Spontaneous Increase of Magnetic Flux and Chiral-Current Reversal in Bosonic Ladders: Swimming against the Tide

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The interplay between spontaneous symmetry breaking in many-body systems, the wavelike nature of quantum particles and lattice effects produces an extraordinary behavior of the chiral current of bosonic particles in the presence of a uniform magnetic flux defined on a two-leg ladder. While noninteracting as well as strongly interacting particles, stirred by the magnetic field, circulate along the system's boundary in the counterclockwise direction in the ground state, interactions stabilize vortex lattices. These states break translational symmetry, which can lead to a reversal of the circulation direction. Our predictions could readily be accessed in quantum gas experiments with existing setups or in arrays of Josephson junctions.

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A charged quantum particle, due to its wavelike nature [1], picks up an increment to its phase proportional to the magnetic flux piercing the area enclosed by the particle's (closed) path. If the particle hops around a plaquette, the accumulated phase is proportional to the magnetic flux $\Phi = Ba^2$, a being the lattice constant. Since the phase of the wave function is defined modulo 2π, the action of a magnetic field on quantum particles in a lattice is periodic: If we define a dimensionless flux $\phi = 2\pi\Phi/\Phi_0$ with the magnetic flux quantum $\Phi_0 = h/q$, where h is Planck's constant and q is the charge, any physical quantity $\mathcal{A}$ obeys $\mathcal{A}(\phi) = \mathcal{A}(\phi + 2\pi)$. The minimal extensive lattice on which this behavior can emerge is a two-leg ladder (see Fig. 1).

In this work, we discuss the intriguing effect of a *reversal* of the circulation direction of the chiral current of interacting bosons on a two-leg ladder, due to the spontaneous formation of a large unit cell *without* changing the external magnetic field. The key ingredients to realize this effect are, first, the wavelike nature of quantum particles defined on a lattice, and second, many-body effects.

The basic idea is sketched in Fig. 1. For a single plaquette, the magnetic fields corresponding to values of the flux $0 < \phi < \pi$ produce a ground-state net current with a counterclockwise chirality. When one assembles these plaquettes into a minimal extensive lattice such as the two-leg ladder, naively, one would expect that the local currents on individual links along the boundary of the ladder add up to produce a net chiral current j_c, also circulating in a counterclockwise direction for $0 < \phi < \pi$. Indeed, this is the case for noninteracting particles [2]. In general, since the chiral current is a ground-state property, the relevant unit cell does not need to be the same as the unit cell of the underlying lattice.

If the unit cell is doubled, then this will result in a doubling of the effective flux $\phi_{\mathrm{eff}} = 2\phi$, piercing the enlarged unit cell of the ground state. The chiral current j_c is an odd function of flux and periodic; thus, $j_c(\phi) = j_c(\phi - 2\pi) = -j_c(2\pi - \phi)$. As a consequence, for values of the flux $\pi/2 < \phi < \pi$ defined with respect to the original cell, the effective flux is in the domain $\phi_{\mathrm{eff}} \in (-\pi, 0)$ modulo 2π, which one would also obtain if the orientation of the magnetic field was inverted. Here, we show that such an increase of the unit cell arises as a consequence of the spontaneous breaking of discrete lattice-translation symmetry in vortex-lattice (VL) phases.

We study the single-band Bose-Hubbard model defined on a two-leg ladder lattice with flux ϕ per plaquette [see Fig. 2(a)]:

$$H = -J\sum_{\substack{r=1\\ \ell=1,2}}^{L-1}(a^\dagger_{\ell,r+1}a_{\ell,r} + \mathrm{H.c.}) + \frac{U}{2}\sum_{\substack{r=1\\ \ell=1,2}}^{L} n_{\ell,r}(n_{\ell,r} - 1) - J_\perp\sum_{r=1}^{L}(e^{-ir\phi}a^\dagger_{1,r}a_{2,r} + \mathrm{H.c.}). \tag{1}$$

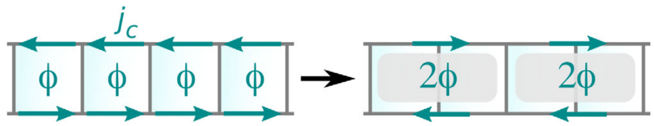

FIG. 1 (color online). A spontaneous doubling of the unit cell leads to an increase of the effective flux as well: $\phi \to 2\phi$. Under the conditions described in the text, the chiral current j_c (indicated by the arrows) reverses its direction.

$a^{\dagger}_{\ell,r}$ creates a boson on the lower ($\ell = 1$) or the upper site ($\ell = 2$) of the rth rung, $n_{\ell,r} = a^{\dagger}_{\ell,r} a_{\ell,r}$, and L is the number of rungs. The hopping matrix elements between the nearest-neighbor sites along the ladder's legs and rungs are denoted by J and $J_{\perp}$, respectively, and U is the repulsive on-site interaction. The filling is $\rho = N/(2L)$, where N is the number of bosons. We carried out density matrix renormalization group (DMRG) simulations [3,4] for $\rho < 2$ and $U \gtrsim J$. For the complementary regime of large densities $\rho \gg 1$ and weak interactions $U \ll J\rho$, we use a mapping to a frustrated XY model and apply a transfer-matrix approach [5,6] (see the Supplemental Material [7] for details on both techniques).

Physical realizations are, for instance, either arrays of Josephson junctions of superconducting islands in a magnetic field [18–20], where the bosonic particles are Cooper pairs of electrons with $q = 2e$, or neutral ultracold bosons in optical lattices [21,22] in the presence of artificial gauge fields, subject to s-wave interactions [23–25]. In the latter type of experiments, using either superlattices [23] or a synthetic lattice dimension [26–28] to realize ladders, chiral currents have recently been measured.

For the two-leg bosonic ladder, the existence of Meissner-like and vortex phases has been firmly established both in the weakly and strongly interacting regime [29–34]. In Fig. 2(b), obtained from a DMRG simulation, we depict the typical behavior of local currents and particle densities in the low-field Meissner phase. The currents are nonzero along the boundary of the two-leg ladder, consisting of the edge rungs and the legs of the ladder, and vanish quickly on the inner rungs away from the first and last rung, producing a net chiral current $j_c = (\sum_r \langle j^{\parallel}_{1,r} - j^{\parallel}_{2,r} \rangle + \langle j^{\perp}_{r=1} - j^{\perp}_{r=L} \rangle)/N$ (see Sec. S1 in Ref. [7] for definitions and properties of j_c). The coherence of the relative phase of bosons on the two legs induces the Meissner phase in bosonic ladders [5,29], and, hence, a chiral current can also emerge in the Mott-insulating regime for fillings with an integer number of particles per rung [30,33].

Upon increasing the flux beyond a critical value, the system enters into a vortex phase where local currents on the inner rungs develop, resulting in a current configuration that is reminiscent of a Meissner phase disordered with vortices [5,29,33]. These vortices interact repulsively with each other [5], yet for generic vortex densities ρ_v, they are distributed in the system without any periodicity. At certain commensurate vortex densities, VLs were predicted to form in the ground state [5,29]. DMRG results for such VL states are depicted in Figs. 2(c) and 2(d). In the VL superfluid (SF) at vortex density $\rho_v = 1/3$ [Fig. 2(c)], the currents on two neighboring plaquettes form a complex [surrounded by a dashed line in Fig. 2(c)] that is a mini copy of the Meissner phase (with "screening" currents circulating around the complex boundary and vanishing currents on its inner rung), such that on every third plaquette a vortex resides, around which the current circulates in the direction opposite to the behavior in the Meissner phase. In Fig. 2(c), the chiral current goes in the counterclockwise direction. In the VL at $\rho_v = 1/2$ ($\mathrm{VL}_{1/2}$ SF), obtained for $\phi \lesssim \pi$, a vortex sits on every other plaquette [Fig. 2(d)] and the direction of the chiral current is reversed. The observation of several stable VLs is a main result of this Letter.

For not too low particle densities and flux values close to π, we generically observe a reversal of the chiral current tuned by the interaction strength U/J: j_c flows in a counterclockwise direction along the ladder's legs for $U = 0$ and $U \to \infty$, yet for a certain range of interaction strengths at low temperatures the circulation direction of the current is reversed, as if the majority of particles swam against the tide.

The interaction-driven chiral-current reversal is evident from the DMRG results presented in Figs. 3(a) and 3(b). There, we plot the chiral current as a function of U/J for $\rho = 0.5$ and $\rho = 0.8$, respectively (see Ref. [7] for data for $\rho = 1$).

We clearly numerically resolve a current reversal, which is tied to the presence of the VL at $\rho_v = 1/2$. This state can be realized both in the superfluid ($\mathrm{VL}_{1/2}$ SF) and the Mott-insulating regime ($\mathrm{VL}_{1/2}$ MI). The $\mathrm{VL}_{1/2}$-SF phase is neighbored by a vortex superfluid (VSF) for small values of U/J. Generally, upon entering into the vortex-liquid phases from the VL phases, in the thermodynamic limit the absolute value $|j_c|$ of the current decreases continuously

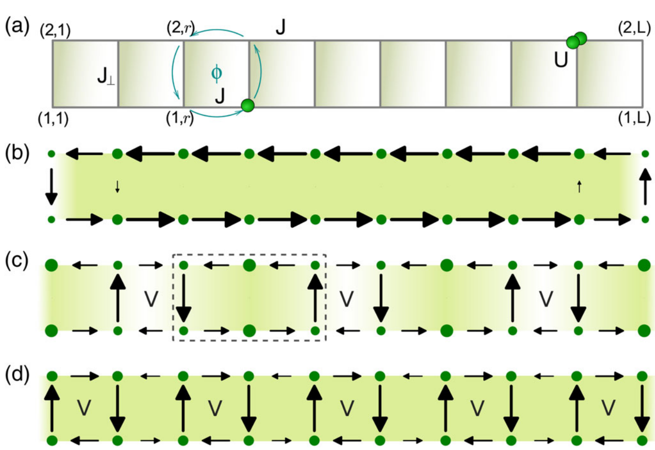

FIG. 2 (color online). (a) Sketch: Two-leg ladder with L rungs and a uniform flux ϕ per plaquette. (b)–(d) Local currents and on-site densities in (b) the Meissner phase, (c) the $\mathrm{VL}_{1/3}$ SF, and (d) the $\mathrm{VL}_{1/2}$ SF [$U = 2J$, $J_{\perp} = 1.6J$, $\rho = 0.8$, and (b) $\phi = 0.6\pi$, $L = 10$, (c) $\phi = 0.8\pi$, $L = 120$, and (d) $\phi = 0.9\pi$, $L = 120$]. The length of the arrows encodes the strength of local currents. The size of the circles and the intensity of the shading are proportional to the on-site density. In (c), a Meissner-like region is indicated by the dashed line. The chiral current is reversed in (d), but not in (b) and (c). The value of $J_{\perp}/J = 1.6$ was chosen since two VLs are realized for these parameters.

(while for finite L, small steps are seen [35]), showing a cusplike behavior at the phase boundary [7]. Other transitions that are crossed in Fig. 3 do not leave a fingerprint in the interaction dependence of j_c for $L \to \infty$. For $\rho = 0.5$, further increasing U/J takes the system from the $\mathrm{VL}_{1/2}$ SF into a $\mathrm{VL}_{1/2}$ MI. Typically, the sign of the chiral current becomes positive again on the large U/J side of the $\mathrm{VL}_{1/2}$ phases. The chiral-current reversal is thus robust against the presence or absence of a mass gap and a variation of density. Moreover, the absolute value of the reversed chiral current can exceed the $U = 0$ value.

The spontaneous symmetry breaking leading to the M-times enlarged unit cell with $\phi_{\mathrm{eff}} = M\phi \in (-\pi, 0)$ modulo 2π ($M = 2$ in Fig. 3) is a vital ingredient for obtaining the current reversal. We argue that yet another requirement is that the dominant contribution to the current comes from particles with a large wavelength experiencing the effective flux. This is exemplified by the behavior of the chiral current in the fully gapped $\mathrm{VL}_{1/2}$ MI [see Fig. 3(a)]: Inside this state with a doubled unit cell, the chiral current, as a function of U/J, passes through zero and returns back to the original direction of rotation. The localization length of bosons in the Mott-insulating phase becomes shorter with increasing U/J, restricting the typical wavelengths of particles, and as a consequence, less particles see the enlarged unit cell with doubled flux. On these grounds, we expect that, in a given phase with a spontaneously enlarged unit cell, the absolute value of the reversed chiral current attains its maximum for the smallest U/J in that phase, consistent with the data presented in Fig. 3. We emphasize that the optimal condition for observing the current reversal in two-leg ladders is a doubling of the unit cell.

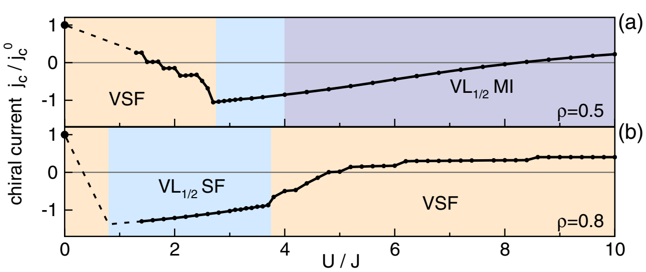

FIG. 3 (color online). j_c versus U/J in the proximity of the VLs at $\rho_v = 1/2$, for (a) $\rho = 0.5$ and (b) $\rho = 0.8$ (DMRG data, $L = 160$, $\phi = 0.9\pi$, $J_\perp = 1.6J$). The exact value for $U = 0$ is $j_c^0 \simeq 0.08J > 0$, independently of ρ. The small steps seen in the VSF phase are finite-size effects (see Sec. S2 in Ref. [7]). The dashed lines serve as a guide to the eye for small values of U/J in the VSF phase, where for large system sizes (such as $L = 160$), it is hard to converge DMRG with respect to both the local number of bosons and the DMRG state space [4,7]. The dashed lines connect the $U = 0$ value to the DMRG data point for the smallest value of U/J at which we have converged results (see Sec. S6 of Ref. [7]), plus we indicated the kink expected at the $\mathrm{VL}_{1/2}$-SF to VSF boundary, based on such data as shown in Fig. S7 of Ref. [7].

Interestingly, one could have deduced the existence of the chiral-current reversal from the flux dependence of the ground-state energy shown in Ref. [36] in the weak-coupling regime. From the bosonization analysis of Ref. [29], valid for $J_\perp \ll J$, one can also obtain the sign change. Yet, neither study actually discussed the reversal effect. DMRG results indicate that the chiral-current reversal driven by interactions also exists on three-leg ladders [37].

The chiral current can also change its sign in fermionic ladders [38,39] as a function of $\phi \in (0, \pi)$. Yet, for fermions, the effect exists already in the noninteracting case as a result of the band structure, while for bosons, the interaction-induced spontaneous flux increase is crucial for the current reversal.

To identify the regions in parameter space spanned by ϕ, U, ρ, and $J_\perp$, in which the VL phases exist, we present a representative ground-state phase diagram in Fig. 4 for $U/J = 2$ and $\rho = 0.8$ as a function of flux and $J_\perp$ (see Ref. [7] for a density versus U/J phase diagram). The figure illustrates the plethora of phases realized in the model: the Meissner superfluid, a VSF, a $\mathrm{VL}_{1/2}$ SF, a $\mathrm{VL}_{1/3}$ SF, and a biased ladder phase (BLP). The BLP, predicted in mean-field theory [31], spontaneously breaks the symmetry between the two legs by imbalancing the density [7]. The $\mathrm{VL}_{1/2}$ SF, which leads to the chiral-current reversal, is stable down to at least $J_\perp = J/2$.

Since the current reversal is connected with spontaneous symmetry breaking in the ground state, a relevant question pertains to the effect of temperature. To obtain a quantitative understanding of $T > 0$, we consider the limit of large particle densities $\rho \gg 1$ and weak interactions $U \ll J\rho$. Using the transfer-matrix approach [6], we obtain the results shown in Fig. 5, where the chiral current [Fig. 5(a)] and vortex density [Fig. 5(b)] versus flux are presented for different temperatures. Upon reducing temperature, VLs appear at $\rho_v = 1/2, 1/3, 2/5, 1/4, 1/5, \ldots$. These have an $M = 2, 3, 5, 4, 5, \ldots$-times enlarged unit cell, respectively [5,6,29,36]. At zero temperature, the vortex

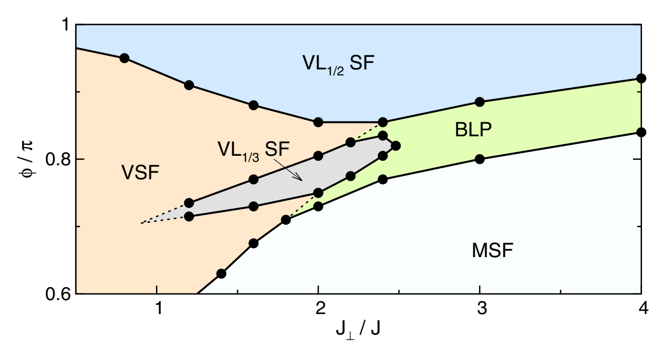

FIG. 4 (color online). ϕ versus $J_\perp/J$ phase diagram at $U/J = 2$ and $\rho = 0.8$. See the text for details.

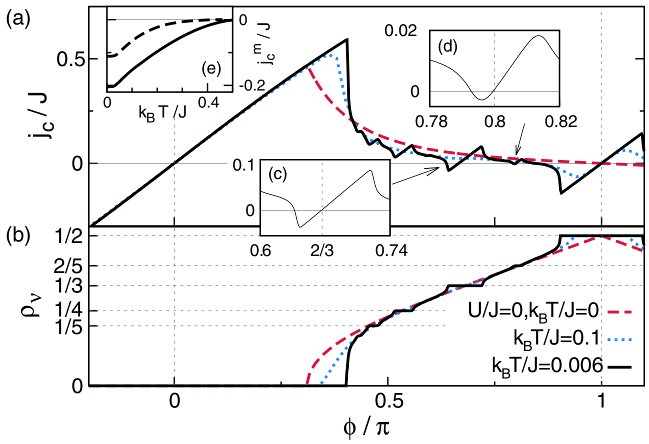

FIG. 5 (color online). (a) Chiral current j_c/J and (b) vortex density versus flux for $J_\perp = J/2$ for $\rho \gg 1$, $U \ll J\rho$. This value of $J_\perp$ is chosen since here, not only the VL at $\rho_v = 1/2$ exhibits a chiral-current reversal at low T, but also those at $\rho_v = 1/3$ and 2/5 [see insets (c) and (d)]. The temperature is $k_B T = 0.006J$ (continuous lines) and $k_B T = 0.1J$ (dotted lines). The dashed lines are for $U = 0$ at $T = 0$. (e) Largest negative value $j_c^m = \min_\phi j_c(\phi)$ of the reversed current for $J_\perp = 0.5J$ (dashed line) and $J_\perp = 1.2J$ (continuous line) as a function of $k_B T/J$. See Ref. [7], Fig. S6, for more data.

density, as a function of flux ϕ, exhibits the famous devil's staircase structure [36,40] with a plateau for each commensurate value of ρ_v. The reversal of the current in the $\mathrm{VL}_{1/2}$-SF phase clearly survives a finite temperature, which also applies (though for lower values of temperatures) to the current reversal in the vicinity of the $\mathrm{VL}_{1/3}$-SF state for $\phi < 2\pi/3$. Thus, the reversal is more stable against thermal fluctuations than the underlying VLs. The optimal parameters for observing the current reversal in the weak-coupling regime are $\phi \simeq 0.9\pi$ and $J_\perp \simeq 1.2J$, where the reversal occurs for $k_B T < J/2$ [7], as is shown in Fig. 5(e) as well as in Sec. S5 in Ref. [7].

At extremely low temperatures and for the parameters of Fig. 5, a reversal of the chiral current is also visible in the $\rho_v = 2/5$ VL state with $M = 5$ for $\phi \simeq 4\pi/5$ [see Fig. 5(d)]. Other VLs [corresponding to the regions in which plateaus are formed in the $\rho_v = \rho_v(\phi)$ curve in Fig. 5(b)] do not induce a reversal of the chiral current even at $T = 0$ since they are stable at values of fluxes for which $M\phi \in (0, \pi)$ modulo 2π.

A striking feature in the weak-coupling regime is the self-similar structure of the $j_c(\phi)$ curve for $k_B T \ll J$: After the spontaneous M-fold increase of the unit cell, the chiral current exhibits a behavior that is similar to the one in the Meissner phase. For instance, for the $\mathrm{VL}_{1/2}$, this is visible in Fig. 5(a) at a flux value of $\phi = \pi$ and for the VLs at $\rho_v = 1/3$ and 2/5 [see Figs. 5(c) and 5(d)] at $\phi = 2\pi/3$ and $4\pi/5$, respectively. As a result, the absolute values of the reversed current for $\phi \sim \pi$ are several times larger than in the $U = 0$ case at the same value of ϕ [compare the continuous black curve and the red dashed curve in Fig. 5(a) near $\phi = \pi$].

We studied the behavior of interacting bosonic particles in the presence of a uniform magnetic flux on a two-leg ladder. Our work has three main results. First, we determined the range of stability of VLs predicted in Refs. [5,29] and analyzed their microscopic structure. We obtained representative phase diagrams, showing that the system is very rich, also realizing the BLP phase [31]. Second, we observed the interaction-driven reversal of the chiral current in the vicinity of certain VLs. Third, we proposed an intuitive interpretation of the reversal via the spontaneous increase of the effective flux due to the enlarged unit cell in the VLs. We expect all of these results to influence experimental work on low-dimensional bosonic systems in the presence of gauge fields.

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