Capítulo 6

Concept of finite limit of a function at a point: meanings and specific terms

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Abstract
In this paper, we present some results of an exploratory study performed with students aged 16-17. We investigate the different uses that these students make of terms such as 'to approach', 'to tend', 'to reach', 'to exceed' and 'limit' that describe the basic notions related to the concept of the finite limit of a function at a point. We use the interpretive framework of conceptual analysis to infer the meanings that students associate with these specific terms in connection with the effective use of terms in their answers.
Keywords: finite limit of a function at a point; pecific terms; effective use of terms; conceptions; conceptual analysis; non-compulsory secondary education

6.1. Introduction

This paper presents an exploratory, descriptive study that focuses on the meanings that Spanish students in Bachillerato[1] (16–17 years old) associate with the concept of the finite limit of a function at a point (Fernández-Plaza, 2011).

We base our study on prior research on cognitive conflicts related to the concepts of real number limit, notion of infinity and continuity of a function (Cornu, 1991; Davis & Vinner, 1986; Monaghan, 1991; Tall & Vinner, 1981).

In contrast to the everyday meanings, we analyse conceptually both the mathematical meaning and the students' use of the key terms. This analysis provides a productive way to interpret the understanding that the subjects have of the concept of finite limit of a function at a point.

6.2. Research problem

We will describe:

- How students express verbally their intuitive conceptions of the notion of finite limit of a function at a point.
- How students interpret this concept and perform tasks related to it by analysing the meaning of specific terms that express different facets of the concept of limit.

6.3. Background

This study forms part of the research agenda for *Advanced Mathematical Thinking* in the international research group on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006, pp. 147-172). We know about the difficulty of defining the transition from elementary to advanced mathematical thinking.

Azcárate and Camacho (2003) stress the importance of the definitions in advanced mathematics as a characteristic that distinguishes elementary from advanced mathematics. In elementary mathematics, descriptions built on students' experience are enough.

The educational stage analysed assumes a period of transition from elementary techniques to advanced mathematical contents.

We assume that the meaning of a mathematical concept is given by its sign, sense and reference, as is developed by Rico (2001, 2012). We analyse the *Conceptual Structure* (given by concepts and properties, propositions or theorems, with their criteria of veracity), *Systems of Representation* (given by sets of signs, graphics and rules to present the concept and establish relationships with other sets) and *Phenomenology* (including phenomena in which the concept originates and that give sense to it). Our model of the meaning of a mathematical concept is different from some cognitive models, such as the *Concept image/Concept definition* (Tall & Vinner, 1981; Vinner, 1983) or *APOS Theory* (Cottrill et al., 1996).

Following Chantal (2002) on the conceptual distinction between 'terms' and 'words', we understand *terms* as words whose meaning is valid for use in a specific disciplinary or technical context. These terms can be *specific* to the discipline itself, *common*, if they are used in several disciplines but with different meaning, or *imported* from the general and colloquial vocabulary of a language. In this paper, we use *specific terms* generically to indicate those with a technical use in calculus, regardless of their origin. We use *effective terms* to indicate the terms used by students, admitting the possibility that these may coincide with the specific terms.

To achieve our study goals, we specify the uses of some terms associated with limiting processes. The terms are 'to approach', 'to tend', 'to reach' and 'to exceed'. All of these terms are linked to the concept of limit and contribute to defining and understanding its meaning, due to the several senses they give to the concept. The definition of these terms contributes to the conceptual analysis of the concept of limit. *Conceptual analysis* is a procedure to establish the mathematical meaning and usefulness of a concept; not only formally, but also in the institutional, educational context and in its historical development, as opposed to its everyday uses (Rico, 2001, 2012; Scriven, 1998). To achieve this analysis of the concept of 'limit', we explore the conceptions that the subjects have of these terms, even though we may eventually need to provide additional information.

We describe the common uses of the specific terms chosen in order to establish how students employ them and to contrast them with their mathematical use or their use in other disciplines. Our review provides different conceptions of these terms that students are likely to consider.

6.3.1. Review of the uses of specific terms

We will describe the terms 'to approach', 'to tend', 'to reach', and 'to exceed'; we also provide the colloquial meaning of the term 'limit'. We chose these terms for the following reasons:

- They are terms with a technical and formal meaning in mathematics, but they also have ordinary colloquial uses not connected to their mathematical meanings.
- They appear frequently in the literature, both in the definition of the concept of limit and in the characterization of the associated difficulties and errors; they illustrate conflicts between formal and colloquial uses.
- The subjects in this study used these terms, as well as synonyms, to express different interpretations of the concept of limit, both technically (terminology acquired through instruction mediated by the profesor, the textbook or the instrument for data collection), and informally (in their own personal and coloquial interpretation).
- In the historical development of the concept of limit, Zeno's Paradoxes of *Dichotomy* and of *Achilles and the Tortoise* considered some properties of motion related to the specific terms 'to reach' and 'to exceed' (Cajori, 1915).
- Each of the terms refers in part to properties and modes of usage associated with the concept of limit.

We follow the dictionaries of the Spanish Royal Academy (Real Academia Española [RAE], 2001), the Spanish Royal Academy of Science [RAC] (1996) and the *Oxford Dictionary* (Oxford University Press [OUP], 2011) to establish the accepted, common and mathematical meanings of the following terms in Spanish: 'to approach', 'to tend', 'to exceed', 'to reach' and 'limit'.

'To tend' means *to approach gradually but never reach the value* [15] and expresses a very specific form of approach. Blázquez, Gatica and Ortega (2009) argue that a sequence of numbers approaches a number *if the error decreases gradually*, but they argue that a sequence 'tends toward a limit' *if the limit can be measured by the terms in the sequence, that is, for any approximation of the limit there exists one term of the sequence, after which all the terms are closer to the limit than that approximation.* We establish a distinction between these two terms.

The correct use of the term 'to tend toward' should be determined using the variable x and not $f(x)$, since the expression '$f(x)$ tends toward L, when x tends toward a' can cause cognitive conflicts. That is, since x never equals a, students may generalize this property to the relationship between $f(x)$ and L, that is contrary to the formal definition in the case of $f(x)$ is constantly L, as Tall and Vinner (1981) note.

'To reach' is intuitively *to arrive at or to come to touch* (RAE, 2001; OUP, 2011). We interpret 'reach' mathematically to mean that a function reaches the limit *if the limit value is the image of the point at which the limit is studied – continuity*; by extension, *the limit can be the value of any other point in the domain.*

Colloquially, 'to exceed' means *to be above a limit*, (RAE, 2001) excluding the meaning *to be below a lower bound.* We say that the limit of a function is exceeded *if we can construct two successive monotones of images that converge at the limit, one ascending and the other descending, for appropriate sequences of x-values that converge at the point at which the limit is studied.*

The reachability and exceedability of the finite limit of a function can be easily interpreted as global or local concepts, but there is no logical derivation between both concepts.

The term 'limit' has colloquial meanings that interfere with students' conceptions of this term, such as ideas of ending, boundary and what cannot be exceeded (RAE, 2001; OUP, 2011). The term's scientific-technical use is related in some disciplines to a subject matter or extreme state in which the behaviour of specific systems changes abruptly (RAC, 1996).

6.3.2. Prior Research

Monaghan (1991) studies the influence of language on the ideas that students have about the terms 'to tend', 'to approach', 'to converge' and 'limit', as these terms are employed in conjunction with different graphs of functions provided by the researcher and examples provided by school students. We stress as a limitation that the specific terms that the students were asked to use were defined *a priori*, instead of enabling students to use their own words freely and spontaneously and to infer the appropriate nuances *a posteriori*.

6.4. Method

This is a descriptive study based on a survey method with semi-open response questions, whose design is summarized below.

6.4.1. Subjects

The sample was composed of 36 Spanish students in the first year of non-compulsory secondary education, 16-17 years of age, who were taking Mathematics for the Science and Technology track. The students were chosen deliberately based on their availability.

6.4.2. Instrument

We used a questionnaire of three semi-open response questions, adapted and translated from (Lauten, Graham, & Ferrini-Mundy, 1994). Two different versions of the questionnaire were called A and B. The respondent was asked to evaluate as true (T) or False (F) the statement of a property related to the concept of the limit of a function at a point and then to justify the option chosen. The questions are described below:

> General instruction: *Circle T or F for each of the following statement, depending on whether it is true or false. Use the box to explain your choice:*
>
> (A.1) *A limit describes how a function moves as x moves towards a certain point.*[2]
>
> (A.2) *A limit is a number or point past which the function cannot go.*
>
> (A.3) *A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.*
>
> (B.1) *A limit is a number or point the function gets close to but never reaches.*
>
> (B.2) *A limit is an approximation that can be made as accurate as you wish*

(B.3) *A limit is a number that the y-values of a function can be made arbitrarily closet o by restricting x-values.*

The survey was administered in the middle of the 2010/2011 academic year. The subjects had received prior instruction on the concept of limit. Of the total of 36 subjects, 18 answered questionnaire A and the other 18 answered questionnaire B. (This was because both questionnaires included four more tasks, and we did not wish to tire the students). The survey was administered during a regular session of their math class. We allowed the students to use their own words freely and spontaneously to infer the appropriate nuances a posteriori in reference to the specific terms.

6.5. Results

We analysed the students' answers to the tasks described above in two phases. The next section describes the first phase. The second phase consisted of characterizing the categories of response, available at (Fernández-Plaza, 2011; Fernández-Plaza, Ruiz-Hidalgo, & Rico, 2012).

6.5.1. Use and counting of effective terms in the written records

We identified and tabulated the different uses of the effective terms in the students' written answers, without making inferences from their meaning. The groupings of effective terms were developed from the review described above, as shown in Table 6.1. Since we did not require the students to define the specific terms, we focus on the presence/absence of these terms or synonyms, and on the use the students make of the terms as they articulate their decisions. For the terms 'to approach' and 'to tend', Figures 6.1 and 6.2 show answers where the related effective terms are used.

In focusing on the terms 'to exceed' and 'to reach' (those directly related to questions A.2 and B.1), we see in Table 6.2 the frequency of some effective terms related to reachability and/or exceedability to characterize the value of the limit. The answers may also include references to the process of convergence through terms related to 'to approach' and 'to tend'. We consider three natural groups of answers: *reachability, exceedability and mixed.*

Table 6.1. Specific terms and groupings of related effective terms associated[a]

Specific terms	Effective terms associated from written records
To approach	*Aproximarse* [to approach] *dirigirse* [to head] *acercarse* [to get close] *moverse, desplazarse* [to move]
To tend	*Tender* [to tend][b]
To exceed	*Rebasar, exceder* [to exceed] *sobrepasar* [to surpass] *limitar* [to limit] *tope numérico* [numerical bound] *máximo* [maximum]
To reach	*Alcanzar* [to reach] *llegar* [to arrive] *tocar* [to touch] *exacto* [exact]

[a] We include the effective terms in their original language (Spanish) together with a non-univocal translation into English, in the following form: Spanish [English translation].
[b] The term 'to tend' has a technical use in mathematics, so it appears as the only effective term associated.

We show three examples provided by the students from different groups that involve their uses of the effective terms associated with 'to reach' and 'to exceed'.

- First, we give a sample answer from *Reachability group*, where the underlined expression includes the effective terms *approach/not reach*:
  Example 1 (Answer to question A.2). '*True. Because a limit is a point that a function approaches infinitely without reaching it'*.[3]

- Second, we present a sample answer within the Exceedability group with effective term *surpass*:
  Example 2 (Answer to question B.1). '*False. A function can indeed surpass a limit, since in many cases to find out the limit we have to give x-values that correspond to bigger images*'.[4]

- Finally, we give the following sample answer from the group classified as *Mixed* due to the uses of the effective terms *reach/not surpass*:

Example 3 (Answer to question B.1). '*False. The function reaches the limit, but it cannot surpass it* '.[5]

Table 6.2. Frequencies of use of the effective terms connected to 'to reach' and 'to exceed' for questions A.2 and B.1

Groups	Términos efectivos asociados	A.2	B.1
	to reach		2
	to arrive	1(Affirm.)/2 (Neg.)[a]	
	to get close/not reach		4
Reachability	to get close/not arrive		1
	to get close/not touch		1
	to approach/not reach		1
	to approach/to be inexact	2	2
	to tend/ to be inexact		1
Exceedability	to exceed	1	
	to surpass	1	
Mixed	to reach/not touch/not exceed	1	
	to reach/not surpass		2
Otras/no answer		10	4
Total		18	18

[a]Affirm. and neg. mean affirmative and negative forms of the effective term 'to arrive' in the sentences.

Questions A.2 and B.1 are not the only ones in which students used effective terms related to 'to reach' and 'to exceed'. These terms also appear in a few answers to other questions, such as A.1, A.3 and B.2. We can thus infer additional meanings of the specific terms 'to reach' and 'to exceed', as follows:

(A.1.) *A limit describes how a function moves as x moves towards a certain point.*

This question tries to find out how students interpret the concept of limit, whether as a process (students accept that the limit describes the movement of the function) or as an object (students refuse the statement and consider the limit only as the point toward which the function moves, and it does not say anything about the movement itself). The following answer states that the unreachability of the limit is a reason that the limit cannot describe the movement of the function:

Example 4. '*False. A limit is an approximate number which a function gets close to without an exact result*'.[6]

The underlined expression 'without an exact result' establishes a particular connection with the unreachability of the limit.

(A.3.) *A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.*

This question tries to determine, first, whether the plugging in process is finite or infinite, and, second, whether or not the limit can be reached. It is significant that only 2 out of 18 valid answers considered the unreachability of the limit. For example:

Example 5. **'***False. The limit cannot be reached, but only approximated and from those approximations we can get the limit'*.[7]

The underlined expression 'cannot be reached' establishes a particular connection with the unreachability of the limit and the infinite process of plugging in.

B.2.) *A limit is an approximation that can be made as accurate as you wish.*

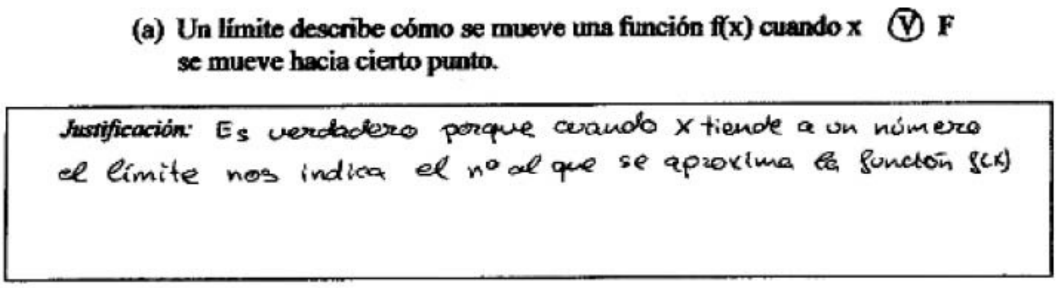
This question, like A.3, tries to determine whether the process of approximation is finite or infinite, and whether or not the subjects consider the approximate nature of the limit value. Two examples of answers are:

Example 6. '*False. A limit is a numerical bound and it is not approximate, but concrete*'.[8]

The underlined expression 'numerical bound' establishes a particular connection with the non-exceedability of the limit. The underlined expression 'concrete' is used to reject the approximate character of the limit.

Example 7. **'***True. The line determined by the function can approach infinitely but it will never arrive at the limit, for example, 0.5; 0.05; 0.005; 0.0005*'.[9]

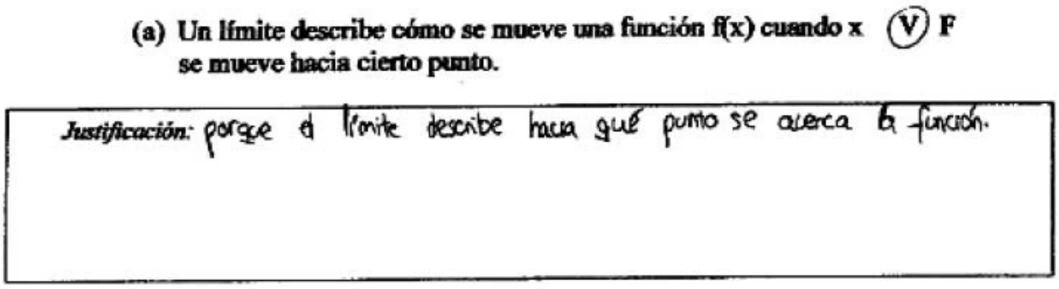
The underlined expressions 'approach infinitely' and 'it will never arrive' establish a particular connection with the unreachability of the limit, and this issue is related to the arbitrary precision of the approximation of the limit value.



(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. (V) F

Justificación: Es verdadero porque cuando x tiende a un número el límite nos indica el nº al que se aproxima la función f(x)

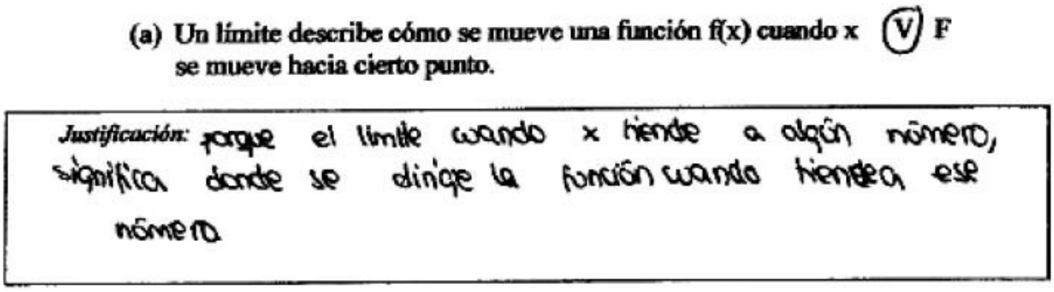
(a) Effective term "to approach." Translation: "It is true because when x tends toward a number the limit shows us the number that the function $f(x)$ approaches."



(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. (V) F

Justificación: porque el límite describe hacia qué punto se acerca la función.

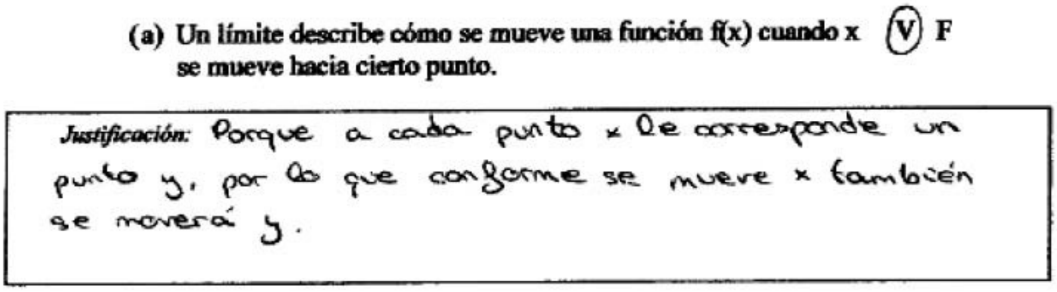
(b) Effective term "to get close." Translation: "True. Because the limit describes the point which the function gets close to."



(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. (V) F

Justificación: porque el límite cuando x tiende a algún número, significa donde se dirige la función cuando tienda a ese número

(c) Effective term "to head." Translation: "True. Because the limit when x tends toward a number means where the function heads when it tends to that number."

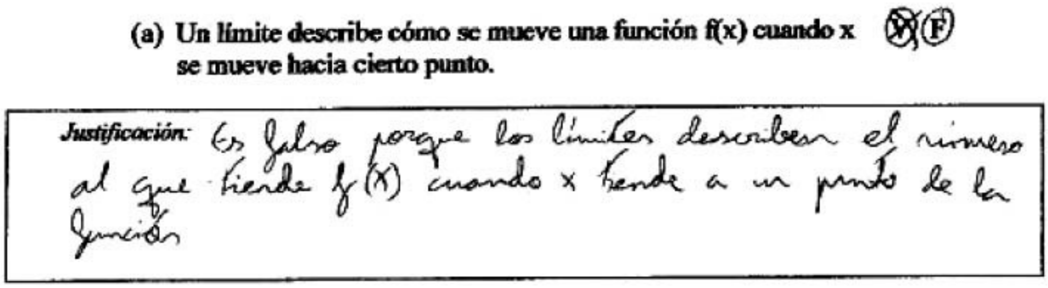


(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. (V) F

Justificación: Porque a cada punto x le corresponde un punto y, por lo que conforme se mueve x también se moverá y.

(d) Effective term "to move." Translation: "True. Because each point 'x' corresponds to a point 'y', so if 'x' moves, 'y' will move too."

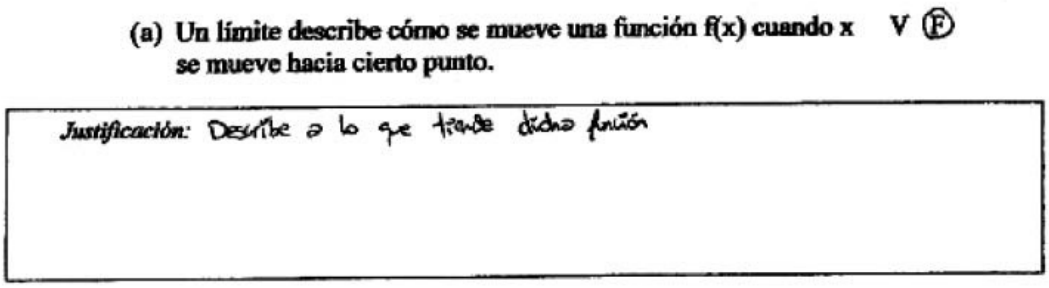
Figure 6.1. Examples of answers from question A.1. including effective terms related to the specific term 'to approach'.



(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. V F

Justificación: Es falso porque los límites describen el número al que tiende f(x) cuando x tiende a un punto de la función

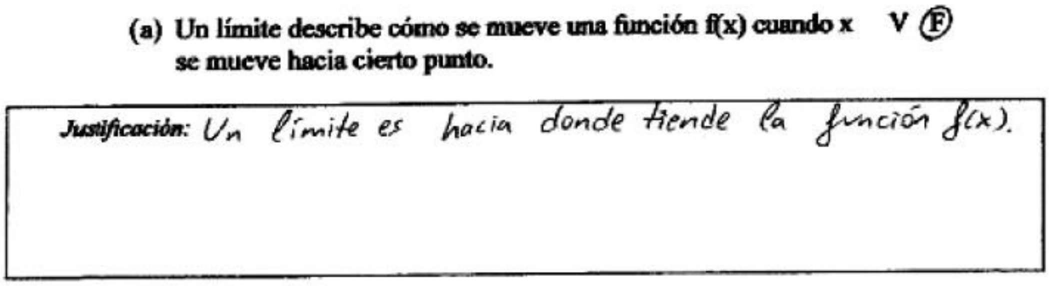
(a) Translation: "It is false because limits describe the number a function $f(x)$ tends toward when x tends toward a point of the function."



(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. V F

Justificación: Describe a lo que tiende dicha función

(b) Translation: "False. It describes what a function tends toward."



(a) Un límite describe cómo se mueve una función f(x) cuando x se mueve hacia cierto punto. V F

Justificación: Un límite es hacia donde tiende la función f(x).

(c) Translation: "False. A limit is where a function $f(x)$ tends toward."

Figure 6.2. Examples of answers from question A.1. including effective term 'to tend'

6.5.2. General discussion

The results provide a great variety of effective terms for interpretation as they relate to the specific terms selected in connection with the concept of limit and its mathematical meaning introduced above.

On the one hand, for question A.2, references to reachability predominate. Table 6.2 shows six out of eight valid answers when the subjects are required to argue about exceedability, whereas two out of eight valid answers refer to exceedability only. On the other hand, for question B.1, only 2 out of 14 valid references refer to the non-exceedability of the limit (both answers state that the limit is reachable), in contrast to 12 out of 14 valid references to reachability. This result shows a connection between the two properties.

Some answers from questions A.1, A.3 and B.2 suggest the following implications for the unreachability or non-exceedability of the limit:

- The impossibility that the limit describe the movement of a function, at least at the point where the study is carried out, due to unreachability.
- The arbitrary precision of the approximation to the limit is due to unreachability.
- The limit is not an approximate but an exact number; however, it cannot be exceeded. We speculate that students suggest this relationship due to the imprecise use of examples, in which convergence is strictly monotone and the value of the limit is, in fact, an upper bound and thus unreachable. Such use of examples excludes from the student's reasoning the image of the point at which the study is made, even when that image coincides with the limit.

6.5.3. Summary of results from the second phase of analysis

From the second phase of analysis, we summarize the findings related to the discussion of the categories of response, which are developed more fully in (Fernández-Plaza, Ruiz-Hidalgo, & Rico, 2012):

- *Discrimination between process conceptions, object conceptions and dual conceptions of the concept of limit.* As *process conceptions*, we consider those examples which suggest that a limit is closely related to the procedures a student uses to find it; as *object conceptions*, those where a student is able to identify the properties of the limit without depending on the process involved; while intermediate conceptions between these two are called *dual conceptions*. Thus, when students were requested to discuss the statement A.1, 'The limit describes how a function *f(x)* moves when *x* moves to certain point', most of their arguments could be classified as one of these three options depending on whether students interpreted the limit as 'how' (process conceptions) or 'where' (object and dual conceptions) a function moves.
- *Persistence of misconceptions related to the limit as a non-exceedable and unreachable value.* Some of students' arguments for questions A.2 and B.1 are consistent with considerations from Cornu (1991) and Monaghan (1991). Our results go beyond these earlier studies, however, in that some students suggested that limit is not exceedable because it is not reachable. Such responses indicate that this kind of misconception

could arise from overgeneralization of the particular case of monotone convergence.

- *Conflicts with the arbitrary precision of approximation to the limit.* The expression 'limit can be approximated as much as you wish' (question B.2) led some students to affirm that as the practical process is finite, so is precision. In question A.3, we can also see that students make a crucial distinction between the potentially infinite character of the process and its implementation in practice. Students conceive the arbitrariness of the process of approximating the limit (question B.3) in different ways. For example: 'False. They approach the limit in an approximate way, but not in an arbitrary way' (arbitrariness implies the reachability of the limit). Or: 'False. The values do not approach to the limit in an arbitrary way, depending on the x-values, the $f(x)$-values get close to the limit or move away from it' (arbitrariness implies that for every x-value, an $f(x)$-value approaches the limit).
- *Conflicts with the exact or indefinite character of the limit value.* Some students considered a limit to be an exact number, whereas others considered the limit an 'approximate' number. We suggest that the latter do not know what the limit is and therefore only think of approximations.

6.6. Conclusions

In analysing the results, we draw the following conclusions concerning the two aims proposed:
Aim 1. *To describe how students express verbally their intuitive conceptions of the notion of finite limit of a function at a point.* The conclusions are as follows:

- Conceptual analysis permits us to recognize possible conceptions that arise from the colloquial and everyday use of specific terms. These uses induce errors in students' understanding of the concept of finite limit of a function at a point. The conceptual analysis helps us to interpret these responses.
- Students use relatively undeveloped and imprecise language, characterized by the use of the terminology provided by the questions, as well as some original synonyms and specific terms. Their characterization of the limit as non-exceedable or unreachable persists, confirming the

influence of older colloquial and informal uses of the word 'limit' in the students' conceptions, as indicated by Cornu (1991).

Aim 2. *To describe how students interpret this concept and perform tasks related to it by analysing the meaning of specific terms that express different facets of the concept of limit.* The conclusions are as follows:

- The unreachability of the limit is considered by most of the students to be a cause of its non-exceedability, and the possibility of exceeding or reaching the limit is deduced through the use of examples. Although we find the use of expressions similar to '$f(x)$ tends toward a number, when x tends toward...', we do not find evidence to verify the presence of the semantic conflicts reported in other Research (Tall & Vinner, 1981; Blázquez, Gatica, & Ortega, 2009).
- Moreover, students occasionally relate these issues to the arbitrary precision of approximation, the impossibility of the limit describing the movement of the function, and the negation of the approximate character of the limit value.

Notes

1. Non-Compulsory Secondary Education.

2. The expression 'A limit describes how a function moves as x moves towards a certain point.' is related to a dynamic conception of function, in which the graph is drawn in the axis of Cartesian coordinates, or the study of phenomena, such as the trajectory of a projectile.

3. Original answer : 'Verdadero. Porque un límite es un punto al que una función se aproxima infinitamente sin llegar a él'

4. Original answer: 'Falso. Una función sí puede sobrepasar un límite, ya que muchas veces para averiguar el límite se dan valores que dan lugar a números más altos'.

5. Original answer: 'Falso. La función alcanza al límite, pero no puede sobrepasarlo'.

6. Original answer: 'Falso. Un límite es un número aproximado al que se acerca una función sin resultado exacto'.

7. Original answer: 'Falso. El límite no se puede alcanzar, pero sí aproximar y a partir de esas aproximaciones sacar el límite'.

8. Original answer: 'Falso. Un límite es un tope numérico y no es aproximativo, sino concreto'.

9. Original answer: 'Verdadero. La línea determinada por la función puede acercarse infinitamente pero nunca llegará, ej: 0.5; 0.05; 0.005; 0.0005'.

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