NEWTON'S METHOD ON BRING-JERRARD POLYNOMIALS

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Abstract. In this paper we study the topology of the hyperbolic component of the parameter plane for the Newton's method applied to n-th degree Bring-Jerrard polynomials given by $P_n(z) = z^n - cz + 1,\ c \in \mathbb{C}$. For $n = 5$, using the Tschirnhaus-Bring-Jerrard non linear transformations, this family controls, at least theoretically, the roots of all quintic polynomials. We also study a bifurcation cascade of the bifurcation locus by considering $c \in \mathbb{R}$.

Keywords: Newton's method, holomorphic dynamics, Julia and Fatou sets, hyperbolic components, bifurcation locus.

1. Introduction

The historical seed of complex dynamics goes back to Ernst Schröder and Arthur Caley who, at the end of the nineteenth century, investigated the global dynamics of Newton's method in $\mathbb{C}$ applied to polynomials of degree two (previous studies did not deal with the complex variable). They were able to see that the two neighborhoods around each root of the quadratic polynomial where Newton's method converges to each root, in fact extend to two half planes and the separation straight line between them is precisely the bisectrix. In other words, any Newton map for a quadratic polynomial is (linearly) conjugate to the map $z \to z^2$ (in McMullen language the family is trivial [15]). With the same aim, Caley also considered the global dynamics of Newton's method applied to cubic polynomials but he was not able to conclude satisfactorily.

Since then, complex dynamics as a whole, that is the study of iterates of holomorphic maps on the complex plane or the Riemann sphere, has become an important issue in dynamical systems. The natural space for iterating a rational map f is the Riemann sphere. So, for a given rational map f, the sphere splits into two complementary domains: the *Fatou set* $\mathcal{F}(f)$ where the family of iterates $\{f^n(z)\}_{n\geq 0}$ is a normal family, and the *Julia set* $\mathcal{J}(f)$ where the family of iterates fails to be a normal family. The Fatou set, when non empty, is given by the union of, possibly, infinitely many open sets in $\hat{\mathbb{C}}$, usually called Fatou components. On the other hand, it is known that the Julia set is a closed, totally invariant, perfect non empty set, and coincides with the closure of the set of repelling periodic points. For a deep and helpful review on iteration of rational maps see [16].

The rational (transcendental meromorphic) family given by the Newton's map applied to a polynomial (transcendental entire) family has become a central subject in complex dynamics. The reason for this special interest is based on the implications of this global analysis on Newton's map as a root finding algorithm. It is very difficult, or, possibly, not possible, to give a short survey on Newton's method and how a better understanding of the whole

The first and fourth authors were partially supported by P11B2011-30 (Convenio Bancaja-Universitat Jaume I) and by the Spanish grant MTM2011-28636-C02-02. The second and third authors were partially supported by the Catalan grant 2009SGR-792, and by the Spanish grant MTM2011-26995-C02-02.

dynamics gives a better understanding of the Newton's map as a root finding algorithm. But we focus on some main observations connected to our work.

A first important observation coming from this global analysis is somehow negative. Newton's method applied to cubic (or higher degree) polynomials $Q_c(z) = z(z-1)(z-a),\ a \in \mathbb{C}$ fails. That is, there are open sets in the a-parameter plane for which there are open sets in dynamical plane converging to neither 0, 1 nor a. The reason for this is the existence of a *free* critical point that, for certain parameters does its own dynamical behavior independently of the attracting basins associated to the roots of Q_c. A remarkable result due to C. McMullen [14] goes deeply in this direction by showing that even though we can substitute Newton's map for another rational root finding algorithm for which the previous limitation is solved, the problem is unsolvable for polynomials of higher degree.

A second relevant consideration is, given P, how to use the Newton's map to find numerically all roots of P; that is, how to choose the initial seeds to ensure we get all roots of P. This important question, from the numerical analysis point of view, was solved using a dynamical system approach in the paper [13], where the authors gave a universal set of initial conditions, with cardinality depending only on the polynomial degree.

A third remark is a topological question which relates the connectivity of the Julia set, or equivalently, the simple connectivity of the Fatou components. It is well known that rational maps, in general, may have non simply connected Fatou components given by either Herman rings (doubly connected components), basins of attraction or parabolic basins with (infinitely many) wholes or preimages of simply connected components which could be multiple connected. Przytycki [17] showed that every root of a polynomial P has a simply connected immediate basin of attraction for its corresponding Newton's method N_P (see below for formal definitions). Later, Meier [22] proved the connectivity of the Julia set of N_P when deg $P = 3$, and later Tan [20] generalized this result to higher degrees of P. However, the deeper result in this line is due to Shishikura [23] who proved that the Julia set of N_P is connected for any non-constant polynomial P. In fact, he obtained this result as a corollary of a much more general theorem for rational functions, namely, the connectedness of the Julia set of rational functions with exactly one weakly repelling fixed point, that is, a fixed point which is either repelling or parabolic of multiplier 1.

Similarly, and in most cases strictly related to this, it is important to study the topology of the hyperbolic components in the parameter plane and, consequently, the structure of the bifurcation locus. A cornerstone example of this is the paper of P. Roesch [18] were she used the Yoccoz Puzzles to prove the simple connectivity of hyperbolic components in parameter as well as the dynamical plane for the family of cubic polynomials.

The main goal of this paper is to study some topological properties of the parameter plane of Newton's method applied to the family

$$P_{n,c} := P_c(z) = z^n - cz + 1, \tag{1}$$

where $n \geq 3$ (to simplify the notation we will assume, throughout the whole paper, that n is fixed; so, we erase the dependence on n unless we need to refer to it explicitly). The interest to consider this family is explained in Section 2 where we show that the general quintic equation $P_5(z) = 0$ can be *transformed* (through a strictly non linear change of variables) to one of the form $P_{5,c} := z^5 - cz + 1 = 0,\ c \in \mathbb{C}$. Letting n as a parameter in (1) allow us to have a better understanding of the problems we are dealing with.

Easily, the expression of the Newton's map applied to (1) writes as

$$N_c(z) = z - \frac{P_c(z)}{P_c'(z)} = z - \frac{z^n - cz + 1}{nz^{n-1} - c} = \frac{(n-1)z^n - 1}{nz^{n-1} - c}. \tag{2}$$

So, the critical points of N_c correspond to the zeroes of P_c, which we denote by $\alpha_j,\ j = 0, \ldots, n-1$, and $z = 0$ which is the unique *free* critical point of N_c of multiplicity $n-2$. We notice that since all critical points except $z = 0$ coincide with the zeroes of P_c they are superattracting fixed points; so, their dynamics is fixed for all $c \in \mathbb{C}$.

For each root $\alpha_j(c) := \alpha_j,\ j = 0, \ldots, n-1$ we define its *basin of attraction*, $\mathcal{A}_c(\alpha_j)$, as the set of points in the complex plane which tend to α_j under the Newton's map iteration. That is

$$\mathcal{A}_c(\alpha_j) = \{z \in \mathbb{C},\ N_c^k(z) \to \alpha_j \text{ as } k \to \infty\}.$$

In general $\mathcal{A}_c(\alpha_j)$ may have infinitely many connected components but only one of them, denoted by $\mathcal{A}_c^*(\alpha_j)$ and called *immediate basin of attraction of* α_j, contains the point $z = \alpha_j$.

Similarly, the hyperbolic components in the c-parameter plane are the open subsets of $\mathbb{C}$ in which the unique *free* critical point $z = 0$ either eventually map to one of the immediate basin of attraction corresponding to one of the roots of P_c or it has its own hyperbolic dynamics associated to an attracting periodic point of period greater than one. Of course, the bifurcation locus corresponds to the union of all boundaries of those components and possible accumulating points (see Section 4 for more precise definitions).

If $N_c^k(0) \in \mathcal{A}_c^*(\alpha_j),\ k \geq 0$ for some $j = 0, \ldots n-1$ (that is the free critical $z = 0$ is eventually trapped by one of the roots of P_c) we say that c is a *capture parameter*. As we will see, the set of all capture parameters has infinitely many connected components depending on the first number $k \geq 0$ and the value of j so that $N_c^k(0) \in \mathcal{A}_c^*(\alpha_j)$. To distinguish among different captured hyperbolic components we use the following notation which takes into account the number of iterates of $z = 0$ to get into the immediate basin of attraction of some of the roots:

$$\begin{aligned} \mathcal{C}_j^0 &= \{c \in \mathbb{C},\ 0 \in A_c^*(\alpha_j)\} \quad \text{and} \\ \mathcal{C}_j^k &= \{c \in \mathbb{C}, N_c^k(0) \in A_c^*(\alpha_j) \text{ and } N_c^{k-1}(0) \notin A_c^*(\alpha_j),\ k \geq 1\}. \end{aligned} \tag{3}$$

Under this notation we prove, in Section 4, some topological results about those stable subsets of the parameter plane.

Theorem A. *The following statements hold.*

(a) $\mathcal{C}_0^0$ *is connected, simple connected and unbounded.*
(b) $\mathcal{C}_j^0,\ 1 \leq j \leq n-1$ *are empty.*
(c) $\mathcal{C}_j^1,\ 0 \leq j \leq n-1$ *are empty.*
(d) $\mathcal{C}_j^k,\ 0 \leq j \leq n-1$ *and* $k \geq 2$ *are simply connected as long as they are non empty.*

The proof of statements (a) and (b) follows directly from Proposition 4.3 while (c) and (d) follows from Proposition 4.5. Apart from the captured components we also observe the presence of Generalized Mandelbrot sets $\mathcal{M}_k$ (the bifurcation locus of the polynomial families $z^k + c,\ c \in \mathbb{C}$). As an application of a result of C. McMullen [15], we can show that for a fixed n, all non-captured hyperbolic components correspond to $n-1$ Generalized Mandelbrot sets. Precisely, we can prove the following (see Proposition 4.7)

Corollary B. *Fix $n \geq 3$. The bifurcation locus $\mathcal{B}(N_n)$ is non empty and contains the quasi-conformal image of $\partial\mathcal{M}_{n-1}$ and $\mathcal{B}(N_n)$ has Hausdorff dimension two. Moreover, small copies of $\partial\mathcal{M}_{n-1}$ are dense in $\mathcal{B}(N_n)$.*

Finally, we turn the attention to real parameters. Because of the symmetries in the parameter plane, to have a good understanding of real positive values of c is quite important to describe the bifurcation locus. In Section 5 we show the existence of different sequences of c-real values tending to 0 corresponding to centers of capture components, preperiodic parameters and centers of the main cardioids of $\mathcal{M}_{n-1}$ sets.

Theorem C. *Fixed $n \geq 3$ and let c be a positive real parameter. Denote by $c^* = n/(n-1)^{\frac{n-1}{n}}$. The following statements hold:*

(a) *If $c > c^*$ then $c \in \mathcal{C}_0^0$.*
(b) *If $c < c^*$ there are two different decreasing sequences of parameters tending to 0 for which the free critical point $z = 0$ is (i) a superattracting periodic point (with increasing period) or (ii) a preperiodic point (in fact pre-fix, with increasing pre-periodicity). Moreover,*
   (b.1) *If n is odd, there is a decreasing sequence of parameters tending to 0 for which the free critical point $z = 0$ is the center of a capture component C_j^k for some j, and*
   (b.2) *If n is even, $\mathcal{C}_j^k \cap \mathbb{R} = \emptyset$ for any $j = 0, \ldots, n-1$ and $k \geq 2$.*

The proof of statement (a) follows from Lemma 4.4. The rest of the statements follows from Proposition 5.1.

The paper is organized as follows. In Section 2 we briefly explain the reduction of a general quintic equation to its Bring-Jerrard form. In section 3.1 we give some results on the dynamical plane of the Newton's map N_c. In Section 4 we state and prove the topological properties of the hyperbolic components on the parameter plane. Finally, in Section 5 we study real parameters and prove Theorem C.

2. Tschirnhaus', Bring's and Jerrard's transformations

As we have already explained in the introduction we study the Newton's method applied to the family of n-degree polynomials (1) defined by

$$P_{n,c}(z) = z^n - cz + 1.$$

Easily, any polynomial of degree 5 can be linearly conjugate through $\eta(z) = \eta_1 z + \eta_2$ to one monic polynomial without degree 4 term. Using this idea any quadratic polynomial $az^2 + bz + c,\ a, b, c \in \mathbb{C}$ can be reduced to a polynomial of the family $z^2 + \lambda,\ \lambda \in \mathbb{C}$. Of course, via a linear transformation, we cannot expect to reduce (in the sense of getting a conjugacy) all quintic polynomial to a one parameter family, concretely to the family (1), like in the quadratic case.

However, using non linear transformations, it is possible to actually reduce all quintic polynomials (in a weaker sense only preserving certain information of the roots of the original polynomial) to the family (1) for $n = 5$. Consequently, the interest of applying Newton's method to (1) is due to Tschirnhaus' (Bring's and Jerrard's) transformations applied to 5-degree polynomials. For a good explanation of all these transformations see the translation of the original paper of Tschirnhaus [24], the short review in [1] and references therein. For completeness we give here a brief summary.

In his original paper in 1683, Tschirnhaus proposed a method for solving $P_n(z) = 0$, where P_n is a polynomial of degree n, by simplifying it to a polynomial $Q_n(y)$ where Q_n is a (simpler) polynomial of degree n with less coefficients (trivially, the linear change of variables allows to eliminate the coefficient z^{n-1}). His idea was to introduce the new variable y in the form $y = T_k(z)$ with $k < n$. Tschirnhaus' original idea was used later by Bring and Jerrard to move forward in the simplification process. Although Tschirnhaus' method works for general polynomials of degree n, here we present $n = 5$.

Precisely, we want to reduce the general expression of a quintic equation

$$z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0, \; a_i \in \mathbb{C} \tag{4}$$

to one of the form

$$z^5 + c_1 z + c_0 = 0, \tag{5}$$

in such a way that the roots of (4) can be recuperated once the roots of (5) are obtained. To do so, we first reduce the general quintic equation to its *principal form*, that is,

$$z^5 + b_2 z^2 + b_1 z + b_0 = 0. \tag{6}$$

The n-th power-sums of the roots, x_j's of (4) are given by

$$S_n = S_n(x_j) = \sum_{j=1}^{5} x_j^n, \quad n = 1, \ldots \tag{7}$$

which satisfy (Newton's formulae [9])

$$S_n = -n a_{5-n} - \sum_{j=1}^{n-1} S_{n-j} a_{5-j}$$

where $a_j = 0$ if $j < 0$. For equation (4) we have, for instance, $S_1 = -a_4$ and $S_3 = -a_4^3 + 3a_3 a_4 - 3a_2$. The key idea is to assume (and prove) that the roots x_j's of (4) are related to the roots y_j's of (6) through a quadratic (Tschirnhaus) transformation

$$y_j = x_j^2 + \alpha x_j + \beta, \quad \alpha, \beta \in \mathbb{C}. \tag{8}$$

That is, we want to see that α and β can be expressed algebraically in terms of the coefficients. From Newton's formulas, the power sums for equation (6) gives

$$S_1 = S_2 = 0, \; S_3 = -3b_2, \; S_4 = -4b_1, \; S_5 = -5b_0. \tag{9}$$

Hence, from $S_1 = S_2 = 0$, we obtain

$$\begin{aligned} \alpha a_4 - 5\beta + 2a_3 - a_4^2 &= 0, \\ a_3 \alpha^2 - 10\beta^2 + (3a_2 - a_3 a_4)\,\alpha + 2a_1 - 2a_2 a_4 + a_3^2 &= 0, \end{aligned} \tag{10}$$

and from those equations we can solve for α and β, algebraically, in terms of the coefficients a_k's (indeed the equations are quadratic in α and β and we are free to choose either of the solutions). In turn, it is an exercise to see (but involve some computations) that from the three later equations in (9), we may obtain $b_j, \; j = 0, 1, 2$ as functions of a_k's, α and β.

Once we have reduced the general equation (4) into its principal form (6) we want also to eliminate the quadratic coefficient b_2 of the later expression to get its Bring-Jerrard form (5).

A first attempt (the one Tschirnhaus had in mind) may be to impose the cubic equation (so getting an extra parameter)

$$r_j = y_j^3 + \alpha y_j^2 + \beta y_j + \gamma, \quad \alpha, \beta, \gamma \in \mathbb{C}. \tag{11}$$

for the roots of (5), denoted by r_j's, and the roots y_j's of (6). If we argue as before, Newton's formulae for the power sums for equation (5) gives

$$S_1 = S_2 = S_3 = 0, \; S_4 = -4c_1, \; S_5 = -5c_0. \tag{12}$$

However to determine α, β and γ using $S_1 = S_2 = S_3 = 0$ one gets a sixth degree polynomial for α, so not being solvable by radicals.

The new ingredient introduced by Bring and Jerrard was to add an extra parameter so that equation (11) becomes

$$r_j = y_j^4 + \alpha y_j^3 + \beta y_j^2 + \gamma y_j + \delta, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}. \tag{13}$$

Using the three first equations in (12), equation (13) and Newton's formulas applied to the principal form (6) we get three new equations from which it is possible to write α, β, γ and δ as algebraic functions of the b_j's coefficients. From the first of those equation we obtain

$$\delta = \frac{1}{5}(4b_1 + 3\alpha b_2),$$

which we substitute in the second equation to get

$$-10\alpha\beta b_0 - 4\beta^2 b_1 + \frac{4}{5}b_1^2 + 8b_0 b_2 + \frac{46}{5}\alpha b_1 b_2 + \frac{6}{5}\alpha^2 b_2^2 + 6\beta b_2^2 - 2\gamma\,(5b_0 + 4\alpha b_1 + 3\beta b_2) = 0$$

If we (cleverly) choose β to cancel out the γ coefficient in the above equation, the expression becomes quadratic in α, so algebraically solvable. Finally, substituting δ, β and α in the third of those mentioned equations we obtain a cubic equation for the later coefficient γ. As a final step in this process we use the fourth and fifth equations in (12) to determine (linearly) the coefficients c_0 and c_1 in terms of the b_j's.

All this process allows to reduce the original equation (4) to the simpler equation (5). Assuming you know the five solutions of the equation (5) you should inverse the process to find out the solutions of your original equation (4). Since the transformations you have applied are not linear at all, what happens is that you have twenty candidates for the five zeros of (4). As far as we know there are no non-numerical tests to determine which one are the *correct* ones, but theoretically you could write the solutions of (4) in terms of the solutions of (5).

On the other hand it is easy to show that the Newton's method applied to the polynomial

$$P(z) = z^5 + c_1 z + c_2, \; c_1, c_2 \in \mathbb{C} \tag{14}$$

is either conjugate to the Newton's method applied to $q_a(z) = z^5 + az$ (which in turns is conjugate to either the Newton's method applied to $q_-(z) = z(z^4-1)$, or the Newton's method applied to $q_+(z) = z(z^4+1)$ or conjugate to the Newton's method applied to $q_0(z) = z^5$), or conjugate to one of the family

$$P_c(z) = z^5 - cz + 1, \; c \in \mathbb{C}$$

Consequently there is a formal connection between the use of Newton's method for the general quintic equation and its Bring-Jerrard form.

3. Dynamical plane: Distribution of the roots and attracting basins

In this section we prove some estimates, that we will need in next sections, for the relative distribution of the roots $\alpha_j,\ j=0,\dots n-1$ of the polynomials in family (1), assuming they are all different roots.

Fix $c\in\mathbb{C}$ and denote by $D(z_0,r)$ the disc centered at $z=z_0$ of radius $r>0$. Let $w_j := w_j(c),\ j=0,\dots,n-1$ be the n different solutions of $z(z^{n-1}-c)=0$. In particular, we set $w_0=0$. Next lemma shows that if $|c|$ is large enough we have $\alpha_j\in D(w_j,1),\ j=0,\dots,n-1$. In particular if $|c|$ is large enough we set α_0 to be the root of the corresponding polynomial such that $\alpha_0\in D(0,1)$.

Lemma 3.1. *The following statements hold:*

(a) *For all c in the parameter space, the roots $\alpha_0,\dots,\alpha_{n-1}$ of (1) belong to the set*

$$\mathcal{D}=\bigcup_{j=0}^{n-1} D(w_j,1)$$

(b) *Let $c\in\mathbb{C}$ such that*

$$|c|>\max\{2^{n-1},\frac{1}{\sin^{n-1}\left(\frac{\pi}{n-1}\right)}\}. \tag{15}$$

Then, $D(w_j,1)\cap D(w_k,1)=\emptyset,\ j\neq k$. Moreover, each $D(w_j,1)$ contains one and only one of the roots of (1).

(c) *If c is large enough, there exists $M:=M(n)>0$ such that*

$$|\alpha_0-N_c(0)|<M|c|^{-(n+1)} \tag{16}$$

Proof. Let α be any of the solutions of the equation $z^n-cz+1=0$ (that is $\alpha=\alpha_j$ for some $j=0,\dots,n-1$). Easily α should satisfy $|\alpha|\cdot|\alpha^{n-1}-c|=|\alpha|\cdot|\alpha-w_1|\cdot\ldots\cdot|\alpha-w_{n-1}|=1$. If $\alpha\notin\mathcal{D}$ we have that $|\alpha|\cdot|\alpha-w_1|\cdot\ldots\cdot|\alpha-w_{n-1}|>1$, a contradiction. Thus, statement (a) is proved.

By definition, the set $\mathcal{D}$ is formed by n discs of radius 1 and centered at the point $w_j,\ j=0,\dots n-1$. Notice that the $w_j,\ j=1,\dots,n-1$ are the vertices of a regular polygon of $n-1$ sides centered at 0 (lying on the circle centered at the origin and radius $|c|^{\frac{1}{n-1}}$) and hence the distance between two consecutive vertices is exactly $2|c|^{\frac{1}{n-1}}\sin\left(\frac{\pi}{n-1}\right)$ while the distance from each of them to the origin is $|c|^{\frac{1}{n-1}}$.

In order to prove that these discs are disjoint we only need to check that the distance between any pair of centers is bigger that 2. Taking into account the previous discussion this happens precisely if (15) is satisfied.

To finish the proof of statement (b) we should show that if the discs are disjoints each of them carries a unique root of (1). Fix c satisfying (15) or in other words, so that $\mathcal{D}$ is formed by n disjoints discs of radius 1 and centered at $w_j,\ j=0,\dots,n-1$. Define $h_1(z)=z(z^{n-1}-c)$ and $h_2(z)\equiv 1$. We claim that $|h_2(z)|<|h_1(z)|$ for all $z\in\partial D(w_j,1),\ j=0,\dots,n-1$. So, Rouche's Theorem implies that $h_1(z)$ and $h_1(z)+h_2(z)=z^n-cz+1$ have the same number of zeros in each $D(w_j,1)$. But clearly $h_1(z)$ has one and only one zero in each of the discs.

To see the claim we observe that

$$|h_1(z)|=|z(z^{n-1}-c)|=|z-w_0|\cdot|z-w_1|\cdot\ldots\cdot|z-w_{n-1}|.$$

If $z \in \partial D(w_j, 1)$ the factor $|z - w_j|$ is equal to 1 and the rest of the factors are bigger than 1, since by assumption $\mathcal{D}$ is formed by n disjoint dics. Obtaining thus that $|h_1(z)| > 1 = |h_2(z)|$ when $z \in \partial D(w_j, 1)$ for $j = 0, \ldots, n-1$.

Finally, we prove statement (c). Easily we have that $N_c(0) = 1/c$. Fix again c large enough so that $\mathcal{D}$ is formed by n disjoint dics, in particular we have that α_0 is in $D(0,1)$. Notice that since $c\alpha_0 = 1 + \alpha_0^n$ we have $|c\alpha_0| = |1 + \alpha_0^n| \leq 1 + |\alpha_0|^n \leq 2$ and so $|c\alpha_0|^n \leq 2^n$. Consequently,

$$|\alpha_0 - \frac{1}{c}| = \frac{1}{|c|}|c\alpha_0 - 1| = \frac{1}{|c|}|\alpha_0^n| = \frac{1}{|c|^{n+1}}|c\alpha_0|^n \leq 2^n \frac{1}{|c|^{n+1}}.$$

□

In particular, for a fixed n, as c goes to infinity the *small* root of (1) tends to $1/c$ (exponentially) faster than c approaching infinity. As we will state in Section 4.1, statement (c) of Lemma 3.1 is equivalent to say that for c outside a certain disc in the parameter plane, the free critical point $z = 0$ always belongs to the same immediate basin of attraction, the one of $\alpha_0 \sim 1/c$.

The following quite general topological properties of the basins of attraction and hyperbolic components of the Julia are well known (see, for instance, [13], where they studied Newton's method for a general polynomial, and Shishikura [19]).

Proposition 3.2. *The following statements hold:*

(a) $\mathcal{A}_c^*(\alpha_j)$ *is unbounded.*
(b) *The number of accesses to infinity of* $\mathcal{A}_{n,c}^*(\alpha_j)$ *is either 1 or* $n-1$*.*
(d) $\mathcal{J}(N_c)$ *is connected. So, any connected component of the Fatou set is simply connected.*

The classical Böttcher Theorem provides a tool related to the behavior of holomorphic maps near a superattracting fixed point [4], which we apply to make a detailed description of the superattracting basin of each simple root α_j for $j = 0, \ldots, n-1$ of N_c.

Theorem 3.3. *Suppose that f is an holomorphic map, defined in some neighborhood U of* 0*, having a superattracting fixed point at* 0*, i.e.,*

$$f(z) = a_m\, z^m + a_{m+1}\, z^{m+1} + \cdots \quad \textit{where } m \geq 2 \textit{ and } a_m \neq 0.$$

Then, there exists a local conformal change of coordinate $w = \varphi(z)$, called Böttcher coordinate at 0 *(or Böttcher map), such that $\varphi \circ f \circ \varphi^{-1}$ is the map $w \to w^m$ throughout some neighborhood of $\varphi(0) = 0$. Furthermore, φ is unique up to multiplication by an $(m-1)$-st root of unity.*

Assume that α_j is one of the simple roots of N_c for $j = 0, \ldots, n-1$. Applying Böttcher's Theorem near α_j the map N_c is conformally conjugate to $z \to z^2$ near the origin and we notice that this Böttcher map is unique since $m = 2$. As explained before, we will use a linear change of coordinates in order to have a monic expansion of N_c near α_j. Near α_j we have that

$$N_c(z) = \alpha_j + \frac{N_c''(\alpha_j)}{2!}(z - \alpha_j)^2 + \frac{N_c'''(\alpha_j)}{3!}(z - \alpha_j)^3 + \ldots$$

Using the conformal map $\tau(z) = \frac{N_c''(\alpha_j)}{2}(z - \alpha_j)$ we obtain that the map

$$\hat{N}_c(z) = (\tau \circ N_c \circ \tau^{-1})(z) = z^2 + \sum_{n\geq 3} \frac{2^{n-1}}{n!} \cdot \frac{N^{(n)}(\alpha_j)}{[N''(\alpha_j)]^{n-1}} \cdot z^n \tag{17}$$

is monic. For each $j = 0, \dots, n-1$, we denote by φ_j its corresponding Böttcher map (so $\varphi_j(\hat{N}_c)(z) = \varphi_j(z)^2$) such that $\varphi_j(0) = 0$, $\varphi_j'(0) = 1$. From equation (17) we deduce that

$$(\varphi_j \circ \tau) \circ N_c = D_2 \circ (\varphi_j \circ \tau)\,, \tag{18}$$

where $D_2(z) = z^2$. Hence $\varphi_j \circ \tau$ is the Böttcher map conjugating N_c near α_j to z^2 to 0.

Before going to the parameter plane we state a result we will use later which allows us to know when a rational map is the Newton's method of a certain polynomial. Precisely, we will use it at the end of the surgery construction in Proposition 4.5.

Lemma 3.4 ([12, 21]). *Any rational map R of degree d having d different superattracting fixed points is conjugate by a Möbius transformation to N_P (Newton's method) for a polynomial P of degree d. Moreover, if ∞ is not superattracting for R and R fixes ∞, then $R = N_P$ for some polynomial P of degree d.*

4. Hyperbolic components in the parameter plane of N_c

As we stated in the introduction the hyperbolic components in the parameter plane correspond to open subsets of $\mathbb{C}$ in which the unique *free* critical point $z = 0$ either eventually map to one of the immediate basin of attraction corresponding to one of the roots of P_c (those were denoted by $\mathcal{C}_j^k$ where j explains the *catcher* root and $k-1$ the number of iterates of $z = 0$ before it is *captured*, that is before reaches $\mathcal{A}_c(\alpha_j)$), or it has its own hyperbolic dynamics associated to an attracting periodic point of period strictly greater than one (black components in Figure 1). We use the following notation:

$$\begin{aligned} \mathcal{H} =& \{c \in \mathbb{C},\ 0 \text{ is attracted by an attracting cycle of period } p \geq 2\} \\ \mathcal{B} =& \{c \in \mathbb{C}\,,\ \text{the Julia set } \mathcal{J}(N_c) \text{ does not move continuously (in the Hausdorff topology)} \\ & \text{over any neighborhood of } c\}. \end{aligned}$$

The first lemma removes from our parameter plane those c-values for which the roots of P_c are not simple and so the Newton's method is not a rational map of degree n.

Lemma 4.1. *Fix $n \neq 2$. The Newton's map N_c is a degree n rational map if and only if*

$$c \neq c_k^* := \frac{n}{(n-1)^{\frac{n-1}{n}}} e^{2k\pi i/n},\ \ k = 0, \dots, n-1.$$

Proof. The rational map N_c has degree n as long as all roots of P_c are simple. Otherwise, the pair (z, c) should be a solution of the polynomial system

$$\begin{aligned} P_n(z) &= z^n - cz + 1 = 0, \\ P_n'(z) &= nz^{n-1} - c = 0. \end{aligned} \tag{19}$$

Solving system (19) we have

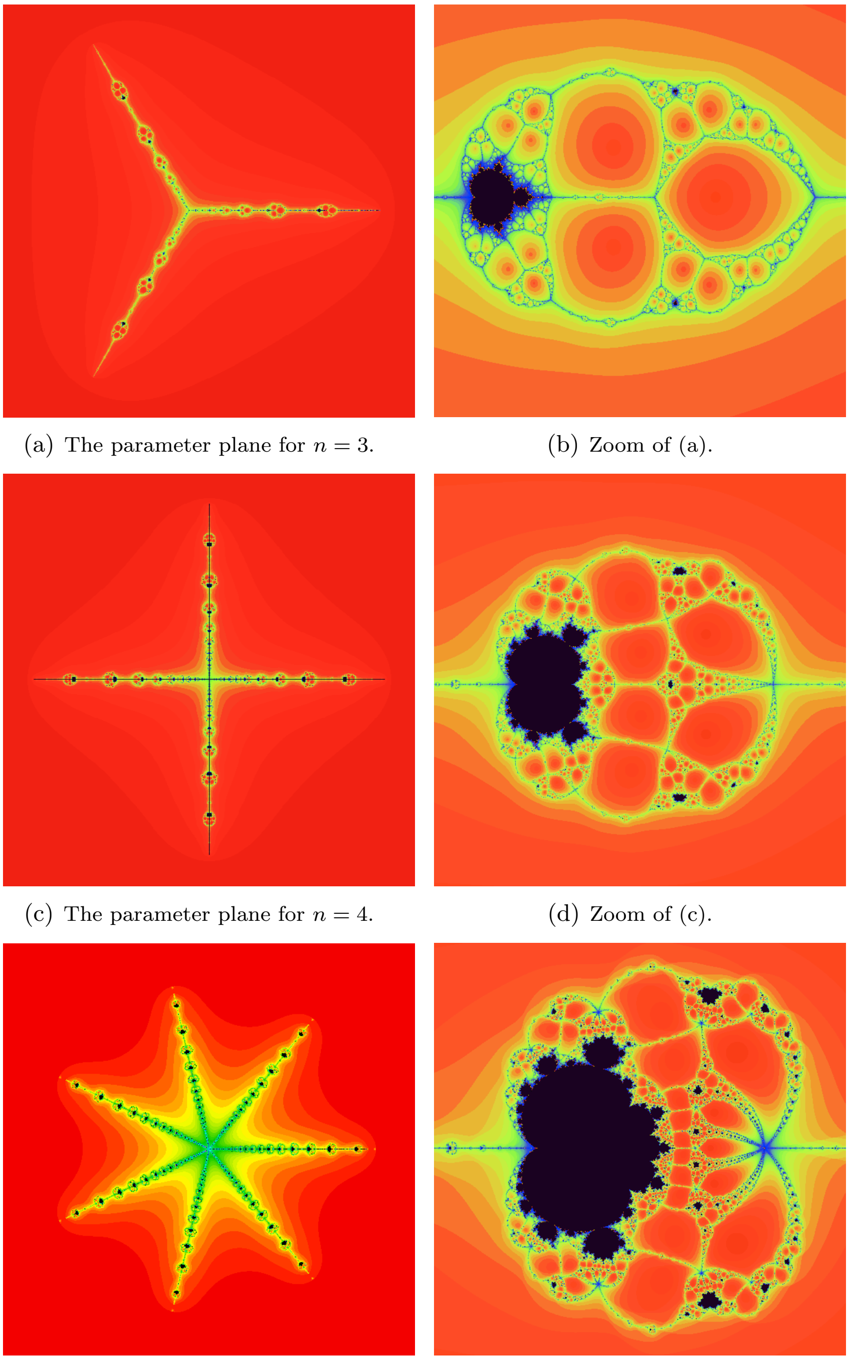
$$(z_k^*, c_k^*) = \left(z^* \exp(\frac{2k\pi i}{n}), c^* \exp(\frac{2k\pi i}{n})\right),\ \ k = 0 \dots, n-1.$$

where

$$(z^*, c^*) = \left(\left(\frac{1}{n-1}\right)^{\frac{1}{n}}, \frac{n}{(n-1)^{\frac{n-1}{n}}}\right), \tag{20}$$

denote the positive real values of the corresponding roots. □

In the next lemma we prove that we can focus on a *sector* in the parameter plane due to the following symmetries.



(a) The parameter plane for $n = 3$. (b) Zoom of (a).

(c) The parameter plane for $n = 4$. (d) Zoom of (c).

(e) The parameter plane for $n = 7$. (f) Zoom of (e).

FIGURE 1. Different parameter planes as n varies. From these pictures we can easily see the symmetries rigorously proven in Lemma 4.2 .

Lemma 4.2. *Let $n \geq 3$. The following symmetries on the c-parameter plane hold:*

(a) *The maps $N_c(z)$ and $N_{\hat{c}}(z)$ with $\hat{c} = e^{\frac{2\pi i}{n}} c$, are conjugate through the holomorphic map $h(z) = e^{\frac{2\pi i}{n}} z$.*

(b) *The maps $N_c(z)$ and $N_{\bar{c}}(z)$ are conjugate through the anti–holomorphic map $h(z) = \bar{z}$.*

Proof. We first prove (a). We take $h(z) = e^{\frac{2\pi i}{n}} z$. Then

$$\begin{aligned}\left(h^{-1} \circ N_c \circ h\right)(z) &= h^{-1}\left(N_c\left(e^{\frac{2\pi i}{n}} z\right)\right) = h^{-1}\left(e^{\frac{2\pi i}{n}} z - \frac{\left(e^{\frac{2\pi i}{n}} z\right)^n - c\left(e^{\frac{2\pi i}{n}} z\right) + 1}{n\left(e^{\frac{2\pi i}{n}} z\right)^{n-1} - c}\right) = \\ &= e^{\frac{-2\pi i}{n}}\left(e^{\frac{2\pi i}{n}} z - \frac{e^{\frac{2\pi n i}{n}} z^n - c e^{\frac{2\pi i}{n}} z + 1}{n e^{\frac{2\pi(n-1)i}{n}} z^{n-1} - c}\right) = N_{\hat{c}}(z),\end{aligned}$$

where $\hat{c} = e^{\frac{2\pi i}{n}} c$.

To see (b) we take $h(z) = \bar{z}$ and argue as above.

$$\begin{aligned}\left(h^{-1} \circ N_c \circ h\right)(z) &= h^{-1}\left(\frac{(n-1)\bar{z}^n - 1}{n\bar{z}^{n-1} - c}\right) = \overline{\frac{(n-1)\bar{z}^n - 1}{n\bar{z}^{n-1} - c}} \\ &= \frac{(n-1)z^n - 1}{nz^{n-1} - \bar{c}} = N_{\bar{c}}(z)\end{aligned}$$

□

In the following subsections we describe the topology of the different hyperbolic components. In subsection 4.1 we study the capture components $\mathcal{C}_j^0$ showing that only $\mathcal{C}_0^0$ is non empty. Moreover, we show that it is unbounded, it contains the complement of a disc of radius 4 and it is simply connected (see Proposition 4.3). In subsection 4.2 we investigate the rest of the capture components showing that every connected component is simply connected (see Proposition 4.5). Finally, in subsection 4.3 we show that the bifurcation locus for N_c contains quasiconformal copies of the bifurcation locus of the maps $z^{n-1} + c$.

4.1. **The hyperbolic components $\mathcal{C}_j^0$ for $0 \leq j \leq n-1$.** The first result determines that one of the roots, α_0, is playing a differentiated role, since for all c outside a certain ball around the origin the free critical point $z = 0$ lies on its immediate basin of attraction. This is due to the fact that the free critical point is $z = 0$ for all $n \geq 3$ and for all c in the parameter space. As a consequence, any other capture component should be bounded (see Figure 1), which in turn implies that $\mathcal{C}_j^0$, $j = 1, \ldots, n-1$ are empty.

Proposition 4.3. *Fix $n \in \mathbb{N}$.*

(a) *$\mathcal{C}_0^0$ is unbounded. In fact we have $\mathcal{C}_0^0 \supset \{c \in \mathbb{C}, |c| > 4\}$.*

(b) *$\mathcal{C}_0^0$ is connected and simply connected.*

(c) *$\mathcal{C}_j^0 = \emptyset$ for all $j \geq 1$.*

Proof. We first prove that there is an unbounded connected component of $\mathcal{C}_0^0$. Let us denote by $B = B(0, 1/2)$ the closed ball of radius $1/2$ centered at $z = 0$. We claim that if $|c| > 4$, the map N_c maps B strictly inside itself. Hence, the Denjoy-Wolf Theorem implies that there must be a unique point $\eta \in B$ such that for all $z \in B$ $N_c^n(z) \to \eta$ as $n \to \infty$ (in other words B

belongs to the immediate basin of attraction of the fixed point η). In particular we have that $N_c^n(0) \to \eta$ as $n \to \infty$. Of course η must be one of the roots α_j of P_c. Since for c large enough we know that $\alpha_0 \in B$, we use continuity of the roots of P_c with respect to the parameter c to conclude $\eta = \alpha_0$ and hence $c \in \mathcal{C}_0^0$.

To see the claim we notice that if $|c| > 4$ the following inequalities follow easily:

$$|N_c(z)| = \left|\frac{(n-1)\,z^n - 1}{nz^{n-1} - c}\right| < \frac{(n-1)\,|z|^n + 1}{\left||c| - n\,|z|^{n-1}\right|} < \frac{(n-1)\,(1/2)^n + 1}{|c| - n(1/2)^{n-1}} < \frac{n-1+2^n}{2^{n+2}-2n} < 1/2,$$

for all $n \geq 3$.

We second prove that $\mathcal{C}_0^0$ is conformally a disc. Since N_c has a superattracting fixed point at α_0, we can use the Böttcher coordinate near the origin to define a suitable representation map in $\mathcal{C}_0^0$. The idea is the same as in the uniformization of the complement of the Mandelbrot set for the quadratic family, see [6, 7] for the original construction. Using a suitable linear change of variables we obtain a new family of maps, so that the superattracting fixed point is now located at $z = 0$ and the functions can be written as $z^2 + O(z^3)$, and thus having a preferred Böttcher coordinate in this region (see equation 17).

It is well known that the Böttcher map cannot be analytically continued to the whole immediate basin of attraction of α_0 since the critical point 0, by assumption, belongs to it. However as in the parametrization of the cubics maps given in [5] by Branner and Hubbard we can use the co-critical point. Observe that N_c is a rational map of degree n, with n critical points of degree 1 located at α_j and the critical point of degree $n-2$ located at 0. So there exists a unique point, denoted by w_c and called the *co-critical point*, such that $N_c(w_c) = N_c(0)$. Indeed a computation shows that $w_c = \frac{n}{(n-1)c}$ and

$$N_c(0) = N_c\left(\frac{n}{(n-1)c}\right) = \frac{1}{c}.$$

Using this co-critical point we define

$$\begin{aligned} \Phi :\ \mathcal{C}_0^0 &\to \mathbb{C} \setminus \overline{\mathbb{D}} \\ c &\to \left[\varphi_0\left(\tfrac{N''(\alpha_0)}{2}\left(\tfrac{n}{(n-1)c} - \alpha_0\right)\right)\right]^{-1} \end{aligned} \tag{21}$$

where φ_0 is the Böttcher coordinate defined in the immediate basin of attraction of $z = 0$ for the monic map $\hat{N}_c = \tau \circ N_c \circ \tau^{-1}$ where $\tau(z) = \frac{N_c''(\alpha_j)}{2}(z - \alpha_j)$.

We claim that Φ is a proper analytic map from $\mathcal{C}_0^0$ onto the exterior of the unit disc. In fact, it is a covering of degree n with a ramified point at ∞. To see the claim we mimic the Douady-Hubbard technique [6, 7] for the uniformization of the exterior of the Mandelbrot set.

A brief computation shows that

$$N_c''(\alpha_0) = \frac{P_c''(\alpha_0)}{P_c'(\alpha_0)} = \frac{n(n-1)\alpha_0^{n-2}}{n\alpha_0^{n-1} - c}.$$

Using now that $\alpha_0 = \frac{1}{c} + \mathcal{O}(1/|c|^{n+1})$ (see Lemma 3.1) we have that

$$\begin{aligned}\frac{N''(\alpha_0)}{2}\left(\frac{n}{(n-1)c} - \alpha_0\right) &= \frac{\frac{n(n-1)}{c^{n-2}} + \mathcal{O}(\frac{1}{|c|^{2n-2}})}{\frac{2}{c^{n-1}}(n-c^n) + \mathcal{O}(\frac{1}{|c|^{2n-1}})}\left(\frac{n}{(n-1)c} - \frac{1}{c} + \mathcal{O}(\frac{1}{|c|^{n+1}})\right)\\ &= \frac{\frac{n}{c^{n-1}} + \mathcal{O}(\frac{1}{|c|^{2n-1}})}{\frac{2}{c^{n-1}}(n-c^n) + \mathcal{O}(\frac{1}{|c|^{2n-1}})} = \frac{n + \mathcal{O}(\frac{1}{|c|^n})}{2(n-c^n) + \mathcal{O}(\frac{1}{|c|^n})}\\ &= \frac{n}{2(n-c^n)}\left(1 + \mathcal{O}(\frac{1}{|c|^n})\right) = \frac{n}{2(n-c^n)} + \mathcal{O}(\frac{1}{|c|^{2n}}).\end{aligned}$$

As mentioned before, $\varphi'(0) = 1$ (or, equivalently $\lim_{z\to 0}\varphi_0(z)/z = 1$). So, as $c \to \infty$ we obtain that

$$\varphi_0\left(\frac{N''(\alpha_0)}{2}\left(\frac{n}{(n-1)c} - \alpha_0\right)\right) \approx \frac{n}{2(n-c^n)} + \mathcal{O}(\frac{1}{|c|^{2n}})$$

.

Thus, the map $c \to \Phi(c)$ is holomorphic and $|\Phi(c)| > 1$, while $|\Phi(c)| \to 1$ as $c \to \partial\mathcal{C}_0^0$. Therefore, Φ is a proper map ramified at ∞ and the above computations show that $\Phi(c) \approx Kc^n + \mathcal{O}(|c|^{2n}),\ K \in \mathbb{C}$.

To prove that $\mathcal{C}_0^0$ is formed by the unique unbounded connected component (the one we just proved it exists), we argue by contradiction. If there were another component U, by the arguments above, would be bounded. When approaching its center, we would have $\alpha_0(c) \to 0$, a contradiction since this only can happen if $c \to \infty$. In fact, the same argument also shows that C_j^0 is empty for all $j \geq 1$. Observe that if there were any (bounded) component C_j^0 its center should satisfy that $\alpha_j(c) \to 0$ but this implies $j = 0$ and $c \to \infty$ again. So the lemma is proved. □

In the next lemma we show that there exist some semi straight lines in the parameter plane joining $c = 0$ (excluding this value) to infinity for which the Newton map has an invariant straight line in the dynamical plane. Once this is proven it is easy to conclude that, for those parameters, $z = 0$ belongs to the immediate basin of attraction of α_0 or equivalently those semi straight lines in parameter plane belong to $\mathcal{C}_0^0$. We also show that all real parameters $c > c^*$ also belong to $\mathcal{C}_0^0$.

We denote by $L_\theta^+ = \{|w|e^{i\theta},\ |w| > 0\}$ and by $L_\theta = L_\theta^+ \cup L_{\theta+\pi}^+ \cup \{0\} \cup \{\infty\}$.

Lemma 4.4. *Fix $n \geq 3$.*

(a) *If $c \in L_{\pi/n}^+$ then $c \in \mathcal{C}_0^0$. Moreover $L_{-\pi/n}$ is a forward invariant straight line for the map N_c with*

$$N_n(|z|e^{-\pi i/n}) = \frac{(n-1)|z|^n + 1}{n|z|^{n-1} + |c|}e^{-\pi i/n}, \tag{22}$$

(b) *If $c \in L_0^+$ with $c > c^\star := \frac{n}{\sqrt[n]{(n-1)^{n-1}}}$ then $c \in \mathcal{C}_0^0$.*

Remark 1. *Taking into account the symmetries described in Lemma 4.2 it is clear that the previous lemma also applies to the corresponding lines of the parameter plane after applying the symmetry.*

Proof. When $c \in L_{\pi/n}^+$ we have that $c = |c|e^{\pi i/n}$. Hence (2) becomes

$$N_c(z) = \frac{(n-1)z^n - 1}{nz^{n-1} - |c|e^{\pi i/n}}. \tag{23}$$

First we assume $z \in L^+_{-\pi/n}$. Hence (23) writes as

$$N_c(|z|e^{-\pi i/n}) = \frac{(n-1)|z|^n e^{-\pi i} - 1}{n|z|^{n-1}e^{-(n-1)\pi i/n} - |c|e^{\pi i/n}} = \frac{-(n-1)|z|^n - 1}{-n|z|^{n-1} - |c|}\frac{1}{e^{\pi i/n}} = \frac{(n-1)|z|^n + 1}{n|z|^{n-1} + |c|}e^{-\pi i/n}.$$

So $L^+_{-\pi/n} \subset L_{-\pi/n}$ is forward invariant. Secondly, we take $z \in L^+_{[-\pi/n+\pi]} = L^+_{[(n-1)/n]\pi}$. Calculating as above we have

$$N_c(|z|e^{\frac{n-1}{n}\pi i}) = \frac{(n-1)|z|^n e^{(n-1)\pi i} - 1}{n|z|^{n-1}e^{(n-1)^2/n\pi i} - |c|e^{\pi i/n}} = -\frac{(n-1)|z|^n e^{n\pi i} + 1}{n|z|^{n-1}e^{n\pi i} - |c|}e^{-\pi i/n},$$

and so

$$N_c(|z|e^{\frac{n-1}{n}\pi i}) = \begin{cases} -\frac{(n-1)|z|^n+1}{n|z|^{n-1}-|c|}e^{-\pi i/n} & \text{if } n \text{ is even} \\ \\ \frac{-(n-1)|z|^n+1}{n|z|^{n-1}+|c|}e^{-\pi i/n} & \text{if } n \text{ is odd} \end{cases} \tag{24}$$

From the formulae it is easy to see that in both cases (n even and n odd) a point $z \in L^+_{[(n-1)/n]\pi}$ maps to one point in $L_{-\pi/n}$. Hence all together conclude that $L_{-\pi/n}$ is a forward invariant straight line for the map N_c.

To see that in fact $c \in \mathcal{C}_0^0$ we write (22) as a map F_c from the positive real line to itself such that

$$F_c(x) = \frac{(n-1)x^n + 1}{nx^{n-1} + |c|} \quad \text{and} \quad F_c'(x) = \frac{(n-1)nx^{n-2}}{(nx^{n-1} + |c|)^2}(x^n + |c|x - 1).$$

From those formulas and knowing that F_c is the restriction of a Newton map on an invariant straight line, we easily get that $F(0) = 1/|c|$, $F(x) \sim \frac{n-1}{n}x$ as $x \to \infty$, there exists a unique positive fixed point $0 < \hat{x_c} < 1/|c|$ of F_c such that $F_c'(\hat{x_c}) = 0$ and $F_c'(x) < 0$ for all $x \in (0, \hat{x_c})$. Therefore it is clear that $[0, \hat{x_c}]$ belongs to the immediate basin of attraction of $\hat{x_c}$. Using the continuously dependence of $\hat{x_c}$ with respect to c we know that $\hat{x_c}$ tends to 0 as $|c|$ tends to ∞. Going back to the map N_c we deduce that $\hat{x_c}e^{-\pi i/n}$ is one of the α_j-roots of P_c and that the segment joining $z = 0$ and $z = \hat{x_c}e^{-\pi i/n}$ belongs to its immediate basin of attraction. Since $\hat{x_c}e^{-\pi i/n}$ should tend to 0 as $c \to \infty$ we conclude that $\hat{x_c}e^{-\pi i/n} = \alpha_0$ and that $\alpha_0 \in \mathcal{C}_0^0$, as desired.

Now we take $c \in \mathbb{R}^+$. The restriction of the Newton map to the real line, which is forward invariant, writes as

$$N_c(x) = \frac{(n-1)x^n - 1}{nx^{n-1} - c}, \quad c \in \mathbb{R}^+.$$

Easily, $N_c(0) = 1/c$, $N_c(x) = 0$ if and only if $x = \sqrt[n]{1/(n-1)}$ and N_c has a vertical asymptote at $x = \sqrt[n-1]{c/n}$. Moreover, the map N_c is a continuous (analytic) map on the interval $[0, \sqrt[n-1]{c/n})$. We claim that if $c > c^*$ then there is a unique $x^* \in (0, \sqrt[n-1]{c/n})$ such that $N_c(x^*) = x^*$ and $N_c'(x^*) = 0$.

To see the claim we show first that the unique positive zero of N_c happens to be before the asymptote if and only if the condition of the statement is satisfied.

$$\sqrt[n]{1/(n-1)} < \sqrt[n-1]{c/n} \iff \sqrt[n]{(\frac{1}{n-1})^{n-1}} < \frac{c}{n} \iff c > \frac{n}{\sqrt[n]{(n-1)^{n-1}}} := c^*.$$

From Bolzano's Theorem we conclude that N_c has (at least) one fixed point (which of course satisfies the equation $x^n - cx + 1 = 0$) on the interval $(0, \sqrt[n-1]{c/n})$, but since N_c is the restriction of a Newton map we know it is unique. We denote it by x_0^c. On the other hand deriving we obtain

$$N_c'(x) = \frac{n(n-1)x^{n-2}}{(nx^{n-1}-c)^2}(x^n - cx + 1),$$

which implies that the map N_c is increasing on the interval $(0, x_0^c)$ and $N_c'(x_0^c) = 0$. From this we see that the close interval $[0, x_0^c]$, and in particular $x = 0$, belongs to the immediate basin of attraction of x_0^c.

Finally, we observe that $x_0^c \to 0$ as $c \to \infty$ and so $x_0^c = \alpha_0$ for large c. Continuity of the roots of a polynomial with respect the parameter concludes statement (b).

□

This section gives a deep understanding of the main hyperbolic component in the parameter space given by the immediate basin of attraction of the special root α_0. As a corollary we obtain that the rest of hyperbolic components are all bounded. In next section we prove that if they are not empty then they are all simply connected.

4.2. **The capture components: $\mathcal{C}_j^k$, for $0 \le j \le n-1,\ k \ge 1$.** In the next proposition we prove the main topological properties of the capture components $\mathcal{C}_j^k$ for $0 \le j \le n-1,\ k \ge 1$. These open sets in the parameter plane contains all the parameters such that the critical point $z = 0$ is attracted by one the roots of p, see equation (1), but the critical point does not belong to the immediate basin of attraction. Precisely, the index k counts the number of iterates that the origin needs to arrive to the immediate basin of attraction.

Proposition 4.5. *Fix n.*

(a) $\mathcal{C}_j^1 = \emptyset$ *for all* $j = 0, \ldots, n-1$.

(b) *If $\mathcal{C}_j^k \neq \emptyset$, its connected components are simply connected.*

Proof. To prove statement (a) assume otherwise. Let $c \in \mathcal{C}_j^1$ and consider its corresponding dynamical plane. We claim that $f : f^{-1}(A_c^*(\alpha_j)) \to A_c^*(\alpha_j)$ has degree $n+1$, a contradiction since the map has global degree n. To see the claim we notice that by assumption the (simply connected) Fatou component of $z = 0$ maps to $A_c^*(\alpha_j)$ with degree $n-1$ (the number of critical points counting multiplicity plus one) and $A_c^*(\alpha_j)$ maps to itself with degree 2.

To prove statement (b) we use a quasiconformal surgery construction (see [2, 10]). Let U be a connected component of $\mathcal{C}_j^k,\ j = 2, \ldots, n-1,\ k \ge 1$. We consider the following map

$$\begin{aligned} \Phi_U :\ & U && \to \mathbb{D} \\ & c && \to \psi_{j,c}\left(N_c^{\circ k+1}(0)\right) \end{aligned}$$

where $\psi_{j,c}$ denotes the Böttcher map conjugating N_c near α_j to $z \to z^2$ near the origin (see equation (18)). As in Proposition 4.3 (b), the map Φ_U is a proper mapping and we will prove that it is a local homeomorphism.

Let $c_0 \in U$ and $z_0 = \Phi_U(c_0)$. The idea of this surgery construction is the following: for any point z near z_0 we can build a map $N_{c(z)}$ such that $\Phi(c(z)) = z$, or in other words, we can build the inverse map of Φ.

We denote by W_{c_0} the connected component of $A_{c_0}(\alpha_j)$ containing $N_{c_0}^{\circ k}(0)$, preimage of $A_{c_0}^*(\alpha_j)$. Let V_{c_0} be a small open neighborhood of $N_{c_0}^{\circ k+1}(0)$ contained in $A_{c_0}^*(\alpha_j)$ and let $B_{c_0} \subset W_{c_0}$ be the preimage of V_{c_0} containing $N_{c_0}^{\circ k}(0)$.

For any $0 < \epsilon < \min\{|z_0|, 1 - |z_0|\}$ and any $z \in D(z_0, \epsilon)$, we choose a diffeomorphism $\delta_z : B_{c_0} \to V_{c_0}$ with the following properties:

- $\delta_{z_0} = N_{c_0}$;
- δ_z coincides with N_{c_0} in a neighborhood of ∂B_{c_0} for any z;
- $\delta_z(N_{c_0}^{\circ k}(0)) = \psi_{j,c_0}^{-1}(z)$.

We consider, for any $z \in D(z_0, \epsilon)$, the following mapping $G_z : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$:

$$G_z(x) = \begin{cases} \delta_z(x) & \text{if } x \in B_{c_0} \\ N_{c_0}(x) & \text{if } x \notin B_{c_0}. \end{cases}$$

We proceed to construct an invariant almost complex structure, σ_z, with bounded dilatation ratio. Let σ_0 be the standard complex structure of $\hat{\mathbb{C}}$. We define a new almost complex structure σ_z in $\hat{\mathbb{C}}$.

$$\sigma_z := \begin{cases} (\delta_z)^*\sigma_0 & \text{on } B_{c_0} \\ (N_{c_0}^n)^*\sigma & \text{on } N_{c_0}^{-n}(B_{c_0}) \text{ for all } n \geq 1 \\ \sigma_0 & \text{on } \hat{\mathbb{C}} \setminus \bigcup_{n\geq 1} N_{c_0}^{-n}(B_{c_0}). \end{cases}$$

By construction σ is G_z-invariant, i.e., $(G_z)^*\sigma = \sigma$, and it has bounded distortion since δ_z is a diffeomorphism and N_{c_0} is holomorphic. If we apply the Measurable Riemann Mapping Theorem (see Section 1.4 in [10]) we obtain a quasiconformal map $\phi_z : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that ϕ_z integrates the complex structure σ_z, i.e., $(\phi_z)^*\sigma = \sigma_0$, normalized so that $\phi(0) = 0$ and $\phi(\infty) = \infty$. Finally, we define $R_z = \phi_z \circ G_z \circ \phi_z^{-1}$, which is analytic, hence a rational function.

We claim that this resulting mapping R_z is the Newton's method applied to the polynomial $P_n(x) = x^n - c(z)x + 1$. By construction R_z is a rational map of degree n with n distint superattracting fixed points and fixing ∞, hence from Lemma 3.4 we can conclude that R_z is the Newton's method for a polynomial $Q(z)$ of degree n. Moreover, 0 is a critical point of R_z with multiplicity $n-2$ and simple computations show that critical points of R_z are zeros of Q and the zeros of Q''. Hence we have that the only zero of Q'' is $x = 0$. Obtaining, perhaps after a conjugation with a Möbius transformation, that $R_z(x) = \frac{(n-1)x^n - 1}{nx^{n-1} - c(z)}$.

By construction, ϕ_{z_0} is the identity for $z = z_0$; then, there exists a continuous function $z \in D(z_0, \epsilon) \mapsto c(z) \in U$ such that

$$c(z_0) = z_0 \text{ and } N_{c(z)} = \phi_z \circ G_z \circ \phi_z^{-1}.$$

Moreover, ϕ_z is holomorphic on $A_{c_0}^*(\alpha_j)$ conjugating $N_{c_0,m}$ and $N_{c(z),m}$. Hence, from the following commutative diagram

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{z^2} & \mathbb{D} \\ \psi_{j,c_0}\big\downarrow & & \big\downarrow\psi_{j,c_0} \\ A_{c_0}^*(\alpha_j) & \xrightarrow{N_{c_0}} & A_{c_0}^*(\alpha_j) \\ \phi_z\big\uparrow & & \big\uparrow\phi_z \\ A_{c(z)}^*(\alpha_j) & \xrightarrow{N_{c(z)}} & A_{c(z)}^*(\alpha_j) \end{array}$$

we have that $\psi_{j,c(z)} = \psi_{j,c_0} \circ \phi_z^{-1}$ is the Böttcher coordinate of $A^*_{c(z)}(\alpha_j)$. Finally, we conclude that

$$\Phi_U(c(z)) = \psi_{j,c(z)}(N_{c(z)}^{\circ k+1}(0)) = z$$

since $N_{c(z)}^{\circ k+1}(0) = \phi_z \circ G_z^{\circ k+1} \circ \phi_z^{-1}(0) = \phi_z \circ G_z^{\circ k+1}(0) = \phi_z \circ G_z(N_{c_0}^{\circ k}(0)) = \phi_z \circ \psi_{j,c_0}^{-1}(z) = \tau_z \circ \phi_z^{-1} \circ \psi_{j,c(z)}^{-1}(z) = \psi_{j,c(z)}^{-1}(z)$.

□

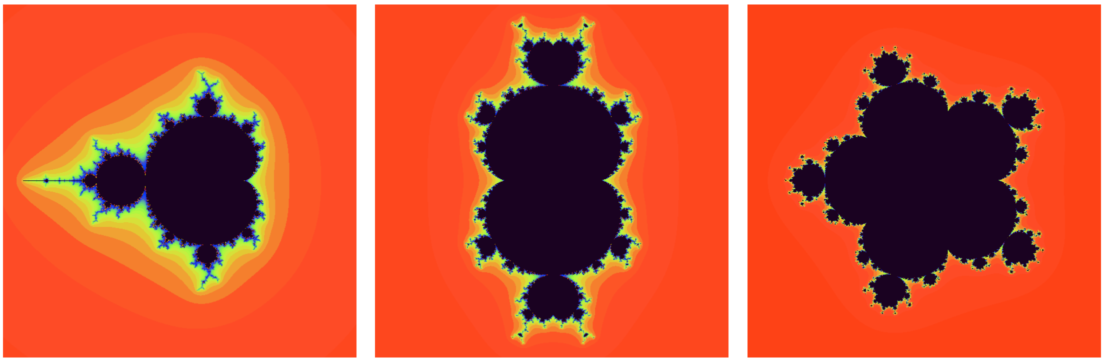
4.3. **Other hyperbolic components and the bifurcation locus.** The theory of polynomial-like maps, developed by Douady and Hubbard [8], explains why pieces of the dynamical and parameter planes of some families of rational, entire or meromorphic maps are so similar to the dynamical and parameter plane of the family of polynomials of the form $z^k + c$, $c \in \mathbb{C}$. Indeed, McMullen [15] showed that small generalized Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps. For a fixed value of $k \geq 2$ the *Generalized Mandelbrot set* is defined as

$$\mathcal{M}_k = \{c \in \mathbb{C}\,, \mathcal{J}(z^k + c) \text{ is connected}\}.$$

We define a holomorphic family of rational maps over the unit disc $\mathbb{D}$ as a holomorphic map

$$f : \mathbb{D} \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

Notice that for each (parameter) $t \in \mathbb{D}$, the map $f_f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map. We also require that $deg(f_t) \geq 2$. The *Bifurcation locus* $\mathcal{B}(f)$ is defined as the set of parameters t such that the Julia set $\mathcal{J}(f_t)$ does not move continuously (in the Hausdorff topology) over any neighbourhood of t. It is known that $\mathcal{B}(f)$ is a closed and nowhere dense subset of $\mathbb{D}$ and its complement is also called the *J-stable* set. In Figure 2 we show the parameter plane of $z^k + c$ for $k = 2, 3$ and 6. The complement of $\mathcal{M}_k$ is called the Cantor set locus and the bifurcation locus is ∂M_d.



(a) The parameter plane of $z^2 + c$. (b) The parameter plane of $z^3 + c$. (c) The parameter plane of $z^6 + c$.

FIGURE 2. Mandelbrot sets of degree 2, 3 and 6.

As we mentioned before in [15], it is showed the universality of the Mandelbrot set, more precisely,

Theorem 4.6 ([15]). *For any holomorphic family of rational maps over the unit disc, the bifurcation $\mathcal{B}(f)$ is either empty or contains the quasiconformal image of $\partial\mathcal{M}_k$ for some k and $\mathcal{B}(f)$ has Hausdorff dimension two. Moreover, small Generalized Mandelbrot sets are dense in $\mathcal{B}(f)$.*

Applying the above result to our family of rational maps $N_c(z) = \frac{(n-1)z^n-1}{nz^{n-1}-c}$ we obtain this quite interesting result.

Proposition 4.7. *Fix $n \geq 3$. The bifurcation locus $\mathcal{B}(N_n)$ is non empty and contains the quasiconformal image of $\partial\mathcal{M}_{n-1}$ and $\mathcal{B}(N_n)$ has Hausdorff dimension two. Moreover, small copies of $\partial\mathcal{M}_{n-1}$ are dense in $\mathcal{B}(N_n)$.*

Proof. With rare exceptions the bifurcation locus of a holomorphic family of rational maps is non empty. One of this exceptions occurs when the family is trivial, or in other words, when all the members in the family are conformally conjugate. This is the case, for example, for the Newton's method applied to polynomials of degree two. For this case all the members in the family are conformally conjugate to the map $z \to z^2$. In our case it is easy to see that $\mathcal{B}(N_n)$ is non empty, since, for example, we have plenty of preperiodic parameters (see Section 5).

The critical points of N_c are $\alpha_j,\ j = 0, \ldots, n-1$ (all simple) and 0 (with multiplicity $n-2$) since $P_c''(z) = n(n-1)z^{n-2}$. Thus, if U is a sufficiently small neighbourhood of the origin the degree of $N_{n,c} : U \to N_{n,c}(U)$ is $n-1$. Hence, in any polynomial-like construction involving the free critical point located at zero we always obtain a member of the family $z^{n-1} + c$. Therefore, applying Theorem 4.6 we obtain that the bifurcation locus of N_c for a certain n contains the quasiconformal image of $\partial\mathcal{M}_{n-1}$. □

5. Real polynomials

Fix $n \in \mathbb{N}$ and $c \in \mathbb{R}$. We notice that we can restrict to parameters $c \in \mathbb{R}^+$; if n is odd, from the symmetry properties of the parameter plane (Lemmas 4.2 and 4.4), we have that $\mathbb{R}^- \setminus \{0\} \subset \mathcal{C}_0^0$, and when n is even N_c, with $c \in \mathbb{R}^-$, is conformally conjugate to N_{-c} (Lemma 4.2).

Because of Lemma 4.4, we only need to deal with $0 < c < c^*$ since otherwise $c \in \mathcal{C}_0^0$. Of course, again, the results we prove for c real also apply to complex parameters after applying the symmetry explained in Lemma 4.2(a). In [11, 3] the authors studied this problem from the real analysis point of view. They characterize the possible combinatorial orbits of $z = 0$ using symbolic dynamics.

We introduce two parameters which will play an important role defining those bifurcations

$$0 < \overline{c} := \sqrt[n]{n-1} < c' := \sqrt[n]{n} < c^*.$$

Proposition 5.1. *For every n, there exists a strictly decreasing sequence of real c-parameters $\{\alpha_k\}_{k\geq 1}$ such that $\alpha_k \to 0$, $0 < \alpha_k < \alpha_1 := \sqrt[n]{n-1}$ and each of those parameters is the center of a free (not captured) hyperbolic component $\mathcal{D}_k$ for which the free critical point $z = 0$ is a super attracting periodic point of period $k+1$ (main black pseudo cardiodides for positive real parameters). Moreover*

1. *If n is odd there also exists a strictly decreasing sequence of real c-parameters $\{\beta_k\}_{k\geq 1}$ such that $\beta_k \to 0$, $\alpha_1 < \beta_1 < c'$, $\alpha_k < \beta_k < \alpha_{k-1}$ for all $k \geq 2$ and each of those parameters is the center of a captured hyperbolic component $\mathcal{C}_j^k$ for some fixed j.*
2. *If n is even $\mathcal{C}_j^k \cap \mathbb{C} = \emptyset$ for any $j = 0, \ldots, n-1$ and $k \geq 0$.*

Proof. We will only consider $c \in (0, c^*)$. The qualitative graph of the Newton's map N_c is represented in Figure 3. From those pictures it is easy to deduce that c' corresponds to the parameter for which the free critical value $x = 1/c$ is equal to the positive zero $x = \sqrt[n]{n-1}$ while $\overline{c}$ corresponds to the parameter for which $x = 1/c$ is equal to the positive vertical asymptote $x = \sqrt[n-1]{c/n}$. We define $G_k(c) = N_c^k(1/c)$, $k \geq 0$, that is G_k is a function of c giving the kth iterate, for the corresponding Newton's map N_c, of the free critical value $x = 1/c$.

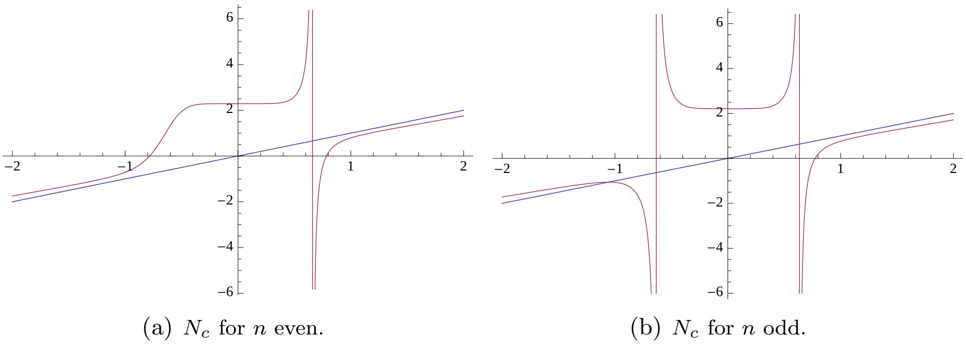


FIGURE 3. The qualitative graph of N_c. The left picture corresponds to $n = 8$ and $c \approx 0.4373$ and the right one corresponds to $n = 7$ and $c \approx 0.4521$. For all $c \in (0, c^*)$ and for all $n \geq 3$ it has a unique (positive) zero at $x = \sqrt[n]{n-1}$ and a unique (positive) vertical asymptote at $x = \sqrt[n-1]{c/n}$. Moreover, $z = 0$ is a minimum for n odd and an inflection point for n even.

From this notation it is clear that the centers of non captured hyperbolic components (intersecting the real line) are given by the solutions of the equation $G_k(c) = 0$, $k \geq 1$ ($(k+1)$ determine the number of iterates used by the critical point $x = 0$ to come back to itself). In particular, $\alpha_1 = \sqrt[n]{n-1}$ is the center of a free hyperbolic component of period 2 since $G_1(\alpha_1) = 0$ (that is $x = 0$ is back to itself after two iterates of the map N_{α_1}). Moreover, it is an exercise to check that G_1 is a (differentiable) strictly decreasing function of c in the interval $[0, \alpha_1]$ whose range is $[0, \infty)$ (notice that, formally, $G_1(0) = \infty$, and $G_1(\alpha_1) = 0$). Hence, we claim that there exists a (unique) real parameter, α_2, in the interval $[0, \alpha_1]$ such that $G_2(\alpha_2) = 0$. To see the claim observe that, on the one hand, $G_2(c) = N_c^2(1/c) = N_c(G_1(c))$ and, on the other hand, $N_c(\sqrt[n]{n-1}) = 0$ and there exist a unique $\alpha_2 \in (0, \alpha_1)$ such that $G_1(\alpha_2) = \sqrt[n]{n-1}$. Clearly for $c = \alpha_2$, $x = 0$ is back to itself after three iterates of the map N_{α_2}.

We repeat the process once again. The map G_2 is a (differentiable) strictly decreasing function of c in the interval $[0, \alpha_2]$ whose range is $[0, \infty)$ (notice that, again, $G_2(0) = \infty$ and $G_2(\alpha_2) = 0$). Hence, as before, we claim that there must exists a (unique) real parameter, α_3, in the interval $[0, \alpha_2]$ such that $G_3(\alpha_2) = 0$. To see the claim observe that, on the one hand, $G_3(c) = N_c^3(1/c) = N_c(G_2(c))$ and, on the other hand, $N_c(\sqrt[n]{n-1}) = 0$ and there exist a unique $\alpha_3 \in (0, \alpha_2)$ such that $G_2(\alpha_3) = \sqrt[n]{n-1}$. Clearly, for $c = \alpha_3$, $x = 0$ is back to itself after four iterates of the map N_{α_3}. Similarly, we may construct the whole sequence $\{\alpha_k\}_{k\geq 1}$ as desired.

We observe that using a similar argument to the one we use to produce the parameter sequence $\{\alpha_k\}_{k\geq 0}$, we claim there exists another (auxiliary) sequence of parameters $\{\delta_k\}_{k\geq 0}$

such that $G_k(\delta_k) = \sqrt[n-1]{\delta_k/n},\ k \geq 0$. To see the claim we observe first that δ_0 is given by the solution of $1/c = \sqrt[n-1]{c/n}$, so $\delta_0 = \sqrt[n]{n}$ and $\delta_0 > \alpha_1$. Secondly, the map $F(c) = \sqrt[n-1]{c/n}$ is a (differentiable) strictly increasing function of c in $[0, c^*]$ such that $F(0) = 0$. So, each of the graphs of the maps $G_k,\ k \geq 1$, on the interval $(0, \alpha_k)$, crosses one and only one time $F(c)$ producing the desired sequence of δ_k's. Moreover, we have $\delta_0 > \alpha_1 > \delta_1 > \alpha_2 > \delta_2 > \ldots$.

Now we turn the attention to n odd to prove statement (1). We construct the sequence β_k of centers of captures parameters using basically the same argument. Those centers, distinguished by the parameters $c = \beta_k$, should be solutions of the equation $G_k(c) = \overline{x}_c$ where $\overline{x}_c$ is the unique negative real solution of $N_c(x) = x$, or, equivalently, of $P_c(x) = 0$.

Clearly $x = \sqrt[n]{n}$ is a pole of the map $N_{\sqrt[n]{n}}$, while $G_1(\alpha_1) = 0$.

Hence, G_1 is a (differentiable) strictly decreasing function of c in the interval $[\alpha_1, \delta_0]$ whose range is $(-\infty, 0)$ (notice that $G_1(\alpha_1) = 0$ and, formally, $G_1(\delta_0) = -\infty$). On the other hand, for $c \in [\alpha_1, \delta_0]$ the value of $\overline{x}_c$ moves continuously in a compact interval $[a, b]$ where $-\infty < a < b < 0$. So, there must be one point $\beta_1 \in (\alpha_1, \delta_0)$ such that $G_1(\beta_1) = \overline{x}_{\beta_1}$. We repeat the argument once again. The map G_2 is a (differentiable) strictly decreasing function of c in the interval $[\delta_1, \alpha_1]$ whose range is $(-\infty, 0)$ (notice that $G_1(\delta_1) = -\infty$ and $G_2(\alpha_1) = 0$); so, there must be one point $\beta_2 \in (\delta_1, \alpha_1)$ such that $G_2(\beta_2) = \overline{x}_{\beta_2}$. And so on.

Finally, assume that n is even and $0 < c < c^*$. We have that the Newton's map and its derivative (restricted to the real line) is given by

$$N_c(x) = \frac{(n-1)x^n - 1}{nx^{n-1} - c} \quad \text{and} \quad N_c'(x) = \frac{n(n-1)x^{n-2}}{(nx^{n-1} - c)^2}(x^n - cx + 1).$$

We claim that $N_c'(x) > 0$ for all $x \in \mathbb{R}$. Since

$$\lim_{x \to \pm\infty} \frac{N_c(x)}{x} = \frac{n-1}{n},$$

it is immediate to see that the graph of $N_c(x)$ does not cross $y = x$ and so $x = 0$ cannot be in any capture hyperbolic component. Hence, $c \notin \mathcal{C}_j^k$ for any $j = 0, \ldots, n-1$ and $k \geq 0$.

To see the claim we observe that $P_{n,c} := x^n - cx + 1$ has no real zeroes as long as n is even and $c \in (0, c^*)$. Assume otherwise. Then $P_{n,c}$ should have two positive real zeroes and one minimum at $x_c = \sqrt[n-1]{\frac{c}{n}}$ with negative image. It is an easy computation that, if $0 < c < c^*$, we get $P_{n,c}(x_c) > 0$, a contradiction. This proves (2).

□

It is worth to notice that taking into account that the bifurcation locus intersects the real line, the above result is not at all surprising since we expect to have all kind of bifurcation parameters. However, we also notice that it would be nice to have a better understanding of this bifurcation cascade in the light of McMullen results in [15]. There it is proved the existence of sequences of mini-generalized Mandelbrot sets approaching Misiurewicz parameters and its size in parameter plane. Our result reproves their existence and shows its relative location in the real line as we approach $c = 0$. However, a deeper study to have a more detailed knowledge of how those cascades are organized is a challenging problem itself.

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