Quantum vacuum energy and the Casimir effect

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Abstract: An overview of the Casimir effect is presented. The area of study is historically introduced, together with a basic presentation on the zero-point energy concept in relation with the quantization of fields. After this, the Casimir force between parallel plates is calculated under two different regularization schemes. The brief overview is completed with a compilation of experimental results and implications of the effect in different fields of physics.

I. INTRODUCTION

It was 65 years ago when Hendrik Casimir published a short paper [1] on the attraction force between two parallel conducting plates due to changes in the quantum electromagnetic zero-point energy, known as the *Casimir effect*.

The concept of zero point energy had been a hot topic since it appeared in the Planck's black-body radiation discussion by Einstein and Stern; a 1913 article on the energy of an hydrogen gas at low temperature, which concluded that a non-zero residual energy at $T = 0$ might exist [2].

In the following decades, the work by Casimir remained relatively unknown, probably because of controversy on the value of the ground state of quantum vacuum energy, which was commonly redefined to zero. It was not until the 70s when experiments started to became precise enough to measure the Casimir force and assure its existence as evident.

Nowadays, the Casimir effect together with quantum vacuum energy is again highly popular in the scope of QFT, together with a set of different applications and consequences that may go from cosmology to nanotechnology.

In the following sections, the basics of zero-point energy in QED are introduced, followed by a first calculation of the Casimir force and a brief overview of infinite integrals regularization. A bibliographical compilation of experiments and applications of the Casimir effect is presented.

II. ZERO-POINT ENERGY AND FIELD QUANTIZATION

During the first decades of the last century, and in the beginnings of quantum theory, a paper by Einstein and Stern [2] on the basis of Planck's previous work stated that the energy of a vibrating unit was

$$\epsilon = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} + \frac{h\nu}{2} \tag{1}$$

for which the energy at lowest temperature was a non-zero term $h\nu/2$. This was the beginning of the concept of zero-point energy, as the lowest possible energy quantum state being different from zero.

In ordinary quantum mechanical systems, the not-null ground state may be seen as a consequence of the Heisenberg uncertainty principle, with the energy of the harmonic oscillator being its clearest example. The term quantum vacuum, although considered equivalent, was coined as a result of the concept of ground state energy within the framework of quantum field theory.

Quantization of fields begins on the postulates of quantum mechanics, replacing the pair $[x, p]$ by the field $\varphi(x, t)$ and conjugate canonical momentum $\pi_i = \partial L/\partial\dot{\varphi}_i$. In the one-dimensional case (by means of what is known as second-quantization), a quantized string $(0, a)$ is considered as a set of quantum harmonic oscillators as

$$\varphi(t, x) = \sum_n \left(a_n \varphi_n^-(t, x) + a_n^+ \varphi_n^+(t, x) \right) \tag{2}$$

with $\varphi_n^\pm$ being the solutions of the usual wave equation. The annihilation and creation operators a_n, a_n^+ are those obeying the commutation relations of QM (or anti-commutation relations for 1/2-spin particles), and so the quantum state is defined by $a_n|0\rangle = 0$. Even though quantum field theory is a giant and expanding field of physics, we may particularly center in our one-dimensional example to study the energy of this vacuum $|0\rangle$ state.

The 00-th component of the energy-momentum tensor is the operator for the energy density, and so what we are basically looking for in the present article is to understand

$$E_0^{vacuum} = \int \langle 0|T_{00}(x)|0\rangle dx. \tag{3}$$

By looking at appendix VI.A, we observe the form of $\langle 0|T_{00}(x)|0\rangle$ in free or bounded states. It is important to remark that this expressions are in terms of sums or integrals over all possible frequency values, which we must not confuse with the integrals among space. The results for bounded and free-space energies are

$$E_0^{bounded}(a) = \int_0^a \langle 0|T_{00}(x)|0\rangle dx = \frac{\hbar}{2}\sum_{n=1}^{\infty} \omega_n, \tag{4}$$

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$$E_0^{free}(-\infty,\infty) = \frac{\hbar L}{2\pi}\int_0^\infty \omega_k dk, \tag{5}$$

(where $L \to \infty$), and their respective frequencies (for a massless field)

$$\omega_n^{bounded} = c\frac{\pi n}{a}, \quad n = 1, 2, ..., \tag{6}$$

$$\omega_k^{free} = ck, \quad -\infty < k < \infty. \tag{7}$$

This is, at a basic point, enough information to approach the comprehension of the vacuum energy of the electromagnetic field.

III. THE CASIMIR EFFECT

A. The quantized electromagnetic field between plates

As mentioned in the introduction, the Casimir effect results from the changes of the presented QED zero-point energy due to external conditions. In this case, the presence of two perfectly conducting parallel plates determines a boundary condition, and so the frequencies of the radiation between the plates are restricted to a discrete set of values.

On the surface S of a perfect conductor, both polarizations $E_{\|}$ and $H_{\perp}$ are zero, and so the existence of two plates with area $S \to \infty$ and separation $z = a$ quantizes the possible frequencies in the three dimensional case to

$$\omega = \omega_{\boldsymbol{k}_\perp,n} = c\sqrt{k_1^2 + k_2^2 + \left(\frac{\pi n}{a}\right)^2} \quad n = 0, 1, 2... \tag{8}$$

.

Now, by the same process done in II and appendix VI.A, we just need to take the three-dimensional case to obtain the vacuum energy

$$E_0 = \iiint d^3x \frac{\hbar}{(2\pi)^2}\iint dk_1 dk_2 \frac{1}{2a}\sum_{n=1}^{\infty}\omega_{k_\perp,n}. \tag{9}$$

Integrating $dxdy$ in area S and dz in $[0, a]$,

$$E_0(a) = S\frac{\hbar}{2}\iint \frac{dk_1 dk_2}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\omega_{\boldsymbol{k}_\perp,n}, \tag{10}$$

where the sum has been extended as to account the two photon polarizations when $n \neq 0$.

This expression diverges for large photon momentum $\boldsymbol{k}$. This ultraviolet divergence is a common problem in QFT, and it is treated through regularization. In the following I will introduce two different regularizations to achieve the *Casimir force* result. These are the damping function (as originally used by Casimir [1]) and the zeta function regularizations. In addition, I will present the equivalence between using the Euler-Maclaurin and the Abel-Plana formulae to proceed with the damping regularization, together with the usefulness of the Abel-Plana formula to understand the independence of the damping function.

B. Regularizations of the ultraviolet divergence

The damping function regularization appears by a physical intuition process. As the conducting plates become transparent to high-frequency waves, the contribution of these to the physical result may be supressed by the use of a damping exponential function

$$E_0(a,\delta) = S\frac{\hbar}{2}\iint \frac{dk_1 dk_2}{(2\pi)^2}\sum_{n=-\infty}^{\infty}\omega_{\boldsymbol{k}_\perp,n}e^{-\delta\omega_{\boldsymbol{k}_\perp,n}}. \tag{11}$$

The expression is finite for $\delta > 0$ and regularization will give the correct result in the limit of $\delta \to 0$.
If we remove the plates, the regularized vacuum energy in the same interval $(0, a)$ accounts from changing $L \to \infty$ to $L = a$

$$E_0^{free}(a,\delta) = aS\frac{\hbar}{(2\pi)^3}\iiint d^3k\omega_{\boldsymbol{k}}e^{-\delta\omega_{\boldsymbol{k}}}S. \tag{12}$$

With these, we may write the regularized potential energy of the system, *i.e.* the energy to bring the plates from a large separation to a, as

$$U_0^{\text{ren}}(a) = \lim_{\delta\to 0}\left(E_0^{bounded}(a,\delta) - E_0^{free}(a,\delta)\right) \tag{13}$$

As in [3], we use $k_\perp^2 = k_1^2 + k_2^2$ and $t = ak_3/\pi$. The expression to evaluate is then

$$U_0^{\text{ren}}(a) = S\frac{c\hbar\pi}{a}\lim_{\delta\to 0}\iint \frac{dk_1 dk_2}{(2\pi)^2}\left(\sum_{n=0}^{\infty}F(n)e^{-\delta\omega_{\boldsymbol{k}_\perp,n}} - \int_0^\infty dtF(t)e^{-\delta\omega_{\boldsymbol{k}}} - \frac{k_\perp a}{2\pi}\right) \tag{14}$$

with

$$F(x) = \sqrt{\left(\frac{k_\perp a}{\pi}\right)^2 + x^2}. \tag{15}$$

We can now recognize the difference between the sum and the integral as a common and useful expression. We may first evaluate it by the Abel-Plana formula, and through this recognize the independence of the damping function $f(\omega_{k_\perp,n},\delta)$ used in Casimir's assumption to compute (13) by the Euler-Maclaurin formula.

Applying the Abel-Plana formula on (14) and using

$y = k_\perp a/\pi$we obtain (see Appendix VI.B)

$$U_0^{\text{ren}}(a) = -S\frac{c\hbar\pi^2}{a^3}\int_y^\infty \frac{dt}{e^{2\pi t}-1}\int_0^t y\sqrt{t^2-y^2}dy$$
$$= -\frac{c\hbar\pi^2}{720a^3}S. \quad (16)$$

The force between the plates is the derivative of the potential energy with respect to their distance

$$F(a) = -\frac{\partial U_0^{\text{ren}}(a)}{\partial a} = -\frac{\pi^2}{240}\frac{\hbar c}{a^4}S. \quad (17)$$

This is the Casimir force between two parallel conducting plates due to QED vacuum energy. We used the Abel-Plana formula, but many authors use Casimir's original calculation ([4]). In the mentioned original paper, instead of introducing an exponential function, Casimir asked for a more general $f(k/k_m)$, unity for $k << k_m$ and tending to zero for $(k/k_m) \to \infty$. With this, he calculated (13) by the means of the Euler-Maclaurin summation formula

$$S - I = \sum_{k=2}^{p}\frac{B_k}{k!}\left(f^{k-1}(m) - f^{k-1}(0)\right) \quad (18)$$

where $m \to \infty$, S states for the sum and I for the integral that appear on (14). The terms $F^k(m \to \infty)$ are zero for the assumptions on $f(k/k_m)$. In our calculations, $F^{(n)}(x) \propto (\pi/ak_m)^K$, with $K > 0$, for $n \geq 4$. With this, the calculations ask for $ak_m >> 1$ to meet the correct result. See Appendix VI.C for an insight on Casimir's calculation.

At last, what is important to understand, at the moment we choose one or another regularization, is that the final result should be independent of the exact shape of $f(\delta k)$. This is a direct conclusion taken from the form of the Abel-Plana formula (Appendix VI.B). Under the assumptions of f being analytic in the positive reals of $\mathbb{C}$, and that

$$\lim_{y\to\infty}\left|f(x \pm iy)\right|e^{-2\pi y} \quad (19)$$

goes to zero uniformly in x, the Abel-Plana formula holds and its right-hand side integral converges, and so the general damping function may just be driven to zero by the limit $\delta \to 0$. It is imporant to remark that the Euler-Maclaurin formula does not ask for an analytic function, but it does need some constraints in the results of the derivatives f^{k-1} in both limits of integration.

Together with the independence of f, the regularization process is also independent of the summation formula. This is, in fact, because under the mentioned constraints of f the Abel-Plana and the Euler-Maclaurin formulae are equivalent [5]. The demonstration of this is not trivial, but it clearly runs through the approximation theorem of Weierstrass to connect the analytic constraints of Abel-Plana and the $f \in C^{(2)}[0,\infty]$ constraint of Euler-Maclaurin.

Before going on to conclusions, a completely different regularization process is presented. This is the zeta function regularization. It is a much modern, commonly used process in QFT, based on the analytic continuation of the Riemann zeta function $\zeta(s)$.

We rewrite (10) with a new regularization parameter s as

$$E_0(a,s) = S\frac{\hbar}{2}\sum_{n=-\infty}^{\infty}\iint\frac{dk_1dk_2}{(2\pi)^2}\omega_{\boldsymbol{k}_\perp,n}^{1-2s}. \quad (20)$$

As in [3], we use polar coordinates and $k_\perp = y(\pi n/a)$ to obtain (Appendix VI.D)

$$E_0(a,s) = S\frac{\hbar c}{2\pi}\int_0^\infty dyy\left(y^2+1\right)^{\frac{1}{2}-s}\sum_{n=1}^{\infty}\left(\frac{\pi n}{a}\right)^{3-2s}. \quad (21)$$

From here, we must observe the Riemann zeta function

$$\zeta_R(t = 2s-3) = \sum_{n=1}^{\infty}\frac{1}{n^t}. \quad (22)$$

However, we need the value $\zeta_R(-3)$ in order to remove the regularization and obtain the unique physical result. Apparently, this value diverges, but we may obtain (see [6]) the analytic continuation to the whole complex plane, described by the functional equation

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \quad (23)$$

at $s = 4$. This gives us the value $\zeta_R(-3) = 1/120$. Performing the integral and removing the regularization by putting $s = 0$ (see Appendix VI.D) we obtain the expected expression (16) for the energy, and applying (17) we obtain the expected Casimir force.

It appears as we did not start on the basic notion of calculating the difference between two energies, but evaluating the energy expression at a given point or distance, which is not what we would physically understand. Besides the *ad hoc* character of the zeta function regularization, a deeper insight - away from the scope of this work - would recall on how the analytic continuation (23) has an additive positive infinity, which accounts for the energy outside the plates and so may give the physical intuitive answer we were expecting.

IV. EXPERIMENTS AND APPLICATIONS

In the preceding section we have seen the particular situation of the effect between two parallel conducting plates. Many further calculations have been done in many different geometries and boundary conditions, together with continuous developments on QFT divergences.

Together with these, experimental tests have been going on and gaining more and more precision since the

Casimir effect started gaining greater importance around the 1970s.

It is important to remark the technical difficulties of measuring this effect. For these, early experiments did not obtain remarkably precise results, but a rather qualitative evaluation. It was not until 1997 when an experiment by S.K. Lamoreaux [7] measured the direct force between plates and was precise enough to consider it, somehow, conclusive.

The difficulties lay on the quantum dimensions of the macroscopic effect to measure, together with complications on obtaining parallel, high conducting, free of impurity plates. By (17), we observe how the force is strongly dependent on the distance and only considerable at values $a \sim 1\mu m$, where it becomes dominant. The force is around 10^{-7}N for two plates with surface of 1cm^2 area and $a = 1\mu m$.

More recent experiments (around year 2000) using the atomic force microscope have become definitive on the measurement of the Casimir force, with claims of a 1% statistical precision [8].

The modern implications of the Casimir effect, together with general developments on quantum fluctuations, occupy a wide spectrum of physics fields, from deep theoretical physics consequences on quantum field theory, nucleon models [4], cosmology and gravitation to modern advances on applied physics and technological uses, such as in modern nanotechnology chips trying to harness the power of the Casimir effect [9].

One of the most remarkable implications of the Casimir effect and the quantum vacuum energy would be its possible relation with Einstein's cosmological constant in general relativity, as to explain the mechanism for the accelerated expansion of the universe. As in the *Casimir effect* calculation, the difference between the zero-point energy of the expanding universe and the one of the flat Minkowski space cancels the common large term and leaves a finite and small contribution, perhaps associable to the Cosmological Constant [10].

V. CONCLUSIONS

• The Casimir effect, and in particular the Casimir force between two parallel conducting plates, exists as proposed, and is nowadays experimentally confirmed with high precision techniques. It is a macroscopical quantum effect, and it is considered a strong pilar for the evidence of zero-point energy in quantum field theory.

• Some important concepts with general use in quantum field theory have been discussed, such as different regularization schemes to calculate divergent integrals. Besides these, the idea of measuring energy differences has been settled as the starting point for a broad range of physical problems.

• Although it was stated originally in 1948, the Casimir effect is a hot topic in nowadays physics, and continuous experimental advances enlighten the paths of a growing number of theoretical developments on the effect. This rapid growth expands to different areas of modern physical science, making the Casimir effect an interesting, mysterious and significant physical phenomenon. In the scope of theoretical physics, its possible implications on the Cosmological Constant problem may be of extraordinary value.

VI. APPENDIX

In order to avoid extense calculations in the text, I remitted to the appendix some important steps on each calculation done. The basics of second-quantization arise from [3] and my university courses. Both damping and zeta function regularizations calculations follow the indications from [3]. Casimir's original Euler-Maclaurin process is calculated from [1]. The connection between the Abel-Plana and the Euler-Maclaurin summation formulae is stated by the author using [5].

VI.A In the standard quantization of the scalar field, starting by the usual equation (for a massive field)

$$\frac{1}{c^2}\partial_t^2\varphi - \partial_x^2\varphi + \frac{m^2c^2}{\hbar^2}\varphi(t,x) = 0 \tag{24}$$

we obtain the bounded and free solutions by means of contour conditions as

$$\varphi_n^{\pm}(t,x) = \left(\frac{c}{a\omega_n}\right)^{1/2} e^{\pm i\omega_n t}\sin k_n x \tag{25}$$

$$\varphi_k^{\pm}(t,x) = \left(\frac{c}{4\pi\omega}\right)^{1/2} e^{\pm i(\omega t - kx)} \tag{26}$$

with respective expressions for their frequencies (6) and (7). Now, the 00-th component operator

$$T_{00}(x) = \frac{\hbar c}{2}\left(\frac{1}{c^2}(\partial_t\varphi)^2 + (\partial_x\varphi)^2\right) \tag{27}$$

brings us, for both bounded and free expressions of (2), to the expressions

$$\langle 0|T_{00}|0\rangle_{bounded} = \frac{\hbar}{2a}\sum_{n=1}^{\infty}\omega_n - \frac{m^2c^4}{2a\hbar}\sum_{n=1}^{\infty}\frac{\cos 2k_n x}{\omega_n}, \tag{28}$$

$$\langle 0|T_{00}|0\rangle_{free} = \frac{\hbar}{2\pi}\int_0^{\infty}\omega dk. \tag{29}$$

The integration of this expressions in $[0,a]$ results in (4) and (5) (The second term of the first expression doesn't contribute to the integral for both massive and massless fields).

VI.B The first change of variables to rewrite equations (11) and (12) in the terms of $F(x)$ is direct. The main step of the damping function regularization presented is on the use of the Abel-Plana formula

$$\sum_{n=0}^{\infty} f(n) - \int_0^{\infty} f(t) - \frac{1}{2} f(0) = i \int_0^{\infty} \frac{\Big(f(it) - f(-it)\Big) dt}{e^{2\pi t} - 1} \tag{30}$$

where we identify $f(x) = F(x) e^{-\delta\omega(x)}$.

The convergence of the right-hand side integral lets us apply the limit $\delta \to 0$ directly. What we need to understand is evaluation of $F(it) - F(-it)$ by the means of complex calculus. The square roots will have branching points $\pm iy$ to be rounded, and so $F(it) - F(-it) = 2i\sqrt{t^2 - y^2}$.

The intermediate step of equation (16) runs through $(x = 2\pi t)$. What remains is the evaluation of a integral by parts and a well known integral to be evaluated by means of complex calculus or the gamma function and the Riemann zeta function.

$$\int_0^{\infty} \frac{dx x^3}{e^x - 1} = \Gamma(4)\zeta(4) = \frac{\pi^4}{15}. \tag{31}$$

VI.C The actual expression used by Casimir was

$$\sum_{(0)1}^{\infty} F(n) dn - \int_0^{\infty} F(x) dx = -\frac{1}{12} F'(0) + \frac{1}{720} F'''(0) + ... \tag{32}$$

where the subindex (0) accounts for the non polarization of the $n = 0$ mode, and so contains the first term $(1/2)F(0)$. Casimir took into account the space integrals and so used an already integrated expression for F

$$F(u) = \int_0^{\infty} dx (x + u^2)^{(1/2)} f(n^2\pi/ak_m) \tag{33}$$

with k_m being in fact the inverse regularization parameter. With this, the only non-vanishing derivative is $F'''(0) = -4$, as higher derivatives are in terms of (π/ak_m), and go to zero as $ak_m = a/\delta >> 1$. From here, obtaining the Casimir force (17) is direct.

VI.D The first step in the zeta function regularization is to rewrite (20) in order to obtain $\zeta(2s-3)$. We use again the polar coordinates $(k_\perp, \varphi_k)$, and the change $k_\perp = y(\pi n/a)$. We may see

$$\left(k_\perp^2 + \left(\frac{\pi n}{a}\right)^2\right)^{\frac{1}{2}-s} = \left(y^2 + 1\right)^{\frac{1}{2}-s} \left(\frac{\pi n}{a}\right)^{1-2s}. \tag{34}$$

The other $(\pi n/a)^2$ come from $dk_1 dk_2 = k_\perp dk_\perp = (\pi n/a) y (\pi n/a) dy$. The demonstration of the functional or reflection equation (23) is not trivial nor short [6]. The key observation of the extension of ζ to the whole complex plane is the observation that in $0 < Re(s) < 1$, $\zeta(s)$ has a symmetry in relation with $\zeta(1-s)$, and together with the expansion of Γ in the complex plane and many other developments we obtain (23).

Once implemented the $\zeta_R(-3)$ value, the sum returns $(\pi/a)^3(1/120)$. The remaining integral converges under the limit as

$$\lim_{s\to 0} \int_0^{\infty} dy y (y^2 + 1)^{(1/2)-s} = -\frac{1}{3} \tag{35}$$

and so the final result for the energy is (16), as expected.

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