Eigenfunctions and fundamental solutions of the fractional Laplace and Dirac operators: the Riemann-Liouville case*

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Abstract

In this paper we study eigenfunctions and fundamental solutions for the three parameter fractional Laplace operator $\Delta_+^{(\alpha,\beta,\gamma)} := D_{x_0^+}^{1+\alpha} + D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}$, where $(\alpha,\beta,\gamma) \in]0,1]^3$, and the fractional derivatives $D_{x_0^+}^{1+\alpha}$, $D_{y_0^+}^{1+\beta}$, $D_{z_0^+}^{1+\gamma}$ are in the Riemann-Liouville sense. Applying operational techniques via two-dimensional Laplace transform we describe a complete family of eigenfunctions and fundamental solutions of the operator $\Delta_+^{(\alpha,\beta,\gamma)}$ in classes of functions admitting a summable fractional derivative. Making use of the Mittag-Leffler function, a symbolic operational form of the solutions is presented. From the obtained family of fundamental solutions we deduce a family of fundamental solutions of the fractional Dirac operator, which factorizes the fractional Laplace operator. We apply also the method of separation of variables to obtain eigenfunctions and fundamental solutions.

Keywords: Fractional partial differential equations; Fractional Laplace and Dirac operators; Riemann-Liouville derivatives and integrals of fractional order; Eigenfunctions and fundamental solution; Laplace transform; Mittag-Leffler function.

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1 Introduction

In the last decades the interest in fractional calculus increased substantially. This fact is due to on the one hand different problems can be considered in the framework of fractional derivatives like, for example, in optics and quantum mechanics, and on the other hand fractional calculus gives us a new degree of freedom which can be used for more complete characterization of an object or as an additional encoding parameter. For more details about fractional partial differential equations, their applications and their numerical solutions see [8] and the references indicated there.

The problems with the fractional Laplace attracted in the last years a lot of attention, due especially to their large range of applications. The fractional Laplace appears e.g. in probabilistic framework as well as in mathematical finance as infinitesimal generators of the stable Lévy processes [1]. One can find problems involving

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the fractional Laplace in mechanics and in elastostatics, for example, a Signorini obstacle problem originating from linear elasticity [4]. The concerning fluid mechanics and hydrodynamics to the nonlocal fractional Laplace appears, for instance, in the quasi-geostropic fractional Navier-Stokes equation [3] and in the hydrodynamic model of the flow in some porous media [16].

The connections between fractional calculus and physics are, in some sense, relatively new but, and more important for the community, a subject of strong interest. In [14] the author proposed a fractional Dirac equation of order 2/3 and established the relation between the corresponding $\gamma_\alpha^\mu-$matrix algebra and generalized Clifford algebras. This approach was generalized in [19], where the author found that relativistic covariant equations generated by taking the $n-$th root of the d'Alembert operator are fractional wave equations with an inherent $\mathrm{SU}(n)$ symmetry. The study of the fractional Dirac operator is important due to its physical and geometrical interpretations. Physically, this fractional differential operator is related with some aspects of fractional quantum mechanics such as the derivation of the fractal Schrödinger type wave equation, the resolution of the gauge hierarchy problem, and the study of super-symmetries. Geometrically, the fractional classical part of this operator may be identified with the scalar curvature in Riemannian geometry.

Clifford analysis is a generalization of classical complex analysis in the plane to the case of an arbitrary dimension. At the heart of the theory lies the Dirac operator D, a conformally invariant first-order differential operator which plays the same role as the Cauchy-Riemann operator in complex analysis. In [10, 17] the authors studied the connections between Clifford analysis and fractional calculus, however, the fractional Dirac operator considered in these works do not coincide with the one used here.

The aim of this paper is to present an explicit expression for the family of eigenfunctions and fundamental solutions of the three-parameter fractional Laplace operator, as well as, a family of fundamental solutions of the fractional Dirac operator. For the sake of simplicity we restrict ourselves to the three dimensional case, however the results can be generalized for an arbitrary dimension. The two dimensional case was studied in [18] without considering the connections with Clifford analysis. The authors would like to point out that the fractional Laplace operator considered in this paper is different from the fractional Laplace operator defined via Fourier transform (see [8]). The deduction of the fundamental solution for the fractional Dirac operator defined via Riemann-Liouville derivatives is a completely new result in the context of fractional Clifford analysis. The fundamental solutions of the fractional Dirac operator obtained in this paper are the basis to develop an operator calculus theory in the context of fractional Clifford analysis.

The structure of the paper reads as follows: in the Preliminaries we recall some basic facts about fractional calculus, special functions and Clifford analysis, which are necessary for the development of this work. In Section 3 we use operational techniques for the two dimensional Laplace transformation and its extension to generalized functions to describe a complete family of eigenfunctions and fundamental solutions of the fractional Laplace operator. In the same section we compute the family of fundamental solutions for the fractional Dirac operator. In Section 4 we obtain the analogous of the results of Section 3, but via the method of separation of variables. Particular cases of solutions can be obtained using the obtained generic formulas and considering $\alpha = \beta = \gamma = 1$.

2 Preliminaries

2.1 Fractional calculus and special functions

Let $\left(D_{a^+}^\alpha f\right)(x)$ denote the fractional Riemann-Liouville derivative of order $\alpha > 0$ (see [11])

$$\left(D_{a^+}^\alpha f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}\, dt, \qquad n = [\alpha]+1, \quad x > a. \tag{1}$$

where $[\alpha]$ means the integer part of α. When $0 < \alpha < 1$ then (1) takes the form

$$\left(D_{a^+}^\alpha f\right)(x) = \frac{d}{dx}\, \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha}}\, dt. \tag{2}$$

The Riemann-Liouville fractional integral of order $\alpha > 0$ is given by (see [11])

$$\left(I_{a^+}^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \quad x > a. \tag{3}$$

We denote by $I^{\alpha}_{a+}(L_1)$ the class of functions f represented by the fractional integral (3) of a summable function, that is $f = I^{\alpha}_{a+}\varphi$, $\varphi \in L_1(a,b)$. A description of this class of functions was given in [15].

Theorem 2.1 *A function $f \in I^{\alpha}_{a+}(L_1), \alpha > 0$ if and only if $I^{n-\alpha}_{a+}f \in AC^n([a,b])$, $n = [\alpha]+1$ and $(I^{n-\alpha}_{a+}f)^{(k)}(a) = 0$, $k = 0, \dots, n-1$.*

In Theorem 2.1 $AC^n([a,b])$ denotes the class of functions f, which are continuously differentiable on the segment $[a,b]$ up to order $n-1$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Removing the last condition in Theorem 2.1 we obtain the class of functions that admits a summable fractional derivative.

Definition 2.2 *(see [15]) A function $f \in L_1(a,b)$ has a summable fractional derivative $\left(D^{\alpha}_{a+}f\right)(x)$ if $\left(I^{n-\alpha}_{a+}\right)(x) \in AC^n([a,b])$, where $n = [\alpha]+1$.*

If a function f admits a summable fractional derivative, then the composition of (1) and (3) can be written in the form (see [15, Thm. 2.4])

$$\left(I^{\alpha}_{a+}\ D^{\alpha}_{a+}f\right)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \ \left(I^{n-\alpha}_{a+}f\right)^{(n-k-1)}(a), \qquad n = [\alpha]+1. \tag{4}$$

We remark that if $f \in I^{\alpha}_{a+}(L_1)$ then (4) reduces to $\left(I^{\alpha}_{a+}\ D^{\alpha}_{a+}f\right)(x) = f(x)$. Nevertheless we note that $D^{\alpha}_{a+}\ I^{\alpha}_{a+}f = f$ in both cases. This is a particular case of a more general property (cf. [13, (2.114)])

$$D^{\alpha}_{a+}\left(I^{\gamma}_{a+}f\right) = D^{\alpha-\gamma}_{a+}f, \qquad \alpha \geq \gamma > 0. \tag{5}$$

It is important to remark that the semigroup property for the composition of fractional derivatives does not hold in general (see [13, Sect. 2.3.6]). In fact, the property

$$D^{\alpha}_{a+}\left(D^{\beta}_{a+}f\right) = D^{\alpha+\beta}_{a+}f \tag{6}$$

holds whenever

$$f^{(j)}(a^+) = 0, \qquad j = 0, 1, \dots, n-1, \tag{7}$$

and $f \in AC^{n-1}([a,b])$, $f^{(n)} \in L_1(a,b)$ and $n = [\beta]+1$.
One important function used in this paper is the two-parameter Mittag-Leffler function $E_{\mu,\nu}(z)$ [7], which is defined in terms of the power series by

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n+\nu)}, \qquad \mu > 0,\ \nu \in \mathbb{R},\ z \in \mathbb{C}. \tag{8}$$

In particular, the function $E_{\mu,\nu}(z)$ is entire of order $\rho = \frac{1}{\mu}$ and type $\sigma = 1$. The exponential, trigonometric and hyperbolic functions are expressed through (8) as follows (see [7]):

$$E_{1,1}(z) = e^z, \qquad E_{2,1}\left(-z^2\right) = \cos(z), \qquad E_{2,1}(z^2) = \cosh(z),$$
$$zE_{2,2}\left(-z^2\right) = \sin(z), \qquad zE_{2,2}\left(z^2\right) = \sinh(z).$$

Two important fractional integral and differential formulae involving the two-parametric Mittag-Leffler function are the following (see [7, p.61,p.87])

$$I^{\alpha}_{a+}\left((x-a)^{\nu-1}E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\alpha+\nu-1}E_{\mu,\nu+\alpha}\left(k(x-a)^{\mu}\right) \tag{9}$$

$$D^{\alpha}_{a+}\left((x-a)^{\nu-1}E_{\mu,\nu}\left(k(x-a)^{\mu}\right)\right) = (x-a)^{\nu-\alpha-1}E_{\mu,\nu-\alpha}\left(k(x-a)^{\mu}\right) \tag{10}$$

for all $\alpha > 0, \mu > 0, \nu \in \mathbb{R}, k \in \mathbb{C}, a > 0, x > a$.
The formal approach presented in Sections 3 and 4 based on the Laplace transform leads to the solution of a linear Abel integral equation of the second kind.

Theorem 2.3 *([7, Thm. 4.2]) Let $f \in L_1[a,b], \alpha > 0$ and $\lambda \in \mathbb{C}$. Then the integral equation*

$$u(x) = f(x) + \frac{\lambda}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}u(t)\ dt, \qquad x \in [a,b]$$

has a unique solution

$$u(x) = f(x) + \lambda \int_a^x (x-t)^{\alpha-1}E_{\alpha,\alpha}(\lambda(x-t)^{\alpha})f(t)\ dt. \tag{11}$$

2.2 Clifford analysis

Let $\{e_1, \cdots, e_d\}$ be the standard basis of the Euclidean vector space in $\mathbb{R}^d$. The associated Clifford algebra $\mathbb{R}_{0,d}$ is the free algebra generated by $\mathbb{R}^d$ modulo $x^2 = -||x||^2\ e_0$, where $x \in \mathbb{R}^d$ and e_0 is the neutral element with respect to the multiplication operation in the Clifford algebra $\mathbb{R}_{0,d}$. The defining relation induces the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij}, \tag{12}$$

where δ_{ij} denotes the Kronecker's delta. In particular, $e_i^2 = -1$ for all $i = 1, \ldots, d$. The standard basis vectors thus operate as imaginary units. A vector space basis for $\mathbb{R}_{0,d}$ is given by the set $\{e_A :\ A \subseteq \{1, \ldots, d\}\}$ with $e_A = e_{l_1} e_{l_2} \ldots e_{l_r}$, where $1 \leq l_1 < \ldots < l_r \leq d$, $0 \leq r \leq d$, $e_\emptyset := e_0 := 1$. Each $a \in \mathbb{R}_{0,d}$ can be written in the form $a = \sum_A a_A\ e_A$, with $a_A \in \mathbb{R}$. The conjugation in the Clifford algebra $\mathbb{R}_{0,d}$ is defined by $\overline{a} = \sum_A a_A\ \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r}\ \overline{e}_{l_{r-1}} \ldots \overline{e}_{l_1}$, and $\overline{e}_j = -e_j$ for $j = 1, \ldots, d$, $\overline{e}_0 = e_0 = 1$. An important subspace of the real Clifford algebra $\mathbb{R}_{0,d}$ is the so-called space of paravectors $\mathbb{R}_1^d = \mathbb{R} \bigoplus \mathbb{R}^d$, being the sum of scalars and vectors. Each non-zero vector $a \in \mathbb{R}_1^d$ has a multiplicative inverse given by $\frac{\overline{a}}{||a||^2}$.

Clifford analysis can be regarded as a higher-dimensional generalization of complex function theory in the sense of the Riemann approach. An $\mathbb{R}_{0,d}-$valued function f over $\Omega \subset \mathbb{R}_1^d$ has the representation $f = \sum_A e_A f_A$, with components $f_A : \Omega \to \mathbb{R}_{0,d}$. Properties such as continuity or differentiability have to be understood componentwise. Next, we recall the Euclidean Dirac operator $D = \sum_{j=1}^d e_j\ \partial_{x_j}$, which factorizes the d-dimensional Euclidean Laplace, i.e., $D^2 = -\Delta = -\sum_{j=1}^d \partial x_j^2$. An $\mathbb{R}_{0,d}$-valued function f is called *left-monogenic* if it satisfies $Du = 0$ on Ω (resp. *right-monogenic* if it satisfies $uD = 0$ on Ω).

For more details about Clifford algebras and basic concepts of its associated function theory we refer the interested reader for example to [5, 9].

3 Operational approach via Laplace transform

3.1 Eigenfunctions and fundamental solution of the fractional Laplace operator

We consider the eigenfunction problem for the fractional Laplace operator

$$\left(D^{1+\alpha}_{x_0^+} u\right)(x,y,z) + \left(D^{1+\beta}_{y_0^+} u\right)(x,y,z) + \left(D^{1+\gamma}_{z_0^+} u\right)(x,y,z) = \lambda\ u(x,y,z), \tag{13}$$

where $\lambda \in \mathbb{C}$, $(\alpha, \beta, \gamma) \in]0,1]^3$, $(x,y,z) \in \Omega = [x_0, X_0] \times [y_0, Y_0] \times [z_0, Z_0]$, $x_0, y_0, z_0 > 0$, $X_0, Y_0, Z_0 < \infty$, and $u(x,y,z)$ admits summable fractional derivatives $D^{1+\alpha}_{x_0^+}$, $D^{1+\beta}_{y_0^+}$, $D^{1+\gamma}_{z_0^+}$. Taking the integral operator $I^{1+\alpha}_{x_0^+}$ from both sides of (13) and taking into account (4) we get

$$\begin{aligned} 0 \quad = \quad & u(x,y,z) - \frac{(x-x_0)^\alpha}{\Gamma(1+\alpha)} \left(D^{\alpha}_{x_0^+} u\right)(x_0,y,z) - \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left(I^{1-\alpha}_{x_0^+} u\right)(x_0,y,z) \\ & + \left(I^{1+\alpha}_{x_0^+} D^{1+\beta}_{y_0^+} u\right)(x,y,z) + \left(I^{1+\alpha}_{x_0^+} D^{1+\gamma}_{z_0^+} u\right)(x,y,z) - \lambda \left(I^{1+\alpha}_{x_0^+} u\right)(x,y,z). \end{aligned}$$

Applying the integral operator $I^{1+\beta}_{y_0^+}$ to both sides of the previous expression and using Fubini's Theorem we get

$$\begin{aligned} 0 \quad = \quad & \left(I^{1+\beta}_{y_0^+} u\right)(x,y,z) - \frac{(x-x_0)^\alpha}{\Gamma(1+\alpha)} \left(I^{1+\beta}_{y_0^+} f_1\right)(y,z) - \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left(I^{1+\beta}_{y_0^+} f_0\right)(y,z) \\ & + \left(I^{1+\alpha}_{x_0^+} u\right)(x,y,z) - \frac{(y-y_0)^\beta}{\Gamma(1+\beta)} \left(I^{1+\alpha}_{x_0^+} D^{\beta}_{y_0^+} u\right)(x,y_0,z) - \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1-\beta}_{y_0^+} u\right)(x,y_0,z) \\ & + \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} D^{1+\gamma}_{z_0^+} u\right)(x,y,z) - \lambda \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} u\right)(x,y,z), \end{aligned} \tag{14}$$

where we denote the Cauchy's fractional integral conditions by

$$f_0(y,z) = \left(I^{1-\alpha}_{x_0^+} u\right)(x_0,y,z), \qquad\qquad f_1(y,z) = \left(D^{\alpha}_{x_0^+} u\right)(x_0,y,z). \tag{15}$$

Finally, we apply $I^{1+\gamma}_{z_0^+}$ to both sides of equation (14) and we use again Fubini's Theorem to get

$$\begin{aligned}
0 \;=\; & \left(I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} u\right)(x,y,z) - \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)} \left(I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} f_1\right)(y,z) - \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left(I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} f_0\right)(y,z) \\
& + \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} u\right)(x,y,z) - \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} h_1\right)(x,z) - \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} h_0\right)(x,z) \\
& + \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} u\right)(x,y,z) - \frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)} \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} D^{\gamma}_{z_0^+} u\right)(x,y,z_0) \\
& - \frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)} \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} I^{1-\gamma}_{z_0^+} u\right)(x,y,z_0) - \lambda \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} u\right)(x,y,z),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \left(I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} u\right)(x,y,z) + \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} u\right)(x,y,z) + \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} u\right)(x,y,z) - \lambda \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} u\right)(x,y,z) \\
& = \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} \left(I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} f_0\right)(y,z) + \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)} \left(I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} f_1\right)(y,z) \\
& \quad + \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} h_0\right)(x,z) + \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} h_1\right)(x,z) \\
& \quad + \frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)} \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} g_0\right)(x,y) + \frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)} \left(I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} g_1\right)(x,y), \qquad (16)
\end{aligned}$$

where we denote the Cauchy's fractional integral conditions by

$$h_0(x,z) = \left(I^{1-\beta}_{y_0^+} u\right)(x,y_0,z), \qquad h_1(x,z) = \left(D^{\beta}_{y_0^+} u\right)(x,y_0,z). \qquad (17)$$

$$g_0(x,y) = \left(I^{1-\gamma}_{z_0^+} u\right)(x,y,z_0), \qquad g_1(x,y) = \left(D^{\gamma}_{z_0^+} u\right)(x,y,z_0). \qquad (18)$$

We observe that the fractional integrals in (16) are Laplace-transformable functions. Therefore, we may apply the two-dimensional Laplace transform to y and z:

$$F(s_1,s_2) = \mathcal{L}\{f\}(s_1,s_2) = \int_{y_0}^{\infty} \int_{z_0}^{\infty} e^{-s_1 y - s_2 z}\, f(y,z)\, dz\, dy.$$

Taking into account its convolution and operational properties [6, 11] we obtain the following relations:

$$\begin{aligned}
& \mathcal{L}\left\{ I^{1+\beta}_{y_0^+} I^{1+\gamma}_{z_0^+} u \right\}(x,s_1,s_2) = s_1^{-1-\beta}\; s_2^{-1-\gamma}\; \mathcal{U}(x,s_1,s_2), \\
& \mathcal{L}\left\{ I^{1+\alpha}_{x_0^+} I^{1+\beta}_{y_0^+} u \right\}(x,s_1,s_2) = \frac{s_1^{-1-\beta}}{\Gamma(1+\alpha)} \int_{x_0}^{x} (x-t)^{\alpha}\, \mathcal{U}(x,s_1,s_2)\, dt, \\
& \mathcal{L}\left\{ \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} h_0\right) \right\}(x,s_1,s_2) = e^{-y_0 s_1}\; s_1^{-\beta}\; s_2^{-1-\gamma} \left(I^{1+\alpha}_{x_0^+} h_0\right)(x,y_0,s_2), \\
& \mathcal{L}\left\{ \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)} \left(I^{1+\alpha}_{x_0^+} I^{1+\gamma}_{z_0^+} h_1\right) \right\}(x,s_1,s_2) = e^{-y_0 s_1}\; s_1^{-1-\beta}\; s_2^{-1-\gamma} \left(I^{1+\alpha}_{x_0^+} h_1\right)(x,y_0,s_2).
\end{aligned}$$

Proceeding in a similar way we obtain the Laplace transform of the remaining terms of (16). Combining all the resulting terms and multiplying by $s_1^{1+\beta} s_2^{1+\gamma}$ we obtain the following second kind homogeneous integral equation of Volterra type:

$$\begin{aligned}
& \mathcal{U}(x,s_1,s_2) + \frac{(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda)}{\Gamma(1+\alpha)} \int_{x_0}^{x} (x-t)^{\alpha}\, \mathcal{U}(t,s_1,s_2)\, dt \\
& \quad = F(x,s_1,s_2) + e^{-y_0 s_1} \left(I^{1+\alpha}_{x_0^+}(s_1 h_0 + h_1)\right)(x,s_1,s_2) + e^{-z_0 s_2} \left(I^{1+\alpha}_{x_0^+}(s_2 g_0 + g_1)\right)(x,s_1,s_2),
\end{aligned} \qquad (19)$$

where

$$\begin{aligned}
F(x,s_1,s_2) &= \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\, F_0(s_1,s_2) + \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\, F_1(s_1,s_2), \\
F_i(s_1,s_2) &= \int_{y_0}^{\infty} \int_{z_0}^{\infty} f_i(y,z)\, e^{-s_1 y}\, e^{-s_2 z}\, dz\, dy, \qquad i = 0,1.
\end{aligned} \qquad (20)$$

Using (11) we have that the unique solution of (19) in the class of summable functions is

$$\begin{aligned}
&\mathcal{U}(x,s_1,s_2) \\
&= F(x,s_1,s_2) + e^{-y_0 s_1}\left(I^{1+\alpha}_{x_0^+}(s_1 h_0 + h_1)\right)(x,s_1,s_2) + e^{-z_0 s_2}\left(I^{1+\alpha}_{x_0^+}(s_2 g_0 + g_1)\right)(x,s_1,s_2) \\
&\quad - \left(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda\right)\int_{x_0}^{x}(x-t)^{\alpha}\ E_{1+\alpha,1+\alpha}\left(-\left(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda\right)\ (x-t)^{1+\alpha}\right) \\
&\quad \left(F(t,s_1,s_2) + e^{-y_0 s_1}\left(I^{1+\alpha}_{x_0^+}(s_1 h_0 + h_1)\right)(t,s_1,s_2) + e^{-z_0 s_2}\left(I^{1+\alpha}_{x_0^+}(s_2 g_0 + g_1)\right)(t,s_1,s_2)\right)\ dt,
\end{aligned} \tag{21}$$

which involves as the kernel the two-parameter Mittag-Leffler function (see (8)). Due to the convergence of the integrals and series that appear in (21), we can interchange them and rewrite (21) in the following way:

$$\begin{aligned}
\mathcal{U}(x,s_1,s_2) \ &= \ (x-x_0)^{\alpha-1}\ E_{1+\alpha,\alpha}\left(-\left(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda\right)\ (x-x_0)^{1+\alpha}\right)\ F_0(s_1,s_2) \\
&\quad + (x-x_0)^{\alpha}\ E_{1+\alpha,1+\alpha}\left(-\left(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda\right)\ (x-x_0)^{1+\alpha}\right)\ F_1(s_1,s_2) \\
&\quad + e^{-y_0 s_1}\sum_{n=0}^{\infty}(-1)^n\ \left(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda\right)^n\ \left(I^{(1+\alpha)(n+1)}_{x_0^+}(s_1 h_0 + h_1)\right)(x,s_1,s_2) \\
&\quad + e^{-z_0 s_2}\sum_{n=0}^{\infty}(-1)^n\ \left(s_1^{1+\beta} + s_2^{1+\gamma} - \lambda\right)^n\ \left(I^{(1+\alpha)(n+1)}_{x_0^+}(s_2 g_0 + g_1)\right)(x,s_1,s_2).
\end{aligned}$$

In order to cancel the Laplace transform we need to take into account its distributional form in Zemanian's space (for more details about generalized integral transforms see [20]) and the following relations:

$$\lim_{r_1,r_2\to\infty}\int_{\sigma_1-ir_1}^{\sigma_1+ir_1}\int_{\sigma_2-ir_2}^{\sigma_2+ir_2} s_1^{n(1+\beta)}\ s_2^{n(1+\gamma)}\ F_i(s_1,s_2)\ e^{s_1 y + s_2 z}\ ds_1\ ds_2 = \left(D^{n(1+\beta)}_{y_0^+} D^{n(1+\gamma)}_{z_0^+} f_i\right)(y,z),$$

$$\lim_{r_1\to\infty}\int_{\sigma_1-ir_1}^{\sigma_1+ir_1} s_1^{n(1+\beta)}\ e^{s_1(y-y_0)}\ ds_1 = \left(D^{n(1+\beta)}_{y_0^+}\delta\right)(y-y_0),$$

$$\lim_{r_2\to\infty}\int_{\sigma_2-ir_2}^{\sigma_2+ir_2} s_2^{n(1+\gamma)}\ e^{s_2(z-z_0)}\ ds_2 = \left(D^{n(1+\gamma)}_{z_0^+}\delta\right)(z-z_0),$$

where $i = 0, 1$, $n \in \mathbb{N}_0$, δ is Dirac's delta function, and the convergence is in $\mathcal{D}'$ Therefore, applying the multinomial theorem and after straightforward calculations we get the following family of eigenfunctions of (13)

$$\begin{aligned}
u_\lambda(x,y,z) &= \sum_{n=0}^{\infty}(-1)^n\ \frac{(x-x_0)^{n(1+\alpha)+\alpha-1}}{\Gamma((1+\alpha)n+\alpha)}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)^n\ f_0(y,z) \\
&\quad + \sum_{n=0}^{\infty}(-1)^n\ \frac{(x-x_0)^{n(1+\alpha)+\alpha}}{\Gamma((1+\alpha)n+1+\alpha)}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)^n\ f_1(y,z) \\
&\quad + \int_{x_0}^{x}\sum_{n=0}^{\infty}(-1)^n\ \frac{(x-t)^{n(1+\alpha)+\alpha}}{\Gamma((1+\alpha)n+1+\alpha)}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)^n \\
&\qquad (\ \delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y)\ )\ dt
\end{aligned} \tag{22}$$

where the convergence of the series is in $\mathcal{D}'$. From the previous calculations we obtain the following theorem, where we describe the eigenfunctions in an operational form using the Mittag-Leffler function (8).

Theorem 3.1 *The generalized eigenfunctions of the fractional Laplace operator $\Delta_+^{(\alpha,\beta,\gamma)}$ are given in the operational form using the Mittag-Leffler function by:*

$$\begin{aligned}
u_\lambda(x,y,z) &= (x-x_0)^{\alpha-1}\ E_{1+\alpha,\alpha}\left(-(x-x_0)^{1+\alpha}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)\right) f_0(y,z) \\
&\quad + \ (x-x_0)^{\alpha}\ E_{1+\alpha,1+\alpha}\left(-(x-x_0)^{1+\alpha}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)\right) f_1(y,z) \\
&\quad + \int_{x_0}^{x}\ (x-t)^{\alpha}\ E_{1+\alpha,1+\alpha}\left(-(x-t)^{1+\alpha}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)\right) \\
&\qquad (\delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y))\ dt,
\end{aligned} \tag{23}$$

where $\lambda \in \mathbb{C}$ and f_0, f_1, h_0, h_1, g_0, g_1 are Cauchy's fractional conditions given by (15), (17), and (18).

Proof: We give a direct proof of the theorem. It is based on the fact that $D^{1+\alpha}_{x_0^+}(x-x_0)^{\alpha-1} = 0$ and $D^{1+\alpha}_{x_0^+}(x-x_0)^{\alpha} = 0$. We use also the fractional analogous formula for differentiation of integrals depending on a parameter where the upper limit also depends on the same parameter (see [13, Section 2.7.4]). Applying the operator $\Delta_+^{(\alpha,\beta,\gamma)}$ to (22) we get

$$\begin{aligned}
\Delta_+^{(\alpha,\beta,\gamma)} u_\lambda(x,y,z) = & \sum_{n=1}^{\infty} (-1)^n \frac{(x-x_0)^{n(1+\alpha)-2}}{\Gamma((1+\alpha)n-1)} \left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)^n f_0(y,z) \\
& + \sum_{n=1}^{\infty} (-1)^n \frac{(x-x_0)^{n(1+\alpha)-1}}{\Gamma((1+\alpha)n)} \left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)^n f_1(y,z) \\
& + \int_{x_0}^{x} \sum_{n=1}^{\infty} (-1)^n \frac{(x-t)^{n(1+\alpha)-1}}{\Gamma((1+\alpha)n)} \left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right)^n \\
& \qquad (\, \delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y)\,)\, dt \\
& + \left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+}\right) u_\lambda(x,y,z).
\end{aligned}$$

Rearranging the terms of the series we obtain

$$\begin{aligned}
\Delta_+^{(\alpha,\beta,\gamma)} u_\lambda(x,y,z) &= -\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+} - \lambda\right) u_\lambda(x,y,z) + \left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+}\right) u_\lambda(x,y,z) \\
&= \lambda\, u_\lambda(x,y,z).
\end{aligned}$$

■

We would like to remark that the eigenfunctions constructed in [18] for the two-parameter Laplace operator have a different structure and do not satisfy the eigenfunction equation, except when $\lambda = 0$. A family of generalized eigenfunctions for the fractional Laplace operator can be obtained considering $\lambda = 0$ as the next theorem states.

Theorem 3.2 *The generalized fundamental solution of the fractional Laplace operator $\Delta_+^{(\alpha,\beta,\gamma)}$ is given by:*

$$\begin{aligned}
u_0(x,y,z) = & (x-x_0)^{\alpha-1}\, E_{1+\alpha,\alpha}\left(-(x-x_0)^{1+\alpha}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+}\right)\right) f_0(y,z) \\
& + (x-x_0)^{\alpha}\, E_{1+\alpha,1+\alpha}\left(-(x-x_0)^{1+\alpha}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+}\right)\right) f_1(y,z) \\
& + \int_{x_0}^{x} (x-t)^{\alpha}\, E_{1+\alpha,1+\alpha}\left(-(x-t)^{1+\alpha}\left(D^{1+\beta}_{y_0^+} + D^{1+\gamma}_{z_0^+}\right)\right) \\
& \qquad (\, \delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y)\,)\, dt. \qquad (24)
\end{aligned}$$

In a similar way, applying in (16) the two-dimensional Laplace transform with respect to x and y we obtain the following generalized fundamental solutions

$$\begin{aligned}
v_0(x,y,z) = & (y-y_0)^{\beta-1}\, E_{1+\beta,\beta}\left(-(y-y_0)^{1+\beta}\left(D^{1+\alpha}_{x_0^+} + D^{1+\gamma}_{z_0^+}\right)\right) h_0(x,z) \\
& + (y-y_0)^{\beta}\, E_{1+\beta,1+\beta}\left(-(y-y_0)^{1+\beta}\left(D^{1+\alpha}_{x_0^+} + D^{1+\gamma}_{z_0^+}\right)\right) h_1(x,z) \\
& + \int_{y_0}^{y} (y-t)^{\beta}\, E_{1+\beta,1+\beta}\left(-(y-t)^{1+\beta}\left(D^{1+\alpha}_{x_0^+} + D^{1+\gamma}_{z_0^+}\right)\right) \\
& \qquad (\delta'(x-x_0)f_0(t,z) + \delta(x-x_0)f_1(t,z) + \delta'(z-z_0)g_0(x,t) + \delta(z-z_0)g_1(x,t))\, dt. \qquad (25)
\end{aligned}$$

Furthermore, a similar result can be obtained when we apply the two-dimensional Laplace transform with respect to x and z.

Remark 3.3 *It is possible to obtain from (24) the fundamental solution for the Euclidean Laplace operator when $\alpha = \beta = \gamma = 1$. Since the fundamental solution of the Euclidean Laplace operator in $\mathbb{R}^3$ is given by $\left((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\right)^{-\frac{1}{2}}$, which corresponds to the following power series*

$$\sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} n-\frac{1}{2} \\ n \end{pmatrix} \frac{(x-x_0)^{2n}}{((y-y_0)^2 + (z-z_0)^2)^{n+\frac{1}{2}}} \qquad (26)$$

defined for $\left|\frac{(x-x_0)^2}{(y-y_0)^2+(z-z_0)^2}\right| < 1$, *we have to consider in (24)* $\alpha=\beta=\gamma=1$, f_1, h_1, g_1, h_0, *and* g_0 *the null functions, and* f_0 *such that*

$$\left(D_y^2+D_z^2\right)^n f_0(y,z) = \begin{pmatrix} n-\frac{1}{2} \\ n \end{pmatrix} \frac{(2n)!}{\left((y-y_0)^2+(z-z_0)^2\right)^{n+\frac{1}{2}}}. \tag{27}$$

The function $f_0(y,z) = \left((y-y_0)^2+(z-z_0)^2\right)^{-\frac{1}{2}}$ *satisfies (27). To see this we recall that the p-dimensional Euclidean Laplace satisfies (see [2, Ch.1])*

$$\Delta^n r^k = \frac{2^{2n}\Gamma\left(\frac{k}{2}+1\right)\Gamma\left(\frac{k+p}{2}\right)}{\Gamma\left(\frac{k}{2}-n+1\right)\Gamma\left(\frac{k+p}{2}-n\right)} r^{k-2n}$$

with $r=\|x\|, x\in\mathbb{R}^p$, $n\in\mathbb{N}$. *Therefore, for* $p=2$ *and* $k=-1$ *we obtain*

$$\Delta^n r^{-1} = \frac{2^{2n}\pi}{\left(\Gamma\left(\frac{1}{2}-n\right)\right)^2} r^{-1-2n} = \frac{\Gamma\left(n+\frac{1}{2}\right)(2n)!}{n!\sqrt{\pi}} r^{-1-2n} = \begin{pmatrix} n-\frac{1}{2} \\ n \end{pmatrix} (2n)!\; r^{-1-2n} \tag{28}$$

which leads immediately to (27) for $f_0(y,z) = \left((y-y_0)^2+(z-z_0)^2\right)^{-\frac{1}{2}}$.

3.2 Fundamental solution of the fractional Dirac operator

In this section we compute the fundamental solution for the three dimensional fractional Dirac operator defined by

$$D_+^{(\alpha,\beta,\gamma)} := e_1\, D_{x_0^+}^{\frac{1+\alpha}{2}} + e_2\, D_{y_0^+}^{\frac{1+\beta}{2}} + e_3\, D_{z_0^+}^{\frac{1+\gamma}{2}}, \qquad (\alpha,\beta,\gamma)\in]0,1]^3. \tag{29}$$

This operator factorizes the fractional Laplace operator $\Delta_+^{(\alpha,\beta,\gamma)}$ for Clifford valued functions f given by $f(x,y,z)=\sum_A e_A f_A(x,y,z)$, where $e_A\in\{1,e_1,e_2,e_3,e_1e_2,e_1e_3,e_2e_3,e_1e_2e_3\}$, $f_A\in AC^2(\Omega)$ and $f_A(x_0,y,z)=f_A(x,y_0,z)=f_A(x,y,z_0)=0$. In fact, for such functions we can apply the semigroup property (6) to obtain

$$D_{x_0^+}^{\frac{1+\alpha}{2}}\left(D_{x_0^+}^{\frac{1+\alpha}{2}} f_A\right) = D_{x_0^+}^{1+\alpha} f_A, \qquad D_{y_0^+}^{\frac{1+\beta}{2}}\left(D_{y_0^+}^{\frac{1+\beta}{2}} f_A\right) = D_{y_0^+}^{1+\beta} f_A, \qquad D_{z_0^+}^{\frac{1+\gamma}{2}}\left(D_{z_0^+}^{\frac{1+\gamma}{2}} f_A\right) = D_{z_0^+}^{1+\gamma} f_A. \tag{30}$$

We remark that in the case of our fractional Dirac operator, condition (7) reduces to $f_A(x_0,y,z)=f_A(x,y_0,z)=f_A(x,y,z_0)=0$ for every component f_A. If $f\in AC^2(\Omega)$ doesn't satisfy this last property it is always possible to define $g(x,y,z)=\sum_A e_A\,(f_A(x,y,z)-f_A(x_0,y,z)-f_A(x,y_0,z)-f_A(x,y,z_0))$ such that $g_A(x_0,y,z)=g_A(x,y_0,z)=g_A(x,y,z_0)=0$ for every component g_A, and therefore, (30) still holds. Moreover, for the mixed fractional derivatives $D_{x_0^+}^{\frac{1+\alpha}{2}}\left(D_{y_0^+}^{\frac{1+\beta}{2}} f_A\right)$, due to the Leibniz's rule for the differentiation under integral sign, Fubini's Theorem and Schwarz's Theorem, we have

$$\begin{aligned}
D_{x_0^+}^{\frac{1+\alpha}{2}}\left(D_{y_0^+}^{\frac{1+\beta}{2}} f_A\right) &= \frac{\partial}{\partial x}\left\{\frac{1}{\Gamma\left(\frac{1-\alpha}{2}\right)}\int_{x_0}^x \frac{1}{(x-w)^{\frac{1+\alpha}{2}}}\left(\frac{\partial}{\partial y}\left\{\frac{1}{\Gamma\left(\frac{1-\beta}{2}\right)}\int_{y_0}^y \frac{f_A(w,t,z)}{(y-t)^{\frac{1+\beta}{2}}}\,dt\right\}\right)dw\right\} \\
&= \frac{\partial}{\partial x}\left\{\frac{\partial}{\partial y}\left\{\frac{1}{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1-\beta}{2}\right)}\int_{x_0}^x \frac{1}{(x-w)^{\frac{1+\alpha}{2}}}\int_{y_0}^y \frac{f_A(w,t,z)}{(y-t)^{\frac{1+\beta}{2}}}\,dt\,dw\right\}\right\} \\
&= \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}\left\{\frac{1}{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{1-\beta}{2}\right)}\int_{y_0}^y \frac{1}{(y-t)^{\frac{1+\beta}{2}}}\int_{x_0}^x \frac{f_A(w,t,z)}{(x-w)^{\frac{1+\alpha}{2}}}\,dw\,dt\right\}\right\} \\
&= \frac{\partial}{\partial y}\left\{\frac{1}{\Gamma\left(\frac{1-\beta}{2}\right)}\int_{y_0}^y \frac{1}{(y-t)^{\frac{1+\beta}{2}}}\left(\frac{\partial}{\partial x}\left\{\frac{1}{\Gamma\left(\frac{1-\alpha}{2}\right)}\int_{x_0}^x \frac{f_A(w,t,z)}{(x-w)^{\frac{1+\alpha}{2}}}\,dw\right\}\right)dt\right\} \\
&= D_{y_0^+}^{\frac{1+\beta}{2}}\left(D_{x_0^+}^{\frac{1+\alpha}{2}} f_A\right).
\end{aligned} \tag{31}$$

In a similar way we conclude that

$$D_{x_0^+}^{\frac{1+\alpha}{2}}\left(D_{z_0^+}^{\frac{1+\gamma}{2}} f_A\right) = D_{z_0^+}^{\frac{1+\gamma}{2}}\left(D_{x_0^+}^{\frac{1+\alpha}{2}} f_A\right) \quad \text{and} \quad D_{y_0^+}^{\frac{1+\beta}{2}}\left(D_{z_0^+}^{\frac{1+\gamma}{2}} f_A\right) = D_{z_0^+}^{\frac{1+\gamma}{2}}\left(D_{y_0^+}^{\frac{1+\beta}{2}} f_A\right). \tag{32}$$

From (30), (31), (32), and the multiplication rules (12) of the Clifford algebra, we finally get

$$D_+^{(\alpha,\beta,\gamma)}\left(D_+^{(\alpha,\beta,\gamma)} f\right) = -\Delta_+^{(\alpha,\beta,\gamma)} f, \tag{33}$$

i.e., the fractional Dirac operator factorizes the fractional Laplace operator. We remark that the factorization property (33) also holds for functions which are continuous on $(x_0, X_0]$ and integrable on any subinterval of $[x_0, X_0]$ since the semigroup property (6) remains valid in this case (see [12]).

In order to get the fundamental solution of $D_+^{(\alpha,\beta,\gamma)}$ we apply this operator to the fundamental solution (24). In the following theorem, to compute $D_{x_0^+}^{\frac{1+\alpha}{2}} u$ we make use of the fractional analogous formula for differentiation of integrals depending on a parameter where the upper limit also depends on the same parameter (see [13, Section 2.7.4]).

Theorem 3.4 *A family of fundamental solutions for the fractional Dirac operator* $D_+^{(\alpha,\beta,\gamma)}$ *is given by*

$$U_0(x,y,z) = e_1\; U_1(x,y,z) + e_2\; U_2(x,y,z) + e_3\; U_3(x,y,z), \tag{34}$$

where

$$\begin{aligned} U_1(x,y,z) &= \left(D_{x_0^+}^{\frac{1+\alpha}{2}} u_0\right)(x,y,z) \\ &= (x-x_0)^{\frac{\alpha-3}{2}} E_{1+\alpha,\frac{\alpha-1}{2}}\left(-(x-x_0)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) f_0(y,z) \\ &\quad + (x-x_0)^{\frac{\alpha-1}{2}} E_{1+\alpha,\frac{1+\alpha}{2}}\left(-(x-x_0)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) f_1(y,z) \\ &\quad + \int_{x_0}^{x} (x-t)^{\frac{\alpha-1}{2}}\; E_{1+\alpha,\frac{1+\alpha}{2}}\left(-(x-t)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) \\ &\qquad (\delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y))\; dt \end{aligned} \tag{35}$$

$$\begin{aligned} U_2(x,y,z) &= \left(D_{y_0^+}^{\frac{1+\beta}{2}} u_0\right)(x,y,z) \\ &= (x-x_0)^{\alpha-1}\left(E_{1+\alpha,\alpha}\left(-(x-x_0)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) D_{y_0^+}^{\frac{1+\beta}{2}}\right) f_0(y,z) \\ &\quad + (x-x_0)^{\alpha}\left(E_{1+\alpha,1+\alpha}\left(-(x-x_0)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) D_{y_0^+}^{\frac{1+\beta}{2}}\right) f_1(y,z) \\ &\quad + \int_{x_0}^{x} (x-t)^{\alpha}\;\left(E_{1+\alpha,1+\alpha}\left(-(x-t)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) D_{y_0^+}^{\frac{1+\beta}{2}}\right) \\ &\qquad (\delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y))\; dt \end{aligned} \tag{36}$$

$$\begin{aligned} U_3(x,y,z) &= \left(D_{y_0^+}^{\frac{1+\beta}{2}} u_0\right)(x,y,z) \\ &= (x-x_0)^{\alpha-1}\left(E_{1+\alpha,\alpha}\left(-(x-x_0)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) D_{z_0^+}^{\frac{1+\gamma}{2}}\right) f_0(y,z) \\ &\quad + (x-x_0)^{\alpha}\left(E_{1+\alpha,1+\alpha}\left(-(x-x_0)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) D_{z_0^+}^{\frac{1+\gamma}{2}}\right) f_1(y,z) \\ &\quad + \int_{x_0}^{x} (x-t)^{\alpha}\;\left(E_{1+\alpha,1+\alpha}\left(-(x-t)^{1+\alpha}\left(D_{y_0^+}^{1+\beta} + D_{z_0^+}^{1+\gamma}\right)\right) D_{z_0^+}^{\frac{1+\gamma}{2}}\right) \\ &\qquad (\delta'(y-y_0)h_0(t,z) + \delta(y-y_0)h_1(t,z) + \delta'(z-z_0)g_0(t,y) + \delta(z-z_0)g_1(t,y))\; dt. \end{aligned} \tag{37}$$

The functions $f_0, f_1, g_0, g_1, h_0, h_1$ *are Cauchy's fractional conditions given by (15), (17), and (18).*

Remark 3.5 *It is possible to obtain from (34) the fundamental solution for the Euclidean Dirac operator when* $\alpha = \beta = \gamma = 1$. *Indeed, since the fundamental solution of the Euclidean Dirac operator in* $\mathbb{R}^3$ *is given up to a constant by*

$$-\frac{(x-x_0)\; e_1 + (y-y_0)\; e_2 + (z-z_0)\; e_3}{((x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2)^{\frac{3}{2}}} \tag{38}$$

which corresponds to the following vector power series

$$
\begin{aligned}
&-\left(\sum_{n=0}^{\infty}(-1)^n \begin{pmatrix} n+\frac{1}{2} \\ n \end{pmatrix} \frac{(x-x_0)^{2n+1}}{((y-y_0)^2+(z-z_0)^2)^{n+\frac{3}{2}}}\right) e_1 \\
&-\left(\sum_{n=0}^{\infty}(-1)^n \begin{pmatrix} n+\frac{1}{2} \\ n \end{pmatrix} \frac{(x-x_0)^{2n}(y-y_0)}{((y-y_0)^2+(z-z_0)^2)^{n+\frac{3}{2}}}\right) e_2 \\
&-\left(\sum_{n=0}^{\infty}(-1)^n \begin{pmatrix} n+\frac{1}{2} \\ n \end{pmatrix} \frac{(x-x_0)^{2n}(z-z_0)}{((y-y_0)^2+(z-z_0)^2)^{n+\frac{3}{2}}}\right) e_3
\end{aligned}
$$

we have to consider in (34) f_1, h_1, g_1, h_0, and g_0 the null functions, and f_0 such that

$$
\begin{aligned}
\left(D_y^2+D_z^2\right)^{n+1} f_0(y,z) &= \frac{(2n+1)! \begin{pmatrix} n+\frac{1}{2} \\ n \end{pmatrix}}{((y-y_0)^2+(z-z_0)^2)^{n+\frac{3}{2}}} \\
D_y\left(D_y^2+D_z^2\right)^{n} f_0(y,z) &= -\frac{(2n)! \begin{pmatrix} n+\frac{1}{2} \\ n \end{pmatrix}(y-y_0)}{((y-y_0)^2+(z-z_0)^2)^{n+\frac{3}{2}}} \\
D_z\left(D_y^2+D_z^2\right)^{n} f_0(y,z) &= -\frac{(2n)! \begin{pmatrix} n+\frac{1}{2} \\ n \end{pmatrix}(z-z_0)}{((y-y_0)^2+(z-z_0)^2)^{n+\frac{3}{2}}}.
\end{aligned}
\tag{39}
$$

The function f_0 that satisfies the three conditions in (39) is the same function as in case of the Laplace. In fact, from (28) it is not difficult to see that the function $f_0(y,z) = \left((y-y_0)^2+(z-z_0)^2\right)^{-\frac{1}{2}}$ satisfies conditions (39).

Remark 3.6 *As a final remark we would like to point out that if in the beginning of this section we consider $u(x,y,z)$ admitting a summable fractional derivative $(D^{1+\alpha}_{x_0^+}u)(x,y,z)$ by x and belonging to $I^{1+\beta}_{y_0^+}(L_1)$ and $I^{1+\gamma}_{z_0^+}(L_1)$ by y and z, then in expressions (22), (24), (35), (36), and (37), the last term will not appear due to Theorem 2.1. In this case the expressions obtained can be considered classical eigenfunctions and classical fundamental solutions.*

4 Method of separation of variables

4.1 Eigenfunctions and fundamental solution of the fractional Laplace operator

Let us consider again equation (13)

$$
\left(D^{1+\alpha}_{x_0^+}u\right)(x,y,z)+\left(D^{1+\beta}_{y_0^+}u\right)(x,y,z)+\left(D^{1+\gamma}_{z_0^+}u\right)(x,y,z)=\lambda\, u(x,y,z),
$$

and assume that $u(x,y,z)=u_1(x)\,u_2(y)\,u_3(z)$. Substituting in (16) and taking into account the initial conditions (15), (17), and (18) we obtain

$$
\begin{aligned}
u_1(x)\ &\left(I^{1+\beta}_{y_0^+}u_2(y)\ I^{1+\gamma}_{z_0^+}u_3(z)\right)+u_2(y)\ \left(I^{1+\alpha}_{x_0^+}u_1(x)\ I^{1+\gamma}_{z_0^+}u_3(z)\right) \\
&+u_3(z)\ \left(I^{1+\alpha}_{x_0^+}u_1(x)\ I^{1+\beta}_{y_0^+}u_2(y)\right)(x,y,z)-\lambda\left(I^{1+\alpha}_{x_0^+}u_1\right)(x)\ \left(I^{1+\beta}_{y_0^+}u_2\right)(y)\ \left(I^{1+\gamma}_{z_0^+}u_3\right)(z) \\
= a_1&\frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)}\left(I^{1+\beta}_{y_0^+}u_2(y)\ I^{1+\gamma}_{z_0^+}u_3(z)\right)\ +\ a_2\frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}\left(I^{1+\beta}_{y_0^+}u_2(y)\ I^{1+\gamma}_{z_0^+}u_3(z)\right) \\
&+\ b_1\frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)}\left(I^{1+\alpha}_{x_0^+}u_1(x)\ I^{1+\gamma}_{z_0^+}u_3(z)\right)\ +\ b_2\frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}\left(I^{1+\alpha}_{x_0^+}u_1(x)\ I^{1+\gamma}_{z_0^+}u_3(z)\right) \\
&+\ c_1\frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)}\left(I^{1+\alpha}_{x_0^+}u_1(x)\ I^{1+\beta}_{y_0^+}u_2(y)\right)\ +\ c_2\frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)}\left(I^{1+\alpha}_{x_0^+}u_1(x)\ I^{1+\beta}_{y_0^+}u_2(y)\right),
\end{aligned}
\tag{40}
$$

where $a_i = f_i, b_i = h_i, c_i = g_i \in \mathbb{C}$, $i = 1, 2$, are constants defined by the initial conditions (15), (17), and (18). Supposing that $\left(I^{1+\alpha}_{x_0^+} u_1\right)(x) \left(I^{1+\beta}_{y_0^+} u_2\right)(y) \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) \neq 0$, for $(x, y, z) \in \Omega$, we can divide (40) by this factor. Separating the variables we get the following three Abel's type second kind integral equations:

$$u_1(x) - \mu \left(I^{1+\alpha}_{x_0^+} u_1\right)(x) = a_1 \frac{(x-x_0)^{\alpha-1}}{\Gamma(\alpha)} + a_2 \frac{(x-x_0)^{\alpha}}{\Gamma(1+\alpha)}, \tag{41}$$

$$u_2(y) + \nu \left(I^{1+\beta}_{y_0^+} u_2\right)(y) = b_1 \frac{(y-y_0)^{\beta-1}}{\Gamma(\beta)} + b_2 \frac{(y-y_0)^{\beta}}{\Gamma(1+\beta)}, \tag{42}$$

$$u_3(z) + (\mu - \lambda - \nu) \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) = c_1 \frac{(z-z_0)^{\gamma-1}}{\Gamma(\gamma)} + c_2 \frac{(z-z_0)^{\gamma}}{\Gamma(1+\gamma)}, \tag{43}$$

where $\lambda,\ \mu,\ \nu \in \mathbb{C}$ are constants. We observe that the equality

$$\left(I^{1+\alpha}_{x_0^+} u_1\right)(x) \left(I^{1+\beta}_{y_0^+} u_2\right)(y) \left(I^{1+\gamma}_{z_0^+} u_3\right)(z) = 0,$$

for at least one point (ξ, η, θ) agrees with (40), (41), (42), and (43). Solving the latter equations using (11) in Theorem 2.3 and after straightforward computations we obtain the following family of eigenfunctions $u_{\lambda,\mu,\nu}(x, y, z) = u_1(x)\ u_2(y)\ u_3(z)$,

$$u_1(x) = a_1 (x-x_0)^{\alpha-1}\ E_{1+\alpha,\alpha}\left(\mu(x-x_0)^{1+\alpha}\right) + a_2 (x-x_0)^{\alpha}\ E_{1+\alpha,1+\alpha}\left(\mu(x-x_0)^{1+\alpha}\right), \tag{44}$$

$$u_2(y) = b_1 (y-y_0)^{\beta-1}\ E_{1+\beta,\beta}\left(-\nu(y-y_0)^{1+\beta}\right) + b_2 (y-y_0)^{\beta}\ E_{1+\beta,1+\beta}\left(-\nu(y-y_0)^{1+\beta}\right), \tag{45}$$

$$\begin{aligned} u_3(z) &= c_1 (z-z_0)^{\gamma-1}\ E_{1+\gamma,\gamma}\left((\mu-\lambda-\nu)(z-z_0)^{1+\gamma}\right) \\ &\quad + c_2 (z-z_0)^{\gamma}\ E_{1+\gamma,1+\gamma}\left((\mu-\lambda-\nu)(z-z_0)^{1+\gamma}\right). \end{aligned} \tag{46}$$

Remark 4.1 *In the special case of $\alpha = \beta = \gamma = 1$ the functions u_1, u_2 and u_3 take the form*

$$\begin{aligned} u_1(x) &= a_1\ \cosh\left(\sqrt{\mu}\ (x-x_0)\right) + \frac{a_2}{\sqrt{\mu}}\ \sinh\left(\sqrt{\mu}\ (x-x_0)\right), \\ u_2(y) &= b_1\ \cos\left(\sqrt{\nu}\ (y-y_0)\right) + \frac{b_2}{\sqrt{\nu}}\ \sin\left(\sqrt{\nu}\ (y-y_0)\right), \\ u_3(z) &= c_1\ \cosh\left(\sqrt{\mu-\lambda-\nu}\ (z-z_0)\right) + \frac{c_2}{\sqrt{\mu-\lambda-\nu}}\ \sinh\left(\sqrt{\mu-\lambda-\nu}\ (z-z_0)\right) \end{aligned}$$

which are the components of the fundamental solution of the Laplace operator in $\mathbb{R}^3$ obtained by the method of separation of variables.

4.2 Fundamental solution of the fractional Dirac operator

Following the procedure presented in Section 3.2, the fundamental solution for the fractional Dirac operator $D_+^{(\alpha,\beta,\gamma)}$ via separation of variables is given by

$$\begin{aligned} U(x,y,z) &= e_1\ u_2(y)\ u_3(z) \left(D^{\frac{1+\alpha}{2}}_{x_0^+} u_1\right)(x) + e_2\ u_1(x)\ u_3(z) \left(D^{\frac{1+\beta}{2}}_{y_0^+} u_2\right)(y) \\ &\quad + e_3\ u_1(x)\ u_2(y) \left(D^{\frac{1+\gamma}{2}}_{z_0^+} u_3\right)(z), \end{aligned} \tag{47}$$

where u_1, u_2, u_3 are given respectively by (44), (45), (46) and

$$\left(D^{\frac{1+\alpha}{2}}_{x_0^+} u_1\right)(x) = a_1\ (x-x_0)^{\frac{\alpha-3}{2}}\ E_{1+\alpha,\frac{\alpha-1}{2}}\left(\mu(x-x_0)^{1+\alpha}\right) + a_2\ (x-x_0)^{\frac{\alpha-1}{2}}\ E_{1+\alpha,\frac{1+\alpha}{2}}\left(\mu(x-x_0)^{1+\alpha}\right) \tag{48}$$

$$\left(D^{\frac{1+\beta}{2}}_{y_0^+} u_2\right)(y) = b_1\ (y-y_0)^{\frac{\beta-3}{2}}\ E_{1+\beta,\frac{\beta-1}{2}}\left(-\nu(y-y_0)^{1+\beta}\right) + b_2\ (y-y_0)^{\frac{\beta-1}{2}}\ E_{1+\beta,\frac{1+\beta}{2}}\left(-\nu(y-y_0)^{1+\beta}\right) \tag{49}$$

$$\begin{aligned} \left(D^{\frac{1+\gamma}{2}}_{z_0^+} u_3\right)(z) &= c_1\ (z-z_0)^{\frac{\gamma-3}{2}}\ E_{1+\gamma,\frac{\gamma-1}{2}}\left((-\mu+\lambda+\nu)(z-z_0)^{1+\gamma}\right) \\ &\quad + c_2\ (z-z_0)^{\frac{\gamma-1}{2}}\ E_{1+\gamma,\frac{1+\gamma}{2}}\left((-\mu+\lambda+\nu)(z-z_0)^{1+\gamma}\right) \end{aligned} \tag{50}$$

Remark 4.2 *In the special case of $\alpha = \beta = \gamma = 1$, expressions (48), (49), and (50) take the form*

$$\begin{aligned} &(D_x u_1)(x) = a_1 \sqrt{\mu}\ \sinh\left(\sqrt{\mu}\ (x-x_0)\right) + a_2 \sinh\left(\sqrt{\mu}\ (x-x_0)\right), \\ &(D_y u_2)(y) = b_1 \sqrt{\nu}\ \sin\left(\sqrt{\nu}\ (y-y_0)\right) + b_2 \cos\left(\sqrt{\nu}\ (y-y_0)\right), \\ &(D_z u_3)(z) = c_1 \sqrt{-\mu+\lambda+\nu}\ \sinh\left(\sqrt{\mu-\lambda-\nu}\ (z-z_0)\right) + c_2 \sinh\left(\sqrt{\mu-\lambda-\nu}\ (z-z_0)\right), \end{aligned}$$

which are the components of the fundamental solution of the Dirac operator in $\mathbb{R}^3$ obtained by the method of separation of variables.

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