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SPACE-TIME STATIONARY SOLUTIONS FOR THE BURGERS EQUATION

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ABSTRACT. We construct space-time stationary solutions of the 1D Burgers equation with random forcing in the absence of periodicity or any other compactness assumptions. More precisely, for the forcing given by a homogeneous Poissonian point field in space-time we prove that there is a unique global solution with any prescribed average velocity. These global solutions serve as one-point random attractors for the infinite-dimensional dynamical system associated to solutions to the Cauchy problem. The probability distribution of the global solutions defines a stationary distribution for the corresponding Markov process. We describe a broad class of initial Cauchy data for which the distribution of the Markov process converges to the above stationary distribution.

Our construction of the global solutions is based on a study of the field of action-minimizing curves. We prove that for an arbitrary value of the average velocity, with probability 1 there exists a unique field of action-minimizing curves initiated at all of the Poissonian points. Moreover action-minimizing curves corresponding to different starting points merge with each other in finite time.

1. INTRODUCTION

The Burgers equation is one of the basic hydrodynamic models. It describes the evolution of velocity field of sticky particles that interact with each other only when they collide. In one space dimension, the inviscid Burgers equation is

$$(1.1) \qquad \partial_t u(x,t) + \partial_x \left(\frac{u^2(x,t)}{2} \right) = -\partial_x F(x,t).$$

Here $u(x,t)$ is the velocity of the particle located at $x \in \mathbb{R}$ at time $t \in \mathbb{R}$. The external "forcing" term $f(t,x) = -\partial_x F(x,t)$ describes the accelerations of particles. Although typically solutions of this equation develop discontinuities (shocks) in finite time, one can work with generalized solutions. So called entropy solutions, or viscosity solutions, are globally well-defined and unique for a broad class of initial velocity profiles and forcing terms.

In this paper we study the long term behavior of the Burgers dynamics for the situation where the forcing $f(x,t)$ is a space-time stationary random process. In particular, we construct space-time stationary global solutions for the Burgers equation on the real line and show that they can be viewed as one-point attractors.

The results that we present here are connected with two big streams of research developed in the last twenty years. The first one concerns stationary solutions and invariant measures for the randomly forced Burgers equation in a compact setting such as periodic forcing that effectively reduces the system to dynamics on a circle, or torus in the multidimensional version of equation (1.1), see [EKMS00], [IK03], [GIKP05], or Burgers dynamics with random boundary conditions, see [Bak07].

The main tool in the proof of these results is the Lax–Oleinik variational principle that allows for an efficient analysis of the system via studying the minimizers of the corresponding random Lagrangian system. Namely, the velocity field can be represented as $u(t,x) = \partial_x U(t,x)$, where the potential $U(t,x)$ is a solution of the Hamilton–Jacobi equation

$$(1.2) \qquad \partial_t U(x,t) + \frac{(\partial_x U(x,t))^2}{2} + F(x,t) = 0.$$

The entropy solution to Cauchy problem for this equation with initial data $U(\cdot, t_0) = U_0(\cdot)$ can be written as

$$(1.3) \qquad U(x,t) = \inf_{\gamma:[t_0,t]\to\mathbb{R}} \left\{ U_0(\gamma(t_0)) + \int_{t_0}^t \left[\frac{1}{2}\dot{\gamma}^2(s) - F(\gamma(s),s) \right] ds \right\},$$

where the infimum is taken over all absolutely continuous curves γ satisfying $\gamma(t) = x$.

The long-term behavior of the Burgers dynamics thus can be understood by studying this variational problem over long time intervals. For example, to construct stationary solutions, one has to study the behavior of Lagrangian action minimizers on an interval $[-T,t]$ as $T \to \infty$. In particular, it was proved in [EKMS00] for the one-dimensional case and in [IK03] for the multi-dimensional case that in the compact situation, for each space point x, the finite-time minimizers on $[-T,t]$ with endpoint x stabilize to a unique infinite one-sided minimizer. Moreover, there is a unique global minimizer which is a hyperbolic trajectory of the stochastic Lagrangian flow. The uniqueness of one-sided minimizers implies the following One Force – One Solution (1F1S) principle for the Burgers equation: for almost every realization of the forcing, there is a unique global solution with given average velocity, and at any given time it depends only on the history of the forcing up to that time.

The compactness condition in the above papers was extremely important. Very few results have been available in noncompact setting. These results were dealing with the case where the forcing was spatially nonhomogeneous and effectively compact, see [HK03] and [Bak12]. The aim of our paper is to consider forcing with homogeneous probability distribution in space and

time. In other words, our goal is to replace the exact periodicity (compactness) with statistical homogeneity, namely, translational invariance in distribution.

Although the 1F1S principle is a simple conceptual fact in the compact situation, it is far from obvious that stabilization of solutions occurs in the noncompact setting. Indeed, one can easily imagine a scenario where the solutions corresponding to different T will be mostly influenced by the realization of the forcing in completely different spatio-temporal patches. The main challenge in the problem, is to show that this is not the case. To tackle this difficulty, we use the second stream of research mentioned above, the theory of one-sided infinite geodesics for first passage and last passage percolation models.

A simple model of that kind can be introduced in the following way: consider a random potential on the lattice $\mathbb{Z}^2$. For a finite path on the lattice its action is defined as the sum of all values of the potential along the path. A path is called a geodesic between two points if it minimizes the action among all paths connecting these two points. An infinite path is called an infinite geodesic if every finite part of it is a geodesic between its endpoints. The theory developed by Howard and Newman in [New95],[HN97],[HN99],[HN01] describes the structure of such infinite length geodesics. It turns out that each infinite geodesic has an asymptotic direction and for a fixed direction the geodesic *tree* spans the entire lattice. These type of results can be extended to a Poissonian point field setting where the disorder is due to random distances between configuration points. The problem is also closely related to the study of Hammersley process (see [AD95],[Joh00],[Wüt02],[CP11],[CP]) which is based on optimal upright paths through Poissonian clouds. In all cases the existence of infinite geodesics is proved using techniques going back to the work of Kesten [Kes93].

In the present paper we use in a systematic way both Lagrangian methods developed in Aubry–Mather theory for action-minimizing trajectories and weak KAM theory for Hamilton–Jacobi equation (see [Fat12],[E99]), and probabilistic techniques related to the first passage percolation problem. We consider equation (1.1) and assume that the random forcing $f(x,t) = -\partial_x F(x,t)$ is associated with a space-time homogeneous Poisson point field. One can think of every Poissonian point as a source of a localized delta-type potential. This means that $\int_{t_0}^t F(\gamma(s),s)ds$ in (1.3) is equal to the number of Poissonian points visited by the curve $\gamma(s), s \in [t_0, t]$. Such singular random potentials were first introduced in the context of Burgers equation in [Bak12].

For systems with this type of forcing, shocks are created at the Poissonian points. One can say that the main focus of analysis is to study forward dynamics of shocks and their merging and the backward dynamics of minimizers (action-minimizing curves) and their coalescence. The crucial difference between [Bak12] and the present paper is that the Poissonian field in [Bak12] is assumed to be very nonhomogeneous in space with density decaying to zero at infinity. This results in quasi-compactness for the problem,

and the asymptotic analysis of the model is related to the one in the compact case. In particular there exists a unique global minimizer and all other minimizers coalesce with it (see also [HK03]). By imposing the homogeneity condition on the density of the Poissonian point field, we are forced to deal with a completely noncompact situation. Despite the fact that the global behavior here is quite different (in particular, a global minimizer does not exist), one still can prove almost sure existence and uniqueness of global solutions for fixed "average" velocity. The global solutions are stationary in time and space which reflects the translation-invariance of the probability distribution for the forcing. The construction is based on a study of a global tree of action-minimizing curves associated with every Poissonian point.

We finish the introduction with a discussion of future directions. We believe that the theory we discuss in this paper opens a wide research area and below we formulate the most interesting and important open problems. We plan to address some of the problems listed below in the future.

The first step in developing a general theory is elimination of the singular character of the forcing. There are several settings where such elimination looks plausible. Consider a smooth nonnegative potential $F(x, t)$ vanishing outside of a disk of small radius $\varepsilon > 0$ and equal to ε^{-1} everywhere in the disk except for a thin boundary layer. We then consider a homogeneous Poissonian point field and assume that every Poissonian point (x_i, t_i) contributes $F(x - x_i, t - t_i)$ to the external potential, so that the total potential is given by the sum of these contributions over all configuration points. In a certain sense the system considered in the present paper corresponds to the limit $\varepsilon \to 0$. We believe that similar results hold for the model with small positive $\varepsilon > 0$. Namely, for ε small enough, there exists a stationary global solution that approaches the global solution constructed in our paper as $\varepsilon \to 0$.

Another interesting generalization concerns systems with kick forcing. The corresponding variational problem will look the following way.

$$U_v(x_0, 0) = \min_{(x_i)_{i=-\infty}^0} \left[\sum_i A_i(x_i, x_{i+1}) + \frac{(x_{i+1} - x_i - v)^2}{2} \right].$$

Here the right-hand side can be defined up to an additive constant. This means that the increments $U(x_0, 0) - U(0, 0)$ are well-defined provided that minimizing orbits for different starting points x_0 are asymptotic to each other. We shall assume that random kernels A_i are independent copies of a double stationary process $A(x, y)$ (i.e., the distribution of the process $A(x+a, y+b)$ does not depend on a, b) with fast enough decay of correlations. The condition of double stationarity is important since it guarantees that the shape function (see Section 4) is quadratic. However, it is natural to conjecture that existence and uniqueness of a global solution only requires that the distribution of $A_i(x + a, y + a)$ does not depend on a.

Of course, the most general result will apply to any reasonable stationary forcing potential $F(x, t)$ in equation (1.2) with fast decay of correlations. However, this problem is technically too hard at present. It is also not clear

how far this program can be pushed in higher dimensions. We believe that extension of our results in this direction will use Lagrangian methods in a very essential way.

Another set of problems is connected to the viscous case. In the case of singular potential corresponding to Poissonian points, positive viscosity makes them irrelevant. However, for the smoothed problems discussed above, the interplay between the viscosity and potential is nontrivial. It will be interesting to construct global stationary solutions for positive viscosity ν and show that they approach the inviscid ones as $\nu \to 0$. In the case of positive viscosity, the solution on a finite time interval is determined not by a single action minimizing path, but by a random probability distribution on the space of paths of fixed length (directed polymers). It is tempting to make a conjecture that with probability 1 these random probability distributions have a random limit as the time interval $[-T, t]$ converges to $(-\infty, t]$ and the endpoint of the paths $\gamma(t)$ stays fixed. The limiting probability distributions can be considered as Gibbs measures on the interval $(-\infty, t]$. These Gibbs measures most probably cannot be extended to $(-\infty, +\infty)$. The obstruction to such an extension is related to strong fluctuations of Gibbs measure as $t \to \infty$. This is similar to the mechanism preventing existence of global minimizers.

Finally we briefly discuss connection with a very active area of KPZ scalings. The global solution to the random Hamilton–Jacobi equation $U(x, t)$ constructed in the present paper is a process with stationary increments. If the increments are also weakly dependent for distant intervals then one should expect to have a Gaussian behavior with usual CLT scalings for increments on large intervals. Namely, for some $\sigma > 0$,

$$(1.4) \qquad \frac{U(x, 0) - U(0, 0)}{\sqrt{|x|}} \xrightarrow{distr} \mathcal{N}(0, \sigma^2).$$

In fact, an even stronger statement of the type of the invariance principle should hold true. It is quite easy to see that such a "non-exotic" asymptotic behavior will imply KPZ scalings for the limiting minimizer. To make a precise statement, denote by $\gamma(t), t \in (-\infty, 0]$ the one-sided minimizer with zero average velocity which originates at the origin at time zero. Then fluctuations of $\gamma(-t)$ and $U(\gamma(-t), -t) - U(0, 0)$ must be of the order $t^{2/3}$ and $t^{1/3}$ respectively. This follows from (1.4) and the balance condition between $U(\gamma(-t), -t) - U(0, -t)$ and the contribution coming from the shape function. It is expected that the probability distribution for

$$\left(\frac{\gamma(-t)}{t^{2/3}}, \frac{U(\gamma(-t), -t) - U(0, 0)}{t^{1/3}} \right)$$

converges to the universal limit which is related to Tracy–Widom distribution for the GOE random matrix ensemble. The exact form of the limiting distribution cannot be derived from the qualitative argument above. However, the universality of the limit law may be a more realistic target. We believe that the limiting behavior is determined by the asymptotic properties of stationary random point field of shocks, i.e., the points where $U(x, 0)$

is not smooth. Notice, however, that the field of shocks at a given time is equipped with a random parameter, namely age of a shock, attached to every point. The age is a time interval for which a given shock can be traced in the past. It looks plausible that asymptotic statistical properties of the point field of "aged" shock determines the KPZ-type limits completely.

2. The setting and notation

The forcing in our system is given by a Poissonian point field ω on space-time $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ with Lebesgue intensity measure. Throughout the paper we adopt the picture where the space axis of $\mathbb{R}^2$ is horizontal and the time axis is vertical and directed upward.

The configuration space Ω_0 is the space of all locally finite point sets in space-time. For a Borel set $A \subset \mathbb{R}^2$, we use $\omega(A)$ to denote the number of Poissonian points in A, and introduce the σ-algebra $\mathcal{F}_0$ generated by maps $\omega \mapsto \omega(A)$ with A running through all bounded Borel sets. The probability measure $\mathbb{P}_0$ is such that for any bounded Borel set A, $\omega(A)$ is a Poisson random variable with mean equal to the Lebesgue measure of A, and for disjoint bounded Borel sets $A_1, \ldots, A_m$, the random variables $\omega(A_1), \ldots, \omega(A_m)$ are independent.

Often, we treat the point configuration ω as a locally bounded Borel measure with a unit atom at each point of the configuration. For background on Poisson point fields, also called Poisson processes, we refer to [DVJ03].

There is a natural family of time-shift operators $(\theta^t)_{t \in \mathbb{R}}$ on the Poisson configurations: the configuration $\theta^t \omega$ is obtained from ω by shifting each point $(x, s) \in \omega$ to $(x, s - t)$.

The space of velocity potentials that we will consider will be $\mathbb{H}$, the space of all locally Lipschitz functions $W : \mathbb{R} \to \mathbb{R}$ satisfying

$$\liminf_{x \to +\infty} \frac{W(x)}{x} > -\infty,$$

$$\limsup_{x \to -\infty} \frac{W(x)}{x} < +\infty.$$

Although it is possible to work with weaker conditions, some restrictions on the growth rate of $W(x)$ as $x \to \pm\infty$ are necessary to control velocities of particles coming from $\pm\infty$.

Let us define random Hamilton–Jacobi–Hopf–Lax–Oleinik (HJHLO) dynamics on $\mathbb{H}$. For a function $W \in \mathbb{H}$, a Poissonian configuration ω, and an absolutely continuous trajectory (path) γ defined on $[s, t]$, we introduce the action

(2.1) $A_\omega^{s,t}(W, \gamma) = W(\gamma(s)) + S^{s,t}(\gamma) - \omega^{s,t}(\gamma),$

where

$$S^{s,t}(\gamma) = \frac{1}{2} \int_s^t \dot{\gamma}^2(r) dr$$

is the kinetic action, and $\omega^{s,t}(\gamma) = \omega(\{(\gamma(r), r) : r \in [s, t)\})$ denotes the number of configuration points that γ passes through. The last term in (2.1)

is responsible for the interaction with the external forcing potential corresponding to the realization of the Poissonian field.

We now consider the following minimization problem:

$$(2.2) \qquad A_\omega^{s,t}(W, \gamma) \to \inf,$$
$$\gamma(t) = x.$$

Notice that the optimal trajectories are given by straight lines for any time interval on which the trajectory stays away from the configuration points. Since Poissonian configurations are locally finite, it is sufficient to take the minimum over broken lines with vertices at configuration points.

Lemma 2.1. *There is a set $\Omega_1 \in \mathcal{F}_0$ with $\mathbb{P}(\Omega_1) = 1$ such that for all $t \in \mathbb{R}$ $\theta^t \Omega_1 = \Omega_1$, and for any $\omega \in \Omega_1$, for any $W \in \mathbb{H}$, any $x \in \mathbb{R}$ and any s, t with $s < t$ there is a path γ^* that realizes the minimum in (2.2). The path γ^* is a broken line with finitely many segments, all its vertices belong to ω.*

The proofs of this and other statements of this section are given in Section 9.

We denote the restrictions of $\mathcal{F}_0$ and $\mathbb{P}_0$ onto Ω_1 by $\mathcal{F}_1$ and $\mathbb{P}_1$. Since this restriction $\mathbb{P}_1$ still defines a Poisson point field with the same intensity measure, from now on for convenience we remove from Ω_0 the zero measure complement to Ω_1 and work with the probability space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$.

We denote the infimum (minimum) value in (2.2) by $\Phi_\omega^{s,t} W(x)$. The family of random nonlinear operators $(\Phi_\omega^{s,t})_{s \leq t}$ is the main object in this paper. Our main goal is to understand the asymptotics of $\Phi_\omega^{s,t}$ as $t - s \to \infty$.

We will need several properties of the random nonlinear operator $\Phi_\omega^{s,t}$ defined on $\mathbb{H}$. We begin with a lemma that shows that $\Phi_\omega^{s,t} W$ can be understood as the potential of the velocity field given by the terminal velocities of minimizers and that it produces a tessellation of space-time into the domains of influence of configuration points.

For a Borel set $B \subset \mathbb{R}^2$, we denote the restriction of ω to B by $\omega\big|_B$.

Lemma 2.2. *For any $\omega \in \Omega_1$, $W \in \mathbb{H}$, $s, t \in \mathbb{R}$ with $s < t$, the following holds true:*

(1) *For any $p \in \omega\big|_{\mathbb{R} \times [s,t)}$, the set O_p of points $x \in \mathbb{R}$ such that p is the last configuration point visited by a unique minimizer for problem (2.2), is open. Also, the set of points with a unique minimizer that does not pass through any configuration points is open. The union of these open sets is dense in $\mathbb{R}$.*

(2) *If x_0 belongs to one of these open sets, $\gamma(t) = x_0$, and $A_\omega^{s,t}(W, \gamma) = \Phi_\omega^{s,t} W(x_0)$, then $\Phi_\omega^{s,t} W(x)$ is differentiable at x_0 w.r.t. x, and*

$$\frac{d}{dx} \Phi_\omega^{s,t} W(x)\big|_{x=x_0} = \dot\gamma(t).$$

At a boundary point x_0 of any of the open sets introduced above, the right and left derivatives of $\Phi_\omega^{s,t} W(x)$ w.r.t. x are well defined. They are equal to the slope of, respectively, the leftmost and rightmost minimizers realizing $\Phi_\omega^{s,t} W(x_0)$.

(3) *For any $p \in \omega\big|_{\mathbb{R} \times [s,t)}$, the function $x \mapsto \frac{d}{dx} \Phi_\omega^{s,t} W(x)$ is linear in O_p.*

(4) *The function $x \mapsto \Phi_\omega^{s,t}W(x)$ is locally Lipschitz.*

The following statement is the cocycle property for the operator family $(\Phi_\omega^{s,t})$. It is a direct consequence of Bellman's principle of dynamic programming.

Lemma 2.3. *If $\omega \in \Omega_1$, then for any $W \in \mathbb{H}$, any s, r, t satisfying $s < r < t$, $\Phi_\omega^{r,t}\Phi_\omega^{s,r}W$ is well-defined and equals $\Phi_\omega^{s,t}W$. If γ is an optimal path realizing $\Phi_\omega^{s,t}W(x)$, then the restrictions of γ on $[s,r]$ and $[r,t]$ are optimal paths realizing $\Phi_\omega^{s,r}W(\gamma(r))$ and $\Phi_\omega^{r,t}(\Phi_\omega^{s,r}W)(x)$.*

Introducing $\Phi_\omega^t = \Phi_\omega^{0,t}$ we can rewrite the cocycle property as
$$\Phi_\omega^{0,t+s}W = \Phi_{\theta^t\omega}^s\Phi_\omega^tW, \quad s,t > 0, \quad \omega \in \Omega_1.$$

Note that potentials are naturally defined up to an additive constant. It is thus convenient to work with $\hat{\mathbb{H}}$, the space of equivalence classes of potentials from $\mathbb{H}$. The cocycle Φ can be projected on $\hat{\mathbb{H}}$ in a natural way. We denote the resulting cocycle on $\hat{\mathbb{H}}$ by $\hat{\Phi}$.

Let us now explain how the dynamics that we consider is connected to the classical Burgers equation. One way to describe this connection is to introduce a mollification of the Poisson integer-valued measure. We give the corresponding statement without a proof. Let us take smooth kernels $\phi, \psi : \mathbb{R} \to [0, \infty)$ with bounded support, satisfying $\int_\mathbb{R} \phi(t)dt = 1$ and $\max_{x \in \mathbb{R}} \psi(x) = 1$, and for each $\varepsilon > 0$ consider the potential of shot-noise type:
$$F_\varepsilon(x,t) = -\frac{1}{\varepsilon} \sum_{(y,s) \in \omega} \phi\left(\frac{t-s}{\varepsilon}\right) \psi\left(\frac{x-y}{\alpha(\varepsilon)}\right),$$
where α is any function satisfying $\lim_{\varepsilon \downarrow 0} \alpha(\varepsilon) = 0$.

Lemma 2.4. *With probability 1, for all $s, t, x \in \mathbb{R}$ and $W \in \mathbb{H}$, the entropy solution $U_\varepsilon(x,t)$ of the Cauchy problem for the Hamilton–Jacobi equation with smooth forcing potential $F_\varepsilon(\cdot, \cdot)$ converges, as $\varepsilon \to 0$, to $U(x,t) = \Phi_\omega^{s,t}W(x)$.*

The next statement shows that away from the Poissonian points the system we consider behaves as unforced Burgers dynamics.

Lemma 2.5. *For all $\omega \in \Omega_1$, $s \in \mathbb{R}$, $W \in \mathbb{H}$, the function $U(x,t) = \Phi_\omega^{s,t}W(x)$ is an entropy solution of the Hamilton–Jacobi equation*

(2.3)
$$\partial_t U + \frac{(\partial_x U)^2}{2} = 0.$$

in $((s,\infty) \times \mathbb{R}) \setminus \omega$.

Equivalently, $u(x,t) = \partial_x U(x,t)$ is an entropy solution of the Burgers equation (1.1) with $f \equiv 0$ in $((s,\infty) \times \mathbb{R}) \setminus \omega$. Of course, for each t, $u(x,t)$ is a piecewise continuous function of x, and at each of the countably many discontinuity points it makes a negative jump. Since a velocity field determines its potential uniquely up to an additive constant we can also introduce dynamics on velocity fields. We can introduce the space $\mathbb{H}'$ of functions w

(actually, classes of equivalence of functions since we do not distinguish two functions coinciding almost everywhere) such that for some function $W \in \mathbb{H}$ and almost every x, $w(x) = W'(x)$. This allows us to introduce the Burgers dynamics. We will say that $w_2 = \Psi_\omega^{s,t} w_1$ if $w_1 = W_1'$, $w_2 = W_2'$, and $W_2 = \Phi_\omega^{s,t} W_1$ for some $W_1, W_2 \in \mathbb{H}$.

Often in the context of the Burgers dynamics, the functions in $\mathbb{H}'$ will have negative jump discontinuities, so called shocks. Although it is not essential, we can require the functions in $\mathbb{H}'$ to be right-continuous.

Let us denote $\mathbb{H}(v_-, v_+) = \{W \in \mathbb{H} : \lim_{x \to \pm\infty}(W(x)/x) = v_\pm\}$. The spaces $\hat{\mathbb{H}}(v_-, v_+)$ are defined as classes of potentials in $\mathbb{H}(v_-, v_+)$ coinciding up to an additive constant.

The following result shows that these spaces are invariant under HJHLO dynamics. Along with Lemma 2.3 it allows to treat the dynamics as a random dynamical system with perfect cocycle property (see, e.g., [Arn98, Section 1.1]).

Lemma 2.6. *There is a set $\Omega \in \mathcal{F}_1$ with the following properties: $\mathbb{P}_1(\Omega) = 1$; for any $t \in \mathbb{R}$, $\theta^t \Omega = \Omega$;   if $\omega \in \Omega$ then for any s, t with $s < t$,*

(1) *If $W \in \mathbb{H}$, then $\Phi_\omega^{s,t} W \in \mathbb{H}$.*
(2) *If $W \in \mathbb{H}(v_-, v_+)$ for some v_-, v_+, then $\Phi_\omega^{s,t} W \in \mathbb{H}(v_-, v_+)$.*

We denote the restrictions of $\mathcal{F}_1$ and $\mathbb{P}_1$ onto Ω by $\mathcal{F}$ and $\mathbb{P}$. Since this restriction $\mathbb{P}$ still defines a Poisson point field with the same intensity measure, from now on for convenience we remove from Ω_1 the zero measure complement to Ω and work with probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In nonrandom setting the family of operators $(\Phi^{s,t})$ constructed via a variational problem of type (2.2) is called a HJHLO evolution semi-group, see [Vil09, Definition 7.33], [Fat12], but in our setting it would be more precise to call it a HJHLO cocycle.

Having defined the dynamics, we now turn to the main results.

3. Main results

We say that $u(x,t) = u_\omega(x,t)$ is a global solution for the cocycle Ψ if there is a set Ω' with $\mathbb{P}(\Omega') = 1$ such that for all $\omega \in \Omega'$, all s and t with $s < t$, we have $\Psi_\omega^{s,t} u_\omega(\cdot, s) = u_\omega(\cdot, t)$. We can also introduce the global solution as a skew-invariant function: $u_\omega(x)$ is called skew-invariant if there is a set Ω' with $\mathbb{P}(\Omega') = 1$ such that for any $t \in \mathbb{R}$, $\theta^t \Omega' = \Omega'$, and for any $t > 0$ and $\omega \in \Omega'$, $\Psi_\omega^t u_\omega = u_{\theta^t \omega}$. If $u_\omega(x)$ is a skew-invariant function, then $u_\omega(x,t) = u_{\theta^t \omega}(x)$ is a global solution. One can naturally view the potentials of $u_\omega(x)$ and $u_\omega(x,s)$ as a skew-invariant function and global solution for the cocycle $\hat{\Phi}$.

Our first result is the description of global solutions.

Theorem 3.1. *For every $v \in \mathbb{R}$ there is a unique (up to zero-measure modifications) skew-invariant function $u_v : \Omega \to \mathbb{H}'$ such that for almost every $\omega \in \Omega$, the potential $U_{v,\omega}$ defined by $U_{v,\omega}(x) = \int^x u_{v,\omega}(y)dy$ belongs to $\hat{\mathbb{H}}(v, v)$.*

The potential $U_{v,\omega}$ is a unique skew-invariant potential in $\hat{\mathbb{H}}(v,v)$. The skew-invariant functions $U_{v,\omega}$ and $u_{v,\omega}$ are measurable w.r.t. $\mathcal{F}|_{\mathbb{R}\times(-\infty,0]}$, i.e., they depend only on the history of the forcing. With probability 1, the realizations of $(u_{v,\omega}(y))_{y\in\mathbb{R}}$ are piecewise linear with negative jumps between linear pieces. The spatial random process $(u_{v,\omega}(y))_{y\in\mathbb{R}}$ is stationary and mixing.

Remark 3.2. Notice that this theorem can be interpreted as a 1F1S Principle: for any velocity value v, the solution at time 0 with mean velocity v is uniquely determined by the history of the forcing: $u_{v,\omega} \overset{\text{a.s.}}{=} \chi_v(\omega|_{\mathbb{R}\times(-\infty,0]})$ for some deterministic functional χ_v of the point configurations on half-plane $\mathbb{R}\times(-\infty,0]$ (we actually describe χ_v in the proof). Since the forcing is stationary in time, we obtain that $u_{v,\theta^t\omega}$ is a stationary process in t, and the distribution of $u_{v,\omega}$ is an invariant distribution for the corresponding Markov semi-group, concentrated on $\mathbb{H}'(v,v)$.

The next result shows that each of the global solutions constructed in Theorem 3.1 plays the role of a one-point pullback attractor. To describe the domains of attraction we will make assumptions on initial potentials $W \in \mathbb{H}$. Namely, we will assume that there is $v \in \mathbb{R}$ such that W and v satisfy one of the following sets of conditions:

$$v = 0,$$

(3.1)
$$\liminf_{x\to+\infty} \frac{W(x)}{x} \geq 0,$$

$$\limsup_{x\to-\infty} \frac{W(x)}{x} \leq 0,$$

or

$$v > 0,$$

(3.2)
$$\lim_{x\to-\infty} \frac{W(x)}{x} = v,$$

$$\liminf_{x\to+\infty} \frac{W(x)}{x} > -v,$$

or

$$v < 0,$$

(3.3)
$$\lim_{x\to+\infty} \frac{W(x)}{x} = v,$$

$$\limsup_{x\to-\infty} \frac{W(x)}{x} < -v.$$

Condition (3.1) means that there is no macroscopic flux of particles from infinity toward the origin for the initial velocity profile W'. In particular, any $W \in \mathbb{H}(0,0)$ or any $W \in \mathbb{H}(v_-,v_+)$ with $v_- \leq 0$ and $v_+ \geq 0$ satisfies (3.1). It is natural to call the arising phenomenon a rarefaction fan. We will see that in this case the long-term behavior is described by the global solution u_0 with mean velocity $v = 0$.

Condition (3.2) means that the initial velocity profile W' creates the influx of particles from $-\infty$ with effective velocity $v \geq 0$, and the influence of the particles at $+\infty$ is not as strong. In particular, any $W \in \mathbb{H}(v, v_+)$ with $v \geq 0$ and $v_+ > -v$ (e.g., $v_+ = v$) satisfies (3.2). We will see that in this case the long-term behavior is described by the global solution u_v.

Condition (3.3) describes a situation symmetric to (3.2), where in the long run the system is dominated by the flux of particles from $+\infty$.

The following precise statement supplements Theorem 3.1 and describes the basins of attraction of the global solutions u_v in terms of conditions (3.1)–(3.3).

Theorem 3.3. *There is a set $\Omega'' \in \mathcal{F}$ with $\mathbb{P}(\Omega'') = 1$ such that if $\omega \in \Omega''$, $W \in \mathbb{H}$, and one of conditions (3.1),(3.2),(3.3) holds: then $w = W'$ belongs to the domain of pullback attraction of u_v in the following sense: for any $t \in \mathbb{R}$ and any $R > 0$ there is $s_0 = s_0(\omega) < t$ such that for all $s < s_0$*

$$\Psi_\omega^{s,t} w(x) = u_{v,\omega}(x, t), \quad x \in [-R, R].$$

In particular,

$$\mathbb{P}\left\{ \Psi_\omega^{s,t} w\big|_{[-R,R]} = u_{v,\omega}(\cdot, t)\big|_{[-R,R]} \right\} \to 1, \quad s \to -\infty.$$

Remark 3.4. The last statement of the theorem implies that for every $v \in \mathbb{R}$, the invariant measure on $\mathbb{H}'(v, v)$ described in Remark 3.2 is unique and for any initial condition $w = W' \in \mathbb{H}'$ satisfying one of conditions (3.1),(3.2), and (3.3), the distribution of the random velocity profile at time t converges to the unique stationary distribution on $\mathbb{H}'(v, v)$ as $t \to \infty$. However, our approach does not produce any convergence rate estimates.

Remark 3.5. Using space-time Galilean transformations, it is easy to obtain a version of Theorem 3.3 for attraction in a coordinate frame moving with constant velocity, but we omit it for brevity.

The proofs of Theorems 3.1 and 3.3 are given in Sections 7 and 8, but most of the preparatory work is carried out in Sections 4, 5, and 6.

The long-term behavior of the cocycles Φ and Ψ defined through the optimization problem (2.2) depends on the asymptotic behavior of the action minimizers over long time intervals. The natural notion that plays a crucial role in this paper is the notion of backward one-sided infinite minimizers or geodesics. A curve $\gamma : (-\infty, t] \to \mathbb{R}$ with $\gamma(t) = x$ is called a backward minimizer if its restriction onto any time interval $[s, t]$ provides the minimum to the action $A_\omega^{s,t}(W, \cdot)$ defined in (2.1) among paths connecting $\gamma(s)$ to x.

It can be shown (see Lemma 6.7) that any backward minimizer γ has an asymptotic slope $v = \lim_{t \to -\infty} (\gamma(t)/t)$. On the other hand, for every space-time point (x, t) and every $v \in \mathbb{R}$ there is a backward minimizer with slope v and endpoint (x, t). The following theorem describes the most important properties of backward minimizers associated with the Poisson point field.

Theorem 3.6. *For every $v \in \mathbb{R}$ there is a set of full measure Ω' such that for all $\omega \in \Omega'$ and any $(x, t) \in \omega$ there is a unique backward minimizer with asymptotic slope v. For any $(x_1, t_1), (x_2, t_2) \in \omega$ there is a time $s \leq$*

$\min\{t_1, t_2\}$ *such that both minimizers coincide before* s, *i.e.,* $\gamma_1(r) = \gamma_2(r)$ *for* $r \leq s$.

The proof of this core statement of this paper is spread over Sections 4 through 6. In Section 4 we apply the sub-additive ergodic theorem to derive the linear growth of action. In Section 5 we prove quantitative estimates on deviations from the linear growth. We use these results in Section 6 to analyze deviations of optimal paths from straight lines and deduce the existence of infinite one-sided optimal paths and their properties.

4. Optimal action asymptotics and the shape function

In this section we study the asymptotic behavior of the optimal action between space-time points (x, s) and (y, t) denoted by

$$(4.1) \qquad A^{s,t}(x,y) = A_\omega^{s,t}(x,y) = \min_{\gamma:\gamma(s)=x,\gamma(t)=y} (A_\omega^{s,t}(0,\gamma))$$
$$= \min_{\gamma:\gamma(s)=x,\gamma(t)=y} \left(S^{s,t}(\gamma) - \omega^{s,t}(\gamma) \right)$$

(the minimum is taken over all absolutely continuous paths γ or, equivalently, over all piecewise linear paths with vertices at configuration points). Although to construct stationary solutions for the Burgers equation, we will need the asymptotic behavior as $s \to -\infty$, it is more convenient and equally useful (due to the obvious symmetry in the variational problem) to formulate most results for the limiting behavior as $t \to \infty$, and so we will here and in the next two sections.

We begin with some simple observations on Galilean shear transformations of the point field.

Lemma 4.1. *Let* $a, v \in \mathbb{R}$ *and let* L *be a transformation of space-time defined by* $L(x, s) = (x + a + vs, s)$.

(1) *Suppose that* γ *is a path defined on a time interval* $[t_0, t_1]$ *and let* $\bar{\gamma}$ *be defined by* $(\bar{\gamma}(s), s) = L(\gamma(s), s)$. *Then*

$$S^{t_0,t_1}(\bar{\gamma}) = S^{t_0,t_1}(\gamma) + (\gamma(t_1) - \gamma(t_0))v + \frac{(t_1 - t_0)v^2}{2}.$$

(2) *Let* $L(\omega)$ *be the point configuration obtained from* $\omega \in \Omega$ *by applying* L *pointwise. Then* $L(\omega)$ *is also a Poisson process with Lebesgue intensity measure.*

(3) *Let* $\omega \in \Omega$. *For any time interval* $[t_0, t_1]$ *and any points* $x_0, x_1, \bar{x}_0, \bar{x}_1$ *satisfying* $L(x_0, t_0) = (\bar{x}_0, t_0)$ *and* $L(x_1, t_1) = (\bar{x}_1, t_1)$,

$$A_{L(\omega)}^{t_0,t_1}(\bar{x}_0, \bar{x}_1) = A_\omega^{t_0,t_1}(x_0, x_1) + (x_1 - x_0)v + \frac{(t_1 - t_0)v^2}{2},$$

and L *maps minimizers realizing* $A_\omega^{t_0,t_1}(x_0, x_1)$ *onto minimizers realizing* $A_{L(\omega)}^{t_0,t_1}(\bar{x}_0, \bar{x}_1)$.

(4) *For any points* $x_0, x_1, \bar{x}_0, \bar{x}_1$ *and any time interval* $[t_0, t_1]$,

$$A^{t_0,t_1}(\bar{x}_0, \bar{x}_1) \stackrel{distr}{=} A^{t_0,t_1}(x_0, x_1) + (x_1 - x_0)v + \frac{(t_1 - t_0)v^2}{2},$$

where

$$v = \frac{(\bar{x}_1 - x_1) - (\bar{x}_0 - x_0)}{t_1 - t_0}.$$

PROOF: The first part of the Lemma is a simple computation:

$$S^{t_0,t_1}(\bar{\gamma}) = \frac{1}{2} \int_{t_0}^{t_1} (\dot{\gamma}(s) + v)^2 ds$$

$$= \frac{1}{2} \int_{t_0}^{t_1} \dot{\gamma}^2(s) ds + \int_{t_0}^{t_1} \dot{\gamma}(s) v ds + \frac{1}{2} \int_{t_0}^{t_1} v^2 ds.$$

The second part holds since L preserves the Lebesgue measure. The third part follows from the first one since the images of paths transformed by L are also paths passing through the L-images of configuration points. The last part is a consequence of the previous two parts, since the appropriate Galilean transformation sending (x_0, t_0) to $(\bar{x}_0, t_0)$ and (x_1, t_1) to $(\bar{x}_1, t_1)$ preserves the Lebesgue measure and the distribution of the Poisson process. □

The next useful property is the sub-additivity of action along any direction: for any velocity $v \in \mathbb{R}$, and any $t, s \geq 0$, we have

$$A^{0,t+s}(0, v(t+s)) \leq A^{0,t}(0, vt) + A^{t,t+s}(vt, v(t+s)).$$

This means that we can apply Kingman's sub-additive ergodic theorem to the function $t \mapsto A^{0,t}(0, vt)$ if we can show that $-\mathbb{E}A^{0,t}(0, vt)$ grows at most linearly in t. We claim this linear bound in the following proposition:

Lemma 4.2. *Let $v \in \mathbb{R}$. There exist constants $C = C(v) > 0$ and $t_0 > 0$ such that for all $t \geq t_0$*

$$\mathbb{E}|A^{0,t}(0, vt)| \leq Ct.$$

PROOF: Lemma 4.1 implies that it is enough to prove this for $v = 0$. So in this proof we work with $A^t = A^t(0, 0)$.

Let $\gamma : [0, t] \to \mathbb{R}$ be a path realizing A^t. We have $\gamma(0) = \gamma(t) = 0$. Let us split up $\mathbb{R}^2$ into unit blocks $B_{i,j} = [i, i+1) \times [j, j+1)$, for $i, j \in \mathbb{Z}$. We define $\mathcal{A}$ as the union of all indices (i, j) such that γ passes through $B_{i,j}$. The set $\mathcal{A}$ is a lattice animal, i.e., it is a connected set that contains the origin $(0, 0) \in \mathbb{Z}^2$ (see, e.g.,[GK94]). Let us introduce the event $E_{n,t} = \{\#\mathcal{A} = n\}$.

Lemma 4.3. *There are constants $C_1, C_2, R, t_0 > 0$ such that if $t \geq t_0$ and $n \geq Rt$, then*

$$\mathbb{P}(E_{n,t}) \leq C_1 \exp(-C_2 n^2/t).$$

PROOF: We define $X_{i,j} = \omega(B_{i,j})$, the number of Poisson points in $B_{i,j}$. Define the weight of the animal $\mathcal{A}$ as

$$N_{\mathcal{A}} = \sum_{\nu \in \mathcal{A}} X_\nu.$$

Clearly, the number of Poisson points picked up by γ between 0 and t, is upper bounded by $N_{\mathcal{A}}$. Define $k_j = \#\{i \in \mathbb{Z} : (i, j) \in \mathcal{A}\}$, the number of blocks hit on the jth row. These blocks will form a connected row of

length k_j, and the kinetic action accumulated between j and $(j+1) \wedge t$ can therefore be bounded by

$$\frac{1}{2} \int_j^{(j+1)\wedge t} \dot{\gamma}^2(s)\, ds \geq \frac{1}{2} (k_j - 2)_+^2 .$$

Here, $a_+ = \max(0, a)$. This leads to the following bound on the action:

$$A^t \geq \frac{1}{2} \sum_{0 \leq j < t} (k_j - 2)_+^2 - N_{\mathcal{A}}.$$

On $E_{n,t}$ we have $\sum_{0 \leq j < t} k_j = n$. Since $a \mapsto (a-2)_+^2$ is convex, we can use Jensen's inequality to see that

$$\frac{1}{2} \sum_{0 \leq j < t} (k_j - 2)_+^2 \geq \frac{1}{2} \lceil t \rceil \left(\frac{n}{\lceil t \rceil} - 2 \right)_+^2 .$$

Therefore,

$$(4.2) \qquad A^t \geq \frac{1}{2} \lceil t \rceil \left(\frac{n}{\lceil t \rceil} - 2 \right)_+^2 - N_{\mathcal{A}}.$$

We also know that $A^t \leq 0$ since we can use the identical zero path on $[0, t]$. Hence, on $E_{n,t}$ we have

$$N_{\mathcal{A}} \geq \frac{1}{2} \lceil t \rceil \left(\frac{n}{\lceil t \rceil} - 2 \right)_+^2 .$$

Furthermore, if N_n is the weight of the greedy animal of size n (i.e., the animal of size n with greatest weight), then $N_n \geq N_{\mathcal{A}}$, and

$$E_{n,t} \subset \left\{ N_n \geq \frac{1}{2} \lceil t \rceil \left(\frac{n}{\lceil t \rceil} - 2 \right)_+^2 \right\}.$$

Let us recall that the reasoning in [CGGK93] after equation (2.12) implies that, due to standard large deviation estimates and the exponential growth of the number of lattice animals as a function of size n, there are constants $K_1, K_2, y_0 > 0$, such that if

$$(4.3) \qquad\qquad\qquad y \geq y_0,$$

then

$$(4.4) \qquad\qquad \mathbb{P}\{N_n \geq yn\} \leq K_1 \exp(-K_2 n y).$$

We now need to make sure that (4.3) holds for $y = \frac{1}{2n} \lceil t \rceil \left(\frac{n}{\lceil t \rceil} - 2 \right)_+^2$. If we require $n \geq \max(4, 8y_0) \lceil t \rceil$, then

$$\frac{1}{2n} \lceil t \rceil \left(\frac{n}{\lceil t \rceil} - 2 \right)_+^2 = \frac{1}{2n} \lceil t \rceil \left(\frac{n - 2\lceil t \rceil}{\lceil t \rceil} \right)_+^2 \geq \frac{1}{2n} \lceil t \rceil \left(\frac{n - \frac{n}{2}}{\lceil t \rceil} \right)^2 \geq \frac{1}{8} \frac{n}{\lceil t \rceil} \geq y_0,$$

and the lemma follows from (4.4). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 4.4. We will choose the constant R to be an integer, making it larger if needed.

From (4.2) we already know that on $E_{n,t}$ we have $0 \geq A^t \geq -N_n$. We wish to use this to estimate $\mathbb{E}|A^t|$, but we need an extension of (4.4).

Lemma 4.5. *For any $k \geq 1$, there is $c_k > 0$ such that for all $n \geq 1$,*

$$\mathbb{E}N_n^k \leq c_k n^k.$$

PROOF: Clearly,

$$\mathbb{E}N_n^k = \sum_{i=0}^{\lfloor y_0 n \rfloor} i^k \mathbb{P}\{N_n = i\} + \sum_{i=\lfloor y_0 n \rfloor + 1}^{\infty} i^k \mathbb{P}\{N_n = i\}.$$

We can bound the first term simply by

$$\sum_{i=0}^{\lfloor y_0 n \rfloor} i^k \mathbb{P}\{N_n = i\} \leq (y_0 n)^k.$$

For the second term we can use (4.4):

$$\sum_{i=\lfloor y_0 n \rfloor + 1}^{\infty} i^k \mathbb{P}\{N_n = i\} \leq \sum_{i=\lfloor y_0 n \rfloor + 1}^{\infty} K_1 i^k \exp(-K_2 i).$$

The right-hand side is bounded in n and the proof is complete. □

Lemma 4.2 now follows from Lemmas 4.3 and 4.5:

$$\begin{aligned}
\mathbb{E}|A^t| &= \sum_{n \leq Rt} \mathbb{E}|A^t| \mathbf{1}_{E_{n,t}} + \sum_{n > Rt} \mathbb{E}|A^t| \mathbf{1}_{E_{n,t}} \\
&\leq \mathbb{E}N_{[Rt]} + \sum_{n > Rt} \mathbb{E}N_n \mathbf{1}_{E_{n,t}} \\
&\leq Rc_1 t + \sum_{n > Rt} \sqrt{\mathbb{E}N_n^2} \sqrt{\mathbb{P}(E_{n,t})} \\
&\leq Rc_1 t + \sqrt{c_2} \sum_{n > Rt} \sqrt{C_1} n \exp(-C_2 n^2 / (2t)) \\
&\leq Ct,
\end{aligned}$$

for C big enough. □

In fact, we can use the last calculation to obtain the following generalization of Lemma 4.2 for higher moments of A^t:

Lemma 4.6. *Let $k \in \mathbb{N}$. Then there are constants $C(k), t_0(k) > 0$ such that*

$$\mathbb{E}(|A^t|^k) \leq C(k)t^k, \quad t \geq t_0(k).$$

Now a standard application of the sub-additive ergodic theorem shows that there exists a *shape function* $\alpha(v)$ such that

(4.5) $$\frac{A^{0,t}(0, vt)}{t} \to \alpha(v), \quad \text{a.s. and in } L^1, \quad t \to \infty.$$

Furthermore, $\alpha(0) < 0$, since $A^t \leq 0$ and $\alpha(0) \leq \mathbb{E}(A^t) < 0$. It turns out that the shape function $\alpha(v)$ is quadratic in v:

Lemma 4.7. *The shape function satisfies*

$$\alpha(v) = \alpha(0) + \frac{v^2}{2}, \quad v \in \mathbb{R}.$$

PROOF: The Galilean shear map $(x,t) \mapsto (x + vt, t)$ transforms the paths connecting $(0,0)$ to $(0,t)$ into paths connecting $(0,0)$ to (vt,t). Lemma 4.1 implies that under this map the optimal action over these paths is altered by a deterministic correction $v^2 t/2$. Since α is a constant almost surely we obtain the statement of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We know now from (4.5) that $A^{0,t}(0, vt) \sim \alpha(v)t$ as $t \to \infty$ with probability 1. However, this is not enough for our purposes since we need quantitative estimates on deviations of $A^{0,t}(0, vt)$ from $\alpha(v)t$. This is the material of the next section.

5. CONCENTRATION INEQUALITY FOR OPTIMAL ACTION

The goal of this section is to prove a concentration inequality for $A^t(vt) = A^t(0, vt) = A^t_\omega(0, vt) = A^{0,t}_\omega(0, vt)$:

Theorem 5.1. *There are positive constants c_0, c_1, c_2, c_3, c_4 such that for any $v \in \mathbb{R}$, all $t > c_0$, and all $u \in (c_3 t^{1/2} \ln^2 t, c_4 t^{3/2} \ln t]$,*

$$\mathbb{P}\{|A^t(0, vt) - \alpha(v)t| > u\} \le c_1 \exp\left\{-c_2 \frac{u}{t^{1/2} \ln t}\right\}.$$

Due to the invariance under shear transformations (Lemmas 4.1 and 4.7), it is sufficient to prove this theorem for $v = 0$. We will first derive a similar inequality with $\alpha(0)t$ replaced by $\mathbb{E}A^t$, and then we will have to estimate the corresponding approximation error. We recall that $A^t \le 0$.

Lemma 5.2. *There are positive constants b_0, b_1, b_2, b_3 such that for all $t > b_0$ and all $u \in (0, b_3 t^{3/2} \ln t]$,*

$$\mathbb{P}\{|A^t - \mathbb{E}A^t| > u\} \le b_1 \exp\left\{-b_2 \frac{u}{t^{1/2} \ln t}\right\}.$$

The method of proof is derived from that for the generalized Hammersley's process in [CP11], but we have to take into account that the optimal paths are allowed to travel arbitrarily far within any bounded time interval in search for areas rich with configuration points. However, the situation where they decline too far from the kinetically most efficient path is not typical. In the remaining part of this section we will often use the following lemma showing that with high probability the minimizer γ connecting $(0,0)$ to $(0,t)$ stays within distance Rt from the origin, where R was introduced in Lemma 4.3.

Lemma 5.3. *There is a constant C_3 such that if $t \ge t_0$ and $u \ge Rt$ then*

$$\mathbb{P}\left\{\max_{s \in [0,t]} |\gamma(s)| > u\right\} \le C_3 \exp(-C_2 u^2/t),$$

where constants C_2, R, t_0 were introduced in Lemma 4.3.

PROOF: If $\max\{|\gamma(s)| : s \in [0,t]\} > u$, then the size of the lattice animal $\mathcal{A}$ traced by γ is at least u. Lemma 4.3 implies

$$\mathbb{P}\left\{\max_{s\in[0,t]} |\gamma(s)| > u\right\} \leq \sum_{n\geq u} C_1 \exp(-C_2 n^2/t) \leq C_3 \exp(-C_2 u^2/t)$$

for a constant C_3, since the first term of the series is bounded by $C_1 \exp(-C_2 u^2/t)$ and the ratio of two consecutive terms is bounded by $\exp(-C_2 R)$. □

Having Lemma 5.3 in mind, we define $\tilde{A}^t$ to be the optimal action over all paths connecting $(0,0)$ to $(0,t)$ and staying within $[-Rt, Rt]$.

Lemma 5.4. *Let constants t_0, R, C_2, C_3 be defined in Lemmas 4.3 and 5.3. For any $t > t_0$,*

$$\mathbb{P}\{A^t \neq \tilde{A}^t\} \leq C_3 \exp(-R^2 C_2 t).$$

PROOF: It is sufficient to notice that

$$\mathbb{P}\{A^t \neq \tilde{A}^t\} \leq \mathbb{P}\left\{\max_{s\in[0,t]} |\gamma(s)| > Rt\right\}$$

and apply Lemma 5.3. □

Lemma 5.5. *There is a constant D_1 such that for all $t > t_0$,*

$$0 \leq \mathbb{E}\tilde{A}^t - \mathbb{E}A^t \leq -\mathbb{E}A^t \mathbf{1}_{\left\{\sup_{s\in[0,t]} |\gamma(s)|>Rt\right\}} \leq D_1.$$

PROOF: The first two inequalities are obvious, since we have that $0 \geq \tilde{A}^t \geq A^t$. For the last one, we have

$$\begin{aligned}
-\mathbb{E}A^t \mathbf{1}_{\left\{\sup_{s\in[0,t]} |\gamma(s)|>Rt\right\}} &\leq \sum_{n>Rt} \mathbb{E}(N_n \mathbf{1}_{E_{n,t}}) \\
&\leq \sum_{n>Rt} \sqrt{\mathbb{E}N_n^2}\sqrt{\mathbb{P}(E_{n,t})} \\
&\leq \sum_{n>Rt} \sqrt{c_2}n\sqrt{C_1} \exp(-C_2 n^2/(2t)),
\end{aligned}$$

where we used Lemmas 4.3 and 4.5. The statement follows since the last series is uniformly convergent for $t > t_0$. □

To obtain a concentration inequality for $\tilde{A}$, we will apply the following lemma by Kesten [Kes93]:

Lemma 5.6. *Let $(\mathcal{F}_k)_{0\leq k\leq N}$ be a filtration and let $(U_k)_{0\leq k\leq N}$ be a family of nonnegative random variables measurable with respect to $\mathcal{F}_N$. Let $(M_k)_{0\leq k\leq N}$ be a martingale with respect to $(\mathcal{F}_k)_{0\leq k\leq N}$. Assume that for some constant $c > 0$ the increments $\Delta_k = M_k - M_{k-1}$ satisfy*

$$|\Delta_k| < c, \quad k = 1, \ldots, N,$$

and

$$\mathbb{E}(\Delta_k^2| \mathcal{F}_{k-1}) \leq \mathbb{E}(U_k| \mathcal{F}_{k-1}).$$

*Assume further that for some positive constants c_1, c_2 and some $x_0 \geq e^2 c^2$
we have*

$$\mathbb{P}\left\{\sum_{k=1}^{N} U_k > x\right\} \leq c_1 \exp(-c_2 x), \quad x \geq x_0.$$

Then

$$\mathbb{P}\{M_N - M_0 \geq x\} \leq c_3 \left(1 + c_1 + \frac{c_1}{c_2 x_0}\right) \exp\left(-c_4 \frac{x}{x_0^{1/2} + c_2^{-1/3} x^{1/3}}\right), \quad x > 0,$$

*where c_3, c_4 are universal positive constants that do not depend on N, c, c_1, c_2, x_0,
nor on the distribution of $(M_k)_{0 \leq k \leq N}$ and $(U_k)_{0 \leq k \leq N}$. In particular,*

$$\mathbb{P}\{M_N - M_0 \geq x\} \leq c_3 \left(1 + c_1 + \frac{c_1}{c_2 x_0}\right) \exp\left(-c_4 \frac{x}{2\sqrt{x_0}}\right), \quad x \leq c_2 x_0^{3/2}.$$

To use this lemma in our framework, we must introduce an appropriate martingale. For a given t we consider the rectangle $Q(t) = [-Rt, Rt] \times [0, t]$ and partition it into $N = 2Rt \cdot t = 2Rt^2$ disjoint unit squares: $Q(t) = \bigcup_{k=1}^{N} B_k$. The order of enumeration is not important. Here we assume that $t \in \mathbb{N}$, but it is easy to adapt the reasoning to the case of non-integer t.

We introduce a filtration $(\mathcal{F}_k)_{0 \leq k \leq N}$ in the following way. We set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and

$$\mathcal{F}_k = \sigma\left(\omega|_{\bigcup_{j=1}^{k} B_j}\right), \quad k = 1, \ldots, N.$$

We introduce a martingale $(M_k, \mathcal{F}_k)_{0 \leq k \leq N}$ by

$$M_k = \mathbb{E}(\tilde{A}^t | \mathcal{F}_k), \quad 0 \leq k \leq N.$$

We denote by P_k the distribution of $\omega|_{B_k}$ on the sample space Ω_k of finite point configurations in B_k. For $\omega, \sigma \in \prod_{k=1}^{N} \Omega_k$ we write

$$[\omega, \sigma]_k = (\omega_1, \ldots, \omega_k, \sigma_{k+1}, \ldots, \sigma_N) \in \prod_{k=1}^{N} \Omega_k.$$

Then

$$\Delta_k(\omega_1, \ldots, \omega_k) := M_k - M_{k-1}$$

$$= \int \tilde{A}^t_{[\omega, \sigma]_k} \prod_{j=k+1}^{N} dP_j(\sigma_j) - \int \tilde{A}^t_{[\omega, \sigma]_{k-1}} \prod_{j=k}^{N} dP_j(\sigma_j)$$

$$= \int \left(\tilde{A}^t_{[\omega, \sigma]_k} - \tilde{A}^t_{[\omega, \sigma]_{k-1}}\right) \prod_{j=k}^{N} dP_j(\sigma_j).$$

Lemma 5.7. *Let I_k denote the indicator that the minimizer connecting $(0, 0)$ to $(0, t)$ and staying in $[-Rt, Rt]$ passes through a Poissonian point in B_k. Then*

$$|\tilde{A}^t_{[\omega, \sigma]_k} - \tilde{A}^t_{[\omega, \sigma]_{k-1}}| \leq \max\{I_k([\omega, \sigma]_k), I_k([\omega, \sigma]_{k-1})\} \max\{\omega(B_k), \sigma(B_k)\}.$$

PROOF: Suppose we delete the points of ω in B_k. When we then consider the minimizer for $[\omega, \sigma]_k$, we decrease the number of Poissonian points contributing to the action by at most $\omega(B_k)$, and only decrease the kinetic action. Comparing the resulting path with the minimizer for $[\omega, \sigma]_{k-1}$, we obtain

$$\tilde{A}^t_{[\omega,\sigma]_{k-1}} \le \tilde{A}^t_{[\omega,\sigma]_k} + \omega(B_k).$$

Similarly, we get

$$\tilde{A}^t_{[\omega,\sigma]_k} \le \tilde{A}^t_{[\omega,\sigma]_{k-1}} + \sigma(B_k).$$

This shows that

$$|\tilde{A}^t_{[\omega,\sigma]_k} - \tilde{A}^t_{[\omega,\sigma]_{k-1}}| \le \max\{\omega(B_k), \sigma(B_k)\}.$$

Now remark that if none of the two minimizers (for $[\omega, \sigma]_k$ and $[\omega, \sigma]_{k-1}$) passes through a Poissonian point inside B_k, then $\tilde{A}^t_{[\omega,\sigma]_k}$ and $\tilde{A}^t_{[\omega,\sigma]_{k-1}}$ coincide. This completes the proof. $\square$

The next step is to define a truncated Poissonian configuration $\bar{\omega}$ by erasing all Poissonian points of ω in each block B_j with $\omega(B_j) > b \ln t$, where $b > 0$ will be chosen later. The restrictions of $\bar{\omega}$ to B_j, $j = 1, \ldots, N$ are jointly independent. Lemma 5.7 applies to truncated configurations as well and we obtain

$$|\tilde{A}^t_{[\bar{\omega},\bar{\sigma}]_k} - \tilde{A}^t_{[\bar{\omega},\bar{\sigma}]_{k-1}}| \le b \ln t \max\{I_k([\bar{\omega}, \bar{\sigma}]_k), I_k([\bar{\omega}, \bar{\sigma}]_{k-1})\},$$

where $\bar{\sigma}$ is obtained from σ in the same way as $\bar{\omega}$ from ω. Therefore,

$$|\Delta_k(\bar{\omega}_1, \ldots, \bar{\omega}_k)| \le b \ln t \int \max\{I_k([\bar{\omega}, \bar{\sigma}]_k), I_k([\bar{\omega}, \bar{\sigma}]_{k-1})\} \prod_{j=k}^{N} dP_j(\sigma_j) \le b \ln t.$$

We must now estimate the increments of the martingale predictable characteristic. This estimate is a straightforward analogue of Lemma 4.3 of [CP11].

Lemma 5.8. *Let* $U_k = 2(b \ln t)^2 I_k$. *Then, with probability 1,* $|U_k(\bar{\omega})| \le 2(b \ln t)^2$ *and*

$$\mathbb{E}(\Delta_k^2(\bar{\omega}_1, \ldots, \bar{\omega}_k)|\mathcal{F}_{k-1}) \le \mathbb{E}(U_k(\bar{\omega})|\mathcal{F}_{k-1}).$$

PROOF:

$$\mathbb{E}(\Delta_k^2(\bar{\omega}_1, \ldots, \bar{\omega}_k) | \mathcal{F}_{k-1}) = \int \left(\int \left(\tilde{A}_{[\bar{\omega},\bar{\sigma}]_k}^t - \tilde{A}_{[\bar{\omega},\bar{\sigma}]_{k-1}}^t \right) \prod_{j=k}^N dP_j(\sigma_j) \right)^2 dP_k(\omega_k)$$

$$\leq \int \left(\int \max\{I_k([\bar{\omega},\bar{\sigma}]_k), I_k([\bar{\omega},\bar{\sigma}]_{k-1})\} \cdot b \ln t \prod_{j=k}^N dP_j(\sigma_j) \right)^2 dP_k(\omega_k)$$

$$\leq \int \int \max\{I_k([\bar{\omega},\bar{\sigma}]_k), I_k([\bar{\omega},\bar{\sigma}]_{k-1})\} \cdot (b \ln t)^2 \prod_{j=k}^N dP_j(\sigma_j) dP_k(\omega_k)$$

$$\leq \int \int (I_k([\bar{\omega},\bar{\sigma}]_k) + I_k([\bar{\omega},\bar{\sigma}]_{k-1})) \cdot (b \ln t)^2 \prod_{j=k}^N dP_j(\sigma_j) dP_k(\omega_k)$$

$$= \mathbb{E}(U_k(\bar{\omega}) | \mathcal{F}_{k-1}).$$

$\square$

We have

$$(5.1) \qquad \sum_{k=1}^N U_k(\bar{\omega}) = 2(b \ln t)^2 \sum_{k=1}^N I_k(\bar{\omega}).$$

Since

$$\sum_{k=1}^N I_k(\bar{\omega}) \leq \#\mathcal{A}(\bar{\omega}),$$

we can write

$$\mathbb{P}\left\{ \sum_{k=1}^N U_k(\bar{\omega}) > x \right\} \leq \mathbb{P}\left\{ \#\mathcal{A}(\bar{\omega}) > \frac{x}{2(b \ln t)^2} \right\} \leq \sum_{n > x/(2(b \ln t)^2)} \mathbb{P}\{\bar{\omega} \in E_{n,t}\}.$$

It is easy to see that Lemma 4.3 applies to $\bar{\omega}$ as well as to ω, since its proof depends only on the tail estimate for the number of configuration points in each block. We can conclude that

$$\mathbb{P}\{\bar{\omega} \in E_{n,t}\} \leq C_1 \exp(-C_2 n^2/t), \quad n \geq Rt, \ t \geq t_0,$$

where C_1, C_2, R, t_0 were introduced in Lemma 4.3.

Combining the last two inequalities and choosing $x_0 = 2Rt(b \ln t)^2$, we can write for $x > x_0$

$$\mathbb{P}\left\{ \sum_{k=1}^N U_k(\bar{\omega}) > x \right\} \leq C_1 \sum_{n > x/(2(b \ln t)^2)} \exp(-C_2 n^2/t)$$

$$\leq C_4 \exp(-C_2 x^2/(4t(b \ln t)^4))$$

$$\leq C_4 \exp(-C_2 x x_0/(4t(b \ln t)^4))$$

$$\leq C_4 \exp(-C_5(b \ln t)^{-2} x).$$

The above estimates on $\Delta_k(\bar{\omega})$ and $U_k(\bar{\omega})$ allow to apply Kesten's lemma with $c = 2b \ln t$, $c_1 = C_4$, $c_2 = C_5(b \ln t)^{-2}$, $x_0 = 2Rt(b \ln t)^2$ and obtain the following statement:

Lemma 5.9. *There are constants $C_6, C_7, C_8, t_0 > 0$ such that for $t > t_0$ and $x \leq C_8 b t^{3/2} \ln t$,*

$$\mathbb{P}\{|\tilde{A}^t(\bar{\omega}) - \mathbb{E}\tilde{A}_t(\bar{\omega})| > x\} \leq C_6 \exp\left(-C_7 \frac{x}{bt^{1/2} \ln t}\right).$$

Lemma 5.10. *With probability 1,*

$$\tilde{A}^t(\omega) \leq \tilde{A}^t(\bar{\omega}).$$

Also, we can choose b and t_0 such that for all $t > t_0$ and $x > 0$,

$$\mathbb{P}\{\tilde{A}^t(\bar{\omega}) - \tilde{A}^t(\omega) > x\} \leq 2e^{-x}.$$

PROOF: The first statement of the lemma is obvious, and we have

$$0 \leq \tilde{A}^t(\bar{\omega}) - \tilde{A}^t(\omega) \leq \sum_{k=1}^{N} \omega(B_k) \mathbf{1}_{\{\omega(B_k) > b \ln t\}}.$$

By Markov's inequality and mutual independence of $\omega|_{B_k}$, $k = 1, \ldots, N$,

$$\mathbb{P}\left\{\sum_{k=1}^{N} \omega(B_k) \mathbf{1}_{\{\omega(B_k) > b \ln t\}} > x\right\} \leq e^{-x} \left[\mathbb{E}e^{\omega(B_1)\mathbf{1}_{\{\omega(B_1) > b \ln t\}}}\right]^N.$$

The lemma will follow from

$$(5.2) \qquad \lim_{t \to \infty} \left[\mathbb{E}e^{\omega(B_1)\mathbf{1}_{\{\omega(B_1) > b \ln t\}}}\right]^{2Rt^2} = 1,$$

which is implied by

$$\mathbb{E}e^{\omega(B_1)\mathbf{1}_{\{\omega(B_1) > b \ln t\}}} \leq 1 + \frac{\mathbb{E}e^{2\omega(B_1)}}{e^{b \ln t}} \leq 1 + \frac{\mathbb{E}e^{2\omega(B_1)}}{t^b},$$

if we choose $b > 2$.   □

The only missing part in the proof of Lemma 5.2 is the following corollary of Lemma 5.10:

Lemma 5.11. *There is a constant D_2 such that for all $t > t_0$,*

$$0 \leq \mathbb{E}\tilde{A}^t(\bar{\omega}) - \mathbb{E}\tilde{A}^t(\omega) < D_2.$$

PROOF OF LEMMA 5.2:   Lemmas 5.5 and 5.11 imply that for $u > D_1 + D_2$

$$\begin{aligned}
\mathbb{P}\{|A^t(\omega) - \mathbb{E}A^t(\omega)| > u\} \leq &\mathbb{P}\{|A^t(\omega) - \tilde{A}^t(\omega)| > (u - D_1 - D_2)/3\} \\
&+ \mathbb{P}\{|\tilde{A}^t(\omega) - \tilde{A}^t(\bar{\omega})| > (u - D_1 - D_2)/3\} \\
&+ \mathbb{P}\{|\tilde{A}^t(\bar{\omega}) - \mathbb{E}\tilde{A}^t(\bar{\omega})| > (u - D_1 - D_2)/3\}.
\end{aligned}$$

The Lemma follows from the estimates of the three terms provided by Lemmas 5.4, 5.9, and 5.10.   □

The following lemma quantifies how the growth of $-\mathbb{E}A^t$ deviates from the linear one under argument doubling. We will use this lemma to find an estimate on $\mathbb{E}A^t - \alpha(0)t$ which makes it possible to fill the gap between Lemma 5.2 and Theorem 5.1.

Lemma 5.12. *There is a number $b_0 > 0$ such that for any $t > t_0$,*

$$0 \leq 2\mathbb{E}A^t - \mathbb{E}A^{2t} \leq b_0 t^{1/2} \ln^2 t.$$

PROOF: The first inequality follows from $A^{0,2t}(0,0) \leq A^{0,t}(0,0) + A^{t,2t}(0,0)$. Let us prove the second one.

Let γ be the minimizer from $(0,0)$ to $(0,2t)$. Then

$$A^{2t} \geq \min_{|x| \leq 2Rt} A^{0,t}(0,x) + \min_{|x| \leq 2Rt} A^{t,2t}(x,0) + A^{2t} \mathbf{1}_{\{\max_{s \in [0,2t]} |\gamma(s)| > 2Rt\}}.$$

Therefore, by symmetry with respect to t and Lemma 5.5,

$$(5.3) \qquad \mathbb{E}A^{2t} \geq 2\mathbb{E} \min_{|x| \leq 2Rt} A^t(0,x) - D_1.$$

For $k \in I_t = \{-2Rt, \dots, 2Rt - 2, 2Rt - 1\}$, we define a unit square $B_k = [k, k+1] \times [t-1, t]$.

Let now γ be the minimizer from $(0,0)$ to (x,t), with $x \in [k, k+1]$ for some $k \in I_t$. Denote $t' = \sup\{s \leq t : \gamma(s) \notin B_k\}$ and $x' = \gamma(t')$.

If $x' < k+1$, then by reconnecting (x', t') to (k, t) we obtain

$$A^t(k) \leq A^{t'}(x') + 1/2 \leq A^t(x) + \omega(B_k) + 1/2.$$

If $x' = k+1$, then by reconnecting (x', t') to $(k+1, t)$ we obtain

$$A^t(k+1) \leq A^{t'}(x') \leq A^t(x) + \omega(B_k).$$

Therefore,

$$A^t(x) \geq \min\{A^t(k), A^t(k+1)\} - \omega(B_k) - 1/2,$$

and (5.3) implies

$$\mathbb{E}A^{2t} \geq 2\mathbb{E} \min_{k \in I_t} A^t(k) - \mathbb{E} \max_{k \in I_t} \omega(B_k) - 1/2 - D_1.$$

The second term grows logarithmically in t. Hence, for some constant $c > 0$,

$$\mathbb{E}A^{2t} \geq 2\mathbb{E} \min_{k \in I_t} A^t(k) - c(\ln t + 1).$$

Lemma 4.1 implies

$$\min_x \mathbb{E}A^t(x) = \mathbb{E}A^t(0).$$

Therefore,

$$\mathbb{E}A^{2t} \geq 2\mathbb{E} \min_{k \in I_t} A^t(k) - c(\ln t + 1)$$

$$\geq 2 \min_{k \in I_t} \mathbb{E}A^t(k) - 2\mathbb{E}X_t - c(\ln t + 1)$$

$$(5.4) \qquad \geq 2\mathbb{E}A^t - 2\mathbb{E}X_t - c(\ln t + 1),$$

where

$$X_t = \max_{k \in I_t}\{(\mathbb{E}A^t(k) - A^t(k))_+\}.$$

For a constant r to be determined later, we introduce the event

$$E = \{X_t \leq r(\ln^2 t)\sqrt{t}\}.$$

Then

$$X_t \leq r(\ln^2 t)\sqrt{t}\mathbf{1}_E + X_t \mathbf{1}_{E^c}.$$

Therefore,

$$(5.5) \qquad \mathbb{E}X_t \leq r(\ln^2 t)\sqrt{t} + \sqrt{\mathbb{E}(X_t)^2 \mathbb{P}(E^c)}.$$

Let us estimate the second term. According to Lemma 4.1, the random variables $A^t(k) - \mathbb{E}A^t(k)$, $k \in I_t$ have the same distribution, so replacing the maximum in the definition of X_t^2 with summation we obtain

$$(5.6) \qquad \mathbb{E}X_t^2 \leq 4Rt\mathbb{E}(A^t - \mathbb{E}A^t)_+^2 \leq 4Rt\mathbb{E}(A^t)^2 \leq Ct^3,$$

for some $C > 0$ and all t exceeding some t_0, where we used Lemma 4.6 in the last inequality.

Also, Lemma 5.2 shows that

$$\mathbb{P}(E^c) \leq \sum_{k \in I_t} \mathbb{P}\left\{A^t(k) - \mathbb{E}A^t(k) > r(\ln^2 t)\sqrt{t}\right\}$$
$$(5.7) \qquad \qquad \leq 4Rtb_1 \exp\{-b_2 r \ln t\}.$$

We can now finish the proof by choosing r to be large enough and combining estimates (5.4)–(5.7). $\qquad \square$

With this lemma at hand we can now use the following statement ([HN01, Lemma 4.2]):

Lemma 5.13. *Suppose the functions* $a : \mathbb{R}_+ \to \mathbb{R}$ *and* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy the following conditions:* $a(t)/t \to \nu \in \mathbb{R}$ *and* $g(t)/t \to 0$ *as* $t \to \infty$, $a(2t) \geq 2a(t) - g(t)$ *and* $\psi \equiv \limsup_{t\to\infty} g(2t)/g(t) < 2$. *Then, for any* $c > 1/(2 - \psi)$, *and for all large* t,

$$a(t) \leq \nu t + cg(t).$$

Taking $a(t) = \mathbb{E}A^t$, $\nu = \alpha(0)$, $g(t) = b_0 t^{1/2} \ln^2 t$, $\psi = \sqrt{2}$, $c = 2$, we conclude that for $b_0' = 2b_0$ and large t,

$$0 \leq \mathbb{E}A^t - \alpha(0)t \leq b_0' t^{1/2} \ln^2 t,$$

and Theorem 5.1 follows from this estimate, Lemma 5.2, and the shear invariance established in Lemma 4.1. $\qquad \square$

6. Existence and uniqueness of semi-infinite minimizers

In this section we will study the properties of geodesics: these are paths $\gamma : [t_1, t_2] \to \mathbb{R}$ such that for all $s \leq t \in [t_1, t_2]$, $\gamma|_{[s,t]}$ is the path that minimizes the action $A^{s,t}(\gamma(s), \gamma(t))$. We will closely follow ideas by Howard and Newman in [HN01] and by Wüthrich in [Wüt02], adapting them to our specific situation, as was done in [CP11].

6.1. δ**-Straightness.** The goal of this section is to estimate deviations of geodesics from straight lines. We will need the curvature of the shape function found in Section 4. Remember that

$$\lim_{t\to\infty} \frac{A^{0,t}(0, vt)}{t} = \alpha(v) = \alpha(0) + \frac{1}{2}v^2.$$

Define $\alpha_0 = \alpha(0)$. We will extend α to a function of $\mathbb{R}^2$:

$$\alpha(p) := \alpha_0 p_2 + \frac{1}{2}\frac{p_1^2}{p_2} = p_2\left(\alpha_0 + \frac{1}{2}\left(\frac{p_1}{p_2}\right)^2\right)$$

(in this section we often denote the space and time coordinates of a space-time point $p \in \mathbb{R}^2$ by p_1 and p_2, respectively). This means that for all $p \in \mathbb{R}^2$ with $p_2 > 0$,

$$\lim_{t \to \infty} \frac{A^{0, t p_2}(0, t p_1)}{t} \stackrel{\text{a.s.}}{=} \alpha(p).$$

We need a convexity estimate of this function α. Define for $p \in \mathbb{R} \times \mathbb{R}^+$ and for $L > 0$

$$\mathcal{C}(p, L) := \left\{ q \in \mathbb{R}^2 : \ q_2 \in (p_2, 2p_2] \text{ and } \left| \frac{q_2}{p_2} p_1 - q_1 \right| \le L \right\}.$$

So $\mathcal{C}(p, L)$ is a parallelogram of width $2L$ along $[p, 2p]$ (for any two points p, q on the plane, $[p, q]$ denotes the straight line segment connecting these two points). We need to consider the side-edges of this parallelogram:

$$\partial_S \mathcal{C}(p, L) := \left\{ q \in \mathbb{R}^2 : \ q_2 \in (p_2, 2p_2] \text{ and } \left| \frac{q_2}{p_2} p_1 - q_1 \right| = L \right\}.$$

For a speed $v > 0$ we define

$$\mathrm{Co}(v) = \{ p \in \mathbb{R} \times \mathbb{R}^+ : \ |p_1| \le p_2 v \}.$$

The following lemma will play the role of Lemma 2.1 in [Wüt02].

Lemma 6.1. *Let $N, v > 0$, $\delta \in (0, 1)$ and $M > N$. There is a constant $C = C(N, M, v, \delta) > 0$ such that if points p, p', q, q' satisfy $p \in \mathrm{Co}(v)$, $p_2 > M$, $q \in \partial_S \mathcal{C}(p, p_2^{1-\delta})$, $p_1' = p_1$, $|p_2' - p_2| \le N$, $q_1' = q_1$, $|q_2' - q_2| \le N$, and $p_2' < q_2'$, then*

(6.1) $$\alpha(q' - p') + \alpha(p') - \alpha(q') \ge C p_2^{1 - 2\delta}.$$

PROOF: Using the definition of α and identities $p_1' = p_1$, $q_1' = q_1$, we get

$$\alpha(q' - p') + \alpha(p') - \alpha(q') = \frac{1}{2} \frac{(q_1 - p_1)^2}{q_2' - p_2'} + \frac{1}{2} \frac{p_1^2}{p_2'} - \frac{1}{2} \frac{q_1^2}{q_2'}.$$

So, $\alpha(q' - p') + \alpha(p') - \alpha(q')$ is a quadratic function in q_1. The minimum of this function equals 0, and is attained at $\tilde{q}_1$, the number defined by

$$\frac{\tilde{q}_1 - p_1}{q_2' - p_2'} = \frac{\tilde{q}_1}{q_2'}.$$

This means that $(\tilde{q}_1, q_2')$ is a multiple of p and

(6.2) $$\alpha(q' - p') + \alpha(p') - \alpha(q') = \frac{1}{2} \frac{p_2'}{q_2'(q_2' - p_2')} (q_1 - \tilde{q}_1)^2.$$

Now note that

$$\tilde{q}_1 = \frac{q_2'}{p_2'} p_1 = \frac{q_2}{p_2} p_1 + \left(\frac{q_2'}{p_2'} - \frac{q_2}{p_2} \right) p_1.$$

By the definition of $\partial_S \mathcal{C}(p, p_2^{1-\delta})$, we know that $|q_1 - q_2 p_1/p_2| = p_2^{1-\delta}$. So,

$$
\begin{aligned}
|\tilde{q}_1 - q_1| &\geq p_2^{1-\delta} - \frac{|p_2 q_2' - p_2' q_2|}{p_2 p_2'} |p_1| \\
&\geq p_2^{1-\delta} - \frac{p_2 |q_2' - q_2| + |p_2 - p_2'| q_2}{p_2 p_2'} |p_1| \\
&\geq p_2^{1-\delta} - N \frac{|p_1|}{p_2'} - 2N \frac{|p_1|}{p_2'}.
\end{aligned}
$$

Since $p_2' \geq p_2 - N$ and $|p_1| \leq v p_2$, we see that if $M > N$ and $p_2 > M$, then

(6.3)
$$
|\tilde{q}_1 - q_1| \geq p_2^{1-\delta} - c_1 \geq c_2 p_2^{1-\delta},
$$

for some constants $c_1 = c_1(M, N, v)$, $c_2 = c_2(M, N, v, \delta)$. Also,

(6.4)
$$
\frac{p_2'}{q_2'(q_2' - p_2')} \geq \frac{p_2 - N}{(2p_2 + N)(p_2 + 2N)} \geq \frac{c_3}{p_2}.
$$

for some $c_3 = c_3(M)$. The lemma now follows from (6.2), (6.3),(6.4). $\qquad \square$

   This deterministic convexity lemma, together with the concentration bound of Section 5, will help us to show that geodesics cannot make large deviations from a straight line. For $p \in \mathbb{R}^2$, we define

(6.5) $\quad K(p, R) := \left\{ q \in \mathbb{R}^2 : |q_1 - p_1| \leq R \text{ and } |q_2 - p_2| \leq R \right\}.$

For $p, q \in \mathbb{R}^2$ satisfying $p_2 < q_2$, we denote by $\gamma_{p,q}$ the optimal path from p to q. For $p, z \in \mathbb{R}^2$ with $0 < p_2 < z_2$, we define the event

$$
G(p, z) = \left\{ \exists \tilde{0} \in K((0,0), 1), \tilde{z} \in K(z, 1) : \gamma_{\tilde{0}, \tilde{z}} \cap K(p, 1) \neq \emptyset \right\}.
$$

This says that a geodesic starting close to $(0,0)$ and ending near z, passes close to p. To bound the probability of the event $G(p, z)$, we first have to control the action to and from points close to p. For $p, z \in \mathbb{R}^2$ with $p_2 < z_2$, we denote

$$
A(p, z) = A^{p_2, z_2}(p_1, z_1).
$$

Lemma 6.2. *Suppose* $p, z \in \mathbb{R}^2$. *Let* $\tilde{p} \in K(p, 1)$, $\tilde{z} \in K(z, 1)$ *with* $\tilde{p}_2 < \tilde{z}_2$. *Define* $\underline{p} = (p_1, p_2 - 2)$, $\bar{p} = (p_1, p_2 + 2)$ *and similarly* $\underline{z}$ *and* $\bar{z}$. *Then*

$$
A(\tilde{p}, \tilde{z}) \geq A(\underline{p}, \bar{z}) - 1.
$$

If $p_2 + 2 < z_2 - 2$, *then*

$$
A(\bar{p}, \underline{z}) + 1 \geq A(\tilde{p}, \tilde{z}).
$$

PROOF: Let $\tilde{\gamma}$ be the optimal path (i.e., the path that picks up the least action) from $\tilde{p}$ to $\tilde{z}$. Define γ as the path that starts at $\underline{p}$, moves at constant speed to $\tilde{p}$, then follows $\tilde{\gamma}$, and then moves at constant speed to $\bar{z}$. Then, denoting $A[\gamma]$ for the action picked up by a path γ, we get

$$
\begin{aligned}
A(\underline{p}, \bar{z}) &\leq A[\gamma] \\
&\leq \frac{1}{2} + A(\tilde{p}, \tilde{z}) + \frac{1}{2}.
\end{aligned}
$$

We use that the first part of this path picks up at most $v^2/2$ action, where the speed $v \leq 1$. For the path from $\tilde{z}$ to $\bar{z}$ we get the same upper bound.

For the second inequality, note that

$$A(\tilde{p}, \tilde{z}) \le A(\tilde{p}, \bar{p}) + A(\bar{p}, \underline{z}) + A(\underline{z}, \tilde{z}).$$

Clearly, as described above, we have $A(\tilde{p}, \bar{p}) \le 1/2$ by taking a similar path from $\tilde{p}$ to $\bar{p}$. Likewise, $A(\underline{z}, \tilde{z}) \le 1/2$. This proves the lemma. $\square$

Lemma 6.3. *Fix $\delta \in (0, 1/4)$ and $v > 0$. There exist constants $c_1, c_2, M > 0$ (independent of δ), such that for all $p \in \text{Co}(v)$ with $p_2 > M$ and $z \in \partial_S \mathcal{C}(p, p_2^{1-\delta})$, we have*

$$\mathbb{P}(G(p, z)) \le c_1 \exp\left(-c_2 p_2^{1/2 - 2\delta} / \log(p_2)\right).$$

PROOF: Let us choose any $M > 8$. Take $z \in \partial_S \mathcal{C}(p, p_2^{1-\delta})$. Define $\underline{p} = (p_1, p_2 - 2)$, $\bar{p} = (p_1, p_2 + 2)$, and likewise $\underline{0}$, $\bar{0}$, $\underline{z}$ and $\bar{z}$. The event $G(p, z)$ implies that there exist three points $\tilde{0} \in K(0, 1)$, $\tilde{p} \in K(p, 1)$ and $\tilde{z} \in K(z, 1)$ with $\tilde{p}_2 < \tilde{z}_2$ such that

$$A(\tilde{0}, \tilde{z}) = A(\tilde{0}, \tilde{p}) + A(\tilde{p}, \tilde{z}).$$

It follows from Lemma 6.2 that

$$A(\bar{0}, \underline{z}) \ge A(\tilde{0}, \tilde{z}) - 1, \quad A(\underline{0}, \bar{p}) \le A(\tilde{0}, \tilde{p}) + 1 \text{ and } A(\underline{p}, \bar{z}) \le A(\tilde{p}, \tilde{z}) + 1.$$

Therefore,

$$(6.6) \qquad A(\bar{0}, \underline{z}) \ge A(\underline{0}, \bar{p}) + A(\underline{p}, \bar{z}) - 3.$$

Furthermore, consider $\alpha(\underline{z} - \bar{0})$. Since $p_2^{1-\delta} < p_2$, we have that $\bar{z} \in \text{Co}(v+1)$.

$$
\begin{aligned}
\alpha(\underline{z} - \bar{0}) &= \alpha_0(z_2 - 4) + \frac{1}{2}\frac{z_1^2}{z_2 - 4} \\
&= \alpha(\bar{z}) - 6\alpha_0 + \frac{3z_1^2}{(z_2 + 2)(z_2 - 4)} \\
&\le \alpha(\bar{z}) - 6\alpha_0 + 6(v + 1)^2.
\end{aligned}
$$

Here we use that $8 < M \le z_2$ and $|z_1| \le (v + 1)z_2$. Therefore, following the same reasoning, we can choose $L > 0$ independent of $p \in \text{Co}(v)$ such that

$$(6.7) \qquad \alpha(\underline{z} - \bar{0}) \le \alpha(\bar{z}) + L \quad \text{and} \quad \alpha(\bar{p} - \underline{0}) \ge \alpha(\underline{p}) - L.$$

Also, Lemma 6.1 implies

$$\alpha(\bar{z} - \underline{p}) + \alpha(\underline{p}) \ge \alpha(\bar{z}) + Cp_2^{1-2\delta}.$$

Combining this with (6.6) and (6.7), we obtain

$$|A(\bar{0}, \underline{z}) - \alpha(\underline{z} - \bar{0})| + |A(\underline{0}, \bar{p}) - \alpha(\bar{p} - \underline{0})| + |A(\underline{p}, \bar{z}) - \alpha(\bar{z} - \underline{p})| \ge -3 - 2L + Cp_2^{1-2\delta}.$$

Define the event corresponding to the third term on the l.h.s.

$$E_3 = \left\{|A(\underline{p}, \bar{z}) - \alpha(\bar{z} - \underline{p})| \ge \frac{C}{4}p_2^{1-2\delta}\right\},$$

and likewise E_1 and E_2. By enlarging M, we can make sure that $G(p, z)$ implies at least one of these three events. We will bound the probability of E_3, which is slightly more complicated than the other two, since $\bar{z}_2 - \underline{p}_2$ cannot be made arbitrarily large by increasing M.

Using the shear transformation we know that

$$A(\underline{p}, \bar{z}) - \alpha(\bar{z} - \underline{p}) \overset{distr}{=} A((0,0), (0, \bar{z}_2 - \underline{p}_2) = A^{\bar{z}_2 - \underline{p}_2}.$$

Clearly, if $s \geq t$, then $|A^s| \geq |A^t|$. Therefore,

$$\mathbb{P}(E_3) \leq \mathbb{P}\left\{|A^{2p_2+4}| \geq \frac{C}{4} p_2^{1-2\delta}\right\}.$$

Using $t = 2p_2 + 4$ and $u = \frac{C}{4}p_2^{1-2\delta}$, we see that Theorem 5.1 guarantees the existence of constants $C_1, C_2 > 0$ such that for M big enough and all p with $p_2 \geq M$,

$$\mathbb{P}(E_3) \leq C_1 \exp\left\{-C_2|p_2|^{1/2-2\delta}/\log(p_2)\right\}.$$

Similar bounds hold for $\mathbb{P}(E_1)$ and $\mathbb{P}(E_2)$, proving the lemma. $\qquad\square$

The above Lemma can be used to show that a minimal path starting close to the origin and passing close to p, with high probability will not exit the slanted cylinder $\mathcal{C}(p, p_2^{1-\delta})$ through the sides. Define the event

$$G(p) = \left\{\exists\, \tilde{0} \in K((0,0),1)\, \exists\, z \in \partial_S \mathcal{C}(p, p_2^{1-\delta}):\ \gamma_{\tilde{0},z} \cap K(p,1) \neq \emptyset\right\}.$$

Lemma 6.4. *Fix $\delta \in (0, 1/4)$ and $v > 0$. There exist constants $c_1, c_2, \kappa, M > 0$, such that for all $p \in \mathrm{Co}(v)$ with $p_2 > M$ we have*

$$\mathbb{P}(G(p)) \leq c_1 \exp(-c_2 p_2^\kappa).$$

PROOF: Suppose $p \in \mathbb{R}^2$ with $p_2 > M$. There exists a constant $c > 0$ (depending on v) and points $z_1, \ldots, z_L \in \partial_S \mathcal{C}(p, p_2^{1-\delta})$ with $L \leq cp_2$, such that

$$\partial_S \mathcal{C}(p, p_2^{1-\delta}) \subset \bigcup_{i=1}^{L} K(z_i, 1).$$

This implies that

$$G(p) \subset \bigcup_{i=1}^{L} G(p, z_i).$$

Therefore, by choosing M large enough, $\kappa < 1/2 - 2\delta$ and using Lemma 6.3, there exist $C_1, C_2, c_1, c_2 > 0$ such that

$$\mathbb{P}(G(p)) \leq LC_1 \exp(C_2 p_2^{1/2-2\delta}/\log(p_2)) \leq c_1 \exp(-c_2 p_2^\kappa).$$

$\qquad\square$

Now we are ready to prove δ-straightness of geodesics, as was introduced by Newman in [New95]. For a path γ and $t \in \mathbb{R}$, we define

$$\gamma^{\mathrm{out}}(t) = \{(\gamma(s), s)\ :\ s \geq t\},$$

which is the set of all points in the path γ (more precisely, the *graph* of γ), that are reached after time t. We also consider the following cone for all $x \in \mathbb{R} \times \mathbb{R}^+$ and $\eta > 0$:

$$(6.8) \qquad \mathrm{Co}(x, \eta) = \{z \in \mathbb{R} \times \mathbb{R}^+\ :\ |z_1/z_2 - x_1/x_2| \leq \eta\},$$

which is the cone starting in the origin of all points z that have a corresponding speed closer than η to the speed of x.

Lemma 6.5 (δ-straightness). *For $\delta \in (0, 1/4)$ and $v > 0$ we have with probability one that there exists $M > 0$ (depending on v and δ) and $R > 0$ (depending only on δ), such that for all $\tilde{0} \in K((0,0), 1)$, for all $z \in \mathbb{R} \times \mathbb{R}^+$ and for all $p \in \gamma(\tilde{0}, z) \cap \mathrm{Co}(v)$ with $p_2 > M$, we have*

$$\gamma^{\mathrm{out}}(p_2) \subset \mathrm{Co}(p, Rp_2^{-\delta}),$$

for $\gamma = \gamma_{\tilde{0}, z}$.

This lemma states that if a geodesic starting near $(0,0)$ passes through a remote point p, it has to stay in a narrow cone around the ray $\mathbb{R}^+ \cdot p$.

PROOF: Consider the events $G(\bar{p})$ for all $\bar{p} \in \mathbb{Z} \times \mathbb{Z}^+ \cap \mathrm{Co}(v')$, with $v' > v$. Using Lemma 6.4 and Borel–Cantelli Lemma, we can choose M big enough, such that for all $\bar{p} \in \mathbb{Z} \times \mathbb{Z}^+ \cap \mathrm{Co}(v')$ with $\bar{p}_2 \geq M$ the event $G(\bar{p})$ does not happen. Increase M if necessary to ensure that if $p_2 \geq M$,

$$K(p, 1 + (p_2 + 1)^{1-\delta}) \subset \mathrm{Co}(p, 2p_2^{-\delta}).$$

So for any $p \in \mathbb{R} \times \mathbb{R}^+$ with $p_2 \geq M$, we now know that if $\bar{p} \in K(p, 1)$, then

$$\mathcal{C}(\bar{p}, \bar{p}_2^{1-\delta}) \subset \mathrm{Co}(p, 2p_2^{-\delta}).$$

Let $\bar{0} \in K((0,0), 1)$. Now suppose there exists $z \in \mathbb{R} \times \mathbb{R}^+$ and $p \in \gamma_{\bar{0}, z} \cap \mathrm{Co}(v)$ with $p_2 > M + 1$. Define $\bar{p} = (\lfloor p_1 \rfloor, \lfloor p_2 \rfloor)$. Suppose z lies outside of the slanted tube $\mathcal{C}(\bar{p}, \bar{p}_2^{1-\delta})$. We know that $G(\bar{p})$ does not happen, and since $p \in K(\bar{p}, 1)$, this implies that we can define $p^{(1)}$ as the crossing of $\gamma^{\mathrm{out}}(p_2)$ with the top edge of the tube $\mathcal{C}(\bar{p}, \bar{p}_2^{1-\delta})$, and that $\gamma_{p, p^{(1)}}$ lies inside this tube. Define $\bar{p}^{(1)} = (\lfloor p_1^{(1)} \rfloor, \lfloor p_2^{(1)} \rfloor)$. We can proceed in a similar way to construct $p^{(2)}, \bar{p}^{(2)}, p^{(3)}, \ldots, \bar{p}^{(m)}$, where we finish whenever $z \in \mathcal{C}(\bar{p}^{(m)}, (\bar{p}_2^{(m)})^{1-\delta})$. We have to check that for all $k \leq m$, $\bar{p}^{(k)} \in \mathrm{Co}(v')$, but this will follow from the following considerations.

Note that for $1 \leq k \leq m$, where we define $p^{(0)} = p$ and $\bar{p}^{(0)} = \bar{p}$,

$$\mathcal{C}(\bar{p}^{(k)}, (\bar{p}_2^{(k)})^{1-\delta}) \subset \mathrm{Co}(p^{(k)}, 2(p_2^{(k)})^{-\delta}) \quad \text{and} \quad p_2^{(k)} \geq 2(p_2^{(k-1)} - 1) \geq \frac{3}{2} p_2^{(k-1)}.$$

This implies that the average speed of any vector in $y \in \gamma^{\mathrm{out}}(p_2)$ satisfies

$$|y_1/y_2 - p_1/p_2| \leq \sum_{k=0}^{m} 2(p_2^{(k)})^{-\delta} \leq \sum_{k=0}^{m} 2\left(\frac{3}{2}\right)^{-\delta k} p_2^{-\delta} \leq Rp_2^{-\delta},$$

if we choose $R > 0$ large enough (depending only on δ). This also shows that we have to choose $v' > v + RM^{-\delta}$. $\square$

Corollary 6.6. *For $\delta \in (0, 1/4)$ and $v > 0$ there exists $M, R, \kappa, C_1, C_2 > 0$, such that when we define the event*

$$G_n = \left\{ \exists \tilde{0} \in K((0,0), 1), z \in \mathbb{R} \times \mathbb{R}^+, p \in \gamma(\tilde{0}, z) \text{ with } p_2 > n \text{ and } p \in \mathrm{Co}(v) : \right.$$
$$\left. \gamma^{\mathrm{out}}(p_2) \not\subset \mathrm{Co}(p, Rp_2^{-\delta}) \right\},$$

we have for $n \geq M$

$$\mathbb{P}(G_n) \leq C_1 e^{-C_2 n^{\kappa}}.$$

PROOF: It follows directly from the proof of Lemma 6.5 that the event G_n is a subset of the event that there exists $\bar{p}^{(n)} \in \mathbb{Z} \times \mathbb{Z}^+ \cap \mathrm{Co}(v')$ with $\bar{p}_2^{(n)} \geq n - 1$ such that the event $G(\bar{p}^{(n)})$ does happen. Here we choose $v' > v + RM^{-\delta}$. The probability of this event is clearly bounded by $C_1 e^{-C_2 n^\kappa}$, for an appropriate choice of constants, simply by Lemma 6.4. $\square$

6.2. **Existence and uniqueness of semi-infinite minimizers.** With δ-straightness in hand, we can prove some important properties of minimizing paths. A semi-infinite minimizer starting at $(x, t) \in \mathbb{R}^2$ is a path $\gamma : [t, \infty) \to \mathbb{R}$ such that $\gamma(t) = x$ and the restriction of γ to any finite time interval is a minimizer. We call (x, t) the endpoint of γ.

Lemma 6.7. *With probability one, all semi-infinite minimizers have an asymptotic slope (velocity, direction): for every minimizer γ there exists $v \in \mathbb{R} \cup \{\pm\infty\}$ depending on γ such that*

$$\lim_{t \to \infty} \frac{\gamma(t)}{t} = v.$$

PROOF: Let us fix a sequence $v_n \to \infty$. Using the translation invariance of the Poisson point field, with probability one, for any $q \in \mathbb{Z}^2$ we can choose a corresponding sequence of constants $M_n(q) > 0$ such that the statement in Lemma 6.5 holds for the entire sequence, for paths starting in $K(q, 1)$.

Let us take some one-sided minimizer γ. If $\gamma(t)/t \to +\infty$ or $-\infty$, then the desired statement is automatically true. In the opposite case we have

$$\liminf_{t \to \infty} \frac{|\gamma(t)|}{t} < \infty.$$

This implies that there exist $n \geq 1$ and a sequence $t_m \to \infty$ such that $|\gamma(t_m)|/t_m \leq v_n$. We define $y_m = (\gamma(t_m), t_m)$ and choose $q \in \mathbb{Z}^2$ such that $y_1 \in K(q, 1)$. For m large enough, we will have that $t_m > M_n(q)$ and, therefore,

$$\gamma^{\mathrm{out}}(y_m) \subset q + \mathrm{Co}(y_m - q, R|y_m - q|^{-\delta}),$$

for some constant $R > 0$ and m large enough. Clearly this implies that γ must have a finite asymptotic slope. $\square$

Lemma 6.8. *With probability one, for every $v \in \mathbb{R}$ and for every sequence $(y_n, t_n) \in \mathbb{R}^2$ with $t_n \to \infty$ and*

$$\lim_{n \to \infty} \frac{y_n}{t_n} = v,$$

and for every $x \in \mathbb{R}^2$, there exists a subsequence (n_k) such that the minimizing paths $\gamma_{x,(y_{n_k}, t_{n_k})}$ are an increasing collection of paths that converge to a semi-infinite minimizer starting at x and with asymptotic speed equal to v.

PROOF: Without loss of generality, we can assume that $x \in K((0,0), 1)$. We take a sequence $v_m \to \infty$ and then choose $M_m \to \infty$ and $R > 0$ such that the statement of Lemma 6.5 holds for every triplet (v_m, M_m, R). From these triplets we choose a triplet (v_0, M, R) with $|v| < v_0 - 2RM^{-\delta}$ (note that R only depends on δ).

By going to a subsequence, we make sure that for all $n \geq 1$, $t_n \geq M$, $t_n \uparrow \infty$ and for all $k > n$ we have $(y_k, t_k) \in \mathrm{Co}((v,1), Rt_n^{-\delta})$. Consider the paths $\gamma_n = \gamma_{x,(y_n,t_n)}$. We claim that for each $n \geq 1$ and $k > n$, the path γ_k lies in the cone $\mathrm{Co}((v,1), 2Rt_n^{-\delta})$ for times larger than t_n. In fact, if γ_k visits a point p outside this cone (but inside $\mathrm{Co}(v_0)$) at time $p_2 \geq t_n$, then γ_k violates the δ-straightness condition (the relevant cone through p will not intersect $\mathrm{Co}((v,1), Rt_n^{-\delta})$, and therefore it does not contain y_k). In particular, this means that there exists a $C > 0$ (independent of n) such that all paths γ_k with $k > n$ cross the segment

$$J_n = [vt_n - Ct_n^{1-\delta}, vt_n + Ct_n^{1-\delta}] \times \{t_n\}.$$

We claim that there exists a point $x_n \in J_n$ visited by an infinite number of paths γ_k. With this claim in hand, we first choose x_1 and a subsequence $k_1(n)$ such that every $\gamma_{k_1(n)}$ passes through x_1, then we choose x_2 in the segment

$$[vt_{k_1(1)} - Ct_{k_1(1)}^{1-\delta}, vt_{k_1(1)} + Ct_{k_1(1)}^{1-\delta}] \times \{t_{k_1(1)}\}$$

such that an infinite number (a subsequence $k_2(n) \subset k_1(n)$) of the paths $\gamma_{k_1(n)}$ pass through x_2, and so on. The paths $\{\gamma_{x,x_n} : n \geq 1\}$ are then an increasing collection of paths, and their union will be a semi-infinite minimizer starting at x, with asymptotic speed v.

What remains is to show that an infinite number of the paths $\gamma_k, k > n$, pass through the same point of the segment

$$[vt_n - Ct_n^{1-\delta}, vt_n + Ct_n^{1-\delta}] \times \{t_n\}.$$

We already know that all these paths remain in the cone $\mathcal{C} = \mathrm{Co}((v,1), Rt_n^{-\delta})$. We use the following fact about the Poisson process, which is easily checked using the Borel–Cantelli Lemma: with probability one, for all $K > 0$ there exists $T_0 > 0$ depending on K, such that for all $T \geq T_0$ and all $x \in [-KT, KT] \times [-1, 2T]$, there is at least one Poisson point in the ball of radius $T^{1/4}$ around x. Let us choose $K > 0$ and $T > t_n$ such that $\mathcal{C} \cap (\mathbb{R} \times [0, t_n + T]) \subset [-KT, KT] \times [-1, 2T]$. Then we choose $T_0 > t_n$ according to this K and take $T \geq T_0$.

Consider k such that $t_k > t_n + 2T$. Suppose γ_k did not pick up any Poisson point in the time interval $[t_n, t_n + 2T]$. That implies that it would have some constant speed u in that interval. We also know that within distance $T^{1/4}$ of $(\gamma_k(t_n + T), t_n + T)$ there will be some Poisson point, since $(\gamma_k(t_n + T), t_n + T) \in \mathcal{C}$. Define the path $\tilde{\gamma}_k$ that picks up this Poisson point by moving to this point with constant velocity from the point $(\gamma_k(t_n), t_n)$ and then moving with constant speed to $(\gamma_k(t_n + 2T), t_n + 2T)$; the rest of the time $\tilde{\gamma}_k$ coincides with γ_k. When we consider the difference in action picked up by the two paths, we note that they start and end at the same point in the time interval $[t_n, t_n + 2T]$. This means that if we define $\delta(t) = \dot{\tilde{\gamma}}_k(t) - \dot{\gamma}_k(t)$, then $\int_{t_n}^{t_n+2T} \delta(s)\,ds = 0$. Furthermore, there exists a constant $d > 0$ depending only on v and the cone $\mathcal{C}$ such that $|\delta(s)| \leq dT^{-3/4}$. If we choose $T > d^4$,

this leads to

$$
\begin{aligned}
A[\tilde{\gamma}_k] - A[\gamma_k] &= -1 + \frac{1}{2} \int_{t_n}^{t_n + 2T} ((u + \delta(s))^2 - u^2)\, ds \\
&= -1 + \frac{1}{2} \int_{t_n}^{t_n + 2T} \delta(s)^2\, ds \\
&\geq -1 + d^2 T^{-1/2} \\
&< 0.
\end{aligned}
$$

This contradicts the optimality of γ_k, and we conclude that γ_k must pick a Poisson point in the time interval $[t_n, t_n + 2T]$. The number of paths γ_k is infinite, and there are only finitely many Poisson points in the set $\mathcal{C} \cap \mathbb{R} \times [t_n, t_n + 2T]$. Therefore, at time t_n, an infinite number of paths γ_k cross J_n at the same point. Here we use that when two minimizers meet at two Poisson points at distinct times, they actually will coincide for all times between these two times (since minimizing paths between two Poisson points are almost surely unique). $\qquad\square$

Lemma 6.9. *With probability 1, the following statement holds: if γ_1 and γ_2 are two (finite-time) geodesics, starting at the same Poisson point p, and for some $t > p_2$ we have $\gamma_1(t) < \gamma_2(t)$, then for all (relevant) $s > t$ we have $\gamma_1(s) < \gamma_2(s)$.*

PROOF: The probability that there are two Poisson point connected by two distinct geodesics is zero. So we only have to consider the situation where two paths with vertices $p, p_1, \ldots, p_n$ and $p, q_1, \ldots, q_m$ intersect transversally, i.e., for some k, j, $[p_j, p_{j+1}] \cap [q_k, q_{k+1}] = \{x\}$ for some point $x \notin \omega$.

In this case, the total actions of the two paths can be improved by switching to paths with vertices $p, p_1, \ldots, p_j, q_{k+1}, \ldots, q_m$ and, respectively, $p, q_1, \ldots, q_k, p_{j+1}, \ldots, p_n$. Therefore, we obtain a contradiction with the optimality of the original paths. $\qquad\square$

Lemma 6.10. *Let $v \in \mathbb{R}$. With probability one, every Poisson point belongs to at most one semi-infinite minimizer with asymptotic slope v.*

PROOF: Let $\mathcal{U}(v)$ be the event that two distinct semi-infinite minimizers with asymptotic speed v pass through a common Poissonian point p. Let

$$
\mathcal{S} = \{v \in \mathbb{R} \; : \; \mathbb{P}(\mathcal{U}(v)) > 0\}.
$$

It is sufficient to show that $\mathcal{S} = \emptyset$. The invariance under shear transformations implies that either $\mathcal{S} = \emptyset$ or $\mathcal{S} = \mathbb{R}$. The latter will be excluded as soon as we prove that $\mathcal{S}$ is at most countable.

A triple of distinct Poisson points (p, q_1, q_2) is called a bifurcation triple for v if there exist two distinct semi-infinite minimizers γ_1 and γ_2 with asymptotic slope v that both start at p, then one goes directly (at constant velocity) to q_1 and the other goes directly to q_2. We choose q_1 such that γ_1 lies to the left of γ_2.

Lemma 6.9 implies that γ_1 and γ_2 will not meet again after p with probability one.

Clearly, $\mathcal{U}(v)$ implies the existence of such a bifurcation triple for v. If $\mathbb{P}(\mathcal{U}(v)) > 0$, then there exists $L = L(v) \in \mathbb{N}$ such that the event

$$\mathcal{T}_L(v) = \{\exists \text{ bifurcation triple } (p, q_1, q_2) \text{ for } v \text{ inside the cube } K((0,0), L)\}$$

has positive probability (using translation invariance).

Now suppose that $\mathcal{S}$ contains an uncountable number of asymptotic slopes. This implies that there exist $m, L \in \mathbb{N}$ such that for uncountably many v we would have that

$$(6.9) \qquad\qquad\qquad \mathbb{P}(\mathcal{T}_L(v)) > 1/m.$$

Now note that three Poisson points can only form a bifurcation triple for one $v \in \mathbb{R}$, since otherwise two distinct semi-infinite minimizers, both starting at some p, would have to split up at p and cross again at a later time, which contradicts Lemma 6.9. Suppose $v_1, v_2, \dots$ satisfy (6.9). Denote by N_L the number of Poisson points in the cube $K((0,0), L)$. Then

$$\sum_{n \geq 1} 1_{\mathcal{T}_L(v_n)} \leq N_L^3.$$

Taking the expectation on both sides and using (6.9) leads to a contradiction.

Therefore, there can be only countably many v's in $\mathcal{S}$ which completes the proof.   □

With uniqueness in hand we can strengthen Lemma 6.8:

Lemma 6.11. *With probability one, for every $v \in \mathbb{R}$ and for every sequence $(y_n, t_n) \in \mathbb{R}^2$ with $t_n \to \infty$ and*

$$\lim_{n \to \infty} \frac{y_n}{t_n} = v,$$

and for every Poissonian point $p \in \mathbb{R}^2$, the minimizing paths $\gamma_{p,(y_n, t_n)}$ converge to a unique semi-infinite minimizer $\gamma_{p,v}$ starting at p and with asymptotic speed equal to v.

PROOF: Let us assume that the convergence does not hold, i.e., there is a sequence (n') such that the restrictions of $\gamma_{p,(y_{n'}, t_{n'})}$ and $\gamma_{p,v}$ on some finite time interval I do not coincide for all n'. Lemma 6.8 allows to choose a subsequence (n'') from (n') such that for sufficiently large n'' the restrictions of $\gamma_{p,(y_{n''}, t_{n''})}$ on I coincide with the restrictions of some infinite one-sided geodesics γ'. The uniqueness established in Lemma 6.10 guarantees that γ' coincides with $\gamma_{p,v}$, and the resulting contradiction shows that our assumption was false, completing the proof.   □

6.3. **Coalescence of minimizers.** Here we prove that any two one-sided minimizers with the same asymptotic slope coalesce.

Lemma 6.12. *With probability one it holds that for every $v \in \mathbb{R}$ and for every pair of semi-infinite minimizers, starting at different Poisson points, with asymptotic speed v, these minimizers either do not touch, or they coalesce at some Poisson point.*
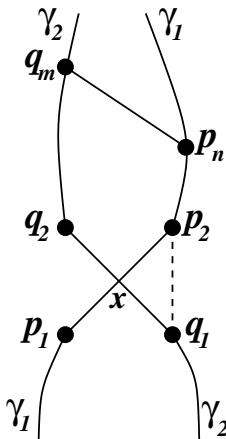
FIGURE 1. Proof of Lemma 6.12.

PROOF: Suppose for some $v \in \mathbb{R}$, there do exist two semi-infinite minimizers with asymptotic speed v that touch, but do not coalesce. If the two minimizers γ_1 and γ_2 contain the same Poisson point p, then they must stay together for all times above p according to Lemma 6.10. Therefore, the only option is that γ_1 and γ_2 cross, i.e., they consecutively visit Poissonian points $p_1, p_2, \ldots$, and, respectively, $q_1, q_2, \ldots$, and $[p_1, p_2] \cap [q_1, q_2] = \{x\}$, for some $x \in \mathbb{R}^2$.

The sequence $\{q_m : m \geq 1\}$ satisfies the conditions of Lemma 6.11, which means that the minimizers γ_{p_1, q_m} converge to γ_1. However, we claim that with probability 1, none of the minimizers γ_{p_1, q_m} contain any of the p_n ($n \geq 2$). In fact, if this claim is violated for some m, n, then due to a.s.-uniqueness of a geodesic between any two Poisson points, we know that γ_{p_1, q_m} passes through p_2 and x. This implies that action picked up by γ_2 between x and q_m must be equal to the action picked up by γ_{p_1, q_m} between x and q_m. However, this contradicts the optimality of γ_2 as the comparison with the path connecting q_1 directly to p_2 and then following γ_{p_1, q_m} shows, see Figure 1. The proof is complete.   □

For every $v \in \mathbb{R}$, coalescence of one-sided geodesics with asymptotic slope v generates an equivalence relation on Poissonian points. We call each equivalence class a coalescence component.

Lemma 6.13. *Let $v \in \mathbb{R}$. With probability 1, every coalescence component is unbounded below in time.*

PROOF: Due to shear invariance, it is sufficient to consider only $v = 0$. Suppose that the probability of existence of a coalescence component bounded from below is positive. Then there is $(i, j) \in \mathbb{Z}^2$ such that with positive probability the earliest (i.e., having the minimal time coordinate) point in some coalescence component belongs to $[i, i + 1] \times [j, j + 1]$. Due to stationarity, this probability is positive for any $(i, j) \in \mathbb{Z}^2$. The Tempelman multi-parametric ergodic theorem (see, e.g., [Kre85, Chapter 6]) implies that there is a constant $c > 0$ and a family of random variables $(n_r)_{r \in \mathbb{N}}$ such that

with probability 1,

$$\text{(6.10)} \qquad \sum_{i,j \in \Delta(r,n)} I_{ij} > cn^2, \quad n \geq n_r.$$

where $\Delta(r, n) = \{(i, j) : r \leq j \leq n, -j \leq i \leq j\}$ and I_{ij} is the indicator of event that there is a tree with earliest point within $[i, i+1] \times [j, j+1]$.

Let $\gamma_+ = \gamma_{3,0}$ and $\gamma_- = \gamma_{-3,0}$ be the one-sided optimal paths emitted from the origin 0 with asymptotic slopes 3 and -3, respectively. Then there is a random variable m such that for all $t > m$, one has $-4t < \gamma_-(t) < -2t$ and $2t < \gamma_+(t) < 4t$. In particular, the set

$$\bigcup_{n \geq m} \bigcup_{(i,j) \in \Delta(m,n)} [i, i+1] \times [j, j+1]$$

is bounded on the left by γ_- and on the right by γ_+.

Let us denote the coalescence component giving rise to $I_{ij} = 1$ by C_{ij}. If there are several such components we choose the one containing the earliest point. All the components C_{ij}, $j > m$ are disjoint and contained in the set $\{(x, t) : t > m, -2t \leq x \leq 2t\}$.

Combining this with (6.10), we can conclude that for every $n \geq n_m$, the segment $J_n = [\gamma_-(n), \gamma_+(n)] \times \{n\}$ is crossed by geodesics from at least cn^2 disjoint coalescence components. Each of these components has to have an edge connecting two Poissonian points and crossing J_n.

Therefore, either at least cn^2 Poissonian points are contained in $R_n = [-4n, 4n] \times [n, n - \sqrt{n}]$, or there is an edge connecting two vertices of one component and passing through a point on $H_n = [-4n, 4n] \times \{n - \sqrt{n}\}$ and a point on $L_n = [-4n, 4n] \times \{n\}$. Let us denote the former event by D_n and the latter by E_n. We have

$$\mathbb{P}(D_n) \leq e^{-cn^2} \mathbb{E} e^{\omega(R_n)} \leq e^{-cn^2} e^{8n^{3/2}(e-1)},$$

and the Borel–Cantelli Lemma implies that with probability 1 only finitely many events D_n occur. To prove an analogous statement for the events E_n, we need the following lemma:

Lemma 6.14. *Let $x, y \in \mathbb{R}$. Suppose there is an optimal path γ connecting a point $x' \in [x, x+n^{1/5}] \times \{n - \sqrt{n}\}$ straight to a point $y' \in [y, y+n^{1/5}] \times \{n\}$, avoiding all Poissonian points between $n - \sqrt{n}$ and n. Then the parallelogram $\Pi_n(x, y)$ that is obtained by intersection of $\mathbb{R} \times [n - \sqrt{n} - 1, n - \sqrt{n} + 1]$ and the parallelogram with vertices $x, x + n^{1/5}, y + n^{1/5}, y$, does not contain any Poissonian points.*

PROOF: Suppose that $\Pi_n(x, y)$ contains a Poissonian point $p = (z, s)$. Then the original straight path is not optimal, which follows from the comparison with the path $\tilde{\gamma}$ consisting of two segments connecting x' to p and p to y'. In fact, an elementary calculation (see also the proof of Lemma 6.5 in [Bak12]) shows that

$$S^{n-\sqrt{n},n}(\tilde{\gamma}) - S^{n-\sqrt{n},n}(\gamma) = \frac{\sqrt{n}}{2(s - n + \sqrt{n})(n - s)} \delta^2,$$

where δ is the distance from p to γ measured along the spatial axis. Noticing that $\delta \leq n^{1/5}$ and $s \in [n - \sqrt{n} - 1, n - \sqrt{n} + 1]$, we obtain that for large n, the right-hand side is less than 1, so increasing the number of points that the path passes through by 1 overweighs the increase of kinetic action. $\square$

To finish the proof of Lemma 6.13, we notice that each H_n and L_n can be covered by $8n^{1-1/5} + 1$ intervals of length $n^{1/5}$. The probability of E_n can be then bounded by the sum over all pairs of these intervals such that there is a straight line minimizer connecting points from these two intervals. Lemma 6.14 implies that

$$\mathbb{P}(E_n) \leq (8n^{1-1/5} + 1)^2 \mathbb{P}\{\omega(\Pi_n(0,0)) = 0\}$$
$$\leq (8n^{1-1/5} + 1)^2 e^{-2n^{1/5}}.$$

since the area of $\Pi_n(x, y)$ equals $2n^{1/5}$ and does not depend on x, y. The Borel–Cantelli Lemma implies that with probability 1, only finitely many events E_n can happen. This completes the proof of Lemma 6.13. $\square$

Lemma 6.15. *Let $v \in \mathbb{R}$. With probability one, every two semi-infinite minimizers with asymptotic slope v coalesce.*

PROOF: Due to shear invariance, it is enough to prove the lemma for $v = 0$. Also, it is enough to prove it for minimizers starting at Poisson points. Lemma 6.12 shows that if two of these semi-infinite minimizers do not coalesce, then they must be disjoint. We call the event that this happens E, and suppose $\mathbb{P}(E) > 0$. Now we define E_k ($k \in \mathbb{Z}$) as the event that there exist two disjoint minimizers with asymptotic slope 0 that both start at Poisson points with times smaller than k. Clearly, there exists $k \in \mathbb{Z}$ with $\mathbb{P}(E_k) > 0$, and using translation invariance, we conclude that $\mathbb{P}(E_{-1}) > 0$. Let us now define, for $x_1, x_2 \in \mathbb{R} \times (-\infty, -1)$ and $\delta \in (0, 1)$, $E(x_1, x_2; \delta)$ as the event that there exist two disjoint minimizers γ_1 and γ_2 with asymptotic slope 0, such that γ_1 starts within distance $\delta > 0$ from x_1 and γ_2 starts within distance δ from x_2. Again it follows that there exist $x_1, x_2 \in \mathbb{R} \times (-\infty, -1)$ and $\delta \in (0, 1)$ such that $\mathbb{P}(E(x_1, x_2; \delta)) > 0$. Let W_i be the crossing of the minimizer starting near x_i with the x-axis. Clearly, there exists $K > 0$ such that the event $E(x_1, x_2; \delta, K)$, consisting of all $\omega \in E(x_1, x_2; \delta)$ with $|W_i - x_{i1}| < K$, has positive probability. Now we choose $h > |x_1 - x_2| + 2\delta + 2K$ and define $z = (h, 0)$. Using the ergodicity of the Poisson point field with respect to translations, we obtain that there are $m, n \in \mathbb{Z}$, $m \neq n$ such that

$$\mathbb{P}(E(x_1 + mz, x_2 + mz; \delta, K) \cap E(x_1 + nz, x_2 + nz; \delta, K)) > 0.$$

Again using translation invariance, we can take $m = 0$ and $n > 0$. Let $x_3 = x_1 + nz$ and $x_4 = x_2 + nz$. Because of our choice of h, we know that $\max(W_1, W_2) < \min(W_3, W_4)$. Now relabel (if necessary) to ensure that $W_1 < W_2 < W_3 < W_4$. We denote the minimizer starting near x_i by γ_i. By construction, γ_1 and γ_2 do not cross, nor do γ_3 and γ_4. This implies that the three minimizers γ_1, γ_2 and γ_4 do not cross for times larger than 0, since γ_3 lies between γ_2 and γ_4 (in principle, γ_2 and γ_3 could still coalesce). Our conclusion is that there is a positive probability of having three non-intersecting

minimizers with asymptotic speed 0, all three starting below time 0.

We can now conclude that for $\delta > 0$ small enough and $K, N > 0$ big enough, there exist $R > 0$, $x \in \mathbb{R} \times \{-\delta\}$, $y_1, \ldots y_n \in \mathbb{R} \times (-\infty, 0)$ and $z_1, \ldots, z_m \in \mathbb{R} \times (-\infty, 0)$ with $y_{n,2} \leq -K - \delta$ and $z_{m,2} \leq -K - \delta$, such that with positive probability:

- There exist three disjoint semi-infinite minimizers γ_1, γ_2 and γ_3 with asymptotic slope 0 and $\gamma_1(0) < \gamma_2(0) < \gamma_3(0)$, where γ_1 starts at a Poisson point p_1 in $y_n + [0, \delta]^2$, γ_3 starts at a Poisson point p_3 in $z_m + [0, \delta]^2$ and γ_2 starts at a Poisson point p_2 in $x + [0, \delta]^2$.
- If $p_{1,2} < p_{3,2}$, then $|p_{3,1} - \gamma_1(p_{3,2})| < N$; if $p_{1,2} \geq p_{3,2}$, then $|p_{1,1} - \gamma_3(p_{1,2})| < N$.
- There is exactly one Poisson point in $x + [0, \delta]^2$ and the next Poisson point of γ_2 (above x) has a strictly positive time.
- There is exactly one Poisson point in $y_i + [0, \delta]^2$ and in $z_j + [0, \delta]^2$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$.
- All Poisson points of γ_1 with negative time lie in $\cup_{i=1}^n (y_i + [0, \delta]^2)$.
- All Poisson points of γ_3 with negative time lie in $\cup_{j=1}^m (z_j + [0, \delta]^2)$.
- γ_1 and γ_3 cross the x-axis in the interval $[-R, R]$.

Let us call this event $A_{\delta,K,N}$. We choose $M > R$ and $L > K + \delta$ such that $[-M, M - \delta] \times [-L, 0]$ contains all y_i's, all z_j's and x. If $\omega \in A_{\delta,K,N}$, then we can modify the Poisson configuration ω by deleting all Poisson points in

$$[-M, M] \times [-L, 0] \setminus \left(\bigcup_{i=1}^n (y_i + [0, \delta]^2) \cup \bigcup_{j=1}^m (z_j + [0, \delta]^2) \cup (x + [0, \delta]^2) \right),$$

and call this new configuration $\tilde{\omega}$. Clearly, γ_1, γ_2 and γ_3 are still semi-infinite minimizers under $\tilde{\omega}$. Furthermore, Lemma 8 of [HN97] shows that there exists a measurable event $\tilde{A}_{\delta,K,N}$ with positive probability, such that

$$\tilde{A}_{\delta,K,N} \subset \{\tilde{\omega} \ : \ \omega \in A_{\delta,K,N}\}.$$

We claim that on the event $\tilde{A}_{\delta,K,N}$, any semi-infinite minimizer γ, starting at a Poisson point p in the set $(\mathbb{R} \times (-\infty, 0]) \setminus ([-M, M] \times [-L, 0])$ cannot coalesce with γ_2. Therefore, the coalescence component of γ_2 is bounded below by $-L$ which contradicts Lemma 6.13 and finishes the proof.

Suppose our claim is wrong and γ connects to γ_2. Since γ cannot cross γ_1 or γ_3, this implies that γ passes *between* γ_1 and γ_3, and thus at most at distance N from either p_1 or p_3, when going with constant velocity from the boundary of $[-M, M] \times [-L, 0]$ to p_2 or a point on the x-axis between γ_1 and γ_3. Therefore, fixing N, choosing K large enough, and increasing M and L if necessary, we can guarantee that if we modify γ and let it pick up one additional Poissonian point p_1 or p_3, we will decrease its action which contradicts the optimality of γ. The proof is completed.   □

6.4. **Domains of influence of Poisson points.** Let us fix a mean velocity value $v \in \mathbb{R}$ for the rest of this section. By a one-sided optimal path we will always mean a one-sided optimal path of asymptotic slope v. In this section

we describe the random partition of the real line $R \times \{0\}$ (or, equivalently, $R \times \{t\}$ for any $t \in \mathbb{R}$) into domains of influence of Poisson points. The domain of influence of $p \in \omega$ consists of all points $x \in \mathbb{R}$ such that the first Poisson point visited by a unique one-sided minimizer with endpoint $(x, 0)$ is p. With a little more effort one can give a description of a space-time tessellation of $\mathbb{R}^2$ of the same kind, but we omit it for brevity.

Lemma 6.16. *With probability one the following holds true. For any convergent sequence of real numbers $x_n \to x_\infty$, and any sequence γ_n of one-sided minimizers with endpoints $(x_n, 0)$, if there is a point q such that the first Poissonian point visited by each γ_n is q, then (i) after q all these paths are uniquely defined and coincide, and (ii) the path γ_∞ starting at $(x_\infty, 0)$ going straight to q and coinciding with paths γ_n after q, is also a one-sided optimal path.*

PROOF: After q, the one-sided optimal path is uniquely defined due to Lemma 6.10. Assuming that there is a finite improvement of γ_∞, i.e., another path starting at $(x_\infty, 0)$ connecting to γ_1 above q, one can also construct an improvement of a path γ_n for sufficiently large n, i.e., for $(x_n, 0)$ sufficiently close to $(x_\infty, 0)$, a contradiction.   □

Lemma 6.17. *With probability one, for every bounded set $S \in \mathbb{R}^2$ there is a finite set P of Poissonian points such that for every one-sided optimal path originating at any $p \in S$, the first Poissonian point it visits belongs to P.*

PROOF: It is easily verified that for any $v > 0$ there there is a random $t_0 > 0$ such that no Poissonian point (x, t) satisfying $|x| \le vt$ and $t \ge t_0$ can be the first point visited by a one-sided optimal path originating within $K((0, 0), 1)$. Also, for any $R, T > 0$, there are only finitely many Poissonian points (x, t) satisfying $|x| \le R$ and $t < T$. Therefore, if there are countably many one-sided minimizers originating from a point within $K((0, 0), 1)$, then for any $R > 0$ and any $v > 0$ there is a Poissonian point (x, t) satisfying $|x| > \max\{R, vt\}$ visited first by a one-sided optimal path originating in $K((0, 0), 1)$.

Since one-sided minimizers cannot cross each other, we see that on the above event, there is a point $p = (y, s) \in K((0, 0), 1)$ such that one of the rays $[y, +\infty) \times \{s\}$, or $(-\infty, y] \times \{s\}$ cannot be crossed by one-sided minimizers originating at negative times. So, if this happens with positive probability, then, due to stationarity and ergodicity, with probability one, there are infinitely many Poissonian points $(y_n, s_n)_{n \in \mathbb{Z}}$ satisfying $s_n \in [0, 1]$ and $\lim_{n \to \pm\infty} y_n = \pm\infty$ such that one of the rays $[y_n, +\infty) \times \{s_n\}$, or $(-\infty, y_n] \times \{s_n\}$ cannot be crossed by any one-sided optimal path starting at a negative time. Therefore, no one-sided optimal path can originate at a negative time. This contradiction proves the lemma for $S = K((0, 0), 1)$. The lemma in full generality follows due to stationarity, since one can cover any bounded S by finitely many translates of $K((0, 0), 1)$.   □

Lemma 6.18. *The following holds with probability one for all $x \in \mathbb{R}$ simultaneously. Suppose there are $n \in \mathbb{N}$ one-sided optimal paths originating at $(x, 0)$. Let us denote by $(x_1, t_1), \ldots, (x_n, t_n)$ the first Poissonian points*

visited by these paths, ordering them so that $(x_1 - x)/t_1 < \ldots < (x_n - x)/t_n$ (note that we still allow $n = 1$). Then there is $\varepsilon > 0$ such that (i) for all $y \in (x - \varepsilon, x)$, there is a unique one-sided optimal path originating at $(y, 0)$, and the first Poissonian point visited by it is (x_1, t_1), and (ii) for all $y \in (x, x + \varepsilon)$, there is a unique one-sided optimal path originating at $(y, 0)$, and the first Poissonian point visited by it is (x_n, t_n).

PROOF: We will prove only part (ii), since the proof of part (i) is the same.

Suppose that there is a sequence $y_k \downarrow x$ and a sequence (γ_k) of one-sided minimizers originating at y_k with first visited Poissonian point $q_k \neq (x_n, t_n)$. Lemma 6.17 implies that the set $\{q_k\}$ is finite, and we can choose a Poissonian point $q = (x', t')$ from this set and a subsequence $y_{k'}$ such that the first Poissonian point visited by the corresponding paths $\gamma_{k'}$ is q. According to Lemma 6.16, there is an optimal path connecting $x = \lim y_{k'}$ to q. However, optimal paths cannot cross, and it follows from our construction that $(x' - x)/t' > (x_n - x)/t_n$, and thus q cannot belong to the set of n Poissonian points in the statement of the Theorem. The resulting contradiction finishes the proof. $\qquad\square$

Lemma 6.19. *With probability 1, there is an increasing doubly infinite sequence of points $(x_k)_{k \in \mathbb{Z}}$ on the real line $\mathbb{R}$ satisfying $\lim_{k \to \pm\infty} x_k = \pm\infty$ with the following properties:*

(1) *For every $k \in \mathbb{Z}$ there is a Poisson point p_k such that for all $x \in (x_k, x_{k+1})$, there is a unique one-sided minimizer $\gamma_{(x,0),v}$ originating at $(x, 0)$ and the first Poissonian point it visits is p_k.*

(2) *For every $k \in \mathbb{Z}$ there are at least two one-sided optimal paths originating at x_k, they pass through p_k and p_{k-1}, respectively.*

(3) *For any $k \in \mathbb{Z}$ any $x \in (x_k, x_{k+1})$ and any sequence $(y_n, t_n) \in \mathbb{R}^2$ with $t_n \to \infty$ and*

$$\lim_{n \to \infty} \frac{y_n}{t_n} = v,$$

and for every $x \in (x_k, x_{k+1})$, the minimizing paths $\gamma_{(x,0),(y_n,t_n)}$ converge to $\gamma_{(x,0),v}$.

PROOF: The first two parts follow from Lemma 6.18. In particular, that lemma shows that each $x \in \mathbb{R}$ has a neighborhood of points (excluding x) with a unique minimizer. The set of all $x \in \mathbb{R}$ such that $(x, 0)$ has two or more one-sided minimizers must therefore be discrete (it has no accumulation points); this set will be $\{x_k : k \in \mathbb{Z}\}$. The proof of the last part repeats that of Lemma 6.11. $\qquad\square$

7. BUSEMANN FUNCTIONS AND STATIONARY SOLUTIONS OF THE BURGERS EQUATION

In this Section we use the one-sided minimizers to construct global solutions of the Burgers equation, thus proving the existence part of Theorem 3.1.

Let us summarize some facts on one-sided backward minimizers that follow from Section 6. For any velocity $v \in \mathbb{R}$, the following holds with probability 1. For every point $p = (x, t)$ there is a non-empty set $\Gamma_{v,p}$ of one-sided action minimizers $\gamma : (-\infty, t] \to \mathbb{R}$ with asymptotic slope v

$$\lim_{s \to -\infty} \frac{\gamma(s)}{s} = v,$$

ending at p. They all coalesce, i.e., they coincide on $(-\infty, t_{v,p}]$ for some $t_{v,p} < t$. For most points $p \in \mathbb{R}^2$, $\Gamma_{v,p}$ consists of a unique minimizer $\gamma_{v,p}$, but even if the uniqueness does not hold, there is the right-most minimizer $\gamma_{v,p} \in \Gamma_{v,p}$ such that $\gamma_{v,p}(s) \geq \gamma(s)$ for $s \leq t$ and any other minimizer $\gamma \in \Gamma_{v,p}$.

For every two points $p_1 = (x_1, t_1)$ and $p_2 = (x_2, t_2)$, all their one-sided minimizers coalesce, i.e., there is a time $t_v = t_v(p_1, p_2)$ such that $\gamma_{v,p_1}(s) = \gamma_{v,p_2}(s)$ for all $s \leq t_v$.

This allows us to define Busemann functions for slope v:

$$B_v(p_1, p_2) = B_{v,\omega}(p_1, p_2) = A_\omega^{t_v(p_1,p_2),t_2}(\gamma_{v,p_2}) - A_\omega^{t_v(p_1,p_2),t_1}(\gamma_{v,p_1}), \ p_1, p_2 \in \mathbb{R}^2.$$

Although t_v is not defined uniquely, the definition clearly does not depend on a concrete choice of t_v or $\gamma_{v,p_1}, \gamma_{v,p_2}$. One can also choose t_v to be the maximal of all possible coalescence times.

Some properties of Busemann functions are summarized in the following lemma:

Lemma 7.1. *Let B_v be defined as above for $v \in \mathbb{R}$.*

(1) *The distribution of B_v is translation invariant: for any $\Delta \in \mathbb{R}^2$,*

$$B_v(\cdot + \Delta, \cdot + \Delta) \overset{distr}{=} B_v(\cdot, \cdot).$$

(2) *B_v is antisymmetric:*

$$B_v(p_1, p_2) = -B_v(p_2, p_1), \quad p_1, p_2 \in \mathbb{R}^2,$$

in particular $B_v(p, p) = 0$ for any $p \in \mathbb{R}^2$.

(3) *B_v is additive:*

$$B_v(p_1, p_3) = B_v(p_1, p_2) + B_v(p_2, p_3), \quad p_1, p_2, p_3 \in \mathbb{R}^2.$$

(4) *For any $p_1, p_2 \in \mathbb{R}^2$, $\mathbb{E}|B_v(p_1, p_2)| < \infty$.*

PROOF: The first three parts of the Lemma are straightforward. Let us prove part 4.

Using the additivity and translation invariance of Busemann functions, we see that it is sufficient to consider points $(0, 0)$ and (x, t) with $t < 0$. Since the effect of shear transformations on Busemann functions is easily computable, it is sufficient to assume that $v = 0$.

If $s < t < 0$, and $x, y \in \mathbb{R}$ we have

$$A^{s,0}(y, 0) \leq A^{s,t}(y, x) + \frac{x^2}{2|t|},$$

so

$$B_0((0,0), (x,t)) \geq -\frac{x^2}{2|t|},$$

and all we need is an upper bound on $\mathbb{E}B_0((0,0),(x,t))$.

Since for any $s < t - 1$ and any $y \in \mathbb{R}$ we have

$$A^{s,t}(y,x) \leq A^{s,t-1}(y,0) + \frac{x^2}{2},$$

it is sufficient to assume $x = 0$. Also, if $t \in (-1,0)$, then

$$A^{s,t}(y,0) \leq A^{s,-1}(y,0),$$

so it is sufficient to consider $t \leq -1$. If we prove finiteness of expectation for $t = -1$, then it will also follow for $t = -2, -3, \ldots$ by additivity, and for all intermediate times by

$$A^{s,[t]+1}(y,0) \leq A^{s,t}(y,0) \leq A^{s,[t]}(y,0).$$

Therefore, it remains to prove that

(7.1) $$\mathbb{E}(A^{s,-1}(y,0) - A^{s,0}(y,0)) < \infty.$$

where (y,s) is the space-time point of coalescence of one-sided minimizers with zero asymptotic slope for the points $(0,0)$ and $(0,-1)$.

Let $H = \{((x,t) : t \leq -1 - |x|)\}$. Notice that $(0,-1)$ is the vertex of the right angle formed by ∂H.

Since we are considering minimizers with zero asymptotic slope, we can define

$$\tau = \inf\{t \leq 0 : \ (\gamma^*(t),t) \notin H\},$$

where we introduced $\gamma^* = \gamma_{0,(0,0)}$ for brevity. Let us denote $z = \gamma^*(\tau) = \pm(\tau + 1)$. If $s \in [\tau, 0]$, then

$$A^{s,0}(y,0) - A^{s,-1}(y,0) = A^{\tau,0}(z,0) - A^{\tau,-1}(z,0).$$

If $s < \tau$, then

$$\begin{aligned} A^{s,-1}(y,0) - A^{s,0}(y,0) &\leq & A^{s,\tau}(y,z) + A^{\tau,-1}(z,0) - (A^{s,\tau}(y,z) + A^{\tau,0}(z,0)) \\ &\leq & A^{\tau,-1}(z,0) - A^{\tau,0}(z,0). \end{aligned}$$

Combining last two relations with (7.1), we see that we need to establish

(7.2) $$\mathbb{E}A^{\tau,0}(z,0) > -\infty$$

and

(7.3) $$\mathbb{E}A^{\tau,-1}(z,0) < \infty.$$

To prove (7.2) it is sufficient to show that

$$\sum_{r=1}^{\infty} \mathbb{P}\left\{-A^{\tau,0}(z,0) > r\right\} < \infty.$$

Let us choose a small number $\varepsilon > 0$ and write
(7.4)
$$\mathbb{P}\left\{-A^{\tau,0}(z,0) > r\right\} \leq \mathbb{P}\{|z| > \varepsilon r\} + \mathbb{P}\left\{|z| \leq \varepsilon r; \ -A^{-1-|z|,0}(z,0) > r\right\}.$$

The second term can be estimated via Theorem 5.1. Estimating the action of the motion with unit speed from $(\pm\varepsilon r, -\varepsilon r - 1)$ to (z, τ), we obtain that for $|z| \leq \varepsilon r$,

$$A^{-1-\varepsilon r,0}(\pm\varepsilon r, 0) \leq \frac{1}{2}(-|z| - 1 - (-\varepsilon r - 1)) + A^{-1-|z|,0}(z, 0),$$

and therefore

$$\mathbb{P}\left\{|z| \leq \varepsilon r;\ A^{-1-|z|,0}(z,0) < -r\right\}$$

$$\leq \mathbb{P}\left\{A^{-1-\varepsilon r,0}(\varepsilon r, 0) < -r + \frac{\varepsilon r}{2}\right\} + \mathbb{P}\left\{A^{-1-\varepsilon r,0}(-\varepsilon r, 0) < -r + \frac{\varepsilon r}{2}\right\}$$

$$= 2\mathbb{P}\left\{A^{\varepsilon r+1} + \frac{1}{2}\frac{(\varepsilon r)^2}{1+\varepsilon r} < -r + \frac{\varepsilon r}{2}\right\}$$

$$\leq 2\mathbb{P}\left\{A^{\varepsilon r+1} - \alpha(0)(\varepsilon r + 1) < -r + \frac{\varepsilon r}{2} - \alpha(0)(\varepsilon r + 1)\right\}$$

$$\leq 2c_1(\varepsilon)\exp\left\{-c_2(\varepsilon)\frac{r^{1/2}}{\ln r}\right\},$$

for $r > c_0(\varepsilon)$, where ε is chosen so small that $1 - \varepsilon/2 + \varepsilon\alpha(0) > 0$. The right-hand side is summable in r.

Let us now estimate $\mathbb{P}\{z > \varepsilon r\}$, one half of the first term in the r.h.s. of (7.4).

Let $p_s = (s, -2s)$, $s > 0$. Let us find $n_0 > 0$ such that $\mathrm{Co}(p_s, (2s)^{-\delta})$ does not intersect $\{(x, -3x) : \ x > 0\}$ for all $s > n_0$. Since the asymptotic slope of γ^* is zero, it will eventually cross $\{(x, -3x) : \ x > 0\}$. Therefore, if $z > \varepsilon r > n_0$, then there must be $s > \varepsilon r$ and a point $q = (q_1, q_2)$ outside of the cone $\mathrm{Co}(p_s, (2s)^{-\delta})$ such that γ^* connects q to $(0,0)$ and passes through p_s. Using Corollary 6.6, we see that if $\varepsilon r > n_0 \wedge M$,

(7.5) $$\mathbb{P}\{z > \varepsilon r\} \leq c_1 e^{-c_2(\varepsilon r)^\kappa},$$

which is summable in r.

This finishes the proof of (7.2). To prove (7.3), it is sufficient to notice that

$$\mathbb{E}A^{\tau,-1}(z, 0) \leq \mathbb{E}\frac{|z|}{2}$$

and apply (7.5). The proof of part 4 of Lemma 7.1 is completed.   □

Having the Busemann function at hand, one can define

$$U_v(x, t) = B((0,0), (x,t)), \quad (x,t) \in \mathbb{R}^2.$$

The main claim of this Section is that thus defined U_v is skew invariant under of the HJHLO cocycle, and its space derivative is the global solution of the Burgers equation.

Let us recall that the HJHLO evolution is given by

(7.6) $$\Phi^{s,t}W(y) = \inf_{x \in \mathbb{R}}\{W(x) + A^{s,t}(x, y)\}, \quad s \leq t, \quad y \in \mathbb{R},$$

where $A^{s,t}(x, y)$ has been defined in (4.1).

Lemma 7.2. *Function U_v defined above is a global solution of the Hamilton–Jacobi equation. If $s \leq t$, then*

$$\Phi^{s,t} U_v(\cdot, s)(x) = U_v(x, t).$$

PROOF: Let γ_v be a minimizer through (x, t) with slope v. Then

$$U_v(x, t) = U_v(\gamma_v(s), s) + (U_v(x, t) - U_v(\gamma_v(s), s))$$
$$= U_v(\gamma_v(s), s) + A^{s,t}(\gamma_v(s), x).$$

We need to show that the right-hand side is the infimum of $U_v(y, s) + A^{s,t}(y, x)$ over all $y \in \mathbb{R}$. Suppose that for some $y \in \mathbb{R}$,

$$(7.7) \qquad U_v(y, s) + A^{s,t}(y, x) < U_v(\gamma_v(s), s) + A^{s,t}(\gamma_v(s), x).$$

Let us take any minimizer $\bar\gamma_v$ originating at (y, s) and denote by $\tau < s$ the time of coalescence of $\bar\gamma_v$ and γ_v. We claim that

$$(7.8) \qquad A^{\tau,s}(\bar\gamma_v) + A^{s,t}(y, x) < A^{\tau,t}(\gamma_v(\tau), x) = A^{\tau,t}(\gamma_v),$$

which contradicts the minimizing property of γ. In fact, (7.8) is a consequence of

$$A^{\tau,s}(\bar\gamma_v) - A^{\tau,s}(\gamma_v) = U_v(y, s) - U_v(\gamma_v(s), s)$$
$$< A^{s,t}(\gamma_v(s), x) - A^{s,t}(y, x).$$

where the second inequality follows from (7.7).   □

Another way to approach the Burgers equation is to consider, for $p = (x, t)$,

$$u_v(x, t) = \dot\gamma_{v,p}(t).$$

Then $U_v(x, t) - U_v(0, t) = \int_0^x u_v(y, t) dy$. We recall that $\Psi^{s,t} w$ denotes the solution at time t of the Burgers equation with initial condition w imposed at time s.

Lemma 7.3. *The function u_v defined above is a global solution of the Burgers equation. If $s \leq t$, then*

$$\Psi^{s,t} u_v(\cdot, s) = u_v(\cdot, t), \quad s \leq t.$$

PROOF: This statement is a direct consequence of Lemmas 2.2, 7.2, and the definition of the Burgers cocycle Ψ.   □

The function $u_v(\cdot, t)$ is clearly piecewise linear with respect to the space coordinate, with downward jumps, each linear regime corresponding to the configuration point visited last by one-sided minimizers, see Lemma 6.19.

To prove that $U_v(\cdot, t) \in \mathbb{H}(v, v)$ for all t, we will compute the expectation of its spatial increments (we already know that it is well defined due to part 4 of Lemma 7.1), and prove that $u_v(\cdot, t)$ is mixing with respect to the spatial variable.

Lemma 7.4. *For any $(x, t) \in \mathbb{R}^2$,*

$$\mathbb{E}(U_v(x + 1, t) - U_v(x, t)) = \mathbb{E} B_v((x, t), (x + 1, t)) = v.$$

PROOF: First, we consider the case $v = 0$. Due to the distributional invariance of Poisson process under reflections,

$$\mathbb{E}B_0((x,t),(x+1,t)) = \mathbb{E}B_0((x+1,t),(x,t)).$$

Combining this with the anti-symmetry of B_0, we obtain $\mathbb{E}B_0((x+1,t),(x,t)) = 0$, as required.

In the general case, we can apply the shear transformation $L : (y,s) \mapsto (y+(t-s)v,s) = (y+vt-vs,s)$. Due to Lemma 4.1, the one-sided minimizers of slope v will be mapped onto one-sided minimizers of slope 0 for the new Poissonian configuration $L(\omega)$. We already know that

$$\mathbb{E}B_{0,L(\omega)}((x+1,t),(x,t)) = 0,$$

and a direct computation based on Lemma 4.1 gives

$$B_{0,L(\omega)}((x,t),(x+1,t)) = B_{v,\omega}((x,t),(x+1,t)) + v,$$

and our statement follows since L preserves the distribution of Poisson process.   □

Lemma 7.5. *Let $v \in \mathbb{R}$. For any t, the process $u_v(\cdot,t)$ is mixing.*

PROOF: Due to translation invariance it is sufficient to consider $t = 0$. Using the shear invariance we can restrict ourselves to the case $v = 0$. Notice that the values of $u_v(\cdot,0)$ on an interval are determined by the increments of $U_v(\cdot,0)$ on that interval. Therefore, in this proof we can work with these increments.

Let us fix $a > 0$ and $\varepsilon > 0$. For any $h > 2a$, we consider the processes

$$L_0(x) = B_0((0,0),(x,0)) = U_0(x,0) - U_0(0,0), \quad x \in [-a,a],$$

and

$$L_h(x) = B_0((h,0),(h+x,0)) = U_0(x+h,0) - U_0(h,0), \quad x \in [-a,a].$$

Next we define the processes

$$L_0^t(x) = A_*^{-t,0}(0,0) - A_*^{-t,0}(0,x), \quad x \in [-a,a],$$

and

$$L_h^t(x) = A_*^{-t,0}(h,h) - A_*^{-t,0}(h,x+h), \quad x \in [-a,a],$$

where A_* is defined as the usual optimal action A, with the restriction that the path cannot cross the vertical line χ through $(h/2,0)$. This means that L_0^t and L_h^t are by definition independent, since the first depends on the Poisson process left of the vertical line, and L_h^t depends on the Poisson process right of the vertical line.

Note that if the one-sided minimizers for $(-a,0)$ and $(a,0)$ coalesce above $-t$, and if the minimizing paths connecting $(0,-t)$ to points $(-a,0)$ and $(a,0)$ coincide with the respective one-sided minimizers above their coalescing point and do not cross χ, then $L_0(x) = L_0^t(x)$ for all $x \in [-a,a]$. By choosing t large enough and then h large enough, we can ensure that

$$\mathbb{P}\{\forall \ x \in [-a,a] \ : \ L_0(x) = L_0^t(x)\} \geq 1 - \varepsilon.$$

Since our set-up is symmetric around χ, we also have

$$\mathbb{P}\{\forall\ x \in [-a, a]\ :\ L_h(x) = L_h^t(x)\} \geq 1 - \varepsilon.$$

Define

$$C = C_{t,h} = \{\forall\ x \in [-a, a]\ :\ L_0(x) = L_0^t(x)\ \text{and}\ L_h(x) = L_h^t(x)\}.$$

For events E, F for the process L_0, we define $\tau_h(F)$ as the "translated" event for L_h, and E^t and $\tau_h(F)^t$ as the corresponding events for L_0^t and L_h^t. Then,

$$
\begin{aligned}
|\mathbb{P}(E \cap \tau_h(F)) - \mathbb{P}(E)\mathbb{P}(F)| &\leq |\mathbb{P}(E \cap \tau_h(F) \cap C) - \mathbb{P}(E)\mathbb{P}(F)| + 2\varepsilon \\
&= |\mathbb{P}(E^t \cap \tau_h(F)^t \cap C) - \mathbb{P}(E)\mathbb{P}(F)| + 2\varepsilon \\
&\leq |\mathbb{P}(E^t \cap \tau_h(F)^t) - \mathbb{P}(E)\mathbb{P}(F)| + 4\varepsilon \\
&= |\mathbb{P}(E^t)\mathbb{P}(\tau_h(F)^t) - \mathbb{P}(E)\mathbb{P}(F)| + 4\varepsilon \\
&\leq |\mathbb{P}(E^t \cap C)\mathbb{P}(\tau_h(F)^t \cap C) - \mathbb{P}(E)\mathbb{P}(F)| + 8\varepsilon \\
&= |\mathbb{P}(E \cap C)\mathbb{P}(\tau_h(F) \cap C) - \mathbb{P}(E)\mathbb{P}(F)| + 8\varepsilon \\
&\leq 12\varepsilon,
\end{aligned}
$$

and mixing follows due to the arbitrary choice of ϵ. $\qquad\square$

Combining Lemmas 7.4 and 7.5, we conclude that the Birkhoff space averages of u_v have a well-defined, deterministic limit v, so $U_v \in \mathbb{H}(v, v)$.

8. STATIONARY SOLUTIONS: UNIQUENESS AND BASINS OF ATTRACTION

In this section we prove Theorem 3.3 and the uniqueness part in Theorem 3.1

The key step in the proof of Theorem 3.3 is the following observation.

Lemma 8.1. *Let $t \in \mathbb{R}$ and suppose that an initial condition W satisfies one of the conditions (3.2),(3.3),(3.1). With probability one, the following holds true for every $y \in \mathbb{R}$. Let $y^*(s)$ be a solution of the optimization problem (7.6). Then*

$$\lim_{s \to -\infty} \frac{y^*(s)}{s} = v.$$

PROOF OF THEOREM 3.3:   Let us take any rectangle $Q = [-R, R] \times [t_0, t_1]$ and set $t = t_1$. We use Lemma 6.19 to find points $a, b \in \mathbb{R}$ satisfying $a < -R < R < b$, not coinciding with any of the points $x_k, k \in \mathbb{Z}$ and such that one-sided backward minimizers $\gamma_{(a,t),v}$ and $\gamma_{(b,t),v}$ do not cross Q.

Applying Lemma 8.1 to $x = a, b$, we see that the corresponding points $a^*(s)$ and $b^*(s)$ satisfy $a^*(s)/s \to v$ and $b^*(s)/s \to v$ as $s \to -\infty$.

Let $p = (x_0, \tau_0)$ be the point of coalescence of the one-sided minimizers $\gamma_{(a,t),v}$ and $\gamma_{(b,t),v}$. We automatically have $\tau_0 < t_0$. Lemma 6.19 then implies that there is $\tau_1 < \min\{\tau_0, 0\}$ such that for $s < \tau_1$, the restrictions of the finite minimizers connecting $(a^*(s), s)$ to (a, t) and $(b^*(s), s)$ to (b, t) on $[\tau_0, t]$ coincide with the restrictions of $\gamma_{(a,t),v}$ and $\gamma_{(b,t),v}$ (this also implies that we can choose $a^*(s) = b^*(s)$).

Since Q is trapped between $\gamma_{(a,t),v}$ and $\gamma_{(b,t),v}$, and minimizing paths cannot cross each other, we conclude that for any $s < \tau_1$, and any $(x, t) \in Q$, the minimizers connecting $(x^*(s), s)$ to (x, t) (where x^* is a solution of the

optimization problem (7.6)) have to pass through p. In particular, the slopes of these minimizers determining the evolution of the Burgers velocity field in $[-R, R]$ throughout $[t_0, t_1]$ do not change (and coincide with the slopes of one-sided backward minimizers) as long as $s < \tau_1$, which completes the proof.   □

PROOF OF LEMMA 8.1:   We will only prove the sufficiency of condition (3.1). The proof of sufficiency of conditions (3.2) and (3.3) follows the same lines and we omit it.

Let us also restrict ourselves to $t = 0$ for simplicity. The proof does not change for other values of t.

Since y^* is increasing in y, it is sufficient to show that the conclusion of the lemma holds with probability 1 for fixed y. The stationarity of Poisson point field implies that we can assume $y = 0$.

We must show that for any $\varepsilon > 0$ it is extremely unlikely for a path γ with $\gamma(0) = 0$ and $|\gamma(-r)| > \varepsilon r$ to be optimal if r is large. For definiteness, let us work with paths satisfying $\gamma(-r) > \varepsilon r$.

For any $\delta_0 \in (0, -\alpha(0)/3)$ and for sufficiently large r,

$$W(0) + A^{-r,0}(0,0) < (\alpha(0) + \delta_0)r.$$

Let us introduce

$$Q_{ij} = [\varepsilon j + i, \varepsilon j + i + 1] \times [-j - 1, -j], \quad i, j \geq 0.$$

If $(x, -r) \in Q_{ij}$ and $W(x) + A^{-r,0}(x,0) < (\alpha(0) + \delta_0)r$, which would be necessary if $x = 0^*(-r)$, then

$$\inf_{z \in [\varepsilon j + i, \varepsilon j + i + 1]} W(z) + A^{-j-2,0}(\varepsilon j + i + 1, 0) < (\alpha(0) + \delta_0)j + \frac{1}{2}.$$

The condition (3.1) at $+\infty$ implies that there is j_0 such that for $j > j_0$ and all i,

$$\inf_{z \in [\varepsilon j + i, \varepsilon j + i + 1]} W(z) > -(j + i)\delta_0.$$

so there is j_1 such that for $j > j_1$,

$$A^{-j-2,0}(\varepsilon j + i + 1, 0) < (\alpha(0) + 2\delta_0)j + \delta_0 i + \frac{1}{2} < j\left(\alpha(0) + 3\delta_0 + \delta_0\frac{i}{j}\right).$$

Let us denote by B_{ji} the event described by this inequality. Due to the Borel–Cantelli lemma, to show that with probability 1, events B_{ji} can happen only for finitely many values of j, it suffices to show that for some $\beta > 0$ and $c > 0$,

(8.1) $$\sum_{j \geq c} \sum_{i \leq \beta j} \mathbb{P}(B_{ji}) < \infty,$$

and

(8.2) $$\sum_{j \geq c} \sum_{i > \beta j} \mathbb{P}(B_{ji}) < \infty.$$

Denoting $\alpha_{ji} = \alpha\left(\frac{\varepsilon j + i + 1}{j+2}\right)$, using shear and translation invariance, we obtain

$$\mathbb{P}(B_{ji}) = \mathbb{P}\left\{A^{j+2} - \alpha(0)(j+2) < j\left(\alpha(0) + 3\delta_0 + \delta_0\frac{i}{j} - \frac{j+2}{j}\alpha_{ji}\right)\right\}.$$

If δ_0 was chosen sufficiently small, then, using Lemma 4.7, we can find j_2 such that for all $j > j_2$ and all i,

$$j\left(\alpha(0) + 3\delta_0 + \delta_0\frac{i}{j} - \frac{j+2}{j}\alpha_{ji}\right) < -(j+2)\left(\frac{\varepsilon^2}{3} + \frac{i^2}{2j^2}\right),$$

so that

$$(8.3)\qquad \mathbb{P}(B_{ji}) \leq \mathbb{P}\left\{A^{j+2} - \alpha(0)(j+2) < -(j+2)\left(\frac{\varepsilon^2}{3} + \frac{i^2}{3j^2}\right)\right\}.$$

Now (8.1) follows (with any $c \geq j_2$ and with arbitrary choice of β) from Theorem 5.1.

To prove (8.2) we need an auxiliary lemma. In its statement and proof we use the notation introduced in Section 4.

Lemma 8.2. *There are constants* $c_1, c_2, c_3, X_0, T_0 > 0$ *such that for* $t > T_0$, $x > X_0$,

$$\mathbb{P}\{A^t \leq -xt\} \leq c_2 e^{-c_3 xt}.$$

Notice that this lemma directly implies (8.2) and thus completes the proof of Lemma 8.1 if we choose $c > T_0$ and β satisfying

$$\frac{\varepsilon^2}{3} + \frac{\beta^2}{3} - \alpha(0) > X_0.$$

It remains to prove Lemma 8.2.

PROOF OF LEMMA 8.2:   It is sufficient to prove the lemma for $t \in \mathbb{N}$, although the values of constants may need adjustment for general t. Let us take $c_4 > 0$ and write

$$(8.4)\qquad \mathbb{P}\{A^t \leq -xt\} \leq \mathbb{P}\{\#\mathcal{A} \leq c_4 xt,\ A^t \leq -xt\} + \mathbb{P}\{\#\mathcal{A} > c_4 xt\}.$$

To estimate the first term on the r.h.s., we can use (4.4) and derive that if c_4 is chosen small enough to ensure $y_0 c_4 < 2$, where y_0 was introduced before condition (4.3), then for some constants $c_5, c_6 > 0$, any $x > 1$, and sufficiently large t,

$$\mathbb{P}\{\#\mathcal{A} \leq c_4 xt,\ A^t \leq -xt\} \leq \mathbb{P}\{N_{\lceil c_4 xt \rceil} \geq xt\} \leq c_5 e^{-c_6 xt}.$$

The second term on the r.h.s. of (8.4) can be estimated using Lemma 4.3. If $c_4 x \geq R$ and t is sufficiently large, then

$$\mathbb{P}\{\#\mathcal{A} > c_4 xt\} \leq \sum_{n \geq c_4 xt} \mathbb{P}(E_{n,t})$$

$$\leq \sum_{n \geq c_4 xt} C_1 \exp(-C_2 n^2/t)$$

$$\leq C_1' \exp(-C_2' x^2 t).$$

for some constants $C_1', C_2' > 0$, which completes the proof. $\square$

PROOF OF UNIQUENESS IN THEOREM 3.1: We will prove that any skew-invariant function u with average velocity v coincides with the global solution u_v at time 0.

Let us take an arbitrary interval $I = (a, b)$. Lemma 8.1 implies that for any W satisfying $\mathbb{H}(v, v)$, there is a time $T_0(a, b, W) \geq 0$ such that if $s < -T_0$, then there is a point $a^* \in \mathbb{R}$ that solves the optimization problem (7.6) for $t = 0$ and for all points $y \in I$ at once, and the respective finite minimizers on $[s, 0]$ have the same velocity at time 0 as the infinite one-sided minimizers of asymptotic slope v.

Suppose now that $U_\omega(x, t) = U_{\theta^t \omega}(x)$ is a global solution in $\hat{\mathbb{H}}(v, v)$. Then $T_0(a, b, U_{\theta^t \omega}(\cdot)) > 0$ is a stationary process. In particular, this means that with probability 1, there is $R > 0$ and a sequence of times $s_n \downarrow -\infty$ such that $T_0(a, b, U_{\theta^{s_n} \omega}(\cdot)) < R$ for all $n \in \mathbb{N}$. Therefore, there is n such that $s_n < -T_0(a, b, U_{\theta^{s_n} \omega}(\cdot))$. This and the fact that U at time 0 is the solution of problem (7.6) for $t = 0$, $s = s_n$ and initial condition $W = U_{\theta^{s_n} \omega}(\cdot)$, we conclude that U and the global solution U_v coincide on I at time 0, and the proof is complete. $\square$

9. BASICS ON BURGERS WITH POISSONIAN FORCING

PROOF OF LEMMA 2.1: Let us take $N \in \mathbb{N}$, $x \in [-N, N]$, $W \in \mathbb{H}$, and $M \in \mathbb{N}$ satisfying $W(y) \geq -M(|y| + 1)$ for all $y \in \mathbb{R}$ and $|W(0)| < M$. Let us take any $m, n \in \mathbb{Z}$ satisfying $m < n$ and any $t, s \in \mathbb{R}$ satisfying $m < s < t < n$.

Suppose now that a path $\gamma : [s, t] \to \mathbb{R}$ and a constant $L \in \mathbb{N}$ satisfy $\gamma(t) = x$ and $\sup_{r \in [s,t]} |\gamma(r)| \in [L, L+1]$. Let us compare γ with the straight path $\gamma_0(r) = x(r - s)/(t - s)$, $r \in [s, t]$. Assuming that

$$A_\omega^{s,t}(W, \gamma) \leq A_\omega^{s,t}(W, \gamma_0),$$

we obtain

$$
\begin{aligned}
M + \frac{x^2}{2(t - s)} &\geq W(0) + \frac{x^2}{2(t - s)} \\
&\geq A_\omega^{s,t}(W, \gamma_0) \\
&\geq A_\omega^{s,t}(W, \gamma) \\
&\geq -M(L + 1 + 1) + \frac{(x - L)^2}{2(t - s)} - \omega([-(L+1), (L+1)] \times [s, t]).
\end{aligned}
$$

Therefore, for sufficiently large L,

$$\omega([-(L+1), (L+1)] \times [m, n]) \geq -M(L+3) + \frac{L^2 - 2Lx}{2(t - s)} \geq -M(L+3) + \frac{L^2 - 2LN}{2(n - m)}.$$

Since the r.h.s. is quadratic in L, and the l.h.s. grows linearly in L with probability 1 due to the strong law of large numbers, we conclude that this inequality can be true only for finitely many values of $L \in \mathbb{N}$. Therefore, under the imposed restrictions on x, W, s, t, there is $L_0 = L_0(\omega, m, n) < \infty$ such that the variational problem (2.2) will not change if supplied with an additional restriction $\sup_{r \in [s,t]} |\gamma(r)| \leq L_0$.

Since with probability 1 there are finitely many Poissonian points in $[-L_0, L_0] \times [m, n]$, it is useful to split the variational problem into two parts: optimization over paths that do not pass through any Poisson points and over paths passing through some configuration points.

The existence of an optimal path among those not passing through any Poisson points is guaranteed by the theory of unforced Burgers equation $u_t + u u_x = 0$.

Also, there are only finitely many broken line paths between Poissonian points, and, for the same reason as above, for each Poisson point p in $[-L_0, L_0] \times [s, t)$ there is an optimal path among paths not passing through any other Poisson points and terminating at p. Gluing these paths together, we see that the extremum in the optimization problem for paths containing Poissonian points is also attained.

Combining the two cases leads to our claim holding a.s. for all x, W, s, t, satisfying constraints specified by integers M, N, m, n. The countable intersection of full measure sets over all M, N, m, n still produces a full measure set. Its time invariance follows since if $L_0(\omega, m, n)$ is finite for all m, n, then $L_0(\theta^t \omega, m, n)$ is finite for all m, n and all $t \in \mathbb{R}$. The proof is complete. $\square$

PROOF OF LEMMA 2.2: Part 1 follows from the observation that the action depends on the path continuously if the family of configuration points visited by the path is fixed.

In the situation where γ does not pass through any configuration points, Part 2 follows from the theory for unforced Burgers equation, see e.g., [Lio82, Section 1.3]. If the minimizer γ for x has a straight line segment connecting a configuration point $p = (y, r)$ to (x, t), then

$$(9.1) \qquad \frac{d}{dx}(\Phi_\omega^{s,t} W)(x) = \frac{d}{dx} \frac{(x-y)^2}{2(t-r)} = \frac{x-y}{t-r} = \dot{\gamma}(t),$$

and Part 2 is proven for points with unique minimizers. The case of the boundary points where the minimizers are not unique is considered similarly.

Part 3 is a direct consequence of Part 2. Part 4 follows from local boundedness of the slope of minimizers. For minimizers not passing through any configuration points, this is a consequence of the classical theory of unforced Burgers equation, and for minimizers passing through some configuration points it follows from (9.1). $\square$

PROOF OF LEMMA 2.5: Let us take a point $(x, t) \in (\mathbb{R} \times (s, \infty)) \setminus \omega$ and a small neighborhood $O \subset (\mathbb{R} \times ((s+t)/2, \infty)) \setminus \omega$ of (x, t). There is a time $s_0 > s$ such that for all $(x', t') \in O$, no minimizers realizing $\Phi^{s,t'} W(x')$ pass through any configuration point after s_0. Lemma 2.3 implies now that the restriction of U to O coincides with the entropy solution of (2.3) with initial data $\Phi_\omega^{s,s_0} W$ imposed at time s_0. $\square$

PROOF OF LEMMA 2.6: The Lipschitz property required in the definition of $\mathbb{H}$ follows from Lemma 2.2, so we only have to control the behavior of $\Phi_\omega^{s,t} W(x)$ as $x \to \pm\infty$. It is easy to see that for our almost sure statement, it is sufficient to construct exceptional sets for integer times s, t and the limits in definition of $\mathbb{H}$ and $\mathbb{H}(v_-, v_+)$ only along integer values of x.

We will prove that if there is a constant M such that if $W(x) > -M(|x| + 1)$ for all $x \in \mathbb{R}$, then for any $\varepsilon > 0$ and for sufficiently large integer values of $|n|$, $\Phi_\omega^{s,t} W(n) > -(M + \varepsilon)|n|$. Suppose, for some n this inequality is violated and for some $L \geq 0$, an optimal path γ realizing $\Phi_\omega^{s,t} W(n)$ satisfies $\sup_{r \in [s,t]} |\gamma(r) - n| \in [L, L+1]$. Since

$$-(M+\varepsilon))|n| \geq \Phi_\omega^{s,t} W(n) \geq -M(|n-L|+1) + \frac{L^2}{2(t-s)} - \omega([n-L, n+L] \times [s,t]),$$

we obtain

$$\omega([n-L, n+L] \times [s,t]) \geq \varepsilon|n| - M(L+1) + \frac{L^2}{2(t-s)}.$$

The l.h.s. has the Poisson distribution with mean $2L(t-s)$, and the Borel–Cantelli Lemma implies that this can hold only for finitely many values of n and L, which proves the first part of the lemma.

The proof of the second part follows the same lines, and we omit it. □

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