# Hyperdeterminant and an integrable partial differential equation

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**Abstract.** We discuss an integrable partial differential equation arising from the hyperdeterminant.

*Key words:* Hyperdeterminant, Legendre Transformation, Integrable Partial Differential Equation

It is well known [1,2,3,4] that the Bateman equation in two dimensions

$$\frac{\partial^2 u}{\partial x_1^2}\left(\frac{\partial u}{\partial x_2}\right)^2 + \frac{\partial^2 u}{\partial x_2^2}\left(\frac{\partial u}{\partial x_1}\right)^2 - 2\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0$$

can be derived from the condition

$$\det\begin{pmatrix} 0 & \partial u/\partial x_1 & \partial u/\partial x_2 \\ \partial u/\partial x_1 & \partial^2 u/\partial x_1^2 & \partial^2 u/\partial x_1 \partial x_2 \\ \partial u/\partial x_2 & \partial^2 u/\partial x_1 \partial x_2 & \partial^2 u/\partial x_2^2 \end{pmatrix} = 0\,.$$

It is also well known (see for example [5,6]) that the Bateman equation can be linearized by a Legendre transformation and the general implicit solution is given by

$$x_1 f(u(x_1,x_2)) + x_2 g(u(x_1,x_2)) = c$$

where $f$ and $g$ are smooth functions. This partial differential equation plays a central role in studying the integrability of partial differential equation using the Painlevé test [5,7]. For example the differential equation appears in the Painleve analysis of the inviscid Burgers equation, double sine-Gordan, discrete Boltzmann equation.

Here we generalize the condition given above from the determinant to the $2 \times 2 \times 2$ hyperdeterminant of a $2 \times 2 \times 2$ hypermatrix and derive the nonlinear partial differential equation and discuss its properties. The extension to $2 \times 2 \times 2 \times 2$ hyperdeterminants will be straightforward.

Cayley [8] in 1845 introduced the hyperdeterminant. Gelfand et al [9] give an in debt discussion of the hyperdeterminant. The hyperdeterminant arises as entanglement measure for three qubits [10,11,12], in black hole entropy [13]. The

Nambu-Goto action in string theory can be expressed in terms of the hyperdeterminant [14]. A computer algebra program for the hyperdeterminant is given by Steeb and Hardy [11]

Let $\epsilon_{00} = \epsilon_{11} = 0$, $\epsilon_{01} = 1$, $\epsilon_{10} = -1$, i.e. we consider the $2 \times 2$ matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

Then the determinant of a $2 \times 2$ matrix $A_2 = (a_{ij})$ with $i, j = 0, 1$ can be defined as

$$\det A_2 := \frac{1}{2} \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{\ell=0}^{1} \sum_{m=0}^{1} \epsilon_{ij} \epsilon_{\ell m} a_{i\ell} a_{jm} .$$

Thus $\det A_2 = a_{00}a_{11} - a_{01}a_{10}$. In analogy the hyperdeterminant of the $2 \times 2 \times 2$ array $A_3 = (a_{ijk})$ with $i, j, k = 0, 1$ is defined as

$$\mathrm{Det} A_3 := -\frac{1}{2} \sum_{ii'=0}^{1} \sum_{jj'=0}^{1} \sum_{kk'=0}^{1} \sum_{mm'=0}^{1} \sum_{nn'=0}^{1} \sum_{pp'=0}^{1} \epsilon_{ii'} \epsilon_{jj'} \epsilon_{kk'} \epsilon_{mm'} \epsilon_{nn'} \epsilon_{pp'} a_{ijk} a_{i'j'm} a_{npk'} a_{n'p'm'} .$$

There are $2^8 = 256$ terms, but only 24 are nonzero. We find

$$\begin{aligned} \mathrm{Det} A_3 \quad = \quad & a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{001}^2 a_{101}^2 + a_{100} a_{011}^2 \\ & -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} \\ & +a_{000}a_{100}a_{011}a_{111} + a_{001}a_{010}a_{101}a_{110} \\ & +a_{001}a_{100}a_{011}a_{110} + a_{010}a_{100}a_{011}a_{101}) \\ & +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}) . \end{aligned}$$

The hyperdeterminant $\mathrm{Det}(A)$ of the three-dimensional array $A_3 = (a_{ijk}) \in \mathbb{R}^{2\times 2\times 2}$ can also be calculated as follows

$$\begin{aligned} \mathrm{Det}(A_3) \quad = \quad & \frac{1}{4} \left( \det \left( \begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} + \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix} \right) \right. \\ & \left. - \det \left( \begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} - \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix} \right) \right)^2 \\ & -4 \det \begin{pmatrix} a_{000} & a_{010} \\ a_{001} & a_{011} \end{pmatrix} \det \begin{pmatrix} a_{100} & a_{110} \\ a_{101} & a_{111} \end{pmatrix} . \end{aligned}$$

If only one of the coefficients $a_{ijk}$ is nonzero we find that the hyperdeterminant of $A_3$ is 0.

Given a $2 \times 2 \times 2$ hypermatrix $A_3 = (a_{jk\ell})$, $j, k, \ell = 0, 1$ and the $2 \times 2$ matrix

$$S = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} .$$

The multiplication $A_3 S$ which is again a $2 \times 2$ hypermatrix is defined by

$$(A_3 S)_{jk\ell} := \sum_{r=0}^{1} a_{jkr} s_{r\ell} .$$

If $\det(S) = 1$, i.e. $S \in SL(2, \mathbb{C})$, then $\mathrm{Det}(A_3 S) = \mathrm{Det}(A_3)$.

In analogy with the Bateman equation we set

$$a_{000} \mapsto 0, \quad a_{001} \mapsto \frac{\partial u}{\partial x_3}, \qquad a_{010} \mapsto \frac{\partial u}{\partial x_2}, \qquad a_{100} \mapsto \frac{\partial u}{\partial x_1},$$

$$a_{011} \mapsto \frac{\partial^2 u}{\partial x_2 \partial x_3}, \quad a_{101} \mapsto \frac{\partial^2 u}{\partial x_1 \partial x_3}, \quad a_{110} \mapsto \frac{\partial^2 u}{\partial x_1 \partial x_2}, \quad a_{111} \mapsto \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3}$$

and obtain the nonlinear partial differential equation

$$\left(\frac{\partial u}{\partial x_1}\right)^2 \left(\frac{\partial^2 u}{\partial x_2 \partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2$$
$$-2 \left( \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_3} \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_3} \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_3} \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right)$$
$$+4 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_3} \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = 0 .$$

The partial differential equation is invariant under the permutations of $x_1, x_2, x_3$. The group of symmetries is $SL(3, \mathbb{R})$. The equation can also be linearized by a Legendre transformation.


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