THE HERGLOTZ VARIATIONAL PROBLEM ON SPHERES AND ITS OPTIMAL CONTROL APPROACH

LÍGIA ABRUNHEIRO, LUÍS MACHADO, NATÁLIA MARTINS

Abstract. The main goal of this paper is to extend the generalized variational problem of Herglotz type to the more general context of the Euclidean sphere S^n. Motivated by classical results on Euclidean spaces, we derive the generalized Euler–Lagrange equation for the corresponding variational problem defined on the Riemannian manifold S^n. Moreover, the problem is formulated from an optimal control point of view and it is proved that the Euler–Lagrange equation can be obtained from the Hamiltonian equations. It is also highlighted the geodesic problem on spheres as a particular case of the generalized Herglotz problem.

1. Introduction

One of the main interests in studying the generalized variational problem of Herglotz type is that, unlike the classical variational approach, it provides a variational description of nonconservative processes even when the Lagrangian is autonomous [5]. Due to its importance on Euclidean spaces and motivated by the fact that Riemannian manifolds are becoming more popular in modern applications, in this paper we extend the generalized variational problem of Herglotz to the more general context of the Euclidean unit $n-$sphere S^n. Typically, S^n is seen as a Riemannian manifold equipped with the metric induced by the embedding space $\mathbb{R}^{n+1}$.

The generalized variational problem of Herglotz, proposed by Gustav Herglotz in 1930 (see [9]), can be formulated as follows.

Problem ($\mathcal{P}$): Determine the trajectories $x \in C^2([0,T],\mathbb{R}^n)$ and $z \in C([0,T],\mathbb{R})$ that minimize the value of the functional $z(T)$:

$$\min_{(x,z)} z(T),$$

where the pair (x,z) satisfies the differential equation

$$\dot{z}(t) = L\left(t, x(t), \dot{x}(t), z(t)\right), \quad t \in [0,T], \tag{1.1}$$

subject to the initial condition

$$z(0) = z_0$$

and the boundary conditions

$$x(0) = x_0 \text{ and } x(T) = x_T,$$

for some $x_0, x_T \in \mathbb{R}^n$ and $z_0, T \in \mathbb{R}$.

The Lagrangian L is assumed to satisfy the following assumptions:

(1) $L \in C^1([0, T] \times \mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$;

(2) The functions $t \mapsto \dfrac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t))$, $t \mapsto \dfrac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t))$ and $t \mapsto \dfrac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t))$ are differentiable for any admissible trajectory (x, z).

Notice that (1.1) represents a family of differential equations, since for each x, a new differential equation is obtained. This dependence can be made explicit by writing $z(t, x(t), \dot{x}(t))$, but, for simplicity of notation, the explicit dependence on x is suppressed and we write $z(t)$ only.

Herglotz proved that a necessary condition for a trajectory (x, z) to be a solution of problem $(\mathcal{P})$ is that it satisfies

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} = 0. \tag{1.2}$$

Equation (1.2) is known as the generalized Euler–Lagrange equation for the variational problem $(\mathcal{P})$ (see [8]).

It is clear that Herglotz's variational problem reduces to the classical fundamental problem of the calculus of variations if the Lagrangian L does not depend on the variable z. In this case,

$$z(T) = \int_0^T \left(L(t, x(t), \dot{x}(t)) + \frac{z_0}{T} \right) dt$$

and the differential equation (1.2) reduces to the classical Euler–Lagrange equation:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0.$$

This kind of problem was poorly understood until 1996. The situation changed with the publication of the book [8] and the PhD thesis of Bogdana Georgieva [3], and nowadays is a subject of current research (see [4–7, 12–16]).

The structure of the paper is as follows. In Section 2 we present the straightforward formulation of the generalized Herglotz problem for an arbitrary Riemannian manifold M. Since the main goal of the paper is to study the Herglotz problem for the particular case when M is the Euclidean unit $n-$sphere S^n, we recall, in Section 3, some geometric properties of S^n that are required to derive our main results stated in Sections 4 and 5. In Section 4, the generalized variational problem of Herglotz on S^n is formulated as a constrained variational problem in the embedding space $\mathbb{R}^{n+1}$ and the corresponding generalized Euler–Lagrange equation is derived. The geodesic problem on S^n arises naturally as a particular case of the Herglotz problem. In Section 5, the optimal control theory is put to use and an alternative approach for defining the generalized Herglotz problem on S^n is presented. It is also proved how the Euler–Lagrange equation can be obtained from the Hamiltonian equations. Finally, some concluding remarks and ideas for future work are carried out in Section 6.

2. The Herglotz variational principle on a Riemannian manifold

In what follows, M denotes a locally complete connected Riemannian manifold equipped with the metric $\langle \cdot, \cdot \rangle$. Given a point $p \in M$, denote, as usual, the tangent space of M at p by $T_p M$. The disjoint union of the tangent spaces of M, called the tangent bundle, will be denoted by TM. So, TM can be thought as the set

$$TM = \big\{ (p, v) \ : \ p \in M \ \wedge \ v \in T_p M \big\}.$$

If M is $n-$dimensional, then the tangent bundle TM is a differentiable manifold of dimension $2n$.

Let ∇ be the unique connection on M that is compatible with the metric. Therefore, if $Y : M \to TM$ is a smooth vector field on M along a curve $x : I \subset \mathbb{R} \to M$, the covariant derivative of Y with respect to t is simply given by $\dfrac{DY}{dt} = \nabla_{\dot{x}} Y$, where $\dot{x}$ stands for the velocity vector field $\frac{dx}{dt}$.

A vector field Y along x is said to be parallel if and only if $\frac{DY}{dt} = 0$. By definition, a geodesic on M is a smooth curve x in M such that its velocity vector field is parallel, that is, if $\frac{D\dot{x}}{dt} = 0$.

For more details about geometric concepts we refer to classical books of differential geometry [2, 11].

Following the approach given in [8] for Euclidean spaces, we define the Herglotz problem on M by the following:

> **Problem ($\mathcal{P}_V$):** Determine the trajectories $x \in C^2([0, T], M)$ and $z \in C^1([0, T], \mathbb{R})$ that minimize the value of the functional $z(T)$:
>
> $$\min_{(x,z)} z(T),$$
>
> where the pair (x, z) satisfies the differential equation
>
> $$\dot{z}(t) = L\big(t, x(t), \dot{x}(t), z(t)\big), \ \ t \in [0, T],$$
>
> subject to the initial condition
>
> $$z(0) = z_0$$
>
> and the boundary conditions
>
> $$x(0) = x_0 \quad \text{and} \quad x(T) = x_T, \tag{2.1}$$
>
> for some $x_0, x_T \in M$ and $z_0, T \in \mathbb{R}$.

The Lagrangian L is assumed to satisfy the following assumptions:

(1) $L \in C^1([0, T] \times TM \times \mathbb{R}, \mathbb{R})$;

(2) The functions $t \mapsto \dfrac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t))$, $t \mapsto \dfrac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t))$ and $t \mapsto \dfrac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t))$ are differentiable for any admissible trajectory (x, z).

Solving problem ($\mathcal{P}_V$) in the general setting of the Riemannian manifold M represents, from the perspective of the authors, a huge challenging. The main purpose of this paper is to study problem ($\mathcal{P}_V$) for the particular case when M is the Euclidean $n-$sphere S^n. Before doing that, we recall in the next section the most important facts about the geometry of this particular Riemannian manifold.

3. The geometry of the Euclidean sphere

The unit $n-$sphere
$$S^n := \left\{ p \in \mathbb{R}^{n+1} \; : \; \langle p, p \rangle = 1 \right\}$$
can be naturally seen as an $n-$dimensional Riemannian submanifold of the Euclidean space $\mathbb{R}^{n+1}$, with Riemannian metric induced by the usual inner product in $\mathbb{R}^{n+1}$, denoted by $\langle \cdot, \cdot \rangle$.

The tangent space of S^n at a point p belonging to S^n is simply defined as
$$T_p S^n := \left\{ v \in \mathbb{R}^{n+1} \; : \; \langle v, p \rangle = 0 \right\}$$
and its orthogonal complement is therefore given by
$$T_p^\perp S^n = \left\{ \alpha \, p \in \mathbb{R}^{n+1} \; : \; \alpha \in \mathbb{R} \right\}.$$
Any vector $u \in \mathbb{R}^{n+1}$ can be uniquely written as
$$u = u - \langle u, p \rangle p + \langle u, p \rangle p,$$
where $u - \langle u, p \rangle \, p \in T_p S^n$ and $\langle u, p \rangle \, p \in T_p^\perp S^n$.

Since S^n is embedded in an Euclidean space, the covariant derivative of a smooth vector field Y along a curve x in S^n is simply obtained by projecting the usual derivative of Y, $\dot{Y}$, orthogonally onto $T_{x(t)} S^n$. Hence,
$$\frac{DY}{dt}(t) = \dot{Y}(t) - \left\langle \dot{Y}(t), x(t) \right\rangle x(t). \tag{3.1}$$

For the particular case where Y is the velocity vector field $\dot{x}$, its covariant derivative, $\frac{D\dot{x}}{dt}$, called covariant acceleration and denoted by $\frac{D^2 x}{dt^2}$, is simply given by
$$\frac{D^2 x}{dt^2} = \ddot{x} - \langle \ddot{x}, x \rangle x.$$
According to the above characterization of the covariant acceleration and to the definition of geodesics given in Section 2, one concludes that a smooth curve x is a geodesic on S^n if and only if $\ddot{x} - \langle \ddot{x}, x \rangle x = 0$.

4. The Herglotz variational problem on S^n

The generalized Herglotz variational problem on S^n can be easily adapted from problem $(\mathcal{P}_V)$ if one replaces the Riemannian manifold M by the Euclidean sphere S^n.

Since S^n is naturally embedded in the Euclidean space $\mathbb{R}^{n+1}$, the proposed problem can be seen as a constrained optimization problem on the Euclidean space $\mathbb{R}^{n+1}$, if one looks to the trajectory x as a curve in $\mathbb{R}^{n+1}$ satisfying the constraint
$$\langle x(t), x(t) \rangle = 1, \; \forall t \in [0, T]. \tag{4.1}$$

In what follows, when there is no possibility of ambiguity, we sometimes suppress arguments of the functions.

Theorem 4.1. *If (x, z) is a solution of problem $(\mathcal{P}_V)$ with $M = S^n$, then it satisfies the generalized Euler-Lagrange equation*
$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} - \left\langle \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}}, x \right\rangle x = 0. \tag{4.2}$$

Proof. Suppose that (x, z) is a solution of problem $(\mathcal{P}_V)$ with $M = S^n$ and let $h \in C^2([0, T], \mathbb{R}^{n+1})$ be such that $h(0) = h(T) = 0$ and $\dot{h}(0) = 0$. In order to find first-order necessary conditions for the optimization problem, let us consider the functional defined by

$$J(t, x(t), \dot{x}(t), z(t), \lambda(t), \mu(t))$$
$$= z(T) + \lambda(t)\big(\dot{z}(t) - L\big(t, x(t), \dot{x}(t), z(t)\big)\big) + \mu(t)\big(\langle x(t), x(t)\rangle - 1\big),$$

where the scalar functions λ and μ are Lagrange multipliers.

Let us introduce the following notation for the first variation of z:

$$\xi(t) = \frac{d}{d\epsilon}z\big(t, x(t) + \epsilon h(t), \dot{x}(t) + \epsilon \dot{h}(t)\big)\bigg|_{\epsilon=0}.$$

Note that since $h(0) = \dot{h}(0) = 0$, then $\xi(0) = 0$. Moreover, since (x, z) is a solution of problem $(\mathcal{P}_V)$, we have $z\big(T, x(T) + \epsilon h(T), \dot{x}(T) + \epsilon \dot{h}(T)\big) \geq z(T, x(T), \dot{x}(T))$ and then $\xi(T) = 0$. Furthermore, we have

$$\dot{\xi}(t) = \frac{d}{d\epsilon}L\big(t, x(t) + \epsilon h(t), \dot{x}(t) + \epsilon \dot{h}(t), z(t)\big)\bigg|_{\epsilon=0}.$$

Note that

$$\frac{d}{d\epsilon}J(t, x + \epsilon h, \dot{x} + \epsilon \dot{h}, z, \lambda, \mu)\bigg|_{\epsilon=0}$$

$$= \xi(T) + \lambda\Big(\frac{d\xi}{dt} - \Big\langle\frac{\partial L}{\partial x}, h\Big\rangle - \Big\langle\frac{\partial L}{\partial \dot{x}}, \dot{h}\Big\rangle - \frac{\partial L}{\partial z}\xi\Big) + 2\mu\langle x, h\rangle.$$

Since $\lambda \neq 0$[1] and using the fact that $\xi(T) = 0$, the first–order necessary optimization condition

$$\frac{d}{d\epsilon}J\big(t, x + \epsilon h, \dot{x} + \epsilon \dot{h}, z, \lambda, \mu\big)\bigg|_{\epsilon=0} = 0$$

is equivalent to

$$\frac{d\xi}{dt} - \frac{\partial L}{\partial z}\xi = \Big\langle\frac{\partial L}{\partial x}, h\Big\rangle + \Big\langle\frac{\partial L}{\partial \dot{x}}, \dot{h}\Big\rangle - 2\frac{\mu}{\lambda}\langle x, h\rangle,$$

which is a first-order linear differential equation. Multiplying both members of the above equation by

$$I(t) = e^{-\int_0^t \frac{\partial L}{\partial z} d\tau},$$

we get

$$\frac{d}{dt}\big(I(t)\xi(t)\big) = I(t)\Big(\Big\langle\frac{\partial L}{\partial x}, h\Big\rangle + \Big\langle\frac{\partial L}{\partial \dot{x}}, \dot{h}\Big\rangle - 2\frac{\mu}{\lambda}\langle x, h\rangle\Big).$$

Integrating both sides of the equation from 0 to t, one gets

$$I(t)\xi(t) - \xi(0) = \int_0^t I(\tau)\Big\langle\frac{\partial L}{\partial x} - \frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{x}}\Big) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} - 2\frac{\mu}{\lambda}x, h\Big\rangle d\tau + \Big\langle I(\tau)\frac{\partial L}{\partial \dot{x}}, h\Big\rangle\Big|_0^t.$$

Now, evaluating the above equation for $t = T$ and taking into account that $\xi(0) = \xi(T) = h(0) = h(T) = 0$, it follows that

$$\int_0^T I(\tau)\Big\langle\frac{\partial L}{\partial x} - \frac{d}{d\tau}\Big(\frac{\partial L}{\partial \dot{x}}\Big) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} - 2\frac{\mu}{\lambda}x, h\Big\rangle d\tau = 0. \tag{4.3}$$

---

[1]If $\lambda(t) = 0$ for some $t \in [0, T]$, then one must have $\mu(t) = 0$ or $x(t) = 0$, which none of them makes sense.

Since the above condition has to be fulfilled for all curves h and since I is a positive real function, condition (4.3) is equivalent to

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} - 2\frac{\mu}{\lambda}x = 0. \qquad (4.4)$$

In order to find the scalar function $\frac{\mu}{\lambda}$, take the inner product of the above with x and use the fact that $\langle x, x\rangle = 1$ to conclude that

$$\frac{\mu}{\lambda} = \frac{1}{2}\left\langle \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}}, x\right\rangle. \qquad (4.5)$$

The result now follows inserting (4.5) into (4.4). $\qquad\square$

Remark. *Note that the generalized Euler-Lagrange equation for the Herglotz problem* $(\mathcal{P}_V)$ *defined on the Euclidean sphere S^n means that the projection of the vector field on the left hand side of (1.2) on T_xS^n must vanish.*

Observe that, by (3.1), the generalized Euler-Lagrange (4.2) can be written in the following form:

$$\frac{D}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} - \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} + \left\langle \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{x}} + \frac{\partial L}{\partial x}, x\right\rangle x = 0. \qquad (4.6)$$

Corollary 4.2. *The Euler-Lagrange equation for the classical variational problem*

$$\min_{x \in C^2([0,T],S^n)} \int_0^T L(t, x(t), \dot{x}(t))\, dt$$

is

$$\frac{D}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \left\langle \frac{\partial L}{\partial x}, x\right\rangle x = 0.$$

Proof. The Euler-Lagrange equation for the classical variational problem is simply obtained by considering $\dfrac{\partial L}{\partial z} = 0$ in the generalized Euler-Lagrange equation (4.6). $\qquad\square$

Example 1. The particular case of geodesics on $\mathbf{S^n}$.
It is well known that geodesics on a Riemannian manifold are locally the shortest paths between points. Therefore, their expressions can be found by minimizing the arc length of a curve using the calculus of variations. For the case of the Euclidean sphere S^n endowed with the metric induced by the usual inner product in $\mathbb{R}^{n+1}$, geodesics are obtained by minimizing the functional

$$E(x) = \frac{1}{2}\int_0^T \langle \dot{x}, \dot{x}\rangle\, dt,$$

subject to boundary conditions $x(0) = x_0$ and $x(T) = x_T$, where $\cos^{-1}\langle x_0, x_T\rangle \in \,]0, \pi[$. This problem is just a particular case of the Herglotz problem considered above, with Lagrangian

$$L(t, x, \dot{x}, z) = \frac{1}{2}\langle \dot{x}, \dot{x}\rangle.$$

Notice that in this case, $\dfrac{\partial L}{\partial x} = \dfrac{\partial L}{\partial z} = 0$ and $\dfrac{\partial L}{\partial \dot{x}} = \dot{x}$. Therefore, the Euler-Lagrange equation (4.2) for this particular case is simply given by

$$\ddot{x} - \langle \ddot{x}, x\rangle x = 0.$$

The solution of the problem is indeed a geodesic on S^n that can be written explicitly as

$$x(t) = x_0 \frac{\sin\big((T-t)\alpha\big)}{\sin(T\alpha)} + x_T \frac{\sin(t\alpha)}{\sin(T\alpha)},$$

where $\alpha = \cos^{-1}\langle x_0, x_T \rangle$ (see [11]).

5. OPTIMAL CONTROL VIEWPOINT

The generalized Herglotz problem on the unit $n-$sphere defined in Section 4 can be formulated in a perspective of optimal control by the following:

> **Problem ($\mathcal{P}_C$):** Determine the control bundle $u \in C^1([0,T], TS^n)$ that minimizes the value of the functional $z(T)$:
>
> $$\min_u z(T),$$
>
> where the trajectory $x \in C^2([0,T], S^n)$ associated to u and the trajectory $z \in C^1([0,T], \mathbb{R})$ satisfy the control system
>
> $$\dot{x}(t) = u(t)$$
> $$\dot{z}(t) = L(t, x(t), u(t), z(t)),$$
>
> subject to the initial condition
>
> $$z(0) = z_0$$
>
> and boundary conditions
>
> $$x(0) = x_0 \quad \text{and} \quad x(T) = x_T,$$
>
> for some $x_0, x_T \in S^n$, $z_0, T \in \mathbb{R}$.

From a geometric point of view, the state space of the control problem is $S^n \times \mathbb{R}$ and the control bundle is the tangent bundle TS^n.

Analogously to what has been done in the variational approach, one will look at the above optimal control problem as a pure constrained optimal control problem in the Euclidean space $\mathbb{R}^{n+1}$ subject to (4.1):

> **Problem ($\bar{\mathcal{P}}_C$):** Determine the control $u \in C^1([0,T], \mathbb{R}^{n+1})$ that minimizes the value of the functional $z(T)$:
>
> $$\min_u z(T),$$
>
> where the trajectory $x \in C^2([0,T], \mathbb{R}^{n+1})$ associated to u and the trajectory $z \in C^1([0,T], \mathbb{R})$ satisfy the control system
>
> $$\dot{x}(t) = u(t)$$
> $$\dot{z}(t) = L(t, x(t), u(t), z(t)),$$ $\qquad (5.1)$
>
> subject to the initial condition
>
> $$z(0) = z_0,$$
>
> the boundary conditions
>
> $$x(0) = x_0 \quad \text{and} \quad x(T) = x_T$$
>
> and the constraint
>
> $$\langle x(t), x(t) \rangle = 1, \ \forall t \in [0,T],$$

for some $x_0, x_T \in S^n$, $z_0, T \in \mathbb{R}$.

Note that condition $\langle u(t), x(t) \rangle = 0$, $\forall t \in [0, T]$, is a control constraint that comes naturally from the pure state constraint and the control system considered.

In this case, the total Hamiltonian is defined by

$$H(t, x, u, z, p_x, p_z, \lambda) = \langle p_x, u \rangle + p_z L(t, x, u, z) + \lambda(\langle x, x \rangle - 1), \qquad (5.2)$$

where x, u, $p_x \in \mathbb{R}^{n+1}$ and z, p_z, $\lambda \in \mathbb{R}$.

Theorem 5.1. *If u is a solution of problem $(\bar{\mathcal{P}}_C)$ and (x, z) is the associated optimal state trajectory, then there exist a trajectory $\lambda \in C^1([0, T], \mathbb{R})$ and a costate trajectory $(p_x, p_z) \in C^1([0, T], \mathbb{R}^{n+1} \times \mathbb{R})$ such that*

$$\dot{x} = u$$
$$\dot{z} = L(t, x, u, z) \qquad (5.3)$$

and

$$\dot{p}_x = -p_z \frac{\partial L}{\partial x} - 2\lambda x$$
$$\dot{p}_z = -p_z \frac{\partial L}{\partial z}. \qquad (5.4)$$

Moreover,

$$\langle x, x \rangle - 1 = 0,$$

$$p_x + p_z \frac{\partial L}{\partial u} = 0 \qquad (5.5)$$

and it is also satisfied the transversality condition

$$p_z(T) = -1. \qquad (5.6)$$

Proof. Equations (5.3) are the control system (5.1) of the problem which are obviously satisfied by the optimal trajectory of the problem.

Let us augment the functional $z(T)$ by the dynamical constraints, using the costate trajectories p_x, p_z and λ, in the following way:

$$J(u) = z(T) + \int_0^T \left[\langle p_x, \dot{x} - u \rangle + p_z \left[\dot{z} - L(t, x, u, z) \right] - \lambda[\langle x, x \rangle - 1] \right] dt$$

$$= z(T) + \int_0^T \left[-\langle p_x, u \rangle - p_z L(t, x, u, z) - \lambda(\langle x, x \rangle - 1) \right] dt$$

$$+ \int_0^T \left(\langle p_x, \dot{x} \rangle + p_z \dot{z} \right) dt$$

$$= z(T) - \int_0^T H(t, x, u, z, p_x, p_z, \lambda) \, dt + \int_0^T \left(\langle p_x, \dot{x} \rangle + p_z \dot{z} \right) dt.$$

Let us consider admissible variations of x, z, u and λ, that is

$$x + \epsilon \delta x, \quad z + \epsilon \delta z, \quad u + \epsilon \delta u \quad \text{and} \quad \lambda + \epsilon \delta \lambda,$$

where ϵ is a real parameter and $\delta x \in C^2([0, T], \mathbb{R}^{n+1})$, $\delta u \in C^1([0, T], \mathbb{R}^{n+1})$ and δz, $\delta \lambda \in C^1([0, T], \mathbb{R})$ are such that $\delta x(0) = \delta x(T) = \delta z(0) = 0$. The first variation

of J is given by

$$\delta J = \delta z(T) - \int_0^T \left[\left\langle \frac{\partial H}{\partial x}, \delta x \right\rangle + \left\langle \frac{\partial H}{\partial u}, \delta u \right\rangle + \frac{\partial H}{\partial z}\delta z + \frac{\partial H}{\partial \lambda}\delta \lambda \right] dt$$
$$+ \int_0^T \left(\left\langle p_x, \frac{d(\delta x)}{dt} \right\rangle + p_z \frac{d(\delta z)}{dt} \right) dt.$$

Integrating by parts,

$$\delta J = - \int_0^T \left[\left\langle \frac{\partial H}{\partial x}, \delta x \right\rangle + \left\langle \frac{\partial H}{\partial u}, \delta u \right\rangle + \frac{\partial H}{\partial z}\delta z + \frac{\partial H}{\partial \lambda}\delta \lambda \right] dt$$
$$- \int_0^T \left(\langle \dot{p}_x, \delta x \rangle + \dot{p}_z \delta z \right) dt + p_z(T)\delta z(T) + \delta z(T)$$
$$= - \int_0^T \left[\left\langle \frac{\partial H}{\partial x} + \dot{p}_x, \delta x \right\rangle + \left(\frac{\partial H}{\partial z} + \dot{p}_z \right)\delta z + \frac{\partial H}{\partial \lambda}\delta \lambda \right] dt$$
$$- \int_0^T \left\langle \frac{\partial H}{\partial u}, \delta u \right\rangle dt + \left[p_z(T) + 1 \right]\delta z(T).$$

The first-order necessary conditions for optimality are obtained from $\delta J = 0$, for all variations of x, z, u and λ. Choosing adequate variations, one gets

$$\dot{p}_x = -\frac{\partial H}{\partial x}$$
$$\dot{p}_z = -\frac{\partial H}{\partial z}$$
$$0 = \frac{\partial H}{\partial \lambda},$$

the transversality condition $p_z(T) + 1 = 0$ and the optimality condition $\dfrac{\partial H}{\partial u} = 0$. Now, from the expression of the Hamiltonian (5.2), the result follows. $\square$

The equations (5.3)-(5.4) are called Hamiltonian equations. Let us now deduce the Euler-Lagrange equation derived in Theorem 4.1 from the necessary conditions of optimality described in Theorem 5.1.

From differentiating (5.5) with respect to t, one has

$$\dot{p}_x + \dot{p}_z \frac{\partial L}{\partial u} + p_z \frac{d}{dt}\frac{\partial L}{\partial u} = 0. \tag{5.7}$$

Now plugging (5.4) into (5.7), one gets

$$-p_z \frac{\partial L}{\partial x} - 2\lambda x - p_z \frac{\partial L}{\partial z}\frac{\partial L}{\partial u} + p_z \frac{d}{dt}\frac{\partial L}{\partial u} = 0. \tag{5.8}$$

Take now the inner product of the above equality with x to obtain

$$-p_z \langle \frac{\partial L}{\partial x}, x \rangle - p_z \langle \frac{\partial L}{\partial z}\frac{\partial L}{\partial u}, x \rangle + p_z \langle \frac{d}{dt}\frac{\partial L}{\partial u}, x \rangle = 2\lambda. \tag{5.9}$$

Finally, inserting (5.9) into (5.8), we end up with

$$-p_z \frac{\partial L}{\partial x} + p_z \langle \frac{\partial L}{\partial x}, x \rangle x - p_z \frac{\partial L}{\partial z}\frac{\partial L}{\partial u} + p_z \langle \frac{\partial L}{\partial z}\frac{\partial L}{\partial u}, x \rangle x + p_z \frac{d}{dt}\frac{\partial L}{\partial u} - p_z \langle \frac{d}{dt}\frac{\partial L}{\partial u}, x \rangle x = 0,$$

which, using (5.6), implies that

$$\frac{d}{dt}\frac{\partial L}{\partial u} - \frac{\partial L}{\partial x} - \frac{\partial L}{\partial z}\frac{\partial L}{\partial u} - \Big\langle \frac{d}{dt}\frac{\partial L}{\partial u} - \frac{\partial L}{\partial x} - \frac{\partial L}{\partial z}\frac{\partial L}{\partial u}, x \Big\rangle x = 0.$$

This equation is equivalent to the Euler-Lagrange equation (4.2) since $u = \dot{x}$.

Without loss of generality, the regularity conditions considered here can be relaxed by adapting some conclusions to that situation.

There are many different ways of addressing an optimal control problem with pure state constraints. We recommend the reader to go through, for example, references [1, 10], for more details about the subject.

6. Concluding Remarks

We formulated the generalized variational problem of Herglotz on the Euclidean unit $n-$sphere S^n as a constrained variational problem in the embedding space $\mathbb{R}^{n+1}$. The correspondent Euler–Lagrange equation has been derived in Theorem 4.1. Moreover, an alternative approach to formulate the variational problem of Herglotz on S^n in terms of optimal control has been provided in Section 5. We have proven that the Euler–Lagrange equation can be easily obtained from the Hamiltonian equations stated in Theorem 5.1. We also noticed that the referred Euler–Lagrange equation can be interpreted in terms of a projection on the tangent space of S^n at a certain point of a vector field emanating from the Euler–Lagrange equation of the classical variational problem of Herglotz in $\mathbb{R}^{n+1}$. This is certainly not the case when one considers the generalized variational problem of Herglotz involving higher order covariant derivatives on S^n. This topic is currently under investigation by the authors.

References

1. B. Bonnard, L. Faubourg, G. Launay and E. Trélat, *Optimal control with state constraints and the space shuttle re-entry problem*, J. Dynam. Control Syst. **9** (2003), no. 2, 155–199.
2. M. P. do Carmo, *Riemannian Geometry*, Mathematics: Theory and Applications, Birkäuser, Boston, MA, 1992.
3. B. Georgieva, *Noether-type theorems for the generalized variational principle of Herglotz*, ProQuest LLC, Ann Arbor, MI, 2001.
4. B. Georgieva, *Symmetries of the Herglotz variational principle in the case of one independent variable*, Ann. Sofia Univ., Fac. Math and Inf. **100** (2010) 113–122.
5. B. Georgieva and R. Guenther, *First Noether-type theorem for the generalized variational principle of Herglotz*, Topol. Methods Nonlinear Anal. **20** (2002), no. 2, 261–273.
6. B. Georgieva, R. Guenther and T. Bodurov, *Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem*, J. Math. Phys. **44** (2003), no. 9, 3911–3927.
7. B. Georgieva and R. Guenther, *Second Noether-type theorem for the generalized variational principle of Herglotz*, Topol. Methods Nonlinear Anal. **26** (2005), no. 2, 307–314.
8. R. B. Guenther, C. M. Guenther and J. A. Gottsch, *The Herglotz Lectures on Contact Transformations and Hamiltonian Systems*, Lecture Notes in Nonlinear Analysis, Vol. 1, Juliusz Schauder Center for Nonlinear Studies, Nicholas Copernicus University, Torún, 1996.

9. G. Herglotz, *Berührungstransformationen*, Lectures at the University of Göttingen, Göttingen, 1930.

10. D. H. Jacobson and M. M. Lee, *New Necessary Conditions of Optimality for Control Problems with State-Variable Inequality Constraints*, J. Math. Anal. Appl. **35** (1971) 255–284.

11. J. Jost, Riemannian Geometry and Geometric Analysis, Universitext, Springer, 6th ed., 2011.

12. J. C. Orum, R. T. Hudspeth, W. Black and R. B. Guenther, *Extension of the Herglotz Algorithm to Nonautonomous Canonical Transformations*, SIAM Rev. **42** (2000), no. 1, 83–90.

13. S. P. S. Santos, N. Martins and D. F. M. Torres, *Higher-Order Variational Problems of Herglotz Type*, Vietnam J. Math. **42** (2014), no. 4, 409–419.

14. S. P. S. Santos, N. Martins and D. F. M. Torres, *An Optimal Control Approach to Herglotz Variational Problems*, Optimization in the Natural Sciences, Communications in Computer and Information Science (CCIS) **499** (2015) 107–117.

15. S. P. S. Santos, N. Martins and D. F. M. Torres, *Variational problems of Herglotz type with time delay: Dubois-Reymond condition and Noether's first theorem*, Discrete Contin. Dyn. Syst. **35** (2015), no. 9, 4593–4610.

16. S. P. S. Santos, N. Martins and D. F. M. Torres, *Noether's theorem for higher-order variational problems of Herglotz type*, 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications, Vol. 2015, AIMS Proceedings (2015), 990–999.

Lígia Abrunheiro
Center for Research and Development in Mathematics and Applications (CIDMA) and Higher Institute of Accounting and Administration, University of Aveiro Portugal
E-mail address: `abrunheiroligia@ua.pt`

Luís Machado
Department of Mathematics, University of Trás-os-Montes e Alto Douro (UTAD)
Vila Real, Portugal
and Institute of Systems and Robotics (ISR), University of Coimbra, Portugal
E-mail address: `lmiguel@utad.pt`

Natália Martins
Center for Research and Development in Mathematics and Applications (CIDMA)
Department of Mathematics, University of Aveiro, Portugal
E-mail address: `natalia@ua.pt`