An Alternative Starting Point for Fraction Instruction

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We analyze the results of a study conducted for the purpose of assessing the viability of an alternative starting point for teaching fractions. The alternative is based on Freudenthal's insights about fraction as comparison. It involves portraying the entities that unit fractions quantify as always being apart from the reference unit, instead of as parts of an equally divided whole. The study consisted of interviewing 16 third-grade students on a series of fraction tasks that embody the proposed alternative starting point. The analysis supports regarding the proposed starting point as viable.

Introduction

Fractions are one of the most cognitively challenging, difficult to teach, and mathematically complex topics in the elementary school curriculum (Lamon, 2007). In this paper, we analyze the results of a study conducted for the purpose of assessing the viability of an alternative starting point for teaching this concept. One of the most important aspects of the proposed alternative approach is that the *initial* understandings it aims to support entail conceiving unit fractions as entities that are separate from the reference unit.

The study was part of the planning phase of a classroom design experiment (Cobb, 2003; Gravemeijer & Cobb, 2006). In this phase, a *hypothetical learning trajectory* (HLT) on fractions was designed and conceptually justified, prior to being tested and refined in classrooms. Empirically, the study is centered in interviews with 16 third-grade students (ages 8 and 9).

We start the paper by describing the theoretical perspective we assumed in conducting the study. It is consistent with the design experiment methodology as developed by Cobb, Gravemeijer and colleagues (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer & Cobb, 2006; Stephan, Bowers, & Cobb, 2003). We clarify how, from the perspective we assumed, the learning paths that pupils follow are regarded as being socially and culturally situated, through and through. Consequently, classroom instruction is not seen simply as a means by which to influence the pace and breadth of pupils' cognitive development. Instead, instruction, as a socially and culturally situated practice, is considered to strongly influence the very nature of the mathematical understandings that students develop.

We then describe the HLT that we formulated, justifying the initial design by drawing on relevant research literature. We start with the overarching learning goals, which center on Thompson and Saldanha's (2003) ideas about *reciprocal relationship of relative size*. Next, drawing on the key tenets of Realistic Mathematics Education (RME) theory (Cobb, Zhao, & Visnovska, 2008; Gravemeijer, 1994), we discuss our design considerations regarding the instructional starting point. We first analyze the viability of the problem situations most commonly used in early fraction instruction; namely, those that aim at engaging students in reasoning about the equal partitioning and sharing of a divisible object. Based on a review of the research literature and, in particular, on Freudenthal's

(1983) phenomenological analysis of fractions, we argue that these problem situations provide a rather precarious instructional starting point. We then propose an alternative, which involves approaching the entities that unit fractions quantify as being apart from the reference unit. To complete the HLT description, we list the main shifts in students' reasoning that we *anticipate* would take place as a result of the proposed instructional design.

In the second part of the paper we address the empirical questions that our instructional alternative raised with respect to the viability of the conjectured starting point. We describe and analyze individual student interviews conducted to assess the problem situations that aim at engaging students in reasoning about the amount of an attribute that is separate from the reference unit, and is defined in terms of fulfilling an iterative condition relative to the reference unit. Our analysis shows that these problem situations meet the criteria for a viable starting point[1], indicating that it is worth advancing the research agenda and testing the whole HLT in a classroom design experiment.

Theoretical and Methodological Perspective: Design Experiments

Design experiments, as developed by Cobb, Gravemijer and colleagues (Cobb et al., 1997; Gravemeijer & Cobb, 2006; Stephan et al., 2003), are a research methodology of mathematics education that involves developing instructional designs to support particular forms of learning. These forms of learning are systematically studied within the context defined by the means of supporting them (Cobb, 2003, p. 1). Individual thought processes, and social and cultural processes are considered to be reflexively related, with neither attributed absolute priority over the other (Cobb et al., 1997). Hence, individual cognition is regarded as being thoroughly social, and social processes as being thoroughly cognitive.

Researchers using this version of the design experiment methodology typically formulate their instructional designs following the RME theory (Gravemeijer, 1994), as it provides them with design heuristics that are consistent with the way in which they construe mathematical learning. They thus design for supporting the progressive reorganization of students' activity where, "with the teacher's guidance, students' *models of* their informal mathematical activity can evolve through their use into *models for* more general mathematical reasoning" (Cobb et al., 2008, p. 109; emphasis in the original). Instructional designs in which such progression breaks down, and where some models need to be abandoned when they can no longer soundly support more general mathematical reasoning, would not be seen as adequate.

Hypothetical Learning Trajectory

A design experiment starts with the formulation of an HLT that consists of an envisioned learning process together with conjectures about possible means of supporting it. The means of support are regarded as aspects of the mathematical practices in which teacher and students participate in a classroom. They include instructional activities, tools for representing mathematical ideas (inscriptions and manipulatives), the nature of classroom discourse, and the organization of classroom activities (Cobb, 2003; Cobb et al., 2008).

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[1] An ultimate test for a viable starting point would be a complete HLT accompanied by evidence of student learning. The analysis of such HLT is underway but is beyond the scope of this paper. Here, we explore viability of starting points using analytical tools that were developed for this purpose.

It is worth clarifying that, in the design experiment methodology, learning trajectories are not regarded as accounts of rather autonomous cognitive processes, by which students come to develop particular mathematical understandings. Instead, they are considered to be tightly related to the social and cultural situations in which they emerge. Instructional means of support, like those previously mentioned, are considered to have a strong influence both on the mathematical understandings that students develop, and on the processes by which they develop them (Cobb, 2000).

In the formulation of an HLT, the mathematical learning goals are first clarified. Next, a viable starting point for instruction is chosen. The final step in formulating an HLT involves developing conjectures about the main shifts in students' learning, together with the means that will support those shifts. Such shifts need not resemble those that previous research have documented, if the means used to support the learning are substantially different from the means that shaped the previously documented forms of learning. From the theoretical perspective assumed in the design experiment methodology, the emergence of uncommon developmental paths is to be expected when the means that support learning are atypical.

HLT: Learning Goals

Clarifying the mathematical learning goals of an HLT usually involves drawing on and synthesizing the prior research literature to identify central organizing ideas for the domain (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). In the case of fractions, we found Thompson and Saldanha's (2003) discussion of *what it means to understand fractions well* to be particularly comprehensive and insightful. For these authors, sophisticated fraction reasoning involves "conceiving two quantities as being in a reciprocal relationship of relative size" (p. 107). We created Figure 1 to better explain our interpretation of Thompson and Saldanha's (2003) assertion.

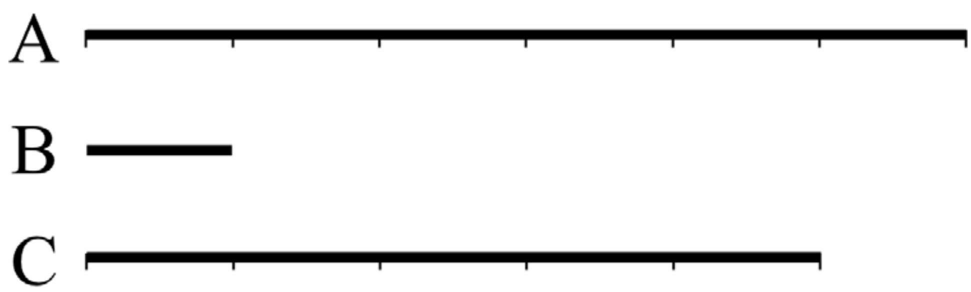

Figure 1. Lengths in a reciprocal relation of relative size.

The kind of understanding that Thompson and Saldanha (2003) describe would involve conceiving *segment B* as being 1/6 as long as *segment A*, and 1/5 as long as *segment C*, given that *A* is 6 times as long as *B*, and *C* is 5 times as long as *B*. In addition, it would involve conceiving *segment A* as being 6/5 as long as *segment C* (i.e., 6 times as long as 1/5 of *C*), and *segment C* as being 5/6 as long as *segment A* (i.e., 5 times as long as 1/6 of *C*).

For Thompson and Saldanha (2003) this kind of understanding would draw heavily on multiplication, division, measurement, and ratio. Among other things, it would involve conceiving the amount of some attribute of an object as being segmented (e.g., the *length* of *segment A*; Figure 1), and realizing that the segmentation is in comparison with the amount of that attribute that serves as a reference (e.g., the *length* of *segment B*). In addition, it would involve imagining the amount that serves as a reference apart from what is measured.

We regard Thompson and Saldanha's (2003) account of *reciprocal relations of relative size* as useful for organizing fraction instruction. It frames fractions as a single, although multi-faceted, mathematical notion around which to orient instructional efforts (Cobb, 1999). In addition, it approaches fractions squarely within the context of quantitative reasoning. As such, this account allows for establishing connections between fractions and the ways in which rational numbers are used in multiple disciplines for measuring and comparing phenomena.

In Thompson and Saldanha's (2003) account of reciprocal relationships of relative size, we recognize four features of unit fractions that students would have to come to make sense of; namely: (a) that they are numbers that account *not* for an object, but for an amount of an attribute present in the object, (b) that the amount they account for is in a direct iterative relation with the amount embodied by a reference unit (B is $1/n$ as big as A when A is n times as big as B), (c) that they can be imagined as being apart from the amount with which they are in an iterative relation, and (d) that the amount they account for is susceptible of being iterated unrestrictedly (m/n where m can be bigger and even much bigger than n). In formulating our HLT, we chose as the overarching learning goal understanding unit fractions in a way that included these four features.

HLT: Starting Point

Within the tenets of RME (Cobb et al., 2008; Gravemeijer, 1994), a viable starting point for instruction consists of problem situations and tools for representing mathematical ideas that need to have the potential of fulfilling three conditions: (a) becoming experientially real to students, (b) triggering informal ways of reasoning that can be a basis for developing increasingly sophisticated mathematical ways of knowing in a particular domain, and (c) serving as paradigmatic cases in which to "anchor students' increasingly abstract mathematical activity" (Cobb et al., 1997, p. 159).

Problem situations and tools that are *experientially real* are those with which students can immediately engage in personally meaningful mathematical activity (Cobb et al., 1997). In psychological terms, these would be problems and representations with which students can readily become imagistically involved (Thompson, 1996). Consequently, for a problem situation or a tool to be regarded as experientially real, it does not necessarily need to come from or be relevant to all students' everyday experiences. However, it has to be possible for all students to construe the problem and representations as personally meaningful and mathematically engaging, with teacher guidance and support.[2]

For a starting point to be viable, it is not enough to use problem situations and tools with which students can engage in personally meaningful mathematical activity. The activities also need to trigger *informal ways of reasoning*, on which students can be

[2] Notice that fictional scenarios can be regarded as being experientially real, if established in a classroom in ways that are meaningful for students, and lead them to engage in mathematical activity.

supported to build and develop more sophisticated understandings about specific mathematical ideas. In other words, the problems and inscriptions need to be a means of achieving learning goals in the short run.

In addition, the problem situations and tools need to *serve as anchors* for students' reasoning, as learning evolves. For this to occur, the representations emerging in students' minds (i.e., metonymies, prototypes, and metaphors; Presmeg, 1992), as they engage in problem solving at the beginning of the learning trajectory, need to be reconcilable with the sequence of mathematical understandings that students are expected to develop throughout the learning trajectory, including those that are the end goals of instruction. Consequently, students' initial activity has to have the potential of serving as a referent as pupils make sense, in the long run, of increasingly sophisticated mathematical ideas.

Equal Partition

The prevalent image of a unit fraction that instructional designers aim to foster in the initial phases of instruction is that of a single piece of an equally divided whole. Most commonly, *the whole* is portrayed as a divisible object—such as a pizza, a pie, a tortilla, a candy bar, or a French loaf—and the unit fraction as the piece that a person would get when fairly sharing the object among a certain number of people. The literature review we discuss below led us to question the viability of instructional activities aimed at fostering this image of a unit fraction as a starting point for instruction. Given the broad acceptance that this way of introducing fractions has had among instructional designers, we amply discuss the considerations that led us to its rejection as a starting point for our HLT.

With respect to viability as a starting point, there is much evidence that students can readily and meaningfully engage with partitioning situations, even from an early age (Pitkethly & Hunting, 1996; Steffe & Olive, 2010). In addition, these problem situations have proven useful in triggering informal ways of reasoning, on which students can be supported to make sense of some important fraction notions (Behr, Wachsmuth, Post, & Lesh, 1984; Empson, Junk, Dominguez, & Turner, 2006; Lamon, 2007; Steffe & Olive, 2010). For instance, these situations can be used to help pupils to overcome the whole number dominance rationale for gauging fraction quantities (Behr et al., 1984), and instead reason in ways consistent with the *inverse order relation among unit fractions* (Tzur, 2007). Students can be encouraged to think about the size of the pieces that would result from, for instance, fairly sharing a cake with more rather than fewer recipients (Sophian, Garyantes, & Chang, 1997).

It is thus clear that situations that aim to foster the image of a unit fraction as a single piece of an equally divided whole can fulfill the first two criteria of a viable starting point for instruction. Whether these situations would also meet the third criterion, and serve as an anchor for the development of increasingly sophisticated understandings of fractions, is more difficult to determine.

Several authors have identified, in theoretical terms, significant limitations of the equal partitioning model of fraction (e.g., Freudenthal, 1983; Kieren, 1976; Thompson & Saldanha, 2003), and some have expressed reservations about its contribution to children's fraction knowledge (e.g., Davis, 2003). Empirically speaking, research studies have shown that many students face great challenges[3] in making sense of fractions as numbers that

[3] Elsewhere (Cortina, Višňovská, & Zúñiga, 2015) we have discussed how meeting these challenges need not be regarded as an unavoidable step in making sense of the fraction concept. Instead, building on

account for the amount of a given attribute (e.g., a length, an area, a mass). Specifically, it has been documented that pupils commonly draw on their understanding of natural numbers when interpreting fractions, which leads them to erroneously judge fraction inscriptions composed of big numbers as necessarily accounting for large amounts (e.g., Behr et al., 1984; Kieren, 1993; Post, Carmer, Behr, Lesh, & Harel, 1993; Streefland, 1991). Among the students who overcome this way of interpreting fractions, many tend to construe them as numbers that account solely for the numerosity of discrete entities ("so many out of so many"; Thompson & Saldanha, 2003), and not for the amount of a continuous attribute (Saxe, Taylor, McIntosh, & Gearhart, 2005). Finally, few pupils come to conceive fractions as numbers that can soundly account for the amount of an attribute that is bigger than one (Norton & Wilkins, 2009).

It is reasonable to presume that instructional activities that depict fractions as pieces of an equally partitioned whole would have been prominent in the instructional experiences of almost all the students who undergo such difficulties. However, the existing research does not firmly support regarding these difficulties as a function of the use of partitioning activities in early fraction instruction. It has been, indeed, documented that students who are oriented to construe fractions as equal sized parts of a whole *can* overcome the difficulties and develop understandings about unit fractions akin to those proposed as the overarching learning goals of the HLT (e.g., Behr et al., 1984). We examined the instructional approaches used by researchers who documented the learning paths of students who successfully overcame these difficulties.

Assuming a constructivist perspective, Steffe and his colleagues (Hackenberg, 2007; Norton & Wilkins, 2009; Olive & Steffe, 2001; Steffe, 2002; Steffe & Olive, 2010; Tzur, 1999) conducted a series of teaching experiments[4], in which they accounted for students' learning in terms of the development of increasingly sophisticated fraction schemes (Hackenberg, 2007; Norton & Wilkins, 2009; Olive & Steffe, 2001; Steffe, 2002; Steffe & Olive, 2010; Tzur, 1999). In the interventions they conducted to encourage students' cognitive development, they started by engaging pupils in situations that involved reasoning about the equal partitioning and fair sharing of food items, such as candy bars, cakes, and French fries.

During the teaching experiments, students were supported to reason about the size of the pieces produced by equally partitioning a whole, in increasingly sophisticated ways. Pupils were first encouraged to conceive the part of a whole as a physical entity that embodied an attribute (i.e., a length), which could be iterated in its own right. Next they were encouraged to reason about the relative size of a single piece of an equally divided whole, in terms of the number of iterations that would be necessary to produce something as big as the whole. For instance, pupils were encouraged to reason about the size of one of the three equal pieces of a whole as being of such a size that three iterations of its length would be necessary to cover the length of the whole.

Brousseau's (1997) insights about *didactical obstacles*, we have explained how the emergence of these challenges can be a function of how the concept is typically taught.

[4] It is worth mentioning a key difference between *teaching experiments* (Steffe & Thompson, 2000) and *design experiments*. Although both methodologies involve engaging students in instruction for extended periods of time, the main purpose of teaching experiments is to account for students' construction of mathematical concepts and operations. Therefore, teaching experiments are different from design experiments in that instructional design—understood as the systematic testing and refinement of heuristics and resources for supporting learning—does not play a central role in the former ones (Cobb, 2003).

The next step involved engaging pupils in thinking about the relative size of lengths produced by iterating the equal size part of a whole. Students were first presented with situations in which the lengths that were to be produced would be shorter than the whole. For instance, they were asked to reason about the length that would be produced by two iterations of one of three equal parts of a whole (i.e., 2/3). They were then encouraged to think about lengths longer than the whole, such as the one produced by four iterations of one of three equal parts (i.e., 4/3).

The research conducted by Steffe and his colleagues clearly shows that students can develop relatively sophisticated understandings of fractions by reorganizing their conceptions about the equal partitioning and fair sharing of a divisible object. However, the path they must follow is not free of obstacles. As Norton and Hackenberg (2010) recognized, to develop the envisioned understandings, pupils have to clear at least two *developmental hurdles*.

The first hurdle involves "moving from part-whole to partitive conceptions" (Norton & Hackenberg, 2010, p. 345). Such a move requires conceiving the partitions made to a whole not only as being actual pieces of an object, but also as being of such a nature that when connected, both their numerosity and the amount of a certain attribute (i.e., a length) would accumulate. As can be noticed, clearing this first hurdle is tantamount to overcoming the previously mentioned difficulty of construing fractions as numbers that account for the amount of a continuous attribute.

The second hurdle specified by Norton and Hackenberg (2010) lays in "moving from partitive conceptions of proper fractions to iterative conceptions of proper and improper fractions" (p. 345). In this case, an equal size part of a whole (i.e., the unit fraction) must come to be conceived as the amount of an attribute susceptible of being iterated *unrestrictedly*, "regardless of how this unit was produced (e.g., dividing a whole into six parts) or which operations were performed on it" (Tzur, 1999, p. 410). Clearing this second hurdle is thus tantamount to overcoming another of the previously mentioned difficulties, making sense of fractions as numbers that can soundly account for the amount of an attribute that is bigger than one.

For the purpose at hand, it is significant that Norton and Hackenberg (2010) identified these hurdles in the developmental processes of students whose learning was carefully supported and monitored by researchers. Hence, the emergence of these difficulties cannot be regarded as a function of oversights or carelessness in the teaching received by pupils.

When reinterpreted within the theoretical perspective we assume, the reviewed findings are consistent with regarding the student difficulties as being a function of the means of support commonly used in early fraction instruction. Specifically, these difficulties can be a function of the limitations of the equal partitioning model as an anchor for making sense of fractions in increasingly sophisticated ways. The program of research, the planning phase of which we report in this paper, intends to explore whether this is indeed the case. In the following sections we draw on Freudenthal's (1983) phenomenological analysis of fractions to first solidify our reasons for not using the equal partitioning model as a starting point in our work, and then introduce an instructional alternative.

Fraction as Fracturer

In his didactical phenomenology, Freudenthal (1983) assumed a strong instructional perspective. He considered that students should experience mathematical concepts as means for organizing phenomena. He thus sought to identify, as broadly as possible, the phenomena that would *"beg to be organized"* (p. 32) by different concepts. He viewed the mathematizing of such phenomena as the basis from which students would reinvent a particular mathematical idea.

In the case of fractions, Freudenthal distinguished between two different kinds of everyday phenomena that would *beg to be organized* with the fraction concept. In the first kind, he recognized the need for fractions in situations in which the size of a part is gauged with respect to a whole: "fractions present themselves if a whole has been or is being split, cut, sliced, broken, coloured in equal parts or if it is experienced, imagined, thought as such" (Freudenthal 1983, p. 140). Freudenthal coined the term *fraction as fracturer* to categorize instructional activities that would fit into this phenomenological way of engaging with the concept.

He also recognized a significantly different kind of situations in which fractions are necessary: "fractions also serve in comparing objects which are separated from each other or are experienced, imagined, thought as such" (Freudenthal 1983, p. 145). He called this second phenomenological approach *fraction as comparer*.

Freudenthal regarded the *fraction as fracturer* approach as being of a "convincing and fascinating concreteness" (p. 147), but also as phenomenologically much too restricted. For this author, the most basic case of fraction as fracturer would involve the breaking of a divisible object into equal parts. He noticed that, in this case, the main action takes place on the object itself, and that the amounts of a given attribute play a secondary role; they serve to "check the fairness of the distributive procedure" (p. 149).

Freudenthal's account thus suggests that situations involving the equal partitioning of a divisible object, when used as a situation to be mathematized with fractions, could easily lead pupils to associate this concept with the most salient quantifiable product of the partitioning act: the numerosity of the pieces that are produced. As a consequence, these situations would not straightforwardly lead to the use of fractions as a means to quantify the amount of a continuous attribute.

Freudenthal's considerations about fraction as fracturer are also useful for understanding why situations that involve the equal partitioning of a whole could be an inadequate anchor for students to make sense of fractions as numbers that can soundly account for the size of something that is bigger than one. To explain this point, Freudenthal zoomed in on the *mathematical model of magnitude*, within which continuous attributes, such as *length*, *area*, and *weight* are susceptible of unrestricted accumulation. He underlined four aspects of this model:

> To constitute a magnitude in a system of quantities requires:
> - an equivalence relation, which describes the conditions for replacing objects (for instance quantities of a certain substance) with each other and which leads to *equality* within the magnitude,
> - a way of taking together objects (quantities), which leads to an *addition* in the magnitude,
> - the unrestricted availability of objects with the same magnitude value (that is, in the same equivalence class), which makes *addition unrestrictedly possible*,
> - the possibility of dividing an object into an arbitrary number of partial objects that replace each other, which leads to *division by natural numbers* (p. 146; emphasis added).

In analyzing the fraction as fracturer approach, Freudenthal recognized in it "a quite restricted equivalence concept" (p. 147), since the number of identical things that are produced by dividing a whole into n equal parts is limited to n. As a consequence, this approach would not lead to regarding the addition of objects with the same magnitude value (i.e., the individual size of each of the equal pieces of a whole) as unrestrictedly possible. Instead, it would present a clear boundary for the addition: the number of pieces into which the whole was divided. Freudenthal thus concluded that, in phenomenological terms, the fraction as fracturer approach "leads to proper fractions (<1) only" (p. 147).

Taken together (see Table 1), Freudenthal's analysis supports regarding situations that involve the equal partitioning of a divisible object as an ill suited anchor for making sense of the mathematical ideas proposed as overarching learning goals of the HLT. These situations might prompt students to regard fractions solely as a resource for mathematizing the numerosity of discrete elements in sets and subsets. More importantly, they could lead pupils to develop metonymies and other mental representations (Presmeg, 1992) that are incongruent with regarding fractions as numbers that can soundly account for the size of an attribute that is bigger than one.

Table 1

Suitability of problem situations and tools—based on the fraction as fracturer approach—as a starting point for instruction, as supported by empirical evidence (e) or Freudenthal's phenomenological analysis (ph).

Aspect of a viable starting point	**Fraction as fracturer**	
	e	**ph**
Becoming experientially real to students	✓	
Triggering informal ways of reasoning that can be a basis for developing increasingly sophisticated mathematical ways of knowing	✓	
Serving as paradigmatic cases in which to anchor students' increasingly abstract mathematical activity	✗	✗

Fraction as Comparer

As mentioned before, Freudenthal (1983) identified a second kind of everyday phenomena that would beg to be organized with the fraction concept, a kind in which the sizes of objects that are separated from each other are compared. For instance, fractions become necessary when comparing the length of a ping-pong paddle to the length of a tennis racquet (1/3 as long).

Freudenthal recognized this alternative phenomenological approach to fractions (i.e., fraction as comparer) to be more consistent with the mathematical model of magnitude. Among other things, it would be consistent with regarding the amount of an attribute quantified by a fraction as susceptible of being iterated unrestrictedly. As an illustration, in the case of the paddle and racquet example, it is noticeable that there would be no clear phenomenological boundary for how many times 1/3 of the length of the tennis racquet could be iterated, since the object embodying this length (the ping-pong paddle) is separate from the reference unit. Hence, it is sensible to interpret the expression "10/3 of the length of the racquet" as ten iterations of the length of the paddle, each iteration being 1/3 as long as the length of the racquet.

Based on Freudenthal's insights about the fraction as comparer approach, we envisioned an alternative starting point for fraction instruction. It involves the use of problem situations and tools aimed at cultivating an image of unit fractions as the amount of an attribute that is (a) separate from a reference unit, and (b) fulfills an iterative condition with respect to the reference unit.

The basic image of unit fractions we chose to work with is shown in Figure 2. It involves a set of rods that would embody lengths of the reference unit and of the unitary fractional amounts. The reference unit would be the length of a wooden stick (about 24 cm long) and the unitary fractional amounts would be the lengths of plastic drinking-straws.

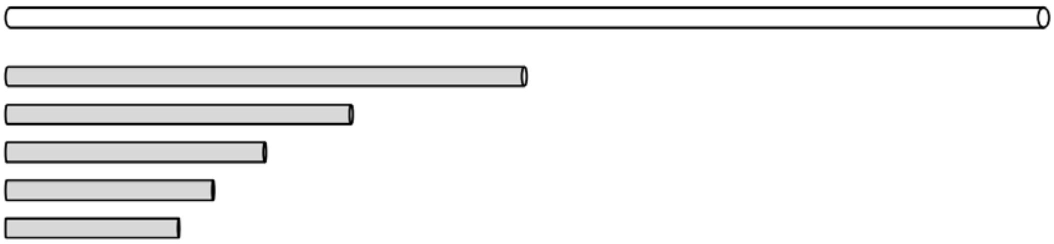

Figure 2. Unit fractions as the lengths of rods.

Students would judge a straw to be 1/2 as long as the stick (reference unit) when being of such a size that two iterations of the length of the straw would be necessary to cover the exact length of the stick (see Figure 3). A straw would be judged as being 1/3 as long as the reference unit when being of such a size that three iterations of its length would be necessary to cover the exact length of the stick, and so on.

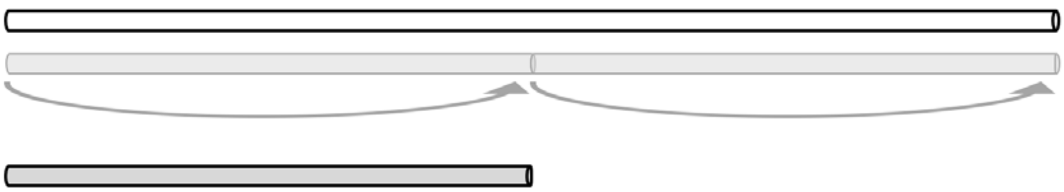

Figure 3. A half conceived as the length of something that requires two iterations to cover the length of the reference unit.

We deemed the proposed starting point as phenomenologically consistent with the learning goals that we chose for the HLT, and specifically with all four features of unit fractions that students need to encounter (cf., Thompson & Saldanha, 2003). It would present fractions squarely as a resource for mathematizing the amount of a continuous attribute; namely, the length of a straw that is apart from the reference unit. In addition, an object separated from the reference unit would embody the amount of an attribute quantified by a unit fraction. This, in turn, would allow for regarding the iteration of unit fractions as unrestrictedly possible.

HLT: Expected Shifts in Students' Reasoning

The Instructional Sequence

To cultivate the image of unit fractions just described, we expected to use a series of activities based on the reinvention of linear measurement. In the initial phase of the instructional sequence, these activities would resemble those used by Stephan et al. (Stephan et al., 2003) to support students' development of measuring conceptions. We conjectured that it would be necessary to start at this stage, given the limitations of students' prior mathematical experiences.

The overarching narrative of the instructional activities would involve learning about the ways in which a fictitious group of people, living in pre-colonial Mesoamerica, measured. Using this scenario, the first issue we expected to problematize with the students was the need for establishing a standard unit of measurement. We expected to achieve this goal by exploring with the children what would happen if a group of people used only the parts of their bodies (hands, feet, thumbs, etc.) to measure. Students would then see the use of sticks with a unified length as a sensible solution to a problem.

The second issue we aimed at problematizing was the fact that the length of many things would not correspond to a whole number of iterations of the stick. For instance, the height of a basket could be more than one stick but less than two. Students would then recognize the need for creating and using units of measurement shorter than the length of one stick. The production of straws that fulfilled specific iterative conditions would be introduced as the solution developed by the ancient people.

In producing the straws, students would be encouraged to make sure that each fulfilled a specified condition. For instance, they would be asked to produce a straw of such a length that two iterations of it would exactly cover the length of the stick. Students would be given a drinking straw about 15 cm long and asked to iterate it along the stick. If the two iterations did not exactly cover the length of the stick, students would be asked to manipulate the length of the straw so that it did, either by reducing its size using a pair of scissors, or by using a longer straw. By repeated trials, students would home in on the specified length (see Figure 4).

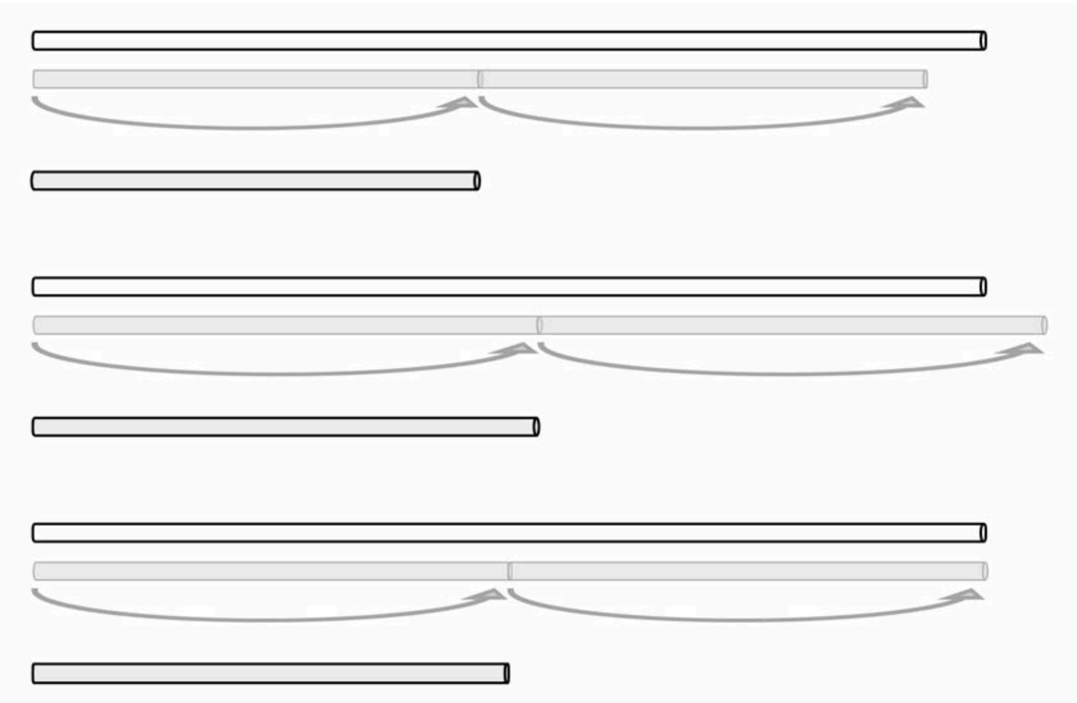

Figure 4. Adjusting the size of a drinking straw so that two iterations of it cover exactly the length of a reference unit.

It is worth clarifying some of the similarities and differences of the activities that we chose to use, with those used by other fraction learning researchers. For instance, Davydov (1969/1991) conducted a series of classroom teaching experiments on fractions in which students were oriented, from the start, to reason about the measurement of lengths and to represent them on the number line. As in our case, an important issue in the instructional agenda involved establishing ratios between different units of measure. However, Davydov's experiments included orienting pupils to reason about unit fractions as parts of a whole (e.g., 1/4 as "one part of a whole divided equally into four parts"; p. 34), whereas our aim is to find out whether avoiding such reasoning altogether, in initial instruction, can be a productive way to encourage powerful fraction learning.

In addition, researchers working in the Freudenthal Institute (van Galen et al., 2008) developed an instructional sequence on fractions, called "the Amsterdam stick," aimed (as in our case) at the reinvention of linear measurement. However, this sequence is not meant to be used at the start of fraction instruction. Instead, students are expected to engage in it after having had ample experiences with partitioning and sharing activities. Furthermore, unit fractions are represented as segments of the stick—like centimeters are on a traditional ruler—instead of as separate rods. Given these differences, this research did not provide answers to the questions we were asking.

Finally, the activity of producing the straws resembles one with which Steffe and colleagues (e.g., Steffe, 2010) encouraged students to reason about the size of a part relative to the size of the whole. However the two activities are significantly different in that the one we just described does not intend for iteration to be a means of gauging the size of *a part of a whole*.[5] Instead, the lengths that are iterated belong to objects (straws) that are presented as always being apart from the reference unit (the stick).

To summarize, the problem situations and tools that we chose as a starting point have some similarities with those used by other fraction learning researchers. However, in phenomenological terms, they also entail a significant difference—they frame fractions squarely within the fraction as comparer approach. Determining whether this difference leads to differences in student learning is at the core of our research program.

Students' Reasoning

We expected to support two initial shifts in students' reasoning. The first would involve making sense of the inverse order relation of unit fractions (Tzur, 2007). We anticipated that the activity of the straws could serve as the basis for engaging students in reasoning about the relation between the size of a straw and the number of times its length has to be iterated to cover exactly the length of the stick. For instance, pupils could be asked to discuss what would be longer, a straw that needed four iterations to cover the length of the stick or one that needed five (i.e., 1/4 vs. 1/5).

The second shift would involve using the drinking straws as units of measure in their own right. The straws could be used to produce paper strips of different sizes; for instance, use the 1/3 straw (i.e., the one that fits three times in the stick) to produce a strip that is four times as long as the straw (4/3). The production of these strips could support the orchestration of whole class conversations about the outcomes of iterating unit fractions: "Would the paper strip be shorter, as long as, or longer than the stick?" We anticipated that

[5] In fact, the proposed activity is intended to substitute activities based on the notion of equipartition (i.e., fraction as fracturer) in initial fraction instruction.

students would then be able to assess the size of any fraction as being smaller, as big as, or bigger than 1. Generally speaking, their activity would resemble that of children who Steffe and colleagues (e.g., Steffe & Olive, 2010) describe as having developed iterative conceptions of proper and improper fractions.

HLT: Summary

It is worth noticing that the envisaged HLT seems to call for more sophisticated quantitative reasoning than situations that entail partitioning a whole into *n* equal parts. Consequently, although apparently a better suited anchor for supporting the emergence of sophisticated understandings of fractions, the possibility exists that the proposed starting point would not meet the first two aspects of a viable starting point: (a) becoming experientially real to students and, (b) triggering informal ways of reasoning about important fraction notions (see Table 2). In the remainder of this paper, we report on the empirical study we conducted to explore this concern and to justify the outlined HLT.

Table 2

Suitability of problem situations and tools as a starting point for instruction, as supported by empirical evidence (e) or Freudenthal's phenomenological analysis (ph). Comparison between the two approaches.

Aspect of a viable starting point	**Fraction as fracturer**		**Fraction as comparer**	
	e	**ph**	**e**	**ph**
Becoming experientially real to students	✓		?	
Triggering informal ways of reasoning that can be a basis for developing increasingly sophisticated mathematical ways of knowing	✓		?	
Serving as paradigmatic cases in which to anchor students' increasingly abstract mathematical activity	✗	✗		✓

Background of the Study

As part of the planning phase of a design experiment, we carried out a study aimed at exploring the viability of the alternative starting point. We interviewed 16 students who formed the only third grade classroom of the public school in which we planned to conduct the design experiment. This school was in a poor neighborhood of a city in southern Mexico. The interviews were conducted eight months into the school year. At the time of the interviews, six of the students were eight years old and ten were nine. Seven were girls and nine were boys.

During the interviews, we used problem situations consistent with the alternative starting point. These were activities in which the amount of an attribute was defined in terms of it fulfilling an iterative condition, relative to a reference unit. We explored whether novice fraction learners would readily construe these problem situations as experientially real, and whether they would, informally, reason about them in ways consistent with the inverse order relation and basic fraction equivalencies.

Students' Prior Instructional Experiences

To document students' prior instructional experiences, we collected data on the group's performance on the national standardized test (ENLACE), and inspected the pupils' mathematics notebooks. In addition, we included questions in the interviews to document how students interpreted conventional fraction inscriptions (see *Chocolate Bar* task).

The students' results on the standardized test in mathematics were low. The group's mean was in the 9th percentile, nationwide. Regarding the four categories of the test achievement, 11 of the participating pupils were categorized as *below basic* (*insuficiente*), 4 as *basic* (*elemental*), one as *proficient* (*bueno*) and none as *advanced* (*excelente*).

The inspection of the notebooks suggested that the students' prior instructional experiences in mathematics had been rather poor. It was noticeable that the notebooks had been used in 45 different activities. Twelve of these involved rote exercises such as: (a) writing series of numbers, from 1 to 100, with increments of one (1, 2, 3...), two (2, 4, 6...), three (3, 6, 9...), and so on, up to ten; and (b) writing all the numbers, one by one, from 6700 to 6900. Fourteen activities involved carrying out mechanical operations such as: five additions and five subtractions with four and five-digit numbers (e.g., 16376 + 24931 + 36184), 23 multiplications of four digit numbers by a one digit number (e.g., 7 × 2554), and 14 divisions of four digit numbers by a one digit number (e.g., 2879 ÷ 8). Other activities seemed to involve copying information from the blackboard, such as copying the fractional equivalences of the weights in grams to one kilogram (1000 g = 1 kg, 750 g = 3/4 kg, 500 g = 1/2 kg, 250 g = 1/4 kg, and 125 g = 1/8 kg).

In the interviews, we included a *Chocolate Bar* task intended to document how students interpreted conventional fraction inscriptions as a result of their prior instructional experiences. This task was presented at the end of the interviews and intentionally called on a fraction as fracturer situation to provide a comparison to other tasks.[6] It involved a rectangle that represented a chocolate bar and cards with conventional fraction inscriptions (see Figure 5). Students were told that the cards represented an amount of chocolate to be eaten. They were asked to read the cards out loud and to indicate on the picture how much chocolate it would involve eating. Pupils were also asked to express whether the amount written on the card would be more, less, or as much as half of the chocolate.

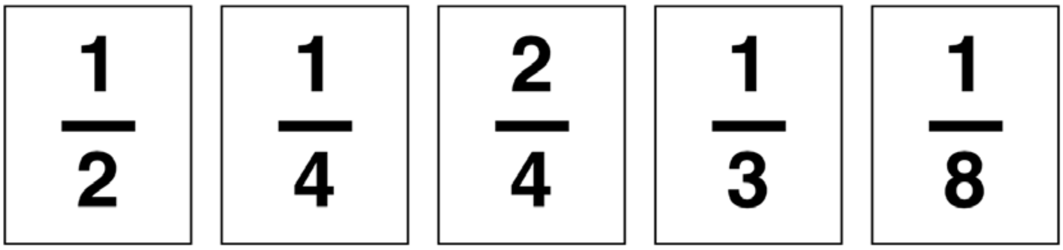

Figure 5. Cards with fraction inscriptions in the order in which they were presented, one by one.

Results from this task are presented in Table 3. They indicate that the students had not yet learned much about how to read and interpret fraction inscriptions. Only eleven of the

[6] Our introduction of this task and the analysis of students' responses followed the method we describe in detail in the next section.

sixteen children both correctly read the card with 1/2 inscription (see Figure 5), and identified the amount of chocolate it represented. The card with the 1/4 inscription was correctly read and interpreted by eight of the students. Six of the remaining eight students were unaware of what the inscription could mean. The other two read it as "one fourth" but considered it to represent a quantity bigger than 1/2. In the case of the 2/4 card, only five students read and interpreted it correctly. The 1/3 and 1/8 cards were not read and interpreted correctly by any of the students.

Responses to the *Chocolate Bar* task indicated that students' prior instructional experiences had contributed little to helping them make sense of the quantitative meaning of common fraction notations. For the purposes of our study, it was therefore reasonable to consider these students to be novice fraction learners.

Table 3

Conventional fraction inscriptions correctly and incorrectly read and interpreted by students.

	1/2	**1/4**	**2/4**	**1/3**	**1/8**
Correct	11	8	5	0	0
Incorrect	5	8	11	16	16

Methodology

The students' interviews were similar to those used in the tradition of the constructivist teaching experiment (Steffe & Thompson, 2000). However, they were conducted in a way consistent with the design experiment methodology. The main purpose was to document the different ways in which the participating students would engage with problem situations consistent with the chosen starting point, given their prior instructional experiences (Stephan et al., 2003).

Interview Protocol

The interview protocol included the *Chocolate Bar* task, and three more problem situations designed to test the viability of the alternative starting point. These were designed to encourage students to reason about amounts of a given attribute as multiplicands that satisfy a certain iterative condition with respect to a reference unit.

The first problem, *Milk Carton*, involved asking students to make judgments about the relative capacity of different kinds of cups based on how many cups of each kind can be filled with the milk contained in a carton. In the problem, cards were used (Figure 6) that showed the number of cups that, in each case, could be filled with the milk in a carton that was physically present (Figure 7). Children were asked both to make general judgments about the change in the level of the milk in the carton after serving one cup of each kind, and to compare which of two kinds of cups would hold more milk (paper cups vs. foam cups; i.e., sevenths vs. ninths). No cups were shown to the children, only the cards.

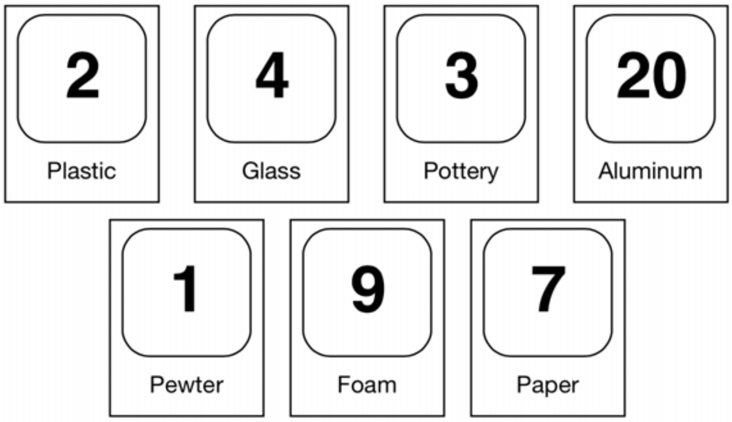

Figure 6. Cards used in the *Milk Carton* problem to show how many cups of each kind could be filled with the milk contained in a carton.

The comparisons that the students were asked to make were the following:

- Plastic cups vs. glass cups (halves vs. fourths)
- Pottery cups vs. glass cups (thirds vs. fourths)
- Paper cups vs. foam cups (sevenths vs. ninths)

In addition, students were asked to judge how much milk (plenty or little) the aluminum (twentieths) and the pewter cups (ones) would hold. In this part of the interview, students were given a whiteboard marker to make marks on the carton, and show the approximate level of the milk after serving different numbers of different kinds of cups.

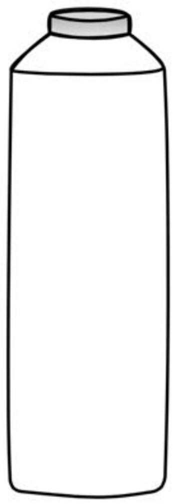

Figure 7. Image of the milk carton that is physically presented to students.

This problem situation included several aspects of the proposed alternative starting point. First, it directly asked for acting on, and reasoning about, amounts of a specific attribute; namely, *amounts of milk*. Second, it asked for gauging the size of unitary amounts (the capacity of the different kinds of cups) in terms of them satisfying an iterative condition. Lastly, it presented the objects embodying the unitary amounts (i.e., the paper cups) as being apart from the reference unit (i.e., the milk carton). Therefore, no clear boundary for the accumulation of the unitary amounts was set, since it would be sensible to imagine an accumulated amount resulting from how much milk would someone drink if he or she drank a pottery-cup of milk every day during a week (7/3), a month (31/3), or a year (365/3).

In the second problem, *Kangaroos*, the students were presented with the picture shown in Figure 8. They were told that the number at the end of each line indicated how many equal-size jumps it would take for each of the kangaroos to cover the whole distance. Students were asked to determine which of the three kangaroos would make the longest jumps and which the shortest (i.e., 1/2 vs. 1/4 vs. 1/5).

In terms of the proposed starting point, the amounts of an attribute that this problem encouraged the students to reason about were *lengths of jumps*. The size of the unitary amounts was defined in terms of them fulfilling an iterative condition: equal-size jumps of such a length that so many of them would be exactly enough to cover a given distance. It is worth noticing that, also in this case, the unitary amounts of an attribute (i.e., the size of the jumps) were portrayed as being apart from the reference unit (i.e., the length of the whole line). Consequently, there was no evident boundary in how many times the unitary amounts could be iterated. This made questions such as the following to be sensible: "Which of the kangaroos would travel the farthest distance after jumping 100 times?"

Figure 8. The *Kangaroos* problem.

In the third problem, *Water Tanks*, the students were shown a picture of three water tanks of equal size (see Figure 9), and told that the number below each tank indicated the time in hours it took for the tank to fill up.[7] Students were first asked to determine which of the three tanks would have more water after one hour of being filled (i.e., 1/2 vs. 1/4 vs. 1/8).

The problem was then used to elicit informal ways of reasoning about basic fraction equivalencies. Pupils were asked to determine how long would it take for the second tank to have the same amount of water that the first tank would have after one hour (i.e., 2/4 = 1/2). Similar questions were asked to help students identify the time it would take for the 8-hour tank to have as much water as:

- the 2-hour tank after 1 hour (4/8 = 1/2)
- the 4-hour tank after 1 hour (2/8 = 1/4)
- the 4-hour tank after 3 hours (6/8 = 3/4)

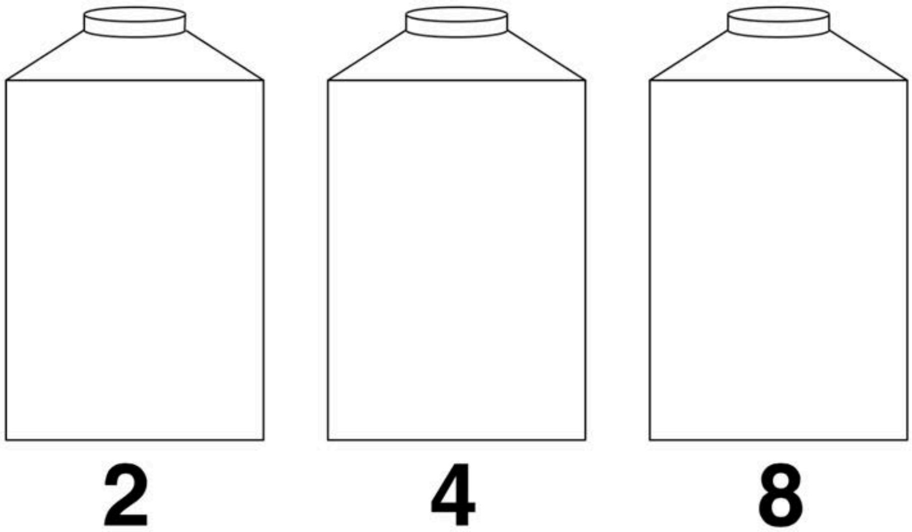

Figure 9. The *Water Tanks* problem.

In terms of the proposed starting point, the amounts of an attribute in this problem situation were *water levels in a tank*. Unitary amounts were defined in terms of them satisfying an iterative condition: An amount of water flowing out of the specific pipe in one hour is such that it would take exactly a certain number of hours to fill the entire tank. In addition, when it is assumed that someone can be consuming the water constantly, there was no apparent restriction on how much water could flow out of the pipes. This made questions such as the following to be sensible: “Which household would receive more water in a 14 hour period?”

Students were first asked to answer the *Kangaroos* and *Water Tanks* questions without marking the sheet. However, when they expressed difficulties in responding to the questions, the interviewer encouraged them to use a pencil and make inscriptions (e.g., showing the level of the water). When the inscriptions made by the students did not meet the stated

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7 In many Mexican households, water tanks are placed on the roofs. It is common for these tanks to fill up during the night hours. The water stored in them is consumed during the day.

conditions (e.g., equal accumulation of water in each hour), the interviewer made it noticeable to the children, and allowed them to correct their estimations.

It is important to clarify that not all the questions in the protocol were presented to all the students. There were many cases in which students appeared to have considerable difficulties in responding to less demanding questions. In some of those cases, the more demanding questions were not addressed to them. For instance, in the *Water Tanks* problem, students who did not recognize that the third tank would need four hours to have as much water as the first tank after one hour (4/8 = 1/2) were not presented with further questions.

Presenting the Problems

In line with the design experiment methodology, as the interviewers presented the problems, they tried to support students in interpreting the situations as experientially real. Before the questions in the protocol were addressed to the students, brief conversations were held about the general setting and the context of each of the problems. The following extract from the *Water Tanks* conversation illustrates how the problems were introduced to the students.

Interviewer: [Shows the sheet with the image of three water tanks, Figure 9]. What are these?

Teresita:[8] Water tanks [*tinacos*]?

Interviewer: Water tanks. Right? And what are they used for?

Teresita: To fill water.

Interviewer: And where are they put?

Teresita: In a house with two floors?

Interviewer: In what part of the house?

Teresita: Where the washing place [*lavadero*] is?

Interviewer: In the terrace roof [*azotea*]? In the roof [*techo*]?

Teresita: Yes.

Interviewer: Look [pointing at the drawings on the sheet], they belong to different houses. How long does it take for this one to fill up?

Teresita: Two hours.

Interviewer: This one?

Teresita: Four hours.

Interviewer: And why do you think that is, if they are the same size?

Teresita: Because the water is slower?

Interviewer: It is slower. Where is the water faster?

Teresita: In the two hours one.

The interviewers supported students in construing the drawings and numbers shown to them as representing specific things and events and, in doing so, used types of guidance (e.g., questions and prompts) that could conceivably be provided by a classroom teacher.

[8] Names used in this paper are pseudonyms.

Teresita was a student who readily developed helpful images. She recognized the drawings as representations of tanks that are used to store water. She also seemed to have a clear image of why it would take longer for some tanks to fill up.

In some cases, the negotiation was not as straightforward as in the extract above. However, in every case, the interviewers worked to support students to make sense of the general setting and the context of each of the problems.

Data Analysis

We analyzed the interviews in the way typically followed in the planning phase of a design experiment (Stephan et al., 2003). We sought to create a coherent account of students' reasoning, considering the social nature of the situation in which it emerged (Cobb & Yackel, 1996). We attended to two issues in particular: (a) the extent to which the problem situations became experientially real to the pupils, and (b) the ways of reasoning that emerged as the students dealt with the problem situations. In the remainder of this paper, we first document how the problems became experientially real for the novice fraction learners with relative ease. We then document that the students' informal mathematical reasoning, evoked when solving those problems, allowed them to meaningfully explore two core notions of fraction reasoning: *inverse order relation* and *basic equivalencies*.

Construing Problems as Experientially Real

The evidence we used for determining that a problem had been construed as experientially real consisted of verbal expressions and gestures that suggested that a student was reasoning about the quantities involved in the problem, and not just about the numbers. For instance, in the *Milk Carton* problem, this type of evidence included students referring to amounts of milk or making gestures with their hands indicating the size of cups. The following extract illustrates the latter case:

Interviewer: When I serve one cup of milk, how far does the milk carton empty?

Marilu: To here [marking the carton at about the middle].

Interviewer: Why?

Marilu: Because they are this size [gesturing with her fingers the size of a cup].

Gestures referring to the actual size of cups indicated to us that a student like Marilu had interpreted the problem as involving actual quantities. In other words, it indicated that a student was imagistically involved with the problem at hand and, thus, engaging in personally meaningful mathematical activity.

Given these evidence criteria, all 16 students construed the three problems as experientially real. Naturally, the process of doing so was not always the same. We have already illustrated how Teresita, understanding where the water was "faster", construed the *Water Tanks* problem as experientially real relatively seamlessly. With other students, like Marilu, it took longer.

Collectively, the results from this component of the analysis support the viability of the proposed starting point. They suggest that it would be possible to support a group of third grade students, like those with which we planned to work, to readily construe problems based on the comparer approach as experientially real.

Triggering Informal Ways of Reasoning

We now turn to the analysis of the students' informal ways of reasoning about the two basic aspects of fraction relations underlying the interview problems: inverse order relation of unit fractions, and basic fraction equivalences. Rather than claiming that students' mathematical reasoning was solid at the time of the interviews, we assessed whether these initial forms of reasoning appear suitable for developing increasingly sophisticated mathematical ways of knowing in the short run.

Comparing the Size of Unitary Fractional Amounts

All of the students applied some intuitions consistent with the inverse order relation when they compared the sizes of unitary fractional amounts. However, there were significant qualitative differences among them. As we explain below, while some students readily reasoned in ways consistent with the inverse order relation, others seemed to rely strongly on what Behr et al. (1984) called the *whole-number-dominance strategy*, when judging the relative size of the amounts.

We created four categories that account for the different ways in which the students reasoned about the relative size of the unitary amounts (see Table 4). We first describe each category in descending order of sophistication of reasoning and later return to discuss the instructional implications.

Table 4

Number of students in different categories, according to how they reasoned about the relative size of unit fractions.

	Category 1 Readily anticipating that the more iterations, the smaller the amount of an attribute	**Category 2** Coming to anticipate	**Category 3** Strong reliance on visual evidence	**Category 4** Initial insight in extreme cases
N	6	5	3	2

Category 1: Readily anticipating that the more iterations, the smaller the amount of an attribute. The students in this category (*N*=6) readily assessed that the plastic cups (halves) would hold more milk than the glass cups (fourths). They also readily assessed that the pottery cups (thirds) would hold more milk than the glass ones (fourths), and that the paper cups (sevenths) would hold more milk than the foam ones (ninths). In the *Kangaroos* problem, they anticipated that the first kangaroo (halves) would make the longest jumps, followed by the second (fourths) and the third (fifths). Finally, they anticipated that the first tank (halves) would be the one with the most water after one hour, followed by the second tank (fourths) and then the third one (eights).

Category 2: Coming to anticipate the amount of the attribute. The students in this category (*N*=5) differentiated themselves from those in the previous one in that it was during the course of interviews that they came to anticipate that the more iterations of an amount of an attribute that were needed to make as much as a reference unit, the smaller the amount had to be. These students initially assessed the size of the cups following the whole-number-dominance strategy, thus judging that the cups with the bigger number would hold more milk. However, after estimating the place where the milk would be after serving the different cups, they came to consistently make comparisons in accord with the inverse order relation.

Zaide was one of the students in this category. She initially anticipated that the plastic cups (halves) would be smaller than the glass cups (fourths). After marking on the carton sensible estimates of where the levels of the milk would be if one plastic cup and one glass cup were served, respectively, she changed her mind and considered that the plastic cups would be bigger. When comparing the glass cups (fourth) and the pottery cups (thirds), she followed a similar path. Finally, she anticipated that the foam cups (ninths) would hold less milk than the paper cups (sevenths), as illustrated by the following excerpt:

Interviewer: Which would be bigger?

Zaide: The foam cups [ninths].

Interviewer: Bigger?

Zaide: Oh, no, the paper ones [sevenths].

Interviewer: The paper or the foam ones?

Zaide: Paper.

Interviewer: Why the paper ones?

Zaide: Because if they put nine cups it goes down less. And if you put seven it goes down faster.

Interviewer: Which can hold more?

Zaide: The seven ones [paper cups].

Interviewer: The seven ones can hold more?

Zaide: Because they are not many cups.

In the excerpt it is noticeable that Zaide reasoned about the capacity of the paper cups based on how the levels of the milk in the carton would change as these cups were served. Her previous estimations seemed to have helped her imagine that when more cups were to be served, the level of the milk in the carton would drop less with one serving ("it goes down less"). This imagery allowed her to correctly anticipate that the foam cups (ninths) would be smaller than the paper cups (sevenths).

In the *Kangaroos* situation, Zaide also started by making comparisons following the whole-number-dominance strategy (i.e., that the second kangaroo, fourths, would make longer jumps than the first one, halves). She then made reasonable estimates for the landing spots of the two kangaroos, and changed her mind. Finally, she correctly anticipated that the third kangaroo (fifths) would make shorter jumps than the second one (fourths). In the *Water Tanks* problem, she readily anticipated that, after one hour, the first tank (halves) would have more water than either the second (fourths) or the third (eights).

Zaide's case illustrates how a group of students seemed to develop imagery, during the interviews, which allowed them to soundly reason in ways consistent with the inverse order relation. Within the quantitative context of the problems, they anticipated that the more iterations of the amount of an attribute that were needed to produce as much as a reference unit, the smaller the amount would have to be.

Category 3: Comparisons based on visual evidence. The students in this category (N=3) consistently started by judging the relative size of the amounts following the whole-number-dominance strategy. Then, like the students in the previous category, they made reasonable estimates for the places where the milk would be after serving the different kinds of cups, and used them to judge the size of the amounts.

These students differentiated themselves from those of the previous group in that, during the interviews, they did not seem to have come to understand this principle sufficiently well to consider that it would apply in every case. They always appeared to follow the whole-number-dominance strategy when anticipating the difference in size between two kinds of cups (or the length of the jumps of two kangaroos). Only after marking estimates they judged otherwise.[9]

Category 4: Initial insight in extreme cases. The two students in this group showed the least sophisticated reasoning. They strongly followed the whole-number-dominance strategy when making the comparisons. They considered that the glass cups (fourths) would hold more milk than the plastic cups (halves), even after they had made marks on the milk carton that appeared to indicate that the opposite was true. They also considered that the second kangaroo (fourths) would make longer jumps than the first one (halves), even after they had made reasonable marks of where the two kangaroos would land.

Nonetheless, these students seemed to have some intuitions consistent with the inverse order relation. They judged the aluminum cups (twentieths) as being small, and the pewter cups (ones) as being big. In addition, with the guidance of the interviewer, they created reasonable estimates for where the level of the milk would be in the carton after serving different kinds of cups (see Figure 10).

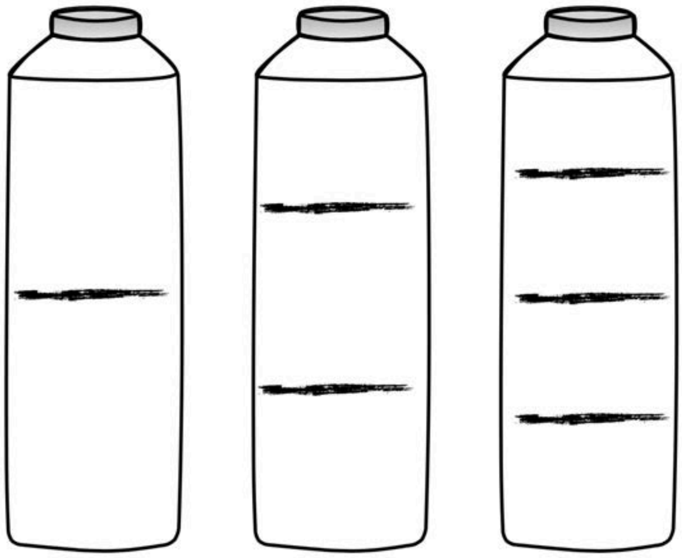

Figure 10. Estimates made by students on the *Milk Carton* problem.

Collectively, the results from this component show that problem situations like those used in the interviews can be a well-suited means of eliciting students' reasoning about comparing size of unit fractions, in informal ways. They suggest that a group of third grade students could productively participate in the analysis and discussion of problem situations that ask for comparing the amounts of a certain attribute, given the number of iterations it

[9] This was the case when comparisons involved, as well as when they did not involve, one half.

would require to make as much as a reference unit. It would be sensible to expect that, during the analysis and discussion of the problem situations, some of the students would produce solutions consistent with the inverse order relation, while others would produce solutions that followed the whole-number-dominance strategy. Whole class conversations could be orchestrated in which the two kinds of solutions were analyzed and contrasted.

Given that even the students who showed the least sophisticated reasoning seemed to have some intuitions consistent with the inverse order relation, it would also be reasonable to expect that, as the two kinds of solutions were collectively analyzed and discussed, those consistent with the inverse order relation would eventually become treated by the class as routine and beyond need of justification (Cobb, 2000). The important number of students who, in the course of a single interview, seemed to have progressed significantly in making sense of solutions consistent with the inverse order relation (Category 2), further allows regarding this expectation as reasonable.

Basic Equivalencies

As previously mentioned, the *Water Tanks* problem was also intended to engage students in reasoning informally about basic equivalencies. It is worth mentioning that, in addition to the inverse order relation, this notion is considered to be of much importance in early fraction instruction (National Council of Teachers of Mathematics, 2000).

As shown in Table 5, there were three students who did not recognize any of the equivalencies. In contrast, five students recognized them all. Of the remaining eight students, two identified only one of the equivalencies (2/4=1/2), four children identified the two equivalencies between the tanks when being half full (2/4=1/2; 4/8=1/2), and the other two pupils identified those plus the one between two hours in the four-hour tank and one hour in the two-hour tank (2/8 = 1/4).

Table 5

Number of students that identified the different equivalencies in the water levels of the tanks.

	2/4 = 1/2 4/8 = 1/2 2/8 = 1/4 6/8 = 3/4	**2/4 = 1/2 4/8 = 1/2 2/8 = 1/4**	**2/4 = 1/2 4/8 = 1/2**	**2/4 = 1/2**	**No equivalence recognized**
N	5	2	4	2	3

When establishing the equivalencies, it was noticeable that several of the students based their answers on numeric patterns. However, they seemed not to lose track of the quantitative meaning of the numbers involved. For instance, Andres justified the equivalence between six hours in the eight-hour tank and three hours in the four-hour tank (6/8 = 3/4) in the following way:

> Andres: Because here. This one [pointing at the number eight below the third tank] is twice as this one [pointing at the number four below the second tank]. If you add four plus four you get eight, and three plus three, six.

As can be noticed, Andres' solution was of a calculational nature. It is unclear whether he was thinking about water levels as he referred to the numbers. Nonetheless, in a conversation that took place briefly after, it was noticeable that he had not lost track of the quantitative meaning of the numbers involved in the problem.

Interviewer: And in which household would you rather live?

Andres: [readily points at the first water tank (two-hour tank)]

Interviewer: Why?

Andres: Because it fills up faster, in two hours.

This conversation indicates that Andres was mindful of the quantitative meaning of the numbers in the problem, and of what they would imply for the users of each of the water tanks.

Similar to the previous component, the results from this component show that problem situations like those used in the interviews can be a well-suited means of eliciting students' reasoning about basic fraction equivalencies, in informal ways. Although we did not recognize informal ways of reasoning about equivalencies in all the students, the children's collective performance was favorable when compared to the one they had in the *Chocolate Bar* situation.[10] The reader will recall that only five of the sixteen students recognized the equivalence between 2/4 and 1/2 (see Table 3). In contrast, thirteen children recognized the equivalencies between the level of the water in the four-hour tank after two hours, and the level in the two-hour tank after one hour (2/4 = 1/2). In addition, five students seemed to have clearly recognized the equivalence in the level of the water in the eight-hour tank after six hours, and the level in the four-hour tank after three hours (6/8 = 3/4; see Table 5).[11]

Discussion and Conclusions

Researchers have widely viewed the development of early fraction notions as stemming from ideas and intuitions of equal partitioning and fair sharing of divisible objects (e.g., Piaget, 1965; Steffe & Olive, 2010; Streefland, 1991). In our work, we call this view into question. Adopting a sociocultural approach, we have conjectured that the widely documented relation between the emergence of early fraction-like intuitions and notions, and situations that entail equally dividing and fairly sharing an object, need not be regarded as a function of a somewhat natural psychogenetic process. We explore the possibility that, instead, this relation is a function of a particular socially and culturally situated way of introducing children into the fraction realm.

With this consideration in mind, we sought alternative activities from which basic fraction notions could emerge. Given the significant difficulties that students typically face when making sense of the concept in increasingly sophisticated ways, we consider this a worthy endeavor. In our explorations, we found Freudenthal's (1983) observations about fraction as fracturer and fraction as comparer to be particularly insightful and useful. We also found it intriguing that, in his outline of how fractions could be taught, Freudenthal

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[10] The *Chocolate Bar* task was used as the last one in the series, after the *Water Tanks* problem.

[11] It is important to clarify that we are not claiming that the students learned to work with basic fraction equivalencies in the course of a single interview. After all, the data on the numbers of students who did not recognize the equivalence between 2/4 and 1/2 in the *Chocolate Bar* situation are direct evidence refuting such claim. Instead, what our data shows is that instructional activities based on the proposed starting point can serve as a means of eliciting informal ways of reasoning about simple equivalencies in classrooms with novice fraction learners.

himself did not propose specific problem situations, based on the fraction as comparer approach, that could become the starting point for fraction instruction.

Other researchers have used problem situations and tools consistent with the fraction as comparer approach. For instance, imagery consistent with this approach is present in the instructional efforts used by Steffe and colleagues to encourage students to develop what they call iterative conceptions of fractions (Hackenberg, 2007; Steffe & Olive, 2010; Tzur, 1999). However, problem situations and tools consistent with the fraction as comparer approach have always been used as an improvement of, or as an addition to, the imagery based in fraction as fracturer situations. We had to ask whether eliminating the fracturer imagery from initial fraction instruction, altogether, could lead to learning paths that introduce fewer hurdles for the learner.

In this paper, we have described an alternative starting point for teaching fractions, and explained why it could be better suited, in the long run, for supporting students' development of increasingly sophisticated fraction notions. We have also analyzed data that show how novice fraction learners can construe problem situations and tools, designed consistently with this starting point, as experientially real. In addition, we documented that such problem situations and tools can become a means of eliciting informal ways of reasoning about the inverse order relation and about basic fraction equivalencies.

In terms of the criteria for the viability of a starting point for fraction instruction, outlined in Table 2, our paper thus presents a reasoned argument for the fraction as comparer approach, and completes the planning phase of our design experiment. Whether specific problem situations and tools consistent with fraction as comparer approach would actually lead to a more beneficial fraction learning requires further empirical research. Our use of the outlined HLT in classroom experiments (Cortina, Visnovska, & Zuniga, 2014) provides initial indications that the proposed alternative starting point might open clearer paths for pupils to learn fractions and, thus, enhance their opportunities to understand this concept.

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