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Cohomology of Product Spaces from a Categorical Viewpoint

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Introduction

One of the aims of Algebraic Topology is to study complex problems within Topology by translating them into the more workable world of Algebra. This is usually done by defining *invariants* such as *singular homology* and *singular cohomology*. Both of them help us find properties of spaces. In many cases, homology is not enough and it is desirable to know the cohomology of spaces. For instance, although a product of two Klein Bottles and $(S^1 \vee \mathbb{R}\mathrm{P}^2) \times (S^1 \vee \mathbb{R}\mathrm{P}^2)$ have isomorphic homology modules, their respective cohomologies have different ring structures, as shown in Example 3.15, which can also be found in [8]. All other examples and counterexamples given in this project are due to the author.

These invariants are very powerful since they contain much information of the spaces that are being studied. The first goal of the present work is to prove a formula that relates the homology of a product of spaces with the homologies of the factors. This formula was obtained in the first half of the past century after the work of H. Künneth, who found in 1923 a relation between the Betti numbers of the product of two spaces and the Betti numbers of each of its factors; see [3, Chapter II, Section 5]. In this work, we want to study Künneth formulas for both homology and cohomology. Although the homology Künneth formula is a very standard result, the cohomology Künneth formula is harder to be found in modern literature, and this is why its study is interesting. Our bibliographic sources have mainly been [4], [9], [10], [12] and [13].

The simplest case in homology is an isomorphism $H(X \times Y) \simeq H(X) \otimes H(Y)$ if the ground ring R is a field. Under more general hypotheses, this relation can be a bit more complex. There are two main steps towards the construction of the homology Künneth formula. The first one is algebraic, and consists in finding a relation between $H(C \otimes D)$ and the graded modules $H(C)$ and $H(D)$ for two free chain complexes C and D. This is proved by using the Universal Coefficient Theorem, which relates the homology of $H(C; M)$ with $H(C)$ and M. We prove this under the condition that R is a principal ideal domain. Also in Example 1.20 we see that if the ground ring is not a principal ideal domain, then the Universal Coefficient Theorem is false in general. After that we proceed to prove the Algebraic Künneth Formula in 1.22. Both theorems are expressed as natural short exact sequences which split but not naturally, as shown in Examples 2.12 and 2.13 by using cellular homology and Moore spaces.

The second step is to establish an isomorphism $H(X \times Y) \simeq H(S(X) \otimes S(Y))$ called the Eilenberg-Zilber Theorem, where S denotes the singular functor. This was first published by S. Eilenberg and J. A. Zilber [5] in 1953 using the method of acyclic models; see [3, Chapter IV, Section F]. In Section 2.2 we study the general version of this method in terms of Category Theory, which is called the Acyclic Model

Theorem. After that, we proceed to prove the Eilenberg-Zilber Theorem in Section 2.3, and get the Künneth Theorem for homology in Theorem 2.9. The Acyclic Model Theorem is a very powerful tool, and it can be used to prove many facts, such as the homotopy invariance of homology as shown in Theorem 2.10 and its corollary.

In Chapter 3 we use all the previous theory in order to find the cohomology Künneth formula from Theorem 3.8. The difficulty in this case lies in the algebra and has two main steps. The first consists in generalizing the previous Algebraic Künneth Formula to non-necessarily free chain complexes. This is done in Section 1.6 and gives a result which holds whenever a given condition on the torsion is true. Example 1.28 shows the importance of this condition. The second step consists in actually proving this Künneth formula. This time the key lies in Lemma 3.4, which imposes some finiteness assumptions in the formula. Also it may be worth looking at counterexample 3.5.

Finally, we gather all the previous results to define the graded ring $H^*(X)$, and give a Künneth formula $H^*(X) \otimes H^*(Y) \simeq H^*(X \times Y)$, which holds under some conditions of finiteness and torsion; see 3.12. Example 3.15 illustrates the importance of the condition on the torsion.

In the last chapter we study the first homology and cohomology modules of an infinite product of circles. The higher homology groups of an infinite product of circles are difficult to compute since $\varinjlim_N \prod_{k=0}^N S^1 \not\simeq \prod_{k=0}^\infty S^1$ and moreover $\prod_{k=0}^\infty S^1$ does not admit a cellular structure. We also observe that, surprisingly, the inclusion $\bigvee_{k=0}^\infty S^1 \hookrightarrow \prod_{k=0}^\infty S^1$ induces $\bigoplus_{k=0}^\infty \mathbb{Z} \to \prod_{k=0}^\infty \mathbb{Z}$ in both H^1 and H_1. For this, we use Specker Duality, which is a well-known phenomenon in Infinite Abelian Group Theory.

Is there a Künneth formula for an infinite product of spaces? So far very little information can be found in the literature. Nonetheless the study of an infinite product of circles shows that it is probably much more complex than the formulation for the finite case.

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Contents

Chapter 1

Algebraic Künneth Formula

1.1 Categories

This section is a review of categorical notions which will be used throughout this work. In particular, a key concept for the development of the Künneth formula is the naturality of constructions, which motivated the early history of Category Theory [3, Chapter 5, Section E].

Definition 1.1. A *category* $\mathcal{C}$ consists of the following items:

i) A class of *objects*, which we denote by Ob $\mathcal{C}$. When there is no danger of confusion we usually write $X \in \mathcal{C}$ instead of $X \in \text{Ob } \mathcal{C}$.

ii) For every ordered pair of objects $X, Y \in \mathcal{C}$, a set of *morphisms* from X to Y, denoted by $\mathcal{C}(X, Y)$.

iii) For every triple of objects X, Y, Z, a map from $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z)$ to $\mathcal{C}(X, Z)$, called *composition*. The image of (f, g) is denoted by $g \circ f$ or gf. Composition has to be associative, that is, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ are composable morphisms, then $h \circ (g \circ f) = (h \circ g) \circ f$.

iv) For every object $X \in \mathcal{C}$, an *identity* morphism $\mathcal{I}_X \in \mathcal{C}(X, X)$ such that $f \circ \mathcal{I}_X = f = \mathcal{I}_Y \circ f$ for every $f \in \mathcal{C}(X, Y)$.

In this project we will mainly deal with the following categories:

- The category Top of topological spaces and continuous maps.
- The category R-mod of R-modules and R-homomorphisms for a commutative ring R with unit.

- The category $(R\text{-mod})^*$ of graded R-modules and degree zero R-homomorphisms for a commutative ring R with unit.
- The category R-ChCpx of chain complexes of R-modules and chain maps over a commutative ring R with unit.

Definition 1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant* (resp. *contravariant*) *functor* F from $\mathcal{C}$ to $\mathcal{D}$, denoted as $F\colon \mathcal{C} \to \mathcal{D}$, consists of

i) a map $F\colon \text{Ob}\,\mathcal{C} \to \text{Ob}\,\mathcal{D}$, and

ii) maps $F = F_{XY}\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ for all X, $Y \in \mathcal{C}$, such that

   (a) $F(g\circ f) = Fg\circ Ff$ (resp. $F(g\circ f) = Ff\circ Fg$) for all composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$;

   (b) $F(\mathcal{I}_X) = \mathcal{I}_{FX}$ for all $X \in \mathcal{C}$.

In the following chapter we will study the covariant functor $H_n\colon \text{Top} \to R\text{-mod}$, which is the n^{th} homology R-module for each integer n. We will study some results, that we want to be natural in the following sense.

Definition 1.3. Let $\mathcal{C}$, $\mathcal{D}$ be two categories, and suppose that we have two functors $F,G\colon \mathcal{C} \longrightarrow \mathcal{D}$. A *natural transformation* $\eta\colon F \longrightarrow G$ assigns

i) a morphism $\eta_X \in \mathcal{D}(FX,GX)$ to every object $X \in \mathcal{C}$;

ii) a commutative diagram

$$\begin{array}{ccc} FX & \xrightarrow{\eta_X} & GX \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle Gf} \\ FY & \xrightarrow{\eta_Y} & GY \end{array}$$

to every morphism $f \in \mathcal{C}(X,Y)$. Note that the composite of two natural transformations is also a natural transformation.

Later on, in Chapter 3, we will introduce the cohomology graded ring $H^*\colon \text{Top} \longrightarrow (R\text{-mod})^*$, which is a contravariant functor. An ultimate aim of this work is to relate $H^*(X\times Y)$ with $H^*(X)$ and $H^*(Y)$ for any two topological spaces X and Y. This relation is natural in the sense that, for any pair of maps $(f,g)\colon (X,Y) \to (X',Y')$, the following diagram commutes:

$$\begin{array}{ccc} H^*(X';R)\otimes H^*(Y';R) & \xrightarrow{\times_1} & H^*(X'\times Y';R) \\ \big\downarrow{\scriptstyle (f\otimes g)^*} & & \big\downarrow{\scriptstyle (f\times g)^*} \\ H^*(X;R)\otimes H^*(Y;R) & \xrightarrow{\times_1} & H^*(X\times Y;R) \end{array}$$

1.2 Chain Complexes

We will suppose that R is a commutative ring with unit. In fact we will be mainly interested in rings such that any submodule of a free R-module is free. These rings are called *hereditary*. In particular we will deal with principal ideal domains, which have the property of being hereditary rings; see [2, Chapter 13, Proposition 1.8].

Definition 1.4. A *chain complex* C is a collection of R-modules C_n for $n \in \mathbb{Z}$, together with *boundary maps* $\partial_n \colon C_n \longrightarrow C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. We can depict a chain complex as a sequence of R-modules and R-homomorphisms

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \tag{1.1}$$

If C_n is a free R-module for all $n \in \mathbb{Z}$, we say that C is a *free chain complex*. If $C_n = 0$ for $n < 0$, we say that C is a *positive chain complex*. We often write (C, ∂) for a chain complex C with boundary maps ∂_n.

Definition 1.5. Given two chain complexes (C, ∂) and (D, ϵ), we define a *chain map* f from C to D as a collection of R-homomorphisms $f_n \colon C_n \to D_n$ such that the following diagrams commute:

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{\epsilon_n} & D_{n-1} \end{array} \tag{1.2}$$

Chain complexes and chain maps form a category, which we denote by R-ChCpx or simply ChCpx if there is no confusion about R. Hence we can interpret the diagram (1.2) as a natural transformation $\eta \colon F_n \longrightarrow F_{n-1}$ between two functors $F_n, F_{n-1} \colon \text{ChCpx} \to R\text{-mod}$ such that for any chain complex C, the images are $F_n(C) = C_n$ and $F_{n-1}(C) = C_{n-1}$. Then η assigns the R-homomorphisms $\eta_C = \partial_n$ and $\eta_D = \epsilon_n$ to the respective chain complexes C and D.

As a consequence of Definition 1.4 we have an inclusion $\text{Im}(\partial_{n+1}) \subset \text{Ker}(\partial_n)$ for all $n \in \mathbb{Z}$.

Definition 1.6. We define the n^{th} *homology module* of a chain complex (C, ∂) as

$$H_n(C) = \frac{\text{Ker}(\partial_n)}{\text{Im}(\partial_{n+1})}. \tag{1.3}$$

The elements of $\mathrm{Ker}(\partial_n)$ are called *n-cycles* and the elements of $\mathrm{Im}(\partial_{n+1})$ are called *n-boundaries*. Usually we write $Z_n(C)$ and $B_n(C)$ for the respective modules of n-cycles and n-boundaries. Even sometimes we will just write B_n and Z_n if there is no confusion about the underlying chain complex C. In particular, given a chain map $f\colon C \to D$, from the commutativity of the diagram (1.2) we have inclusions $f(Z_n(C)) \subseteq Z_n(D)$ and $f(B_n(C)) \subseteq B_n(D)$. Hence f induces homomorphisms $f_*\colon H_n(C) \to H_n(D)$ for every $n \in \mathbb{Z}$. Therefore we have a covariant functor $H_n\colon \mathrm{ChCpx} \to R\text{-mod}$.

Definition 1.7. We say that a chain complex C is *exact* if its homology modules vanish. On the other hand, if C is a positive chain complex and its homology modules vanish for $n > 0$, we say that C is *acyclic*.

Therefore in some sense the homology modules measure how a chain complex fails to be exact. In fact the names *cycles* and *boundaries* are very convenient when we study the homology modules of a topological space, since these indicate whether a "hole" is a boundary or not. In order to use homology in this sense, we need first to define functors $F\colon \mathrm{Top} \to \mathrm{ChCpx}$. There are many such functors. In Section 2.1 we will introduce the singular functor $S\colon \mathrm{Top} \to \mathrm{ChCpx}$.

The following definitions are analogous to the concepts of homotopy between maps and homotopy equivalence from Topology.

Definition 1.8. Given two chain complexes (C, ∂) and (D, ϵ) and two chain maps $f, g : C \to D$, a *chain homotopy* T from f to g is a collection of R-homomorphisms $T_n : C_n \to D_{n+1}$ such that

$$T_{n-1} \circ \partial_n + \partial_{n+1} \circ T_n = f_n - g_n$$

holds for all n.

Observe that two chain homotopic maps induce the same morphisms in homology since $\partial_{n+1} \circ T_n$ is a boundary and $T_{n-1} \circ \partial_n$ vanishes on cycles. Hence we deduce that $f_* = g_*\colon H_n(C) \to H_n(D)$ for all n.

Definition 1.9. Let C and D be two chain complexes, and let $f\colon C \to D$ be a chain map. We say that f is a *chain homotopy equivalence* if there is a chain map $g\colon D \to C$ such that the compositions $f \circ g$ and $g \circ f$ are chain homotopic to the respective identity maps in D and C. In particular, if $f\colon C \to D$ is a homotopy equivalence then it induces isomorphisms in homology $f\colon H_n(C) \simeq H_n(D)$.

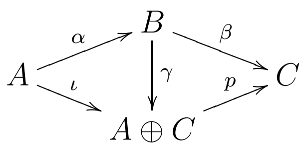
The following three lemmas are of key importance for this chapter. These are related to *short exact sequences*, which are exact chain complexes that vanish everywhere but in three consecutive non-zero entries.

Lemma 1.10 (Splitting Lemma)**.** *If* $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is a short exact sequence of R-modules, then the following three conditions are equivalent:*

1. *α has a left inverse α'.*

2. *β has a right inverse β'.*

3. *There exists an isomorphism $\gamma\colon B \to A \oplus C$ such that the diagram*



commutes, where $\iota\colon A \to A \oplus C$ is the inclusion into the first factor and $p\colon A \oplus C \to C$ is the projection onto the second factor.

A proof of this lemma can be found in many text books, for example [10, Chapter 5, Section 1, Lemma 11]. If the equivalent conditions stated in Lemma 1.10 are satisfied, we say that the exact sequence *splits*. Note that if C is a free R-module, then we can define β' on each free generator k of C so that $\beta'(k) \in \beta^{-1}(k)$. Hence the sequence splits if C is a free R-module.

Lemma 1.11 (Five Lemma)**.** *Consider the following diagram of R-modules and R-homomorphisms*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\phi} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array} \tag{1.4}$$

where the rows are exact and the squares commute up to sign. If all the vertical arrows but γ are isomorphisms, then γ is also an isomorphism.

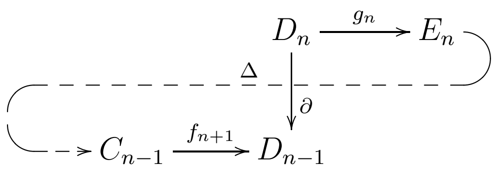
For a proof, see [4, Chapter I, Proposition 2.9].

Definition 1.12. A sequence of chain complexes is said to be *exact* if it is an exact sequence of R-modules in all dimensions.

Lemma 1.13 (Snake Lemma)**.** *Given a short exact sequence of chain complexes* $0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0$ *, the sequence*

$$\cdots \xrightarrow{f_*} H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\Delta} H_{n-1}(C) \xrightarrow{f_*} H_{n-1}(D) \xrightarrow{g_*} \cdots$$

is exact, where Δ is defined as shown in the diagram below:



(1.5)

For a proof see [4, Chapter II, Proposition 2.9].

Definition 1.14. For an R-module M, a *free resolution of* M is an exact sequence

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the R-modules F_n are free.

If the ring R is a principal ideal domain, then every R-module admits a free resolution where $F_n = 0$ for $n > 1$. To prove this we define F_0 as the free R-module on a set of generators of M, and $\mu\colon F_0 \to M$ sending each generator to itself. Since R is a principal ideal domain, $\mathrm{Ker}(\mu)$ is also a free R-module and we can put $F_1 = \mathrm{Ker}(\mu)$. This is the main reason why in most of the results in this study we need the base ring to be a principal ideal domain.

1.3 Tensor and Torsion Products

This section is a brief explanation of tensor and torsion products. A more extensive discussion can be found in [4, Chapter VI, Section 5].

Definition 1.15. Given two R-modules A and B, we define the *tensor product* $A \otimes_R B$ as the quotient of the free R-module generated by the elements of $A \times B$ by the relations

1. $(a_1 + a_2, b) \sim (a_1, b) + (a_2, b)$
2. $(a, b_1 + b_2) \sim (a, b_1) + (a, b_2)$
3. $r(a, b) \sim (ra, b) \sim (a, rb)$

for all elements a, a_1, a_2 in A, and b, b_1, b_2 in B, and $r \in R$. For convenience, we will write $\otimes$ instead of $\otimes_R$ when the underlying ring R cannot be confused. For $a \in A$ and $b \in B$, $a \otimes b$ denotes the class of (a, b) in $A \otimes B$.

The tensor product of R-modules has the following properties, which we will only state here. For a proof, see [4].

i) There is an isomorphism $\phi\colon A \otimes B \simeq B \otimes A$ such that $\phi(a \otimes b) = b \otimes a$.

ii) Given homomorphisms $f\colon A \to A'$ and $g\colon B \to B'$ there exists a unique homomorphism $f \otimes g\colon A \otimes B \to A' \otimes B'$ with $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$.

iii) Given any index set J, if $A \simeq \bigoplus_{j\in J} A_j$ then $A \otimes B \simeq \bigoplus_{j\in J}(A_j \otimes B)$.

iv) For any R-module A, there is an isomorphism $\mu\colon R \otimes A \simeq A$ with $\mu(r \otimes a) = ra$.

v) For any integers p and q we have an isomorphism $\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}_q \simeq \mathbb{Z}_{(p,q)}$, where (p, q) denotes the greatest common divisor of p and q.

Note that $\otimes_R\colon R\text{-mod} \times R\text{-mod} \to R\text{-mod}$ is a covariant functor in each variable. We are interested in knowing how $\otimes$ behaves with exact sequences. Let

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \tag{1.6}$$

be a short exact sequence of R-modules. For any R-module M, the sequence

$$A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact; see [12, Page 78]. In particular if the exact sequence (1.6) splits, then there is a map α' such that $\alpha' \circ \alpha = id\colon A \to A$. Hence, by property iii) of tensor products, the map $\alpha \otimes id$ is injective and the sequence

$$0 \longrightarrow A \otimes M \xrightarrow{\alpha\otimes id} B \otimes M \xrightarrow{\beta\otimes id} C \otimes M \longrightarrow 0$$

is exact. In order to generalize this result to non-splitting short exact sequences, we need to define the *torsion functor*. We will define it only for R a Principal Ideal Domain. Let M and D be two R-modules and let $0 \to F_1 \to F_0 \to M \to 0$ be a free resolution of M. We define $\operatorname{Tor}_R(M, D)$ as the R-module such that the sequence

$$0 \longrightarrow \operatorname{Tor}_R(M, D) \longrightarrow F_1 \otimes D \longrightarrow F_0 \otimes D \longrightarrow M \otimes D \longrightarrow 0$$

is exact. As shown in [6, Lemma 3A.2], the torsion product does not depend on the free resolution. We usually write Tor instead of Tor_R when there is no danger of confusion about the ground ring R. The torsion functor has the following properties; see [10, Chapter 5, Section 2].

1. $\operatorname{Tor}(A, B) \simeq \operatorname{Tor}(B, A)$
2. Given any index set J, if $A \simeq \bigoplus_{j\in J} A_j$ then $\operatorname{Tor}(A, B) \simeq \bigoplus_{j\in J} \operatorname{Tor}(A_j, B)$.
3. If A or B is a free R-module, then $\operatorname{Tor}(A, B) = 0$.

4. If $R = \mathbb{Z}$, and if $T(A)$ is the torsion subgroup of A, then we have an isomorphism $\mathrm{Tor}_{\mathbb{Z}}(A, B) \simeq \mathrm{Tor}_{\mathbb{Z}}(T(A), B)$.

5. $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}_n, A) \simeq \mathrm{Ker}(A \xrightarrow{n} A)$. This implies that $\mathrm{Tor}_{\mathbb{Z}}(\mathbb{Z}_p, \mathbb{Z}_q) = \mathbb{Z}_{(p,q)}$, where (p, q) is the greatest common divisor of the integers p and q.

Note that $\mathrm{Tor}_R \colon R\text{-mod} \times R\text{-mod} \to R\text{-mod}$ is a covariant functor in each variable.

Definition 1.16. Given chain complexes (C, ∂) and (D, ϵ), we define the *tensor chain complex* $C \otimes D$ in each dimension n as follows:

$$(C \otimes D)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

with boundary maps $\partial^{\otimes}$ defined as

$$\partial_n^{\otimes}(c \otimes d) = (\partial_p c \otimes d) + ((-1)^{|c|} c \otimes \epsilon_q d)$$

for all $c \otimes d \in C_p \otimes D_q$ and integers $p + q = n$. Also $|c|$ denotes the dimension of c, in this case $|c| = p$. These boundary maps satisfy the condition $\partial_n^{\otimes} \circ \partial_{n+1}^{\otimes} = 0$ for all n. In order to simplify notation we may write ∂ for both boundary maps ϵ and $\partial^{\otimes}$ if there is no confusion.

The following definition is very similar to the definition of tensor chain complex 1.16. Nevertheless, given its importance later on, we decided to state the definition again.

Definition 1.17. Given two chain complexes (C, ∂) and (D, ϵ) we define the *torsion chain complex* $\mathrm{Tor}(C, D)$ as the chain complex whose R-module in dimension n is

$$\mathrm{Tor}_n(C, D) = \bigoplus_{p+q=n} \mathrm{Tor}(C_p, D_q)$$

with boundary maps $\partial^T \colon \mathrm{Tor}(C_p, D_q) \to \mathrm{Tor}(C_{p-1}, D_q) \oplus \mathrm{Tor}(C_p, D_{q-1})$ determined by $\mathrm{Tor}(C_p, D_q) \xrightarrow{\partial_p} \mathrm{Tor}(C_{p-1}, D_q)$ and $\mathrm{Tor}(C_p, D_q) \xrightarrow{(-1)^p \epsilon_q} \mathrm{Tor}(C_p, D_{q-1})$ for all integers $p + q = n$, and they satisfy $\partial_n^T \circ \partial_{n+1}^T = 0$ for all n. This torsion chain complex is very important for the development of the cohomology Künneth formula; see [10, Chapter 5, Section 3].

1.4 Universal Coefficient Theorem

Let C be a chain complex and let M be an R-module. Consider the sequence:

$$\cdots \longrightarrow C_{n+1} \otimes M \xrightarrow{\partial_{n+1} \otimes id} C_n \otimes M \xrightarrow{\partial_n \otimes id} C_{n-1} \otimes M \longrightarrow \cdots$$

The homology groups of this chain complex are the *homology of* C *with coefficients in* M, usually written as $H_n(C; M)$. In this section we find an expression of $H_n(C; M)$ in terms of M and $H_n(C)$. Note that this is a clear step towards finding an expression for the homology modules of the tensor chain complex $C \otimes D$, for chain complexes C and D.

Theorem 1.18 (Universal Coefficient Theorem)**.** *Suppose that R is a principal ideal domain, (C, ∂) is a free chain complex, and M an arbitrary R-module. Then we have natural short exact sequences*

$$0 \longrightarrow H_n(C) \otimes M \longrightarrow H_n(C \otimes M) \longrightarrow \operatorname{Tor}(H_{n-1}(C), M) \longrightarrow 0$$

which split, but not in a natural way.

Proof. Let Z be the chain complex which contains $Z_n(C)$ in dimension n, and consider the chain map $j \colon Z \to C$ whose components are the inclusions $j_n \colon Z_n(C) \longrightarrow C_n$ for every $n \in \mathbb{Z}$. Composing j with the boundary map ∂, we obtain an exact sequence of chain complexes

$$0 \longrightarrow Z \xrightarrow{j} C \xrightarrow{\partial} B^- \longrightarrow 0$$

where B^- is the chain complex formed by $B_{n-1}(C)$ in dimension n. Since C_n is a free R-module and R is a principal ideal domain, B_n is also a free R-module. Hence the chain complex above splits in all dimensions, and we deduce the short exact sequences

$$0 \longrightarrow Z_n \otimes M \xrightarrow{j_n \otimes id} C_n \otimes M \xrightarrow{\partial_n \otimes id} B_{n-1} \otimes M \longrightarrow 0.$$

By considering their boundary maps, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n+1}\otimes M & \xrightarrow{j_{n+1}\otimes id} & C_{n+1}\otimes M & \xrightarrow{\partial_{n+1}\otimes id} & B_n\otimes M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_{n+1}\otimes id} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n}\otimes M & \xrightarrow{j_{n}\otimes id} & C_{n}\otimes M & \xrightarrow{\partial_{n}\otimes id} & B_{n-1}\otimes M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_{n}\otimes id} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1}\otimes M & \xrightarrow{j_{n-1}\otimes id} & C_{n-1}\otimes M & \xrightarrow{\partial_{n-1}\otimes id} & B_{n-2}\otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{1.7}$$

where the boundary maps of the chain complexes $Z_n \otimes M$ and $B_n \otimes M$ vanish for all n, since the respective boundary maps for the chain complexes Z and B^- are zero. By the Snake Lemma, we have connecting morphisms between subsequent rows $\Delta_n \colon B_n \otimes M \to Z_n \otimes M$, induced by the diagrams

$$\begin{array}{ccccc}
 & & C_{n+1}\otimes M & \xrightarrow{\partial_{n+1}\otimes id} & B_n\otimes M \\
 & & \downarrow & & \\
Z_n\otimes M & \xrightarrow{j_n\otimes id} & C_n\otimes M & &
\end{array} \qquad \Delta\colon B_n\otimes M \to Z_n\otimes M \tag{1.8}$$

where, by following the arrows, we see that Δ_n are the maps in homology induced by embeddings $\iota_n \otimes id \colon B_n \otimes M \longrightarrow Z_n \otimes M$, with the inclusion maps $\iota_n \colon B_n \longrightarrow Z_n$. Additionally by the Snake Lemma we get an exact sequence

$$\cdots \longrightarrow H_n(B\otimes M) \xrightarrow{\iota_n\otimes id} H_n(Z\otimes M) \xrightarrow{j_n\otimes id} H_n(C\otimes M) \xrightarrow{\partial_n\otimes id} H_{n-1}(B\otimes M) \longrightarrow \cdots$$

Since the the chain complexes $Z \otimes M$ and $B \otimes M$ have null boundary maps, we deduce that the sequence

$$\cdots \longrightarrow B_n\otimes M \xrightarrow{\iota_n\otimes id} Z_n\otimes M \xrightarrow{j_n\otimes id} H_n(C\otimes M) \xrightarrow{\partial_n\otimes id} B_{n-1}\otimes M \longrightarrow \cdots \tag{1.9}$$

is exact. We claim that this sequence is natural. Suppose that we have another chain complex (K, ϵ) and a chain map $f \colon C \to K$. We want to see that the diagram

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & B_n(C)\otimes M & \xrightarrow{\iota_n\otimes id} & Z_n(C)\otimes M & \xrightarrow{j_n\otimes id} & H_n(C\otimes M) & \xrightarrow{\partial_n\otimes id} & B_{n-1}(C)\otimes M & \longrightarrow \cdots \\
 & \downarrow{\scriptstyle f\otimes id} & \mathbf{(1)} & \downarrow{\scriptstyle f\otimes id} & \mathbf{(2)} & \downarrow{\scriptstyle (f\otimes id)_*} & \mathbf{(3)} & \downarrow{\scriptstyle f\otimes id} & \\
\cdots \longrightarrow & B_n(K)\otimes M & \xrightarrow{\iota'_n\otimes id} & Z_n(K)\otimes M & \xrightarrow{j'_n\otimes id} & H_n(K\otimes M) & \xrightarrow{\epsilon_n\otimes id} & B_{n-1}(K)\otimes M & \longrightarrow \cdots
\end{array}$$

commutes, where $\iota'_n \colon B_n(K) \to Z_n(K)$ and $j'_n \colon Z_n(K) \to K_n$ are the inclusions. For this purpose we only need to show that the squares (1), (2) and (3) commute for all integers n. This follows directly from the fact that f is a chain map. For the commutativity of (1), note that given an n-boundary b_n, its image $f_n(b_n)$ is also an n-boundary, hence $f_n\iota_n(b_n) = f_n(b_n) = \iota'_n f_n(b_n)$. Therefore we deduce that $(f_n \otimes id) \circ (\iota_n \otimes id) = (\iota'_n \otimes id) \circ (f_n \otimes id)$. Similarly, for an arbitrary n-cycle z_n, its image $f_n(z_n)$ is also an n-cycle and $f_n j_n(z_n) = f_n(z_n) = j'_n f_n(z_n)$. Hence $(f_n \otimes id)_* \circ (j_n \otimes id) = (j'_n \otimes id) \circ (f_n \otimes id)$, and the square (2) commutes. Finally (3) commutes since $(f_n \otimes id) \circ (\partial_n \otimes id) = (\epsilon_n \otimes id) \circ (f_n \otimes id)$. From this discussion we deduce the natural short exact sequences:

$$0 \longrightarrow \mathrm{Coker}(\iota_n \otimes id) \xrightarrow{j_n \otimes id} H_n(C \otimes M) \xrightarrow{\partial_n \otimes id} \mathrm{Ker}(i_{n-1} \otimes id) \longrightarrow 0. \tag{1.10}$$

Therefore we need to know what the terms $\mathrm{Coker}(\iota_n \otimes id)$ and $\mathrm{Ker}(\iota_{n-1} \otimes id)$ are. Consider the projections $\pi_n \colon C_n \to H_n(C)$, and the short exact sequence

$$0 \longrightarrow B_n \xrightarrow{\iota_n} C_n \xrightarrow{\pi_n} H_n(C) \longrightarrow 0.$$

Note that these sequences are free resolutions of the homology modules $H_n(C)$. Hence, by definition of Tor, the sequences

$$0 \longrightarrow \mathrm{Tor}(H_n(C), M) \longrightarrow B_n \otimes M \xrightarrow{\iota_n \otimes id} C_n \otimes M \xrightarrow{\pi_n \otimes id} H_n(C) \otimes M \longrightarrow 0$$

are exact, and give us an expression for the terms $\mathrm{Coker}(\iota_n \otimes id)$ and $\mathrm{Ker}(\iota_{n-1} \otimes id)$. Using this in (1.10) produces natural short exact sequences:

$$0 \longrightarrow H_n(C) \otimes M \xrightarrow{(j_n)_* \otimes id} H_n(C \otimes M) \xrightarrow{(\partial_n \otimes id)_*} \mathrm{Tor}(H_{n-1}(C), M) \longrightarrow 0. \tag{1.11}$$

Finally, we proceed to prove that this sequence splits. As in the beginning of the proof, we consider the exact sequences

$$0 \longrightarrow Z_n \xrightarrow{j_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0,$$

which split since B_{n-1} is a free R-module. By the Splitting Lemma, there must exist R-homomorphisms $l_n \colon C_n \longrightarrow Z_n$ such that $l_n \circ j_n = id \colon Z_n \to Z_n$ for all $n \in \mathbb{Z}$. Hence the compositions $(\pi_n \circ l_n)$ induce maps in homology

$$((\pi_n \circ l_n) \otimes id)_* \colon H_n(C \otimes M) \longrightarrow H_n(C) \otimes M$$

which are such that $((\pi_n \circ l_n) \otimes id)_* \circ ((j_n)_* \otimes id) = (id \otimes id)_*$. As a consequence, the sequences (1.11) split, and we obtain the isomorphisms

$$H_n(C \otimes M) \simeq H_n(C) \otimes M \oplus \mathrm{Tor}(H_{n-1}(C), M).$$

In Chapter 2 we will see in Example 2.12 that the splitting is unnatural. □

Example 1.19. Suppose that $R = \mathbb{Z}$, and let p, $q \in \mathbb{Z}$. Consider the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow 0$$

where the only non-zero homology module is $H_0(C) = \mathbb{Z}_p$. We want to calculate the homology modules of the chain complex

$$0 \longrightarrow \mathbb{Z} \otimes \mathbb{Z}_q \xrightarrow{p \otimes id} \mathbb{Z} \otimes \mathbb{Z}_q \longrightarrow 0 .$$

Using the Universal Coefficient Theorem, we find the non-zero homology groups

$$H_0(C \otimes \mathbb{Z}_q) \simeq \mathbb{Z}_p \otimes \mathbb{Z}_q = \mathbb{Z}_{(p,q)}, \text{ and } H_1(C \otimes \mathbb{Z}_q) \simeq \mathrm{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) = \mathbb{Z}_{(p,q)}.$$

Example 1.20. Does the Universal Coefficient Formula hold for chain complexes over a ring which is not a principal ideal domain? In general this is false if the base ring is not hereditary. For instance, consider the ring $R = \mathbb{Z}[X] \,/\, (X^2 - 1)$, which is not even an integral domain, since the product $(X + 1)(X - 1)$ of non-zero elements is zero. Hence we can construct a free chain complex C defined as

$$\cdots \longrightarrow R \xrightarrow{1+X} R \xrightarrow{1-X} R \xrightarrow{1+X} R \xrightarrow{0} 0 .$$

Note that it is acyclic since the greatest common divisor of the polynomials $1 + X$ and $1 - X$ is 1. On the other hand we consider the R-module $M = \mathbb{Z}$, with trivial action of R, that is, $X \cdot n = n$ for all $n \in \mathbb{Z}$. By the property iv) of tensor products, there is an isomorphism $R \otimes_R M \simeq M$. Hence, we obtain the following tensor chain complex $C \otimes_R M$:

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} 0,$$

where the second homology group is $H_2(C \otimes_R M) \simeq \mathbb{Z}_2$. If the Universal Coefficient Theorem was true for the ring R, there would be a short exact sequence

$$0 \longrightarrow H_2(C) \otimes_R M \longrightarrow H_2(C \otimes_R M) \longrightarrow \mathrm{Tor}_R(H_1(C), M) \longrightarrow 0 .$$

Nonetheless this is false since the first and last terms vanish, as C is acyclic.

1.5 Algebraic Künneth Formula

Let C and D be two free chain complexes. Consider the inclusion maps $j_n \colon Z_n(D) \to D_n$ for all n, and the short exact sequences

$$0 \longrightarrow Z_n(D) \xrightarrow{j_n} D_n \xrightarrow{\partial_n} B_{n-1}(D) \longrightarrow 0 . \tag{1.12}$$

Since R is a principal ideal domain, we deduce that $B_{n-1}(D)$ is a free R-module and the chain complex (1.12) splits. Hence, by the Splitting Lemma, there are maps $l_n\colon D_n \to Z_n(D)$ such that $l_n \circ j_n = id$. On the other hand, consider the projections $\pi_n\colon Z(D_n) \to H_n(D)$, and let Φ be the chain map such that $\Phi_n = \pi_n \circ l_n$.

Lemma 1.21. *Let C and D be two free chain complexes. There is a chain map $\Phi\colon C \to H(D)$ that induces isomorphisms $(id \otimes \Phi)_*\colon H_n(C \otimes D) \simeq H_n(C \otimes H(D))$.*

Proof. Since (1.12) is exact for all n, the sequence of chain complexes

$$0 \longrightarrow Z(D) \xrightarrow{j} D \xrightarrow{\partial} B^-(D) \longrightarrow 0$$

is exact. Here Z and B^- are the chain complexes whose n^{th} modules are $Z_n(D)$ and $B_{n-1}(D)$ respectively. Since it splits in all dimensions, if we tensor with the R-module C_n, exactness is preserved. Hence adding this for all integers n, we deduce the exactness of the sequence of chain complexes

$$0 \longrightarrow C \otimes Z(D) \xrightarrow{id\otimes j} C \otimes D \xrightarrow{id\otimes\partial} C \otimes B^-(D) \longrightarrow 0. \tag{1.13}$$

On the other hand, let $\iota\colon B(D) \to Z(D)$ be the chain map whose components are the inclusions $\iota_n\colon B(D)_n \to Z(D)_n$ for all integers n. Then the sequence

$$0 \longrightarrow B(D) \xrightarrow{\iota} Z(D) \xrightarrow{\pi} H(D) \longrightarrow 0$$

is exact. Since C is free, by property 3 of Tor the sequence

$$0 \longrightarrow C \otimes B(D) \xrightarrow{id\otimes\iota} C \otimes Z(D) \xrightarrow{id\otimes\pi} C \otimes H(D) \longrightarrow 0 \tag{1.14}$$

is also exact. Therefore if we apply the Snake Lemma to both sequences (1.13) and (1.14), we obtain two long exact sequences with respective connecting morphisms Δ and Δ'. This yields the following diagram:

$$\begin{array}{ccccc}
H_n(C \otimes B(D)) & \xrightarrow{\Delta} & H_n(C \otimes Z(D)) & \xrightarrow{(id\otimes j)_*} & H_n(C \otimes D) \\
\downarrow = & & \downarrow = & & \downarrow (id\otimes(\pi\circ l))_* \\
H_n(C \otimes B(D)) & \xrightarrow{(id\otimes\iota)_*} & H_n(C \otimes Z(D)) & \xrightarrow{(id\otimes\pi)_*} & H_n(C \otimes H(D))
\end{array} \tag{1.15}$$

$$\begin{array}{ccccc}
\xrightarrow{(id\otimes\partial)_*} & H_{n-1}(C \otimes B(D)) & \xrightarrow{\Delta} & H_{n-1}(C \otimes Z(D)) \\
& \downarrow = & & \downarrow = \\
\xrightarrow{\Delta'} & H_{n-1}(C \otimes B(D)) & \xrightarrow{(id\otimes\iota)_*} & H_{n-1}(C \otimes Z(D))
\end{array}$$

which we claim is commutative. By construction, from the Snake Lemma, the connecting morphism Δ equals $(id \otimes \iota)_*$ and therefore the upper left and lower right rectangles commute. Since $(\pi \circ l) \circ j = \pi$ the upper right rectangle commutes. In order to show that the lower left rectangle commutes, consider an n-cycle $\sum_{j \in J} c_j \otimes d_j$, for an index set J and $c_j \in C_{p_j}$, $d_j \in D_{q_j}$ for integers $p_j + q_j = n$. By hypothesis,

$$\partial(\sum_{j \in J} c_j \otimes d_j) = \sum_{j \in J} \partial_{p_j} c_j \otimes d_j + \sum_{j \in J} (-1)^{p_j} c_j \otimes \partial_{q_j} d_j = 0$$

and if we compose with $(id \otimes l)$ we get the following equality in $H_{n-1}(C \otimes D)$:

$$\{\sum_{j \in J} \partial_{p_j} c_j \otimes l_{q_j} d_j\} = \{-\sum_{j \in J} (-1)^{p_j} c_j \otimes l_{q_j - 1} \partial_{q_j} d_j\} = \{-\sum_{j \in J} (-1)^{p_j} c_j \otimes \partial_{q_j} d_j\}$$

which implies that $\Delta'(id \otimes (\pi \circ l))_* = \pm(id \otimes \partial)_*$, and the whole diagram (1.15) is commutative up to sign. Therefore by the Five Lemma we deduce that the induced $(id \otimes (\pi \circ l))_*$ is an isomorphism, and the lemma is true for $\Phi = \pi \circ l$. □

Note that the inverse of $(id \otimes \Phi)_*$ sends the class $\{z_p \otimes \{z_q\}\}$ to $\{z_p \otimes z_q\}$, for elements $z_p \in Z_p(C)$ and $z_q \in Z_q(D)$, and integers $p + q = n$. On the other hand, from the proof of the Universal Coefficient Theorem, the injective homomorphisms $(j_p)_* \otimes id\colon H_p(C) \otimes H_q(D) \to C_p \otimes H_q(D)$ have $((j_p)_* \otimes id)(\{z_p\} \otimes \{z_q\}) = \{z_p \otimes \{z_q\}\}$. Hence the composition of these induce a well-defined injective homomorphism

$$\begin{aligned} \alpha\colon H(C) \otimes H(D) &\longrightarrow H(C \otimes D) \\ \{z_p\} \otimes \{z_q\} &\longrightarrow \{z_p \otimes z_q\}. \end{aligned}$$

The algebraic Künneth theorem gives us an expression for the cokernel of α.

Theorem 1.22 (Algebraic Künneth Formula)**.** *Let R be a principal ideal domain, and let C and D be two free chain complexes. There are short exact sequences*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \longrightarrow H_n(C \otimes D) \longrightarrow \bigoplus_{r+s=n-1} \mathrm{Tor}(H_r(C), H_s(D)) \longrightarrow 0$$

which split, but not naturally.

Proof. We start by applying, for each $q \in \mathbb{Z}$, the Universal Coefficient Theorem 1.18 to the chain complex $C \otimes H_q(D)$. Hence we obtain sequences

$$0 \longrightarrow H_p(C) \otimes H_q(D) \longrightarrow H_n(C_p \otimes H_q(D)) \longrightarrow \mathrm{Tor}(H_{p-1}(C), H_q(D)) \longrightarrow 0$$

that are exact, natural and admit a splitting. Summing over for integers $p+q=n$, we get the sequences

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \longrightarrow \bigoplus_{p+q=n} H_n(C_p \otimes H_q(D)) \longrightarrow \bigoplus_{r+s=n-1} \operatorname{Tor}(H_r(C), H_s(D)) \longrightarrow 0 \quad (1.16)$$

which are exact, natural and split, since these are the direct sum of natural split exact sequences. Therefore by applying Lemma 1.21 we get the result. In particular we have an isomorphism

$$H_n(C \otimes D) \simeq \sum_{p+q=n} H_p(C) \otimes H_q(D) \bigoplus \sum_{r+s=n-1} \operatorname{Tor}(H_r(C), H_s(D))$$

which is unnatural, as will be shown in Example 2.13. □

1.6 Free Approximation

In this section we want to give a more general Künneth formula which works well with not necessarily free chain complexes. This formula will be used in Chapter 3, in order to construct a Künneth formula for cohomology.

Definition 1.23. Given a chain complex C over a ring R, a *free approximation of* C is a chain map $\tau : \bar{C} \to C$ such that

1. $\bar{C}$ is a free chain complex over R;
2. τ is an epimorphism;
3. τ induces an isomorphism $\tau_* \colon H(\bar{C}) \simeq H(C)$.

Lemma 1.24. *Let (C, ∂) be a chain complex of R-modules over a principal ideal domain R. Then there exists a free approximation of C.*

Proof. Let F_n be a free chain complex generated by the generators of C_n for all $n \in \mathbb{Z}$, and let $\pi_n \colon F_n \to C_n$ be the projection onto the generators. Consider a chain complex $\mathcal{F}$ defined as

$$\cdots \longrightarrow F_{n+2} \oplus F_{n+1} \xrightarrow{\tau_{n+1}} F_{n+1} \oplus F_n \xrightarrow{\tau_n} F_n \oplus F_{n-1} \longrightarrow \cdots \quad (1.17)$$

where $\tau_n(f_{n+1}, f_n) = (f_n, 0)$ for all $f_{n+1} \in F_{n+1}$ and $f_n \in F_n$. Hence (1.17) is exact by definition. On the other hand, consider a chain map $P \colon \mathcal{F} \to C$ defined as

$P_n = (\partial_n \pi_{n+1}, \pi_n)$. Then we have diagrams of exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_n & \longrightarrow & F_{n+1} \oplus F_n & \xrightarrow{P_n} & C_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \tau_n} & & \downarrow{\scriptstyle \partial_n} & & \\ 0 & \longrightarrow & K_{n-1} & \longrightarrow & F_n \oplus F_{n-1} & \xrightarrow{P_{n-1}} & C_{n-1} & \longrightarrow & 0 \end{array}$$

where $K_n = \mathrm{Ker}(P_n)$, and by the Snake Lemma we have isomorphisms $\Delta_n \colon H_n(C) \simeq H_{n-1}(K)$ for all $n \in \mathbb{Z}$. Since R is a principal ideal domain, we can define a free chain complex $\bar{C}$ as $\bar{C}_n = K_{n-1}$. Hence we have an isomorphism in homology $\Delta^{-1} \colon H_n(\bar{C}) \simeq H_n(C)$. Therefore, since $\bar{C}$ is free, there is a chain map $\phi \colon \bar{C} \to C$ which induces Δ^{-1} in homology. For details, see [4, Chapter II, Proposition 4.6]. □

Lemma 1.25. *Suppose that C and D are chain complexes over a principal ideal domain. If C is free and either C or D is exact, then $C \otimes D$ is exact.*

Proof. Consider a free chain approximation $\kappa \colon \bar{D} \to D$. Then we have the following exact sequence of chain complexes:

$$0 \longrightarrow \mathrm{Ker}(\kappa) \xrightarrow{\iota} \bar{D} \xrightarrow{\kappa} D \longrightarrow 0\,.$$

Note that $\mathrm{Ker}(\kappa)$ is exact since $H(\bar{D}) \simeq H(D)$, also it is free since R is a principal ideal domain. Hence by definition of torsion, we deduce the exact sequence

$$0 \longrightarrow \mathrm{Tor}(C, D) \longrightarrow C \otimes \mathrm{Ker}(\kappa) \xrightarrow{id \otimes \iota} C \otimes \bar{D} \xrightarrow{id \otimes \kappa} C \otimes D \longrightarrow 0$$

where the torsion term vanishes since C is free. Using the Künneth formula on the second and third terms we see that both are exact if either C or D is exact. Hence we can conclude that the chain complex $C \otimes D$ is exact. □

Proposition 1.26. *Given two free approximations $\tau \colon \bar{C} \to C$ and $\kappa \colon \bar{D} \to D$, if* $\mathrm{Tor}(C, D)$ *is exact, then the map $\tau \otimes \kappa \colon \bar{C} \otimes \bar{D} \to C \otimes D$ is a free approximation of $C \otimes D$.*

Proof. Since τ and κ are free approximations of C and D respectively, we deduce that $\tau \otimes \kappa$ satisfies conditions (1) and (2) from Definition 1.23. We only have to prove that $\tau \otimes \kappa$ induces an isomorphism in homology. First consider the following short exact sequence of chain complexes:

$$0 \longrightarrow \mathrm{Ker}(\tau) \xrightarrow{\iota} \bar{C} \xrightarrow{\tau} C \longrightarrow 0$$

where $\operatorname{Ker}(\tau)$ is exact since $H(\bar{C}) \simeq H(C)$. If we apply the tensor product with the chain complex D, we get the exact sequence

$$0 \longrightarrow \operatorname{Tor}(C, D) \longrightarrow \operatorname{Ker}(\tau) \otimes D \xrightarrow{\iota \otimes id} \bar{C} \otimes D \xrightarrow{\tau \otimes id} C \otimes D \longrightarrow 0$$

where the first term is exact by hypothesis, and $\operatorname{Ker}(\tau) \otimes D$ is exact by Lemma 1.25. Hence there is an isomorphism $(\tau \otimes id)_*\colon H(\bar{C} \otimes D) \simeq H(C \otimes D)$.

On the other hand consider the exact sequence of chain complexes

$$0 \longrightarrow \operatorname{Ker}(\kappa) \xrightarrow{\iota} \bar{D} \xrightarrow{\kappa} D \longrightarrow 0$$

where $\operatorname{Ker}(\kappa)$ is exact as $H(\bar{D}) \simeq H(D)$. If we tensor this with $\bar{C}$ we get

$$0 \longrightarrow \operatorname{Tor}(\bar{C}, D) \longrightarrow \bar{C} \otimes \operatorname{Ker}(\kappa) \xrightarrow{id \otimes \iota} \bar{C} \otimes \bar{D} \xrightarrow{id \otimes \kappa} \bar{C} \otimes D \longrightarrow 0$$

where $\operatorname{Tor}(\bar{C}, D)$ is exact since $\bar{C}$ is a free chain complex. Using again Lemma 1.25, we see that the term $\bar{C} \otimes \operatorname{Ker}(\kappa)$ is exact. Therefore there is an isomorphism $(id \otimes \kappa)_*\colon H(\bar{C} \otimes \bar{D}) \simeq H(\bar{C} \otimes D)$. As a consequence we deduce the induced isomorphism in homology $(\tau \otimes \kappa)_* = (\tau \otimes id)_*(id \otimes \kappa)_*$, and $\tau \otimes \kappa$ is a free approximation. □

If we apply this result from Proposition 1.26 to Theorem 1.22, we deduce a more general Künneth formula.

Theorem 1.27 (Algebraic Künneth Formula)**.** *Suppose that C and D are two chain complexes such that* $\operatorname{Tor}(C, D)$ *is exact and R is a principal ideal domain. Then the sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \longrightarrow H_n(C \otimes D) \longrightarrow \bigoplus_{r+s=n-1} \operatorname{Tor}(H_r(C), H_s(D)) \longrightarrow 0$$

is natural, is exact and splits, but the splitting is not natural.

By Lemma 1.24, there are two free approximations $\tau\colon \bar{C} \to C$ and $\kappa\colon \bar{D} \to D$. Furthermore, since by hypothesis $\operatorname{Tor}(C, D)$ is exact, we make use of Proposition 1.26 to obtain a free approximation $\tau \otimes \kappa\colon \bar{C} \otimes \bar{D} \to C \otimes D$. Finally, using the Künneth Formula from 1.22, we obtain the diagrams

$$\begin{array}{ccccccccc}
0 \longrightarrow & \bigoplus_{p+q=n} H_p(\bar{C}) \otimes H_q(\bar{D}) & \longrightarrow & H_n(\bar{C} \otimes \bar{D}) & \longrightarrow & \bigoplus_{r+s=n-1} \operatorname{Tor}(H_r(\bar{C}), H_s(\bar{D})) & \longrightarrow 0 \\
& \simeq \downarrow & & \simeq \downarrow (\tau\otimes\kappa)_* & & \simeq \downarrow & \\
0 \longrightarrow & \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) & \longrightarrow & H_n(C \otimes D) & \longrightarrow & \bigoplus_{r+s=n-1} \operatorname{Tor}(H_r(C), H_s(D)) & \longrightarrow 0
\end{array} \tag{1.18}$$

where the maps in the lower row are induced by the maps in the Künneth sequence for $\bar{C}$ and $\bar{D}$.

Example 1.28. It is important that the condition on the torsion holds. For instance consider two equal chain complexes C and D defined as:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}_2 \longrightarrow 0.$$

In Example 3.15 we see that these chain complexes correspond to the cohomology modules of the Klein Bottle and the wedge $\mathbb{R}\mathrm{P}^2 \vee S^1$. From the tensor $C \otimes D$ and its properties we get

$$\cdots \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow 0.$$

On the other hand, the Künneth Theorem 1.27 gives an isomorphism

$$H_1(C \otimes D) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

since we have the isomorphisms

$$H_0(C) \otimes H_1(D) \simeq \mathbb{Z}_2$$
$$H_1(C) \otimes H_0(D) \simeq \mathbb{Z}_2$$
$$\mathrm{Tor}(H_0(C), H_0(D)) \simeq \mathbb{Z}_2.$$

Hence the Künneth formula from 1.27 is not true in general if the term $\mathrm{Tor}(C, D)$ is not exact.

Chapter 2

Topological Künneth Formula

2.1 Singular Homology

Here we proceed to define singular homology, as it can be found in many standard books; see [6, Chapter 2, Section 1]. In order to study singular homology, we first define an *n-simplex*, which is the subspace of $\mathbb{R}^{n+1}$ defined by

$$\Delta_n := \{(t_0, t_1, ..., t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1 \text{ and } t_i \geq 0 \text{ for all } 0 \leq i \leq n\}.$$

Similarly we define the i^{th} face of the n^{th} simplex as

$$\Delta_n^i := \{(t_0, t_1, ..., t_n) \in \Delta_n \mid t_i = 0\}.$$

Let ChCpx^+ be the category of positive chain complexes of R-modules and chain maps for a commutative ring R with unit. We are going to define a functor which will allow us to apply all the machinery developed in the previous chapter to topological spaces.

Definition 2.1. We define the *singular functor* $S\colon \text{Top} \to \text{ChCpx}^+$ as a covariant functor such that, for any topological space X, the following properties are satisfied:

i) For any integer $n \geq 0$, the R-module $S_n(X)$ is the free R-module generated by continuous maps $\phi\colon \Delta_n \to X$, which are called *singular n-simplices.*

ii) The boundary maps $\partial_n\colon S_n(X) \to S_{n-1}(X)$ are defined as $\partial_n(\phi) = \sum_{i=0}^{n}(-1)^i\phi_i$, where ϕ_i is the i^{th} face $\phi_i = \phi|\Delta_n^i$. Note that $\partial_n \circ \partial_{n+1} = 0$ for all n.

iii) For $Y \in \text{Top}$, let $f\colon X \to Y$ be a continuous map. Then the image $S(f)$ is a chain map $S(f)\colon S(X) \to S(Y)$ defined as $S(f)(\phi) = f \circ \phi$, for any ϕ in $S(X)$.

For an arbitrary topological space X, we define the *singular homology modules of* X as the homology modules $H_n(S(X))$ of the chain complex $S(X)$. These are denoted as $H_n(X)$. Later in this chapter we will see that H_n is invariant under homotopy. The following is a very simple, but useful result which we will be using in the following sections.

Proposition 2.2. *The homology module $H_0(X)$ is a free R-module generated by the set of path components of X.*

Proof. Since the boundary of an arbitrary path equals either zero or the difference between two points, we deduce that $H_0(X)$ is a free R-module. If given two points p and q there is a path $\gamma\colon p \to q$, then $\partial\gamma = q - p$ and both p and q are in the same homology class. □

2.2 Acyclic Model Theorem

The aim of this chapter is to prove the Eilenberg-Zilber theorem, namely that the homology graded R-modules $H(X \times Y)$ and $H(S(X) \otimes S(Y))$ are isomorphic for spaces X and Y. Nevertheless we will approach the problem in a categorical way by proving the Acyclic Model Theorem first. This approach is very convenient since we can use this theorem to prove other useful results, such as the homotopy invariance of singular homology.

Definition 2.3. A *category with models* is a category in which a collection of objects $\mathfrak{M} = \{M_\alpha\}_{\alpha\in\Lambda}$ is distinguished; the distinguished objects are the *models*.

Definition 2.4. Let $\mathcal{C}$ be a category with models $\mathfrak{M}$ and let ChCpx^+ be the category of positive chain complexes and chain maps. Also suppose that we have a functor $F\colon \mathcal{C} \longrightarrow \mathrm{ChCpx}^+$, with some distinguished elements $e_\alpha^n \in F_n(M_\alpha)$, for any integer n and $\alpha \in \Lambda$. If F is such that the set $\{F(f)(e_\alpha^n) \mid \alpha \in \Lambda, f \in \mathrm{Hom}(M_\alpha, X)\}$ forms a basis for $F_n(X)$, then we say that F is *free with respect to the models* $\mathfrak{M}$.

Definition 2.5. A functor $G\colon \mathcal{C} \longrightarrow \mathrm{ChCpx}^+$ is *acyclic with respect to the models* $\mathfrak{M}$ if $H_n(G(M_\alpha)) = 0$ for $n > 0$, and for all α.

Theorem 2.6 (Acyclic Model Theorem)**.** *Let $\mathcal{C}$ be a category with models $\mathfrak{M} = \{M_\alpha\}_{\alpha\in\Lambda}$. Let F and G be covariant functors $F, G\colon \mathcal{C} \longrightarrow \mathrm{ChCpx}^+$ such that F is free with models $\mathfrak{M}$ and G is acyclic with respect to the models $\mathfrak{M}$.*

1. *Every natural transformation $\Phi\colon H_0(F) \longrightarrow H_0(G)$ is induced by some natural transformation $\phi\colon F \longrightarrow G$.*

2. *If there is another natural transformation $\psi\colon F \longrightarrow G$ which also induces Φ in homology, then ψ and ϕ are naturally chain homotopic.*

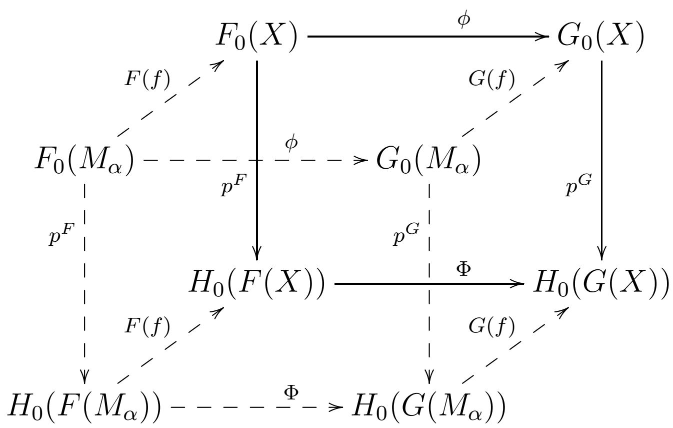
Proof. To prove this we define first the natural transformation ϕ inductively. Note that the elements of $F_0(M_\alpha)$ and $G_0(M_\alpha)$ must be cycles, since the range category is ChCpx$^+$. Hence for an arbitrary model $M_\alpha \in \mathfrak{M}$ we have the projections

$$p^F\colon F_0(M_\alpha) \longrightarrow H_0(F(M_\alpha)) \tag{2.1}$$

$$p^G\colon G_0(M_\alpha) \longrightarrow H_0(G(M_\alpha)). \tag{2.2}$$

Since F is free with respect to the models $\mathfrak{M}$, there are some prescribed elements $e^0_\alpha \in F_0(M_\alpha)$ for all $\alpha \in \Lambda$ such that $\{F(f)(e^0_\alpha) \mid \alpha \in \Lambda$, and $f \in \mathcal{C}(M_\alpha, X)\}$ forms a basis of the R-module $F_0(X)$. In order to ensure that ϕ induces Φ, we choose an element $\phi(e^0_\alpha) \in G_0(M_\alpha)$ such that the equality in homology $\{\phi(e^0_\alpha)\} = \Phi(\{e^0_\alpha\})$ holds, where $\{.\}$ denotes the homology class projected by p^F or p^G respectively.

Given an arbitrary $X \in \mathcal{C}$ and a map $f \in \mathcal{C}(M_\alpha, X)$, we can extend ϕ to a morphism $\phi\colon F_0(X) \to G_0(X)$ by defining it on the basis elements as $\phi(F(f)(e^0_\alpha)) = G(f)(\phi(e^0_\alpha))$. We proceed by checking that ϕ induces Φ on zero-dimensional homology. To prove it consider the diagram



(2.3)

where if we prove that the front face commutes, then the map ϕ certainly induces Φ. To prove this, we only need to show that the front face commutes on the basis elements $F(f)(e^0_\alpha)$ for $\alpha \in \Lambda$.

The bottom face commutes because Φ is a natural transformation. Also from our previous definition of ϕ, we have that the back and top faces commute. Both right and left faces commute since $F(f)$ and $G(f)$ are chain maps. Therefore by commutativity we deduce the equality $p^G \circ \phi \circ F(f) = \Phi \circ p^F \circ F(f)$, which leads us to the equality $\Phi(\{F(f)e^0_\alpha\}) = \{\phi(F(f)(e^0_\alpha))\}$ on the basis elements, and therefore ϕ induces Φ.

Since F is free with respect to the models $\mathfrak{M}$, there are some prescribed elements $e^1_\alpha \in F_1(M_\alpha)$ for all $\alpha \in \Lambda$. As we have already defined ϕ in dimension 0, the element $\phi(\partial e^1_\alpha) \in G(M_\alpha)$ is well defined. In fact it is a boundary since $\{\phi(\partial e^0_\alpha)\} = \Phi(\{\partial e^0_\alpha\}) = \{0\}$, and therefore we can define $\phi(e^1_\alpha)$ to be the element in $G_1(M_\alpha)$ such that $\partial\phi(e^1_\alpha) = \phi(\partial e^1_\alpha)$. As before we extend ϕ to a morphism $\phi\colon F_1(X) \to G_1(X)$, by defining it on the basis elements as $\phi(F(f)(e^1_\alpha)) = G(f)(\phi(e^1_\alpha))$.

Given an integer $n > 1$, suppose that the chain map $\phi\colon F(X) \to G(X)$ is defined for all dimensions $i < n$. Since F is free with respect to the models $\mathfrak{M}$, there are some prescribed elements $e^n_\alpha \in F_n(M_\alpha)$ for all $\alpha \in \Lambda$. Also as by hypothesis ϕ is defined in dimension $n-1$, the image $\phi(\partial e^n_\alpha) \in G(M_\alpha)$ is well defined, and $\partial\phi(\partial e^n_\alpha) = \phi(\partial\partial e^n_\alpha) = 0$ so ∂e^n_α is a cycle. In fact it is a boundary since by hypothesis G is acyclic in $\mathfrak{M}$, hence we can define $\phi(e^n_\alpha)$ to be the element in $G_n(M_\alpha)$ such that $\partial\phi(e^n_\alpha) = \phi(\partial e^n_\alpha)$. As before we extend ϕ to a morphism $\phi : F_n(X) \to G_n(X)$, by defining it on the basis elements as $\phi(F(f)(e^n_\alpha)) = G(f)(\phi(e^n_\alpha))$. Note that by definition ϕ is a natural transformation since $\phi F(f) = G(f)\phi$, for any $f \in \mathcal{C}(M_\alpha, X)$.

Now suppose that we have another natural transformation $\psi\colon F \longrightarrow G$ that induces Φ in the 0^{th} homology R-module. For each object X in $\mathcal{C}$ we must construct a chain homotopy $\mathcal{T}\colon F(X) \to G(X)$ such that

$$\partial \circ \mathcal{T} + \mathcal{T} \circ \partial = \phi - \psi \tag{2.4}$$

and $\mathcal{T}$ is natural with respect to the morphisms in $\mathcal{C}$. As before we define $\mathcal{T}$ inductively. For any $e^0_\alpha \in F_0(M_\alpha)$,

$$\{\phi(e^0_\alpha) - \psi(e^0_\alpha)\} = \Phi(\{e^0_\alpha\}) - \Phi(\{e^0_\alpha\}) = \{0\}$$

and therefore this difference is a boundary. Hence there exists some $\mathcal{T}(e^0_\alpha)$ in $G_1(M_\alpha)$ such that $\partial\mathcal{T}(e^0_\alpha) = \phi(e^0_\alpha) - \psi(e^0_\alpha)$. In particular, equation (2.4) holds. As we did with ϕ, we extend $\mathcal{T}$ to a morphism $\mathcal{T}\colon F(X) \to G(X)$ by the equality $\mathcal{T}(F(f)(e^0_\alpha)) = G(f)(\mathcal{T}e^0_\alpha)$. We proceed by supposing that for an integer $n > 0$, the homotopy $\mathcal{T}$ has been defined for all dimensions less than n. The element $\phi(e^n_\alpha) - \psi(e^n_\alpha) - \mathcal{T}(\partial e^n_\alpha)$ is a cycle by equation (2.4). Hence

$$\begin{aligned}\partial\phi(e^n_\alpha) - \partial\psi(e^n_\alpha) - \partial\mathcal{T}(\partial e^n_\alpha)& \\ = \phi(\partial e^n_\alpha) - \psi(\partial e^n_\alpha) - (-\mathcal{T}\partial\partial(e^n_\alpha) + \phi(\partial e^n_\alpha) - \psi(\partial e^n_\alpha)) = 0.&\end{aligned} \tag{2.5}$$

Since G is acyclic in the models $\mathfrak{M}$, we may conclude that there is some $\mathcal{T}(e^n_\alpha)$ in $G_n(M_\alpha)$ such that equation (2.4) holds. Also we extend the homotopy $\mathcal{T}$ to a map $\mathcal{T}\colon F(X) \to G(X)$ as we did in the zero case. In particular, this definition implies that $\mathcal{T}$ is natural with respect to the morphisms in $\mathcal{C}$. □

2.3 Eilenberg-Zilber Theorem

In Algebraic Topology there are some situations where it is more convenient to work with the *cubical functor* $S^{\square}\colon \mathrm{Top} \to \mathrm{ChCpx}^{+}$ which is similarly defined as in Definition 2.1. In this case, instead of singular n-simplices we have *singular n-cubes* which are continuous maps $\phi : I_n \to X$, and their boundary maps are much more complicated than in the singular case; see [7, Chapter 1]. Thus, given two topological spaces X and Y, for two given $\varphi_p \in S_p^{\square}(X)$ and $\psi_q \in S_q^{\square}(Y)$, we have a corresponding element in $S_{p+q}^{\square}(X \times Y)$ defined as $\varphi_p \times \psi_q \colon I_{p+q} \to X \times Y$. As a consequence of this fact, we can define chain maps between the complexes $S^{\square}(X) \otimes S^{\square}(Y)$ and $S^{\square}(X \times Y)$ in a straightforward way. Unfortunately this is not so simple with singular homology. Through all this section we solve this problem.

Lemma 2.7. *Let $X \subset \mathbb{R}^n$ for some integer $n \geq 0$. If X is a convex space then $S(X)$ is acyclic.*

The proof can be found in [12, Proposition 1.8]. We will use this fact for the n-simplices Δ_n.

Consider the product $\mathrm{Top} \times \mathrm{Top}$ whose objects are the ordered pairs (X_1, X_2) of topological spaces $X_1, X_2 \in \mathrm{Top}$, and whose morphisms are ordered pairs (f, g) with $f\colon X_1 \to Y_1$ and $f\colon X_2 \to Y_2$. We choose the models $\mathfrak{M} = (\Delta_p, \Delta_q)$, $p, q \geq 0$, and define two functors F, $G\colon \mathrm{Top} \times \mathrm{Top} \to \mathrm{ChCpx}^{+}$ such that $F(X, Y) = S(X \times Y)$ and $G(X, Y) = S(X) \otimes S(Y)$.

We define the *diagonal map* $d_n\colon \Delta_n \to \Delta_n \times \Delta_n$ as $d_n(x) = (x, x)$ for any element x in Δ_n and any integer $n \geq 0$. Also let p_1 and p_2 be the projections of $X \times Y$ onto X and Y respectively. Then for any singular n-simplex $\sigma\colon \Delta_n \to X \times Y$ we have that σ is equal to the composition

$$\Delta_n \xrightarrow{d_n} \Delta_n \times \Delta_n \xrightarrow{\sigma_1 \times \sigma_2} X \times Y$$

where $\sigma_1 = p_1 \sigma$ and $\sigma_2 = p_2 \sigma$. Note that we have some distinguished elements d_n in $S_n(\Delta_n \times \Delta_n)$, and also $F(\sigma) = (\sigma_1 \times \sigma_2)$. As a consequence, since $\{(\sigma_1 \times \sigma_2) d_n\}$ forms a basis for $S_n(X \times Y)$, the functor F is free with respect to the models $\mathfrak{M}$. Therefore, since both Δ_p and Δ_q are convex for all non-negative integers p, q, we have that $\Delta_p \times \Delta_q$ is also convex, and using Lemma 2.7 we have that the chain complex $S(\Delta_p \times \Delta_q)$ is acyclic.

On the other hand, the singular functor $S\colon \mathrm{Top} \to \mathrm{ChCpx}^{+}$ has distinguished elements $id_n \in S(\Delta_n)$. These are such that for any space X, the continuous functions $S_n(f)(id_n)\colon \Delta_n \to X$ generate $S_n(X)$. Therefore we deduce that the functor G has distinguished elements $id_p \otimes id_q \in S(\Delta_p) \otimes S(\Delta_q)$, and is also free with models $\mathfrak{M}$. Furthermore by using Lemma 2.7 we deduce that S is acyclic in $\mathfrak{M}$ which implies, by referring to the Künneth formula (1.22), that the functor G is also acyclic with respect to $\mathfrak{M}$. We proceed to prove the Eilenberg-Zilber theorem.

Theorem 2.8 (Eilenberg-Zilber)**.** *Let two functors F, $G\colon \mathrm{Top}\times\mathrm{Top}\to\mathrm{ChCpx}^+$ be such that $F(X,Y)=S(X\times Y)$ and $G(X,Y)=S(X)\otimes S(Y)$. There is a natural transformation $\phi\colon F\to G$ that induces an isomorphism in homology $\phi_*: H_k(X\times Y)\to H_k(S(X)\otimes S(Y))$, for all integers k and all topological spaces X and Y.*

Proof. By Proposition 2.2 both $H_0(X)$ and $H_0(Y)$ are isomorphic to a sum of copies of R, where each copy corresponds to a connected component in X or Y respectively. In particular both $H_0(X)$ and $H_0(Y)$ are torsion-free, and using the Künneth formula from 1.22, we deduce that $H_0(S(X)\otimes S(Y))\simeq H_0(X)\otimes H_0(Y)$. Therefore $H_0(S(X)\otimes S(Y))$ is isomorphic to a free R-module whose generators correspond to the elements $C\otimes D$ for C and D connected components of X and Y respectively. Since the connected components of $X\times Y$ are of the form $C\times D$, we have as a result a natural isomorphism

$$\Phi\colon H_0(X\times Y)\to H_0(S(X)\otimes S(Y))$$

which is natural since continuous maps preserve connected components. Therefore if we apply the Acyclic Model Theorem to the functors F and G and to the natural transformations Φ and Φ^{-1}, we obtain two natural transformations

$$\phi\colon S(X\times Y)\to S(X)\otimes S(Y)$$

and

$$\bar{\phi}\colon S(X)\otimes S(Y)\to S(X\times Y)$$

that induce Φ and Φ^{-1} in dimension zero. Hence we have that the compositions $\phi\circ\bar{\phi}$ and $\bar{\phi}\circ\phi$ induce identity morphisms in the 0^{th} homology modules. Since the respective identity chain maps also have this property, we can deduce by using again the Acyclic Model Theorem that $\phi\circ\bar{\phi}$ and $\bar{\phi}\circ\phi$ are chain homotopic to the identity morphisms in $S(X)\otimes S(Y)$ and $S(X\times Y)$ respectively. ☐

The natural transformations ϕ and $\bar{\phi}$ are usually called Eilenberg-Zilber (EZ) maps. Define the maps $t\colon X\times Y\to Y\times X$ by $t(x,y)=(y,x)$, and also $\tau\colon X\otimes Y\to Y\otimes X$ defined as $\tau(x\otimes y)=(-1)^{|x||y|}y\otimes x$. In the next corollary we prove that ϕ is commutative in the sense that the diagram

$$\begin{array}{ccc} S(X\times Y) & \xrightarrow{\phi} & S(X)\otimes S(Y) \\ \downarrow{\scriptstyle S(t)} & & \downarrow{\scriptstyle \tau} \\ S(Y\times X) & \xrightarrow{\phi} & S(Y)\otimes S(X) \end{array} \tag{2.6}$$

commutes in homology. That is, if we denote $\phi = \phi_1 \otimes \phi_2$ and by $\{\}$ the homology classes, we have the equality

$$\{\phi_1(y,x) \otimes \phi_2(y,x)\} = (-1)^{|x||y|}\{\phi_2(x,y) \otimes \phi_1(x,y)\}$$

for any class $\{x \times y\}$ in $H^n(X \times Y)$.

Corollary. *The Eilenberg-Zilber maps are commutative up to chain homotopy.*

Proof. We want to prove that the diagram (2.6) is commutative up to chain homotopy. First we prove that τ is a chain map.

$$\begin{aligned} \partial\tau(x \otimes y) &= \partial((-1)^{|x||y|}(y \otimes x)) \\ &= (-1)^{|x||y|}\partial y \otimes x + (-1)^{|x||y|+|y|} y \otimes \partial x \end{aligned}$$

$$\begin{aligned} \tau\partial(x \otimes y) &= \tau(\partial x \otimes y + (-1)^{|x|} x \otimes \partial y) \\ &= (-1)^{(|x|-1)|y|} y \otimes \partial x + (-1)^{(|y|-1)|x|+|x|}\partial y \otimes x \end{aligned}$$

Then we have two natural transformations $\phi \circ S(t)$ and $\tau \circ \phi$. These two induce the same natural transformation in zero homology, which sends connected components $C \times D$ in $X \times Y$ to tensor products of components $C \otimes D$, with C and D connected components from X and Y respectively. Therefore using the Acyclic Model Theorem we deduce that both $\phi \circ S(t)$ and $\tau \circ \phi$ are naturally chain homotopic. □

Corollary. *The Eilenberg-Zilber maps are associative up to chain homotopy.*

Proof. Consider the diagram

$$\begin{array}{ccc} S(X \times Y \times Z) & \xrightarrow{EZ} & S(X \times Y) \otimes S(Z) \\ \downarrow{\scriptstyle EZ} & & \downarrow{\scriptstyle EZ \otimes id} \\ S(X) \otimes S(Y \times Z) & \xrightarrow{id \otimes EZ} & S(X) \otimes S(Y) \otimes S(Z) \end{array} \tag{2.7}$$

Note that connected components are sent to connected components. Hence by Proposition 2.2 the induced natural maps in zero homology are isomorphisms. By a similar argument to the commutativity of EZ maps we deduce the result. □

Theorem 2.9 (Topological Künneth Formula)**.** *Let X and Y be two topological spaces. Then there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{r+s=n-1} \mathrm{Tor}(H_r(X), H_s(Y)) \longrightarrow 0$$

which splits, but not naturally.

Proof. By the Eilenberg-Zilber theorem, we have an EZ natural transformation

$$\phi\colon S(X \times Y) \to S(X) \otimes S(Y)$$

which induces a natural isomorphism in homology. Hence since by definition $S(X)$ and $S(Y)$ are free chain complexes, we use the Algebraic Künneth Formula 1.22 and get the result. □

Theorem 2.10. *If $f, g\colon X \to Y$ are homotopic continuous functions, then $f_* = g_*$.*

Proof. Since f and g are homotopic, there must exist a homotopy $F\colon X \times I \to Y$ such that $F(-,0) = f$ and $F(-,1) = g$. Define two maps $p, q\colon X \to X \times I$ as $p(x) = (x, 0)$ and $q(x) = (x, 1)$ for all $x \in X$. Hence we have the equalities $f = F \circ p$ and $g = F \circ q$. Suppose that the chain maps $S(p)$ and $S(q)$ are chain homotopic. Then there is a chain homotopy $T\colon S(p) \Rightarrow S(q)$ such that

$$\partial \circ T + T \circ \partial = S(p) - S(q).$$

Hence considering the chain map $S(F)$, we have that

$$\partial S(F)T + S(F)T\partial = S(F)S(p) - S(F)S(q) = S(f) - S(g)$$

finding therefore a chain homotopy between $S(f)$ and $S(g)$. Hence we only need to show that p and q are chain homotopic. Consider the n-simplices Δ_n as the models $\mathfrak{M}$ for the category Top. By Definition 2.1, the functor $S\colon \text{Top} \to \text{ChCpx}$ is free on the models $\mathfrak{M}$. On the other hand define a functor $F\colon \text{Top} \to \text{ChCpx}$ as $F(X) = S(X \times I)$ for any topological space X. Since the space $\Delta_n \times I$ is convex, by Lemma 2.7 it is acyclic in the models $\mathfrak{M}$. Also note that both $S(p)$ and $S(q)$ induce the same map $\bar{G}$ in the 0^{th} homology module:

$$\begin{aligned} \bar{G} : H_0(\Delta^n) &\longrightarrow H_0(\Delta^n \times I) \\ \{x\} &\longmapsto \{(x,0)\} = \{(x,1)\} \end{aligned}$$

Hence we can apply the Acyclic Model Theorem and conclude that both $S(p)$ and $S(q)$ are chain homotopic. □

Corollary. *Singular homology is a homotopical invariant.*

Proof. Let X and Y be two topological spaces and let $f\colon X \to Y$ be a homotopy equivalence. Then there exists a continuous map $f\colon Y \to X$ such that $g \circ f$ and $f \circ g$ are homotopic to the respective identities in X and Y. Thus by Theorem 2.10 we have the equalities $g_* \circ f_* = id_{X*}$ and $f_* \circ g_* = id_{Y*}$. Hence X and Y have isomorphic homology modules. □

Although singular homology is very useful in order to prove general results, it turns out that it is not very practical when we want to use it in specific examples. A very useful tool is *cellular homology*, which is explained in many introductory books of Algebraic Topology. One reference is [6, Section 2.2]. I shall be using this in the following two examples.

Definition 2.11. Let D be an R-module. We define a *Moore space*, denoted $M(D,n)$, to be a space whose homology R-modules are D in dimension n and equal to 0 in all other non-zero dimensions. Its 0^{th} homology R-module is R.

We proceed to prove that the sequence in the Universal Coefficient Theorem 1.18 splits unnaturally.

Example 2.12. Consider the Moore space $M(\mathbb{Z}_m, n)$ for two given natural numbers n, $m > 0$. This space has a CW-structure having three cells e_0, e_n and e_{n+1} in dimensions 0, n and $n+1$ respectively, where the cell e_{n+1} is attached to e_n by a map of degree m. Now consider the sphere S^{n+1} formed by two cells d_0 and d_{n+1} in dimensions 0 and $n+1$. Then we define a map

$$f\colon M(\mathbb{Z}_m, n) \to S^{n+1}$$

such that $f(e_{n+1}) = d_{n+1}$, $f(e_n) = d_0$ and $f(e_0) = d_0$. Hence this map preserves the cell e_{n+1} and collapses e_n. We proceed by applying the Universal Coefficient Theorem (1.18) to the cellular spaces $M(\mathbb{Z}_m, n)$ and S^{n+1} with coefficients in $\mathbb{Z}_m$. Hence in dimension $n+1$ we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_{n+1}M(\mathbb{Z}_m,n)\otimes\mathbb{Z}_m & \longrightarrow & H_{n+1}(M(\mathbb{Z}_m,n);\mathbb{Z}_m) & \longrightarrow & \mathrm{Tor}(H_nM(\mathbb{Z}_m,n),\mathbb{Z}_m) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H_{n+1}(S^{n+1})\otimes\mathbb{Z}_m & \longrightarrow & H_{n+1}(S^{n+1};\mathbb{Z}_m) & \longrightarrow & \mathrm{Tor}(H_n(S^{n+1}),\mathbb{Z}_m) & \longrightarrow & 0
\end{array}$$

where the vertical maps are induced by f. Therefore, we consider the diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & 0\oplus\mathbb{Z}_m & \xleftarrow[j]{\sim} & \mathbb{Z}_m \\
\downarrow & & \sim\downarrow f_* & & \downarrow \\
\mathbb{Z}_m & \xrightarrow[\sim]{i} & \mathbb{Z}_m\oplus 0 & \longleftarrow & 0
\end{array} \qquad (2.8)$$

where the right-hand horizontal arrows are the splitting maps from the Universal Coefficient Formula, and the upper and lower rows correspond to $M(\mathbb{Z}_m, n)$ and S^{n+1} respectively. The map f_* is the $(n+1)$th induced homology map, and it is an isomorphism since it sends the $(n+1)$-cell from the Moore space to the $(n+1)$-cell of the sphere. Hence since $f_* \circ j$ is an isomorphism, the right square in (2.8) is not commutative. As a consequence we deduce that the splitting in the Universal Coefficient Theorem is unnatural.

In the following example we see that the splitting in the Künneth formula from Theorem 1.22 is not natural either.

Example 2.13. The procedure is very similar to the previous example. Consider the map $f\colon M(\mathbb{Z}_m, n) \to S^{n+1}$ from Example 2.12. Then define

$$f \times id\colon M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n) \to S^{n+1} \times M(\mathbb{Z}_m, n)$$

such that $(f \times id)(a, b) = (f(a), b)$, for all $(a, b) \in M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n)$.

If we calculate the terms of the Künneth Formula in dimension $2n + 1$, we get the diagram of horizontal splitting maps

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 \oplus \mathbb{Z}_m & \underset{j}{\overset{\sim}{\longleftarrow}} & \mathbb{Z}_m \\ \downarrow & & \sim \downarrow (f\times id)_* & & \downarrow \\ \mathbb{Z}_m & \underset{\sim}{\overset{i}{\longrightarrow}} & \mathbb{Z}_m \oplus 0 & \longleftarrow & 0 \end{array} \tag{2.9}$$

where the upper row corresponds to the space $M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n)$, whereas the lower row corresponds to $S^{n+1} \times M(\mathbb{Z}_m, n)$. The vertical map $(f \times id)_*$ is the $(2n+1)$th induced homology map by $f \times id$. On the other hand, since $f \times id$ sends the cell $e_{n+1} \times e_n$ to the cell $d_{n+1} \times e_n$, we have that $(f \times id)_*$ is an isomorphism. This is because $e_{n+1} \times e_n$ and $d_{n+1} \times e_n$ generate the respective homology modules $H_{2n+1}(M(\mathbb{Z}_m, n) \times M(\mathbb{Z}_m, n); \mathbb{Z}_m)$ and $H_{2n+1}(S^{n+1} \times M(\mathbb{Z}_m, n); \mathbb{Z}_m)$. This can be seen by using the following boundary formula for products of cells

$$\partial(a^j \times b^i) = (\partial_j a^j) \times b^i + (-1)^j a^j \times \partial_i b^i$$

for cells a^j and b^i of two cellular spaces X and Y; see [6, Proposition 3B.1]. Thus the right hand rectangle in diagram (2.9) is not commutative.

Chapter 3

Cohomology of Product Spaces

3.1 Künneth Formula for Cohomology Modules

Suppose that R is a commutative ring with unit and let A and B be two R-modules. We denote by $\mathrm{Hom}_R(A, B)$ the abelian group of R-homomorphisms from A to B. Given two R-homomorphisms $\alpha\colon A' \to A$ and $\beta\colon B \to B'$, we have a map

$$\mathrm{Hom}_R(\alpha, \beta)\colon \quad \mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Hom}_R(A', B')$$

such that $\mathrm{Hom}_R(\alpha, \beta)(f) = \beta \circ f \circ \alpha$. Hence $\mathrm{Hom}_R\colon R\text{-mod} \times R\text{-mod} \to R\text{-mod}$ is a functor of two variables, contravariant in the first variable and covariant in the second variable. If there is no confusion about the base ring R we will just write Hom instead of Hom_R. We proceed by noting some of the properties of Hom; for a proof, see [4, Chapter VI, Section 6].

(i) $\mathrm{Hom}(R, L) \simeq L$, for all L.

(ii) For every index set J, $\mathrm{Hom}(\bigoplus_{j\in J} A_j, B) \simeq \prod_{j\in J} \mathrm{Hom}(A_j, B)$.

(iii) For every index set J, $\mathrm{Hom}(A, \prod_{j\in J} B_j) \simeq \prod_{j\in J} \mathrm{Hom}(A, B_j)$. Note that if J is finite, then $\bigoplus_{j\in J} B_j \simeq \prod_{j\in J} B_j$.

(iv) If $A \to B \to C \to 0$ is a short exact sequence, then the sequence $0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M)$ is exact for any R-module M.

(v) If $0 \to A \to B \to C \to 0$ is a short exact sequence, then the sequence $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C)$ is exact for any R-module M. Furthermore, if M is a free R-module, then $\mathrm{Hom}(M, -)$ is an exact functor. That is, the sequence $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact.

Definition 3.1. Let (C, ∂) be a free chain complex, and let M be an arbitrary R-module. We define the *Hom cochain complex* as the sequence

$$\cdots \longrightarrow \mathrm{Hom}(C_{n-1}, M) \xrightarrow{\delta_{n-1}} \mathrm{Hom}(C_n, M) \xrightarrow{\delta_n} \mathrm{Hom}(C_{n+1}, M) \longrightarrow \cdots$$

where $\delta_n(f) = f \circ \partial_{n+1}$ for any map $f \colon C_n \to M$. We can view this cochain complex as a chain complex by defining (D, ϵ) such that $D_n = \mathrm{Hom}(C_{-n}, M)$ and $\epsilon_n = \delta_{-n}$ for all $n \in \mathbb{Z}$. Then we define the *cohomology* modules of (C, ∂) as the homology modules of (D, ϵ). Usually the n^{th} cohomology module is written as $H^n(C; M)$.

Similarly as for tensor products, if R is a principal ideal domain, then there is a functor $\mathrm{Ext} \colon R\text{-mod} \times R\text{-mod} \to R\text{-mod}$ such that, using property (iv) of Hom, for any exact sequence of R-modules and R-homomorphisms $0 \to A \to B \to C \to 0$ we have the following sequence

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}(C, M) \to 0$$

which is exact. There is also a Universal Coefficient Theorem for cohomology, which is stated below. A proof can be found in [6, Section 3.1.].

Theorem 3.2. *Let C be a free chain complex with homology groups $H_n(C)$, and let M be an arbitrary R-module, where R is a principal ideal domain. Then the cohomology modules of the cochain complex* $\mathrm{Hom}(C, M)$ *are determined by split exact sequences*

$$0 \to \mathrm{Ext}(H_{n-1}(C), M) \to H^n(C; M) \to \mathrm{Hom}(H_n(C), M) \to 0.$$

This result is very useful, since in general it is easier to compute homology modules than cohomology modules. I shall be using it in some examples. Although it seems a quite important result to be included in this text, I decided to omit it because it is not necessary in order to prove the Künneth theorem that I want to prove here.

Definition 3.3. Let (C, ∂) and (D, ϵ) be two chain complexes. Suppose given two R-modules M and L. We define the *tensor cochain complex* as the cochain complex whose n^{th} component is equal to $\mathrm{Hom}((C \otimes D)_n; M \otimes L)$. The boundary morphisms are defined as $\delta(f \otimes g) = (\delta f) \otimes g + (-1)^{|f|} f \otimes (\delta g)$ for all $f \otimes g \colon (C \otimes D)_n \to M \otimes L$. This holds because

$$\delta(f \otimes g)(x \otimes y) = (f \otimes g)\partial(x \otimes y) = (f \otimes g)(\partial x \otimes y + (-1)^p x \otimes \partial y)$$

for all $x \otimes y$ in $C_p \otimes D_q$.

In this section we want to find an expression of the cohomology modules $H^n(C \otimes D; M \otimes L)$ in terms of $H^n(C; M)$ and $H^n(D; L)$. The following lemma gives a relation in the minimal case, where both C and D have a unique non-zero R-module.

Lemma 3.4. *If the ring R is a principal ideal domain, and the R-modules A and B are free, we define a map*

$$\gamma\colon \mathrm{Hom}(A,M)\otimes\mathrm{Hom}(B,L)\to\mathrm{Hom}(A\otimes B,M\otimes L) \tag{3.1}$$

as $\gamma(f\otimes g)(x\otimes y)=f(x)\otimes g(y)$, for any $f\in\mathrm{Hom}(A,M)$, $g\in\mathrm{Hom}(B,L)$, and for all $x\otimes y\in A\otimes B$. If A and B are finitely generated, A and M are finitely generated, or B and L are finitely generated, then γ is a natural isomorphism.

Proof. First we prove the naturality of γ. Let A', B', M' and L' be R-modules, and consider the R-homomorphisms $\alpha\colon A'\to A$, $\mu\colon M\to M'$, $\beta\colon B'\to B$ and $\lambda\colon L\to L'$. We get the following diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(A,M)\otimes\mathrm{Hom}(B,L) & \xrightarrow{\gamma} & \mathrm{Hom}(A\otimes B,M\otimes L)\\
\Big\downarrow{\scriptstyle \mathrm{Hom}(\alpha,\mu)\otimes\mathrm{Hom}(\beta,\lambda)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}(\alpha\otimes\beta,\mu\otimes\lambda)}\\
\mathrm{Hom}(A',M')\otimes\mathrm{Hom}(B',L') & \xrightarrow{\gamma'} & \mathrm{Hom}(A'\otimes B',M'\otimes L')
\end{array}$$

which is commutative. To actually see it, note that if we start from $f\otimes g$ in $\mathrm{Hom}(A,M)\otimes\mathrm{Hom}(B,L)$, the result in $\mathrm{Hom}(A'\otimes B',M'\otimes L')$ that we obtain going in both ways of the square is equal to $(\mu\circ f\circ\alpha)\otimes(\lambda\circ g\circ\beta)$. Hence γ is natural.

Now we proceed to prove that γ is an isomorphism. First suppose that both A and B are finitely generated. Then $A\simeq\bigoplus_N R$ and $B\simeq\bigoplus_{N'} R$ for positive integers N and N', i.e., both are isomorphic to finite direct sums of the base ring R. By property (ii) of the functor Hom, we have that $\mathrm{Hom}(\bigoplus_N R,M)\simeq\prod_N\mathrm{Hom}(R,M)$. Furthermore, since it is a finite product, and $\mathrm{Hom}(R,M)\simeq M$ we have that $\prod_N\mathrm{Hom}(R,M)\simeq\bigoplus_N M$. Similarly we obtain that $\mathrm{Hom}(B,M)\simeq\bigoplus_{N'} L$. Hence the left hand side in equation (3.1) is isomorphic to the product $(\bigoplus_N M)\otimes(\bigoplus_{N'} L)$, which by properties of tensor products is isomorphic to $\oplus_{N+N'}M\otimes L$. Again using the properties of tensor products, we have $A\otimes B\simeq\oplus_{N+N'}R$ and the right hand side is isomorphic to $\oplus_{N+N'}M\otimes L$. Moreover, the isomorphism is induced by γ since the expression that defines γ is additive.

On the other hand suppose that A and M are finitely generated. Note that if we suppose that $A=M=R$, both sides of equation (3.1) are isomorphic to $\mathrm{Hom}(B,L)$. Hence if M is free, both sides are isomorphic to a finite sum $\bigoplus\mathrm{Hom}(B,L)$, and γ is still an isomorphism. Suppose now that M is an arbitrary finitely generated R-module. Since R is a principal ideal domain, we can take a finite free resolution of M, $0\longrightarrow F_1\longrightarrow F_0\longrightarrow M\longrightarrow 0$, where F_1 and F_0 are free and finitely generated R-modules. If we replace M by each of the modules F_1 or F_0, the morphism γ

becomes an isomorphism. Then by naturality of γ we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(A, F_1) \otimes \mathrm{Hom}(B, L) & \xrightarrow{\cong} & \mathrm{Hom}(A \otimes B, F_1 \otimes L) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(A, F_0) \otimes \mathrm{Hom}(B, L) & \xrightarrow{\cong} & \mathrm{Hom}(A \otimes B, F_0 \otimes L) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(A, M) \otimes \mathrm{Hom}(B, L) & \xrightarrow{\gamma} & \mathrm{Hom}(A \otimes B, M \otimes L) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

where the columns are exact. This is because $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(A \otimes B, -)$ are exact functors, which is shown using property (v) of Hom, together with the fact that both A and $A \otimes B$ are free R-modules. Therefore we deduce that γ is an isomorphism. The case where B and L are finitely generated follows directly from commutativity of the tensor product. ☐

Example 3.5. In general $\mathrm{Hom}(A, M) \otimes \mathrm{Hom}(B, L) \ncong \mathrm{Hom}(A \otimes B, M \otimes L)$ as shown in this example. Let $R = \mathbb{Z}$. In this example we use direct and inverse limits; see [4, Chapter VIII, Section 5]. Let p be a prime number. We define the group of p-adic integers as $\widehat{\mathbb{Z}_p} := \varprojlim \mathbb{Z}_{p^n}$. Recall that $\varinjlim \mathbb{Z}_{p^n} = \mathbb{Z}_{p^\infty}$, where $\mathbb{Z}_{p^\infty}$ is the p-primary part of $\mathbb{Q} \,/\, \mathbb{Z}$. Then

$$\mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) = \mathrm{Hom}(\varinjlim_n \mathbb{Z}_{p^n}, \mathbb{Z}_{p^\infty}) = \varprojlim_n \mathrm{Hom}(\mathbb{Z}_{p^n}, \mathbb{Z}_{p^\infty}) = \varprojlim_n \mathbb{Z}_{p^n} = \widehat{\mathbb{Z}_p}$$

where the second equality is deduced from the universal property of inverse limits. For the third, recall that a homomorphism $f \colon \mathbb{Z}_{p^n} \to \mathbb{Z}_{p^\infty}$ has image $f(1) \in \mathbb{Z}_{p^\infty}$ of order less than or equal to p^n. Note that $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \otimes \mathrm{Hom}(\mathbb{Z}_{p^\infty}, \mathbb{Z}_{p^\infty}) \simeq \mathbb{Q} \otimes \widehat{\mathbb{Z}_p} \simeq \widehat{\mathbb{Q}_p}$, while $\mathrm{Hom}(\mathbb{Q} \otimes \mathbb{Z}_{p^\infty}, \mathbb{Q} \otimes \mathbb{Z}_{p^\infty}) = 0$ since $\mathbb{Q} \otimes \mathbb{Z}_{p^\infty} = 0$.

Definition 3.6. A chain complex C is *of finite type* if C_n is finitely generated for all natural numbers n.

Lemma 3.7. *Given a free chain complex C such that $H(C)$ is of finite type, there exists a chain complex $\bar{C}$ of finite type such that $H(\bar{C}) \cong H(C)$.*

Proof. Since R is a principal ideal domain, by the explanation after Definition 1.14, there is a free resolution $0 \to P_1 \to P_0 \to H_n(C) \to 0$ for $H_n(C)$, where the R-modules P_1 and P_0 are finitely generated. Then define the chain complex $C(n)$ as the one which has P_1 and P_0 in positions $n+1$ and n, and zero elsewhere. As a consequence $C(n)$ is a free chain complex of finite type, $H_n(C(n)) \cong H_n(C)$ and $H_k(C(n)) = 0$ for $k \neq n$. Therefore if we define the free chain complex $\bar{C} = \bigoplus_n C(n)$, we have that $\bar{C}$ is free and of finite type. Furthermore, $H(\bar{C}) \cong H(C)$ as wanted. □

Consider two chain complexes C and D and two modules M and L. If C and D are of finite type, or C is of finite type and M is finitely generated, then by Lemma 3.4 we have isomorphisms:

$$\gamma_{pq} \colon \mathrm{Hom}(C_p, M) \otimes \mathrm{Hom}(D_q, L) \longrightarrow \mathrm{Hom}(C_p \otimes D_q, M \otimes L).$$

On the other hand, let the chain complexes K and B be defined by $K_p = \mathrm{Hom}(C_{-p}, M)$ and $B_q = \mathrm{Hom}(D_{-q}, L)$ as in Definition 3.9. Considering the first arrow α, we deduce well-defined monomorphisms in cohomology $\Theta_{pq} = \gamma_{pq} \circ \alpha$ as

$$\Theta_{pq} \colon H^p(C; M) \otimes H^q(D; L) \longrightarrow H^{p+q}(C \otimes D; M \otimes L) \tag{3.2}$$

where $\Theta_{pq}(\{f\} \otimes \{g\}) = \{f \otimes g\}$. Then we define Θ as the direct sum of Θ_{pq} for $p+q=n$. This is an isomorphism under some conditions, for example when the ring R is a field. The following theorem gives a general formulation.

Theorem 3.8 (Algebraic Künneth Formula for Cohomology)**.** *Let R be a principal ideal domain. Let M and L be two R-modules such that* $\mathrm{Tor}(M, L) = 0$*, and suppose given two positive free chain complexes (C, ∂) and (D, ϵ). If $H(C)$ and $H(D)$ are of finite type, or if $H(C)$ is of finite type and M is finitely generated, then there is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H^p(C; M) \otimes H^q(D; L) \longrightarrow H^n(C \otimes D; M \otimes L) \longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}(H^r(C; M), H^s(D; L)) \to 0$$

which splits unnaturally.

Proof. First we define two chain complexes K and B such that $K_n = \mathrm{Hom}(C_{-n}, M)$ and $B_n = \mathrm{Hom}(D_{-n}, L)$ respectively. And define their boundary maps as $\delta_n = \partial^*_{-n}$ and $\mu_n = \epsilon^*_{-n}$, where ∂^*_n and ϵ^*_n are the respective boundary maps of $\mathrm{Hom}(C_{-n}, M)$ and $\mathrm{Hom}(D_{-n}, L)$. We want to use the Homology Künneth Formula for the tensor product of the chain complexes K and B. To proceed we need to show first that $\mathrm{Tor}(K, B)$ is exact. Using Lemma 3.7 we can suppose that the conditions for $H(C)$ and $H(D)$ are true for the chain complexes C and D. Now consider a free resolution $0 \to F_1 \to F_0 \to M \to 0$ where the modules F_1 and F_0 are finitely generated if M

is finitely generated. Since by hypothesis $\operatorname{Tor}(M, L) = 0$, tensoring with L we get a free exact sequence

$$0 \longrightarrow F_1 \otimes L \longrightarrow F_0 \otimes L \longrightarrow M \otimes L \longrightarrow 0\,.$$

Also by hypothesis the R-modules C_p and D_q are free for all integers $p+q=n$. This leads us to the exact sequences

$$0 \longrightarrow \operatorname{Hom}(C_p \otimes D_q, F_1 \otimes L) \longrightarrow \operatorname{Hom}(C_p \otimes D_q, F_0 \otimes L). \tag{3.3}$$

On the other hand, since C_p is free, we have exact sequences

$$0 \longrightarrow \operatorname{Hom}(C_p, F_1) \longrightarrow \operatorname{Hom}(C_p, F_0) \longrightarrow \operatorname{Hom}(C_p, M) \longrightarrow 0\,.$$

Tensoring with $\operatorname{Hom}(D_q, L)$, we get exact sequences

$$0 \longrightarrow \operatorname{Tor}(\operatorname{Hom}(C_p, M), \operatorname{Hom}(D_q, L)) \longrightarrow \operatorname{Hom}(C_p, F_1) \otimes \operatorname{Hom}(D_q, L) \longrightarrow \operatorname{Hom}(C_p, F_0) \otimes \operatorname{Hom}(D_q, L). \tag{3.4}$$

Using Lemma 3.4 the second and third terms of this sequence are respectively isomorphic to the first and second terms from the sequence (3.3). Thus the first term from the chain complex (3.3) is zero.

We proceed to apply the Künneth formula from Theorem 1.27, and get the following natural short exact sequence:

$$0 \to \bigoplus_{p+q=n} H_p(K) \otimes H_q(B) \longrightarrow H_n(K \otimes B) \longrightarrow \bigoplus_{r+s=n-1} \operatorname{Tor}(H_r(K), H_s(B)) \to 0.$$

Note that by definition $K_p \otimes B_q = \operatorname{Hom}(C_{-p}, M) \otimes \operatorname{Hom}(D_{-q}, L)$, and as a consequence we have that $H_n(K \otimes B) = H_{-n}(\operatorname{Hom}(C, M) \otimes \operatorname{Hom}(D, L))$. Therefore we can deduce the following natural short exact sequence:

$$\begin{aligned} 0 \to \bigoplus_{p+q=n} H^p(C; M) \otimes H^q(D; L) &\longrightarrow H_n(\operatorname{Hom}(C, M) \otimes \operatorname{Hom}(D, L)) \\ &\longrightarrow \bigoplus_{r+s=n+1} \operatorname{Tor}(H^r(C; M), H^s(D; L)) \to 0 \end{aligned}$$

which is not precisely what we wanted, since we would like to have $H^n(C \otimes D; M)$ as the central term. We proceed by defining a chain map

$$\Gamma\colon \operatorname{Hom}(C, M) \otimes \operatorname{Hom}(D, L) \to \operatorname{Hom}(C \otimes D, M \otimes L)$$

where Γ is defined by the equality

$$(\Gamma(f \otimes g))(x \otimes y) = \begin{cases} f(x) \otimes g(y) & \text{if } x \in C_p \text{ and } y \in D_q \\ 0 & \text{otherwise} \end{cases}$$

for all R-homomorphisms $f \in \mathrm{Hom}(C_p, M)$ and $g \in \mathrm{Hom}(D_q, L)$, and for all integers $p+q = n$. We proceed to check that Γ is indeed a chain map. Let $x \otimes y$ in $(C \otimes D)_{n+1}$. Then we get the following equalities:

$$(\delta\gamma(f \otimes g))(x \otimes y) = \gamma(f \otimes g)(\partial x \otimes y + (-1)^{|x|} x \otimes \partial y)$$
$$= \begin{cases} f\partial x \otimes gy & \text{if } |x| = p+1 \\ (-1)^{|x|} fx \otimes g\partial y & \text{if } |x| = p \\ 0 & \text{otherwise;} \end{cases} \tag{3.5}$$

$$(\gamma\delta(f \otimes g))(x \otimes y) = \gamma(\delta f \otimes g + (-1)^{|f|} f \otimes \delta g)$$
$$= \begin{cases} \delta f x \otimes gy & \text{if } |x| = p+1 \\ (-1)^{|f|} fx \otimes \delta gy & \text{if } |x| = p \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Since the right hand sides agree, we deduce that $\delta\gamma = \gamma\delta$, and Γ is a chain map. Using Lemma 3.7 we can suppose that the conditions for $H(C)$ and $H(D)$ are true for the chain complexes C and D. Therefore using Lemma 3.4 we have isomorphisms

$$\gamma_{(p,q)} \colon \mathrm{Hom}(C_p, M) \otimes \mathrm{Hom}(D_q, L) \simeq \mathrm{Hom}(C_p \otimes D_q; M \otimes L)$$

for all integers $p+q = n$. Note that $\Gamma = \bigoplus_{p+q=n} \gamma_{(p,q)}$, which implies that Γ must be an isomorphism. Hence the result follows. Since we have used the Künneth Formula from Theorem 1.27, we can deduce that the sequence is natural and splits. Hence we have an isomorphism

$$H^n(C \otimes D; M \otimes L) \cong \bigoplus_{p+q=n} H^p(C; M) \otimes H^q(D; L) \bigoplus_{r+s=n+1} \mathrm{Tor}(H^r(C; M), H^s(D; L)) \quad \square$$

Definition 3.9. Let M be an R-module and let X be a topological space. We define the singular cochain complex of X with coefficients in M as the cochain complex whose n^{th} component is $\mathrm{Hom}(S_n(X), M)$. Its n^{th} cohomology module is $H^n(X; M)$.

Theorem 3.10 (Cohomology Künneth Theorem)**.** *Suppose that the ring R is a principal ideal domain, and let M be a free R-module. Let X and Y be two topological spaces. If $H(X; M)$ is of finite type, and either $H(Y; M)$ is of finite type or M is finitely generated, then there are natural short exact sequences*

$$0 \to \bigoplus_{p+q=n} H^p(X; M) \otimes H^q(Y; L) \longrightarrow H^n(X \times Y; M \otimes L)$$
$$\longrightarrow \bigoplus_{r+s=n+1} \mathrm{Tor}(H^r(X; M), H^s(Y; L)) \to 0$$

which split, but not naturally.

Proof. By the Eilenberg-Zilber Theorem we have a natural homotopy equivalence $\phi\colon S(X\times Y)\to S(X)\otimes S(Y)$. The map ϕ induces isomorphisms between cohomology groups $\Phi^*\colon H^n(S(X)\otimes S(Y);M\otimes L)\longrightarrow H^n(S(X\times Y);M\otimes L)$, defined by $\Phi^*(f)=f\circ\Phi$. Therefore the result follows by applying the previous theorem on the chain complexes $S(X)$ and $S(Y)$. ☐

3.2 Cross Product and Cup Product

Let X and Y be two topological spaces, and consider two R-modules M and L. Consider the morphisms $\alpha\colon S_p(X)\to M$ and $\beta\colon S_q(Y)\to L$ for all integers $n\geq 0$. Then the first arrow from the Künneth Formula from Theorem 3.10 is the injection Θ from equation 3.2. Composing this with an Eilenberg-Zilber map $\phi\colon S(X\times Y)\to S(X)\otimes S(Y)$, we get a morphism

$$\begin{aligned}\times\colon\qquad H^p(X;M)\otimes H^q(Y;L)&\longrightarrow H^{p+q}(X\times Y;M\otimes L)\\ \{\alpha\}\otimes\{\beta\}&\longrightarrow\{\Theta(\alpha\otimes\beta)\circ\phi\}\end{aligned}$$

This map is called *cross product*, and by construction it is essentially the same as the first arrow in the previous Künneth theorem. We can also define a product for the case where $M=L=R$ and $X=Y$. From properties of tensor products there is an isomorphism $\mu\colon R\otimes R\to R$. We define the interior cross product

$$\begin{aligned}\times_1\colon\qquad H^p(X;R)\otimes H^q(X;R)&\longrightarrow H^{p+q}(X\times X;R)\\ \{\alpha\}\otimes\{\beta\}&\longrightarrow\{\mu\circ\Theta(\alpha\otimes\beta)\circ\phi\}\end{aligned}$$

Now we want to see that $\times_1$ is commutative up to sign. Recall the maps $t\colon S(X\times Y)\to S(Y\times X)$ defined as $(x,y)=(y,x)$. Also recall $\tau\colon S(X)\otimes S(Y)\to S(Y)\otimes S(X)$, defined as $\tau(x\otimes y)=(-1)^{|x||y|}y\otimes x$. Using the first corollary from the Eilenberg-Zilber Theorem we deduce that $\phi\circ t$ and $\tau\circ\phi$ are chain homotopic. Hence

$$\begin{aligned}(\{\beta\}\times_1\{\alpha\})\circ t&=\{\mu\circ\Theta(\beta\otimes\alpha)\circ\phi\circ t\}\\ &=\{\mu\circ\Theta(\beta\otimes\alpha)\circ\tau\circ\phi\}\\ &=(-1)^{|\alpha||\beta|}\{\mu\circ\Theta(\alpha\otimes\beta)\circ\phi\}\\ &=(-1)^{|\alpha||\beta|}(\{\alpha\}\times_1\{\beta\}).\end{aligned}$$

Similarly by the second corollary of the Eilenberg-Zilber Theorem we have that $\times_1$ is associative. Also we can define the cup product as

$$\begin{aligned}\smile\colon\qquad H^p(X;R)\otimes H^q(X;R)&\longrightarrow H^{p+q}(X;R)\\ \{\alpha\}\otimes\{\beta\}&\longrightarrow\{\mu\circ\Theta(\alpha\otimes\beta)\circ\phi\circ d\}\end{aligned}$$

where $d\colon X \to X \times X$ is the diagonal map defined as $d(x) = (x, x)$. Thus the cup product is commutative and associative by the commutativity and associativity of $\times_1$. We can also define the interior cross product in terms of the cup product as $f \times_1 g = ((f \circ p_1) \times (g \circ p_2))d = p_1^*(f) \smile p_2^*(g)$.

Proposition 3.11. *Let X be a topological space. Let R be a commutative ring with unit, and let $H^*(X;R)$ be the direct sum of the cohomology modules in all dimensions equipped with the cup product. Then $H^*(X;R)$ is a graded ring with unit.*

Proof. This is contained in the above discussion. Distributivity follows from distributivity of R. The unit element $1_X \in H^0(X;R)$ is the class obtained by augmenting the chain complex $S(X)$ with a map $\alpha\colon S_0(X) \to R$ such that $\alpha(\sigma) = 1_R$ for any $\sigma \in S_0(X)$. □

3.3 Künneth Formula for Cohomology Rings

If K and L are graded algebras, then $K \otimes_R L$ is also a graded algebra with respect to the multiplication $(k_1 \otimes l_1) \cdot (k_2 \otimes l_2) = (-1)^{|l_1||k_2|}(k_1 k_2) \otimes (l_1 l_2)$.

Theorem 3.12 (Cohomology Künneth Formula)**.** *If R is a principal ideal domain and X, Y are spaces such that $H(X;R)$ is of finite type and all torsion R-modules* $\operatorname{Tor}(H^i(X;R), H^j(X;R))$ *vanish, then the cross product*

$$\times_1\colon H^*(X;R) \otimes H^*(Y;R) \longrightarrow H^*(X \times Y;R)$$

is an isomorphism of algebras.

Proof. By hypothesis we can use the Künneth Formula for cohomology (3.10) and deduce that the product $\times_1$ induces a group isomorphism. To prove that it is a ring isomorphism we see that

$$\begin{aligned}
(x_1 \times y_1) \smile (x_2 \times y_2) &= p_1^*(x_1) \smile p_2^*(y_1) \smile p_1^*(x_2) \smile p_2^*(y_2) \\
&= (-1)^{|x_2||y_1|} p_1^*(x_1) \smile p_1^*(x_2) \smile p_2^*(y_1) \smile p_2^*(y_2) \\
&= (-1)^{|x_2||y_1|} p_1^*(x_1 \smile x_2) \smile p_2^*(y_1 \smile y_2) \\
&= (-1)^{|x_2||y_1|} (x_1 \smile x_2) \times (y_1 \smile y_2)
\end{aligned}$$

for all x_1 and x_2 in $S(X)$, and for all y_1 and y_2 in $S(Y)$. □

Example 3.13. Does every graded commutative R-algebra occur as a cup product algebra of some topological space? This is a very difficult problem; see [6, Page 227]. In this example we will study some cases using the Künneth formula. The graded ring $H^*(S^1;\mathbb{Z})$ has only one generator of dimension 1. The Künneth Formula gives us the ring for a finite product of circles:

$$H^*(\prod_{k=0}^{N} S^1; R) \simeq \bigotimes_{k=0}^{N} H^*(S^1;R) \simeq \bigotimes_{k=0}^{N} (R[x_k]/(x_k^2)) \simeq \Lambda_R[x_0,\dots,x_N]$$

where $|x_i| = 1$ for all $0 \leq i \leq N$ and $\Lambda_R[x_0,\dots,x_N]$ denotes an exterior algebra.

The cohomology rings of the complex projective spaces are $H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]/(\alpha^{n+1})$ and also $H^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \simeq \mathbb{Z}[\alpha]$ where $|\alpha| = 2$; see [6, Theorem 3.12]. Using the Künneth formula 3.12 we have

$$H^*(\prod_{k=0}^{N} \mathbb{C}\mathrm{P}^{n_k};\mathbb{Z})) \simeq \bigotimes_{k=0}^{N} H^*(\mathbb{C}\mathrm{P}^{n_k};\mathbb{Z}) \simeq \bigotimes_{k=0}^{N} \mathbb{Z}[\alpha_k]/(\alpha_k^{n_k+1})$$

with $n_k \in \mathbb{N} \cup \infty$, and $|\alpha_k| = 2$ for all integers $0 \leq k \leq N$.

Example 3.14. $\bigvee_{k=0}^N S^1$ is not a deformation retract of $\prod_{k=0}^N S^1$, for integers $N > 0$. First we start by a calculation of the cohomology ring of $\bigvee_{k=0}^N S^1$. As we know, this wedge sum is $\bigvee_{k=0}^N S^1 \simeq \bigsqcup_{k=0}^N S^1 \big/ \bigsqcup_{k=0}^N \{(0,0)\}$. Hence we have a short exact sequence of singular complexes:

$$0 \longrightarrow S(\textstyle\bigsqcup_{k=0}^N \{(0,0)\}) \longrightarrow S(\bigsqcup_{k=0}^N S^1) \longrightarrow S(\bigvee_{k=0}^N S^1) \longrightarrow 0.$$

Hence using the Snake Lemma we get the exact sequence

$$\cdots \longrightarrow H^n(\textstyle\bigvee_{k=0}^N S^1;R) \longrightarrow H^n(\bigsqcup_{k=0}^N S^1;R) \longrightarrow H^n(\bigsqcup_{k=0}^N \{(0,0)\};R) \longrightarrow \cdots$$

which leads us to the isomorphism $H^n(\bigvee_{k=0}^N S^1;R) \simeq H^n(\bigsqcup_{k=0}^N S^1;R)$ for any integer $n > 0$. Therefore the only nonzero cohomology modules are $H^1(\bigvee_{k=0}^N S^1;R) \simeq \prod_{k=0}^N H^1(S^1;R) \simeq \prod_0^N R$, and $H^0(\bigvee_{k=0}^N S^1;R) \simeq R$. Therefore we get an expression for the cohomology graded ring of the wedge sum $\bigvee_{k=0}^N S^1$, namely

$$H^*(\bigvee_{k=0}^{N} S^1;R) \simeq R[x_0,\dots,x_N]/(x_i^2, x_i x_j), \text{ where } |x_i| = 1 \text{ for all } 0 \leq i \leq N,$$

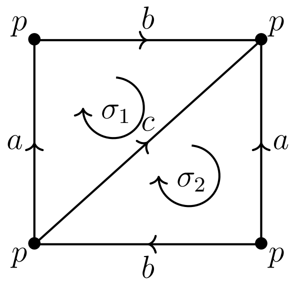
which is not isomorphic to the cohomology ring of $\prod_{k=0}^N S^1$ for $N > 0$. Therefore we deduce the result, since otherwise $\bigvee_{k=0}^N S^1$ and $\prod_{k=0}^N S^1$ would have the same cohomology ring.

In the following example we use the *Alexander-Whitney diagonal approximation*, which expresses the cup product as

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|[t_0,\dots,t_p])\psi(\sigma|[t_p,\dots,t_{p+q}])$$

for two singular cochains $\varphi \in \text{Hom}(S_p(X), R)$ and $\psi \in \text{Hom}(S_q(X), R)$. This is equivalent to our definition of cup products, and can be shown by using the Acyclic Model Theorem. More details can be found in [12, Pages 113-114].

Example 3.15. In 1957, F. P. Palermo submited his thesis on the cohomology ring of product complexes; see [8]. He studied the cohomology of product spaces by the use of Bockstein morphisms. Also he included an example which highlights the importance of the condition on the torsion in 3.12. We proceed to study it. Consider the Klein Bottle KB, and the wedge $\mathbb{R}\text{P}^2 \vee S^1$. We will see that although they have isomorphic cohomology rings, their respective products $\text{KB} \times \text{KB}$ and $\mathbb{R}\text{P}^2 \vee S^1 \times \mathbb{R}\text{P}^2 \vee S^1$ do not have isomorphic cohomology rings. This is because the algebraic Künneth theorem fails in this case, as shown in Example 1.28. Given $y \in S_n(X)$, we denote by y^* the dual map of y so that $y^*(y) = 1$ and $y^*(x) = 0$ for any $x \in S_n(X) - x$. Now we proceed to calculate the respective cohomology rings of each space. Consider the Klein Bottle with the structure pictured below.



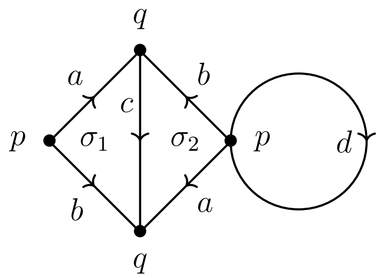
The 2-simplices σ_1 and σ_2 are such that $\partial(\sigma_1) = a+b-c$ and $\partial(\sigma_2) = b+c-a$. Note that the only non-zero coboundaries are $\delta(a^*) = \sigma_1^* - \sigma_2^* = \delta(c^*)$, and $\delta(b^*) = \sigma_1^* + \sigma_2^*$. Thus the only non-zero cohomology groups are

$$H^n(\text{KB}; \mathbb{Z}) = \begin{cases} (p^*) \simeq \mathbb{Z} & \text{if } n = 0 \\ (a^* + c^*) \simeq \mathbb{Z} & \text{if } n = 1 \\ (\sigma_1^*)/(2\sigma_1^*) \simeq \mathbb{Z}_2 & \text{if } n = 2. \end{cases}$$

Considering the Alexander-Whitney diagonal approximation, we deduce that p^* is the neutral element, and the only product that we need to check is $(a^*+c^*) \smile (a^*+c^*)$ which is zero, by commutativity up to sign of cup products. Hence the cohomology ring is

$$H^*(\text{KB}; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2, y^2, xy, 2y)$$

where $x = (a^* + c^*)$ and $y = (\sigma_1^*)$. Now we proceed to compute the cohomology ring for $\mathbb{R}\text{P}^2 \vee S^1$, which is pictured below.



In dimension two the coboundaries are $\partial(\sigma_1) = a + c - b$ and $\partial(\sigma_2) = b + c - a$. Whereas in dimension zero $\delta(q^*) = a^* + b^* = -\delta(p^*)$ and in dimension one $\delta(a^*) = \sigma_1^* - \sigma_2^* = -\delta(b^*)$, $\delta(c^*) = \sigma_1^* + \sigma_2^*$, and $\delta(d^*) = 0$. Thus the only non-zero cohomology modules are

$$H^n(\mathbb{R}\mathrm{P}^2 \vee S^1; \mathbb{Z}) = \begin{cases} (p^* + q^*) \simeq \mathbb{Z} & \text{if } n = 0 \\ (d^*) \simeq \mathbb{Z} & \text{if } n = 1 \\ (\sigma_1^*)/(2\sigma_1^*) \simeq \mathbb{Z}_2 & \text{if } n = 2. \end{cases}$$

Note that $(p^* + q^*)$ is the neutral element of the ring, and $(d^*) \smile (d^*) = 0$. Hence the cohomology ring is

$$H^*(\mathbb{R}\mathrm{P}^2 \vee S^1; \mathbb{Z}) = \mathbb{Z}[x, y]/(x^2, y^2, xy, 2y)$$

where $x = d^*$ and $y = \sigma_1^*$.

Finally, we proceed to study the cohomology rings of $\mathrm{KB} \times \mathrm{KB}$ and $(\mathbb{R}\mathrm{P}^2 \vee S^1) \times (\mathbb{R}\mathrm{P}^2 \vee S^1)$. Since KB and $\mathbb{R}\mathrm{P}^2 \vee S^1$ have isomorphic cohomology R-modules, we already know from the Künneth Formula 3.10 for cohomology that their respective R-modules will be isomorphic. In particular, we deduce that their cohomology R-modules have two generators in dimension one, and only one generator in dimension three. Using the Eilenberg-Zilber theorem, we only need to study the cohomology of their tensor product chain complexes.

First, in $S(\mathrm{KB}) \otimes S(\mathrm{KB})$ the generators of the first cohomology module are $\alpha = (a^* + c^*) \otimes p^*$, and $\beta = p^* \otimes (a^* + c^*)$. On the other hand $\gamma = \sigma_1^* \otimes (a^* + b^*) - (a^* + b^*) \otimes \sigma_1^*$ generates cohomology in dimension three. Since the product $(a^* + b^*) \smile (a^* + c^*)$ is equal to σ_2^*, we get that both products $\gamma \cdot \alpha$ and $\gamma \cdot \beta$ are equal to the nonzero element $\sigma_1 \otimes \sigma_1$.

On the other hand, we proceed to study $S(\mathbb{R}\mathrm{P}^2 \vee S^1) \otimes S(\mathbb{R}\mathrm{P}^2 \vee S^1)$. The generators for the cohomology in dimension one are $\phi = d^* \otimes (p^* + q^*)$, and $\psi = (p^* + q^*) \otimes d^*$, whereas in dimension three the generator is $\tau = \sigma_1^* \otimes (a^* + c^*) - (a^* + c^*) \otimes \sigma_1^*$, and the products $\tau \cdot \phi$ and $\tau \cdot \psi$ vanish. Therefore the cohomologies of the spaces $\mathrm{KB} \times \mathrm{KB}$ and $(\mathbb{R}\mathrm{P}^2 \vee S^1) \times (\mathbb{R}\mathrm{P}^2 \vee S^1)$ have different ring structures, even though they are isomorphic as graded R-modules.

Chapter 4

Infinite Products of Circles

Definition 4.1. A *directed set* Λ is a set with a partial order relation $\leq$ such that for any two a and b in Λ there exists an element c in Λ such that $a \leq c$ and $b \leq c$.

Definition 4.2. Let $\mathcal{C}$ be a category. A *directed system of objects* in $\mathcal{C}$ is a family of objects $\{X_\alpha\}_{\alpha \in \Lambda}$ in $\mathcal{C}$, where Λ is a directed set, and morphisms $f_a^b \colon X_a \to X_b$ whenever $a \leq b$ satisfying the following requirements:

(i) For each a in Δ, the morphism f_a^a is the identity of X_a.

(ii) If $a \leq b \leq c$ in Λ, then $f_b^c \circ f_a^b = f_a^c$.

We will denote directed systems as triples $\{X_a, f_a^b, \Lambda\}$.

Example 4.3. Let $\mathcal{C} = R$-mod, and let M_a and M_b two R-modules from a directed system of R-modules $\{M_a, f_a^b, \Lambda\}$. Then for $A \in M_a$ and $B \in M_b$, we have $\phi_a(A) = \phi_b(B)$ if and only if the system contains an R-module M_k, and two R-homomorphisms f_a^k and f_b^k such that $f_a^k(A) = f_b^k(B)$. This is because by Definition 4.2 we have

$$\phi_a(A) = \phi_k(f_a^k(A)) = \phi_k(f_b^k(B)) = \phi_b(B).$$

Definition 4.4. Let $\{X_a, f_a^b, \Lambda\}$ be a directed system in a category $\mathcal{C}$. The *direct limit* of this system is an object X in $\mathcal{C}$ with morphisms $\phi_a \colon X_a \to X$ satisfying:

(i) $\phi_a = \phi_b \circ f_a^b$ for all $a \leq b$ in Λ.

(ii) (Universal Property) For each object $Y \in \mathcal{C}$ together with morphisms $\psi_a \colon X_a \to Y$ satisfying $\psi_a = \psi_b \circ f_a^b$ for all $a \leq b$, there is a unique morphism $g \colon X \to Y$ such that $g \circ \phi_a = \psi_a$ for all a.

The direct limit is written as $X = \varinjlim X_a$. The universal property characterizes direct limits up to isomorphism, provided that they exist. Suppose that $\{X, \phi_i\}_{i\in\Lambda}$ and $\{Y, \psi_i\}_{i\in\Lambda}$ satisfy condition (i) from Definition 4.4. Then, by (ii), there exist unique morphisms $f\colon X \to Y$ and $f'\colon Y \to X$ such that $f \circ \phi_i = \psi_i$ and $f' \circ \psi_i = \phi_i$. In particular $(f' \circ f) \circ \phi_i = \phi_i$ and since $id_X \circ \phi_i = \phi_i$, using the universal property once more we deduce that $f' \circ f = id_X$. Similarly $f \circ f' = id_Y$ and f is an isomorphism.

Example 4.5. In the case where $\mathcal{C} = \text{Top}$, the direct limit of a directed system $\{X, \phi_i\}_{i\in\Lambda}$ has the final topology with respect to the morphisms $\phi_a\colon X_a \to X$ for all $a \in \Lambda$. An example of a direct limit is $\bigvee_{k=0}^{\infty} S^1 \cong \varinjlim \bigvee_{k=0}^{N} S^1$, since the inclusions $\bigvee_{k=0}^{N} S^1 \hookrightarrow \bigvee_{k=0}^{\infty} S^1$ satisfy the universal property that defines a direct limit.

Proposition 4.6. *If a space X is the union of a directed set of subspaces X_a with the property that each compact set in X is contained in some X_a, then*

(a) *the natural map $\mu\colon \varinjlim H_i(X_a; M) \to H_i(X; M)$ is an isomorphism in all dimensions i and for all M;*

(b) *also if X has a base point x_0 and all the subspaces X_a contain x_0, then the natural map $\nu\colon \varinjlim \pi_1(X_a, x_0) \to \pi_1(X, x_0)$ is an isomorphism.*

Proof. Given a cycle f in $H_i(X; M)$, we can represent it by a finite sum of singular simplices. Since the image of a singular simplex $\phi\colon \Delta_n \to X$ is compact in X, we have that the image of f is a compact subset of X, hence it must be in some X_a by hypothesis. Therefore the map μ is surjective. On the other hand, if g is a boundary in $H_i(X; M)$, we have that there exists a finite sum of singular simplices such that their boundary is g. Since the image of this sum is compact, it must lie in some X_a and hence g is zero in $\varinjlim H_i(X_a; M)$. The second part is analogous since a loop is a compact subspace of X, and it is nullhomotopic in some X_a if and only if it is also nullhomotopic in X, since the square $I \times I$ is compact. □

In general homology works very well with wedge sums, even infinite. Recall that the homology modules of a finite wedge of circles are

$$H_i(\bigvee_{k=0}^{N} S^1) \cong \begin{cases} \mathbb{Z} \oplus \overset{N}{\cdots} \oplus \mathbb{Z} & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

We apply Proposition 4.6 to deduce that

$$H_i(\bigvee_{k=0}^{\infty} S^1) \simeq H_i(\bigcup_{N=0}^{\infty} \bigvee_{k=0}^{N} S^1) \simeq \varinjlim_N H_i(\bigvee_{k=0}^{N} S^1) \simeq \varinjlim_N \bigoplus_{k=0}^{N} H_i(S^1) \simeq \bigoplus_{k=0}^{\infty} H_i(S^1).$$

Note that $\bigvee_{k=0}^{\infty} S^1$ is not compact while $\prod_{k=0}^{\infty} S^1$ is compact by Tychonoff's Theorem; see [1, Chapter I, Section 9.5]. We next study the induced homomorphisms in homology and cohomology by the inclusion $\bigvee_{k=0}^{\infty} S^1 \hookrightarrow \prod_{k=0}^{\infty} S^1$. By the Künneth Formula,

$$H_i(\prod_{k=0}^{N} S^1) \cong \begin{cases} \mathbb{Z} \oplus \overset{\binom{N}{i}}{\cdots} \oplus \mathbb{Z} & \text{if } 0 \leq i \leq N \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$H_1(\prod_{k=0}^{\infty} S^1) \cong (\pi_1(\prod_{k=0}^{\infty} S^1))_{ab} \cong \prod_{k=0}^{\infty} \mathbb{Z}.$$

However, the higher homology groups $H_i(\prod_{k=0}^{\infty} S^1)$ are much more difficult to compute. Although homology commutes with direct limits, the infinite product $\prod_{k=0}^{\infty} S^1$ is not the direct limit of $\prod_{k=0}^{N} S^1$, as we next show.

Proposition 4.7. $\prod_{k=0}^{\infty} S^1 \ncong \varinjlim \prod_{k=0}^{N} S^1$.

Proof. To prove this we use the fundamental group. First note that since the fundamental group commutes with Cartesian products we have $\pi_1(\prod_{k=0}^{\infty} S^1) \cong \prod_{k=0}^{\infty} \mathbb{Z}$. On the other hand, if we calculate the fundamental group of the direct limit

$$\pi_1(\varinjlim_{N} \prod_{k=0}^{N} S^1) \simeq \varinjlim_{N} \pi_1(\prod_{k=0}^{N} S^1) \simeq \varinjlim_{N}(\prod_{k=0}^{N} \mathbb{Z}) \simeq \bigoplus_{k=0}^{\infty} \mathbb{Z},$$

which is not isomorphic to $\prod_{k=0}^{\infty} \mathbb{Z}$. □

Note that $\prod_{k=0}^{N} S^1$ is a retract of $\prod_{k=0}^{\infty} S^1$ for all N. Since the i^{th} singular homology $H_i \circ S\colon \text{Top} \to R\text{-mod}$ is a covariant functor, we have that $H_i(\prod_{k=0}^{N} S^1)$ is also a retract of $H_i(\prod_{k=0}^{\infty} S^1)$. Therefore $H_i(\prod_{k=0}^{\infty} S^1)$ contains free abelian groups of arbitrary finite rank, since $H_i(\prod_{k=0}^{N} S^1) \simeq \mathbb{Z} \oplus \overset{\binom{N}{i}}{\cdots} \oplus \mathbb{Z}$ for $0 \leq i \leq N$. As a consequence we can state the following fact:

Proposition 4.8. $\prod_{k=0}^{\infty} S^1$ *does not admit a cellular structure.*

Proof. A cell complex is compact if it has finitely many cells; see [4, Chapter V, 2.1]. Since $\prod_{k=0}^{\infty} S^1$ is compact, if it admitted a cellular structure then $H_i(\prod_{k=0}^{\infty} S^1) = 0$ for sufficiently large values of i, and it would be finitely generated for all i, which is not the case, as we just observed. □

Since $\bigvee_{k=0}^{\infty} S^1$ is a retract of $\prod_{k=0}^{\infty} S^1$, the homology $H_i(\bigvee_{k=0}^{\infty} S^1)$ is also a retract of $H_i(\prod_{k=0}^{\infty} S^1)$ and we have the inclusion

$$\bigoplus_{k=0}^{\infty} \mathbb{Z} \to \prod_{k=0}^{\infty} \mathbb{Z}$$

in the first homology group. We conclude by observing that, unexpectedly, it induces the same homomorphism in cohomology.

Proposition 4.9. *The homomorphism* $H^1(\prod_{k=0}^{\infty} S^1) \to H^1(\bigvee_{k=0}^{\infty} S^1)$ *induced by the inclusion is equal to* $\bigoplus_{k=0}^{\infty} \mathbb{Z} \to \prod_{k=0}^{\infty} \mathbb{Z}$.

Proof. First we calculate the cohomology modules of $\bigvee_{k=0}^{\infty} S^1$. Using the Universal Coefficient Theorem for cohomology, together with property (ii) of Hom, we deduce the isomorphism

$$H^1(\bigvee_{k=0}^{\infty} S^1) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(\bigvee_{k=0}^{\infty} S^1), \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{k=0}^{\infty} \mathbb{Z}, \mathbb{Z}) \cong \prod_{k=0}^{\infty} \mathbb{Z}.$$

On the other hand, we make use of a surprising result by Specker,

$$\mathrm{Hom}(\prod_{k=0}^{\infty} \mathbb{Z}, \mathbb{Z}) \cong \bigoplus_{k=0}^{\infty} \mathbb{Z},$$

which can be found in [11, pages 131-140]. Combining this with the Universal Coefficient Theorem for cohomology, we obtain

$$H^1(\prod_{k=0}^{\infty} S^1) \simeq \mathrm{Hom}_{\mathbb{Z}}(H_1(\prod_{k=0}^{\infty} S^1), \mathbb{Z}) \simeq \mathrm{Hom}_{\mathbb{Z}}(\prod_{k=0}^{\infty} \mathbb{Z}, \mathbb{Z}) \simeq \bigoplus_{k=0}^{\infty} \mathbb{Z}.$$

Therefore the inclusion $\bigvee_{k=0}^{\infty} S^1 \hookrightarrow \prod_{k=0}^{\infty} S^1$ induces the homomorphism $\bigoplus_{k=0}^{\infty} \mathbb{Z} \to \prod_{k=0}^{\infty} \mathbb{Z}$ in both H^1 and H_1. □

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