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THE (IN)FINITE HORIZON OPEN-LOOP NASH LQ GAME: AN APPLICATION TO EMU

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The (In)finite Horizon Open-loop Nash LQ-game: an Application to EMU

by

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Abstract
In this paper we consider a generalization of the linear-quadratic differential game studied in Engwerda (1997). We show that the theoretical results obtained in that paper w.r.t. existence of open-loop Nash equilibria for the finite-planning-horizon, the infinite-planning horizon as well as some convergence properties can be straightforwardly generalized. Particular attention is paid to computational aspects. To motivate the study of this generalized game, we show how the results can be applied and study macroeconomic stabilization in the Economic and Monetary Union (EMU).

Keywords: Linear quadratic games, open-loop Nash equilibrium, solvability conditions, Riccati equations, EMU

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I. Introduction

The last decade there has been an increasing interest to study several problems in economics using dynamic game theory. In particular in the area of environmental economics and macro-economic policy coordination dynamic games are a very natural framework to model problems (see e.g. de Zeeuw and van der Ploeg (1991), Mäler (1992), Kaitala, Pohjola and Tahvonen (1992) and Dockner, Feichtinger and Jorgensen (1985), Tabellini (1986), Fershtman and Kamien (1987), Petit (1989), Levine and Brociner (1994), van Aarle, Bovenberg and Raith (1995), Neck and Dockner (1995), Douven and Engwerda (1995)). Particularly in macro-economic policy coordination problems, the open-loop Nash strategy is often used as a benchmark to evaluate outcomes under different control strategies such as the Nash feedback and the open-loop and feedback Stackelberg equilibria. In Engwerda (1997) several aspects of open-loop Nash equilibria are studied of the standard linear-quadratic differential game as considered by Starr and Ho in (1969). Both necessary and sufficient conditions for existence of a unique solution for the finite-planning horizon case are given, and it is shown that there exist situations where the set of associated Riccati differential equations has no solution, whereas the problem does have an equilibrium. Furthermore, conditions are given under which this strategy converges if the planning horizon expands, and a detailed study of the infinite planning horizon case is given. In particular it is shown that, in general, the infinite horizon problem has no unique equilibrium and that the limit of the above mentioned converged strategy may be not an equilibrium for the infinite planning horizon problem. The purpose of this paper is twofold. First, we consider an extension of the standard game. We show that under an invertibility condition similar results can be obtained. In particular we show how for the infinite planning horizon game all equilibria can be easily obtained from the eigenspace structure of a Hamiltonian matrix that is associated with the game. Second, we use these results to study an example of a dynamic macro-economic policy game. We analyse the conflict between macro-economic policymakers on output stabilization that arises in the Economic Monetary Union (EMU), to which the European Union countries have committed themselves in the Maastricht Treaty of 1991. Basically, the analysed model is an extension of the two-country linear-quadratic dynamic stabilization game studied by Neck and Dockner (1995) and Turnovsky, Basar and d'Orey (1988) to a two-country EMU setting.

The outline of the paper is as follows. In section 2 we start by stating the problem analysed in this paper and show the equivalent results of Engwerda (1997). Since, from an application point of view, the infinite planning horizon is of particular interest, we present in section 3 an algorithm to compute all equilibria of the infinite planning horizon problem. For the sake of simplicity of presentation, we will consider in both these sections only the two-player game. The generalization to the N-player case is, however, straightforward. This is illustrated in section 4, where we use the algorithm in a simulation study of the differential game on output stabilization in the EMU . The paper ends with some concluding remarks.

II. Analysis of the general model

In this paper we consider the problem where two parties (henceforth called players) try to minimize their individual quadratic performance criterion. Each player controls a different set of inputs to a single system, described by a differential equation of arbitrary order. We are looking for combinations of pairs of strategies of both players which are secure against any attempt by one player to unilaterally alter his strategy. That is, for those pairs of strategies which are such that if one player deviates from his strategy he will only lose. In the literature on dynamic games this problem is well-known as the open-loop Nash non-zero-sum linear quadratic differential game (see e.g. Starr and Ho (1969), Simaan and Cruz (1973), Başar and Olsder (1982) or Abou-Kandil and Bertrand (1986)). Formally the system we consider is as follows:

$$\dot{x} = Ax + B_1u_1 + B_2u_2, \; x(0) = x_0, \tag{1}$$

where x is the n-dimensional state of the system, u_i is an m-dimensional (control) vector player i can manipulate, x_0 is the initial state of the system, A, B_1, and B_2 are constant matrices of appropriate dimensions, and $\dot{x}$ denotes the time derivative of x.
The performance criterium player $i = 1,2$ aims to minimize is:

$$J_i(u_1, u_2) := \frac{1}{2}x(t_f)^T K_{if} x(t_f) + \frac{1}{2}\int_0^{t_f} \{(x(t)^T u_1^T(t) u_2^T(t))^T F_i \begin{pmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{pmatrix}\}dt,$$

where $F_i := \begin{pmatrix} Q_i & P_i & O_i \\ P_i^T & R_{1i} & N_i \\ O_i^T & N_i^T & R_{2i} \end{pmatrix}$ is semi-positive definite, K_{if} is semi-positive definite and R_{ij} positive definite, $i = 1,2$. Note that this performance criterium is more general than the one usually considered in literature, where F_i is assumed to be diagonal.
Throughout this paper we will assume that the following matrix G is invertible:

$$G := \begin{pmatrix} R_{11} & N_1 \\ N_2^T & R_{22} \end{pmatrix}. \tag{2}$$

In this section we consider in detail the existence of open-loop Nash equilibria of this differential game, for both a finite as an infinite planning horizon. Due to the stated assumptions both cost functionals $J_i, i = 1,2$, are strictly convex functions of u_i for all admissible control functions $u_j, j \neq i$ and for all x_0. This implies that the conditions following from the minimum principle are both necessary and sufficient (see e.g. Başar and Olsder (1982, section 6.5)).
Minimization of the Hamiltonian

$$H_i := (x^T u_1^T u_2^T)^T F_i \begin{pmatrix} x(t) \\ u_1(t) \\ u_2(t) \end{pmatrix} + \psi_i^T(Ax + B_1u_1 + B_2u_2), \; i = 1,2$$

with respect to u_i yields the optimality conditions:

$$G\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = -\begin{pmatrix} P_1^T x(t) + B_1^T \psi_1(t) \\ O_2^T x(t) + B_2^T \psi_2(t) \end{pmatrix}. \tag{3}$$

Due to our invertibility condition (2) we can rewrite this as

$$\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = -G^{-1}\begin{pmatrix} P_1^T x(t) + B_1^T \psi_1(t) \\ O_2^T x(t) + B_2^T \psi_2(t) \end{pmatrix}. \tag{4}$$

Here, the n-dimensional vectors $\psi_i(t)$ satisfy

$$\dot{\psi}_i(t) = -(Q_i x(t) + P_i u_1^* + O_i u_2^* + A^T \psi_i(t)), \text{ with } \psi_i(t_f) = K_{if} x(t_f), i = 1, 2.$$

and

$$\dot{x}(t) = Ax(t) + B_1 u_1^*(t) + B_2 u_2^*(t);\ x(0) = x_0.$$

Introducing the matrix $M :=$

$$\begin{pmatrix} -A + (B_1 B_2)G^{-1}\begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} & (B_1 B_2)G^{-1}\begin{pmatrix} B_1^T \\ 0 \end{pmatrix} & (B_1 B_2)G^{-1}\begin{pmatrix} 0 \\ B_2^T \end{pmatrix} \\ Q_1 - (P_1 O_1)G^{-1}\begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} & A^T - (P_1 O_1)G^{-1}\begin{pmatrix} B_1^T \\ 0 \end{pmatrix} & -(P_1 O_1)G^{-1}\begin{pmatrix} 0 \\ B_2^T \end{pmatrix} \\ Q_2 - (P_2 O_2)G^{-1}\begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} & -(P_2 O_2)G^{-1}\begin{pmatrix} B_1^T \\ 0 \end{pmatrix} & A^T - (P_2 O_2)G^{-1}\begin{pmatrix} 0 \\ B_2^T \end{pmatrix} \end{pmatrix}, \tag{5}$$

we see that the problem has a unique open-loop Nash equilibrium if and only if the differential equation

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -M\begin{pmatrix} x(t) \\ \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

with boundary conditions $x(0) = x_0$, $\psi_1(t_f) - K_{1f} x(t_f) = 0$ and $\psi_2(t_f) - K_{2f} x(t_f) = 0$, has a unique solution. Denoting the state variable $(x^T(t)\ \psi_1^T(t)\ \psi_2^T(t))^T$ by $y(t)$, we can rewrite this two-point boundary value problem in the standard form

$$\dot{y}(t) = -My(t), \text{ with } Py(0) + Qy(t_f) = (x_0^T\ 0\ 0)^T, \tag{6}$$

where M is given by (5), $P = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ -K_{1f} & I & 0 \\ -K_{2f} & 0 & I \end{pmatrix}$.

From (6) we have immediately that problem (1) has a unique open-loop Nash equilibrium if and only if

$$(P + Qe^{-Mt_f})y(0) = (x_0^T\ 0\ 0)^T,$$

or, equivalently,

$$(Pe^{Mt_f} + Q)e^{-Mt_f}y(0) = (x_0^T\ 0\ 0)^T, \tag{7}$$

is uniquely solvable for every x_0.
Using the following notation:

$$H(t_f) := W_{11}(t_f) + W_{12}(t_f)K_{1f} + W_{13}(t_f)K_{2f},$$

with $W(t_f) = (W_{ij}(t_f))\ \{i,j = 1,2,3;\ W_{ij} \in R^{n \times n}\} := \exp(Mt_f)$,
elementary matrix analysis then shows that

Theorem 1:
The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state if and only if matrix $H(t_f)$ is invertible. Moreover, the open-loop Nash equilibrium solution as well as the associated state trajectory can be calculated from the linear two-point boundary value problem (6). □

Next, consider the following set of coupled asymmetric Riccati-type differential equations:

$$\begin{pmatrix} \dot{K}_1 \\ \dot{K}_2 \end{pmatrix} = (\begin{pmatrix} -A^T & 0 \\ 0 & -A^T \end{pmatrix} + \begin{pmatrix} P_1 & O_1 \\ P_2 & O_2 \end{pmatrix} G^{-1} \begin{pmatrix} B_1^T & 0 \\ 0 & B_2^T \end{pmatrix}) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} +$$

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \left(-A + (B_1 B_2) G^{-1} \begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} \right) + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} ((B_1 B_2) G^{-1} \begin{pmatrix} B_1^T & 0 \\ 0 & B_2^T \end{pmatrix}) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} +$$

$$(\begin{pmatrix} P_1 O_1 \\ P_2 O_2 \end{pmatrix} G^{-1} \begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}); \begin{pmatrix} K_1(t_f) \\ K_2(t_f) \end{pmatrix} = \begin{pmatrix} K_{1f} \\ K_{2f} \end{pmatrix}. \tag{8}$$

Let $K_i(t)$ satisfy this set of Riccati equations and assume that the players use the strategy

$$\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = -G^{-1} \begin{pmatrix} P_1^T + B_1^T K_1(t) \\ O_2^T + B_2^T K_2(t) \end{pmatrix} \Phi(t,0)x_0, \tag{9}$$

where $\Phi(t,0)$ is the solution of the transition equation

$$\dot{\Phi}(t,0) = \left(A - (B_1 B_2) G^{-1} \begin{pmatrix} P_1^T + B_1^T K_1(t) \\ O_2^T + B_2^T K_2(t) \end{pmatrix} \right) \Phi(t,0); \Phi(0,0) = I.$$

Now, define $\psi_i(t) := K_i(t)\Phi(t,0)x_0$. Then, obviously $\dot{\psi}_i(t) = \dot{K}_i(t)\Phi(t,0)x_0 + K_i(t)\dot{\Phi}(t,0)x_0$. Substitution of $\dot{K}_i$ from (6) and $\dot{\Phi}(t,0)$ into these equations shows then that $x(t) := \Phi(t,0)x_0$, $\psi_1(t)$ and $\psi_2(t)$ satisfy the two-point boundary value problem (6). This proves the following claim:

Theorem 2:
The two-player linear quadratic differential game (1) has a unique open-loop Nash equilibrium for every initial state provided the set of Riccati equations (8) has a solution. Moreover, the equilibrium strategies are then given by (9). □

The following result gives the exact relationship that exists between solvability of the set of Riccati equations and existence of equilibrium strategies for the game. Like the other results that will be presented in this section, a proof can be given along the lines in Engwerda (1997) for the standard case, where F_i is assumed to be diagonal. Detailed proofs are therefore omitted.

Theorem 3:
The following statements are equivalent:
1) For all $t_f \in [0, t_1]$ there exists a unique open-loop Nash equilibrium for the two-player linear quadratic differential game (1) defined on the interval $[0, t_f]$.
2) $H(t)$ is invertible for all $t_f \in [0, t_1]$.
3) The set of Riccati differential equations (8) has a solution on $[0, t_1]$. □

In our analysis of convergence properties of the equilibrium strategy and the infinite horizon case, the set of all M-invariant subspaces plays a crucial role. Therefore we introduce a separate notation for this set:

$$\mathcal{M}^{inv} := \{\mathcal{T} \mid M\mathcal{T} \subset \mathcal{T}\}.$$

In particular, the with (8) associated set of algebraic Riccati equations, (ARE),

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (\begin{pmatrix} -A^T & 0 \\ 0 & -A^T \end{pmatrix} + \begin{pmatrix} P_1 & O_1 \\ P_2 & O_2 \end{pmatrix} G^{-1} \begin{pmatrix} B_1^T & 0 \\ 0 & B_2^T \end{pmatrix}) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} +$$
$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \left(-A + (B_1 B_2) G^{-1} \begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} \right) + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} ((B_1 B_2) G^{-1} \begin{pmatrix} B_1^T & 0 \\ 0 & B_2^T \end{pmatrix}) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} +$$
$$\left(\begin{pmatrix} P_1 & O_1 \\ P_2 & O_2 \end{pmatrix} G^{-1} \begin{pmatrix} P_1^T \\ O_2^T \end{pmatrix} - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \right), \tag{10}$$

can be calculated directly from the following collection of M-invariant subspaces:

$$\mathcal{K}^{pos} := \left\{ \mathcal{K} \in \mathcal{M}^{inv} | \mathcal{K} \oplus Im \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix} = R^{3n} \right\}.$$

The exact result on how all solutions of (ARE) can be calculated is given in the next theorem. In this theorem we use the notation $\sigma(H)$ to denote the spectrum of matrix H.

Theorem 4:
(ARE) has a real solution (K_1, K_2) if and only if $K_1 = YX^{-1}$ and $K_2 = ZX^{-1}$ for some $\mathcal{K} =: Im \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \in \mathcal{K}^{pos}$.

Moreover, if the strategies $\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = -G^{-1} \begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix} \Phi(t,0)x_0$ are used in system (1), the spectrum of the closed-loop matrix $A - (B_1 B_2)G^{-1} \begin{pmatrix} P_1^T + B_1^T K_1(t) \\ O_2^T + B_2^T K_2(t) \end{pmatrix}$ coincides with $\sigma(-M|_{\mathcal{K}})$. □

Next we consider the question how the open-loop equilibrium solution changes when the planning horizon t_f tends to infinity. To study convergence properties for problem (1), it seems reasonable to require that problem (1) has a properly defined solution for every finite planning horizon. Therefore we will make the following well-posedness assumption (see theorem 1),

$$H(t_f) \text{ is invertible for all } t_f < \infty. \tag{11}$$

To derive general convergence results we define first

Definition 5:
M is called dichotomically separable if there exist subspaces V_1 and V_2 such that $MV_i \subset V_i, i = 1,2$, $V_1 \oplus V_2 = \mathbb{R}^{3n}$, where $\dim V_1 = n$, $\dim V_2 = 2n$, and moreover $Re\ \lambda > Re\ \mu$ for all $\lambda \in \sigma(M|V_1), \mu \in \sigma(M|V_2)$. □

Theorem 6:
Assume that the well-posedness assumption (11) holds.
Then, if M is dichotomically separable and Span $\begin{pmatrix} I \\ K_{1f} \\ K_{2f} \end{pmatrix} \oplus V_2 = \mathbb{R}^{3n}$,

$$K_1(0,t_f) \to Y_0 X_0^{-1}, \text{ and } K_2(0,t_f) \to Z_0 X_0^{-1}.$$

Here X_0, Y_0, Z_0 are defined by (using the notation of definition 5) $V_1 =: \text{Span}(X_0^T\ Y_0^T\ Z_0^T)^T$. □

Finally we consider the case that the performance criterion player $i = 1,2$ likes to minimize is given by:

$$\lim_{t_f \to \infty} J_i(u_1, u_2).$$

The information structure is similar to the finite-planning horizon case. Each player only knows the initial state of the system and has to choose a control for the entire infinite time horizon. So, the actions are now described as functions of time, where time runs from zero to infinity. Since we only like to consider those outcomes of the game that yield a finite cost to both players, we restrict ourselves to consider only control functions belonging to the following set,

$$U := \{\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, t \in [0,\infty) | \lim_{t_f \to \infty} J_i(u_1, u_2) < \infty, \ i = 1, 2.\}$$

Note that a necessary condition that must be satisfied by the system is that both (A, B_1) and (A, B_2) are stabilizable. Moreover, we assume that Q_i is positive definite w.r.t. the controllability subspace $< A, B_i >$. We have:

Theorem 7:
The infinite-planning horizon two-player linear quadratic differential game has for every initial state an open-loop Nash equilibrium strategy $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ if and only if there exist K_1 and K_2 that are solutions of the algebraic Riccati equations (ARE) satisfying the additional constraint that the eigenvalues of $A_{cl} := A - (B_1 B_2)G^{-1}\begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix}$ are all situated in the left half complex plane. In that case, the strategy

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = -G^{-1}\begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix} \Phi(t,0)x_0,$$

where $\Phi(t,0)$ satisfies the transition equation $\dot{\Phi}(t,0) = A_{cl}\Phi(t,0)$; $\Phi(0,0) = I$, is an open-loop Nash equilibrium strategy. Moreover, the costs obtained by using this strategy for the players are $\frac{1}{2}x_0^T M_i x_0$, $i = 1,2$, where M_i is the unique positive semi-definite solution of the Lyapunov equation

$$A_{cl}^T M_i + M_i A_{cl} + (I \ - G^{-1}\begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix}) F_i \begin{pmatrix} I \\ (-G^{-1}\begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix})^T \end{pmatrix} = 0.$$

□

An algorithm to calculate all infinite horizon equilibria

In general the question arises how all equilibria can be calculated for the infinite planning horizon game. From the previous analysis one can deduce the following algorithm:

Algorithm 8:

Step 1 : Calculate matrix M from (5).

Step 2 : Calculate the spectrum of matrix M.

If the number of positive eigenvalues (counted with algebraic multiplicities) is less than n, goto Step 5.

Step 3 : Calculate all M invariant subspaces $\mathcal{K}$in $\mathcal{K}^{pos}$ for which $Re\lambda > 0$ for all $\lambda \in \sigma(M|_{\mathcal{K}})$. If this set is empty, goto Step 5.

Step 4 : Let $\mathcal{K}$ be an arbitrary element of the set determined in Step 3.

Calculate 3 nxn matrices X, Y and Z such that $Im \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathcal{K}$.

Denote $K_1 := YX^{-1}$and $K_2 := ZX^{-1}$.

$\begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} = -G^{-1} \begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix} \Phi(t,0)x_0$ is an open – loop Nash equilibrium strategy.

The spectrum of the corresponding closed – loop matrix

$A - (B_1 \ B_2)G^{-1} \begin{pmatrix} P_1^T + B_1^T K_1 \\ O_2^T + B_2^T K_2 \end{pmatrix}$ equals $\sigma(-M|_{\mathcal{K}})$.

If the set determined in step 3 contains more elements one can repeat this step to calculate different equilibria.

Step 5 : End of algorithm.

Note that if the diagonal entries of matrix A are relatively large compared to the other entries of matrix M, one can easily draw conclusions on the inertia of matrix M using Gersgorin's theorem (see e.g. Lancaster et al. (1985)). In particular, this implies that in case the players discount their losses, generically the game has a unique equilibrium, if this discount factor is chosen large enough. In the next section we will use this property and the algorithm to calculate the unique open-loop Nash equilibrium in the three player EMU dynamic stabilization game.

IV. A dynamic stabilization game in the EMU

This section considers an application of the theoretical results of section 2. An interesting application can be considered in the context of the Economic and Monetary Union (EMU) to which the European Union (EU) countries have committed themselves to start on January 1, 1999, with the signing and ratifying of the Maastricht Treaty on EMU in 1991. With the EMU, the EU countries replace their national currencies and monetary policy autonomy by a common currency, the Euro, and a common monetary policy that is designed and implemented by the European Central Bank (ECB). Fiscal management, on the other hand, remains predominantly a national competence under

EMU. With national monetary policy and exchange rate adjustment no longer available under EMU, the stabilization of output fluctuations is likely to require more fiscal flexibility under EMU. The EMU Treaty, however, stipulates a set of fiscal stringency criteria -the fiscal convergence criteria and the 'Excessive Deficit' procedure- that will actually reduce fiscal flexibility in the EMU considerably. The fiscal restrictions are motivated by fears that large fiscal deficits will undermine the credibility of the low-inflation commitment of the ECB.

In a situation of recession in the EU, expansionary monetary and fiscal policies, however, will increase output rather than inflation and a conflict arises between the national fiscal authorities and the ECB to which extent monetary and fiscal policies should adjust to stimulate macroeconomic recovery in the EU. To model such possible dynamic stabilization conflicts that may arise in EMU, we extend a recently developed approach by Neck and Dockner (1995) who analyse the dynamic interaction of the monetary authorities of two symmetric countries. Their analysis considers a dynamic two-country model that builds on an earlier analysis by Turnovsky, Başar and d'Orey (1988) and Turnovsky (1986), which on their turn are two-country model extensions of the Dornbusch (1976) model. In their analyses, the monetary policies of both countries affect short-term output in both the domestic and foreign economies. The interdependencies of both economies, hence, create a dynamic conflict between both monetary authorities. Output, inflation and exchange rate adjustment and their implications for social welfare are calculated for a number of different modes of strategic interaction: (i) cooperative equilibria, (ii) open-loop and feedback Nash equilibria and (iii) open-loop and feedback Stackelberg equilibria.

We extend these two-country models to a setting of a monetary union, implying centralized monetary policy. In addition, we consider the effects of fiscal policy in such a setting of a monetary union. We focus our attention on outcomes in the non-cooperative Nash open-loop setting using the approach developed in the previous section.

Consider a situation where EMU has been fully implemented, implying that national currencies have been replaced by a common currency, national central banks by the ECB and that the exchange rate has disappeared as an adjustment instrument. Capital markets are fully integrated and we abstain from any country-risk premia implying that any interest differential is arbitraged away instantaneously. On the other hand, we assume that there is no labour mobility between both EMU parts and that goods and labour markets adjust sluggishly. Hence, the model displays Keynesian features in the short-run.

The economic structure of the two-country EMU is given by the following equations,

Table 1 A Two-Country EMU

country 1	country 2
(Ia) $y_1(t) = \delta_1 s(t) - \gamma_1 r_1(t) + \rho_1 y_2(t) +$ $+\eta_1 f_1(t)$	(Ib) $y_2(t) = -\delta_2 s(t) - \gamma_2 r_2(t) + \rho_2 y_1(t) +$ $+\eta_2 f_2(t)$
(IIa) $s(t) = p_2(t) - p_1(t)$	(IIb) $s(t) = p_2(t) - p_1(t)$
(IIIa) $r_1(t) = i(t) - \dot{p}_1(t)$	(IIIb) $r_2(t) = i(t) - \dot{p}_2(t)$
(IVa) $m_1(t) - p_1(t) = \kappa_1 y_1(t) - \lambda_1 i(t)$	(IVa) $m_2(t) - p_2(t) = \kappa_2 y_2(t) - \lambda_2 i(t)$
(Va) $\dot{p}_1(t) = \xi_1 y_1(t)$	(Vb) $\dot{p}_2(t) = \xi_2 y_2(t)$

in which, y, denotes real output, p, the output price level, i the nominal interest rate and, r, the real interest rate. s measures competitiveness of country 1 vis-à-vis country 2 as it is defined as the output price differential. f, equals the real fiscal deficit that the fiscal authority sets. m denotes the amount of nominal money balances that the public demands. Except for the nominal interest rate and the rate of inflation, variables are in logarithms and expressed as deviations from their long-run non-inflationary equilibrium. (I) is the aggregate demand function having competitiveness, the real interest rate, foreign output and the fiscal deficit as its arguments. (II) defines the competitiveness of the EMU countries relative to eachother. The definition of the real interest rate is given in (III). The demand for real balances of the common currency is given in (IV). (V), finally, gives the short run trade-off between inflation and output, along the Phillips curve.
The common money market is cleared by the common nominal interest rate, i, and which is found by imposing the condition that the total supply of Euro base money, $m^E(t)$, which is set by the ECB, equals demand, $m_1(t) + m_2(t)$. Money market equilibrium, therefore, results if,

$$(VI) \quad i(t) = \frac{1}{\lambda_1 + \lambda_2}(\kappa_1 y_1(t) + \kappa_2 y_2(t) + p_1(t) + p_2(t) - m^E(t))$$

Both economies are connected by a number of channels through which price and output-fluctuations in one part transmit themselves to the other part of the EMU. Output fluctuations in both economies transmit themselves partly to the other EMU part through the import channel. Therefore, the relative openness of both economies, as measured by ρ_i, implies an important interdependency of both economies. Also through their effect on the demand for the common currency at the common money market, output fluctuations in the domestic economy have repurcussions for the foreign economy. Price differences between the foreign and domestic economy affect relative competitiveness of both EMU parts, as measured by $s(t)$, and therefore production in both economies. In addition, price fluctuations affect output because of the short-run output inflation trade-off and affect money demand and by that the common interest rate.
Combining (I)-(VI) yields after some rewriting, that the model can be rewritten as the following first-order two-dimensional linear differential equation, with as state variables the price level of both countries, $p_i(t)$, and as control variables the policy instruments of

the ECB, $m^E(t)$, and the fiscal deficits, $f_1(t)$ and $f_2(t)$, set by the fiscal authorities:

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t) + B_3u_3(t),\ x(0) = x_0,, \tag{12}$$

$$y(t) = Cx(t) + D_1u_1(t) + D_2u_2(t) + D_3u_3(t), \tag{13}$$

where $x := \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, $u_1 := f_1$, $u_2 := f_2$, $u_3 := m^E$ and $y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. For the specific parameter values we refer to the Appendix.

Having modeled the economies of both EMU countries and derived the adjustment dynamics of output and prices over time, we still need to determine the monetary and fiscal policies and to derive their dynamic adjustment over time as a consequence of interaction between the macroeconomic policymakers. In order to do so, we need to specify objective functions of the players. We assume that the players have quadratic objective functions. Fiscal authorities are assumed to care about stabilization of inflation, output and fiscal deficits, i.e.

$$J_{F_1} := \frac{1}{2}\int_0^{\infty}\{\dot{p}_1^2(t) + \alpha_1 y_1^2(t) + \beta_1 f_1^2(t)\}e^{-\theta t}dt, \tag{14}$$

$$J_{F_2} := \frac{1}{2}\int_0^{\infty}\{\dot{p}_2^2(t) + \alpha_2 y_2^2(t) + \beta_2 f_2^2(t)\}e^{-\theta t}dt, \tag{15}$$

The assumption that the fiscal authorities value budget balance can reflect the notion that high deficits, while beneficial to stimulate output, are not costless: they, to some extent, crowd out private investment and lead to debt accumulation that has to be serviced in the future by lower government spending and higher taxes. Deficits in the loss function also reflect the possibility that excessive deficits in the EMU will be subject to sanctions, as proposed in the 'Excessive Deficits' procedure of the Maastricht Treaty (art. 104c). Therefore, countries will prefer low fiscal deficits to high deficits.

The ECB also features inflation and output objectives. More in particular we assume that it cares about average inflation, $\dot{p}_A(t)$, and average output, $y_A(t)$, in the EU,

$$J_E := \frac{1}{2}\int_0^{\infty}\{\dot{p}_A^2(t) + \alpha_3 y_A^2(t) + \beta_3 m_E^2(t)\}e^{-\theta t}dt, \tag{16}$$

Average inflation and average output are weighted averages of the two countries, using the relative sizes $\{\omega, 1-\omega\}$ of the economies of country 1 and country 2 in the total EU economy, i.e. $\dot{p}_A(t) := \omega\dot{p}_1(t) + (1-\omega)\dot{p}_2(t)$ and $y_A(t) := \omega y_1(t) + (1-\omega)y_2(t)$.

We want to consider the dynamic stabilisation game in the context of a situation where the European countries are in a situation of a recession, implying a negative output gap, i.e. $y_i(0) < 0$, and analyse how policy instruments and output and prices adjust over time as a result of the dynamic interaction between macroeconomic policymakers in the EMU. In this dynamic interaction one can consider a number of different strategical and informational concepts. We restrict here to the open-loop Nash case as outlined in the previous sections.

V. A simulation study

Before we can use algorithm 8 to find the equilibrium of the above stabilization game, we first have to rewrite model (12)-(16) as a standard three player game corresponding with (1).

To that end introduce $\dot{\tilde{x}}(t) := e^{-\frac{1}{2}\theta t}x(t)$, $\tilde{y}(t) := e^{-\frac{1}{2}\theta t}y(t)$ and $\tilde{u}_i(t) := e^{-\frac{1}{2}\theta t}u_i(t)$. Furthermore, let $B := (B_1\ B_2\ B_3)$; $D := (D_1\ D_2\ D_3)$; $e_{5,1} := (1\ 0\ 0\ 0\ 0)^T$, similarly, $e_{5,j}$ denote the j-th standard basis vector in $\mathbb{R}^5$ and, in general, $e_{i,j}$ denote the j-th standard basis vector in $\mathbb{R}^i$; and $g := \omega e_{2,1} + (1-\omega)e_{2,2}$, we see that the above minimization problem (12)-(16) can be rewritten as

$$\min_{\tilde{u}_i} J_i := \frac{1}{2}\int_0^\infty \{(\tilde{x}(t)^T \tilde{u}_1^T(t)\tilde{u}_2^T(t)\tilde{u}_3^T(t))^T F_i \begin{pmatrix} \tilde{x}(t) \\ \tilde{u}_1(t) \\ \tilde{u}_2(t) \\ \tilde{u}_3(t) \end{pmatrix} \}dt, \quad i = F_1, F_2, E, \tag{17}$$

subject to

$$\dot{\tilde{x}} = (A - \frac{1}{2}\theta I)\tilde{x} + B_1\tilde{u}_1 + B_2\tilde{u}_2 + B_3\tilde{u}_3,\ \tilde{x}(0) = x_0, \tag{18}$$

$$\tilde{y}(t) = C\tilde{x}(t) + D_1\tilde{u}_1(t) + D_2\tilde{u}_2(t) + D_3\tilde{u}_3(t), \tag{19}$$

where

$$F_{F_1} = \begin{pmatrix} A^T - \frac{1}{2}\theta I \\ B^T \end{pmatrix} e_{2,1}e_{2,1}^T(A - \frac{1}{2}\theta I\ B) + \alpha_1 \begin{pmatrix} C^T \\ D^T \end{pmatrix} e_{2,1}e_{2,1}^T(C\ D) + \beta_1 e_{5,3}e_{5,3}^T;$$

$$F_{F_2} = \begin{pmatrix} A^T - \frac{1}{2}\theta I \\ B^T \end{pmatrix} e_{2,2}e_{2,2}^T(A - \frac{1}{2}\theta I\ B) + \alpha_2 \begin{pmatrix} C^T \\ D^T \end{pmatrix} e_{2,2}e_{2,2}^T(C\ D) + \beta_2 e_{5,4}e_{5,4}^T;$$

and

$$F_E = \begin{pmatrix} A^T - \frac{1}{2}\theta I \\ B^T \end{pmatrix} gg^T(A - \frac{1}{2}\theta I\ B) + \alpha_3 \begin{pmatrix} C^T \\ D^T \end{pmatrix} gg^T(C\ D) + \beta_3 e_{5,5}e_{5,5}^T.$$

Now, factorize F_i as $\begin{pmatrix} Q_i & P_i & O_i & S_i \\ P_i^T & R_{1i} & N_i & T_i \\ O_i^T & N_i^T & R_{2i} & V_i \\ S_i^T & T_i^T & V_i^T & R_{3i}, \end{pmatrix}$ where Q_i are 2x2 matrices; P_i, O_i and S_i are 2x1 matrices; and all other entries are scalars. With this notation, matrix G in (2) becomes

$$G := \begin{pmatrix} R_{1F_1} & N_{F_1} & T_{F_1} \\ N_{F_2}^T & R_{2F_2} & V_{F_2} \\ T_E^T & V_E^T & R_{3E} \end{pmatrix}. \tag{20}$$

In the following simulations, we assume that the model parameters are such that this matrix G is invertible. Finally, we have to formulate matrix M. It is easily verified from (5) that $M =$

$$\begin{pmatrix} -A+\frac{1}{2}\theta I & 0 & 0 & 0 \\ Q_{F_1} & A^T-\frac{1}{2}\theta I & 0 & 0 \\ Q_{F_2} & 0 & A^T-\frac{1}{2}\theta I & 0 \\ Q_E & 0 & 0 & A^T-\frac{1}{2}\theta I \end{pmatrix} + \begin{pmatrix} B \\ -\tilde{P}_{F_1} \\ -\tilde{P}_{F_2} \\ -\tilde{P}_E \end{pmatrix} G^{-1} \begin{pmatrix} P_{F_1}^T & B_1^T & 0 & 0 \\ O_{F_2}^T & 0 & B_2^T & 0 \\ S_E^T & 0 & 0 & B_3^T \end{pmatrix}, \tag{21}$$

where we used the notation $\tilde{P}_i$ to denote the 2x3 matrix $\tilde{P}_i := (P_i\ O_i\ S_i)$. We are now able to calculate the equilibrium of the game using algorithm 8.
In the following numerical example we consider the case of two symmetric EU countries. We used the following parametervalues:
$\gamma_i = \delta_i = \rho_i = 0.5$, $\lambda_i = 0.3$, $\eta_i = 1$, $\xi_i = 0.25$, $\kappa_i = 1$, $\omega = 0.5$ and $\theta = 0.1$. This yields (see appendix)

$$C_1 = \begin{pmatrix} 1.71 & 0.33 \\ 0.33 & 1.71 \end{pmatrix};\ \Xi = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix};\ \tilde{C} = \begin{pmatrix} -1.33 & -0.33 \\ -0.33 & -1.33 \end{pmatrix};$$

and

$$\tilde{D}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix};\ \tilde{D}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix};\ \tilde{D}_3 = \begin{pmatrix} 0.833 \\ 0.833 \end{pmatrix}.$$

From this we deduce immediately that

$$A = \begin{pmatrix} -0.19 & -0.01 \\ -0.01 & -0.19 \end{pmatrix};\ B_1 = \begin{pmatrix} 0.15 \\ -0.03 \end{pmatrix};\ B_2 = \begin{pmatrix} -0.03 \\ 0.15 \end{pmatrix};\ B_3 = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix};$$

and

$$C = \begin{pmatrix} -0.77 & -0.05 \\ -0.05 & -0.77 \end{pmatrix};\ D_1 = \begin{pmatrix} 0.61 \\ -0.12 \end{pmatrix};\ D_2 = \begin{pmatrix} -0.12 \\ 0.61 \end{pmatrix};\ D_3 = \begin{pmatrix} 0.41 \\ 0.41 \end{pmatrix};$$

Substitution of these parametervalues in the performance weight criteria yields with

$$x_1 := \begin{pmatrix} -0.24 \\ -0.01 \\ 0.15 \\ -0.03 \\ 0.1 \end{pmatrix};\ y_1 := \begin{pmatrix} -0.77 \\ -0.05 \\ 0.61 \\ -0.12 \\ 0.41 \end{pmatrix};\ x_2 := \begin{pmatrix} -0.01 \\ -0.24 \\ -0.03 \\ 0.15 \\ 0.1 \end{pmatrix};\ \text{and } y_2 := \begin{pmatrix} -0.05 \\ -0.77 \\ -0.12 \\ 0.61 \\ 0.41 \end{pmatrix}$$

that

$$F_{F_1} = x_1 x_1^T + \alpha_1 y_1 y_1^T + \beta_1 e_{5,3} e_{5,3}^T,$$

and

$$F_{F_2} = x_2 x_2^T + \alpha_2 y_2 y_2^T + \beta_2 e_{5,4} e_{5,4}^T,$$

For example, choosing $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\beta_1 = \beta_2 = \beta_3 = 1$ yields then

$$Q_{F_1} = \begin{pmatrix} 0.66 & 0.04 \\ 0.04 & 0.00 \end{pmatrix};\ P_{F_1} = \begin{pmatrix} -0.51 \\ -0.03 \end{pmatrix};\ O_{F_1} = \begin{pmatrix} 0.1 \\ 0.01 \end{pmatrix};\ S_{F_1} = \begin{pmatrix} -0.34 \\ -0.02 \end{pmatrix};$$

$$R_{1F_1} = 1.89;\ N_{F_1} = -0.08;\ T_{F_1} = 0.27;\ R_{2F_1} = 0.02;\ V_{F_1} = -0.05;\ R_{3F_1} = 0.18;$$

and

$$Q_{F_2} = \begin{pmatrix} 0.00 & 0.04 \\ 0.04 & 0.66 \end{pmatrix};\ P_{F_2} = \begin{pmatrix} 0.01 \\ 0.1 \end{pmatrix};\ O_{F_2} = \begin{pmatrix} -0.03 \\ -0.51 \end{pmatrix};\ S_{F_2} = \begin{pmatrix} -0.02 \\ -0.34 \end{pmatrix};$$

$$R_{1F_2} = 0.02;\ N_{F_2} = -0.08;\ T_{F_2} = -0.05;\ R_{2F_2} = 1.89;\ V_{F_2} = 0.26;\ R_{3F_2} = 0.18.$$

Finally, we obtain in a similar way by substituting the parametervalues into the above expression for F_E that

$$Q_E = \begin{pmatrix} 0.18 & 0.18 \\ 0.18 & 0.18 \end{pmatrix};\ P_E = \begin{pmatrix} -0.11 \\ -0.11 \end{pmatrix};\ O_E = \begin{pmatrix} -0.11 \\ -0.11 \end{pmatrix};\ S_E = \begin{pmatrix} -0.18 \\ -0.18 \end{pmatrix};$$

$$R_{1E} = 0.06;\ N_E = 0.06;\ T_E = 0.11;\ R_{2E} = 0.06;\ V_E = 0.11;\ R_{3E} = 1.68.$$

Consequently, we find that

$$G = \begin{pmatrix} 1.89 & -0.08 & 0.26 \\ -0.08 & 1.89 & 0.26 \\ 0.11 & 0.11 & 1.68 \end{pmatrix}.$$

We have now specified all entries appearing in matrix M (see (21)). Substitution of these parameters yields

$$M = \begin{pmatrix} 0.195 & 0.008 & 0.012 & -0.002 & 0.001 & -0.002 & 0.005 & 0.005 \\ 0.008 & 0.195 & -0.002 & 0.001 & -0.002 & 0.012 & 0.005 & 0.005 \\ 0.500 & 0.025 & -0.204 & -0.019 & 0.002 & -0.008 & 0.017 & 0.017 \\ 0.028 & 0.001 & -0.009 & -0.243 & 0.000 & -0.000 & 0.001 & 0.001 \\ 0.001 & 0.028 & -0.000 & 0.000 & -0.243 & -0.010 & 0.001 & 0.001 \\ 0.025 & 0.500 & -0.008 & 0.002 & -0.019 & -0.204 & 0.017 & 0.017 \\ 0.138 & 0.138 & 0.008 & -0.002 & -0.002 & 0.008 & -0.234 & -0.002 \\ 0.138 & 0.138 & 0.008 & -0.002 & -0.002 & 0.008 & -0.002 & -0.234 \end{pmatrix}.$$

We can apply now algorithm 8 to determine the equilibrium of the game. Figure 1 displays the adjustment of output, prices, competitiveness, the common money supply and fiscal deficits in the Nash open-loop case that results,

Insert Figure 1 here

Panel (a) gives the adjustment of the output price level in both countries and the relative competitiveness of both countries. It was assumed that the countries feature different initial price levels, $p_1(0) = 0.05$ and $p_2(0) = 0.1$, implying that the economy of country 2 started further out of equilibrium than country 1. Output in country 2 (Panel (c)) therefore is more depressed than output in country 1 initially. The depressed state of the economy induces the policymakers to pursue active stabilization policies as shows the adjustment of fiscal deficits and money supply in Panel (b).

The actual adjusment of policy instruments, and therefore of output and prices depends critically on the weights $\{\alpha_i, \beta_i\}$ that are assumed in the performance criteria of the players. On their turn, these weights could reflect the stringency with which the Maastricht Treaty is adhered to in practice. In particular, a strict interpretation of the Treaty would imply that deficits will be penalized considerably and that the ECB's premise lies almost exclusively with price stability. A loose implementation of the Maastricht Treaty could imply that deficits are hardly penalized and that the ECB puts less emphasis on price stability. Therefore, it seems to interesting to consider outcomes under alternative values of β_i where i={1,2,3}.

Figure 2 considers outcomes for alternative values of β_i and allows us to compare outcomes under our baseline scenario (I) where $\beta_1 = \beta_2 = \beta_3 = 1$ (solid lines) with alternative scenarios (II) where $\beta_1 = \beta_2 = \beta_3 = 0.5$ (dotted lines) and (III) where $\beta_1 = \beta_2 = \beta_3 = 1.5$ (dashed lines). In the scenarios (II) and (III), the fiscal and monetary stringency criteria of the Maastricht Treaty are interpreted and implemented with less resp. more strictness than in the baseline scenario (I). The following adjustment picture emerges,

Insert Figure 2 here

As expected, less (dashed lines) viz. more (dotted lines) fiscal and monetary stringency leads to more viz. less active fiscal (Panel (d) and (e)) and monetary policies (Panel (f)). This leads to a faster viz. slower output adjustment (Panel (g) and (h)).

This simulation exercise shows that - for given values of the structural modelparameters - the actual monetary and fiscal policies, and output and price adjustment depend on the initial state of the EMU economies and the intensity with which the monetary and fiscal stringency criteria of the EMU treaty are applied.

VI. Concluding remarks

In this paper we considered the existence and asymptotic behaviour of open-loop Nash equilibrium solutions in a two-player linear quadratic game which is a generalization of the game that is usually studied in literature.

We formulated both necessary and sufficient conditions for this problem and presented, for the infinite-planning horizon game, an algorithm to compute all equilibria. It will be clear that there are still a number of open problems. More specific, one would like to get more insight into the question under which conditions on the system parameters one may expect that an equilibrium exists or, even more, there will be a unique equilibrium. In particular, for both computational purposes and for a better theoretical understanding of the open-loop problem, it would be necessary to have a global existence result for the set of Riccati differential equations (8) and the set of algebraic Riccati equations (ARE). Up to now this is, however, an unsolved problem. The obtained results can be straightforwardly generalized to the N-player game.

We used the algorithm developed in the theoretical part in a simulation study to analyse macroeconomic stabilization in the EMU. It was analyzed how monetary policy of the ECB and national fiscal policies adjust in the open-loop Nash equilibrium of the dynamic stabilization conflict between the ECB and the national fiscal authorities when the EMU faces a recession. Moreover, it was analyzed how a looser and a stricter interpretation of the monetary and fiscal stringency in the Maastricht Treaty affect outcomes in the EMU.

Appendix

In this appendix, we show how the system matrices A, B_i, C and D_i in (12,13) can be determined from the model equations (Ia)-(VI). By substituting (VI) and (Va) into (IIIa), and next, this result together with $s(t)$ from equation (IIa) into (Ia), we get for y_1

$$y_1(t) = \delta_1(p_2(t) - p_1(t)) -$$
$$\gamma_1(\frac{1}{\lambda_1 + \lambda_2}\{\kappa_1 y_1(t) + \kappa_2 y_2(t) + p_1(t) + p_2(t) - m^E(t)\} - \xi_1 y_1(t)) + \rho_1 y_2(t) + \eta_1 f_1(t),$$

and similarly for y_2

$$y_2(t) = -\delta_2(p_2(t) - p_1(t)) -$$
$$\gamma_2(\frac{1}{\lambda_1 + \lambda_2}\{\kappa_1 y_1(t) + \kappa_2 y_2(t) + p_1(t) + p_2(t) - m^E(t)\} - \xi_2 y_2(t)) + \rho_2 y_1(t) + \eta_2 f_2(t).$$

Elementary calculations shows that both equations can be rewritten as

$$\begin{pmatrix} 1 + \frac{\gamma_1\kappa_1}{\lambda_1+\lambda_2} - \gamma_1\xi_1 & \frac{\gamma_1\kappa_2}{\lambda_1+\lambda_2} - \rho_1 \\ \frac{\gamma_2\kappa_1}{\lambda_1+\lambda_2} - \rho_2 & 1 + \frac{\gamma_2\kappa_2}{\lambda_1+\lambda_2} - \gamma_2\xi_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} =$$
$$\begin{pmatrix} -\delta_1 - \frac{\gamma_1}{\lambda_1+\lambda_2} & \delta_1 - \frac{\gamma_1}{\lambda_1+\lambda_2} \\ \delta_2 - \frac{\gamma_2}{\lambda_1+\lambda_2} & -\delta_2 - \frac{\gamma_2}{\lambda_1+\lambda_2} \end{pmatrix} \begin{pmatrix} p_1(t) \\ p_2(t) \end{pmatrix} + \begin{pmatrix} \frac{\gamma_1}{\lambda_1+\lambda_2} \\ \frac{\gamma_2}{\lambda_1+\lambda_2} \end{pmatrix} m^E(t) + \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} f_1(t) + \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} f_2(t)$$

Furthermore we have from (V):

$$\begin{pmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \end{pmatrix} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

Introducing

$$C_1 := \begin{pmatrix} 1 + \frac{\gamma_1\kappa_1}{\lambda_1+\lambda_2} - \gamma_1\xi_1 & \frac{\gamma_1\kappa_2}{\lambda_1+\lambda_2} - \rho_1 \\ \frac{\gamma_2\kappa_1}{\lambda_1+\lambda_2} - \rho_2 & 1 + \frac{\gamma_2\kappa_2}{\lambda_1+\lambda_2} - \gamma_2\xi_2 \end{pmatrix},\ \tilde{C} := \begin{pmatrix} -\delta_1 - \frac{\gamma_1}{\lambda_1+\lambda_2} & \delta_1 - \frac{\gamma_1}{\lambda_1+\lambda_2} \\ \delta_2 - \frac{\gamma_2}{\lambda_1+\lambda_2} & -\delta_2 - \frac{\gamma_2}{\lambda_1+\lambda_2} \end{pmatrix},$$

$$\tilde{D}_1 := \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix},\ \tilde{D}_2 := \begin{pmatrix} 0 \\ \eta_2 \end{pmatrix} \tilde{D}_3 := \begin{pmatrix} \frac{\gamma_1}{\lambda_1+\lambda_2} \\ \frac{\gamma_2}{\lambda_1+\lambda_2} \end{pmatrix},\ \text{and } \Xi := \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix},$$

the above model can be rewritten as

$$C_1 y(t) = \tilde{C}x(t) + \tilde{D}_1 u_1(t) + \tilde{D}_2 u_2(t) + \tilde{D}_3 u_3(t)$$

$$\dot{x}(t) = \Xi y(t).$$

So, under the assumption that matrix C_1 is invertible (which holds generically), we obtain equations (12,13) if we introduce $C := C_1^{-1}\tilde{C}$, $D_1 := C_1^{-1}\tilde{D}_1$, $D_2 := C_1^{-1}\tilde{D}_2$, $D_3 := C_1^{-1}\tilde{D}_3$, $A := \Xi C$, $B_1 := \Xi D_1$, $B_2 := \Xi D_2$ and $B_3 := \Xi D_3$.

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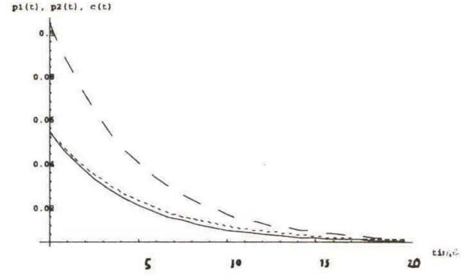
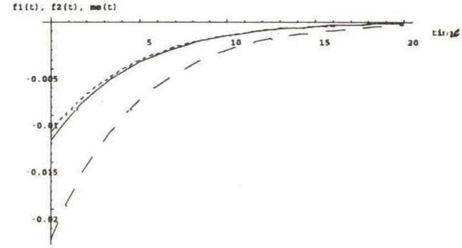
Figure 1



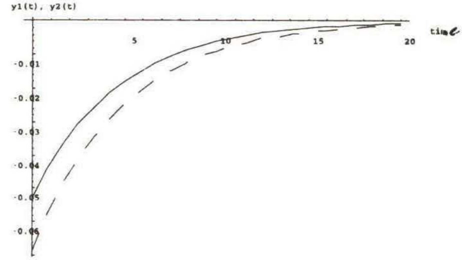
(a)

__ $p_1(t)$ ---- $p_2(t)$ .... $c(t)$



(b)

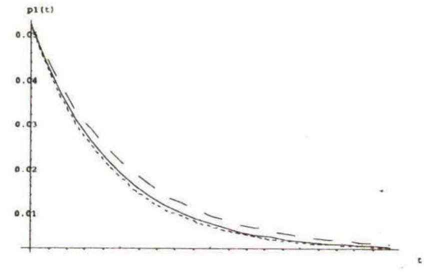
__ $f_1(t)$ ---- $f_2(t)$ .... $m_E(t)$



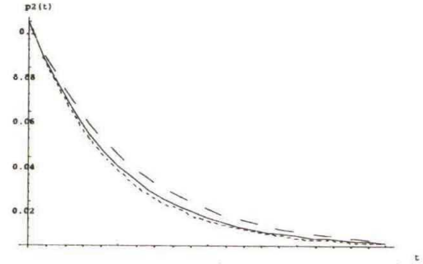
(c)

__ $y_1(t)$ ---- $y_2(t)$

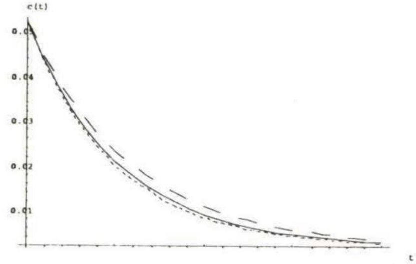
Figure 2



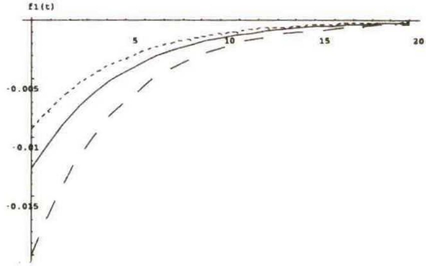
(a)
__ EMU(I) ---- EMU(II) .... EMU(III)



(b)
__ EMU(I) ---- EMU(II) .... EMU(III)

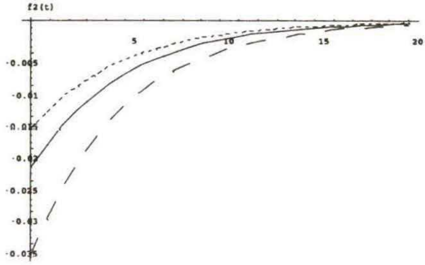


(c)
__ EMU(I) ---- EMU(II) .... EMU(III)



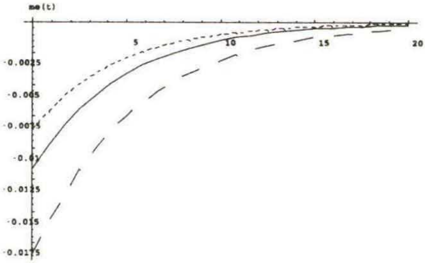
(d)

__ EMU(I) ---- EMU(II) .... EMU(III)



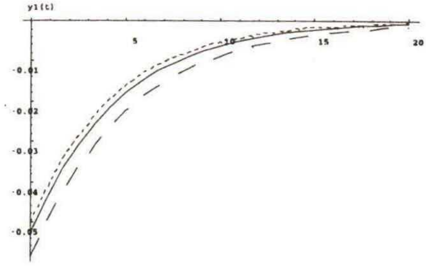
(e)

__ EMU(I) ---- EMU(II) .... EMU(III)



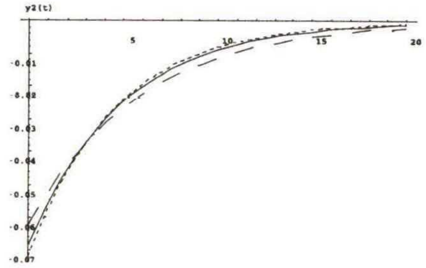
(f)

__ EMU(I) ---- EMU(II) .... EMU(III)



(g)

__ EMU(I) ---- EMU(II) .... EMU(III)



(h)

__ EMU(I) ---- EMU(II) .... EMU(III)

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