

# Riemann-Hilbert problems for poly-Hardy space on the unit ball

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## Abstract

In this paper we focus on a Riemann-Hilbert boundary value problem (for short BVP) with a constant coefficients for the poly-Hardy space on the real unit ball in higher dimensions. We first discuss the boundary behaviour of functions in the poly-Hardy class. Then we construct the Schwarz kernel and the higher order Schwarz operator to study Riemann-Hilbert BVPs over the unit ball for the poly-Hardy class. Finally, we obtain explicit integral expressions for their solutions. As a special case, monogenic signals as elements in the Hardy space over the unit sphere will be reconstructed in the case of boundary data given in terms of functions having values in a Clifford sub-algebra. Such monogenic signals represent the generalization of analytic signals as elements of the Hardy space over the unit circle of the complex plane.

**Keywords:** Hardy space, Riemann-Hilbert problems, monogenic signals, Schwarz kernel

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## 1 Introduction

The Riemann-Hilbert BVP is a classic topic in complex analysis posed by B. Riemann in his famous dissertation in 1851 and studied in the linear case in pioneering papers by D. Hilbert in 1904, 1905 and F. Noether in 1921. A complete solution of the linear Riemann-Hilbert problem was given in the classic works of F.D. Gakhov, I.N. Vekua, N.I. Mishkelishvili, B.V. Khvedekidze, D. A. Kveselava, and others. The question of determining an analytic function by its boundary values is linked to many problems in signal analysis and continuum mechanics,

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such as in hydrodynamics or in studying materials with memory. Later on, it has been extended to the cases of poly-analytic functions and functions belonging to the poly-Hardy class defined in sub-domains of the complex plane. For instance, we refer to Refs. [1, 2, 3, 4, 5, 6, 7] for details. In particular, Schwarz problems on the unit disc and on the half plane are dealing with the reconstruction of analytic signals on the circle and on the real axis, respectively. Nowadays, as a generalization to higher dimensions, Riemann-Hilbert BVPs for poly-monogenic functions defined over domains of the higher-dimensional Euclidean space were studied in several papers, e.g. in [8, 9, 10, 11]. Those studies rely on Clifford analysis as an elegant generalization of the theory of complex analysis to the higher dimensional Euclidean space, e.g., see [12, 13, 14]. Also, compared to analytic signals on the complex plane, monogenic signals in three dimensions were studied in Refs. [15, 16, 17], utilizing methods of Clifford analysis. More related results could be seen in Refs. [18, 19, 20, 21, 22, 23, 24] or elsewhere. However, to our knowledge, monogenic signals have not been uniquely reconstructed. Motivated by this, we aim at the reconstruction of monogenic signals via the study of a Riemann-Hilbert BVP with a constant coefficients for the Hardy class defined in domains of the higher dimensional Euclidean space. Our main idea is to construct a kernel function for the considered BVP on the higher-dimensional unit ball. Explicitly, by applying methods of Clifford analysis, we introduce the poly-Hardy class over the unit ball, characterize the boundary behavior of functions in the poly-Hardy class. Furthermore, we construct the Schwarz kernel and the higher order Schwarz operator to solve Riemann-Hilbert BVPs for the poly-Hardy class over the unit ball. Finally, we get explicit expressions for their solutions. Also, as a byproduct, monogenic signals over the unit sphere are characterized in a unique form.

The paper is organized as follows. In Section 2, we simply recall some basic facts about Clifford analysis which will be needed in the sequel. In Section 3, we will introduce the poly-Hardy space on the unit ball of the higher dimensional space, derive the decomposition theorem and characterize the boundary behaviour of functions in the poly-Hardy class. In Section 4, we introduce the theory of Riemann-Hilbert BVPs for the Hardy class on the unit ball with  $\mathbb{L}_p$  ( $1 < p < +\infty$ )-integral boundary data, and construct the Schwarz kernel to solve it. In the last section we will discuss a kind of Riemann-Hilbert BVPs for the poly-Hardy class on the unit ball with boundary data belonging to  $\mathbb{L}_p$  ( $1 < p < +\infty$ ) space. For this class of Riemann-Hilbert BVP, by applying the decomposition theorem and the higher order Schwarz operator, we get explicit expressions of their solutions in the unique form.

## 2 Preliminaries

In this section we simply review some basic facts about Clifford algebras which will be needed in the sequel. More details can be seen in Refs. [12, 13, 14] or elsewhere.

Let  $\{e_1, e_2, \dots, e_n\}$  be an orthogonal basis of the Euclidean space  $\mathbb{R}^n$  satisfying  $e_i^2 = -1$  for  $i = 1, 2, \dots, n$ ,  $e_i e_j + e_j e_i = 0$  for  $1 \leq i < j \leq n$ , and  $e_{h_1} e_{h_2} \dots e_{h_r} = e_{h_1 h_2 \dots h_r}$  for  $1 \leq h_1 < h_2 < \dots < h_r \leq n$ . Then  $\mathbb{R}_n$

denotes the  $2^n$ -dimensional real Clifford algebra with basis  $\{e_{\mathcal{A}} : \mathcal{A} = \{h_1, \dots, h_r\} \in \mathcal{PN}\}$ , where  $\mathcal{N}$  stands for the set  $\{1, 2, \dots, n\}$  and  $\mathcal{PN}$  denotes for the family of all order-preserving subsets of  $\mathcal{N}$ . We denote  $e_{\emptyset}$  as either  $e_0$  or 1 which is the identity element and  $e_{\mathcal{A}}$  as  $e_{h_1 \dots h_r}$  for  $\mathcal{A} = \{h_1, \dots, h_r\} \in \mathcal{PN}$ . The real  $n+1$ -dimensional linear space  $\mathbb{R}^{n+1}$  spanned by  $e_0, e_1, \dots, e_n$  is embedded into the Clifford algebra  $\mathbb{R}_n$ , whose typical element is denoted by  $x = x_0 + x_1 e_1 + \dots + x_n e_n$ ,  $x_j \in \mathbb{R}$  ( $j = 1, 2, \dots, n$ ). Moreover,  $\mathbb{R}_n = \mathbb{R}_{n-1} \oplus e_n \mathbb{R}_{n-1}$ , where  $\mathbb{R}_{n-1}$  is a sub-algebra of  $\mathbb{R}_n$  constructed by  $\{e_1, e_2, \dots, e_{n-1}\}$ . For arbitrary  $\lambda \in \mathbb{R}_n$ , we define projections  $X^{(n)}(\lambda) = \lambda_1$  and  $Y^{(n)}(\lambda) = \lambda_2$  where  $\lambda = \lambda_1 + e_n \lambda_2$  with  $\lambda_1, \lambda_2 \in \mathbb{R}_{n-1}$ , which is a higher dimensional decomposition corresponding to the decomposition into real and imaginary parts of a complex number, respectively.

The conjugation is defined by  $\bar{a} = \sum_{\mathcal{A}} a_{\mathcal{A}} \bar{e}_{\mathcal{A}}$ ,  $\bar{e}_{\mathcal{A}} = (-1)^{\frac{k(k+1)}{2}} e_{\mathcal{A}}$ ,  $N(\mathcal{A}) = k$ ,  $a_{\mathcal{A}} \in \mathbb{R}$ , hence we have  $\overline{\bar{a}} = a$ .

The inner product  $\langle \cdot, \cdot \rangle$  in  $\mathbb{R}_n$  is defined by putting for arbitrary  $a, b \in \mathbb{R}_n$ ,  $\langle a, \bar{b} \rangle = [a\bar{b}]_0$ . It is easy to see  $\langle a, \bar{b} \rangle = \sum_{\mathcal{A} \in \mathcal{PN}} a_{\mathcal{A}} b_{\mathcal{A}}$  with  $a = \sum_{\mathcal{A} \in \mathcal{PN}} a_{\mathcal{A}} e_{\mathcal{A}}$ ,  $b = \sum_{\mathcal{A} \in \mathcal{PN}} b_{\mathcal{A}} e_{\mathcal{A}}$ ,  $a_{\mathcal{A}}, b_{\mathcal{A}} \in \mathbb{R}$ . This leads to the norm on  $\mathbb{R}_n$  by  $|a| = \left( \sum_{\mathcal{A}} |a_{\mathcal{A}}|^2 \right)^{\frac{1}{2}} = \sqrt{\langle a, \bar{a} \rangle}$ . In the particular case  $x = \sum_{j=0}^n e_j x_j \in \mathbb{R}^{n+1} \subset \mathbb{R}_n$ , we have  $|x|^2 = \left( \sum_{j=0}^n x_j^2 \right)^{\frac{1}{2}} = \langle x, \bar{x} \rangle = x\bar{x}$ . Moreover,  $x^{-1} \triangleq \bar{x}|x|^{-2}$  is the inverse of an arbitrary  $x \in \mathbb{R}^{n+1} \setminus \{0\}$ , i.e.,  $xx^{-1} = x^{-1}x = 1$ . Up to the conjugation it corresponds to the Kelvin inverse in real analysis.

We now introduce the generalized Cauchy-Riemann operator  $\mathcal{D} = \sum_{j=0}^n e_j \partial_{x_j} \triangleq \partial_{x_0} + \underline{\mathcal{D}}$ , where  $\partial_{x_j}$  denotes the partial differential operator  $\frac{\partial}{\partial x_j}$ ,  $j = 0, 1, \dots, n$ , and  $\underline{\mathcal{D}} = \sum_{j=1}^n e_j \partial_{x_j}$ . It is obvious that  $\underline{\mathcal{D}}^2 = -\sum_{j=1}^n \partial_{x_j}^2$  and  $\overline{\mathcal{D}}\mathcal{D} = \mathcal{D}\overline{\mathcal{D}} = \sum_{j=0}^n \partial_{x_j}^2$ , which are the negative Laplacian in  $\mathbb{R}^n$  and the Laplace operator in  $\mathbb{R}^{n+1}$ , respectively.

Let  $\mathbb{B} = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$  be the unit ball centered at the origin with its boundary  $\partial\mathbb{B} = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$ .  $L_p$  ( $1 < p < +\infty$ )-integrability, continuity, differentiability, and so on of the function  $\phi(x) = \sum_{\mathcal{A}} \phi_{\mathcal{A}}(x) e_{\mathcal{A}} : \mathbb{B} \rightarrow \mathbb{R}_n$  are defined by being ascribed to each component  $\phi_{\mathcal{A}}(x) : \mathbb{B} \rightarrow \mathbb{R}$ . Let  $\mathcal{C}^k(\mathbb{B}, \mathbb{R}_n)$  ( $k \geq 1, k \in \mathbb{N}$ ) denote the Banach bi-module of all  $k$ -times continuously differentiable functions defined over  $\mathbb{B}$ . Null-solutions to the higher order generalized Cauchy-Riemann operator  $\mathcal{D}^k$ , i.e.,  $\mathcal{D}^k \phi(x) = 0$ , are called poly-monogenic functions, in particular, we have monogenic functions when  $\phi$  satisfies  $\mathcal{D}\phi(x) = 0$ . We use the notation  $\mathbb{H}_k = \{\phi : \mathbb{B} \rightarrow \mathbb{R}_n \mid \mathcal{D}^k \phi = 0\}$  in this context.

### 3 Boundary behaviour for poly-Hardy space

In this section we introduce the poly-Hardy space over the higher-dimensional unit ball, derive the decomposition theorem, and characterize the boundary behaviour of functions belonging to the poly-Hardy space.

Let  $\phi$  be a function defined over the unit ball  $\mathbb{B}$ . We define

$$\phi_r(\eta) = \phi(r\eta), 0 \leq r < 1. \quad (1)$$

The monogenic Hardy space over the unit ball is defined as

$$\mathbb{H}^p(\mathbb{B}) = \left\{ \phi \in \mathbb{H}_1(\mathbb{B}) : |\phi|_{1,p} < +\infty \right\}, \quad (2)$$

where

$$|\phi|_{1,p} = \sup \{ |\phi_r|_p : 0 \leq r < 1 \} \quad (3)$$

and

$$|\phi_r|_p = \left( \int_{\partial\mathbb{B}} |\phi_r(\eta)|^p dS_\eta \right)^{\frac{1}{p}}, 1 < p < +\infty. \quad (4)$$

We could check that the space  $\mathbb{H}^p(\mathbb{B})$  is a left- or right-linear Banach module under the norm of (3). When  $p = 2$  the space  $\mathbb{H}^2(\mathbb{B})$  is a right-linear Hilbert module under the inner product

$$(f, g) = \int_{\partial\mathbb{B}} \bar{f}(\eta)g(\eta)dS_\eta, f, g \in \mathbb{H}^2(\mathbb{B}).$$

For  $\phi \in \mathbb{H}_k$  ( $k > 1, k \in \mathbb{N}$ ), we define

$$\phi^j(x) = \mathcal{D}^j \phi(x), x \in \mathbb{B}, j = 0, 1, 2, \dots, k-1, \quad (5)$$

with  $\phi^0(x) = \phi(x), x \in \mathbb{B}$ . Moreover, when  $\phi \in \mathbb{H}_k$ , we have the decomposition, c.f. [8, 23, 24]

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j \phi_j(x), x \in \mathbb{B}, \quad (6)$$

where  $\phi_j \in \mathbb{H}_1(\mathbb{B}), j = 0, 1, 2, \dots, k-1$  is  $j$ -component of  $\phi$ .

**Definition 3.1.** Let  $1 < p < +\infty$  and  $k > 1, k \in \mathbb{N}$ . The subset of poly-monogenic functions defined over the unit ball  $\mathbb{B}$

$$\left\{ \phi \in \mathbb{H}_k(\mathbb{B}) : |\phi^j|_{1,p} < +\infty, j = 0, 1, 2, \dots, k-1 \right\} \quad (7)$$

is the poly-Hardy space of order  $k$  over the unit ball, where  $|\cdot|_{1,p}$  is defined as term (3), and  $\phi^j$  ( $j = 0, 1, 2, \dots, k-1$ ) given by term (5). Thus such a poly-Hardy space on the unit ball will be denoted by  $\mathbb{H}_k^p(\mathbb{B})$ .

It is obvious that  $\mathbb{H}_k^p(\mathbb{B})$  ( $1 < p < +\infty$ ) is linear. Define

$$|\phi|_{k,p} = \sum_{j=0}^{k-1} |\phi^j|_{1,p}, \phi \in \mathbb{H}_k^p(\mathbb{B}), \quad (8)$$

where the norm  $|\cdot|_{1,p}$  is presented by term (3). In what follows, whenever no confusion arises we write shortly  $\mathbb{H}_1^p(\mathbb{B}) = \mathbb{H}^p(\mathbb{B})$  for brevity. Moreover, when  $k = 1$ , the norm  $|\cdot|_{k,p}$  in term (8) reduces to that in term (3). From Definition 3.1, it can be seen that  $\mathbb{H}_k^p = \left\{ \phi \in \mathbb{H}_k(\mathbb{B}) : |\phi|_{k,p} < +\infty \right\}$ . Therefore, we have the following decomposition of the poly-Hardy space.

**Theorem 3.2.** *Let  $\mathbb{H}_k^p(\mathbb{B})$ ,  $1 < p < +\infty$ , be the poly-Hardy space of order  $k$  ( $k > 1$ ) on the unit ball. Then*

$$\mathbb{H}_k^p(\mathbb{B}) = \mathbb{H}_1^p(\mathbb{B}) \oplus x_0 \mathbb{H}_1^p(\mathbb{B}) \oplus \dots \oplus x_0^{k-1} \mathbb{H}_1^p(\mathbb{B}), \quad (9)$$

where  $x_0^j \mathbb{H}_1^p(\mathbb{B}) = \left\{ x_0^j \phi(x) : \phi \in \mathbb{H}_1^p(\mathbb{B}) \right\}$ ,  $j = 0, 1, 2, \dots, k-1$ .

*Proof.* First of all, we need to prove that

$$\mathbb{H}_k^p(\mathbb{B}) \subset \mathbb{H}_1^p(\mathbb{B}) \oplus x_0 \mathbb{H}_1^p(\mathbb{B}) \oplus \dots \oplus x_0^{k-1} \mathbb{H}_1^p(\mathbb{B}). \quad (10)$$

In fact, if  $\phi \in \mathbb{H}_k^p(\mathbb{B})$  by applying Definition 3.1 one has

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j \phi_j(x), \phi_j \in \mathbb{H}_1(\mathbb{B}), j = 0, 1, 2, \dots, k-1.$$

Moreover, by the direct calculation, we get

$$\Phi^*(x) = \mathbf{A}(x_0) \Phi_*(x), x \in \mathbb{B}, \quad (11)$$

where

$$\Phi^*(x) = \begin{pmatrix} \phi^0(x) \\ \phi^1(x) \\ \phi^2(x) \\ \vdots \\ \phi^{k-1}(x) \end{pmatrix}, \quad \Phi_*(x) = \begin{pmatrix} \phi_0(x) \\ \phi_1(x) \\ \phi_2(x) \\ \vdots \\ \phi_{k-1}(x) \end{pmatrix}, \quad (12)$$

and

$$\mathbf{A}(x_0) = \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^{k-1} \\ 0 & 1! & 2x_0 & \dots & (k-1)x_0^{k-2} \\ 0 & 0 & 2! & \dots & (k-1)(k-2)x_0^{k-3} \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & (k-1)! \end{pmatrix}_{k \times k}. \quad (13)$$

Due to  $x_0 \in \mathbb{B}$ ,  $\mathbf{A}(x_0)$  has inverse  $\mathbf{A}^{-1}(x_0)$ . This leads to

$$\Phi_*(x) = \mathbf{A}^{-1}(x_0)\Phi^*(x), x \in \mathbb{B}, \quad (14)$$

where the matrix  $\mathbf{A}^{-1}(x_0) = (b_{ij}(x_0))_{k \times k}$  (c.f. [8]) has the form

$$b_{ij}(x_0) = \begin{cases} \frac{(-1)^{i+j}}{(j-i)!(i-1)!}x_0^{j-i}, & j \geq i, \\ 0, & j < i. \end{cases}$$

Term (14) implies  $\phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \dots, k-1$ , which gives the validity of term (10). Thus we get

$$\mathbb{H}_k^p(\mathbb{B}) = \mathbb{H}_1^p(\mathbb{B}) + x_0\mathbb{H}_1^p(\mathbb{B}) + \dots + x_0^{k-1}\mathbb{H}_1^p(\mathbb{B}). \quad (15)$$

Next, consider  $0 = \sum_{j=0}^{k-1} x_0^j \phi_j(x), \phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \dots, k-1$ .

Equation (14) immediately gives  $\phi_j \equiv 0, x \in \mathbb{B}, j = 0, 1, 2, \dots, k-1$ .

The proof of the result is complete.  $\square$

In what follows, we will describe the boundary behavior of functions in the poly-Hardy space  $\mathbb{H}_k^p(\mathbb{B})$ .

**Theorem 3.3.** *If  $\phi \in \mathbb{H}_k^p(\mathbb{B})$  ( $1 < p < +\infty$ ), then  $\phi$  has the non-tangential limit  $\phi^+$  almost everywhere on the sphere  $\partial\mathbb{B}$ , and*

$$\lim_{r \rightarrow 1} |\phi^+ - \phi_r|_p = 0, \quad (16)$$

where  $\phi_r, |\cdot|_p$  are given in terms (1), (4), respectively.

*Proof.* If  $\phi \in \mathbb{H}_k^p(\mathbb{B})$  ( $1 < p < +\infty$ ) by Theorem 3.2 we have

$$\phi(x) = \sum_{j=0}^{k-1} x_0^j \phi_j(x), \phi_j \in \mathbb{H}_1^p(\mathbb{B}), j = 0, 1, 2, \dots, k-1. \quad (17)$$

Taking into account the boundary behavior of functions in the monogenic Hardy space  $\mathbb{H}_1^p(\mathbb{B})$  (see Theorem 7.9 in [14]),  $\phi_j (j = 0, 1, 2, \dots, k-1)$  has non-tangential limit  $\phi_j^+(t)$  almost everywhere on the sphere  $\partial\mathbb{B}$ , and

$$\lim_{r \rightarrow 1} |\phi_j^+ - (\phi_j)_r|_p = 0, \quad (18)$$

where  $(\phi_j)_r (j = 0, 1, 2, \dots, k-1)$  is defined similarly to term (1). Together with the decomposition (17), one gets

$$\phi^+(t) = \sum_{j=0}^{k-1} t_0^j \phi_j^+(t), \text{ a.e. } t \in \partial\mathbb{B}. \quad (19)$$

This implies that  $\phi$  has non-tangential limit  $\phi^+$  almost everywhere on the sphere  $\partial\mathbb{B}$ . Applying terms (18) and

(19), we have

$$|\phi^+ - \phi_r|_p \leq \sum_{j=0}^{k-1} \left[ |\phi_j^+ - (\phi_j)_r|_p + (1 - r^j) |(\phi_j)_r|_p \right], \quad (20)$$

which leads to the validity of (16) by means of condition (18). It follows the result.  $\square$

**Remark 3.4.** *Theorem 3.3 gives the characterization of the boundary behavior of a function in the poly-Hardy space  $\mathbb{H}_k^p(\mathbb{B})$  ( $1 < p < +\infty$ ). The boundary value  $\phi^+$ , given by (19), is the so-called non-tangential boundary value of  $\phi$ . In the underlying context, the symbol  $\phi^+$  will be understood as non-tangential boundary value of  $\phi$ .*

## 4 Riemann-Hilbert BVP for Hardy class

In this section, we introduce the theory of the Riemann-Hilbert BVP for the Hardy class on the unit ball with  $\mathbb{L}_p$  ( $1 < p < +\infty$ )-integrable boundary data. For this kind of Riemann-Hilbert BVP, we obtain explicit expressions of the solutions.

**Definition 4.1.** *Let  $\phi$  be an  $\mathbb{R}_{n-1}$ -valued harmonic function defined on  $\mathbb{B}$ . Any  $\mathbb{R}_{n-1}$ -valued harmonic function  $\psi$  defined on  $\mathbb{B}$  is called harmonic conjugate function of  $\phi$  if and only if  $\phi + e_n\psi$  is monogenic on  $\mathbb{B}$ .*

**Remark 4.2.** *In order to keep the correspondence with the following context, in Definition 4.1, the harmonic conjugate function for the monogenic function is only described for the case of the unit ball  $\mathbb{B}$ . In fact, the harmonic conjugate function for the monogenic function could be defined in arbitrary domains of  $\mathbb{R}^{n+1}$ . Moreover, when the dimension  $n$  of the space equals to 1,  $\mathbb{R}^2 = \{x_0 + x_1e_1 : x_0, x_1 \in \mathbb{R}\} \cong \mathbb{C}$  and  $\mathbb{R}_0 = \mathbb{R}$ , Definition 4.1 reduces to the conjugate harmonic function for an analytic function defined in domains of the complex plane.*

Starting from Definition 4.1, we derive the following lemmas.

**Lemma 4.3.** *Let  $\phi$  be a  $\mathbb{R}_{n-1}$ -valued harmonic function defined on the unit ball  $\mathbb{B}$ , then its conjugate harmonic function is uniquely determined and has the following form*

$$\psi(x) = \int_0^{x_n} \mathcal{D}_1\phi(\underline{x} + e_n u) du - \int_{\mathbb{B}} \frac{\underline{x} - \underline{y}}{|\underline{x} - \underline{y}|^n} (\partial_{x_n}\phi)(\underline{y}) dV_{\underline{y}} + f(\underline{x}), \quad x \in \mathbb{B}, \quad (21)$$

where  $\mathcal{D}_1 = \sum_{j=0}^{n-1} e_j \partial_{x_j}$ ,  $\underline{x} = \sum_{j=0}^{n-1} e_j x_j$ ,  $\overline{\mathcal{D}_1} f = 0$ .

*Proof.* According to Definition 4.1,  $\phi + e_n\psi$  is monogenic over the unit ball  $\mathbb{B}$ . Then we have

$$\begin{cases} \mathcal{D}_1\phi - \partial_{x_n}\psi = 0, \\ \partial_{x_n}\phi(x) + \overline{\mathcal{D}_1}\psi = 0. \end{cases} \quad (22)$$

From the first term in (22), we obtain

$$\psi(x) = \int_0^{x_n} \mathcal{D}_1 \phi(\underline{x} + e_n u) du + f_1(\underline{x}), \quad (23)$$

where  $f_1$  is a function with respect to  $\underline{x} = \sum_{j=0}^{n-1} e_j x_j$ .

Inserting term (23) into the second term of (22), we have

$$\overline{\mathcal{D}}_1 f_1(\underline{x}) = -\partial_{x_n} \phi(\underline{x}, 0). \quad (24)$$

Therefore, we get

$$f_1(\underline{x}) = - \int_{\mathbb{B}} \frac{\underline{x} - \underline{y}}{|\underline{x} - \underline{y}|^n} (\partial_{x_n} \phi)(\underline{y}, 0) dV_{\underline{y}} + f(\underline{x}), \quad \underline{x} \in \mathbb{B}, \quad (25)$$

where  $\overline{\mathcal{D}}_1 f = 0$  with respect to  $\underline{x}$ .

Inversely, if  $\psi$  is given by (21), then  $\phi + e_n \psi$  satisfies the equation (22). It follows the result.  $\square$

**Corollary 4.4.** *Let  $P(x, y) = \frac{1-|x|^2}{\omega_{n+1}|y-x|^{n+1}}$  ( $n \geq 2, n \in \mathbb{N}$ ),  $x \in \mathbb{B}, y \in \partial\mathbb{B}$ , where  $\omega_{n+1}$  denotes the area of unit sphere  $\partial\mathbb{B}$ . Then a conjugate harmonic function  $Q(x, y)$  of  $P(x, y)$  is given by*

$$\begin{aligned} Q(x, y) = & \partial_{n_y} f_3(x, y)|_{y \in \partial\mathbb{B}} + \frac{1}{\omega_{n+1}} \left[ \frac{(1-n)(\underline{x}-\underline{y})}{|\underline{y}-\underline{x}|^n} + \frac{n(\underline{x}-\underline{y}) \langle \underline{x}-\underline{y}, 2\underline{x} \rangle}{|\underline{y}-\underline{x}|^{n+2}} \right. \\ & \left. - \frac{2\underline{x}}{|\underline{y}-\underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y}-\underline{x}|^n}\right) + \frac{2(\underline{x}-\underline{y})}{|y-x|^{n+1}} \left( \frac{(x_n - y_n) \langle \underline{x}-\underline{y}, \underline{x} \rangle}{|\underline{y}-\underline{x}|^2} - x_n \right) \right], \end{aligned} \quad (26)$$

where  $\overline{\mathcal{D}}_1 f_3 = 0$  with respect to  $\underline{x}$ ,  $\partial_{n_y}$  denotes the directional derivative along the outward pointing unit normal vector at  $y \in \partial\mathbb{B}$ , and

$$F(\alpha, t) = \begin{cases} \frac{1}{2\alpha-1} \frac{t}{(1+t^2)^{\alpha-1}} + \frac{2\alpha-3}{2\alpha-2} F(\alpha-1, t), & 2\alpha \in \mathbb{N} + 2, \alpha \neq 1, \\ \frac{1}{2\alpha-2} \frac{t}{(1+t^2)^{\alpha-1}} F\left(\frac{3}{2} - \alpha, 1; 2 - \alpha; 1 + t^2\right), & \alpha \in \mathbb{N} + \frac{3}{2}, \\ \frac{1}{2\alpha-2} \frac{t}{(1+t^2)^{\alpha-1}} \left( \sum_{k=0}^{\alpha-2} \frac{(\frac{3}{2}-\alpha)_k}{(2-\alpha)_k} (1+t^2)^k \right) + \frac{1}{2\alpha-2} \frac{(\frac{3}{2}-\alpha)_{\alpha-2}}{(2-\alpha)_{\alpha-2}} \arctan t, & \alpha \in \mathbb{N} + 1, \end{cases} \quad (27)$$

and

$$(a)_k \triangleq \begin{cases} 1, & k = 0, \\ a(a+1) \dots (a+k-1), & k \in \mathbb{N}, \end{cases} \quad (28)$$

$F(a, b; c; t)$  stands for the ordinary hyper-geometric function [25, 26].



In particular, we have

$$\begin{aligned}
Q(x, y) &= \frac{1}{\omega_{n+1}} \left[ \frac{(1-n)(\underline{x}-\underline{y})}{|\underline{y}-\underline{x}|^n} + \frac{n(\underline{x}-\underline{y})\langle \underline{x}-\underline{y}, 2\underline{x} \rangle}{|\underline{y}-\underline{x}|^{n+2}} - \frac{2\underline{x}}{|\underline{y}-\underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n-y_n}{|\underline{y}-\underline{x}|^n}\right) \right. \\
&\quad \left. + \frac{2(\underline{x}-\underline{y})}{|\underline{y}-\underline{x}|^{n+1}} \left( \frac{(x_n-y_n)\langle \underline{x}-\underline{y}, \underline{x} \rangle}{|\underline{y}-\underline{x}|^2} - x_n \right) \right].
\end{aligned} \tag{29}$$

*Proof.* We will first check that

$$P(x, y) = \left[ \partial_{n_y} G(x, y) \right] \Big|_{y \in \partial \mathbb{B}}, \quad x \in \mathbb{B}, \tag{30}$$

where

$$G(x, y) = \frac{1}{(1-n)\omega_{n+1}} \left( |y-x|^{1-n} - |x|^{1-n} \left| y - \frac{x}{|x|^2} \right|^{1-n} \right), \quad x, y \in \mathbb{B} \ (x \neq y), \tag{31}$$

and  $\partial_{n_y}$  denotes the directional derivative along the outward pointing unit normal vector at  $y \in \partial \mathbb{B}$ .

Denote  $x^* = \frac{x}{|x|^2}$ , then

$$\begin{aligned}
|y-x^*|^2 &= \left\langle y - \frac{x}{|x|^2}, \bar{y} - \frac{\bar{x}}{|x|^2} \right\rangle = \langle y, \bar{y} \rangle - \frac{1}{|x|^2} \langle x, \bar{y} \rangle - \frac{1}{|x|^2} \langle y, \bar{x} \rangle + \frac{1}{|x|^4} \langle x, \bar{x} \rangle \\
&= |y|^2 - \frac{2}{|x|^2} \langle y, \bar{x} \rangle + \frac{1}{|x|^2} = \frac{1}{|x|^2} (|x|^2 - 2 \langle y, \bar{x} \rangle + |y|^2) = \frac{1}{|x|^2} |y-x|^2, \quad y \in \partial \mathbb{B}.
\end{aligned} \tag{32}$$

Therefore, we obtain

$$|y-x^*| = \frac{1}{|x|} |y-x|, \quad y \in \partial \mathbb{B}. \tag{33}$$

Moreover, since

$$\nabla_y |y-x|^{1-n} = \frac{1}{\omega_{n+1}} \frac{y-x}{|y-x|^{n+1}}, \quad \nabla_y |x|^{1-n} |y-x^*|^{1-n} = \frac{1}{\omega_{n+1} |x|^{n-1}} \frac{y-x^*}{|y-x^*|^{n+1}}, \quad x, y \in \mathbb{B} \ (x \neq y), \tag{34}$$

where  $\nabla_y$  is the gradient operator with respect to  $y$ , we arrive at

$$P(x, y) = \frac{1}{\omega_{n+1}} \left( \frac{y-x}{|y-x|^{n+1}} - |x|^{1-n} \frac{y-x^*}{|y-x^*|^{n+1}}, y \right) \Big|_{y \in \partial \mathbb{B}} = \frac{1}{\omega_{n+1}} \frac{1-|x|^2}{|y-x|^{n+1}}, \quad x \in \mathbb{B}, y \in \partial \mathbb{B}, \tag{35}$$

where  $\langle \nabla_y \phi, y \rangle = \sum_{j=0}^n y_j \partial_{y_j} \phi$ .

Now, by applying Lemma 4.3, the conjugate harmonic function of

$$G_1(x, y) = \frac{1}{(1-n)\omega_{n+1}} |y-x|^{1-n},$$

with respect to  $x$ , is given by

$$Q_1(x, y) = \int_0^{x_n} \mathcal{D}_1 G_1(\underline{x} + e_n u, y) du - \int_{\mathbb{B}} \frac{\underline{x} - \underline{z}}{|\underline{x} - \underline{z}|^n} d\sigma_z (\partial_{x_n} G_1)(y, \underline{z}) + f(x, y), \quad x, y \in \mathbb{B} \quad (x \neq y), \quad (36)$$

where  $\overline{\mathcal{D}_1} f = 0$  with respect to  $\underline{x}$ .

Therefore, we have

$$\int_0^{x_n} \mathcal{D}_1 G_1(\underline{x} + e_n u, y) du = \int_0^{x_n} \frac{1}{\omega_{n+1}} \frac{\underline{x} - \underline{y}}{|y - \underline{x} - e_n u|^{n+1}} du \quad (37)$$

$$= \frac{1}{\omega_{n+1}} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} \left( F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right) + F\left(\frac{n+1}{2}, \frac{y_n}{|\underline{y} - \underline{x}|}\right) \right), \quad (38)$$

where  $F(\alpha, t) = \int_0^t \frac{ds}{(1+s^2)^\alpha}$  is a well-defined function for an appropriately chosen  $\alpha \in \mathbb{Q}$ . Explicitly,

$$F(\alpha, t) = \begin{cases} \frac{1}{2\alpha-1} \frac{t}{(1+t^2)^{\alpha-1}} + \frac{2\alpha-3}{2\alpha-2} F(\alpha-1, t), & 2\alpha \in \mathbb{N} + 2, \alpha \neq 1, \\ \frac{1}{2\alpha-2} \frac{t}{(1+t^2)^{\alpha-1}} F\left(\frac{3}{2} - \alpha, 1; 2 - \alpha; 1 + t^2\right), & \alpha \in \mathbb{N} + \frac{3}{2}, \\ \frac{1}{2\alpha-2} \frac{t}{(1+t^2)^{\alpha-1}} \left( \sum_{k=0}^{\alpha-2} \frac{\left(\frac{3}{2} - \alpha\right)_k}{(2-\alpha)_k} (1+t^2)^k \right) + \frac{1}{2\alpha-2} \frac{\left(\frac{3}{2} - \alpha\right)_{\alpha-2}}{(2-\alpha)_{\alpha-2}} \arctan t, & \alpha \in \mathbb{N} + 1, \end{cases} \quad (39)$$

with

$$(a)_k \triangleq \begin{cases} 1, & k = 0, \\ a(a+1) \dots (a+k-1), & k \in \mathbb{N}, \end{cases} \quad (40)$$

and  $F(a, b; c; t)$  stands for the hyper-geometric function as usual (c.f. [25, 26]).

Moreover, we get

$$\begin{aligned} \partial_{x_n} G_1(x, y) \Big|_{x_n=0} &= \frac{1}{(1-n)\omega_{n+1}} \partial_{x_n} |y-x|^{1-n} \Big|_{x_n=0} \\ &= \frac{1}{\omega_{n+1}} \frac{-y_n}{(y_n^2 + |\underline{y} - \underline{x}|^2)^{\frac{n+1}{2}}}, \quad x, y \in \mathbb{B} \quad (x \neq y) \end{aligned} \quad (41)$$

and

$$\begin{aligned} \overline{\mathcal{D}_1} \left( \frac{1}{\omega_{n+1}} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{y_n}{|\underline{y} - \underline{x}|}\right) \right) \\ &= \frac{1}{\omega_{n+1}} \left( \overline{\mathcal{D}_1} \int_0^{\frac{y_n}{|\underline{y} - \underline{x}|}} \frac{ds}{(1+s^2)^{\frac{n+1}{2}}} \right) \cdot \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} \\ &= \frac{1}{\omega_{n+1}} \frac{-y_n}{(y_n^2 + |\underline{y} - \underline{x}|^2)^{\frac{n+1}{2}}}, \quad x, y \in \mathbb{B} \quad (x \neq y). \end{aligned} \quad (42)$$

Then, one has

$$\frac{1}{\omega_{n+1}} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{y_n}{|\underline{y} - \underline{x}|}\right) = \int_{\mathbb{B}} \frac{\underline{x} - \underline{z}}{|\underline{x} - \underline{z}|^n} d\sigma_z (\partial_{x_n} G_1)(y, \underline{z}) + f_2(\underline{x}, y), \quad x, y \in \mathbb{B} \quad (x \neq y), \quad (43)$$

where  $\overline{\mathcal{D}}_1 f_2 = 0$  with respect to  $\underline{x}$ .

Thus, we get the conjugate harmonic function of  $G_1(x, y)$ , with respect to  $x$ ,

$$Q_1(x, y) = \frac{1}{\omega_{n+1}} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right) + f_3(\underline{x}, y), \quad x, y \in \mathbb{B} \quad (x \neq y), \quad (44)$$

where  $\overline{\mathcal{D}}_1 f_3 = 0$  with respect to  $\underline{x}$ .

Especially, we obtain a conjugate harmonic function of  $G_1(x, y)$ , with respect to  $x$ ,

$$Q_1(x, y) = \frac{1}{\omega_{n+1}} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right), \quad x, y \in \mathbb{B} \quad (x \neq y). \quad (45)$$

Hence, the harmonic conjugate function of  $|y|^{1-n} \left| x - \frac{y}{|y|^2} \right|^{1-n}$ , with respect to  $x$ , is given by

$$|y|^{1-n} Q_1\left(x, \frac{y}{|y|^2}\right), \quad x, y \in \mathbb{B} \quad (x \neq y),$$

where  $Q_1(x, y)$  is presented by (44) and (45).

Next, we have a look at the following function

$$\begin{aligned} |x|^{1-n} \left| y - \frac{x}{|x|^2} \right|^{1-n} &= |\overline{x}|^{1-n} \left| y - \frac{x}{|x|^2} \right|^{1-n} = \frac{1}{(1 - 2 \langle \overline{x}, y \rangle + |x|^2 |y|^2)^{\frac{n-1}{2}}} \\ &= \frac{1}{(1 - 2 \langle \overline{y}, x \rangle + |x|^2 |y|^2)^{\frac{n-1}{2}}} = |y|^{1-n} \left| x - \frac{y}{|y|^2} \right|^{1-n}, \quad x, y \in \mathbb{B} \quad (x \neq y), \end{aligned} \quad (46)$$

that is, the conjugate harmonic function of  $|x|^{1-n} \left| y - \frac{x}{|x|^2} \right|^{1-n}$ , with respect to  $x$ , is given by  $|y|^{1-n} Q_1(x, y)$ ,

where  $Q_1\left(x, \frac{y}{|y|^2}\right)$  is presented by (44) and (45).

Furthermore, we get the harmonic conjugate function of  $G(x, y)$ , with respect to  $x$ , given by

$$Q_1(x, y) - |y|^{1-n} Q_1\left(x, \frac{y}{|y|^2}\right), \quad x, y \in \mathbb{B} \quad (x \neq y), \quad (47)$$

where  $Q_1(x, y)$  is given by (44), in particular, (45).

Finally, we get that the conjugate harmonic function of  $P(x, y)$ ,  $x \in \mathbb{B}$ ,  $y \in \partial\mathbb{B}$ , with respect to  $x$ , is given by

$$Q(x, y) = \left[ \partial_{n_y} \left( Q_1(x, y) - |y|^{1-n} Q_1\left(x, \frac{y}{|y|^2}\right) \right) \right] \Big|_{y \in \partial\mathbb{B}}, \quad x \in \mathbb{B}, \quad (48)$$

where  $Q_1(x, y)$  is described as in (44), in particular, in (45).

Explicitly, we can express  $Q$  by

$$\begin{aligned} Q(x, y) &= \partial_{n_y} f_3(\underline{x}, y)|_{y \in \partial \mathbb{B}} + \frac{1}{\omega_{n+1}} \left[ \partial_{n_y} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right) \right. \\ &\quad \left. - \partial_{n_y} |y|^{1-n} \frac{\underline{x} - \frac{y}{|y|^2}}{\left|\frac{y}{|y|^2} - \underline{x}\right|^n} F\left(\frac{n+1}{2}, \frac{x_n - \frac{y_n}{|y|^2}}{\left|\frac{y}{|y|^2} - \underline{x}\right|}\right) \right] \Big|_{y \in \partial \mathbb{B}}, \quad x \in \mathbb{B}, \end{aligned} \quad (49)$$

where  $\overline{D_1} f_3 = 0$  with respect to  $\underline{x}$ . By direct calculations, we obtain

$$\begin{aligned} &\left[ \partial_{n_y} \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right) \right] \Big|_{y \in \partial \mathbb{B}} \\ &- \left[ \partial_{n_y} |y|^{1-n} \frac{\underline{x} - \frac{y}{|y|^2}}{\left|\frac{y}{|y|^2} - \underline{x}\right|^n} F\left(\frac{n+1}{2}, \frac{x_n - \frac{y_n}{|y|^2}}{\left|\frac{y}{|y|^2} - \underline{x}\right|}\right) \right] \Big|_{y \in \partial \mathbb{B}} \\ &= \left\langle \nabla_y \left[ \frac{\underline{x} - \underline{y}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right) \right], y \right\rangle \Big|_{y \in \partial \mathbb{B}} \\ &- \left\langle \nabla_y \left[ |y|^{1-n} \frac{\underline{x} - \frac{y}{|y|^2}}{\left|\frac{y}{|y|^2} - \underline{x}\right|^n} F\left(\frac{n+1}{2}, \frac{x_n - \frac{y_n}{|y|^2}}{\left|\frac{y}{|y|^2} - \underline{x}\right|}\right) \right], y \right\rangle \Big|_{y \in \partial \mathbb{B}} \\ &= \frac{(1-n)(\underline{x} - \underline{y})}{|\underline{y} - \underline{x}|^n} + \frac{n(\underline{x} - \underline{y}) \langle \underline{x} - \underline{y}, 2\underline{x} \rangle}{|\underline{y} - \underline{x}|^{n+2}} - \frac{2\underline{x}}{|\underline{y} - \underline{x}|^n} F\left(\frac{n+1}{2}, \frac{x_n - y_n}{|\underline{y} - \underline{x}|}\right) \\ &+ \frac{2(\underline{x} - \underline{y})}{|y - x|^{n+1}} \left( \frac{(x_n - y_n) \langle \underline{x} - \underline{y}, \underline{x} \rangle}{|\underline{y} - \underline{x}|^2} - x_n \right), \quad x \in \mathbb{B}. \end{aligned} \quad (50)$$

By joining together expressions (49) and (50) the result of Corollary 4.4 is established.  $\square$

**Remark 4.5.** Lemma 4.3 gives a general reconstruction of a conjugate harmonic function of a monogenic function defined over the unit ball of  $\mathbb{R}^{n+1}$ , which is a generalization of the classical Green theorem of two real variables to higher dimensions. As an application, Corollary 4.4 presents a conjugate harmonic function of the classic Poisson kernel over the unit ball of  $\mathbb{R}^{n+1}$  ( $n \in \mathbb{N}, n \geq 2$ ), which will play a very crucial role in solving our Riemann-Hilbert BVPs for the poly-Hardy class defined over the unit ball of  $\mathbb{R}^{n+1}$  in the following. Similar results for the Dirac operator can be also found in Refs. [10, 27, 28, 29, 30] or elsewhere.

In what follows, we introduce the functions

$$\begin{aligned} K(x, y) &= P(x, y) + e_n Q(x, y), \quad x \in \mathbb{B}, y \in \partial \mathbb{B}, \\ z_j &= x_j e_0 + x_0 e_j, \quad j = 0, 1, \dots, n-1, \\ V_{l_1, \dots, l_k}(x) &= \frac{1}{k!} \sum_{\pi(l_1, \dots, l_k)} z_{l_1} \dots z_{l_k}, \quad \{l_1, \dots, l_k\} \subset \{1, \dots, n-1\}, \end{aligned} \quad (51)$$

where  $P(x, y) = \frac{1-|x|^2}{\omega_{n+1}|y-x|^{n+1}}$  ( $n \geq 2, n \in \mathbb{N}$ ),  $Q(x, y)$  is given by (29), the symbol  $\pi$  of the third term in (51)

denotes all distinguishable permutations of  $\{l_1, l_2, \dots, l_k\}$ , and  $V_0(x) = 1$ . Then  $\bar{z}_j \mathcal{D}_1 = 0, j = 0, 1, \dots, n-1$ ,  $\overline{\mathcal{D}_1} \sum_{k=0}^{+\infty} V_{l_1, \dots, l_k}(x) a_{l_1, \dots, l_k} = 0$ , with appropriately chosen  $a_{l_1, \dots, l_k} \in \mathbb{R}_{n-1}$  satisfying the corresponding series convergence, and it is the same in the following context.

**Theorem 4.6.** *If  $f \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ , then*

$$p.v. \int_{\partial\mathbb{B}} Q(x, y) f(y) dS_y, x \in \partial\mathbb{B} \quad (52)$$

is well defined, where *p.v.* is short for the Cauchy principle value. Moreover,

$$Qf(x) \triangleq p.v. \int_{\partial\mathbb{B}} Q(x, y) f(y) dS_y \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}). \quad (53)$$

This leads to

$$p.v. \int_{\partial\mathbb{B}} K(x, y) f(y) dS_y \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_n), \quad (54)$$

and

$$\lim_{\mathbb{B} \ni x \rightarrow y} \int_{\partial\mathbb{B}} K(x, y) f(y) dS_y = f(y) + e_n p.v. \int_{\partial\mathbb{B}} Q(x, y) f(y) dS_y, y \in \partial\mathbb{B}. \quad (55)$$

*Proof.* By applying formula (29) in Corollary 4.4, the singular integral operator (52) is of Calderón-Zygmund type which implies the result.  $\square$

**Remark 4.7.** *Formula (55) characterizes the boundary behavior of the singular integral operator (54), which actually corresponds to the well-known Plemelj-Sokhotzki formula. This allows us to define a mapping*

$$\begin{aligned} \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}) &\rightarrow \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1}), \\ f &\mapsto Qf. \end{aligned}$$

**Problem I.** Let  $f \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ . Find a function  $\phi : \mathbb{B} \rightarrow \mathbb{R}_n$ , satisfying the Riemann-Hilbert boundary value condition

$$\begin{cases} \phi \in \mathbb{H}^p(\mathbb{B}), 1 < p < +\infty, \\ X^{(n)}\{\lambda\phi(t)\} = f(t), \text{ a.e. } t \in \partial\mathbb{B}, \end{cases} \quad (56)$$

where  $\lambda \in \mathbb{R}_{n-1}$  is a constant with its inverse  $\lambda^{-1}$ .

**Theorem 4.8.** *Given a function  $f \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ , Problem (56) has a solution in the unique form, given by*

$$\phi(x) = \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} f(y) dS_y + e_n \sum_{k=0}^{+\infty} V_{l_1, \dots, l_k}(x) a_{l_1, \dots, l_k}, a_{l_1, \dots, l_k} \in \mathbb{R}_{n-1}, x \in \mathbb{B}. \quad (57)$$

*Proof.* From Corollary 4.4 and formula (51), we can state that

$$\phi_1(x) = \int_{\partial\mathbb{B}} K(x, y) \lambda^{-1} f(y) dS_y, \quad x \in \mathbb{B} \quad (58)$$

is a solution to Problem (56).

Let  $\phi$  be also a solution to Problem (56). Then  $\Phi$  is a solution to the homogeneous case of Problem (56)

$$\begin{cases} \Phi \in \mathbb{H}^p(\mathbb{B}), 1 < p < +\infty, \\ X^{(n)}\{\Phi(t)\} = 0, \text{ a.e. } t \in \partial\mathbb{B}. \end{cases} \quad (59)$$

Hence, we have  $\mathcal{D}\Phi = 0, x \in \mathbb{B}$ , and

$$\begin{cases} \Delta(X^{(n)}\Phi) = 0, \quad x \in \mathbb{B}, \\ (X^{(n)}\Phi)(t) = 0, \text{ a.e. } t \in \partial\mathbb{B}. \end{cases} \quad (60)$$

Therefore,  $X^{(n)}\Phi \equiv 0, x \in \mathbb{B}$ . That is,  $\Phi = e_n Y^{(n)}\Phi, x \in \mathbb{B}$ .

By applying Lemma 4.3, the solution in the unique form to Problem (59) is still denoted by  $Y^{(n)}\Phi(x) = f_4(\underline{x})$ , with  $\overline{\mathcal{D}_1}f_4 = 0, x \in \mathbb{B}$ . From the references, e.g. Ref. [12], we know that

$$Y^{(n)}\Phi(x) = \sum_{k=0}^{+\infty} V_{l_1, \dots, l_k}(\underline{x}) a_{l_1, \dots, l_k}, a_{l_1, \dots, l_k} \in \mathbb{R}_{n-1}. \quad (61)$$

Thus, it follows the result. □

**Corollary 4.9.** *If a  $\mathbb{R}_{n-1}$ -valued data  $f$  defined on  $\partial\mathbb{B}$  is given, then, a monogenic signal in the Hardy space  $\mathbb{H}^2(\mathbb{B})$  could be reconstructed, and it is the non-tangential limit of*

$$\phi(x) = \int_{\partial\mathbb{B}} K(x, y) f(y) dS_y + e_n \sum_{k=0}^{+\infty} V_{l_1, \dots, l_k}(\underline{x}) a_{l_1, \dots, l_k}, a_{l_1, \dots, l_k} \in \mathbb{R}_{n-1}, x \in \mathbb{B}. \quad (62)$$

*Proof.* Let  $p = 2$  and  $\lambda = 1$  in Theorem 4.8, applying Theorem 4.6, it follows the Corollary 4.9. □

**Remark 4.10.** *In Theorem 4.8  $K(x, y)$  given by (51) is still named the Schwarz kernel. This is due to the fact that it actually plays the same role as the well-known Schwarz kernel for analytic functions on the disc of the complex plane.*

When  $n = 1, \lambda = 1$ , Problem (56) reduces to the following classical Riemann-Hilbert BVP for the Hardy class over the unit disc of the complex plane

$$\begin{cases} \phi \in \mathcal{H}^p(\mathbb{D}), 1 < p < +\infty, \\ \mathbf{Re}\{\phi(t)\} = f(t), \text{ a.e. } t \in \partial\mathbb{D}, \end{cases} \quad (63)$$



where

$$(\mathbf{X}^{(n)}\Phi)^*(t) = \left( (X^{(n)}\phi_1)(t), (X^{(n)}\phi_2)(t), \dots, (X^{(n)}\phi_{k-1})(t) \right)^T, \quad t \in \partial\mathbb{B}, \quad (68)$$

$$\mathbf{f}_*(t) = (\tilde{f}_1(t), \tilde{f}_2(t), \dots, \tilde{f}_{k-1}(t))^T, \quad t \in \partial\mathbb{B}, \quad (69)$$

$$\tilde{f}_j(t) = \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} t_0^{i-j} \lambda^{-1} f_j(t), \quad j = 0, 1, 2, \dots, k-1, \quad t \in \partial\mathbb{B}$$

as in (14), and  $(\cdot)^T$  denotes the transpose of a vector. Moreover, we have  $\tilde{f}_j(t) \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ ,  $j = 0, 1, 2, \dots, k-1$ .

Due to Theorem 4.8, we get

$$\phi(x) = \sum_{j=0}^{k-1} \int_{\partial\mathbb{B}} x_0^j K(x, y) \lambda^{-1} \tilde{f}_j(y) dS_y + e_n \sum_{j=0}^{k-1} \sum_{k=0}^{+\infty} x_0^j V_{l_1, \dots, l_k}(x) a_{l_1, \dots, l_k}, \quad a_{l_1, \dots, l_k} \in \mathbb{R}_{n-1}, \quad x \in \mathbb{B}, \quad (70)$$

where  $K(x, y)$  is given by (51).

Thus, the proof of the result is complete.  $\square$

**Remark 5.2.** Using Theorem 3.2, Problem (64) is first transferred into Problem (56), and its solution is obtained, explicitly. A similar idea could be seen in Refs. [8, 9], where it is used to solve the Riemann BVPs for null-solutions to the iterated Cauchy-Riemann operator in the higher dimensional Euclidean space. In fact, denote

$$\mathbf{K}(x, y) = \sum_{j=0}^{k-1} x_0^j K(x, y) \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} y_0^{i-j}, \quad x \in \mathbb{B}, y \in \partial\mathbb{B},$$

with  $K(x, y)$  is presented in (51). For  $f_j \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_{n-1})$ ,  $j = 0, 1, 2, \dots, k-1$ , we define the higher order Schwarz operator, also called poly-Schwarz operator,

$$S[f_0, f_1, f_2, \dots, f_{k-1}] = \int_{\partial\mathbb{B}} \sum_{j=0}^{k-1} x_0^j K(x, y) \sum_{i=j}^{k-1} \frac{(-1)^{i+j}}{(i-j)!j!} y_0^{i-j} f_j(y) dS_y, \quad x \in \mathbb{B}. \quad (71)$$

Then, by applying formula (58) in Theorem 4.8, we have

$$\begin{cases} S[f_0, f_1, f_2, \dots, f_{k-1}] \in \mathbb{H}_k^p(\mathbb{B}), \quad 1 < p < +\infty, \\ X^{(n)} \left\{ [\mathcal{D}^j S[f_0, f_1, f_2, \dots, f_{k-1}]]^+(t) \right\} = f_j(t), \quad a.e. \quad t \in \partial\mathbb{B}, \quad j = 0, 1, 2, \dots, k-1. \end{cases} \quad (72)$$

This implies that  $\mathbf{K}(x, y)$ , still called higher order Schwarz kernel or poly-Schwarz kernel, in essence, acts as the well-known Schwarz kernel for poly-monogenic functions defined over the unit ball of  $\mathbb{B}$ , which is an extension of the poly-Schwarz kernel for analytic functions defined on the disc of the complex plane, see Refs. [6, 5], to



the higher dimensional Euclidean space. Moreover, by associating it with Theorem 4.6, we obtain

$$p.v. S[f_0, f_1, f_2, \dots, f_{k-1}] \in \mathbb{L}_p(\partial\mathbb{B}, \mathbb{R}_n). \quad (73)$$

**Corollary 5.3.** *If a vector of  $\mathbb{R}_{n-1}$ -valued data  $f_j, j = 0, 1, 2, \dots, k-1$  ( $k \in \mathbb{N}, k \geq 2$ ) defined on  $\partial\mathbb{B}$  is given, then, a vector of a monogenic signal in the Hardy space  $\mathbb{H}^2(\mathbb{B})$  could be reconstructed, and it is presented by the non-tangential limit of  $\phi$ , given by term (65).*

*Proof.* Applying Theorems 4.6 and 5.1, it follows the Corollary 5.3. □

**Remark 5.4.** *In Theorem 5.1, under the given boundary conditions, we get the solution in the unique form belonging to the poly-Hardy space over the unit ball of the higher-dimensional Euclidean space  $\mathbb{R}^{n+1}$ . When  $n = 1$ , Problem (67) reduces to the Riemann-Hilbert type BVP for the poly-Hardy class on the unit disc of the complex plane, seen in Ref. [6], which are solved in terms of the classical Schwarz kernel defined on the unit disc of the complex plane.*

**Remark 5.5.** *In this context, beginning with boundary data with values in a Clifford sub-algebra Riemann-Hilbert problems with constant coefficients for the poly-Hardy spaces over the unit ball are solved. Riemann-Hilbert problems with variable coefficients are beyond the scope of this papers because they need more analytic techniques and will be discussed in another paper. However, following the same argument used at the moment together with the results obtained in Ref. [32], Riemann-Hilbert problems for the poly-Hardy spaces over the unit ball could be studied when the given data only takes scalar values. Similar to Corollaries 4.9 and 5.3, the non-tangential limits of the solutions to the Riemann-Hilbert problems will result in the reconstruction of monogenic signals on the unit sphere only when the initial data is scalar-valued, which is a more fascinating extension of the case of the analytic signals into higher dimensions from the point of practical applications.*

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