-orbit and tensor force. Moreover these corrections can be calculated almost in closed form.

The potential for the two proton system will be written as (neglecting vacuum polarization):

$$V = V_{electric} + V_{nuclear} \tag{1}$$

This decomposition suggests that the electric and nuclear potential are additive. However, this is only of importance if one considers $V_{nuclear}$ as the potential between protons without charge or when one tries to calculate $V_{nuclear}$ in a non-phenomenological (physical) way (e.g. the OBE prescription). As it stands one could also consider $V_{nuclear}$ as the difference between the total potential V and the electric potential

$V_{electric}$; the only property for $V_{nuclear}$ we need in the following is its finite range (exponential decreasing for $r \to \infty$).

Neglecting spin dependent and magnetic moment dependent terms we obtain from eq. (5.16):

$$V_{electric} = \frac{e^2}{\vec{k}^2}\left(1 + \frac{\vec{q}^2}{M^2} + \frac{q_s^2}{2M^2} + \frac{\vec{k}^2}{4M^2}\right) \tag{2}$$

which becomes in configuration space:

$$V_{electric} = \frac{\alpha\left(1 + \frac{q_s^2}{2M^2}\right)}{r} - \frac{1}{2M^2}\left(\vec{\nabla}^2 \frac{\alpha}{r} + \frac{\alpha}{r}\vec{\nabla}^2\right) \tag{3}$$

The scattering amplitude can be written as:

$$f(\theta) = f_{electric}(\theta) + f_{nuclear}(\theta) \tag{4}$$

with $f_{electric}(\theta)$ the amplitude caused by $V_{electric}$ <u>alone</u>, and $f_{nuclear}(\theta)$ by <u>definition</u> the difference between $f(\theta)$ and $f_{electric}(\theta)$ (<u>not</u> the amplitude caused by $V_{nuclear}$ alone). Due to the finite range of $V_{nuclear}$ one can write $f_{nuclear}(\theta)$ in a partial wave decomposition:

$$f_{nuclear}(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell\, P_\ell(\cos\theta) \tag{5}$$

and one expects that for lower energies only a few of the coefficients f_ℓ will contribute (for $E_{lab} < 5$ MeV only f_0 is appreciable).

For the electric amplitude such a partial wave decomposition does not exist; the scattering amplitude for an $\frac{\alpha}{r}$ potential is [Ta 72]:

$$f_{\frac{\alpha}{r}}(\eta,\theta) = -\frac{\eta}{2k \sin^2\frac{1}{2}\theta} \exp\left[2i\sigma_0 - i\eta \log\left(\sin^2\tfrac{1}{2}\theta\right)\right] \tag{6}$$

with $\eta = \frac{M\alpha}{2k}$ and $\sigma_0 = \arg\Gamma(1 + i\eta)$.

It can be shown that this amplitude does not allow for an expansion in Legendre polynomials in cos θ (uniformly nor pointwise convergent)

[Ta 74]. However, it can be shown that in the sense of distributions*.

$$f_{\frac{\alpha}{r}}(\eta,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \frac{e^{2i\sigma_\ell} - 1}{2ik} P_\ell(\cos\theta) \tag{7}$$

with $\sigma_\ell = \arg \Gamma(\ell+1+i\eta)$.

The "Coulomb phase shifts" σ_ℓ are connected with the asymptotic form of the regular solution of the radial Schrodinger equation with potential $\frac{\alpha}{r}$ through:

$$F_\ell(\eta,kr) \underset{r\to\infty}{\longrightarrow} \sin(kr - \frac{1}{2}\ell\pi + \sigma_\ell - \eta \ln 2kr) + O(r^{-1}) \tag{8}$$

The Schrodinger equation with a potential of the form (3) can be solved easily by a method invented by Green [Gr 63]. Defining u_ℓ as the original radial wave function and ϕ, v_ℓ by:

$$\begin{aligned} \phi &= \frac{\alpha}{2Mr} \\ v_\ell &= (1 + 2\phi)^{1/2} u_\ell \end{aligned} \tag{9}$$

One proves.

$$v_\ell'' + \left(q_s^2 - MW - \ell(\ell+1)/r^2\right) v_\ell = 0 \tag{10}$$

with W the "pseudopotential" reading:

$$W = \frac{\alpha(1 + \frac{q_s^2}{2M^2})}{r(1+2\phi)} - \frac{1}{M}\left(\frac{\phi'}{1+2\phi}\right)^2 + \frac{2\phi}{1+2\phi}\frac{q_s^2}{M} \tag{11}$$

Noting that the electric potential as defined in eq. (3) is derived with help of the $\frac{1}{M}$ expansion up to second order, one observes that only

---

*By this we mean that both sides of the equality (7) have to be integrated with a twice continuously differentiable test function vanishing at $\cos\theta = 1$, in order to obtain convergence of the sum over ℓ [Ta 74].

terms linear in ϕ (produced by the $\frac{"q^2"}{M^2}$ part of eq. (2)) have significance and one could write:

$$W = \frac{\alpha(1 + \frac{3}{2}\frac{q_s^2}{M^2})}{r + \frac{\alpha}{M}} + O(\phi^2) \qquad (12)$$

leaving terms of order φ^2 which correspond with terms of order $\frac{1}{M^4}$ in the $\frac{1}{M}$ expansion. Expanding the denominator one arrives at:

$$W = \frac{\alpha(1 + \frac{3}{2}\frac{q_s^2}{M^2})}{r} - \frac{\alpha^2}{M}\frac{1 + \frac{3}{2}\frac{q_s^2}{M^2}}{r^2} + O(\varphi^2) \qquad (13)$$

and the radial Schrödinger equation ($k^2 = q_s^2$):

$$v_\ell'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} + \frac{\alpha^2(1 + \frac{3}{2}\frac{k^2}{M^2})}{r^2} - \frac{M\alpha(1 + \frac{3}{2}\frac{k^2}{M^2})}{r}\right] v_\ell = 0 \qquad (14)$$

Defining:

$$\eta = \frac{M\alpha(1 + \frac{3}{2}\frac{k^2}{M^2})}{2k} \quad , \quad \mu = \sqrt{(\ell + \frac{1}{2})^2 - \alpha^2(1 + \frac{3}{2}\frac{k^2}{M^2})}$$

$$\equiv \ell' + \frac{1}{2}$$

$$\kappa = -i\eta \quad , \quad z = -2i\rho = -2ikr \qquad (15)$$

eq. (14) can be written as:

$$\frac{\partial^2 v_\ell}{\partial z^2} + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right] v_\ell = 0 \qquad (16)$$

which is Whittaker's equation with regular solution [Ab 70]:

$$M_{\kappa,\mu}(z) = e^{-\frac{1}{2}z} z^{\frac{1}{2}+\mu} M(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z) \qquad (17)$$

In the limit $r \to \infty$, $z \to -i\infty$ this solution behaves like ([Ab 70], eq. 13.5.1):

$$M_{k,\mu}(z) \underset{z\to -i\infty}{\longrightarrow} -2i\,\frac{\Gamma(2+2\ell')e^{+\pi\frac{\eta}{2}}\,e^{-i\pi\frac{\ell'}{2}}}{|\Gamma(\ell'+1+i\eta)|} * \sin\left(\rho-\eta\log 2\rho+\sigma_{\ell'}-\pi\frac{\ell'}{2}\right) \tag{18}$$

Expanding ℓ' in powers of α one obtains:

$$\ell' = \ell - \frac{\alpha^2\left(1+\frac{3}{2}\frac{k^2}{M^2}\right)}{2\ell+1} + O(\alpha^4)$$

and comparing eq. (18) with eq. (8) one obtains for the "relativistic corrected Coulomb phase shifts":

$$\sigma'_\ell = \sigma_{\ell'} - \pi\frac{\ell'-\ell}{2} \simeq \sigma_\ell + \frac{\alpha^2\left(1+\frac{3}{2}\frac{k^2}{M^2}\right)}{2\ell+1}\left(\frac{\pi}{2}-\frac{\partial\sigma_\ell}{\partial\ell}\right) + O(\alpha^4) \tag{19}$$

Writing $\sigma'_\ell = \sigma_\ell + \Delta_\ell$ one obtains for the "relativistic corrected Coulomb amplitude":

$$f_{\text{electric}}(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\,\frac{e^{2i(\sigma_\ell+\Delta_\ell)}-1}{2ik}P_\ell(\cos\theta)$$

$$= -\frac{\eta}{2k\sin^2\frac{1}{2}\theta}\exp\left[2i\sigma_0 - i\eta\log\left(\sin^2\tfrac{1}{2}\theta\right)\right]$$

$$+\sum_{\ell=0}^{\infty}(2\ell+1)\,e^{2i\sigma_\ell}\,\frac{e^{2i\Delta_\ell}-1}{2ik}P_\ell(\cos\theta) \tag{20}$$

(in the sense of distributions).

We did not succeed in evaluating the last term of eq. (20) in closed form. However, for laboratory energies > 1 MeV one can approximate:

$$\Delta_\ell \simeq \frac{\alpha^2\left(1+\frac{3}{2}\frac{k^2}{M^2}\right)}{2\ell+1}\,\frac{\pi}{2} + O(\alpha^2\eta)$$

and

$$\sigma_\ell \simeq \sigma_0 + O(\eta)$$

with an error less than 15%, and obtain:

$$f_{electric} \overset{E_{lab}>1\ MeV}{\simeq} - \frac{\eta}{2k \sin^2 \frac{1}{2}\theta} \exp\left[2i\sigma_0 - i\eta \log\left(\sin^2 \tfrac{1}{2}\theta\right)\right]$$

$$+ e^{2i\sigma_0} \frac{\alpha^2\left(1 + \frac{3}{2}\frac{k^2}{M^2}\right)\pi}{2k} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)$$

$$= - \frac{\eta}{2k \sin^2 \frac{\theta}{2}} \exp\left[2i\sigma_0 - i\eta \log\left(\sin^2 \tfrac{1}{2}\theta\right)\right]$$

$$+ e^{2i\sigma_0} \frac{\alpha^2\left(1 + \frac{3}{2}\frac{k^2}{M^2}\right)\pi}{2k \sin \frac{\theta}{2}} \qquad (21)$$

The differences between eq. (21) and its extreme non-relativistic limit eq. (6) are:

(i) The use of $\eta = M\alpha\left(1 + \frac{3}{2}\frac{k^2}{M^2}\right)/2k$ instead of $\eta = \frac{M\alpha}{2k}$. One can verify that this corresponds with $\eta = \frac{\alpha}{V_{lab}}$, $V_{lab} = P_{lab}/\sqrt{P_{lab}^2 + M^2}$, expanded in M and expressed in the center of mass momentum k. In fact this "ad hoc" substitution for η has already been used during a long time with the purpose to simulate some relativistic effects of the Coulomb interaction.

(ii) An additional correction to be added to the "classical" expression:

$$e^{2i\sigma_0} \frac{\alpha^2\left(1 + \frac{3}{2}\frac{k^2}{M^2}\right)}{2k \sin \frac{\theta}{2}}$$

Its contribution to the total electric amplitude is small, approximately:

$$\sqrt{E_{lab}(MeV)} \times \sin \frac{\theta}{2} \times 1\ ‰ \quad .$$

Unfortunately in the region where the Coulomb interaction dominates (small energies, small angles) this correction is also small (errors on the experimental quantities are in the order of 1 ‰).

In order to obtain a feeling about the magnitude of the second correction in proton-proton scattering, we calculated cross sections from the amplitude:

$$f(\theta) = f_{\frac{\alpha}{r}}(\theta) + e^{2i\sigma_0} \frac{\alpha^2\pi}{2k \sin\frac{\theta}{2}} + f_{nuclear}(\theta) \qquad (22)$$

with the nuclear amplitude approximated by its s-wave partial wave amplitude:

$$f_{nuclear}(\theta) = e^{2i\sigma_0} \frac{e^{2i\delta_0}-1}{2ik} \qquad (23)$$

and used the effective range approximation for δ_0:

$$C_0^2 k \cot\delta_0 + 2\eta k h(\eta) = -\frac{1}{a} + \frac{1}{2} rk^2 \qquad (24)$$

with a = -7.8 fm, r = 2.7 fm.

These cross sections (for energies 0.5 - 10 MeV, CM angles 20^o-90^o) were compared with those of the amplitude:

$$f(\theta) = f_{\frac{\alpha}{r}}(\theta) + f_{nuclear}(\theta) \qquad (25)$$

with $f_{nuclear}(0)$ approximated by an s- and a p-wave contribution:

$$f_{nuclear}(\theta) = e^{2i\sigma_0} \frac{e^{2i(\delta_0+\Delta_0)}-1}{2ik} + 3\, e^{2i\sigma_1} \frac{e^{2i\Delta_1}-1}{2ik} \cos\theta \qquad (26)$$

We searched for Δ_0 and Δ_1 to obtain an impression to what extend the relativistic correction can be simulated by a (nuclear) s- and p-wave correction (table VI.1).

From table VI.1 we conclude that it is quite possible to simulate the relativistic correction on the Coulomb amplitude by a change in the nuclear s- and p-wave (better than 1:10000 in the cross sections). The changes in the phases are in the order of 10^{-4} radians. The errors in the experimental phase-shifts (from single energy analysis) are ranging from 10^{-4} radians below 1 MeV up to $2\cdot10^{-3}$ radians at 10 MeV for the

Energy (MeV)	mean rel.dev. (‰) $\Delta_0=\Delta_1=0$	mean rel.dev. (‰) fitted Δ_0,Δ_1	Δ_0 (rad)	Δ_1 (rad)
0.5	0.73	0.07	$9.4\ 10^{-5}$	$2.7\ 10^{-5}$
1.0	0.35	0.08	$1.1\ 10^{-4}$	$3.9\ 10^{-5}$
2.0	0.27	0.08	$1.2\ 10^{-4}$	$5.7\ 10^{-5}$
3.0	0.25	0.08	$1.2\ 10^{-4}$	$6.7\ 10^{-5}$
4.0	0.25	0.07	$1.3\ 10^{-4}$	$7.3\ 10^{-5}$
5.0	0.24	0.07	$1.3\ 10^{-4}$	$7.6\ 10^{-5}$
6.0	0.24	0.07	$1.3\ 10^{-4}$	$7.8\ 10^{-5}$
7.0	0.24	0.06	$1.3\ 10^{-4}$	$7.8\ 10^{-5}$
8.0	0.24	0.06	$1.3\ 10^{-4}$	$7.8\ 10^{-5}$
9.0	0.23	0.06	$1.3\ 10^{-4}$	$7.8\ 10^{-5}$
10.0	0.23	0.06	$1.3\ 10^{-4}$	$7.8\ 10^{-5}$

Table VI.1: Simulation of the relativistic correction of the Coulomb amplitude by nuclear s- and p-waves (Δ_0 and Δ_1). The second column gives the mean relative deviation between cross sections calculated from eq. (22) and from eq. (25) with the same $f_{nuclear}$. The remaining columns give the corrections on the nuclear s- and p-waves needed to obtain an optimal fit to the relativistic correction of the electric amplitude.

$^1S_0$ phase shift and from 10^{-4} radians up to $3\cdot10^{-4}$ radians for the central p-wave combination. Therefore it is unlikely that the second correction of the Coulomb amplitude is observable in single energy fits of proton-proton scattering data. Perhaps in a multi-energy fit this effect is observable since the corrections on the s- and p-wave

phase shifts are rather independent on the energy (typical for a $\frac{1}{r^2}$ potential) and since the phase shifts of a finite range potential are strongly dependent on the energy. Definite statements, however, cannot be made until such a fit has been performed. In the next section we will study the changes in the effective range functions caused by the "relativistic" Coulomb potential, which can be used in a multi-energy fit.

2. Corrections on the effective range functions

We recall the definitions of the effective range functions for potentials $V_L(r)$ with an $\frac{\alpha}{r}$ tail:

1. The regular solution V_R, with boundary conditions at $r = 0$: $V_R(0)=0$ must be a real analytical function of k^2; in the limit $r \to \infty$ it defines the phase shift Δ_ℓ and the function N:

$$V_R(r) \xrightarrow[r\to\infty]{} N \sin (kr - \eta \ln 2kr - \frac{1}{2}\ell\pi + \sigma_\ell + \Delta_\ell) \tag{27}$$

2. The irregular solution V_I, which is unequal to zero in the origin, also a real analytical function of k^2, satisfying the Wronskian condition $V_R'V_I - V_RV_I' = 1$, defines the function H:

$$V_I(r) \xrightarrow[r\to\infty]{} \frac{1}{Nk} \cos (kr - \eta \ln 2kr - \frac{1}{2}\ell\pi + \sigma_\ell + \Delta_\ell)$$
$$- HN \sin (kr - \eta \ln 2kr - \frac{1}{2}\ell\pi + \sigma_\ell + \Delta_\ell) \tag{28}$$

If the total potential V can be written as:

$$V(r) = V_L(r) + V_S(r) \tag{29}$$

then the total scattering amplitude can be written as:

$$f_{tot}(\theta) = f_{\frac{\alpha}{r}}(\theta) + \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{2i\sigma_\ell}\, \frac{e^{2i\Delta_\ell}-1}{2ik}\, P_\ell(\cos\theta)$$

$$+ \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{2i(\sigma_\ell+\Lambda_\ell)}\, \frac{e^{2i\delta_\ell}-1}{2ik}\, P_\ell(\cos\theta) \qquad (30)$$

with δ_ℓ the phase shift of V with respect to the solutions of $V_L(r)$. Then δ_ℓ satisfies:

$$\frac{1}{N^2k}\cot\delta_\ell + H = \text{analytic function of } k^2 \text{ except for singularities whose positions are determined by } V_s \text{ alone and except for isolated poles.} \qquad (31)$$

In our case we take:

$$V_L(r) = V_{electric}(r)$$

and using Green's trick we transform it to a local potential, arriving at eq. (13). However, in doing so, one also transforms the wave functions according to eq. (9). Since $\phi(r)$ vanishes in the limit $r \to \infty$ this transformation does <u>not</u> affect the $r \to \infty$ limit of the wave functions. Also the "regular" or "irregular" nature and the analyticity in k^2 of a wave functions are not affected (to avoid complications on this point we will assume that the non-local corrections in $V_{electric}$ are small). There is a slight complication in the effective range formalism since for non-local potentials the Wronskian between <u>two</u> independent solutions is not a constant (in r). This condition, usually applied to the wave functions at the origin and for local potentials therefore also holding for $r \to \infty$, must be replaced by the condition:

$$W(V_R, V_I) \xrightarrow[r\to\infty]{} 1$$

Another (and equivalent) method appears to be to first transform the

potential with Green's trick and to calculate the effective range functions with this (local) pseudopotential.

Using the arguments of section 1 one arrives at:

$$W_{electric} = \frac{\alpha(1 + \frac{3}{2}\frac{k^2}{M^2})}{r} - \frac{\alpha^2}{M}\frac{1 + \frac{3}{2}\frac{k^2}{M^2}}{r^2} \tag{32}$$

and using:

$$\eta = \frac{M\alpha(1 + \frac{3}{2}\frac{k^2}{M^2})}{2k} \quad , \quad \mu = \ell' + \frac{1}{2} \approx \sqrt{(\ell + \frac{1}{2})^2 - \alpha^2(1 + \frac{3}{2}\frac{k^2}{M^2})} \quad ,$$

$$\kappa = -i\eta \quad , \quad z = -2i\rho = -2ikr$$

the radial Schrödinger equation for the ℓ'th partical wave can be written as:

$$\frac{\partial^2 V}{\partial z^2} + \left[-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right] V = 0 \tag{33}$$

Two independent solutions are $M_{\kappa,\mu}(z)$ and $M_{\kappa,-\mu}(z)$ ([Ab 70], eq. 13.1.32, 33, 34):

$$\begin{aligned} M_{\kappa,\mu}(z) &= e^{i\rho}\ (-2i\rho)^{\ell'+1}\ M(\ell' + 1 + i\eta,\ 2\ell' + 1,\ -2i\rho) \\ M_{\kappa,-\mu}(z) &= e^{i\rho}\ (-2i\rho)^{-\ell'}\ \ M(-\ell' + i\eta,\ -2\ell',\ -2i\rho) \end{aligned} \tag{34}$$

(the solution $M_{\kappa,-\mu}$ does not exist for ℓ' integer).
A closer inspection of the properties of the Kummer functions at the right hand side of eq. (34) shows that $M_{\kappa,\mu}$ and $M_{\kappa,-\mu}$ are entire and real functions of k^2 and r, except for the factor $(-2i\rho)^{\ell'+1}$ and $(-2i\rho)^{-\ell'}$.

Since M(a,b,0) = 1 we define the regular and irregular solutions V_R and V_I:

$$V_R(r) = \frac{2^{\ell'}\Gamma(\ell'+1)}{\Gamma(2\ell'+1)} \left(\frac{1}{2k}\right)^{\ell'+1} M_{\kappa,\mu}(z)$$

$$V_I(r) = \frac{\Gamma(2\ell')}{2^{\ell'-1}\Gamma(\ell')} \left(\frac{1}{2k}\right)^{-\ell'} M_{\kappa,-\mu}(z) \qquad (35)$$

V_R and V_I are real analytic in k^2 and satisfy:

$$V_R' V_I - V_I' V_R = 1$$

Using eq. 13.5.1 from [Ab 70] one proves:

$$V_R(r) \xrightarrow[r\to\infty]{} \frac{\Gamma(\ell'+1)e^{\pi\frac{\eta}{2}}}{|\Gamma(\ell'+1+i\eta)|k^{\ell'+1}} \sin\left(\rho - \eta\log 2\rho - \pi\frac{\ell'}{2} + \sigma_{\ell'}\right) \qquad (36)$$

therefore:

$$\Delta_\ell = \sigma_{\ell'} - \sigma_\ell - \frac{\pi}{2}(\ell'-\ell)$$

and

$$N = \frac{\Gamma(\ell'+1)e^{\pi\frac{\eta}{2}}}{|\Gamma(\ell'+1\pm i\eta)|k^{\ell'+1}} \qquad (37)$$

Similarly:

$$V_I(r) \xrightarrow[r\to\infty]{} \frac{1}{Nk}\cos\left(\rho - \eta\log 2\rho - \pi\frac{\ell'}{2} + \sigma_{\ell'}\right) - HN\sin\left(\rho - \eta\log 2\rho - \pi\frac{\ell'}{2} + \sigma_{\ell'}\right)$$

with:

$$H = 4\pi k^{2\ell'+1}\frac{\Gamma(2\ell')\Gamma(-2\ell')}{|\Gamma(-\ell'\pm i\eta)|^2\Gamma(\ell'+1)\Gamma(\ell')} \times \left(\frac{1}{e^{-2i\pi\ell'-\pi\eta} - e^{\pi\eta}} + \frac{1}{e^{2i\pi\ell'-\pi\eta} - e^{\pi\eta}}\right) \qquad (38)$$

In this form these functions can be used in the effective range expansion eq. (31). However, one observes that for ℓ' = integer, H does not exist. This corresponds with the fact that for such ℓ' the solution $M_{\kappa,-\mu}(z)$ is not valid anymore and must be replaced by a "logarithmic"

type of solution ([Ab 70]; eq. 13.1.6). In our case ℓ' is very close to an integer and therefore H will be very big, causing large terms in the right hand side of eq. (31). Now we can use the property that the irregular solution is defined up to a multiple of the regular solution. Defining H_I as a real analytic function of k^2 one observes that the substitution: $V_I \to V_I + H_I V_R$ will not affect the analytical properties and boundary conditions of the irregular solution, but would lead to the effective range function:

$H - H_I$ instead of: H .

It is not difficult to prove that:

$$H \underset{\ell' \to \ell}{\sim} \frac{1}{\ell' - \ell} \frac{(2\ell+1)!!(2\ell-1)!!2^{2\ell}}{(2\ell)!(2\ell+1)!} \frac{1}{\eta k} \times (\ell^2 k^2 + (\eta k)^2)((\ell-1)^2 k^2 + (\eta k)^2) \ldots (\eta k)^2 \qquad (39)$$

Note that the right hand side of eq. (39) is an analytic function in k^2 and therefore we define:

$$H_I(k^2) = \frac{1}{\ell' - \ell} \frac{(2\ell+1)!!(2\ell-1)!!2^{2\ell}}{(2\ell)!(2\ell+1)!} \frac{1}{\eta k} \times (\ell^2 k^2 + (\eta k)^2)((\ell-1)^2 k^2 + (\eta k)^2) \ldots (\eta k)^2 \qquad (40)$$

The effective range function:

$$\frac{1}{N^2 k} \cot \delta_\ell + H - H_I \qquad (41)$$

is finite for $\ell' = \ell$ and one can show that for $\ell' = \ell = 0$ it reduces to the well known form:

$$k C_0^2 \cot \delta_0 + 2\eta k\, h(\eta) - 2\eta k \qquad (42)$$

The factor $2\eta k$ is conventionally put into the right hand side of the effective range identity, eq. (31), corresponding with the additional replacement: $H - H_I \to H - H_I - \lim_{k \to 0} (H - H_I)$.

APPENDIX A

Contributions from 6^{th} order diagrams in φ^3 theory to the potential

The 6^{th} order diagrams which contribute to the BS scattering amplitude are (without self energy graphs):

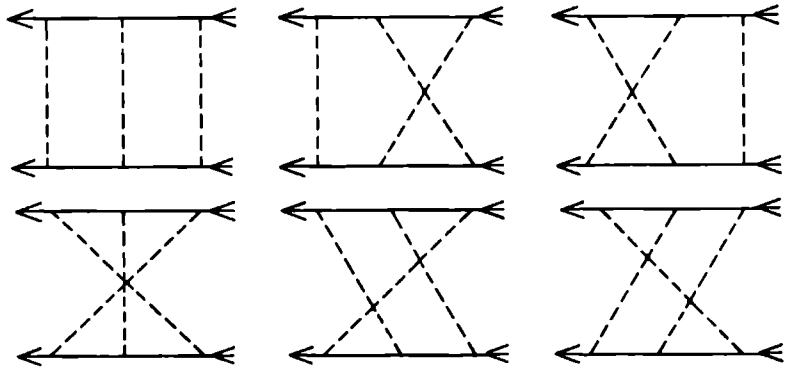


Fig. A1: sixth order contributions to M.

These diagrams, involving the integration in two intermediate four-momenta could be written as a sum of diagrams involving only three dimensional integrations, as done in chapter 3 (see fig. III.6). However, this would require several pages and it is also not necessary for our purposes.

We merely note that the potential of eq. (3.36) in a LS equation will generate, iterated with the LS propagator, already some of the 6^{th} order diagrams (up to second order in the $\frac{1}{M}$ expansion); see fig. A2. We can write these diagrams again as four-dimensional integrations; using:

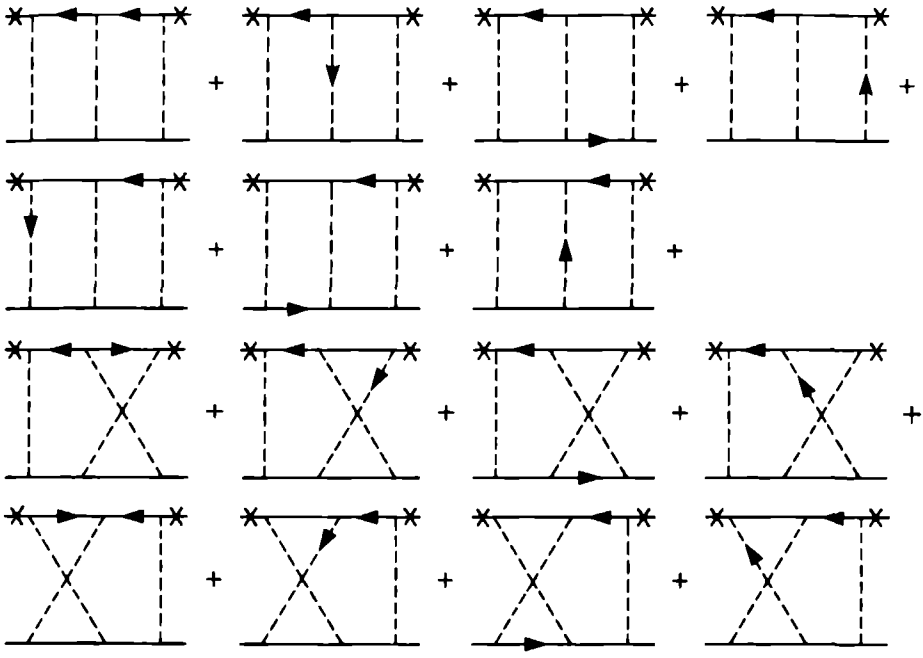


Fig. A2: sixth order contributions generated by the potential of eq. (3.36) in the LS equation.

$$p_a \equiv \int \frac{dp^0}{(2\pi)} \frac{2\pi\delta(p_a^0 - E_{\vec{p}})}{2E_{\vec{p}}\sqrt{s}(\sqrt{s} - 2E_q + i\varepsilon)} = \int \frac{dp^0}{2\pi} \, g \quad .$$

$$p_b = P - p_a$$

The diagrams of fig. A2 become:

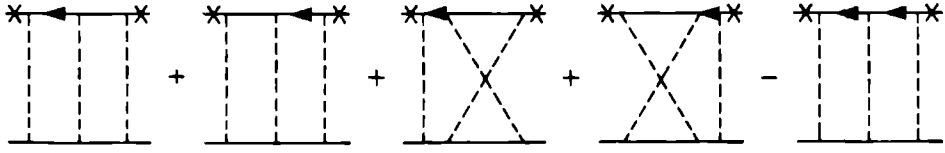


Fig. A3: same as fig. A2; written as four dimensional integrations.

The terms of fig. A2 (fig. A3) do <u>not</u> contribute to $W^{(6)}$, since they

since they are generated by $W^{(2)}$ and $W^{(4)}$. Therefore the total contribution to $W^{(6)}$ is the difference between the diagrams of fig. A1 and those of fig. A3:

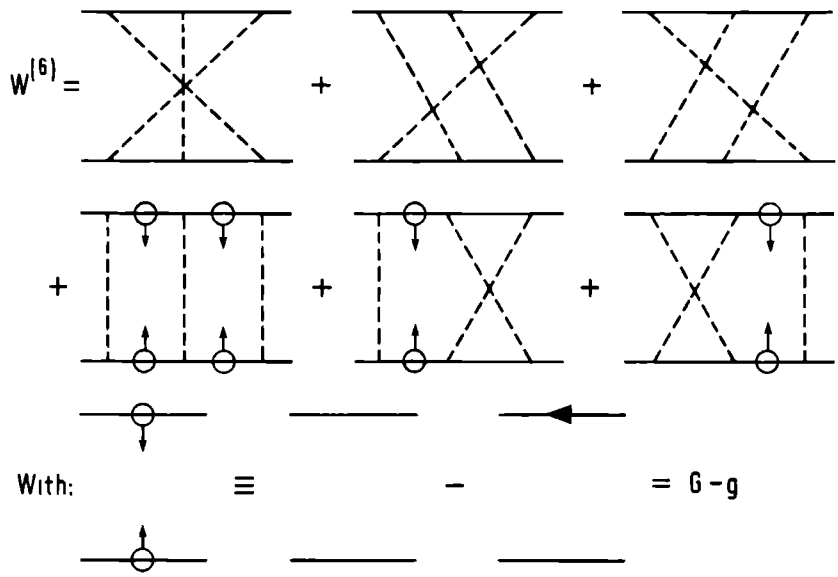


Fig. A4: the sixth order pseudopotential.

Note that fig. A4 is nothing else than the 6^{th} order contribution as derived from the pseudo-potential equation (2.25):

$$W = M^{irr} + M^{irr}(G-g)W \quad .$$

The diagrams of fig. A4 are calculated by [Mu 78].

Using the coupling constant $f^2 = g^2/M$, and using their results one proves:

$$\lim_{\substack{M\to\infty \\ \vec{p}',\vec{p},f^2 \text{ fixed}}} \frac{W^{(6)}}{M} \sim \frac{f^6}{M^3}$$

and therefore:

$$\lim_{\substack{M\to\infty \\ \vec{p}',\vec{p},f^2 \text{ fixed}}} M\, V^{(6)} \sim \frac{f^6}{M^3}$$

which shows that*:

$$V_0^{(6)} = V_1^{(6)} = V_2^{(6)} = 0.$$

*In [Mu 78] is used g^2 and is proven that the diagrams of fig. A4 contribute separately to order $\frac{g^6}{M^4}$; however, in the summation the leading terms cancel giving a contribution of order $\frac{g^6}{M^5}$.

APPENDIX B

Equivalent potentials in the Lippmann-Schwinger equation

We will prove that a potential:

$$V(\vec{p}',\vec{p}) = \sum_i \frac{g_i^2}{\vec{k}^2+m_i^2} \tag{B.1}$$

is equivalent with:

$$V'(\vec{p}',\vec{p}) = \sum_i \frac{g_i^2}{\vec{k}^2+m_i^2} * \left(1 + \frac{\lambda}{2}(\vec{p}^2+\vec{p}'^2-2q_s^2) + \frac{\lambda m_i^2}{\vec{k}^2+m_i^2}(\vec{p}^2+\vec{p}'^2-2q_s^2) - \lambda \frac{(\vec{p}^2-\vec{p}'^2)^2}{\vec{k}^2+m_i^2}\right) \tag{B.2}$$

for on shell momenta ($= q_s^2$) and up to first order in λ. Writing:

$$V' = V + \lambda W \tag{B.3}$$

(note: on shell $V' = V$) and the formal solution of the L.S. equation:

$$\begin{aligned} &T = V + VGV + VGVGV + \ldots \\ \text{and} \qquad &T' = V' + V'GV' + V'GV'GV' + \ldots \end{aligned} \tag{B.4}$$

with: $$G_{(p,q_s)} = \frac{2M_r}{q_s^2-\vec{p}^2+i\varepsilon}$$

it will be clear that it is sufficient to prove:

$$WGVG \ldots V + VGWG \ldots V + \ldots + VGVG \ldots W = 0 \tag{B.5}$$

to obtain:

$$T' = T + O(\lambda^2) \quad .$$

Potentials V and W are sums of subpotentials with different masses and coupling constants. To prove (B.5) it will be sufficient to prove that:

$$\sum_{i,j..k} W_i G V_j G..V_k + V_i G W_j G..V_k + \ldots + V_i G V_j G..W_k = 0 \tag{B.6}$$

with: $V_1 = \frac{1}{\vec{k}^2+m_1^2}$

and $W_1 = \frac{\vec{p}^2+\vec{p}'^2-2q_s^2}{2(\vec{k}^2+m_1^2)} + m_1^2 \frac{\vec{p}^2+\vec{p}'^2-2q_s^2}{(\vec{k}^2+m_1^2)^2} - \frac{(\vec{p}^2-\vec{p}'^2)^2}{(\vec{k}^2+m_1^2)^2}$

A specific term in (B.6) will be denoted with the self explaining diagram:

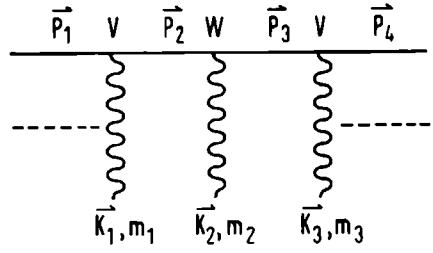


Fig. B.1

Leaving the integration in $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, .. the reduced mass, and using: $\epsilon_1^2 = \vec{k}_1^2 + m_1^2$ one proves that this diagram gives a term:

$$\cdots \frac{1}{\epsilon_1^2}\left(-\frac{1}{q_s^2-\vec{p}_3^2}\frac{1}{2\epsilon_2^2} - \frac{1}{q_s^2-\vec{p}_2^2}\frac{1}{2\epsilon_2^2} - \frac{m_2^2}{\epsilon_2^4}\frac{1}{q_s^2-\vec{p}_3^2} - \frac{m_2^2}{\epsilon_2^4}\frac{1}{q_s^2-\vec{p}_2^2} - \frac{1}{\epsilon_2^4}\frac{\vec{p}_3^2-\vec{p}_2^2}{q_s^2-\vec{p}_3^2} - \frac{1}{\epsilon_2^4}\frac{\vec{p}_2^2-\vec{p}_3^2}{q_s^2-\vec{p}_2^2}\right)\frac{1}{\epsilon_3^2}\cdots \qquad (B.7)$$

Now we can use:

$$\vec{\nabla}_2 \frac{\vec{p}_2-\vec{p}_3}{\epsilon_1^2\epsilon_2^2} = \frac{1}{\epsilon_1^2\epsilon_2^2} + \frac{2m_2^2}{\epsilon_1^2\epsilon_2^4} - \frac{2(\vec{p}_2-\vec{p}_3)\cdot(\vec{p}_2-\vec{p}_1)}{\epsilon_1^4\epsilon_2^2}$$

and

$$\vec{\nabla}_2 \frac{\vec{p}_2-\vec{p}_1}{\epsilon_1^2\epsilon_2^2} = \frac{1}{\epsilon_1^2\epsilon_2^2} + \frac{2m_1^2}{\epsilon_1^4\epsilon_2^2} - \frac{2(\vec{p}_2-\vec{p}_1)\cdot(\vec{p}_2-\vec{p}_3)}{\epsilon_1^2\epsilon_2^4}$$

so integrated with respect to $\vec{p}_2$ we can use:

$$\frac{(\vec{p}_2-\vec{p}_3)\cdot(\vec{p}_2-\vec{p}_1)}{\epsilon_1^4\epsilon_2^2} = \frac{1}{2\epsilon_1^2\epsilon_2^2} + \frac{m_2^2}{\epsilon_1^2\epsilon_2^4} \qquad \text{(B.8.1)}$$

$$\frac{(\vec{p}_2-\vec{p}_1)\cdot(\vec{p}_2-\vec{p}_3)}{\epsilon_1^2\epsilon_2^4} = \frac{1}{2\epsilon_1^2\epsilon_2^2} + \frac{m_1^2}{\epsilon_1^4\epsilon_2^2} \qquad \text{(B.8.2)}$$

Similarly, integrating with respect to $\vec{p}_3$ one proves:

$$\frac{(\vec{p}_3-\vec{p}_4)\cdot(\vec{p}_3-\vec{p}_2)}{\epsilon_2^4\epsilon_3^2} = \frac{1}{2\epsilon_2^2\epsilon_3^2} + \frac{m_3^2}{\epsilon_2^2\epsilon_3^4} \qquad \text{(B.8.3)}$$

$$\frac{(\vec{p}_3-\vec{p}_2)\cdot(\vec{p}_3-\vec{p}_4)}{\epsilon_2^2\epsilon_3^4} = \frac{1}{2\epsilon_2^2\epsilon_3^2} + \frac{m_2^2}{\epsilon_2^4\epsilon_3^2} \qquad \text{(B.8.4)}$$

Using (B.8) one can rewrite (B.7):

$$= \ldots \frac{1}{\epsilon_1^2}\left(-\frac{m_2^2}{\epsilon_2^4} + \frac{m_1^2}{\epsilon_1^2\epsilon_2^2} + \frac{(\vec{p}_2-\vec{p}_3)\cdot(\vec{p}_1+\vec{p}_3)}{\epsilon_2^4}\right)\frac{1}{q_s^2-\vec{p}_3^2}\frac{1}{\epsilon_3^2}\ldots$$

$$+ \ldots \frac{1}{\epsilon_1^2}\frac{1}{q_s^2-\vec{p}_2^2}\left(-\frac{m_2^2}{\epsilon_2^4} + \frac{m_3^2}{\epsilon_2^2\epsilon_3^2} + \frac{(\vec{p}_3-\vec{p}_2)\cdot(\vec{p}_2+\vec{p}_4)}{\epsilon_2^4}\right)\frac{1}{\epsilon_3^2}\ldots \qquad \text{(B.9)}$$

So the diagram of fig. B.1 can be written as the sum of two terms:

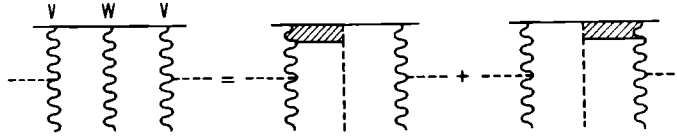
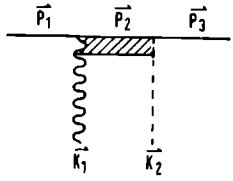


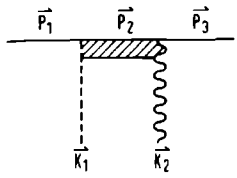
Fig. B.2

with:



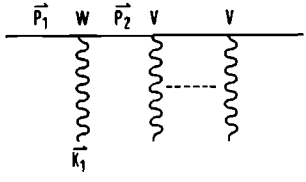
$$= \frac{1}{\epsilon_1^2}\left(-\frac{m_2^2}{\epsilon_2^4} + \frac{m_1^2}{\epsilon_1^2\epsilon_2^2} + \frac{(\vec{p}_2-\vec{p}_3)\cdot(\vec{p}_1+\vec{p}_3)}{\epsilon_2^4}\right) \qquad \text{(B.10.1)}$$

and

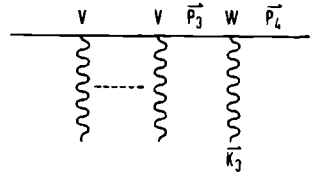


$$- \left(- \frac{m_1^2}{\varepsilon_1^4} + \frac{m_2^2}{\varepsilon_1^2 \varepsilon_2^2} + \frac{(\vec{p}_2 - \vec{p}_1)\cdot(\vec{p}_1 + \vec{p}_3)}{\varepsilon_1^4}\right) \frac{1}{\varepsilon_2^2} \qquad \text{(B.10.2)}$$

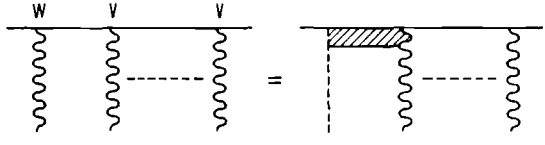
In the summation of eq. (B.6) there are two diagrams which cannot be depicted as in fig. B.1:



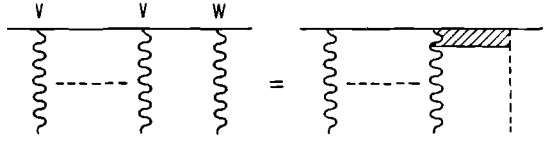
and



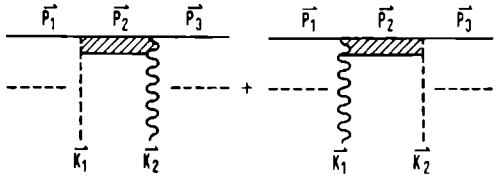
However, in these diagrams $\vec{p}_1$ ($\vec{p}_4$) is an external leg and therefore on shell: $\vec{p}_1^{\,2} = \vec{p}_4^{\,2} = q_s^2$. Also these diagrams can be written with the notation of eq. (B.10); one proves:



and



Now summing over all possible insertions of W in the ladder we will obtain pairs of terms which cancel each other partially:



$$\left(\frac{(\vec{p}_1+\vec{p}_3)\cdot(\vec{p}_2-\vec{p}_1)}{\epsilon_1^4 \epsilon_2^2} + \frac{(\vec{p}_1+\vec{p}_3)\cdot(\vec{p}_2-\vec{p}_3)}{\epsilon_1^2 \epsilon_2^4} \right) \cdot \cdot \qquad \text{(B.11)}$$

Note that the integration with respect to $\vec{p}_2$ still has to be performed; changing variables.

$$\vec{p}_1 = \vec{q} + \vec{\Delta} \quad ; \quad \vec{p}_3 = \vec{q} - \vec{\Delta} \quad ; \quad \vec{p}_2 = \vec{q} - \vec{\delta} \quad ;$$

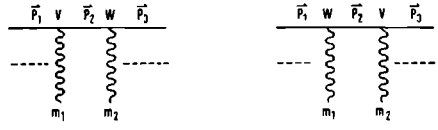
one proves:

$$= - \frac{2\vec{q}\cdot(\vec{\delta}+\vec{\Delta})\,(\Delta-\delta)^2 + 2\vec{q}\cdot(\vec{\delta}-\vec{\Delta})\,(\Delta+\delta)^2}{\epsilon_1^4 \epsilon_2^4} - \frac{2\vec{q}\cdot(\vec{\delta}+\vec{\Delta})\,m_2^2 + 2\vec{q}\cdot(\vec{\delta}-\vec{\Delta})\,m_1^2}{\epsilon_1^4 \epsilon_2^4} \qquad \text{(B.12)}$$

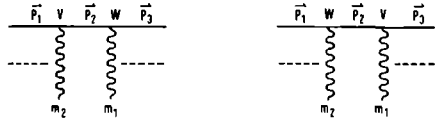
One must distinguish between two cases:

1. The two meson masses happen to be the same: $m_1 = m_2$. In this case the term of eq. (B.12) is obviously zero when integrated with respect to $\vec{\delta}$ since it is odd in $\vec{\delta}$.

2. The two meson masses are not the same. In that case the diagrams:



will have a counterpart with m_1 and m_2 interchanged:



It is not difficult to prove that the sum of these diagrams is also odd in $\vec{\delta}$ and therefore is zero once integrated in $\vec{p}_2$.

This completes the proof. We merely wish to note that the transformation of eq. (B.1) to eq. (B.2) has to be applied to all the subpotentials. Also the proof is not affected when the coupling constants g_1^2 contain terms which are proportional with λ:

$$g_1^2 \to g_1^2 \, (1 + \lambda \, (\text{mom.dep.terms}) + \ldots) \quad .$$

In that case the momentum dependent terms in the coupling constant, together with the transformation which is proportional with λ, will give λ^2 terms, which are neglected from the beginning.

APPENDIX C

The potentials in configuration space

Below we list the Fourier transforms to configuration space for the different potential forms we encountered. The derivation is straightforward but rather tedious and we refer to the literature for the well-known techniques [Na 75, Ho 60, Pa 70]. The transformation is done without cutoff and therefore the δ-functions in the origin are included in the result:

Spin independent potentials

$$\frac{1}{\vec{k}^2+m^2} \quad \leftrightarrow \quad \frac{1}{4\pi}\frac{e^{-mr}}{r} \equiv \phi_1$$

$$\frac{\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2}{\vec{k}^2+m^2} \quad \leftrightarrow \quad -\frac{1}{2}\left(\vec{\nabla}^2\,\phi_1+\phi_1\,\vec{\nabla}^2\right)-q_s^2\,\phi_1$$

$$\frac{m^2}{(\vec{k}^2+m^2)^2} \quad \leftrightarrow \quad \frac{m}{8\pi}e^{-mr} \equiv \phi_2$$

$$\frac{\vec{q}^2+\frac{\vec{k}^2}{4}-q_s^2}{(\vec{k}^2+m^2)^2} \quad \leftrightarrow \quad -\frac{1}{2}\left(\vec{\nabla}^2\,\phi_2+\phi_2\,\vec{\nabla}^2\right)-q_s^2\,\phi_2$$

$$\frac{(\vec{q}\cdot\vec{k})^2}{(\vec{k}^2+m^2)^2} \quad \leftrightarrow \quad \left(\frac{m}{8\pi}\frac{e^{-mr}}{r^2}+\frac{e^{-mr}}{8\pi r^3}\right)\vec{L}^2+\frac{1}{2}\left(\vec{\nabla}^2\,\phi_2+\phi_2\,\vec{\nabla}^2\right)$$

$$-\frac{m^3}{32\pi}e^{-mr}+\frac{1}{4}\,\delta^{(3)}(\vec{r})$$

$$\frac{\vec{k}^2}{\vec{k}^2+m^2} \quad \leftrightarrow \quad -\left[\vec{\nabla}^2\,\phi_1\right]=\delta^{(3)}(\vec{r})-m^2\,\phi_1$$

Spin dependent potentials

$$\frac{i\vec{S}\cdot(\vec{k}\times\vec{q})}{\vec{k}^2+m^2} \quad \leftrightarrow \quad -\frac{m^2}{4\pi}\left(\frac{1}{mr}+\frac{1}{m^2r^2}\right)\frac{e^{-mr}}{r}\,\vec{L}\cdot\vec{S}$$

$$\frac{(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})}{\vec{k}^2+m^2} \quad \leftrightarrow \quad -\frac{m^2}{4\pi}\left(\frac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\,\frac{e^{-mr}}{r}+\frac{1}{3}\left(1+\frac{3}{mr}+\frac{3}{m^2r^2}\right)\times\frac{e^{-mr}}{r}\,S_{12}\right)+\frac{1}{3}\vec{\sigma}_1\cdot\vec{\sigma}_2\,\delta^{(3)}(\vec{r})$$

$$\frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{\vec{k}^2 + m^2} \qquad \leftrightarrow \qquad \phi_1 \, \vec{\sigma}_1 \cdot \vec{\sigma}_2$$

$\vec{L}$ = angular momentum operator.

$S_{12} = 3 \, (\vec{\sigma}_1 \cdot \hat{r})(\vec{\sigma}_2 \cdot \hat{r}) - \vec{\sigma}_1 \cdot \vec{\sigma}_2$ = tensor operator.

$\vec{S} = \frac{1}{2} (\vec{\sigma}_1 + \vec{\sigma}_2)$ = total spin operator.

[Ab 65] | M. Abramowitz, I.A. Stegun; Handbook of Mathematical Functions;
[Ab 70] | Dover, New York (1965)

[Ar 74] R.A. Arndt, R.H. Hackman, L.D. Roper; Phys.Rev. C 9, 555 (1974)

[Ar 80] R.A. Arndt, B.J. Verwest; contributed paper to "Polarization Symposium", Santa Fe; "9th International Conference on the Few Body Problem", Eugene, August 1980, int.nu. DOE/ER/05223-29.

[BD 65] J.D. Björken, S.D. Drell; Relativistic Quantum Mechanics; McGraw-Hill (1965).

[Be 49] H.A. Bethe; Phys.Rev. 76, 38 (1949).

[Bl 48] J. Blatt; Phys.Rev. 74, 92 (1948).

[Bl 49] J. Blatt, J.D. Jackson; Phys.Rev. 76, 18 (1949).

[Bl 66] R. Blankenbecler, R. Sugar; Phys.Rev. 142, 1051 (1966).

[Br 53] K.A. Brueckner, K.M. Watson; Phys.Rev. 92, 1023 (1953).

[Br 69] C.N. Bressel, A.K. Kerman, B. Rouben; Nucl.Phys. A124, 624 (1969).

[BS 51] H.A. Bethe, E.E. Salpeter; Phys.Rev. 84, 1232 (1951), see also [Ge 51].

[BS 57] H.A. Bethe, E.E. Salpeter; Quantum Mechanics of One- and Two-Electron Atoms; Academic Press (1957).

[Ca 67] F. Calogero; Variable Phase Approach to Potential Scattering; Academic Press (1967).

[Che 60] G.F. Chew, S. Mandelstam; Phys.Rev. 119, 467 (1960).

[CFS 59] M. Cini, S. Fubini, A. Stanghellini; Phys.Rev. 114, 1633 (1959).

[Co 62] H. Cornille, A. Martin; Nuov.Cim. 26, 298 (1962).

[Do 79] W. van Dongen; Int.report, Institute for Theoretical Physics, University of Nijmegen; THEF-NYM-79.1 (1979).

[Du 57] L. Durand III; Phys.Rev. 108, 1597 (1957).

[Em 79] A. Emmen; Int.report, Institute for Theoretical Physics, University of Nijmegen; THEF-NYM-79.14 (1979).

[Fl 75] J. Fleischer, J.A. Tjon; Nucl.Phys. B84, 375 (1975).

[Ge 51] M. Gell-Mann, F. Low; Phys.Rev. 84, 350 (1951).

[Ge 79] F. Gesztesy, C.B. Lang; Preprint University Graz, Institute for Theoretical Physics: UTP 09/79 (1979).

[Gr 63] A.M. Green; Nucl.Phys. 33, 218 (1963).

[Gr 69] H. Grotch, D.R. Yennie; Rev.Mod.Phys. 41-2, 350 (1969).

[Gro 69] F. Gross; Phys.Rev. 186, 1448 (1969).

[Ha 62] T. Hamada, I.D. Johnston; Nucl.Phys. 34, 382 (1962).

[Ha 73] J. Hamilton, I. Øverbø, B. Tromborg; Nucl.Phys. B60, 443 (1973).

[He 60] L. Heller; Phys.Rev. 120-2, 627 (1960).

[Ho 60] N. Hoshizaki, S. Machida; Progr.Theor.Phys. 24, 1325 (1960).

[Ja 50] J.D. Jackson, J.M. Blatt: Rev.Mod.Phys. 22, 77 (1950).

[La 44] - L. Landau, J. Smorodinsky; J.Phys. (USSR) 8, 154 (1944); - J. Smorodinsky; J.Phys. (USSR) 11, 195 (1947).

[Le 66] M. Levine, J.A. Tjon, J. Wright; Phys.Rev.Lett. 16, 962 (1966)

[Le 67] M. Levine, J.A. Tjon, J. Wright; Phys.Rev. 154, 1433 (1967).

[Lo 63] A. Logunov, A. Tavkhelidze; Nuov.Cim. 29, 380 (1963).

[Lu 64] M. Luming; Phys.Rev. 136 B4, 1120 (1964).

[Me 61] A. Messiah; Quantum Mechanics; North Holland Publishing Co. (1961).

[MG 68] M.H. MacGregor, R.A. Arndt, R.M. Wright; Phys.Rev. 173, 1272 (1968).

[MG 69] M.H. MacGregor, R.A. Arndt, R.M. Wright; Phys.Rev. 182, 1714 (1969).

[Mu 78] L. Müller, W. Glockle; Preprint July 1978; University Bochum, Institut für Theoretische Physik.

[Na 69] N. Nakanishi; Progr.Theor.Phys.Suppl. 43 (1969).

[Na 75] M.M. Nagels; Thesis University of Nijmegen (1975), unpublished.

[Na 77] J.P. Naisse; Nucl.Phys. A278, 506 (1977)

[Na 78] M.M. Nagels, T.A. Rijken, J.J. de Swart; Phys.Rev. D 17, 768 (1978).

[Ne 66] R.G. Newton; Scattering Theory of Waves and Particles; McGraw-Hill (1966).

[No 64] H. Noyes; C.R. du Congres International de Physique Nucléaire; ed. P. Gugenberger; Vol. II, p. 172-173; Paris; Editions du Ctr-Natl de la Recherche Scientifique; see [No 71, No 72].

[No 71] H.P. Noyes, H.M. Lipinski; Phys.Rev. C 4, 995 (1971).

[No 72] H.P. Noyes, Ann.Rev.Nucl.Sc. 22, 465 (1972).

[Nu 33] D. Numerov; Publ. de l'Observ.Astrophys. Central de Russie, 2, 188 (1933).

[Pa 70] M.H. Partovi, E.L. Lomon; Phys.Rev. D 2, 1999 (1970).

[Re 68] R.V. Reid Jr.; Ann.Phys. (N.Y.) 50, 411 (1968).

[Ry 75] T.A. Rijken; Thesis University of Nijmegen (1975), unpublished.

[Sa 79] W. van der Sanden; Int.report, Institute for Theoretical Physics, University of Nijmegen; THEF-NYM-79.3 (1979).

[Sch 47] J. Schwinger, Harvard Lecture Notes (1947), see [Bl 48].

[Sch 51] J. Schwinger, Proc.Natl.Acad.Sci. (U.S.) 37, 452 and 455 (1951).

[Sh 70] M. Sher, P. Signell, L. Heller; Ann.Phys. (N.Y.) 58, 1 (1970).

[Sw 62] J.J. de Swart, C. Dullemond; Ann.Phys. (N.Y.) 19, 458 (1962).

[Sw 77] J.J. de Swart, M.M. Nagels; Fortschritte der Physik 26, 215 (1978).

[Ta 52] M. Taketani, S. Machida, S. Ohnuma; Progr.Theor.Phys. 7, 45 (1952).

[Ta 72] J.R. Taylor; Scattering Theory; John Wiley & Sons (1972).

[Ta 74] J.R. Taylor; Nuov.Cim. 23B, 313 (1974).

[Th 70] R. Thompson; Phys.Rev. D 1, 110 (1970).

[Vi 74] R. Viollier; Phys.Lett. 48 B, 99 (1974).

[Wi 54] G.C. Wick; Phys.Rev. 96, 1124 (1954).

[Wo 62] D. Wong, H. Noyes; Phys.Rev. 126, 1866 (1962).

[Ya 71] R.J. Yaes; Phys.Rev. D 3, 3086 (1971).

In dit proefschrift wordt een studie gemaakt van twee onderwerpen die een rol spelen bij de bestudering van de nucleon-nucleon interactie. Alhoewel de resultaten in eerste instantie beschouwd worden vanuit het kader van lage energie proton-proton verstrooiing, blijven de methodieken niet alleen tot dit gebied beperkt, maar kunnen in zekere zin overal toegepast worden waar bij de analyse van verstrooiingsdata een parametrisatie naar de energie gezocht wordt of waar de electromagnetische interactie een rol speelt.

Het eerste gedeelte van dit proefschrift gaat over een gegeneraliseerd effectieve drachts formalisme. Met behulp van effectieve drachts methodieken probeert men experimenten op het gebied van twee deeltjes verstrooiing, uitgevoerd bij verschillende energieen, met elkaar in verband te brengen. In wezen is een effectieve drachts formule niets anders dan een interpolatie- of extrapolatietechniek die faseverschuivingen, gemeten op bepaalde energieen, probeert onder te brengen in een formule en die vandaaruit deze faseverschuivingen probeert te voorspellen bij andere energieën.

De kracht van effectieve drachts theorieën ligt in het feit dat men van de interactie die de verstrooiing bepaalt niet veel hoeft te weten om ze toe te kunnen passen. Dit is aardig als men helemaal niets weet; echter in een situatie waarin men bepaalde stukken van de interactie wel meent te kennen zou het toepassen van effectieve drachts formules in hun meest eenvoudige vorm ertoe leiden dat ook deze stukken opnieuw geparametriseerd worden.

We zullen dit aan de hand van lage energie proton-proton verstrooiing proberen toe te lichten:

Men denkt dat de proton-proton verstrooiing bepaald wordt door de volgende mechanismes:

- electromagnetische wisselwerkingen ten gevolge van de ladingen van de protonen;
- vacuum polarizatie wisselwerking ten gevolge van het feit dat een foton een electron-positron paar kan creeren;
- de nucleaire wisselwerking, bestaande uit:
  (a) lange drachts pion uitwisseling,
  (b) korte drachts interacties ten gevolge van de uitwisseling van zwaardere mesonen,
  (c) een korte drachts repulsie, vaak op zuiver fenomenologische manier beschreven door middel van een harde pit, of door middel van Pomeron "uitwisseling".

Bovenstaande volgorde geeft tevens de "bekendheid" van de interactie aan: de kennis van de electromagnetische interactie is het grootst, die van de korte drachts repulsie het kleinst. Past men effectieve drachts methodieken nu toe op de experimenteel gemeten grootheden zonder dat men de kennis die men van de proton-proton interactie heeft, á priori al inbouwt, dan zal dit leiden tot een onbruikbaar geheel, voornamelijk omdat de electromagnetische interactie een oneindige dracht heeft. Dit heeft er in het verleden toe geleid dat het Coulomb stuk van de electromagnetische interactie en de vacuum polarizatie in de effectieve drachts formules werden ingebouwd. Op deze manier hoefde men alleen de nucleaire wisselwerking nog op een fenomenologische wijze weer te geven. Het energiegebied dat op deze

manier bestreken kon worden, lag tussen de 0 en 10 MeV laboratorium energie. In de jaren zestig zijn er pogingen gedaan om ook de lange drachts pion uitwisseling in de formules te verwerken. Echter door de gekozen methode kon dit slechts in benaderde vorm gebeuren; in de praktijk leidde dit ertoe dat er twee, elkaar tegenwerkende, benaderingen werden toegepast:

- de lange drachts pion interactie werd benaderd door een eenvoudige pool in de partiele golf amplitude: dit leidde tot een verzwakking van de attractie.
- er werd geen rekening gehouden met de korte dracht repulsie: dit leidde tot een (schijnbare) versterking van de attractie.

De op deze manier verkregen formules werden toegepast in het gebied tussen 0 en 40 MeV en werden tevens gebruikt om uit s-golven de pion-nucleon koppelingsconstante te bepalen. Door de elkaar tegenwerkende fouten kreeg men een resultaat dat in overeenstemming scheen te zijn met waarden op andere manieren verkregen. Toen echter enkele jaren geleden de korte drachts repulsie werd ingebouwd in de parametrizaties van de effectieve drachts functie leidde dit meteen tot het naar voren springen van de fout ten gevolge van de andere benadering: de gefitte koppelingsconstante bleek meer dan 30% te groot te zijn vergeleken met de conventionele waarden. Dit leidde tot de noodzaak om het gehele effectieve drachts formalisme kritisch te beschouwen.

In het eerste gedeelte van dit proefschrift beschrijven we de fundamenten van algemene effectieve drachts theorie. We gebruiken hiervoor niet de "klassieke" methode van Bethe (zie hoofdstuk I), maar technieken gebaseerd op de analytische eigenschappen van de golffuncties. Met behulp van deze methodes is men in staat om iedere interactie

die expliciet bekend is, exact in de effectieve drachts functie in te bouwen. Aandacht wordt ook besteed aan de keuze van de fenomenologische representatie van deze effectieve drachts functie, speciaal aan de zgn. poolbenaderingen. In de laatste twee hoofdstukken wordt het effectieve drachts formalisme in de praktijk getest: dit gebeurt zowel aan de hand van een potentiaal model als aan de hand van experimentele grootheden.

De resultaten op deze manier verkregen, leiden tot de conclusie dat het gegeneraliseerde effectieve drachts formalisme, met daarin exact ingebouwd de electrische interactie, de vacuum polarizatie en de lange drachts pion wisselwerking, in staat moet zijn om zeer nauwkeurig met slechts drie parameters het gebied tussen 0 en 40 MeV te beschrijven (we refereren hier alleen maar naar de s-golf) en met vier parameters het gebied tussen 0 en 250 MeV. Tevens wordt het formalisme gebruikt om opnieuw de waarde van de pion-proton koppelings-constante te bepalen.

In het tweede gedeelte van dit proefschrift beschouwen we een van de interacties die bij lage energie proton-proton verstrooiing een belangrijke rol spelen: de electromagnetische wisselwerking. Om de resultaten van het eerste deel in de praktijk te kunnen toepassen, moet vooral dit gedeelte van de interactie zeer nauwkeurig bekend zijn en in een dusdanige vorm gegoten worden dat er gemakkelijk mee gewerkt kan worden. In de praktijk blijkt deze vorm een configuratie ruimte potentiaal te zijn. De afleiding echter van een potentiaal beschrijving uit een velden theoretisch model (ons uitgangspunt), gaat gepaard met benaderingen op diverse nivo's. In wezen gebruiken we de bekende procedure uitgaande van Feynman diagrammen, de sommatie hiervan via de Bethe-Salpeter vergelijking, de reductie naar de Blankenbecler-Sugar

vergelijking, uiteindelijk leidend naar de Lippmann-Schwinger vergelijking en de configuratie ruimte potentialen. Deze reductie kan in de praktijk niet exact gedaan worden en het is van belang dat een consequente benadering aangehouden wordt in het verwaarlozen en aanhouden van de diverse bijdragen tot de potentiaal. Daarom kiezen we voor een methode gebaseerd op een expansie in termen van de nucleon massa M ten opzichte van het punt: $M = \infty$ (de statische limiet). Termen tot en met orde $1/M^2$ worden nog meegenomen in de potentiaal. In het laatste hoofdstuk worden de effectieve drachts functies voor deze potentiaal afgeleid.

De resultaten van het tweede gedeelte van dit proefschrift kunnen ook gebruikt worden voor de afleiding van potentialen voor de sterke wisselwerking. Het blijkt dat de electromagnetische potentiaal op deze wijze afgeleid alleen maar gebruikt kan worden in combinatie met een nucleaire potentiaal op dezelfde manier verkregen. Voor de effectieve drachts methodiek is dit van minder belang, omdat het grootste deel van de nucleaire interactie toch op een fenomenologische manier wordt beschreven. Echter in de meer "fysische" modellen, zoals de zgn. OBE-potentialen, zal men rekening hiermee moeten (gaan) houden.

CURRICULUM VITAE

Geboren: 30 maart 1949 te Heerlen.

Lagere School: sept. 1955 - juli 1961 te Heerlen.

Gymnasium β: sept. 1961 - juni 1967; St. Bernardinuscollege te Heerlen.

Katholieke Universiteit Nijmegen:

september 1967: eerste inschrijving

maart 1970: kandidaatsexamen

december 1973: doctoraal examen (theoretische natuurkunde)

dec. 1973 - juli 1981: wetenschappelijk medewerker aan het Instituut voor Theoretische Fysica van de Katholieke Universiteit te Nijmegen.

vanaf oktober 1981: werkzaam bij de afdeling I.V.V. van de Faculteit der Wiskunde en Natuurwetenschappen te Nijmegen.

STELLINGEN

I

Het feit dat T. Sawada de singulariteiten afkomstig van een sterke Van-der-Waals-kracht blijkbaar ook aantreft in de energie-afhankelijke fase-analyse van Sher et al., maakt zijn analyse zeer twijfelachtig.

T. Sawada; Progr.Theor.Phys. 59, 149 (1978)

M.S. Sher, P. Signell, L. Heller; Ann.Phys. (N.Y.) 58, 1 (1970)

II

De invloed van vervalkanalen op het spectrum van een "confining" potentiaal kan zeer groot zijn. Bij de beschrijving van systemen met grote vervalbreedtes wordt hiermee vaak onvoldoende rekening gehouden.

A. Martin; Phys.Lett. 93B, 338 (1980) en de hierin genoemde referenties

D.P. Stanley; A.I.P. Conf.Proc. 74, 392 (1981)

III

Een lokale potentiaal leidt in BSLT-achtige vergelijkingen tot een amplitude die niet voldoet aan de Mandelstam-representatie. Dit kan vermeden worden door retardatie in de potentiaal mee te nemen.

O.I. Zav'yalov, M.K. Polivanov, S.S. Khoruzhii; Sov.Phys. JETP 18, 1136 (1964)

M.K. Polivanov, S.S. Khoruzhii; Sov.Phys. JETP 19, 232 (1964)

IV

Het gebruik van een P_μ, Q_μ combinatie (nodig bij de Sachs-vormfactoren) voor de electromagnetische stroom van het proton, in plaats van een γ_μ, $\sigma_{\mu\nu}k^\nu$ combinatie (bij de Pauli-Dirac-vormfactoren) zal in het waterstofatoom leiden tot ontoelaatbare afwijkingen in de Lamb-shift.

De niet-lokale termen van de Coulombwisselwerking moeten bij de berekening van ladingssymmetriebrekende effecten in het nucleon-nucleon-systeem worden meegenomen. Ze geven een bijdrage in grootte vergelijkbaar met reeds bekende effecten, zoals π-η en ρ-ω menging.

E.M. Henley, T.E. Keliher; Nucl.Phys. A189, 632 (1972)

VI

Bij het gebruik van $\frac{\vec{L}^2}{r^3}$ potentialen in verstrooiingsproblemen moet men erop verdacht zijn dat een deel van de door deze term veroorzaakte faseverschuiving niet waarneembaar is.

VII

De constatering van Simonov, dat het verschil in grootte tussen de effectieve drachten in de $^1S_0$- en $^3S_1$-golven voor nucleon-nucleon-verstrooiing voornamelijk veroorzaakt wordt door verschillen in residuen en posities van P-matrixpolen, is onjuist.

Yu.A. Simonov; Phys.Lett. 107B, 1 (1981)

VIII

Thomson-verstrooiing aan een nucleon schijnt aan te tonen dat tussentoestanden met negatieve energie in nucleon-nucleon-verstrooiing niet onderdrukt zijn. In dit argument wordt voorbijgegaan aan het feit dat het nucleon een samengesteld deeltje is waarin Thomson-verstrooiing veroorzaakt kan worden door creatie van een quark-antiquark-paar in de tussentoestand.

IX

"Computing" neemt een zeer belangrijke plaats in binnen het onderzoek in de fysica. Het verdient daarom aanbeveling om een fysicus-informaticus aan te stellen om de fysici op dit terrein te ondersteunen, zowel in het onderwijs als in het onderzoek.

X

De bedrijfsgezondheidszorg voor de lagere overheid voldoet nog steeds niet aan de wettelijke eisen zoals die gesteld worden aan de bedrijfsgezondheidszorg in het bedrijfsleven. Dit is een van de voorbeelden waarin het spreekwoord "Verbeter de wereld, begin bij Uzelf" op de overheid van toepassing is.

XI

Een adequate gezondheidszorg is alleen te verwezenlijken als iedereen bereid is om, zoveel als in zijn vermogen ligt, zelf kennis te vergaren omtrent en verantwoordelijkheid te dragen voor zijn eigen gezondheid. Het is onjuist om dit over te laten aan de behandelende artsen.
Het begrip "behandelend arts" moet meer worden "adviserend arts".

G.J.M. Austen

18 maart 1982