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# ACCURATE INTEGRATION OF FORCED AND DAMPED OSCILLATORS

Fernando GARCÍA-ALONSO[1*], Mónica CORTÉS-MOLINA[1], Yolanda VILLACAMPA[1], José Antonio REYES[1]

*The new methods accurately integrate forced and damped oscillators. A family of analytical functions is introduced known as T-functions which are dependent on three parameters. The solution is expressed as a series of T-functions calculating their coefficients by means of recurrences which involve the perturbation function. In the T-functions series method the perturbation parameter is the factor in the local truncation error. Furthermore, this method is zero-stable and convergent.*

*An application of this method is exposed to resolve a physic IVP, modeled by means of forced and damped oscillators.*

*The good behavior and precision of the methods, is evidenced by contrasting the results with other-reputed algorithms implemented in MAPLE.*

**Keywords**: Numerical solutions of ODE's, perturbed and damped oscillators, Initial Value Problems (IVP).

## 1. Introduction

In the nineteen seventies, Stiefel and Bettis [1], published the first numerical methods for solving this type of oscillators, that were fixed-step non-linear multistep codes. Stiefel and Scheifele [2] developed methods based on the use of a series of so-called *G*-functions in place of the Taylor series. Both methods, although they differ, share the important property of integrating harmonic oscillations in a frequency without truncation error. Subsequently, other methods have been developed for integrating oscillators by applying different ideas and with different motivations, Deuflhard [3], Neta and Ford [4], Denk [5].

Methods were designed using $\varphi$-function series [6], in place of *G*-function series, which also integrate the homogeneous problem without truncation error. The methods presented in this paper generalize codes based on series of *G*-function and $\varphi$-function, and are designed for the integration of forced and damped oscillators of type:

$$x''(t)+\gamma x'(t)+\alpha x(t)=\varepsilon\cdot f\left(t,x(t),x'(t)\right),\ x(0)=x_0 \text{ and } x'(0)=x'_0, \qquad (1)$$

[1] Dept. of Applied Mathematics, Escuela Politécnica Superior, University of Alicante, Spain.
[*]Corresponding author: Fernando García-Alonso (Fernando.Garcia@ua.es)

with $\varepsilon$ being a small perturbation parameter, and $\alpha$ a known constant frequency and $\gamma$ the damping constant.

It is accepted that the perturbation function $g(t) = f\left(t, x(t, x_0, x_0', t_0), x'(t, x_0, x_0', t_0)\right)$ defined throughout the solution $x(t)$, $x'(t)$ of IVP (Initial Value Problem) (1), admits in [0, T], a power series expansion absolutely convergent, in the following manner $g(t) = \sum_{n=0}^{\infty} c_n \left(t^n / n!\right)$.

In order to carry out the exact integration of the IVP (1), the differential linear operator, $D^2 + \beta^2$, is defined with $\beta$ being a third frequency. In the case of a more complex perturbation, which cannot cancel, simpler expressions of it would be obtained, which would facilitate numerical integration of the IVP (1).

For this purpose new functions are defined, the *T*-functions. For $\gamma = 0$, they coincide with the Ferrándiz $\varphi$-functions [6] and verify equations which relate them to the Scheifele [2] *G*-functions. The numerical algorithm, based on *T*-function series, generalises the original Scheifele method and permits integration of the non perturbed problem without truncation error.

## 2. Basic ideas and formulations

We shall consider the following equations:

$$x''(t) + \gamma x'(t) + \alpha x(t) = \varepsilon \cdot f\left(t, x(t), x'(t)\right), \quad x(0) = x_0 \text{ and } x'(0) = x_0', \tag{2}$$

which formulate an IVP corresponding to a forced and damped oscillator with $\alpha, \gamma \in \mathbb{R}^+$ frequencies, where $\varepsilon$ is a perturbation parameter, usually small.

The solution of (1), $x(t)$ obtained for the initial given conditions, is analytic in the $[0, \mathrm{T}] \subset \mathbb{R}$ interval. In terms of the differential operator *D*, where $D^n$ represents $\left(d^n / dt^n\right)$, (1) may be expressed in the following manner:

$$\left(D^2 + \gamma D + \alpha\right) x(t) = \varepsilon \cdot f\left(t, x(t), x'(t)\right), \ x(0) = x_0 \ , \ x'(0) = x_0'. \tag{3}$$

By applying the differential operator $D^2 + \beta^2$ a (3), in order to cancel perturbation, the following superior order equation is obtained:

$$D^4 x(t) + \gamma D^3 x(t) + \left(\alpha + \beta^2\right) D^2 x(t) + \beta^2 \gamma D x(t) + \alpha \beta^2 x(t) = \varepsilon \left(D^2 + \beta^2\right) f\left(t, x, x'\right) \tag{4}$$

Given that $x(0) = x_0$, $x'(0) = x_0'$ as $x''(t) = -\alpha x(t) - \gamma x'(t) + \varepsilon f\left(t, x(t), x'(t)\right)$ then $x''(0) = -\alpha x_0 - \gamma x_0' + \varepsilon \cdot f\left(0, x_0, x_0'\right) = x_0''$. (5)

In addition $x'''(t) = -\alpha x'(t) - \gamma x''(t) + \varepsilon\ \vec{\nabla} f\left(t, x(t), x'(t)\right)\left(1, x'(t), x''(t)\right)$,

then $x'''(0) = -\alpha x'(0) - \gamma x''(0) + \varepsilon\ \vec{\nabla} f\left(0, x_0, x'_0\right)\left(1, x_0, x'_0\right) = x'''_0$. (6)

Note that the equations (5) and (6) are defined throughout the solution of (1). The equations (4), (5) and (6) permit a new auxiliary IVP to be defined:

$$D^4 x(t) + \gamma D^3 x(t) + \left(\alpha + \beta^2\right) D^2 x(t) + \beta^2 \gamma D x(t) + \alpha\beta^2 x(t) \quad (7)$$

$$= \varepsilon \cdot \left(D^2 + \beta^2\right) f\left(t, x(t), x'(t)\right),\ x(0) = x_0,\ x'(0) = x'_0,\ x''(0) = x''_0,\ x'''(0) = x'''_0,$$

which has the same exact solution as (1) in the [0, T] interval. For the purpose of brevity a more compact expression is introduced, known as:

$$L_4\left(x(t)\right) = \left(\left(D^2 + \beta^2\right)\left(D^2 + \gamma D + \alpha\right)\right) x(t). \quad (8)$$

With the help of Taylor development of $g(t)$, the IVP (7), may be formulated by the equations:

$$L_4\left(x(t)\right) = \varepsilon \sum_{n=0}^{\infty}\left(c_{n+2} + \beta^2 c_n\right) t^n / n!,\ x(0) = x_0, x'(0) = x'_0, x''(0) = x''_0, x'''(0) = x'''_0. \quad (9)$$

As is usual, the solution $x(t)$ of the IVP (9), can split divided into two parts, one corresponding to the solution $x_H(t)$ associated with the homogeneous IVP with the given initial conditions and the other part is the non homogeneous IVP solution in which this and its first three derivatives are cancelled in $t = 0$.

With this last taking into account the principle of superposition of solutions, it may be obtained by calculating the following IVP particulars: $L_4\left(x(t)\right) = t^n / n!$, $x(0) = x'(0) = x''(0) = x'''(0) = 0$, with $n \geq 0$, combining in an adequate form with $\varepsilon$, $c_n$ and $\beta$.

### 3. Definition and properties of the *T*-functions

We consider the following IVP's

$$L_4\left(x(t)\right) = t^n / n!\ ,\quad x(0) = x'(0) = x''(0) = x'''(0) = 0 \ \text{with}\ n \geq 0. \quad (10)$$

**Definition 1:** $\Gamma_n(t) = x_n(t)$, $\forall n \in \mathbb{N}$.

That is, the functions $\Gamma_n(t)$, $\forall n \in \mathbb{N}$, verify:

$L_4\left(\Gamma_n(t)\right) = t^n / n!$, $\Gamma_n(0) = \Gamma'_n(0) = \Gamma''_n(0) = \Gamma'''_n(0) = 0$.

**Proposition 1:** $\Gamma'_n(t) = \Gamma_{n-1}(t)$, $\forall n \in \mathbb{N}$ *with* $n \geq 1$.

**Proposition 2:** *The functions* $\Gamma_n(t)$, $\forall n \in \mathbb{N}$ *verify the following recurrence law:*

$$\Gamma_n(t) + \gamma\ \Gamma_{n+1}(t) + \left(\alpha + \beta^2\right)\Gamma_{n+2}(t) + \beta^2\gamma\ \Gamma_{n+3}(t) + \alpha\beta^2\Gamma_{n+4}(t) = t^{n+4} / (n+4)!$$

We shall consider the homogeneous problem:

$$L_4\left(x(t)\right)=0\,,\ x(0)=x_0\,,\ x'(0)=x'_0\,,\ x''(0)=x''_0\,,\ x'''(0)=x'''_0\,. \tag{11}$$

**Definition 2:** *Let* $T_0,T_1,T_2,T_3$ *be the solutions of* $L_4\left(x(t)\right)=0$ *with initial conditions* $T_i^{(j)}(0)=\delta_{i,j}\,,\ \ i,j=0,1,2,3$.

In its most general case, they are described by means of the equations:

$$T_0=\left(\alpha/L\right)\left(\left(\alpha-\beta^2\right)\cos\left(\beta t\right)+\beta\gamma\sin\left(\beta t\right)\right) \tag{12}$$
$$+\left(\beta^2/\left(2ML\right)\right)\left(\left(N_1+MP\right)e^{-t(\gamma-M)/2}-\left(N_1-MP\right)e^{-t(\gamma+M)/2}\right),$$

$$T_1=\left(1/\left(\beta L\right)\right)\left(\left(\left(\gamma^2-\alpha\right)\beta^2+\alpha^2\right)\sin\left(\beta t\right)-\beta^3\gamma\cos\left(\beta t\right)\right) \tag{13}$$
$$+\left(\beta^2/\left(2ML\right)\right)\left(\left(\gamma M+N_2\right)e^{-t(\gamma-M)/2}+\left(\gamma M-N_2\right)e^{-t(\gamma+M)/2}\right),$$

$$T_2=\left(1/\left(\beta L\right)\right)\left(-\beta P\cos(\beta t)+\alpha\gamma\sin(\beta t)\right) \tag{14}$$
$$+1/\left(2ML\right)\left(\left(MP+N_1\right)e^{-t(\gamma-M)/2}+\left(MP-N_1\right)e^{-t(\gamma+M)/2}\right),$$

$$T_3=\left(1/\left(\beta L\right)\right)\left(\left(\alpha-\beta^2\right)\sin\left(\beta t\right)-\gamma\beta\cos\left(\beta t\right)\right) \tag{15}$$
$$+\left(1/\left(2ML\right)\right)\left(\left(N_2+\gamma M\right)e^{-t(\gamma-M)/2}-\left(N_2-\gamma M\right)e^{-t(\gamma+M)/2}\right).$$

Where $L=\left(\alpha-\beta^2\right)^2+\gamma^2\beta^2$, $M=\left(\gamma^2-4\alpha\right)^{1/2}$, $P=\beta^2+\gamma^2-\alpha$, $N_1=\gamma\left(-3\alpha+\beta^2+\gamma^2\right)$ and $N_2=2\beta^2+\gamma^2-2\alpha$.

**Proposition 3:** *The* $\left\{T_0(t),T_1(t),T_2(t),T_3(t)\right\}$ *is a fundamental system for equation solutions* $L_4\left(x(t)\right)=0$.

**Theorem 1:** *The general solution (11) is* $x_H(t)=x_0T_0(t)+x'_0T_1(t)+x''_0T_2(t)+x'''_0T_3(t)$.

**Theorem 2:** *The general solution for (9) is:*

$x(t)=x_H(t)+\varepsilon\sum_{n=0}^{\infty}\left(c_{n+2}+\beta^2c_n\right)\Gamma_n(t)$.

Using a more compact notation, if

$\boldsymbol{\omega}(t)=\left(T_0(t)\quad T_1(t)\quad T_2(t)\quad T_3(t)\right)$ and $\boldsymbol{x}=\left(x_0\quad x'_0\quad x''_0\quad x'''_0\right)^t$, then $x^{(j)}(0)=\sum_{n=0}^{\infty}\left(c_{n+2}+\beta^2c_n\right)\Gamma_n^{(j)}(0)+\boldsymbol{\omega}^{(j)}(0)\cdot\boldsymbol{x}=x_0^{(j)}$, with $j=0,1,2,3$. Thus $x(t)$ is the solution to (9).

Using the functions $\Gamma_n(t)$, it is possible to extend the functions $T_n(t)$, in the following manner.

**Definition 3:** $T_{n+4}(t) = \Gamma_n(t)$ *with* $n \geq 0$.

Thus the solution of (9) may be written as:

$$x(t) = \boldsymbol{\omega}(t) \cdot \boldsymbol{x} + \varepsilon \sum_{n=0}^{\infty} \left( c_{n+2} + \beta^2 c_n \right) T_{n+4}(t) . \tag{16}$$

It is possible to extend the T-functions, the law of derivation given in proposition 1, obtaining the following relations: $T_0' = -\alpha\beta^2 T_3$, $T_1' = T_0 - \gamma\beta^2 T_3$, $T_2' = T_1 - \left(\alpha + \beta^2\right) T_3$, $T_3' = T_2 - \gamma T_3$, $T_n' = T_{n-1}$, $n \geq 4$.

In [6] to integrate the perturbed oscillator

$$x'' + \alpha^2 x = \varepsilon g(t) \quad x(0) = x_0, x'(0) = x_0', \tag{17}$$

methods based on $\varphi$-functions are applied, which depend on the oscillator frequency $\alpha$ and the free frequency $\beta$. The latter frequency is associated with an differential operator $D^2 + \beta^2$.

If $n \geq 0$, $\varphi$-functions are solutions of IVP's:

$$\left(D^2 + \beta^2\right)\left(D^2 + \alpha^2\right)\varphi_{n+4}(t) = \frac{t^n}{n!}, \ \varphi_{n+4}^{(j)}(0) = 0, \ j = 0,1,2,3. \tag{18}$$

The first four $\varphi$-functions are expressed by:

$$\varphi_0(t) = \left(\alpha^2 \cos(\beta t) - \beta^2 \cos(\alpha t)\right) \Big/ \left(\alpha^2 - \beta^2\right) \tag{19}$$

$$\varphi_1(t) = \left(\left(\alpha^2/\beta\right)\sin(\beta t) - \left(\beta^2/\alpha\right)\sin(\alpha t)\right) \Big/ \left(\alpha^2 - \beta^2\right) \tag{20}$$

$$\varphi_2(t) = \left(\cos(\beta t) - \cos(\alpha t)\right) \Big/ \left(\alpha^2 - \beta^2\right) \tag{21}$$

$$\varphi_3(t) = \left(\sin(\beta t)/\beta - \sin(\alpha t)/\alpha\right) \Big/ \left(\alpha^2 - \beta^2\right). \tag{22}$$

In addition:

$$\varphi_0(t) + \alpha^2\beta^2\varphi_4(t) = 1, \qquad \varphi_1(t) + \alpha^2\beta^2\varphi_5(t) = t \tag{23}$$

$$\varphi_n(t) + \left(\alpha^2 + \beta^2\right)\varphi_{n+2}(t) + \alpha^2\beta^2\varphi_{n+4}(t) = \frac{t^n}{n!}, \ \forall n \geq 2 . \tag{24}$$

Furthermore, relations may be established between the *T*-functions, the *G*-functions and the $\varphi$-functions, for this purpose we introduce the notations $T_n(t,\alpha,\beta,\gamma)$, $\varphi_n(t,\alpha,\beta)$ and $G_n(t,\alpha)$, associated with the IVP´s $x'' + \gamma x' + \alpha x = \varepsilon g(t)$, $x'' + \alpha^2 x = \varepsilon g(t)$ and $x'' + \alpha^2 x = \varepsilon g(t)$ respectively.

Obtaining the following equalities [2, 6, 7]:

$$T_n(t,\alpha^2,\beta,0)=\varphi_n\left(t,\alpha,\beta\right),\text{ with } n\geq 0 \tag{25}$$

$$T_n(t,\alpha^2,\beta,0)=\left(\alpha^2 G_n(t,\alpha)-\beta^2 G_n(t,\beta)\right)\Big/\left(\alpha^2-\beta^2\right),\text{ with } n\geq 2 \tag{26}$$

$$G_n(t,\alpha)=T_n(t,\alpha^2,\beta,0)+\beta^2 T_{n+2}(t,\alpha^2,\beta,0)\text{ with } n\geq 2\,. \tag{27}$$

## 4. *T*-functions series method

We should consider the IVP (1) and with $x(t)$ being the solution, which we consider to be analytical in the interval $\left[0,\mathrm{T}\right]\subset\mathbb{R}$, that is, $x(t)=\sum_{n=0}^{\infty}a_n t^n\big/n!$ and that the perturbation function $g(t)$ admits in [0, T], a development in powers series, which is absolutely convergent in the form of $g(t)=\sum_{n=0}^{\infty}c_n\left(t^n/n!\right)$.

As $x''(t)+\gamma x'(t)+\alpha x(t)=\varepsilon g(t)$, then: $\sum_{n=0}^{\infty}a_{n+2}t^n\big/n!$ $+\gamma\sum_{n=0}^{\infty}a_{n+1}t^n\big/n!+\alpha\sum_{n=0}^{\infty}a_n t^n\big/n!=\varepsilon\sum_{n=0}^{\infty}c_n\left(t^n/n!\right)$, from whence it may be deduced that:

$$a_{n+2}+\gamma a_{n+1}+\alpha a_n=\varepsilon c_n,\forall n\geq 0,\ c_n=D^n g(0),\ a_0=x_0,a_1=x_0'. \tag{28}$$

By (5) and (6), it is possible to propose the following recurrence: $a_0=x_0$, $a_1=x_0'$, $a_2=-\gamma a_1-\alpha a_0+\varepsilon c_0=-\gamma a_1-\alpha a_0+\varepsilon f\left(0,x(0),x'(0)\right)=x_0''$, $a_3=-\gamma a_2-\alpha a_1+\varepsilon c_1=-\gamma a_2-\alpha a_1+\varepsilon f'\left(0,x(0),x'(0)\right)=x_0'''$, $a_n=-\gamma a_{n-1}-\alpha a_{n-2}+\varepsilon c_{n-2}=-\gamma a_{n-1}-\alpha a_{n-2}+\varepsilon f^{(n-2)}\left(0,x(0),x'(0)\right)$, $n\geq 4$.

Through theorem 2 it is known that $x(t)=\boldsymbol{\omega}(t)\cdot\boldsymbol{x}+\varepsilon\sum_{n=0}^{\infty}\left(c_{n+2}+\beta^2 c_n\right)T_{n+4}(t)$, defining: $b_0=a_0$, $b_1=a_1$, $b_2=a_2$, $b_3=a_3$, $b_n=\varepsilon\left(c_{n-2}+\beta^2 c_{n-4}\right)$, with $n\geq 4$, of (28) it is deduced that: $b_n=a_n+\gamma a_{n-1}+\left(\alpha+\beta^2\right)a_{n-2}+\gamma\beta^2 a_{n-3}+\alpha\beta^2 a_{n-4}$, $n\geq 4$, therefore, $x(t)=\sum_{n=0}^{\infty}b_n T_n(t)$, which, dispensing with truncations, is analogous to the expressions obtained for developments in *G*-functions [2, 7] and $\varphi$-functions [6].

If for $x_1$, $x_1'$, $x_1''$ and $x_1'''$ we denote the approximations to $x(h)$, $x'(h)$, $x''(h)$ and $x'''(h)$ respectively, where $h$ is the step size, the approximation to the solution, using $(p+1)$ *T*-functions will be given by:

$$x_1^{(j)} = \sum_{n=0}^{3} b_n T_n^{(j)}(h) + \sum_{n=4}^{p} b_n T_{n-j}(h), \;\; j = 0,1,2,3. \tag{29}$$

We assume the approximation to the solution $x_i = x(ih)$ and $x_i' = x'(ih)$ has been calculated.

$$\begin{aligned}
&a_0 = x_i, a_1 = x_i', a_2 = x_i'', a_3 = x_i''',\\
&a_n = -\gamma a_{n-1} - \alpha a_{n-2} + \varepsilon f^{(n-2)}\left(ih, x(ih), x'(ih)\right), n \ge 2,\\
&b_i = a_i, i = 0,1,2,3\;,\\
&b_n = a_n + \gamma a_{n-1} + \left(\alpha + \beta^2\right) a_{n-2} + \beta^2 \gamma a_{n-3} + \alpha\beta^2 a_{n-4}, n \ge 4,\\
&x_{i+1}^{(j)} = \sum_{n=0}^{3} b_n T_n^{(j)}(h) + \sum_{n=4}^{p} b_n T_{n-j}(h), \; j = 0,1,2,3.
\end{aligned} \tag{30}$$

### 5. Zero-stability and convergence

The solution $x(t)$ to the given IVP (1) admits a Taylor expansion about any point $t_m \in [0, \mathrm{T}]$ of the form, $x(t) = \sum_{n=0}^{\infty} a_n \left(t - t_m\right)^n / n!$.

Let us define a function:

$$\begin{aligned}
g_m(t) &= f\left(t, x\left(t; x(t_m), x'(t_m), t_m\right), x'\left(t; x(t_m), x'(t_m), t_m\right)\right)\\
&= \sum_{n=0}^{\infty} c_n \left(t - t_m\right)^n / n!,
\end{aligned} \tag{31}$$

upon substitution of the last $x(t)$ into (1), we find the recursion relation:

$$a_0 = x\left(t_m\right), \;\; a_1 = x'\left(t_m\right), \; a_n = -\gamma a_{n-1} - \alpha a_{n-2} + \varepsilon c_{n-2}, \;\; n \ge 2\,. \tag{32}$$

By defining auxiliary constants $b_n$ as in the above section, the exact solution can be expanded as $x_n(t) = \sum_{n=0}^{\infty} b_n T_n\left(t - t_m\right)$.

Notice that $b_{n+4}$ $(n \ge 0)$ is the $n$ order derivative of the function $\hat{g}_m(t) = \varepsilon\left(D^2 + \beta^2\right) g_m(t)$, at $t = t_m$. Therefore, the definition of the $T$-functions allows us to obtain the identities for $i = 0,1$:

$$x^{(i)}(t) = \sum_{n=0}^{p} b_n T_n^{i)}\left(t - t_m\right) + \hat{g}_m^{(p+1)}\left(\xi_m\right) T_{p+1}^{i)}\left(t - t_m\right), \; t_m < \xi_m < t\,, \; p \ge 4\,. \tag{33}$$

Given a sequence of points $\{t_m\}_{m=0}^{N}$ with $t_0 = 0$, $t_N = \mathrm{T}$, evenly spaced or not, we can compute approximations $\left(x_m, x_m'\right)$ to the exact values $\left(x\left(t_m\right), x'\left(t_m\right)\right)$ by truncating (33) as:

$$x_{m+1} = \sum_{n=0}^{p} b_n T_n\left(t_{m+1}-t_m\right), \quad x'_{m+1} = \sum_{n=0}^{p} b_n T'_n\left(t_{m+1}-t_m\right), \tag{34}$$

where $b_n$ are computed from the recurrences by setting $a_0 = x_m$, $a_1 = x'_{m+1}$, instead of $a_0 = x_0$, $a_1 = x'_1$. When $t_{m+1} - t_m = h$, $m = 0,\ldots,N-1$ the local truncation error is easily derived from (33) and (34) as

$$x^{(i)}(t_{m+1}) - x^{(i)}_{m+1} = \hat{g}_m^{(p+1)}\left(\xi_m\right) T^{(i)}_{p+1}\left(t-t_m\right) = O\left(\varepsilon h^{p+1-i}\right),\ i = 0,1. \tag{35}$$

The numerical scheme given in (34) is consistent, of order $p$ for the solution $x(t)$. The stability of (34) is easy to prove directly. Taking into account equations (32), the algorithm can be written, for $i = 0,1$, as

$$\begin{aligned} x^{(i)}_{n+1} &= \left(T_0^{(i)} - \alpha T_2^{(i)} + \alpha\gamma T_3^{(i)}\right) x_n + \left(T_1^{(i)} - \gamma T_2^{(i)} + \left(\gamma^2 - \alpha\right) T_3^{(i)}\right) x'_n \\ &+ h\Phi_i\left(t, x_n, x'_n, f\right), \end{aligned} \tag{36}$$

when all $T$-functions are evaluated at $h$, (36) can be transformed into $\left(x_{n+1} \quad x'_{n+1}\right)^t = A\left(x_n \quad x'_n\right)^t + h\left(\Phi_0 \quad \Phi_1\right)^t$,where:

$$A = \begin{pmatrix} e^{-\gamma t/2}\left(\left(\gamma/M\right)\sinh\left(Mt/2\right) + \cosh\left(Mt/2\right)\right) & 2\left(e^{-\gamma t/2}/M\right)\sinh\left(Mt/2\right) \\ -2\alpha\left(e^{-\gamma t/2}/M\right)\sinh\left(Mt/2\right) & e^{-\gamma t/2}\left(\cosh\left(Mt/2\right) - \left(\gamma/M\right)\sinh\left(Mt/2\right)\right) \end{pmatrix}.$$

This equation owns the form of equation (2-4) of Lambert´s book [8], p. 24. We have to notice that the assumptions there are fulfilled, because the function $\Phi = \left(\Phi_0 \quad \Phi_1\right)^t$ vanishes whenever $f\left(t\right) \equiv 0$, since then $c_n = 0$, $\forall n \geq 0$ and it verifies a Lipschitz condition, since $f$ was assumed to be analytic.

On the other hand, the eigenvalues of matrix $A$ are $e^{\left(\pm\left(M \mp \gamma\right)t/2\right)}$, then the root condition holds. Therefore, the application of Lambert´s Theorem 2.1 [8] shows that

**Proposition 4** : *The scheme given by (33) is zero-stable.*

Notice that the Lipschitz condition coming from the differentiability of $f$ implies that the single-step method (34) is regular, and by virtue of its consistency we could have established directly the following

**Proposition 5** : *The method (36) is convergent.*

## 6. Numerical examples

**Example 1.** This stiff problem has been selected in order to demonstrate not only the accuracy of the $T$-functions series method, when the perturbation function is cancelled, but also the fact that the $T$-functions generalize the $\varphi$-functions. Let´s consider the following IVP stiff problem, which appears in [9].

$$\begin{cases} x_1'(t) = -2x_1(t) + x_2(t) + 2\sin(t) \\ x_2'(t) = -(\eta+2)x_1(t) + (\eta+1)(x_2(t) - \cos(t) + \sin(t)) \end{cases}, \quad (37)$$

with initial conditions $x_1(0) = 2$, $x_2(0) = 3$ and solution, independent of $\eta$:

$$x_1(t) = 2e^{-t} + \sin(t), \quad x_2(t) = 2e^{-t} + \cos(t). \quad (38)$$

The eigenvalues of the system are –1 and $\eta$, which enables its degree of stiffness to be regulated. For the case $\eta$ = -1000, the stiff problem (37), proposed in [8] is expressed as an oscillator:

$$\left(D^2 + 1001D + 1000\right)x(t) = 1001\cos(t) + 999\sin(t),\ x(0) = 2,\ x'(0) = -1. \quad (39)$$

Applying the operator $D^2 + \beta^2$, with $\beta = 1$, the following IVP is obtained:

$$\begin{aligned} &x^{(iv)}(t) + 1001x'''(t) + 1001x''(t) + 1001x'(t) + 1000x(t) = 0, \\ &x(0) = 2, x'(0) = -1, x''(0) = 2, x'''(0) = -3, \end{aligned} \quad (40)$$

which it is possible to integrate accurately by means of the *T*-functions series method. Having carried out the integration using the algorithm described in (30), particularized to this problem:

$$a_0 = x_i, a_1 = x_i', a_2 = -1000a_0 - 1001a_1 + \left(1001\cos(ih) + 999\sin(ih)\right), \quad (41)$$

$$a_3 = -1000a_1 - 1001a_2 + \left(999\cos(ih) - 1001\sin(ih)\right),$$

$$a_n = -1000a_{n-2} - 1001a_{n-1} + \left(1001\cos\left(ih + (n-2)\pi/2\right) + 999\sin((n-2)\pi/2)\right),$$

$$b_0 = a_0, b_1 = a_1, b_2 = a_2, b_3 = a_3, b_n = a_n + 1001\left(a_{n-1} + a_{n-2} + a_{n-3}\right) + 1000a_{n-4},$$

$$x_{i+1} = \sum_{j=0}^{3} b_j T_j(h), \quad x_{i+1}' = \sum_{j=0}^{3} b_j T_j'(h), n \geq 4.$$

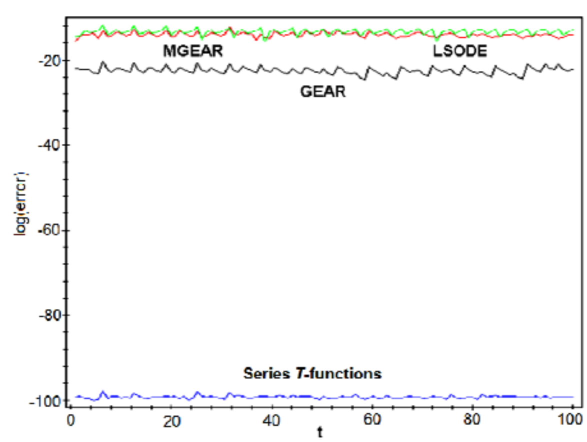


Fig. 1. $x(t)$ position with four *T*-functions.

Fig. 1 shows the graph obtained, with 100 digits of the absolute value logarithm of the relative error of the solution $x(t)$ with $\varepsilon = 1$, calculated by means of (30), with four *T*-functions and step size $h = 0.9$, compared with the graphs corresponding for the logarithm of the absolute value of the relative error of the methods of LSODE[BACKFUNC] with $tol = 10^{-16}$, MGEAR with $errorper = Float(1,-13)$ and GEAR with $errorper = Float(1,-18)$.

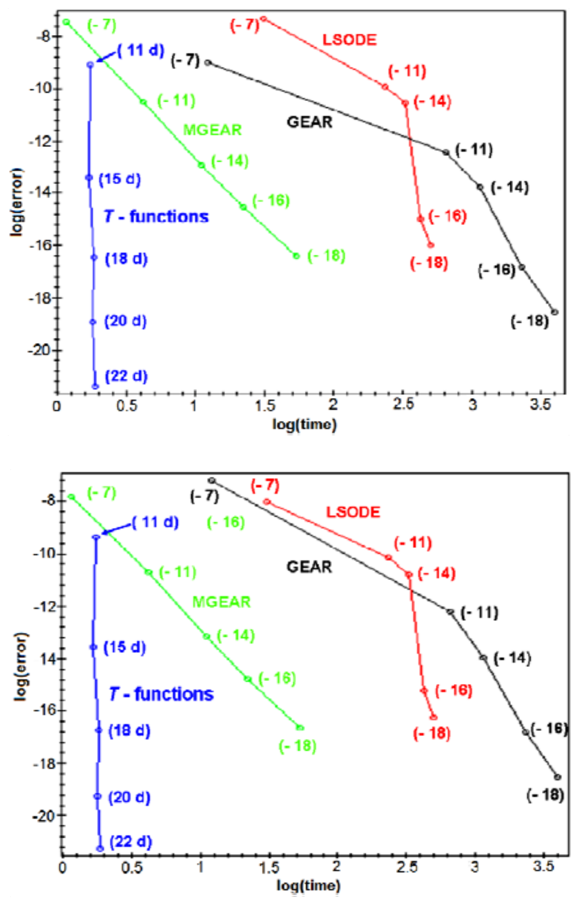


Fig. 2. Efficiency plot for the integration of the coordinate *x* (top graph) and *x*′(bottom graph) at last point *t* = 100 versus computation time for different methods (d = digits).

Fig. 2 shows an efficiency plot where *T*-functions series integrations are compared with integrations using well known general purpose codes. The computation time is represented in the horizontal axis, in logarithmic scale, and the decimal logarithm of the integration error at the last point, *t* = 100, is shown in the vertical axis. The tolerances used in the standard codes are displayed in the figure into parentheses, marking each time-error point.

**Example 2.** The problem proposed by [2], which is a linear oscillator with quadratic perturbation function will be considered:

$$x''(t)+\alpha x(t)=\varepsilon x^2(t)\,,\;\; x(0)=1 \text{ y } x'(0)=0\,, \tag{42}$$

that admits a first integral $H\left(x(t),x'(t)\right)=\left(1/2\right)\left(\alpha x^2(t)+x'^2(t)\right)-\left(\varepsilon/3\right)x^3(t)$.

It has been selected this problem to test the behaviour of the $T$-functions series method when the differential operator $D^2+\beta^2$ does not cancel the perturbation.

For the case $\alpha=1$ and $\varepsilon=10^{-3}$, applying the operator $D^2+\beta^2$ with $\beta=2$ to (42), the next IVP is obtained

$$x^{(iv)}(t)+5x''(t)+4x(t)=\varepsilon\left(D^2+4\right)x^2(t)\,, \tag{43}$$

$$x(0)=1\,,\;\; x'(0)=0\,,\;\; x''(0)=10^{-3}-1 \text{ and } x'''(0)=0\,.$$

After obtaining the value of the $T$-functions, and if $x_i$ and $x_i'$ are the approximations of $x(ih)$ and $x'(ih)$, respectively, each step of integration is completed by following algorithm:

$$a_0=x_i\,,\;\; a_1=x_i'\,,\;\; a_2=-a_0+10^{-3}a_0^2\,,\;\; a_3=-a_1+2\cdot 10^{-3}a_0a_1\,, \tag{44}$$

$$a_n=-a_{n-2}+10^{-3}\sum_{j=0}^{n-2}\binom{k}{j}a_ja_{k-j}\,,\;\; 4\le n\le p\,,$$

$$b_i=a_i\,,\; i=0,1,2,3\,,\;\; b_n=a_n+5a_{n-2}+4a_{k-4}\,,\;\; 4\le n\le p\,,$$

$$x_{i+1}=\sum_{n=0}^{16}b_nT_n(h)\,,\qquad x_{i+1}'=\sum_{n=0}^{16}b_nT_n'(h)\,.$$

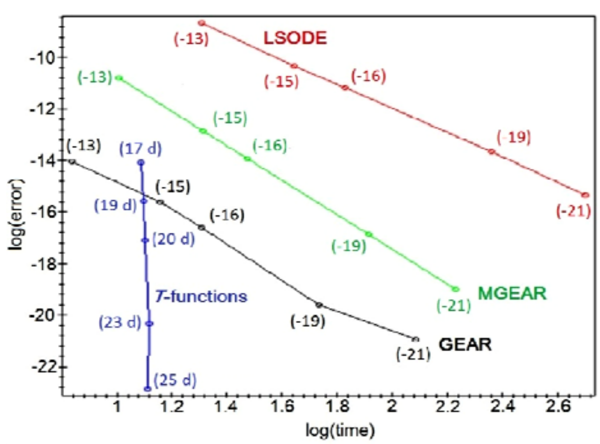


Fig. 3. Efficiency plot for the integration of the coordinate $x$ at last point $t = 100$ versus computation time for different methods (d = digits).

The information in Fig. 3, with step size $h = 0.1$ is organized in the same way as in Fig. 2.

## 7. Conclusions

A family of analytical functions has been defined, known as *T*-functions, dependent on three parameters $\alpha$, $\beta$ and $\gamma$, studying their properties and establishing their relation with the *G*-functions and $\varphi$-functions. Based on the *T*-functions and with certain hypotheses, a series method has been constructed which permits precise numerical integration of a wide range of problems. Amount $T$ -functions series method is zero-stable and convergent. The *T*-functions series method integrates exactly forced and damped oscillators.. The *T*-functions series method may successfully compete with known integrators.

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