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NONCOOPERATIVE GAME THEORY: GENERAL OVERVIEW AND ITS APPLICATION TO THE STUDY OF COOPERATION

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Chapter 1

Introduction

A game, in Game Theory, is a tool that can model any situation in which there are people that interact - taking decisions, making moves, etc - in order to attain a certain goal. This mathematical description of conflicts began in the twentieth century thanks to the work of John Von Neumann, Oskar Morgenstern and John Nash and one of its first motivations was to help military officers design optimal war strategies. Nowadays, however, Game Theory is applied to a wide range of disciplines, like Biology or Political Science, but above all, to Economy. Interestingly, eleven game-theorists have won the Economics Nobel Prize up to date but never has a Fields Medal been awarded to an expert in this field. This shows to what great extent Game Theory is important for Economy and at the same time how mathematicians regard it as a secondary discipline compared to other areas of Mathematics. This undergraduate thesis clearly falls under the category of applied mathematics or mathematical modeling and therefore its goal is far from just accurately proving a series of theorems. Instead, even if the foundations of Game Theory will be laid, I will focus on showing how Game Theory can be applied to solve a great number of different problems, like, for example, the emergence of cooperative dispositions towards strangers.

Bearing this in mind, I will begin this undergraduate thesis by analyzing a military conflict between two countries whose officials will have a symbolic name: Nash and Neumann. To so do, I warn the reader that I will informally explain and use certain results that will be accurately justified later in this thesis. Let's begin!

1.1 Motivation: A First Example

Suppose that a country A and a country B are at war and that the generals of each army are called, respectively, John Nash and John Von Neumann. Every single day, Nash will send a heavily armed bomber and a smaller support plane to attack B. To do so, he will put a bomb in one of the two aircrafts. At the same time, Neumann knows Nash's intentions and decides to embark on military action but judges unnecessary to attack the two planes, mainly for economic reasons. The bomber will survive 60% of the times it suffers an attack and, if it manages to live through the raid, it will always hit the target. The lighter plane, which is not as precise, hits the target 70% of the times and plus only survives Neumann's attack half of the times. There are clearly only four possible results, which come from the combination of Nash choice to put the bomb either in the bomber or in the support plane, and Neumann's call to attack one or the other aircraft. Nash feels that he gains when he hits the target and he does not care about suffering an attack on one of his planes. At the same time, Neumann desperately wants to protect his citizens and therefore he will lose utility when an attack is carried out. Nash's gain for every possible combination of strategies is:

	Bomber attacked	Support attacked
Bomb in bomber	$0.6 \times 1 = 0.6$	1
Bomb in support	0.7	$0,5 \times 0.7 = 0.35$

Table 1.1. Nash's Payoff

where we have assumed an increase in a unit of his utility comes from a successful attack. In an isolated realization of the game, the target can only either be or not be hit, so what do the numbers in the matrix represent? Clearly, the expected values of the outcome of the game when those strategies are employed. Given that this game is repeated every day, the Law of Large Numbers guarantees that the average outcome of the confrontation of two strategies (let's say for example "Bomb in bomber" against "Bomber attacked") will tend to its expected value (for the aforementioned strategies, 0.6). Therefore, the following will be a good long-term analysis.
Bear in mind that we should give the utility of both players, but here Neumann's utility has implicitly been given as it will be the matrix on top with a minus sign in front of all entries.

Remark.

1. Nash can make sure that at least 60% of the attacks are successful by always putting the bomb in the bomber, as we would only be dealing with the matrix' first row.

2. Neumann can make sure that no more than 70% of the attacks are successful by attacking the bomber plane, as we would only be dealing with the matrix' first column.

These facts could determine their strategies. If that was the case, the game would unfold as a constant 0.6 gain for Nash. However, instead of always putting the bomb in the bomber, Nash decides, every now and then, to bluff and to put the bomb in the support plane. How often should he do that? Which is the best percentage of success he can get? How should Neumann adapt to this change? Let's give an answer to all this questions.
Given that Nash will no longer stick to one of the strategies but combine them, he will start to employ a so-called mixed strategy. Let's use a probability distribution $X = (a, b)$

[1] to encapsulate this, where it should be understood that Nash will put the bomb in the bomber with a probability a and in the support plane with a probability b. Using this notation, a pure strategy is then $(1,0)$ -bomb always in the bomber- or $(0,1)$ -bomb always in the support plane. Let us denote by A the matrix of table 1.1. For any strategies $X \in \widehat{S}_1$ and $Y \in \widehat{S}_2$ ($\widehat{S}_i$ should be understood as the set of all possible mixed strategies of player i) we can calculate the expected payoff $\widehat{H}(X,Y)$ by ponderating the payoffs of the pure strategies, i.e., by calculating XAY^T. For any strategy Nash (Neumann) picks, the strategy that minimizes the other's payoff is called the optimal counterstrategy or the best reply.

Theorem 1.1. *If either Nash or Neumann employ a fixed strategy (and it means they will not keep changing the probabilities of how they distribute their choices), then the opponent's best reply is a pure strategy.*

Proof. Assume Nash's strategy is $X = (1-x, x)$ and Neumann's $Y = (1-y, y)$. The expected payoff is the averaged payoff of every situation. For the sake of generality, let's assume a general (a_{ij}) payoff 2×2-matrix for Nash. Thus, for $x \in [0,1]$

$$\widehat{H}(X,Y) = \widehat{H}((1-x,x),(1-y,y)) = (1-x)(1-y)a_{11} + (1-x)ya_{12} + x(1-y)a_{21} +$$
$$+xya_{22} = x(-a_{11} + a_{11}y + a_{21} - a_{12}y - a_{21}y - a_{22}y) + (a_{11} - a_{11}y + a_{21}y)$$

Given that y is fixed, the function $\widehat{H}(x)$ is a straight line, $\widehat{H}(x) = ax + b$, so obviously the maximum and minimum of the function will be attained at the borders (either where $x = 0$ or where $x = 1$). The same result clearly holds if x is fixed and we let y vary. □

1.1.1 Max-Min Strategy

Let Nash have a strategy $X = (1-x, x)$, where x is the probability he will put the bomb in the support plane. As it has just been seen, Neumann's best reply will either be the strategy $(1,0)$ or $(0,1)$, so let's focus only on these cases. We will therefore have two possible payoffs when Nash uses X.

$$r_1(x) = H(X,(1,0)) = (0.7-0.6)x + 0.6 = 0.1x + 0.6$$
$$r_2(x) = H(X,(0,1)) = -0.65x + 1$$

Neumann, in order to protect his citizens, clearly wants to minimize Nash's gain, and therefore between these two possible options, he will always prefer the smaller one, so in the end the payoff will be:

$$\widehat{H}(X,Y) = \widehat{H}(x) = \min\{r_1(x), r_2(x)\}$$

In figure 1.1 we can see the graph of $\widehat{H}(x)$. Since the intersection of $r_1(x)$ and $r_2(x)$ ($x^* = 0,533$ and $\widehat{H}(x^*) = 0,653$) is the higher point of the graph $\widehat{H}(X,Y)$, it is Nash's

[1] Throughout the whole senior thesis the following convention will be used: A vector will be a row-vector and a transposed vector will be a column vector.

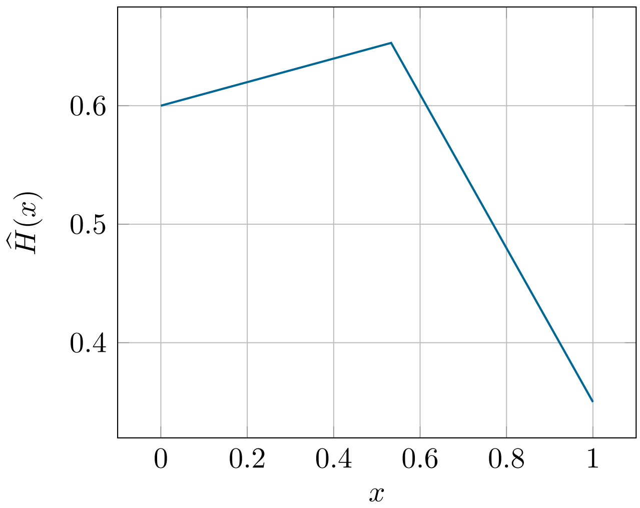

Figure 1.1: Graph of $\min\{H(X,(1,0)),\ H(X,(0,1))\}$

wisest choice. Nash will maximize the function $\min\{r_1(x), r_2(x)\}$, thus he will maximize Neumann's minimal return, and that's why we call it the max-min strategy. Nash then will employ the strategy $X^* = (0.47, 0.53)$ and succeed, at least, 65.3% of the times. Nevertheless, if he does not adhere to these recommendations, it is clear that:

- for $x \leq 0.533$ (when the bomb is in the bombarder more than 53.3% of the time) Neumann should always attack the bombarder.
- for $x \geq 0.533$ (when the bomb is in support plane more than 46.7% of the times) Neumann should attack the support plane.

1.1.2 Min-Max Strategy

Let's focus now on Neumann and let's think of a good strategy for him. Again, for any strategy $Y = (1-y, y)$ that he picks (where y represents the probability of attacking the support plane), Nash's best reply will be either $(0,1)$ or $(1,0)$. As before, we can calculate the expected payoff for these cases:

$$c_1(y) = H((1,0),Y) = (1-0,6)y + 0.6 = 0.4y + 0.6$$
$$c_2(y) = H((0,1),Y) = -0.35y + 0.6$$

Nash will always choose the strategy that yields the greatest payoff, so now, $\widehat{H}(X,Y) = \widehat{H}(y) = \max\{c_1(y), c_2(y)\}$, which is graphed in figure 1.2.
Neumann clearly wants to minimize $\widehat{H}(y) = \max\{c_1(y), c_2(y)\}$, which is attained at $y^* = \frac{4}{30}$ and yields $\widehat{H}(y^*) = 0.653$. This is called a min-max strategy because Neumann is minimizing Nash's maximum payoff. Neumann will then employ the strategy $Y^* = (0.87, 0.13)$ and the attack will succeed no more than 65.3% of the times. If he does not adhere to these recommendations, it is clear that:

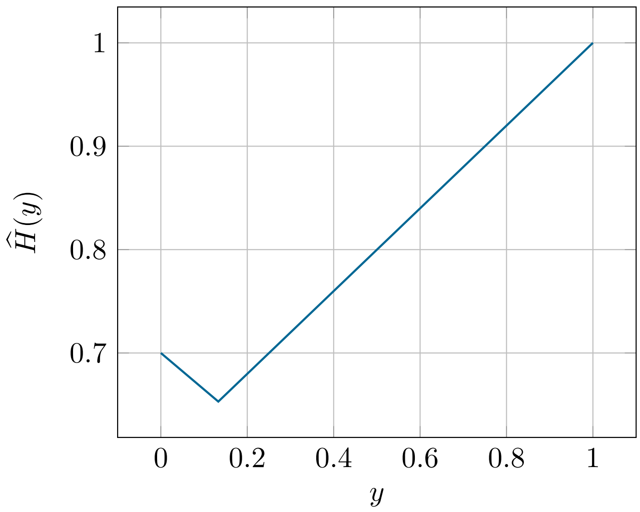

Figure 1.2: Graph of $\max\{H((1,0),Y),\ H((0,1),Y)\}$

- for $y \leq 0.133$ (when Neumann attacks the bombarder more than 86.6% of the times) Nash should place the bomb in the small plane.
- for $y \geq 0.133$ (when Neumann attacks the bombarder less than 86.6% of the times) Nash should put the bomb in the bombarder.

1.1.3 Solution of the game

At the beginning, we said that Nash could guarantee an attack efficiency of 60% and Neumann could make sure the attack success rate didn't exceed 70%. Their guarantees were different. However, if we allow mixed strategies, the guarantees do coincide! This is a central theorem in Game Theory that we will prove in this thesis. When Nash employs his min-max strategy and Neumann his max-min strategy [2], we will see a 65.3% success rate of Nash's attacks. This will be called the value of the game. As we will see, this game is solved as we can give:

$$\begin{aligned}&\text{min-max strategy } X^* = (0.47, 0.53)\\&\text{max-min strategy } Y^* = (0.87, 0.13)\\&\text{value of the game } v = \widehat{H}(X^*, Y^*) = 0.653\end{aligned}$$

Remark. The reader should now wonder: doesn't this contradict Theorem 1.1? It was stated and proven that for any fixed strategy that your opponent picked, the optimal counterstrategy was a pure strategy. When player 1 picks X^*, which is a fixed strategy, why do we suggest player 2 pick Y^*, which is not a pure strategy? The answer is that it

[2] To lighten the notation, we will most of the times refer to these strategies as the max-min strategies.

can be readily checked that:

$$\widehat{H}(X, Y^*) = v \qquad \forall X \in \widehat{S}_1$$
$$\widehat{H}(X^*, Y) = v \qquad \forall Y \in \widehat{S}_2$$

and given this remarkable property, it is convenient for every player to pick a max-min strategy because they guarantee certain results that other strategies fail to assure.

Chapter 2

Definition of Game and First Properties

2.1 What are games?

As said in chapter 1, a game is a tool that can model any situation in which there are people that interact - taking decisions, making moves, etc - in order to attain a certain goal. In this undergraduate thesis, we will focus on noncooperative games, which model those situations in which all players want to maximize their own profit and don't cooperate between each other. We will always assume that the players are rational, i.e., that they know what's best for them and can think ahead of the game no matter how complex that is. We will also assume that the number of players is finite and we will assign a number to each player. Let $I = \{1, 2, ..., N\}$ be the set of all players and let $i \in I$ mean player i. Each of these players has available a set S_i of strategies. Throughout this senior thesis, all players will choose their strategies simultaneously and independently. A game consists in every player choosing a strategy $s_i \in S_i$, thus creating a situation $s = (s_1, ..., s_N)$ which will translate in a certain outcome. The set of all possible situations is clearly $S = S_1 \times ... \times S_N$.
Every player $i \in I$ has preferences over the outcome of the game, which derives directly from the situation. First, we would like to mathematize these predilections. Let's assume the preferences of player i are given by the binary relation $\succcurlyeq \in S \times S$. The expression $s^1 \succcurlyeq s^2$ should be understood as player i either prefers s^1 over s^2 or in indifferent about it. To move from this abstract preference relation to a numeric expression we will use a utility function.

Definition. A utility function representing $\succcurlyeq \in S \times S$ is a function $H : S \longrightarrow \mathbb{R}$ s.t. $\forall a, b \in S$,

$$a \succcurlyeq b \iff H(a) \geq H(b)$$

To define the payoff of player i with a preference relation $\succcurlyeq \in S \times S$ we will use a utility function $H_i : S \longrightarrow \mathbb{R}$ that sends every situation s, to $H_i(s)$, which will denote the payoff that player i gets when s arises.

Under what conditions can a preference relation be modeled by a utility function?

Theorem 2.1. *Let S be a countable set and $\succcurlyeq$ a preference relation over $S \times S$. Then there exists a utility function representing $\succcurlyeq$.*

Proof. Given that S is a countable set, we can write $S = \{s_1, s_2, ...\}$. Let's define $\forall i, j \in \mathbb{N}$

$$h_{ij} = \begin{cases} 1 & s_i, s_j \in S \text{ and } s_i \succcurlyeq s_j \\ \\ 0 & \text{otherwise} \end{cases}$$

We can define the utility function H as:

$$H(s_i) = \sum_{j=1}^{\infty} \frac{1}{2^j} h_{ij} \leq \infty$$

The transitivity of the preference relation (a preference relation is a binary relation and these are always transitive) guarantees that, for this definition, if $s_1 \succcurlyeq s_2 \iff H(s_1) \geq H(s_2)$. □

This theorem will help us in most games, but let's give, without a proof, the most general result:

Theorem 2.2. *Let $\succcurlyeq$ be a preference relation over $S \times S$. Then $\succcurlyeq$ can be represented by a utility function iff there is a countable set $A \in S$ that is order dense in S, i.e., that $\forall s_1, s_2 \in S$ there exists $a \in A$ s.t. $s_1 \succcurlyeq a \succcurlyeq s_2$.*

To define a noncooperative game we need:

- A set of players I
- A set of available strategies to every player ($\forall i \in I \; \exists \, S_i = \{\text{set of strategies of player i}\}$)
- The payoff functions of every player ($\forall i \in I \; \exists \, H_i : S \longrightarrow \mathbb{R}$ that gives the payoff of player i when s is played)

Definition. In a compact way, a noncooperative game Γ is thus defined as:

$$\Gamma = \langle I, \{S_i\}_{i \in I}, \{H_i\}_{i \in I} \rangle$$

2.2 Basic Concepts

Intuitively, a situation is admissible for player i if, when deviating unilaterally from it, he decreases his own payoff. If we have a certain situation $s = (s_1, ..., s_N)$ and i changes his strategy from s_i to some other $s'_i \in S_i$ we will be left with a new situation $(s_1, ..., s_{i-1}, s'_i, s_{i+1}, ..., s_N)$, which, in an attempt to simplify the notation, will be shortened to $s||s'_i$. In mathematical terms, a situation $s \in S$ is admissible for player i if:

$$H_i(s||s'_i) \leq H_i(s) \qquad \forall s'_i \in S_i$$

At the same time, a situation $s^* \in S$ is admissible for everyone if:

$$H_i(s^*||s_i') \leq H_i(s^*) \qquad \forall s_i' \in S_i \qquad \forall i \in I$$

This is either called a Nash equilibrium or an equilibrium situation. If adopted, nobody would want to change his strategy and if the game was repeated, everybody would stick to the same strategy again. In most cases, to reach an equilibrium situation will be referred as to solve the game. As Game Theory is crucial in Economy, let's start with a typical example: the study of a duopoly.

Example. Let $i \in I$ be the manufacturer of a certain good and let $\#I = N$ be the number of manufacturers. Each of them must choose a certain strategy $s_i \in S_i = [0, \infty)$, which denotes the number of units made and put up for sale. Let $c_i(s_i)$ be the cost every producer has to face when manufacturing s_i units of this good. Let's set $c_i(s_i) = \{\text{Cost per unit}\} \times \{\text{number of units manufactured}\} = cs_i$. It is a good approximation to suppose that the price π of every unit, according to the supply and demand model, depends on the number of units that are up for sale, so $\pi\left(\sum_{i\in I} s_i\right)$ We have the set I of players, the strategies S_i of every player and, if we define the payoff of every player when choosing a certain strategy, we will have a game. Let:

$$H_i(s) = \{\text{Total income}\} - \{\text{Total cost}\} = \underbrace{\overbrace{\pi\left(\sum_{i\in I} s_i\right)}^{\text{Unit price}} \overbrace{s_i}^{\#\text{ units}}}_{\text{Total income}} - \underbrace{c_i s_i}_{\text{Total cost}}$$

where it was assumed all manufactured units were sold. Let's find now an equilibrium situation for the case $N = 2$. Let d be a number that accounts for the price of a unit in a noncompetitive market. Let $d > c$ and let's assume that the function π has the form:

$$\pi(s_1, s_2) = \begin{cases} d - (s_1 + s_2) & s_1 + s_2 < d \\ \\ 0 & \text{otherwise} \end{cases}$$

The payoff, using the expressions $H_i(s)$ and $\pi(s_1, s_2)$ is:

$$H_i(s_i) = \begin{cases} s_i(d - s_1 - s_2 - c) & s_1 + s_2 < d \\ \\ -s_i c & \text{otherwise} \end{cases}$$

A pair $(s_1^*, s_2^*) \in S$ is a stable equilibrium if $H_1(s_1^*, s_2^*) \geq H_1(s_1^{'}, s_2^*)\ \forall s_1^{'} \in S_1$ and $H_2(s_1^*, s_2^*) \geq H_2(s_1^*, s_2^{'})\ \forall s_2^{'} \in S_2$. Let's focus on the interesting case, namely $s_1 + s_2 < d$ and so $H_i(s) = s_i(d - s_1 - s_2 - c)$. We look for maximums of the functions H_i, so we will procede as we all know:

$$\frac{\partial H_1}{\partial s_1}(s) = 0 \iff -2s_1 + d - s_2 - c = 0 \iff s_1 = \frac{d - s_2 - c}{2}$$

At the same time, from $\frac{\partial H_2}{\partial s_2}(s) = 0$ we deduce that: $s_2 = \frac{d-s_1-c}{2}$. It is immediate to check that $\frac{\partial^2 H_1}{\partial s_1^2} = \frac{\partial^2 H_2}{\partial s_2^2} = -2$, so we have found maximums. Let's explain a bit what we found. For any strategy $s_1 \in S_1$ that player 1 picks, the greatest payoff that player 2 can get will be attained when playing $s_2 = \frac{d-s_1-c}{2}$. At the same time, for any strategy $s_2 \in S_2$ that player 2 picks, the greatest playoff that player 1 can get will be attained when playing $s_1 = \frac{d-s_2-c}{2}$. These are clearly the best replies to any strategy of the opponent. Thus, to find an equilibrium situation we only need to solve:

$$\begin{cases} s_1^* = \frac{d-s_2^*-c}{2} \\ \\ s_2^* = \frac{d-s_1^*-c}{2} \end{cases}$$

which gives:

$$s^* = \left(\frac{d-c}{3}, \frac{d-c}{3}\right) \qquad H_i(s^*) = \frac{(d-c)^2}{9}, \qquad \forall i$$

and therefore we have solved the game.

All strategic games can be classified and divided into classes, which will facilite their study.

Definition. Two games with the same players and strategies:

$$\Gamma = \langle I, \{S_i\}_{i\in I}, \{H_i\}_{i\in I}\rangle \qquad \Gamma^{'} = \langle I, \{S_i\}_{i\in I}, \{H_i^{'}\}_{i\in I}\rangle$$

are strategically equivalent (and will be written $\Gamma \sim \Gamma^{'}$) if $\exists k > 0$ and $c_i \in \mathbb{R}$ s.t. $H_i(s) = kH_i^{'}(s) + c_i$

Theorem 2.3. *The strategically equivalence relation is an equivalence relation*

Proof. Let's start

- Reflexive: Just set $k = 1$ and $c_i = 0 \,\forall i$ and the result follows.
- Symmetric: If $H_i(s) = kH_i^{'}(s) + c_i$, then clearly:

$$H_i^{'} = \frac{1}{k}H_i - \frac{c_i}{k} = \hat{k}H_i + \hat{c}_i \qquad \hat{k} = \frac{1}{k} > 0 \qquad \hat{c}_i = -\frac{c_i}{k} \in \mathbb{R}$$

- Transitive: We have that $\Gamma \sim \Gamma^{'}$ and $\Gamma^{'} \sim \Gamma^{''}$, so:

$$H_i(s) = kH_i^{'}(s) + c_i \qquad \& \qquad H_i^{'}(s) = k'H_i^{''}(s) + c_i^{'}$$

  It readily follows that:

$$H_i(s) = kH_i^{'}(s) + c_i = k(k'H_i^{''}(s) + c_i^{'}) + c_i = kk'H_i^{''}(s) + (k'c_i + c_i^{'}) = \hat{k}H_i^{''}(s) + \hat{c}_i$$

$$\hat{k} = kk' > 0 \qquad \hat{c}_i = k'c_i + c_i^{'} \in \mathbb{R}$$

□

Theorem 2.4. *Strategic Equivalent games have the same equilibrium situations*

Proof. Let $\Gamma \sim \Gamma'$ and let s^* be an equilibrium situation in Γ. By definition,

$$H(s^*||s_i) \leq H(s^*) \qquad \forall s_i \in S_i \qquad \forall i \in I$$

Using that $\Gamma \sim \Gamma'$, i.e., that $H_i(s) = kH_i'(s) + c_i$, we conclude that:

$$kH_i'(s^*||s_i) + c_i \leq kH_i'(s^*) + c_i \overset{k>0}{\Longrightarrow} H_i'(s^*||s_i) \leq H_i'(s^*) \qquad \forall s_i \in S_i \qquad \forall i \in I$$

Given that $I = I'$ (same players) and $S_i = S_i'\ \forall i \in I$ (players have same strategies) s^* is an equilibrium situation in game Γ'. $\square$

Definition. A non cooperative game $\Gamma = \langle I, \{S_i\}_{i\in I}, \{H_i\}_{i\in I}\rangle$ is a constant-sum game if:

$$\sum_{i\in I} H_i(s) = c \qquad \forall s \in S$$

In particular, a noncooperative game $\Gamma = \langle I, \{S_i\}_{i\in I}, \{H_i\}_{i\in I}\rangle$ is a zero-sum game if

$$\sum_{i\in I} H_i(s) = 0 \qquad \forall s \in S$$

Theorem 2.5. *All noncooperative constant-sum games are strategically equivalent to a certain zero-sum game.*

Proof. Consider a constant-sum game Γ, for which $\sum_{i\in I} H_i(s) = c\ \forall s \in S$. Pick some $\{c_i\}_{i\in I}$ s.t. $\sum_{i\in I} c_i = c$. Let's consider now a strategically equivalent game with $H_i'(s) = H_i(s) - c_i$. It readily follows that

$$\Gamma' = \langle I, \{S_i\}_{i\in I}, \{H_i'\}_{i\in I}\rangle$$

is a zero-sum game strategically equivalent to Γ. $\square$

2.3 Antagonistic Games

Definition. An antagonistic game is a two-player zero-sum game.

These types of games are called antagonistic because everything that player 1 gains results from the loss of player 2. Clearly $H_2(s) = -H_1(s)$ and therefore when dealing with antagonistic games we only need to give the payoff function of one of the players. For this reason, and bearing in mind that $I = \{1, 2\}$, antagonistic games are expressed as:

$$\Gamma = \langle S_1, S_2, H_1\rangle$$

2.3.1 Equilibrium situations in antagonistic games

It is time to talk about equilibrium situations in antagonistic games. As was previously said, we have an equilibrium situation when no player increases his payoff when unilaterally deviating from the situation. Let $\Gamma = \langle S_1, S_2, H_1 \rangle$ be an antagonistic game and $s^* = (s_1^*, s_2^*)$ an equilibrium situation. For player 1, this means $H_1(s_1^*, s_2^*) \geq H_1(s_1, s_2^*)$, $\forall s_1 \in S_1$, and for player 2: $H_2(s_1^*, s_2^*) \geq H_2(s_1^*, s_2)$, $\forall s_2 \in S_2$. Recalling that $H_1 = -H_2$, we can transform the condition for player two into $H_1(s_1^*, s_2^*) \leq H_1(s_1^*, s_2)$, $\forall s_2 \in S_2$. Putting together both inequalities we readily conclude that, if $s^* = (s_1^*, s_2^*)$ is an equilibrium situation, it follows that:

$$H_1(s_1, s_2^*) \leq H_1(s_1^*, s_2^*) \leq H_1(s_1^*, s_2) \qquad s_1 \in S_1 \qquad s_2 \in S_2$$

Recall that (s_1^*, s_2^*) are fixed values here and the variables are s_1 and s_2. Let's try to understand this expression. For s_2 fixed and $s_2 = s_2^*$, the s_1-function $H_1(s_1, s_2^*)$ attains its absolute maximum at s_1^*, according to the first inequality. At the same time, according to the last inequality, if we set $s_1 = s_1^*$, the s_2-function $H_1(s_1^*, s_2)$ attains its absolute minimum at s_2^*. The function $H_1(s_1, s_2)$ therefore has a saddle point in (s_1^*, s_2^*). We deduce:

$$(s_1^*, s_2^*) \text{ is a Nash eq. of } \Gamma = \langle S_1, S_2, H_1 \rangle \iff (s_1^*, s_2^*) \text{ is a saddle point of } H_1(s_1, s_2) \tag{2.1}$$

Remark. In Game Theory, a saddle point is not exactly the same that in Analysis. Typically, the directions in which the function increases and decreases are irrelevant. For us, however, the first variable needs to attain a maximum and the second a minimum, and the inverse option is not acceptable. At the same time, the definition of a saddle point in analysis involves partial derivatives and the Hessian matrix. For us, this regularity is not required and a saddle point can be perfectly defined at the boundary.

Chapter 3

General Results Concerning Equilibrium Situations

3.1 Matrix Games

Definition. A Matrix Game is an antagonistic game where the number of available strategies for every player is finite. In mathematical terms, it's a certain Γ:

$$\Gamma = \langle S_1, S_2, H_1 \rangle \ \ s.t. \ \#S_i < \infty \ \ i = 1, 2$$

Given that $\#S_i < \infty$, we can now numerate the strategies of every player and speak of strategy j of player i, i.e. $j \in S_i$. Let's think of a matrix (a_{ij}) in which each row represents a strategy of player 1 and every column a strategy of player 2. Consequently, each cell corresponds to a situation. Let us write in such cell the payoff of player 1 for such situation, i.e. $a_{ij} = H_1(i, j)$, where $i \in S_1$ and $j \in S_2$. To obtain the payoff of player 2 we only need to recall that for antagonistic games $H_2 = -H_1$. We therefore have a description of the game in the form of $\#S_1 \times \#S_2$-matrix (The matrix of the game) and that's why we speak of matrix games. For what we have seen for antagonistic games, a situation (i^*, j^*) will be an equilibrium situation (or saddle point) of a matrix game $A = (a_{ij})$ if and only if $a_{ij^*} \le a_{i^*j^*} \le a_{i^*j} \ \forall i \ \forall j$. Let us derive now methods to determine whether generic functions have saddle points.

3.2 Requirements for the existence of saddle points

Theorem 3.1. *For any function $f(x, y)$ defined in a certain set the following inequality holds:*

$$\sup_x \inf_y f(x, y) \le \inf_y \sup_x f(x, y)$$

Proof. Obviously $f(x, y) \le \sup_x f(x, y) \ \forall y$. For any function $f(x, y) \le g(y) \ \forall y$, then $\inf_y f(x, y) \le \inf_y g(y)$, so it follows that:

$$\inf_y f(x, y) \le \inf_y \sup_x f(x, y)$$

The $\inf_y \sup_x f(x,y)$ of any function is a number, so the inequality on top of this line could be summarized as $\inf_y f(x,y) \leq C$. For a function s.t. $f(x) \leq C$, using the definition of the supremum, one can assert that $\sup_x f(x) \leq C$, so:

$$\sup_x \inf_y f(x,y) \leq C = \inf_y \sup_x f(x,y)$$

and the result follows. $\square$

Theorem 3.2. *A necessary and sufficient condition for a function $f(x,y)$ to have saddle points (and consequently equilibrium situations if $f(x,y)$ expresses the payoff of one player in an antagonistic game) is the existence of $\max_x \inf_y f(x,y)$ and $\min_y \sup_x f(x,y)$ and the satisfaction of the equality:*

$$\max_x \inf_y f(x,y) = \min_y \sup_x f(x,y)$$

Proof. Let (x^*, y^*) be a saddle point. Clearly then:

$$f(x,y^*) \leq \underbrace{f(x^*,y^*)}_{constant} \leq f(x^*,y)$$

Proceeding as in the previous proof, it follows that $\sup_x f(x,y^*) \leq f(x^*,y^*)$. The quantity $\sup_x f(x,y^*)$ is clearly a value, so trivially, if we let y vary and chose the infimum:

$$\inf_y \sup_x f(x,y) \leq \sup_x f(x,y^*) \leq f(x^*,y^*) \tag{3.1}$$

Proceeding similarly, given that if $k \leq f(y)\ \forall y$ then $k \leq \inf_y f(y)$ it follows that $f(x^*,y^*) \leq \inf_y f(x^*,y)$ and then

$$f(x^*,y^*) \leq \inf_y f(x^*,y) \leq \sup_x \inf_y f(x,y) \tag{3.2}$$

and using the two previous equations:

$$\inf_y \sup_x f(x,y) \leq \sup_x \inf_y f(x,y)$$

According to the previous theorem, however: $\inf_y \sup_x f(x,y) \geq \sup_x \inf_y f(x,y)$, so the only way both inequalities can hold is if:

$$\inf_y \sup_x f(x,y) = \sup_x \inf_y f(x,y)$$

This means that all the above inequalities are equalities, in particular $\inf_y \sup_x f(x,y) = \sup_x f(x,y^*)$ which means that the infimum is attained. At the same time the expression $\inf_y f(x^*,y) = \sup_x \inf_y f(x,y)$ tells us the supremum is also attained. We could thus write:

$$\min_y \sup_x f(x,y) = \max_x \inf_y f(x,y)$$

and we have proved that if (x^*, y^*) is a saddle point of function $f(x, y)$, then the above equality is satisfied.
Let's prove the converse result. Let $\max_x \inf_y f(x, y)$ and $\min_y \sup_x f(x, y)$ exists and be equal to each other. Let x^* and y^* be the points where the extrema of this expressions are attained. By this we mean that:

$$\max_x \inf_y f(x,y) = \inf_y f(x^*, y) \qquad \& \qquad \min_y \sup_x f(x,y) = \sup_x f(x, y^*)$$

By definition, $\inf f(y) \leq f(y^*)$, so $\inf_y f(x^*, y) \leq f(x^*, y^*)$ and then:

$$\max_x \inf_y f(x,y) = \inf_y f(x^*, y) \leq f(x^*, y^*)$$

Analogously we could prove that: $f(x^*, y^*) \leq \sup_x f(x, y^*) = \min_y \sup_x f(x, y)$. As the the minimaxes must be equal, again all inequalities are in fact equalities. In particular:

$$\begin{aligned} \sup_x f(x, y^*) = f(x^*, y^*) \quad &\Longrightarrow \quad f(x, y^*) \leq f(x^*, y^*) \qquad \forall x \\ \inf_y f(x^*, y) = f(x^*, y^*) \quad &\Longrightarrow \quad f(x^*, y^*) \leq f(x^*, y) \qquad \forall y \end{aligned}$$

And it follows that (x^*, y^*) is a saddle point. □

Remark. It must be stressed that throughout this demonstration, a series of useful results were proven. First, given that in the end all the inequalities turned out to be equalities, it is deduced from (3.1) and (3.2) that:

$$\min_y \sup_x f(x,y) = \max_x \inf_y f(x,y) = f(x^*, y^*)$$

and therefore the value of the function at the saddle point is the same as the obtained by computing the expressions $\min_y \sup_x f(x, y)$ or $\max_x \inf_y f(x, y)$. What's more, this shows that we can determine the saddle points treating the variables independently, first maximizing one and the minimizing the other, or the other way round. It therefore follows that if (x_1^*, y_1^*) and (x_2^*, y_2^*) are saddle points, then (x_1^*, y_2^*) and (x_2^*, y_1^*) are also saddle points. The same property then obviously holds for the Nash equilibriums.

Definition. Let $\Gamma = \langle S_1, S_2, H_1 \rangle$ be a matrix game with matrix $A = (a_{ij})$. Then:

$$v_1(A) = \max_i \min_j a_{ij} \qquad \& \qquad v_2(A) = \min_j \max_i a_{ij}$$

Remark. Let $A = (a_{ij})$ be the matrix of a certain game. For what we have seen in Theorem 3.2, such a game has equilibrium situations if, and only if, $v_1(A) = v_2(A)$, where we have used that for matrix games, which let's recall that have a finite number of strategies, the supremum and infimum of each row or column are trivially attained.

Player 1 should think as follows: Assume I chose strategy $i \in S_1$. In the worst case scenario, I will be left with $\min_j a_{ij}$. Then I shall chose a strategy that guarantees me at least $v_1(A)$. Player 2 should think analogously: If I chose strategy $j \in S_2$, what I will lose in the worst case scenario is $\max_i a_{ij}$. Therefore, the intelligent thing is to chose the strategy that minimizes this and yields $v_2(A)$. We will call this optimal strategies the max-min strategies.

3.2.1 Examples

1. Assume we have a game with matrix:

$$\begin{array}{c} \\ 1_1 \\ 1_2 \\ 1_3 \end{array} \begin{array}{c} \begin{array}{ccc} 2_1 & 2_2 & 2_3 \end{array} \\ \begin{pmatrix} 5 & 1 & 4 \\ 3 & 2 & 4 \\ -4 & 0 & 2 \end{pmatrix} \end{array}$$

where I_j denotes the strategy number j of player $I = 1, 2$. Here $\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = 2$, so it has equilibrium situations. When picking a strategy, player 1, in the worst case scenario, will get the following payoff:

$$\begin{array}{c} 1_1 \\ 1_2 \\ 1_3 \end{array} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$$

so it would be advisable for him to pick 1_2. When picking a strategy, player 2, in the worst case scenario will lose the following payoff:

$$\begin{array}{c} \begin{array}{ccc} 2_1 & 2_2 & 2_3 \end{array} \\ \begin{pmatrix} 5 & 2 & 4 \end{pmatrix} \end{array}$$

so it would be advisable for him to pick 2_2. The game is perfectly determined. Even if the players knew the strategy the opponent would employ, they wouldn't be able to do any better than this.

2. Assume we have a game with matrix:

$$\begin{array}{c} \\ 1_1 \\ 1_2 \end{array} \begin{array}{c} \begin{array}{cc} 2_1 & 2_2 \end{array} \\ \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix} \end{array}$$

Now $\max_i \min_j a_{ij} = 2 \neq 3 = \min_j \max_i a_{ij}$, so no equilibrium situation exists. If we proceed like in the previous game we see that player 1 could guarantee a minimal gain of 2 if he choses 1_1 and player 2 could guarantee a maximal loss of 3 when choosing 2_2. However, the game is not strictly determined because if player 1 knew that player 2 was picking 2_2, he would pick 1_2, and also would player 2 change his strategy is he knew his opponent's intentions.

3.2.2 How should we play against an irrational player?

All the theory we have developed is applicable when both players are rational, but I believe this is not a really good approximation, so let's momentarily change this assumption. If we stopped someone by in the street and we convinced him to play the game of example

1 (as player 2) against us, chances are he would not reason as we explained before. How should we play against such an individual? Should we opt for 1_2 again? There is no best strategy, but some interesting things can be said. Could we use our rationality to get something better than a 2? The first interesting observation is that the max-min strategy is a really conservative option: we only focus our attention in minimizing the maximum possible loss. If this is the most important thing for us, we should employ this strategy. However, we might be confident in our chances and like to take risks. Someone like that would like to opt for 1_1, hoping to get the 5.
If we played with a completely irrational player that chose his strategies randomly, a possible thing to do would be to average the payoffs we can obtain and opt for the one that maximizes that. In that case we would have:

$$\begin{array}{cc} & \text{a. p.} \\ \begin{array}{c} 1_1 \\ 1_2 \\ 1_3 \end{array} & \begin{pmatrix} 3,3 \\ 3 \\ -0,6 \end{pmatrix} \end{array}$$

where a.p. stands for average payoff. It would therefore be appropriate to chose 1_1. Now imagine we knew our opponent was a risk lover, so we could foresee he would try to get the -4 and therefore employ 2_1. We could now chose 1_1 and get the 5.
There is no optimal strategy in this kind of situations, but we should make sure to use all the information we know about the opponent and use it in our favor, as we have outlined how to do so here.

Chapter 4

Mixed Extensions of Matrix Games

In such games where $\max_i \min_j a_{ij} \neq \min_j \max_i a_{ij}$, no equilibrium situation will be reached. This won't satisfy the players, and, if the game is repeated, they may try to change their strategy in order to increase their payoff. Let's assume now that the players will chose their strategy among the $s_i \in S_i^{n_i}$ [1] with a certain frequency. The determination of a mixed strategy consists in assigning a probability x_i to every strategy $s_i \in S_i^{n_i}$, so $x_i = P(Y = s_i)$, where Y is the strategy it is picked. This can be thought as a probability distribution X that:

$$X = (x_1, ..., x_{n_i}) \qquad x_i \geq 0 \qquad \sum_{i=1}^{n_i} x_i = 1$$

If I_{n_i} is the n_i-vector $(1, ..., 1)$, we can summarize the last condition to $XI_{n_i}^T = 1$. The totality of all vectors $(x_1, ..., x_{n_i})$ form a n_i-dimensional Euclidian space E^{n_i}. However, these are subject to conditions $x_i \geq 0$ and $\sum_{i=1}^{n_i} x_i = 1$. Taking that into account, it is easy to see that the set of these resulting vectors is a $(n_i - 1)$-dimensional simplex submerged in E^{n_i}. We will call this set, the one of all possible mixed strategies over $S_i^{n_i}$, as $\widehat{S_i^{n_i}}$, and we will denote a mixed strategy by $X \in \widehat{S_i^{n_i}}$. In the case of $n_i = 2$, for example, $\widehat{S_i^2}$ is the segment that joins $(1, 0)$ to $(0, 1)$. The fact that $\sum_{i=1}^{n} x_i = 1$ trivially tells us that $\widehat{S_i^{n_i}}$ is bounded. The simplex $\widehat{S_i^{n_i}}$ is clearly a subset of a hyperplane of E^{n_i}. It is easy to see that is closed and it follows that $\widehat{S_{n_i}}$ is therefore compact.

4.1 Mixed Extension of a game

With a clever trick, we can transform an antagonistic game where players use mixed strategies to a normal game. Let $A = (a_{ij})$ be the matrix of the matrix game $\Gamma = \langle S_1, S_2, H_1 \rangle$.

Definition. A pair $(X, Y) \in \widehat{S}_1 \times \widehat{S}_2$ of mixed strategies is called a situation in mixed games.

[1] From now on, for the sake of clarity, we will always make explicit the cardinal n_i of S_i, so $S_i^{n_i}$ will mean the set of n_i available strategies to player i.

Every situation (i, j) (in the common usage) is a random event that occurs now with a certain probability $x_i y_j$. In such situation the payoff a_{ij} is obtained. We can easily compute the average payoff when X and Y are employed as:

$$\widehat{H_1}(X, Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = XAY^T$$

So basically now we can assign a precise payoff to every situation in mixed strategies because we have eliminated the probabilistic nature of the strategies. Given that we are using the expected value, the Law of Large Numbers accounts for why the following analysis will works when the game is repeated a lot of times. Therefore a good strategy might give bad results in an isolated realization of the game.

Definition. A mixed extension of the game $\Gamma = \langle S_1, S_2, H_1 \rangle$ is the antagonistic game $\widehat{\Gamma} = \langle \widehat{S_1}, \widehat{S_2}, \widehat{H_1} \rangle$.

As condition (2.1) claims, for a general antagonistic game $\Gamma = \langle S_1, S_2, H_1 \rangle$, a situation $(s_1^*, s_2^*) \in S_1 \times S_2$ will be an equilibrium situation of Γ iff (s_1^*, s_2^*) is a saddle point of $H_1(s_1, s_2)$. If we apply this result to the antagonistic game $\widehat{\Gamma} = \langle \widehat{S_1}, \widehat{S_2}, \widehat{H_1} \rangle$, we deduce that a situation $(X^*, Y^*) \in \widehat{S_1} \times \widehat{S_2}$ will be an equilibrium situation of $\widehat{\Gamma}$ if (X^*, Y^*) is a saddle point of the function $\widehat{H}(X, Y)$. We know then that a situation $(X^*, Y^*) \in \widehat{S_1} \times \widehat{S_2}$ is an equilibrium situation / saddle point iff:

$$XAY^{*T} \leq X^*AY^{*T} \leq X^*AY^T \qquad \forall (X, Y) \in \widehat{S_1} \times \widehat{S_2}$$

We will see that for any matrix game Γ with matrix A there exists an equilibrium situation (X^*, Y^*) in the mixed extension of the game $\widehat{\Gamma}$. As we saw in Theorem 3.2, to prove so we have to verify that $\max_X \min_Y XAY^T$ and $\min_Y \max_X XAY^T$ [2] exist and are equal.

Lemma. For any $Y \in \widehat{S_2}$ and a s.t.

$$A_{i\cdot}Y^T \leq a \qquad \forall i$$

then $\forall x \in \widehat{S_1}$ we have that $XAY^T \leq a$. Similar results for the inequalities of the form $A_{i\cdot}Y^T \geq a$, $XA_{\cdot j} \leq a$ and $XA_{\cdot j} \geq a$ follow.

Proof. Clearly, as $x_i \geq 0 \; \forall i$:

$$A_{i\cdot}Y^T \leq a \Longrightarrow x_i A_i Y^T \leq x_i a$$

It readily follows that:

$$XAY^T = \sum_i x_i A_{i\cdot} Y^T \leq \sum_i x_i a = a \sum_i x_i = a$$

[2] For matrix games, given that $X = (1-x, x)$ and $Y = (1-y, y)$, to maximize and minimize for every vector, i.e., to compute for example $\max_X \min_Y XAY^T$, is the same as to compute $\max_x \min_y XAY^T$.

Theorem 4.1. *Let $\widehat{\Gamma}$ be the mixed extension of a game Γ. Then $(X^*, Y^*) \in \widehat{S}_1 \times \widehat{S}_2$ is an equilibrium situation of $\widehat{\Gamma}$ iff $A_{i\cdot}Y^{*T} \leq X^*AY^{*T} \leq X^*A_{\cdot j}$*

Proof. The necessity is trivial, as this is a particular case of $XAY^{*T} \leq X^*AY^{*T} \leq X^*AY^T$ for $X = 1_i$ and $Y = 1_j$, where the notation should be clear. Let's prove the sufficiency. To do so, let's call $X^*AY^{*T} = a$. We just proved that:

$$A_{i\cdot}Y^{*T} \leq a \Longrightarrow XAY^{*T} \leq a$$
$$X^*A_{\cdot j} \geq a \Longrightarrow X^*AY^T \geq a$$

and then clearly $XAY^{*T} \leq X^*AY^{*T} \leq X^*AY^T$. □

Existence of $\max_X \min_Y XAY^T$ **and** $\min_Y \max_X XAY^T$

Lemma. Let $\widehat{\Gamma}$ be the mixed extension of a game Γ. For any $Y_0 \in \widehat{S}_2$ there exists the $\max_x XAY_0^T$ and for any $X_0 \in \widehat{S}_1$ there exists the $\min_y X_0AY^T$.

Proof. It is clear that:

$$XAY_0^T = \sum_i x_i A_{i\cdot}Y_0^T \equiv \text{ linear function } f(x_i)$$

Given that $\widehat{S}_1$ is a compact set (it is trivially closed and it is bounded because we are dealing with matrix games) then the maximum clearly exists. The other statement is proved analogously.

Lemma. Let $\widehat{\Gamma}$ be the mixed extension of a game Γ. For any $X_0 \in \widehat{S}_1$ there exists $j_0(X_0)$ s.t.

$$\min_y X_0AY^T = X_0A_{\cdot j_0}$$

and for any $Y_0 \in \widehat{S}_2$ there exists $i_0(Y_0)$ s.t.

$$\max_x XAY_0^T = A_{i_0\cdot}Y_0$$

Proof. Consider $\{X_0A_{\cdot j}\}_j$ and let $X_0A_{\cdot j_0}$ be the smallest element in the set. Then given that:

$$X_0A_{\cdot j_0} \leq X_0A_j \;\; \forall j \Longrightarrow X_0A_{\cdot j_0} \leq X_0AY^T \;\; \forall Y \in \widehat{S}_2$$

Given that this inequality is valid $\forall Y \in \widehat{S}_2$, using the definition of minimum we can conclude that:

$$X_0A_{\cdot j_0} \leq \min_y X_0AY^T$$

At the same time, $X_0A_{\cdot j_0}$ is a particular case of X_0AY^T (for $Y = 1_j$) so it's also true that:

$$X_0A_{\cdot j_0} = X_0A1_j^T \geq \min_y X_0AY^T$$

and the result follows. The demonstration of $\max_x XAY_0^T = A_{i_0\cdot}Y_0^T$ would be carried out analogously.

Lemma. The y-function $\max_x XAY^T$ and the x-function $\min_y XAY^T$ are continuous

Proof. We are only going to prove the continuity of the first function, the other case is proven analogously. Given that $\max_x XAY^T = \max_i A_{i.}Y^T$, we only need to prove that $\max_i A_{i.}Y^T$ is continuous. $A_{i.}Y^T$ is obviously continuous in Y. Take $\epsilon > 0$ and δ s.t.

for $|Y' - Y''| < \delta \Longrightarrow |A_{i.}Y'^T - A_{i.}Y''^T| < \epsilon \iff A_{i.}Y''^T - \epsilon < A_{i.}Y'^T < A_{i.}Y''^T + \epsilon \;\; \forall Y', Y''$

Recall that if $f(Y') > f(Y'') \, \forall Y', Y''$ then $\max f(Y') > \max f(Y'')$ and so it follows that:

$$\max_i A_{i.}Y''^T - \epsilon < \max_i A_{i.}Y'^T < \max_i A_{i.}Y''^T + \epsilon \Longrightarrow |\max_i A_{i.}Y'^T - \max_i A_{i.}Y''^T| < \epsilon$$

Theorem 4.2. *The quantities* $\max_X \min_Y XAY^T$ *and* $\min_Y \max_X XAY^T$ *exist.*

Proof. It has just been proven that $\max_X XAY^T$ is a continuous function of Y, which is defined in $\widehat{S}_2$, which, as was discussed, is a compact set and hence the minimum is trivially attained. The other result is proven analogously. □

Convex Sets

Definition. In an euclidian space E^n, a certain $S \subset E^n$ is called a convex subset of E^n if $\forall \, U, V \in S$ and $\forall \lambda \in [0, 1]$ then

$$\lambda U + (1 - \lambda)V \in S$$

i.e. the whole segment going from any point of the subset to some other one is also part of the subset.

Lemma. (*cf.[8]*) If A is a closed convex set and $x \in E^n \backslash A$, then A and x can be separated by a hyperplane.

Lemma. For any matrix A one of this two options holds:

1. There exists a vector $X \in \widehat{S}_1^m$ [3] s.t. $XA_{.j} \geq 0 \, \forall j$
2. There exists a vector $Y \in \widehat{S}_2^n$ s.t. $A_{i.}Y^T \leq 0 \, \forall i$

Proof. Recall that $\widehat{S}_1^m$ is the $(m-1)$-simplex. The convex hull or convex envelope of a set S, i.e., the smallest possible convex set C that contains it, can be expressed as:

$$C(S) = \left\{ \sum_i^{\#S} \alpha_i x_i \right\} \text{ where } x_i \in S, \;\; \alpha_i > 0 \; \forall i, \;\; \text{and } \sum_i^{\#S} \alpha_i = 1$$

Let's consider the convex envelope of $\widehat{S}_1^m$ and $A_{.j}$. There are two possibilities: $\vec{0} \in C(\widehat{S}_1^m \cup A_{.j})$ or $\vec{0} \notin C(\widehat{S}_1^m \cup A_{.j})$.

[3] Bear in mind that we are using again the notation explained in the first footstep of this chapter.

- If $\vec{0} \in C(\widehat{S}_1^m \cup A_{\cdot j})$, applying the definition of convex hull it follows that there exist certain $\{\alpha_j\}_j$ and $\{\eta_i\}_i$ s.t.

$$\sum_{j=1}^{n} \alpha_j A_{\cdot j} + \sum_{i=1}^{m} \eta_i s_i = \vec{0} \text{ with } \alpha_i > 0,\ \eta_i > 0 \text{ and } \sum_j \alpha_j + \sum_i \eta_i = 1$$

The above expression can be written as $\sum b_i s_i = \vec{0}$, and therefore:

$$b_i = \sum_{j=1}^{n} \alpha_j a_{\cdot j} + \eta_i = 0$$

and given that $\eta_i > 0\ \forall i$ it follows that $\sum_{j=1}^{n} \alpha_j a_{\cdot j} \leq 0$. The quantity $\alpha = \sum_j^n \alpha_j$ is positive. It is clearly nonnegative, and if it was 0 it would follow that $\alpha_j = 0\ \forall j$, what would mean using the expression above that $\eta_i = 0$, something which would contradict $\sum_j \alpha_j + \sum_i \eta_i = 1$. We can therefore define $y_j = \alpha_j/\alpha$. Given that $y_j \geq 0\ \forall j$ and $\sum_j y_j = 1$, $Y = (y_1, ..., y_n)$ could be thought as a mixed strategy, i.e., as an element of $\widehat{S}_2^n$. If we use now the expression $\sum_{j=1}^{n} \alpha_j a_{\cdot j} \leq 0$ and we divide it for α, the desired result follows:

$$\sum_{j=1}^{n} y_j a_{ij} = A_{i\cdot} Y^T \leq 0 \quad \forall i$$

- For $\vec{0} \notin C(\widehat{S}_1^m \cup A_{\cdot j})$, we can apply the previous lemma[4] to find that $\vec{0}$ can be separated from C by a hyperplane, which we will assume to go through the point $\vec{0}$. Let $Vz = 0$ be its equation. Without any loss of generality, we can assume that:

$$VZ > 0 \quad \forall z \in C(\widehat{S}_1^m \cup A_{\cdot j})$$

In particular, for any pure strategy, it follows that $Vs_i = v_i > 0$ and let's call $v = \sum v_i$. It is time we considered now

$$X = \left(\frac{v_1}{v}, ..., \frac{v_n}{v}\right)$$

The same arguments that justified in the previous case that Y was a mixed strategy apply here to show that $X \in \widehat{S}_1^m$. Consider now:

$$Xz = \left(\sum_i \frac{1}{v} v_i\right) z_i = \frac{1}{v} Vz \geq 0 \quad \forall z$$

apply this to the vectors $A_{\cdot j}$ and the result readily follows.

[4] It can be proved that the convex hull of a compact set is closed.

4.1.1 The MinMax Theorem

Theorem 4.3. *(The MinMax Theorem) For any matrix A and for any $(X,Y) \in \widehat{S}_1 \times \widehat{S}_2$ the equality:*

$$\max_X \min_Y XAY^T = \min_Y \max_X XAY^T$$

holds.

Proof. We apply to A the previous lemma. Let's assume first the first option, i.e. $XA_{\cdot j} \geq 0 \, \forall j$. As it is clear now:

$$XA_{\cdot j} \geq 0 \, \forall j \Longrightarrow X_0 AY^T \geq 0 \, \forall j \quad \forall Y \in \widehat{S}_2$$

As this is valid for every $Y \in \widehat{S}_2$, then

$$\min_Y X_0 AY^T \geq 0 \Longrightarrow \max_X \min_Y XAY^T \geq 0$$

The second option, $A_{i\cdot}Y^T \leq 0 \, \forall i$, leads, proceeding analogously, to $\min_Y \max_X XAY^T \leq 0$. One of the two inequalities must hold, so it is impossible that:

$$\max_X \min_Y XAY^T < 0 < \min_Y \max_X XAY^T$$

If $A = (a_{ij})$, consider $A(t) = (a_{ij} - t)$. Now:

$$XA(t)Y^T = \sum_i \sum_j x_i(a_{ij} - t)y_j = \sum_i \sum_j x_i a_{ij} y_j - t \sum_i \sum_j x_i y_j = XAY^T - t$$

It is impossible that:

$$\max_X \min_Y (XAY^T - t) < 0 < \min_Y \max_X (XAY^T - t) \Longrightarrow \max_X \min_Y XAY^T < t < \min_Y \max_X XAY^T$$

And therefore, it is impossible that $\max_X \min_Y XAY^T < \min_Y \max_X XAY^T$ so it must always be $\max_X \min_Y XAY^T \geq \min_Y \max_X XAY^T$. However, according to Theorem 3.1, and applying it to a matrix game, it must always hold that $\max_X \min_Y XAY^T \leq \min_Y \max_X XAY^T$, so the only way both things can be true is if:

$$\max_X \min_Y XAY^T = \min_Y \max_X XAY^T$$

□

Just as it happened for non-repeated games, player 1 is inclined to choose the strategy that accomplishes $v_1 = \max_i \min_j XAY^T$ and player 2 the strategy that accomplishes $v_2 = \min_j \max_i XAY^T$. But now, the equivalent of this expressions, as it has just been proven, will always be equal, i.e. $v'_1(A) = v'_2(A)$ [5] always! This means that if both players behave rationally the game will always end up in this situation. Mixed extensions of matrix games are therefore predetermined and that's why they are called completely determined games.

[5] where by analogy $v'_1(A) = \max_i \min_j XAY^T$ and $v'_2 = \min_j \max_i XAY^T$

Definition. Let $\widehat{\Gamma}$ be the mixed extension of the matrix game Γ. We define the value of the game (and bear in mind it only makes sense in mixes strategies) as:

$$v(A) = \max_X \min_Y XAY^T = \min_Y \max_X XAY^T$$

or in a shorter way $v(A) = v_1'(A) = v_2'(A)$. Notice that $v(A)$ tells us the mean gain player 1 will get, as long as they both employ their max-min strategies. Trivially then: $v(A) = H_1(X^*, Y^*)$. To solve a game is to determine the value of the game and the min-max strategies.

4.1.2 First Example

It is time now we talked about how to find equilibrium situations in mixed strategies. The first thing it should be said is that this has already been done in the first chapter of this undergraduate thesis. Let's give now a more precise explanation of how we proceeded. To do so, let's focus our attention on the example 2 of section 3.2.1. We had a game with matrix A for which there was no equilibrium situation. We know now that, in mixed strategies, it must have at least one equilibrium situation. Let's assume each player picks a general mixed strategy, i.e. $X = (1-x, x)$ and $Y = (1-y, y)$ [6].

$$\begin{array}{cc} & \begin{array}{cc} 1-y & y \end{array} \\ \begin{array}{c} 1-x \\ x \end{array} & \begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix} \end{array}$$

We therefore have that:

$$\widehat{H}(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = 4(1-x)(1-y) + 2(1-x)y + 3xy = 5xy - 2y - 4x + 4$$

Now we want to compute:

$$\max_{0 \le x \le 1} \min_{0 \le y \le 1} 5xy - 2y - 4x + 4 = \max_{0 \le x \le 1} \min_{y \in \{0,1\}} 5xy - 2y - 4x + 4 = \max_{0 \le x \le 1} \min\{-4x + 4, x + 2\}$$

where we have used Theorem 1.1 in the first equality. The function $\widehat{H}(x) = \min\{-4x + 4, x + 2\}$ is graphed in figure 4.1.
It can readily be checked that it attains its maximum at $x^* = 2/5$ and thus:

$$v(A) = v_I'(A) = \frac{2}{5} + 2 = 2.4$$

If we want to know now the strategy player 2 should employ we only need to compute:

$$\min_{0 \le y \le 1} \max_{0 \le x \le 1} 5xy - 2y - 4x + 4 = \min_{0 \le y \le 1} \max_{x \in \{0,1\}} 5xy - 2y - 4x + 4 = \min_{0 \le y \le 1} \max\{-2y + 4, 3y\}$$

[6] This is arbitrary and we could have perfectly taken $X = (x, 1-x)$ and $Y = (y, 1-y)$. We warn the reader that we will alternate between these throughout the following chapters.

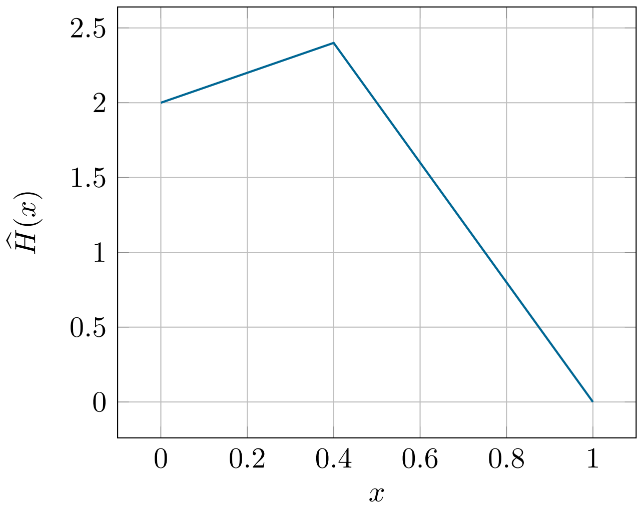

Figure 4.1: Graph of $\min\{-4x+4, x+2\}$

The minimum of $\max\{-2y+4, 3y\}$ is attained when $y^* = 4/5$ and it obviously yields 2.4. We have solved the game:

$$\text{min-max strategy } X^* = \left(\frac{3}{5}, \frac{2}{5}\right)$$
$$\text{max-min strategy } Y^* = \left(\frac{1}{5}, \frac{4}{5}\right)$$
$$\text{value of the game } v(A) = \frac{12}{5} = 2.4$$

Both this and the bomber game were matrix games with 2×2-matrixes, i.e. $n_1 = \#S_1^{n_1} = 2$ and also $n_2 = 2$. Let us say at this point that there is no easy algorithm to solve $n \times m$ games. Everything that can be said is that a Nash Equilibrium will always exist in mixed strategies. However, let's try to go beyond 2×2 games by studying $2 \times m$ games

4.1.3 Two lemmas

Definition. Let Γ be a game with matrix $A = (a_{ij})$. Then a row i dominates over row k if $a_{ij} \geq a_{kj}$ $\forall j$ and column j dominates over column l if $a_{ij} \leq a_{il}$ $\forall i$. Intuitively, a dominated row is not interesting for player 1 and a dominated column is not interesting for player 2, so given that the players are not going to pick them, we can remove them from the game. Let's prove it.

Lemma. If either a dominated row or column is removed from the matrix A, the solution of the remaining game is the same as the one of the original game.

Proof. Let $\Gamma = \langle S_1, S_2, H_1\rangle$ be a matrix game and $\widehat{\Gamma} = \langle \widehat{S_1}, \widehat{S_2}, \widehat{H_1}\rangle$ its mixed extension. Let the second strategy of player 1 dominate over the first one. Consider thus the games $\Gamma' = \langle S_1', S_2, H_1\rangle$ and $\widehat{\Gamma}' = \langle \widehat{S}_1', \widehat{S}_2, \widehat{H_1}\rangle$ where clearly $\widehat{S}_1' = \{X \in \widehat{S}_1 \text{ s.t. } x_1 = 0\}$. Let

(X'^*, Y'^*) be the optimal strategies of $\widehat{\Gamma}'$. Are these the optimal strategies of $\widehat{\Gamma}$ as well? Let v' be the value of $\widehat{\Gamma}'$. This means that:

$$\begin{aligned}
\widehat{H_1}(X'^*, Y') \geq \widehat{H_1}(X'^*, Y'^*) = v' \qquad \forall Y' \in \widehat{S}_2 \\
\widehat{H_1}(X', Y'^*) \leq \widehat{H_1}(X'^*, Y'^*) = v' \qquad \forall X' \in \widehat{S'}_1
\end{aligned}$$

In order to prove the lemma, we want to check if:

$$\begin{aligned}
\widehat{H_1}(X'^*, Y') \geq v' \qquad \forall Y' \in \widehat{S}_2 \\
\widehat{H_1}(X', Y'^*) \leq v' \qquad \forall X' \in \widehat{S}_1
\end{aligned}$$

what will mean that v' is also the value of the game $\widehat{\Gamma}$. The first inequality is trivially true. When it comes to the second:

$$\begin{aligned}
&\widehat{H_1}(X', Y'^*) = \sum_i^n \sum_j^m x'_i a_{ij} y'^*_j = x'_1 \sum_j^m a_{1j} y'^*_j + \sum_{i=2}^n \sum_{j=1}^m x'_i a_{ij} y'^*_j \leq \text{ (domination)} \\
&\leq x'_1 \sum_{j=1}^m a_{2j} y'^*_j + \sum_{i=2}^n \sum_{j=1}^m x'_i a_{ij} y'^*_j = \widehat{H_1}(\tilde{X}', Y'^*) \leq v \qquad \forall X' \in \widehat{S}_1
\end{aligned}$$

where we used that $\tilde{X}' = (0, x_1 + x_2, x_3, ..., x_n) \in \widehat{S}'_1$. □

Definition. A pure strategy is said to be relevant if it is employed with probability greater than zero in a max-min strategy.

Lemma. Any relevant strategy played against a max-min strategy yields the value of the game.

Proof. Let $\Gamma = \langle S_1, S_2, H_1 \rangle$ be a matrix game and $\widehat{\Gamma} = \langle \widehat{S}_1, \widehat{S}_2, \widehat{H_1} \rangle$ its mixed extension. Let $X^* = (x^*_1, ..., x^*_k, 0, ..., 0)$ $(x^*_i \neq 0\ i < k)$ and Y^* be the max-min strategies. Then:

$$v = \widehat{H}_1(X^*, Y^*) = \sum_i \sum_j x^*_i a_{ij} y^*_j = \sum_i^k x^*_i \widehat{H}_1(1_i, Y^*) \implies \sum_i^k x^*_i \frac{\widehat{H}_1(1_i, Y^*)}{v} = 1$$

We know that $\sum_i x^*_i = 1$. Recall that $v = v'_2$, so given that player 2 is employing a max-mix strategy we have that $\frac{\widehat{H}_1(1_i, Y^*)}{v'_2} \leq 1\ \forall i \leq k$. In fact, these inequalities will turn out to be equalities. If $\frac{\widehat{H}_1(1_i, Y^*)}{v} < 1$ for some i and the others were equal to 1, we would have:

$$\sum_i x^*_i \frac{\widehat{H}_1(1_i, Y^*)}{v} < \sum_i x^*_i = 1$$

which can't be true. Therefore:

$$\frac{\widehat{H}_1(1_i, Y^*)}{v} = 1\ \ \forall i \leq k \qquad \implies \qquad \widehat{H}_1(1_i, Y^*) = v\ \ \forall i \leq k$$

□

In the example of section 4.1.2, we found that all strategies were relevant and that the value of the games was 12/5. We can thus check the validity of the lemma:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} \frac{12}{5} \\ \frac{12}{5} \end{pmatrix} \quad \& \quad \begin{pmatrix} \frac{3}{5}, \frac{2}{5} \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{12}{5}, \frac{12}{5} \end{pmatrix}$$

which obviously turns out to be right.

4.1.4 $2 \times m$ games

Let's assume we have a game $\Gamma = \langle S_1, S_2, H_1 \rangle$ with matrix A where $\#S_1 = 2$ and $\#S_2 = m$. Also let's assume no saddle point exists (otherwise we don't need mixed strategies to solve the game). We can solve these games by:

1. Using the domination lemma.

2. Finding relevant strategies.

3. Using the relevant strategies lemma to find the max-min strategies.

Let's see how this works with an example:

$$\begin{pmatrix} 1 & 7 & 5 & 3 \\ 8 & 2 & 6 & 4 \end{pmatrix}$$

We apply 1) and we are left with:

$$\begin{pmatrix} 1 & 7 & 3 \\ 8 & 2 & 4 \end{pmatrix}$$

Now let's move to 2). We assume a strategy $X = (x, 1 - x)$ for player 1 and we confront it all the pure strategies of player 2 ($2_1 = (1, 0, 0)$, $2_2 = (0, 1, 0)$ and $2_3 = (0, 0, 1)$). We do that because if X is a max-min strategy, we know that we only need to confront it against all the pure strategies to obtain the value of the game. We thus have:

$$\begin{aligned} &\text{The combination of } X \text{ with } 2_1 \text{ yields: } r_1(x) = x + 8(1 - x) = -7x + 8 \\ &\text{The combination of } X \text{ with } 2_2 \text{ yields: } r_2(x) = 7x + 2(1 - x) = 5x + 2 \\ &\text{The combination of } X \text{ with } 2_3 \text{ yields: } r_3(x) = 3x + 4(1 - x) = -1x + 4 \end{aligned}$$

Figure 4.2 a) shows the graph of these functions.
Player 1, as always, will choose the strategy that accomplishes $\max\min\{-7x + 8, 5x + 2, -x + 4\}$. See figure 4.2 b), where the function $f(x) = \min\{-7x + 8, 5x + 2, -x + 4\}$ is represented. The maximum (the value of the game), which lays somewhere between $x = 0.2$ and $x = 0.4$, comes from the intersection of $r_2(x)$ and $r_3(x)$. Provided that $r_1(x)$ can't yield the value of the game and applying the relevant strategies lemma, we conclude that 2_1 is an irrelevant strategy, so it can be removed. As a consequence, we are left with:

$$\begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix}$$

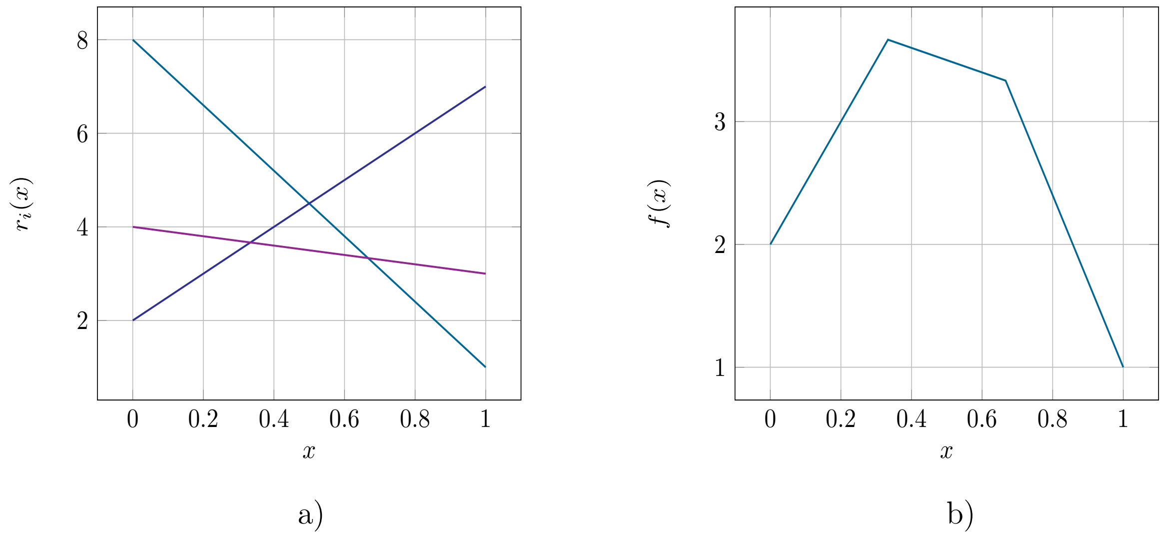

Figure 4.2: a) Graph of $r_1(x)$, $r_2(x)$ and $r_3(x)$. b) Graph of $\min\{r_1(x), r_2(x), r_3(x)\}$

Let $X^* = (x^*, 1-x^*)$ be the max-min strategy of player 1. The relevant strategies lemma asserts that:

$$(x^*, 1-x^*)\begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = v \qquad \& \qquad (x^*, 1-x^*)\begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = v$$

From these equations we conclude that: $5x^* + 2 = -x^* + 4$ and therefore $x^* = 1/3$ and $v = 11/3$. At the same time:

$$(1,0)\begin{pmatrix} 7 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} y^* \\ 1-y^* \end{pmatrix} = v = \frac{11}{3}$$

from where we deduce that: $4y^* + 3 = 11/3$ and therefore $y^* = 1/6$. We have solved the game:

$$\text{min-max strategy } X^* = \left(\frac{1}{3}, \frac{2}{3}\right)$$
$$\text{max-min strategy } Y^* = \left(\frac{1}{6}, \frac{5}{6}\right)$$
$$\text{value of the game } v(A) = \frac{11}{3} = 3.67$$

Chapter 5

General Noncooperative Games

Let us study now those noncooperative games where there are more than two players. We have seen so far how all matrix games have at least one equilibrium situation in mixed strategies. This interesting result can be generalized into a much more powerful statement, originally proved by Nash, that we present now and will be proved later.

Nash's Theorem. *In any noncooperative game there is at least one equilibrium situation in mixed strategies.*

5.1 Mixed Extensions of Noncooperative Games

Let $\Gamma = \langle I, \{S_i\}_{i\in I}, \{H_i\}_{i\in I}\rangle$ be a noncooperative game and let's assume that $\#\{S_i\} < +\infty \quad \forall i$. Let σ_i denote the mixed strategy of player i. We could think of σ_i as a probability distribution that assigns to every strategy $s_i \in S_i$ the probability of being actually employed: $\sigma_i(s_i)$. If player i decides to pick the same strategy $\dot{s}_i$ every time, i.e. to pick a pure strategy, clearly $\sigma_i(s_i) = 0 \ \forall s_i \neq \dot{s}_i$ and $\sigma_i(\dot{s}_i) = 1$. The set of all mixed strategies of player i will be denoted by $\widehat{S}_i$. The probability distributions will be assumed to be independent, so the probability of arriving at the situation $s \in S$ will be:

$$\sigma(s) = \sigma(s_1, ..., s_n) = \sigma_1(s_1) \cdots \sigma_n(s_n)$$

The set of all this probability distributions will be now the set of situations. To define the payoff of this new situations, we will proceed as we did for the mixed extension of matrix games, i.e. averaging the payoff of every pure strategy.

$$\widehat{H}_i(\sigma) = \sum_{s\in S} H_i(s)\sigma(s) = \sum_{s_1\in S_1} \cdots \sum_{s_n\in S_n} H(s_1, ..., s_n) \prod_{i=1}^{n} \sigma_i(s_i)$$

Clearly now we can define the mixed extension of the game $\Gamma = \langle I, \{S_i\}_{i\in I}, \{H_i\}_{i\in I}\rangle$ as:

$$\widehat{\Gamma} = \langle I, \{\widehat{S}_i\}_{i\in I}, \{\widehat{H}_i\}_{i\in I}\rangle$$

Theorem 5.1. *For any situation σ, there is at least one pure strategy $s_i^0 \in S_i$ $\forall i$ s.t.*

$$\sigma_i(s_i^0) > 0 \text{ (It is sometimes employed)} \quad \& \quad \widehat{H}_i(\sigma||s_i^0) \leq \widehat{H}_i(\sigma)$$

Proof. We are going to proceed with a *reductio ad absurdum.* Suppose no such strategy exists. Then, all pure strategies s_i of player i that satisfy $\sigma_i(s_i^0) > 0$ must satisfy as well:

$$\widehat{H}_i(\sigma||s_i) > \widehat{H}_i(\sigma)$$

As for these strategies $\sigma_i(s_i) > 0$, we can readily state that $\widehat{H}_i(\sigma||s_i)\sigma_i(s_i) > \widehat{H}_i(\sigma)\sigma_i(s_i)$. For those $s_i \in S_i$ that are never employed ($\sigma_i(s_i) = 0$) it holds that $\widehat{H}_i(\sigma||s_i)\sigma_i(s_i) = \widehat{H}_i(\sigma)\sigma_i(s_i) = 0$. Clearly:

$$\sum_{s_i \in S_i} \widehat{H}_i(\sigma||s_i)\sigma_i(s_i) > \sum_{s_i \in S_i} \widehat{H}_i(\sigma)\sigma_i(s_i)$$

from where it follows that $\widehat{H}_i(\sigma) > \widehat{H}_i(\sigma)$, which is a contradiction. □

Analogously as we have done before, we will say that $\sigma^* \in \widehat{S}$ is an equilibrium situation in $\widehat{\Gamma}$ if:

$$\widehat{H}_i(\sigma^*||\sigma_i') \leq \widehat{H}_i(\sigma^*) \qquad \forall \sigma_i' \in \widehat{S}_i \qquad \forall i \in I$$

Let's derive now a result we will use to prove Nash's Theorem.

Theorem 5.2. *A necessary and sufficient condition for a situation σ^* in $\widehat{\Gamma}$ to be an equilibrium situation is that:*

$$\widehat{H}_i(\sigma^*||s_i') \leq \widehat{H}_i(\sigma^*) \qquad \forall s_i' \in S_i \qquad \forall i \in I \tag{5.1}$$

Proof. First of all, what is $\widehat{H}_i(\sigma^*||s_i')$? Clearly, as we said before, $\sigma_i(s_i) = \delta_{(s_i=s_i')}$ so using the expression for $\widehat{H}_i(\sigma)$ it's easy to see that:

$$\widehat{H}_i(\sigma^*||s_i') = \sum_{s_1 \in S_1} \cdots \sum_{s_{i-1} \in S_{i-1}} \sum_{s_{i+1} \in S_{i+1}} \cdots \sum_{s_n \in S_n} H(s_1, ..., s_{i-1}, s_i', s_{i+1}, ..., s_n) \prod_{\substack{i=1 \\ i \neq j}}^{n} \sigma_i(s_i)$$

The statement (5.1) is a particular case of the definition of equilibrium situation. Let's focus on the necessity. Let's assume the validity of (5.1) and let's choose an arbitrary strategy $\sigma_i \in \widehat{S}_i$. We have that:

$$\widehat{H}_i(\sigma^*||s_i') \leq \widehat{H}_i(\sigma^*) \iff \widehat{H}_i(\sigma^*||s_i')\sigma_i(s_i') \leq \widehat{H}_i(\sigma^*)\sigma_i(s_i')$$

Trivially:

$$\sum_{s_i' \in S_i} \widehat{H}_i(\sigma^*||s_i')\sigma_i(s_i') \leq \sum_{s_i' \in S_i} \widehat{H}_i(\sigma^*)\sigma_i(s_i') = \widehat{H}_i(\sigma^*) \sum_{s_i' \in S_i} \sigma_i(s_i') = \widehat{H}_i(\sigma^*)$$

But:

$$\sum_{s'_i\in S_i}\widehat{H}_i(\sigma^*||s'_i)\sigma_i(s'_i)=\sum_{s'_i\in S_i}\left(\sum_{s_1\in S_1}\cdots\sum_{s_{i-1}\in S_{i-1}}\sum_{s_{i+1}\in S_{i+1}}\cdots\sum_{s_n\in S_n}H(s||s'_i)\prod_{\substack{i=1\\ i\neq j}}^{n}\sigma_i(s_i)\right)\sigma_i(s'_i)=$$

$$=\sum_{s_1\in S_1}\cdots\sum_{s_{i'}\in S_i}\cdots\sum_{s_n\in S_n}H(s||s'_i)\left(\prod_{\substack{i=1\\ i\neq j}}^{n}\sigma_i(s_i)\right)\sigma_i(s'_i)=\widehat{H}_i(\sigma^*||\sigma_i)$$

and the result follows.

5.2 Nash's Theorem

Theorem 5.3. *(Brower's Fixed Point Theorem)* (cf.[5]) *For any continuous function f mapping a compact convex set into itself there is a fixed point, i.e. a point x_0 such that $f(x_0) = x_0$.*

Theorem 5.4. *(Nash's Theorem) In any noncooperative game $\Gamma = \langle I, \{S_i\}_{i\in I}, \{\ H_i\}_{i\in I}\rangle$ there is at least one equilibrium situation in mixed strategies.*

Proof. Every player has available a set of pure strategies $S_i^{m_i}$. The set $\widehat{S}_i^{m_i}$ is the set of mixed strategies. Geometrically, as previously discussed, it's a $(m_i - 1)$-simplex. Any situation $\sigma = (\sigma_1, ..., \sigma_n)$ can be seen as a point of the cartesian product $\widehat{S}_1^{m_1} \times ... \times \widehat{S}_n^{m_n}$. This is a convex closed subset of the $(m_1 + ... + m_n - n)$-euclidean space. For any situation σ and for any pure strategy of any player, $\forall s_i^j \in S_i$ (it should understood as the j^{th} strategy of player i) let's define

$$\phi_{ij}(\sigma) = \max\{0, \widehat{H}_i(\sigma||s_i) - \widehat{H}_i(\sigma)\} \geq 0$$

For a fixed situation σ, we are calculating the increase of payoff when we change σ_i by a pure strategy s_i^j. Let's define now, $\forall i$ and $\forall j$:

$$\frac{\sigma_i(s_i^j)+\phi_{ij}(\sigma)}{1+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}$$

where $\sigma_i(s_i^j)$ is the probability of employing s_i^j, so it's a nonnegative value. So is $\phi_{ij}(\sigma)$ and consequently the fraction is always positive. Clearly:

$$\sum_{j=1}^{m_i}\frac{\sigma_i(s_i^j)+\phi_{ij}(\sigma)}{1+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}=\frac{\sum_{j=1}^{m_i}\sigma_i(s_i^j)+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}{1+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}=\frac{1+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}{1+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}=1$$

For a fixed situation σ and for a fixed player i, the elements of

$$\left\{\frac{\sigma_i(s_i^j)+\phi_{ij}(\sigma)}{1+\sum_{j=1}^{m_i}\phi_{ij}(\sigma)}\right\}_j$$

given that are positive and add up to one, could be thought as forming a mixed strategy. As this can be done for every player, we would have a mixed strategy for every player and thus a situation. Therefore, given a situation σ we can define a new situation $f(\sigma)$[1] the way we have explained. To be able to apply Brower's fixed theorem, we need to check if f is continuous. Is $\phi_{ij}(\sigma)$ a continuous function of σ? Clearly $\max\{0, x\}$, $H_i(\sigma)$ and the rest function are continuous and provided that the composition of continuous functions is a continuous function the result follows. At the same time, the probability distribution $\sigma_i(s_i^j)$ is also continuous on σ and therefore the numerator is continuous. The same arguments show that the denominator is always continuous and given that it never vanishes, f is a continuous function. All the requirements to apply the Brower Fixed Theorem hold so we conclude that there exists at least one σ_0 s.t. $f(\sigma^0) = \sigma^0$. In other terms:

$$\sigma_i^0(s_i^j) = \frac{\sigma_i^0(s_i^j) + \phi_{ij}(\sigma^0)}{1 + \sum_{j=1}^{m_i} \phi_{ij}(\sigma^0)}$$

According to Theorem 5.1, for every player and applied to the situation σ^0, there exists a pure strategy $s_i^0 \in S_i$ s.t. $\sigma_i(s_i^0) > 0$ and $\phi_{i0}(\sigma^0) = 0$. For this particular strategy, the expression above reads as:

$$\sigma_i^0(s_i^0) = \frac{\sigma_i^0(s_i^0)}{1 + \sum_{j=1}^{m_i} \phi_{ij}(\sigma^0)}$$

from where it follows that $\sum_{j=1}^{m_i} \phi_{ij}(\sigma^0) = 0$. Recalling that $\phi_{ij}(\sigma) = \max\{0, \widehat{H}_i(\sigma || s_i) - \widehat{H}_i(\sigma)\} \geq 0$ it is clear that:

$$\phi_{ij}(\sigma^0) = 0 \ \forall i \forall j \qquad \Longrightarrow \qquad \widehat{H}_i(\sigma^0 || s_i^j) \leq \widehat{H}_i(\sigma^0)$$

and applying Theorem 5.2 we deduce that σ^0 is a Nash Equilibrium. □

Remark. This important theorem is what allows us to say that there always exists, at least, one Nash equilibrium. However, it is not a constructive theorem, so it doesn't tell us how to find such equilibrium situations. This comes from the fact that Brower's fixed theorem, the basis of our demonstration, is not constructive either because it does not explain how to find the fixed points.

In noncooperative games, not all the players have the same payoff at all equilibrium situations. For general noncooperative games, there is no optimal strategy that yields the value of the game, because there is no value of the game, only one or more Nash equilibriums. Also, assuming that every player knows which of his strategies can yield an equilibrium situation, only certain combinations of these strategies will in the end actually lead to a Nash equilibrium. Let us illustrate this with some examples.

---

[1] $f : \widehat{S}_1 \times ... \times \widehat{S}_n \longrightarrow \widehat{S}_1 \times ... \times \widehat{S}_n$ and clearly $f(\sigma) = \frac{\sigma_i(s_i^j) + \phi_{ij}(\sigma)}{1 + \sum_{j=1}^{m_i} \phi_{ij}(\sigma)}$

5.3 Examples

In general noncooperative games, we can also speak of $v_1 = \max_X \min_Y H_1(X, Y)$ and $v_2 = \max_Y \min_X H_2(X, Y)$ [2]. They represent, clearly, the minimum gain each player can assure. As such, we can define the max-min solution (X^*, Y^*) formed by those strategies that satisfy the previous equalities. Every time we do that, we are assuming a zero-sum game for every player. However, now the players do not necessarily have opposite motivations, i.e., the players gain does not come from the opponent's loss as in zero-sum games. This will mean that $H_1(X^*, Y^*)$ and $H_2(X^*, Y^*)$ are not going to be necessarily equal to v_1 and v_2. So even if we can define a max-min solution, this will not correspond to an equilibrium situation. Let's illustrate this with an example.

5.3.1 The gender war

A couple is discussing what to do this evening. The guy, who is passionate about cinema, wants to go to the movies whereas the girl, a big tennis fan, wants to stay in and watch a match on TV. Even if they have opposite preferences, both of them prefer to do something together. If 1_1 and 2_1 stand for going to the cinema and 1_2 and 2_2 for staying in, we can write:

$$\begin{array}{cc} & \begin{array}{cc} 2_1 & 2_2 \end{array} \\ \begin{array}{c} 1_1 \\ 1_2 \end{array} & \begin{pmatrix} (1,4) & (0,0) \\ (0,0) & (4,1) \end{pmatrix} \end{array}$$

where it is clear which player is the guy and which the girl. Let's find the max-min solution for the girl. We assume a zero-sum game and therefore have:

$$\begin{array}{cc} & \begin{array}{cc} y & 1-y \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \end{array}; \quad H_1 = 5xy - 4x - 4y + 4 \implies v_1 = \max_x \min_y (5xy - 4x - 4y + 4) = \frac{4}{5}$$

This is attained at $x^* = \frac{4}{5}$ and the max-min strategy for the girl therefore is $X^* = (\frac{4}{5}, \frac{1}{5})$. Analogously for the boy we have that $Y^* = (\frac{1}{5}, \frac{4}{5})$ and $v_2 = \frac{4}{5}$. If this were the strategies they actually picked, $\frac{16}{25}$ of the times we would end up with the outcome $(0, 0)$. There are two evident equilibrium situations, which are: $(1_1, 2_1)$ and $(1_2, 2_2)$. Are there any more? To answer the question, let's explain a method to find Nash equilibriums in 2×2 games.

5.3.2 The swastika method

Let's assume general strategies $(X, 1-X)$ and $(Y, 1-Y)$ for our players. A equilibrium situation is an admissible situation for every player. We are going to calculate separately for every player the admissible situations and we will then intersect them. Let's focus on

[2] If $H_2 = -H_1$ we have that $v_{II} = \max_Y \min_X -H_1 = -\min_Y \max_X H_1$ which corresponds to the definition we gave of v_{II} for matrix games with a minus sign in front. This is because before we spoke of the loss of player 2 whereas now we talk about his gain.

player 1. For any strategy his opponent picks, i.e. for any Y, we are going to calculate the best reply which yields $\max_X H_1(X,Y)$. Let's pick the same matrix as in the gender war. Starting with H_1 we have:

$$\begin{array}{cc} & \begin{array}{cc} y & 1-y \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \end{array}; \quad H_1 = x(5y-4) - 4y + 4$$

The part that player 1 can alter is the one that depends on x. The maximum of the expression above, given that it is linear on x, is either attained at $x = 0$ or at $x = 1$. Also, there is the option that the linear term vanishes. All in all, to maximize H_1:

- if $y < \frac{4}{5}$, then player 1 should set $x = 0$.
- if $y = \frac{4}{5}$, then the maximum is attained for any x.
- if $y > \frac{4}{5}$, then player 1 should set $x = 1$.

If we adopt now player's 2 perspective, for him:

$$\begin{array}{cc} & \begin{array}{cc} y & 1-y \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \end{array}; \quad H_2 = y(5x-1) - x + 1$$

and proceeding analogously:

- if $x < \frac{1}{5}$, then player 2 should set $y = 0$.
- if $x = \frac{1}{5}$, then the maximum is attained for any y.
- if $x > \frac{1}{5}$, then player 2 should set $y = 1$.

We can graph all this (see figure 5.1) and we find three equilibrium situations:

$$\begin{aligned}
&(X^*, Y^*) = ((0,1),(0,1))\,. \text{ Then } H_1(X^*,Y^*) = 4 \text{ and } H_2(X^*,Y^*) = 1 \\
&(X^*, Y^*) = ((1,0),(1,0))\,. \text{ Then } H_1(X^*,Y^*) = 1 \text{ and } H_2(X^*,Y^*) = 4 \\
&(X^*, Y^*) = \left(\left(\frac{1}{5},\frac{4}{5}\right),\left(\frac{4}{5},\frac{1}{5}\right)\right). \text{ Then } H_1(X^*,Y^*) = \frac{4}{5} \text{ and } H_2(X^*,Y^*) = \frac{4}{5}
\end{aligned}$$

First of all, it is important to remark that the Nash Equilibriums yield different payoffs for the players, something that illustrates that the concept of the value of the game has no sense in general noncooperative games. Also, notice that the equilibrium points are not interchangeable. See for example that $(0,1)$ is one of the strategies of player 1 that yield a Nash equilibrium and so is $(1,0)$ for player 2. However, the combination of these strategies is not an equilibrium situation. As we said before, even if everybody knew which strategies yield Nash equilibriums, only certain combinations of these would in fact actually lead to equilibrium situations.
Could this method be generalized? or in other words: How can we solve a general noncooperative game? The answer is that nobody knows. The research of algorithms to do so is a very active topic of research nowadays.

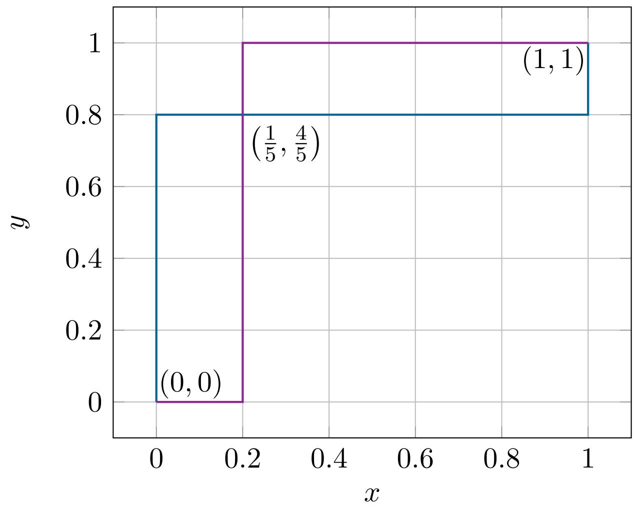

Figure 5.1: Swastika method for the gender war

5.3.3 Prisoner's Dilemma - First approach

Consider a 2×2-game where $S_1 = S_2 = \{C, D\}$ [3] with:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (-\frac{1}{2}, -\frac{1}{2}) & (-10, 0) \\ (0, -10) & (-6, -6) \end{pmatrix} \end{array}$$

We can apply the Swastika Method to this game. Let's see what happens. Proceeding as before, for player 1:

$$\begin{array}{cc} & \begin{array}{cc} y & 1-y \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} -\frac{1}{2} & -10 \\ 0 & -6 \end{pmatrix} \end{array}; \quad H_1 = x\left(\frac{7}{2}y - 4\right) + 6y - 6$$

The part that player 1 can alter is the one that depends on x. However, $\forall y \in [0, 1]$ the term $(\frac{7}{2}y - 4) < 0$. To maximize H_1 player 1 should always set $x = 0$ regardless of the value of y. If we do the same for player 2, we will have $H_2 = y\left(\frac{7}{2}x - 4\right) + 6x - 6$, so the same argument applies. Therefore, the only equilibrium situation is (D, D). But let's take some perspective: This result is awful. Both player would be better off with (C, C), which, even if it is not an equilibrium situation, it is the best outcome of the game. What is going on here?

[3] The meaning of every strategy will be explained later.

Chapter 6

A Mathematical Explanation of Cooperation

The last chapter of this undergraduate thesis is of a different nature. Now we are going to see how Game Theory can be used to help disciplines like Biology, Political Science, Law, Economy, etc. We are going to address a philosophical problem, known as the origin of cooperation, that reads as follows: How can cooperation emerge in a world of selfish individualistic people? As experience suggests, nowadays people normally pursue their self interest, most of the times leaving no room for helping others. From a rational point of view, there's no point in helping strangers. Thomas Hobbes believed that, if left to a complete *laissez faire*, selfishness would compromise communal living and life would become "solitary, poor, nasty, brutish and short". To prevent this, he claimed that a central authority, in the form of a state, was completely necessary. The debate, however, did not finish here, and the question of whether cooperation could emerge without the aid of a central supervision continued. In both the human race and animals, there are certain individuals that challenge these pessimistic views on the human nature by showing clear signs of altruism towards strangers. For example, is it known that during World War One, an admirable tactic agreement was made between the fighters of each side on the western front. Violating the orders of their superiors, the front-line soldiers kept from shooting to kill as long as their enemies did the same, creating a pattern of cooperation that benefited them all.

Darwin, the father of the theory of evolution, was very aware of the need of cooperating dispositions to perpetuate a species. As a consequence, he was concerned at the thought that cooperation was destroyed by natural selection. As he himself explained:

> "*He who was ready to sacrifice his life [...] rather than betray his comrades, would often leave no offspring to inherit his noble nature. The bravest men, who were always willing to come to the front in war, and who freely risked their lives for others, would on an average perish in larger numbers than other men. Therefore, it hardly seems probable that the number of men gifted with such virtues [...] could be increased through natural selection, that is, by the survival of the fittest.*"

Charles Darwin, *The Descent of Man.* Part One, Chapter V.

In the animal kingdom, where there is neither government nor law, theories such as *the Selfish Gene*, by Richard Dawkins, have been put forward to account for the presence of cooperation. However, Game Theory can be used to prove the counterintuitive thesis that pure self interest can lead to the emergence of cooperative dispositions towards strangers. To begin with, let's take another look at the prisoner's dilemma.

6.1 Prisoner's Dilemma

Imagine that two burglars are caught red-handed breaking into a house but in accordance with the need in a civilized world to keep within the legal frameworks, not enough valid evidence is collected to find them guilty of an unlawful break-in. The intelligent police officers proceed to separate the suspected offenders and tell them the exact same thing. If you testify against your friend, you will be automatically released, and he will have to face a 10-year prison sentence. However, if your friend testifies against you too, you will also be found guilty and both of you will face a 6-year prison sentence. If nobody says anything, both of you will be accused of minor charges and will serve a 6 month sentence. The possible strategies are $S_1, S_2 = \{C, D\}$, C standing for Cooperate (with your partner) and D standing for Defect (your partner).

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (-\frac{1}{2}, -\frac{1}{2}) & (-10, 0) \\ (0, -10) & (-6, -6) \end{pmatrix} \end{array}$$

We saw in the previous chapter that there is only one equilibrium situation. We could have reasoned the following way to discover this: If your opponent cooperates, we compare a payoff of $-\frac{1}{2}$ (when you cooperate) to a payoff of 0 (when you defect). If he doesn't cooperate we compare a payoff of -10 to a payoff of -6. This means, by definition, that de situation (D, D) is an equilibrium situation. As we said before, this is an awful result, because it is clear that they would be better off cooperating. The prisoners dilemma, as an example of how the pursuit of self interest by each player leads to a bad outcome for all, goes beyond the mere anecdotal story to raise a fundamental reflection regarding human cooperation. As we are going to see, the prisoner's dilemma can shed some light on why we cooperate with unknown individuals. To study this a bit better, let's consider a generic version of the game:

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} (R, R) & (S, T) \\ (T, S) & (P, P) \end{pmatrix} \end{array}$$

where R is the reward for mutual cooperation, T is the temptation to defect, S is the sucker's payoff and P the punishment for mutual defection [1]. To encapsulate the problems we want to deal with, we need to set $T > R > P > S$. In the repeated prisoner's dilemma

[1] Where I am using Axelrod's terms. See: [4]

a second condition is added: $2R > T + S$, which prevents the players to get out of the dilemma by exploiting each other. We want mutual cooperation to be a better option than betraying and being betrayed in turn.
This game has served to show how cooperation can emerge in a world of selfish players where there is no central authority. Is this, nevertheless, a good approximation of the real world? Many philosophers have argued that human nature is entirely selfish. If we care for others, they claim, it is because our own welfare resides in the welfare of others. This is not a very optimistic perspective of the human being but it is quite realistic. If we adhere to this train of thought, our analysis would also apply to cooperation between friends and family members. At the same time, nowadays we live in a society ruled by a central authority that forces us to cooperate (by prosecuting for failure to provide aid and assistance, for example), so someone could say our analysis would not apply to real life. However, in most cases, like in the story that motivates the prisoner's dilemma, cooperation is an option and not an obligation. But that's not everything. Countries, for example, do interact between each other without any supervision, and therefore topics like the nuclear disarmament or the custom duties would also fall into this description.
These situations are iterated versions of the prisoners dilemma. How should we play in such case? Let's assume we tell the players of the game that we are going to repeat the game 20 times. Can cooperation emerge in this context? As we said, if the game is only played once, the optimal choice is to defect. Let's then consider the last movement of our iterated game. In that stage of the game, given that there will be no further encounter between the players, the wisest choice again is to defect. But bearing that in mind and going one step back in the game, we too should defect in the $19th$ move. Following this train of thought, we can deduce that the game is determined and its outcome is awful.
What happens, though, if we never know how long the game is going to last for? Now, in every move, you have to think about the consequences your decisions might entail. To study this, the mathematician and political scientist Robert Axelrod organized a computer prisoner's dilemma tournament. He invited experts in the field to submit ideas of strategies, and he confronted 14 of them in total. It should be noted that no best strategy exists regardless of the one your opponent employs. What it is clear is that as long as you play against someone that decides how to play before the start of the game, i.e., someone that has a predetermined strategy, the optimal strategy is to defect all the time. Cooperation can only emerge when your movements will have repercussions and when your opponent will take into account your behavior to decide how to play. Intuitively, cooperation will emerge when the chances of having another encounter with the player are high. In this analysis, we will see how mathematics can be helpful to social sciences and at the same time how social sciences and humanities are crucial to understand the results mathematics produce. The outcome of Axelrod's tournament turned out to be very interesting, and it is convenient to know why. The strategy "Tit for tat", the simplest of all the submitted ones, won the tournament. "Tit for tat" starts cooperating in the first move and copies the strategy his opponent employed in the previous step of the game. The key to its success, using Alxelrod terms is that: it is nice, it retaliates but

at the same time is forgiving. Being nice means never being the first player to defect, i.e., starting the game with good intentions. This is a property that clearly determines the success of the strategy, as the eight top-ranking entries were nice and none of the others was. The reason for this is that a nice strategy does very well when confronted with another nice strategy, as they cooperate until the end of the game. However, just as it happens in real life where mean people can take advantage of the kindness of other people, the strategies that were too nice were exploited by some of the others. Retaliation is necessary to avoid this, but how should we punish our opponent? Some of the strategies were nice but resentful, so when the opponent betrayed them, they would punish him defecting ever after. This attitude is not clever because it fosters a defection loop that in then end will be negative for all the players. Even if there has to be retaliation, it has to be proportional. "Tit for tat" punishes a defection with a defection, but it is happy to cooperate again if his opponent rectifies. This is the key to "Tit for tat's" success.
In this undergraduate thesis, I thought it would be interesting to repeat this tournament. To do so, I created a program in C that repeats and slightly modifies Axelrod's confrontations. Before we start making calculations, there are certain parameters that need to be given, which are the number n of iterations and the elements of the matrix of the game: R, P, S and T [2]. Let's thus create a vector:

```
int G[5]= {n,R,P,S,T};
```

We are going to adopt the following convention: a 0 will denote cooperation and a 1 defection. The strategies will therefore be n-vectors of either 0s or 1s. Let s1 be the strategy of player one and s2 the strategy of player 2. Clearly si[j] tells us the strategy player i employs in turn j. To confront them, we create a function that takes the vector G, s1 and s2 and returns the payoff of player 1 after the n iterations. This is the form of our function:

```
int game(int G[5], int s1[G[0]], int s2[G[0]]){
    int a = 0;
    for(int i=0; i<G[0]; i++){
        if((s1[i] == 0)&&(s2[i] == 0)){
            a=a+G[1];
        }
        else if((s1[i] == 1)&&(s2[i] == 1)){
            a=a+G[2];
        }
        else if((s1[i] == 0)&&(s2[i] == 1)){
            a=a+G[3];
        }
        else if((s1[i] == 1)&&(s2[i] == 0)){
            a=a+G[4];
        }
    }
```

[2]Let us recall that R is the reward for mutual cooperation, P the punishment for mutual defection, T is the temptation to defect and S is the sucker's payoff

```
    return a;
}
```

We are going to confront 11 strategies in total. We will therefore create a matrix int b[11][11], where b[i][j] will denote the payoff of player 1 when he plays strategy i against a player that employs strategy j. Let us explain all the strategies we will use:

- Always Cooperation (C)
- Always Defection (D)
- Random Choices (RC)
- Alternate between cooperation and defection (Alt)
- Tit for Tat (TfT)
- Harsh Tit for Tat (HTfT): If in one of the two last iterations my opponent defected, I will punish and defect him too. Otherwise I cooperate.
- Resentful (Res): I cooperate as long as my partner cooperates. If he defects, I shall defect ever after.
- Evil Tit for Tat (ETfT): I defect in the first move and use Tit for Tat ever after.
- Light Resentful (LRes): I always cooperate as long as my partner does not defect twice. In other other, if he defects once I will give him a second chance. If he ever defects again, I will defect ever after.
- Majority Other (Maj): I keep track of the opponent and employ the strategy he most employs. I will start with good intentions, i.e., cooperating.

Let us present the results, for n=1000, R=3, P=1, S=0 and T=5.

Strategies	C	D	RC	Alt	TfT	HTfT	Res	ETfT	LRes	Maj
C	3000	0	1434	1500	3000	2994	3000	2997	3000	3000
D	5000	1000	2912	3000	1004	1004	1004	1000	1004	1004
RC	4044	522	1956	2284	2200	1342	526	2196	526	580
Alt	4000	500	2174	2000	2503	500	2503	2500	503	4000
TfT	3000	999	2200	2498	3000	3000	3000	2500	3000	3000
HTfT	3004	999	2562	3000	3000	3000	3000	1000	3000	3000
Res	3000	999	2911	2498	3000	3000	3000	1003	3000	3000
ETfT	3002	1000	2201	2500	2500	1000	1003	1000	3002	2500
LRes	3000	999	2911	2998	3000	3000	3000	2997	3000	3000
Maj	3000	999	2890	1500	3000	3000	3000	2500	3000	3000

Table 6.1. Tournament Result.

If we now sum all the elements in one row we will obtain the total payoff for the corresponding strategy after all the confrontations. Let us directly present the ranking:

1. Light Resentful (27905)
2. Tit for Tat (26197)

3. Majority other (25889)
4. Harsh Tit for Tat (25565)
5. Resentful (25411)
6. Always cooperation (23925)
7. Alternate (21183)
8. Evil Tit for Tat (19708)
9. Always defection (17932)
10. Random (16176)

Axelrod's analysis can account for this distribution. Notice that the six top-ranking entries are nice and none of the others are. This clearly shows that being nice is extremely important for maximizing your result. But we can say more: among these 6 strategies, 5 retaliate (at some point) and 1 - "always cooperation" - does not. Just as Axelrod suggests, those strategies that retaliate do better and that's why "always cooperation" is the last of the nice strategies. Finally, Axelrod claims a good strategy has to be forgiving. "Resentful", which is the last of the 5 nice and retaliatory strategies, is not forgiving at all. The others, to some extent, do forgive. It is important to stress to what great extent Axelrod's explanations account for the overall ranking on the strategies. The analysis cannot go any further without going into the details of every strategy and I invite the reader to do so. The most important difference between my tournament and Axelrod's is that, even if "Tit for Tat" ranks very good in mine, it does not hold the first position as in Axelrod's. Instead, "Light Resentful", a strategy that was not part of the 14 that Axelrod confronted, won my tournament. This, far from showing that Axelrod was wrong, shows that there is no best strategy regardless of the one you play against. In his tournament, some of the strategies tried to figure out how the opponent played by analyzing how he responded to cooperation and to defection. If the opponent cooperated regardless of whether they defected, they would always defect and exploit him. However, if the opponent retaliated, they cooperated with him as long as he went back to cooperation. Therefore, "Tit for Tat" did good against these strategies. On the contrary, "Light Resentful", given that it is not forgiving after the second defection, did awful as a spiral of defection emerged that was detrimental to both players.

6.2 Evolutionary Game Theory

To continue, let's see how Game Theory can be applied to the field of Biology, specially evolution. In the early seventies, Game Theory, which was centered on the concept of a rational individual, was modified and enriched to be applied to a wide range of biological problems. In classical Game Theory, individuals could choose their strategies out of a certain set and could change them in repeated games. However, evolutionary Game Theory deals with entire populations whose members have fixed strategies - for instance, a type of behavior. A change of strategy is not a decision of a certain player but the replacement of certain individuals by their offspring. In this kind of games, what a player does depends on what everybody else is doing and that's why they are called frequency-dependent games. Here it is not one species evolving against another, but the members of a certain species evolving against one another.

6.2.1 Basic Concepts

Imagine a mutation comes up in a species and modifies the behavior of certain individuals. Is this mutation going to spread across the population? If it increases the reproductive success of those individuals who carry it, it definitely will, and we will call this an invasion. A Evolutionary Stable Strategy (ESS) is a strategy that cannot be invaded. This is typically a problem of Dynamical Systems. Consider a population consisting of n different types of individuals and let x_i be the frequency of type i. We are interested in studying how this frequency changes with time, i.e., in $\dot{x_i}$. If we assume that individuals meet randomly and engage in a game with matrix A, the average payoff for an individual of type i is:

$$1_i A x^T = (Ax^T)_i$$

whereas xAx^T clearly stands for the mean payoff. The rate of change of the frequencies of the different types derives from the reproductive success (fitness) of each of these types. In evolutionary Game Theory, the payoff will be the fitness. This yields the replicator equation:

$$\dot{x_i} = x_i((Ax^T)_i - xAx^T) \qquad \forall i$$

How can we know which alternatives and mutations are there that could invade a species? It is a complicated question. However, if we restrict our attention to a certain finite set of n possible mutations (that let's assume that will generate n different behaviors), it can be proved that there's a bijection between the zeros of the replicator equation and the equilibrium situations of the symmetric game A which results from the confrontation of individuals with one of these behaviors. We therefore have a link between Dynamical Systems and Game Theory (normally called the Folk Theorem of evolutionary Game Theory) and then employing the tools we have developed we can readily solve evolutionary-related questions.

Let's focus on the easiest kind of games, where only two players interact. The first thing it should be said is that we want the players to be interchangeable individuals. As they are part of the same population, they must have available the same strategies and the same payoff matrix. The games where $S_1 = S_2$ and $a_{ij} = H_1(i,j) = H_2(j,i) = b_{ji}$ are called symmetric games. Let $W(I)$ be the mean fitness of the individuals that employ strategy I. Let us assume I is the strategy employed by all the members of a population and J is trying to invade. I will be stable if $W(I) > W(J)$, i.e., if it is more widely spread than the other. At the same time, I will be stable iff $H_1(I,I) > H_1(J,I)$ or if $H_1(I,I) = H_1(J,I)$ then $H_1(I,J) > H_1(J,J)$. If this conditions do not hold, J is going to invade I.

Imagine that two animals contest a resource of value V. When contesting the resource, there are two possible strategies: F: fight for it, and D: display but back off if attacked. Therefore $S_1 = S_2 = \{F, D\}$. If F confronts D, his fitness will be increased by V. D, on the other hand, won't modify his fitness out of the encounter. If two individuals F contest the resource, one of them will get injured and suffer a loss of fitness C. If two individuals D meet, they will cooperate and share the resource. Therefore:

$$\begin{array}{cc} & \begin{array}{cc} F & D \end{array} \\ \begin{array}{c} F \\ D \end{array} & \begin{pmatrix} \left(\frac{V-C}{2}, \frac{V-C}{2}\right) & (V, 0) \\ (0, V) & \left(\frac{V}{2}, \frac{V}{2}\right) \end{pmatrix} \end{array}$$

where we have assumed there's a 50% chance for F to win or get injured and, given that we do a long-term analysis, we have averaged his payoff. We can say some interesting things by looking at the matrix

- D is not an EES. If in a population of Ds one F comes up, it will rapidly spread throughout the population, as it will increase his fitness by V in every encounter.
- F is an EES if $\frac{V-C}{2} > 0$, i.e., if $V > C$. If this conditions holds $H(F, F) > 0 = H(D, F)$. However, if $V < C$, D invades. That's because the fighters basically mutilate each other, whereas a D is a neutral element. D's fitness is not decreased but F's is.

Let's proceed now to find the equilibrium situations of this game using the swastika method and check the folk theorem. Given that the game is symmetric $H_1(x, y) = H_2(y, x)$, so we only need to analyze one of the players. Let's focus on player 1, who has a strategy $(x, 1-x)$ and plays against player 2, who employs $(y, 1-y)$.

$$H_1 = x\left(\frac{V}{2} - \frac{C}{2}y\right) - \frac{V}{2}(y-1)$$

Therefore:

- if $\frac{V}{2} - \frac{C}{2}y < 0 \implies x = 0$.
- if $\frac{V}{2} - \frac{C}{2}y = 0 \implies$ any x.
- if $\frac{V}{2} - \frac{C}{2}y > 0 \implies x = 1$.

The expression $\frac{V}{2} - \frac{C}{2}y = 0$ is equivalent to $y = V/C$. Therefore we can write:

- if $y < \frac{V}{C} \implies x = 1$
- if $y = \frac{V}{C} \implies$ any x
- if $y > \frac{V}{C} \implies x = 0$

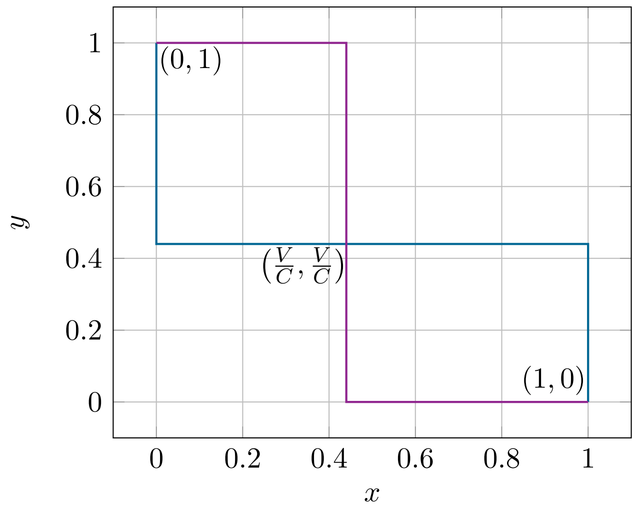

Figure 6.1: Swastika method.

If $V > C$, then clearly $\frac{V}{C} > 1$ and therefore $\forall y \in [0,1]$ it follows than $y < V/C$, so player 1 will set $x = 1$. Thanks to the symmetry of the game we will also have $y = 1$ so the situation (F, F) will be an equilibrium situation.
If $V < C$ the swastika method leads to fig. 6.1, from where we see that we have 3 equilibrium situations.

1. (F, D). Then $H_1(F, D) = V$ and $H_2(F, D) = 0$.

2. (D, F). Then $H_1(D, F) = 0$ and $H_2(D, F) = V$.

3. $(X^*, Y^*) = \left(\left(\frac{V}{C}, 1 - \frac{V}{C}\right), \left(\frac{V}{C}, 1 - \frac{V}{C}\right)\right)$. Then $H_1(X^*, Y^*) = H_2(X^*, Y^*) = \frac{V}{2}\left(1 - \frac{V}{C}\right)$.

Clearly there is a bijection between an EES and a equilibrium situation. The methods we have developed in this undergraduate thesis have enabled us to make a complete analysis of this. As we already said, the situation (D, D) is never an EES and (F, F) is only an EES when $V > C$. However, the following results are new. The first two equilibrium situations for $V < C$ are not symmetric. They come from the fact that D is not tempted to turn into a F (this obviously doesn't make sense in evolutionary Game Theory) because now that $V < C$ he would decrease his payoff. However, in evolutionary Game Theory we are looking for symmetric equilibrium situations where all the individuals are interchangeable. We therefore have to turn to $\left(\frac{V}{C}, 1 - \frac{V}{C}\right)$.
This game, for $V < C$ shows one of the main models of human cooperation. Just like the prisoners dilemma, it confronts us with a situation in which the cost of not cooperating (C) is higher than the gain of betraying your opponent (V), but at the same time betraying you cooperating opponent is more beneficial than cooperating with him. In this case, what Game Theory suggests is betraying V/C of the times and cooperating the rest. Again, even if we are selfish and only want to maximize our payoff, there is room for cooperation.

6.3 The Ultimatum Game

To conclude this chapter, it is convenient to analyze an experiment, the result of which cannot be explained with the tools we have developed. Imagine that we randomly select two people and we tell them to play the following game: Player 1 has to propose a way to divide a certain quantity of money between the two players and then player 2 decides if he accepts or rejects the deal. If he accepts, the distribution is carried out as player 1 suggested, but if he rejects nobody gains anything. According to classical Game Theory, even if player 1 proposes to keep with 99% of the total amount and give only 1% to player 2, he should always accept the deal because something, even if it's a paltry sum of money, is better than nothing. The economist Ernst Fehr, from the University of Zurich, conducted this experiment to find that people do not adhere to the recommendations of Game Theory. In fact, most of the times that player 1 offered player 2 less than 30% of the initial value, the deal was rejected. When you reject a deal, you are sacrificing a certain economic gain in order to punish the other player. Those who renounce to a gain do it for the greater good. The fact that you could increase your payoff is subordinated to a bigger idea: you have to make sure your opponent does not increase his payoff. This concept, known as altruistic punishment, is a crucial element in fostering cooperation. At the moment, nobody has been able to give an explanation for this in terms of self interest. This experiment, and other similar ones, has motivated the introduction of different kinds of players, apart form the selfish player, in Game Theory. This could be a good start to study Cooperative Game Theory. However, it is time for this undergraduate thesis to come to an end.

Conclusions

In this undergraduate thesis the foundations of Noncooperative Game Theory have been laid. From a mathematical standpoint, Nash's Theorem (Theorem (5.4)) represents the cornerstone of this senior thesis. It asserts that in any noncooperative game - regardless of the number of players, of the number of strategies they can employ, etc - there is at least one equilibrium situation. However, Nash's Theorem falls under the category of those theorems which are only really appreciated by mathematicians. This is because, even if its implications are huge, it doesn't give the slightest clue on how to find Nash equilibriums, i.e., it is not constructivist. Therefore, when it comes to applying Game Theory to real life, it is rather useless. The main goal of my senior thesis was to show how Game Theory could be used to tackle a great number of different problems. Throughout this undergraduate thesis, Game Theory has served us to design optimal war strategies, to study a duopoly or to analyze how animals contest a resource. However, from a social standpoint, the cornerstone of this research work was to show that cooperative dispositions can indeed emerge in a world of selfish individuals.

I would consider myself to be satisfied if I managed to convince the reader that Game Theory is a really useful tool to analyze all kinds of situations in which there is some sort of interaction. If I had the opportunity to pursue research in this field, I believe it would be very interesting to study Cooperative Game Theory, as it would enable us to gain a new perspective and tackle a lot of new problems.

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