Restricted three body problems in the Solar System: simulations

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Abstract: In this project the planar, restricted and elliptic three body problem has been analitically studied in order to program an orbit simulator. Then, the simulator has been used to see the accuracy and utility of the problem and its simulator in some well known physical systems.

I. INTRODUCTION

The aim of this project is to use an orbit simulator for the restricted three body problem in order to study some physical systems which can be approximated by it.

The restricted three body problem (R3BP) consists in studying the motion of a body with very low mass under the influence of two other bodies with very big masses relative to the first one. In such a frame of work, the motion of the two massive bodies (known as the primaries, or the primary and the secondary) is Keplerian. When the orbits of the primaries are circular (CR3BP) there exists a first integral called the Jacobi constant, but as soon as we add a positive eccentricity and the orbits become elliptic (ER3BP), we do not have it anymore. Nevertheless, the simulator used in this project computes the orbit of the third body assuming the primaries move in elliptic orbits with an eccentricity e, which can also be 0 (CR3BP).

That being said, we can find several examples of primaries of the ER3BP in our Solar System (SS). The most common one is the Sun-Jupiter system, which can be thought as an approximation of the whole SS because they are the two most massive bodies. In this system we usually take as the third body a comet or a meteorite whose orbit needs to be determined. Two other interesting systems for us due to proximity are the Sun-Earth and the Earth-Moon systems. In these ones we take spaceships, spacecrafts, satellites or even space debris as possible third bodies. A particular example is the Sun-Earth-Moon system, where the Sun and the Earth stand for the primaries and the Moon is the third body. Other examples of primaries out of the SS could be binary star systems among others, although in this project we are going to focus on the three former examples.

II. EQUATIONS OF THE ER3BP

A. Sidereal reference frame

Let P_1 and P_2 be the primaries, which have masses m_1 and m_2 and move around their common mass centre describing elliptic orbits. Before writing the equations of the motion of the third body, we normalize the units in order to make $\mathcal{G}(m_1 + m_2) = 1$ and $a = 1$, where $\mathcal{G}$ is the gravitational constant and a is the semi-major axis of the ellipse described by the orbit of one primary orbiting around the other, with this second one at the focus of such ellipse. Thus, the distance between the primaries is

$$r = \frac{1 - e^2}{1 + e\cos f} = \frac{h^2}{1 + e\cos f},$$

being f the true anomaly[1] and h the angular momentum.

With these dimensionless units, we define the mass parameter

$$\mu = \frac{m_2}{m_1 + m_2}$$

and the dimensionless masses of the primaries

$$\mu_1 = \mathcal{G}m_1 = 1 - \mu, \ \ \mu_2 = \mathcal{G}m_2 = \mu$$

We can assume, without loss of generality that the primaries satisfy $m_1 \geq m_2$, so we have $0 < \mu \leq 0.5$.

Let us consider now an inertial reference frame (RF) $(\mathcal{O}, X, Y, Z)$, where $\mathcal{O}$ is the mass centre of the primaries, (X, Y) is their orbital plane, and Z is the axis perpendicular to this plane. We call this RF sidereal and the equations of the motion of the third body are

$$\ddot{X} = -\frac{\mu_1(X - X_1)}{r_1^3} - \frac{\mu_2(X - X_2)}{r_2^3},$$
$$\ddot{Y} = -\frac{\mu_1(Y - Y_1)}{r_1^3} - \frac{\mu_2(Y - Y_2)}{r_2^3},$$
$$\ddot{Z} = -\frac{\mu_1(Z - Z_1)}{r_1^3} - \frac{\mu_2(Z - Z_2)}{r_2^3},$$

where

$$r_i^2 = (X - X_i)^2 + (Y - Y_i)^2 + (Z - Z_i)^2$$

and

$$(X_1, Y_1, Z_1) = (\mu_2 r\cos f, \mu_2 r\sin f, 0),$$
$$(X_2, Y_2, Z_2) = (-\mu_1 r\cos f, -\mu_1 r\sin f, 0)$$

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[1] The true anomaly is the angle sweeped counting from the angle of pericenter.

are the coordinates of the primaries. We assume that the motion of the third body also takes place in the orbital plane ($Z = 0$), so the third equation, and actually the third component is no longer needed.

B. Synodical reference frame

We move now to the synodical RF $(\mathcal{O}, x, y)$, which is obtained by computing the following coordinate changes.

1. First we fix the primaries to the same axis by making a rotation of angle f at each time.

2. Then we fix the distance between the primaries by dividing the coordinates by r. Thus, the primaries get fixed in the positions

$$(x_1, y_1) = (\mu, 0),$$
$$(x_2, y_y) = (\mu - 1, 0).$$

3. Finally, we no longer consider the time as the independent variable, but the true anomaly, by using the change of variables

$$dt = \frac{r^2}{h} df.$$

In this RF, the equations of the motion are given by

$$x'' - 2y' = \frac{x - \dfrac{(1-\mu)(x-x_1)}{R_1^3} - \dfrac{\mu(x-x_2)}{R_2^3}}{1 + e\cos f},$$

$$y'' + 2x' = \frac{y - \dfrac{(1-\mu)y}{R_1^3} - \dfrac{\mu y}{R_2^3}}{1 + e\cos f},$$

where the symbol $'$ means derivative with respect to f and $R_i = rr_i$. These equations correspond to the following hamiltonian

$$\mathcal{H}(x, y, p_x, p_y) = \frac{1}{2}((p_x + y)^2 + (p_y - x)^2) - \frac{1}{1 + e\cos f}\left(\frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{R_1} + \frac{\mu}{R_2}\right)$$

III. THE SIMULATOR

The simulator has been programmed with the programming language `C` and using `OpenGL` and `FLTK` libraries. It consists of a window where the orbits are displayed in the central part.

In the right side of the window we can find several buttons that let us set the mass parameter μ, the eccentricity e, as well as the initial conditions of the problem, which are the initial true anomaly and angle of pericentre of the primaries, as well as the initial position and velocity of the third body. We can also find the buttons `S-J`, `S-E` and `E-M`, which automatically set the eccentricity and mass parameter of the Sun-Jupiter, Sun-Earth and Earth-Moon systems, respectively.

Finally, in the left side of the window we can find a set of controls of several physical magnitudes, that are refreshed as the motion takes place.

The computation of the orbit is made in the synodical RF, although it is displayed in the sidereal one. This means that the simulator integrates the motion equations in the synodical RF, with the regularized hamiltonian (see VI A) if the third body gets close to one of the primaries. Then it undoes the coordinate changes in order to display the orbit in the sidereal RF, which is physically easier to understand.

IV. SIMULATION RESULTS

In this section we are going to show some results obtained with the simulator, and possible utilities of them. Let us first introduce the three physical systems we have used: the Earth-Moon, Sun-Jupiter and Sun-Earth systems. The physical parameters of these systems are summarized in the TABLE IV

System P_1-P_2	m_1 ($\times 10^{24}$kg)	m_2 ($\times 10^{22}$kg)	e (adim)	T
E-M	$5,9742$	$7,3477$	$0,0549$	$27,32158$d
S-J	1.9891×10^6	$1,8987 \times 10^5$	$0,04839266$	$11,862615$y
S-E	1.9891×10^6	$597,42$	$0,01671022$	1y

TABLE I: Physical parametres of the Earth-Moon, Sun-Jupiter and Sun-Earth systems.

From the period T we can compute the semi-major axis a, using Kepler's third law, which states

$$T^2 = \frac{4\pi^2}{\mathcal{G}(m_1 + m_2)} a^3.$$

Thus, we have

$$a_{E-M} = 3,847 \times 10^5\text{km}\ , a_{S-J} = 7,779 \times 10^8\text{km},$$

$$a_{S-E} = 1,495 \times 10^8\text{km}.$$

As we mentioned former, we are using a system of units with dimensionless times and longitudes, in which we have taken $a = 1$ and $\mathcal{G}(m_1 + m_2) = 1$. This implies $T = 2\pi$, so, in order to get back the physical units, we have to multiply the longitudes by a, and the times by $\frac{T}{2\pi}$.

That being said, we can start now with some simulations.

A. Simulations in the E-M system

One possible utility of the simulator, when considering the Earth and the Moon as the primaries, is to compute the orbits of the spatial debris that lies between them, and see how the gravitational field caused by the Moon can affect these orbits.

With this in mind, we make a scattering of initial positions within the orbit of the Moon, and set an initial velocity such that the orbit is as circular as possible when considering the Moon has mass 0, and then we recompute the orbit setting the correct value of the mass, to see the deviation of the orbit.

In the figures FIG. 1 and FIG. 2 we can see an example of the orbit with and without a neglect of the Moon mass.

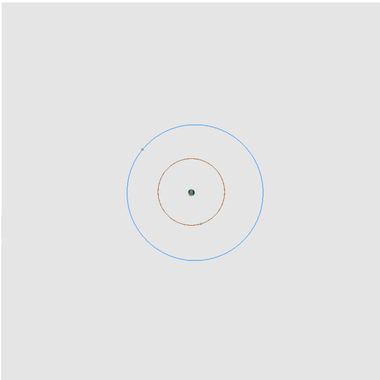

FIG. 1: Orbit of the third body neglecting the effect of the Moon.

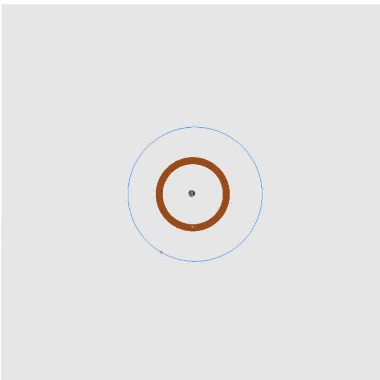

FIG. 2: Orbit of the third body considering the effect of the Moon.

Acting as explained, we obtain that for initial positions such that the distance with the mass center is smaller than 0.685 the orbit goes from being a circumference with radius R to being confined in a circular crown of radius $R \pm \delta R$; and for bigger initial distances the orbit becomes very perturbed, spinning around the Moon occasionally, and sometimes even being expulsed from the system.

In physical units, this means that low mass particles orbiting around the Earth, between the Earth and the Moon, at distances greater than 2.635×10^5 km, can have unpredictable orbits due to the gravitational action of the Moon. Those particles orbiting at smaller distances will keep on orbiting with a precession caused by the gravitational action of the Moon.

B. Simulations in the S-J system

The main application of the ER3BP taking the Sun and Jupiter as the primaries could be to make a first approximation of the SS itself, and computing, for example, the orbit of a comet that goes through it.

The aim of this kind of simulations is to predict whether the comet is going to remain in the SS or not.

Although Jupiter and the Sun are the two heavier bodies in the SS, this approximation is obviously not a very accurate one, as the SS is composed by many more bodies, and this simulation does not even take in account the possible collisions with other planets, satellites or even asteroids.

In order to do these simulations, we set the initial conditions relatively far from the primaries. Then, we see in which cases the orbit looks like an hyperbola or a parabola, and in which cases it has some kind of perturbation in the region of the primaries, or is even captured by one of them. Acting as we made with the E-M simulations, we use initial velocities such that when the mass of Jupiter is neglected the orbit is indeed a parabola or an hyperbola.

Doing these computations, we oserve that the orbits are significantly perturbed only if they get very close to Jupiter.

For initial positions at a distance from the mass center greater than a_{S-J}, the chances of getting close to Jupiter are very small. However, for initial positions with distances than a_{S-J}, the chances are very high.

C. Simulations in the S-E system

The mass of the Earth is so small compared with the mass of the Sun that it is almost negligible, so the orbits computed in the Sun-Earth system are almost Keplerian orbits, unless the third body is very close to the Earth.

Thus, a possible application of the simulator for the Sun-Earth system is to calculate the radius of the sphere of influence of the Earth.

For that, we compute the orbit of the third body with initial positions closer and closer to the Earth until the resulting orbit at least has a visible precession.

We have obtained that the gravitational effect of the Earth can only precessionate a little bit the orbits of the third body when it is at a distance a_{S-E} from the Sun. This means it is almost impossible to compute an orbit like the one of the Moon with this simulator.

D. Simulations in binary star systems

Although the project is supposed to make simulations in the SS, we found it interesting to show that the simulator is not restricted to that.

When taking values of the mass parameter near 0.5, the simulator suits very well in computing the orbits of planets belonging to a binary star system. With these values of μ we can find very interesting orbits.

FIG. 3, FIG. 4 and FIG. 5 are some examples of orbits obtained with $\mu = 0.5$

FIG. 3: In this picture the third body started orbiting around one primary, then the other one captured it and after that it started circling both of them. A few revolutions later it was expelled of the system.

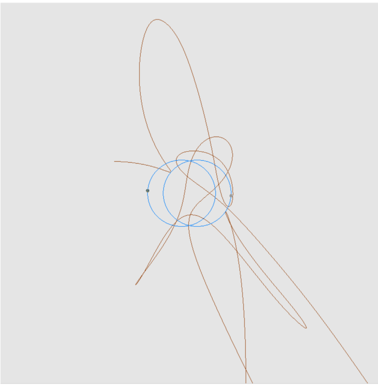

FIG. 4: In this picture we can observe a quite random orbit.

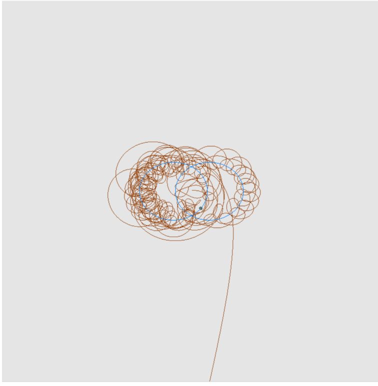

FIG. 5: In this picture the third body was getting captured by a different primary every few revolutions until it finally got expelled of the system.

V. CONCLUSIONS

Let us start saying that the study of the ER3BP via the simulator has been a qualitative study. In order to improve the obtained results, a numerical study could be done. One way to make an exhaustive numerical study is to modify the simulator so that it automatically computes several magnitudes, such as the mean distance with the mass center and its standard deviation.

We could also modify the simulator to directly introduce the parameters in the physical units. That means introducing the masses of the primaries in kg and the value of the semi-major axis in m or km. Thus, the simulator could also compute the physical magnitudes in the right units.

Referring to the simulations we have made, we can conclude that the ER3BP is not a very good approximation for the Sun-Earth system, as the gravitational field due to the Earth is almost neglected even for very small distances with it. However, it is very well known that there are several bodies orbiting around the Earth, such as the Moon, artificial satellites and spatial debris.

We could try to make a better approximation for the SS, by improving the Sun-Jupiter system. We are interested in finding a way to consider the gravitational effect of all the planets and still be working with the R3BP. These could be done considering the Sun as the primary and Jupiter as the secondary, unless the third body got close to another planet. In that case we would then consider the mentioned planet as the secondary.

An orbits simulator that computed this way could be done modifying our simulator, although a lot of improvements and global controls should be done. For example, the dimensionless units and the orbital plane would vary every time we changed the secondary, among other parameters.

It would be interesting to improve the simulator this way, although we have already seen that the ER3BP does not approximate well for the Sun-Earth system, so we would probably have the same problem with the Sun-Mercury, Sun-Venus and Sun-Mars systems.

VI. APPENDIX

A. Levi-Civita regularization

In order to avoid the singularities of the hamiltonian, due to collisions with the primaries, and the hugeness in the errors it provokes in the computation, we use the Levi-Civita regularization.

The Levi-Civita regularization deals with the collision with each primary separately. Let $\mathcal{H}_k(x_k, y_k, p_{x_k}, p_{y_k})$ be the synodical hamiltonian with a translation that brings the origin to the primary P_k.

To regularize $\mathcal{H}_k$, we must first make a change of coordinates such that

$$x_k + iy_k = (\xi_k + i\eta_k)^2.$$

Then, we also change the time (which in this case has already been changed to the true anomaly) to a regularizing time τ as follows

$$\frac{df}{d\tau} = 4(\xi_k^2 + \eta_k^2).$$

The regularizing time makes the motion happen slowlier. After that, for each level $\overline{\mathcal{H}_k}$ of the hamiltonian, we define an extended hamiltonian $\mathscr{H}_k$ in this way

$$\mathscr{H}_k = \frac{df}{d\tau}(\mathcal{H}_k - \overline{\mathcal{H}_k}).$$

In this extended hamiltonian, f is treated as the conjugated momentum of the variable $\overline{\mathcal{H}_k}$.From the regularized hamiltonians, we obtain the regularized equations of motion.

For the collisions with the primary, these equations are

$$\begin{aligned}
\xi_1^* &= p_{\xi_1} + 2(\xi_1^2 + \eta_1^2 - \mu)\eta_1,\\
\eta_1^* &= p_{\eta_1} - 2(\xi_1^2 + \eta_1^2 + \mu)\xi_1,\\
p_{\xi_1}^* &= 4\xi_1\left(2\overline{\mathcal{H}_1} + \xi_1 p_{\eta_1} - \eta_1 p_{\xi_1}\right) + 2(\xi_1^2 + \eta_1^2 + \mu)p_{\eta_1}\\
&\quad + \frac{\mu}{1 + e\cos f}\frac{8\xi_1(1 + \xi_1^2 - 3\eta_1^2)}{((\xi_1^2 + \eta_1^2)^2 + 2(\xi_1^2 - \eta_1^2) + 1)^{\frac{3}{2}}}\\
&\quad - \frac{4\xi_1 e\cos f}{1 + e\cos f}(3(\xi_1^2 + \eta_1^2)^2 + \mu^2 + 4\mu\xi_1^2),\\
p_{\eta_1}^* &= 4\eta_1\left(2\overline{\mathcal{H}_1} + \xi_1 p_{\eta_1} - \eta_1 p_{\xi_1}\right) - 2(\xi_1^2 + \eta_1^2 - \mu)p_{\xi_1}\\
&\quad + \frac{\mu}{1 + e\cos f}\frac{8\eta_1(1 - \eta_1^2 + 3\xi_1^2)}{((\xi_1^2 + \eta_1^2)^2 + 2(\xi_1^2 - \eta_1^2) + 1)^{\frac{3}{2}}}\\
&\quad - \frac{4\eta_1 e\cos f}{1 + e\cos f}(3(\xi_1^2 + \eta_1^2)^2 + \mu^2 - 4\mu\eta_1^2)\\
f^* &= 4(\xi_1^2 + \eta_1^2),
\end{aligned}$$

$$\begin{aligned}
\frac{(1 + e\cos f)^2}{2e\sin f}\overline{\mathcal{H}_1}^* &= -\xi_1^2 + \eta_1^2)^3 - \mu^2(\xi_1^2 + \eta_1^2)\\
&\quad - 2\mu(\xi_1^4 - \eta_1^4) - 2(1 - \mu)\\
&\quad - \mu\frac{2(\xi_1^2 + \eta_1^2)}{\sqrt{(\xi_1^2 + \eta_1^2)^2 - 2(\xi_1^2 - \eta_1^2) + 1}};
\end{aligned}$$

and for the secondary,

$$\begin{aligned}
\xi_2^* &= p_{\xi_2} + 2(\xi_2^2 + \eta_2^2 + (1 - \mu))\eta_2,\\
\eta_2^* &= p_{\eta_2} - 2(\xi_2^2 + \eta_2^2 - (1 - \mu))\xi_2,\\
p_{\xi_2}^* &= 4\xi_2\left(2\overline{\mathcal{H}_2} + \xi_2 p_{\eta_2} - \eta_2 p_{\xi_2}\right) + 2(\xi_2^2 + \eta_2^2 + 1 - \mu)p_{\eta_2}\\
&\quad + \frac{1 - \mu}{1 + e\cos f}\frac{8\xi_2(1 + \xi_2^2 + 3\eta_2^2)}{((\xi_2^2 + \eta_2^2)^2 - 2(\xi_2^2 - \eta_2^2) + 1)^{\frac{3}{2}}}\\
&\quad - \frac{4\xi_2 e\cos f}{1 + e\cos f}(3(\xi_2^2 + \eta_2^2)^2 + (1 - \mu)^2 - 4(1 - \mu)\xi_2^2),\\
p_{\eta_2}^* &= 4\eta_2\left(2\overline{\mathcal{H}_2} + \xi_2 p_{\eta_2} - \eta_2 p_{\xi_2}\right) - 2(\xi_2^2 + \eta_2^2 - (1 - \mu))p_{\xi_2}\\
&\quad + \frac{1 - \mu}{1 + e\cos f}\frac{8\eta_2(1 + \eta_2^2 - 3\xi_2^2)}{((\xi_2^2 + \eta_2^2)^2 - 2(\xi_2^2 - \eta_2^2) + 1)^{\frac{3}{2}}}\\
&\quad - \frac{4\eta_2 e\cos f}{1 + e\cos f}(3(\xi_2^2 + \eta_2^2)^2 + (1 - \mu)^2 + 4(1 - \mu)\eta_2^2),\\
f^* &= 4(\xi_2^2 + \eta_2^2),
\end{aligned}$$

$$\begin{aligned}
\frac{(1 + e\cos f)^2}{2e\sin f}\overline{\mathcal{H}_2}^* &= -(\xi_2^2 + \eta_2^2)^3 - (1 - \mu)^2(\xi_2^2 + \eta_2^2)\\
&\quad + 2(1 - \mu)(\xi_2^4 - \eta_2^4) - 2\mu\\
&\quad - (1 - \mu)\frac{2(\xi_2^2 + \eta_2^2)}{\sqrt{(\xi_2^2 + \eta_2^2)^2 - 2(\xi_2^2 - \eta_2^2) + 1}}.
\end{aligned}$$

In the former equations, the symbol * means derivative with respect to τ.

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