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ON p-CONFIDENCE INTERVALS FOR g-EXPECTATIONS

LEO SHEN AND RODNEY C. WOLFF

Abstract. This paper defines a g-variance and p-confidence interval for a g-expectation. We also determine the p-confidence intervals under some specific assumptions on g.

1. Introduction

The theory and applications of backward stochastic differential equations (BSDEs) have been developing rapidly. Linear BSDEs were introduced by Bismut [5]. Then the concept was generalized by Pardoux and Peng [16] to a fully non-linear setting. Risk measures can be defined using g-expectations which are non-linear expectations given by solutions of BSDEs depending on a driver, or drift function g ([17], [18]).

From the perspective of statistics, g-expectations are estimates of risks. Therefore, the corresponding confidence interval of the estimate should be given. The aim of this paper is first to define g-variance, then to define a confidence interval of a g-expectation, depending on g-variance and the function g. The confidence intervals for g satisfying certain properties are calculated.

We shall first review risk measures and g-expectations. In the following section, the g-variance and p-confidence interval of a g-expectation are defined. We then discuss different confidence intervals corresponding to specific functions g. In the final section, we summarize the results.

2. Risk Measures and Non-linear Expectations

Coherent risk measures were first defined by Artzner, Delbaen, Eber and Heath [1], and later extended by Delbaen [11]. Coherent risk measures satisfy a four-axiom framework: monotonicity, translation invariance, subadditivity and positive homogeneity. In this context, monotonicity means that if a portfolio is always valued higher than another portfolio, then the risk of this portfolio should be less than the risk of the other. Translation invariance implies that adding a constant amount of cash to a position reduces the risk of this position by the same amount. Under subadditivity, the risk of two portfolios together cannot be greater than the sum of the two risks computed separately. Positive homogeneity implies that if a portfolio is scaled by a positive factor, then its risk is changed by the same factor.

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Schied, Föllmer [20] and Frittelli, Rosazza Gianin [14] independently developed the idea of a convex risk measure as an extension of coherent risk measures. A convex risk measure satisfies the monotonicity and translation invariance assumptions of a coherent risk measure, along with the assumption of convexity, which guarantees that the diversification of a portfolio will not increase the risk.

Peng [17] introduced g-expectations using a class of BSDEs. Let $T \in [0,\infty)$ be a fixed time horizon, and let $(W_t)_{0\le t\le T}$ be a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. Suppose $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration generated by $(W_t)_{0\le t\le T}$, i.e., $\mathcal{F}_t = \sigma\{W_s; s\le t\}$. Let $L^2(\Omega,\mathcal{F}_t,\mathbb{P})$ with $0\le t\le T$ denote the space of all real-valued, $\mathcal{F}_t$-measurable random variables Q with $E_{\mathbb{P}}|Q|^2<\infty$, and $L^2(T;\mathbb{R}^d)$ denote the space of all $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted processes $(X_t)_{0\le t\le T}$ with $E_{\mathbb{P}}\int_0^T |X_s|^2 ds<\infty$.

General assumptions on g

Suppose $g:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ satisfies:

(A) g is Lipschitz in (y,z), that is, there exists a constant $C>0$ such that $\forall t\in[0,T]$ and $\forall (y_1,z_1),(y_2,z_2)\in\mathbb{R}\times\mathbb{R}^d$,

$$|g(t,y_1,z_1)-g(t,y_2,z_2)|\le C\left(|y_1-y_2|+|z_1-z_2|\right), \mathbb{P}-a.s.$$

(B) $g(\,\cdot\,,y,z)\in L^2(T;\mathbb{R}), \forall (y,z)\in\mathbb{R}\times\mathbb{R}^d$.

(C) $g(t,y,0)=0, \forall (t,y)\in[0,T]\times\mathbb{R}, \mathbb{P}-a.s.$

Throughout this paper, we assume g always satisfies these general assumptions.

Under assumptions (A) and (B), Pardoux and Peng [16] showed that for any $Q\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ and $t\in[0,T]$, the BSDE

$$Y_t-\int_t^T g(\omega,s,Y_s,Z_s)ds+\int_t^T Z_s dW_s = Q \tag{2.1}$$

has a unique solution pair $(Y_t,Z_t)\in L^2(T;\mathbb{R})\times L^2(T;\mathbb{R}^d)$.

Using the solution Y_t of Equation (2.1), g-expectations were defined by Peng [17].

Definition 2.1. Let (Y_t,Z_t) be the solution of Equation (2.1) with terminal condition $Q\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. The *$g$-expectation* $\mathcal{E}_g$ of Q is defined by

$$\mathcal{E}_g[Q]=Y_0,$$

while, for any $t\in[0,T]$, the *conditional g-expectation* of Q under $\mathcal{F}_t$ (denoted by $\mathcal{E}_g[Q|\mathcal{F}_t]$) is defined by

$$\mathcal{E}_g[Q|\mathcal{F}_t]=Y_t.$$

Now g-expectations and conditional g-expectations extend the notions of mathematical expectation and conditional mathematical expectation to a nonlinear framework. The conditional g-expectation $\mathcal{E}_g[\,\cdot\,|\mathcal{F}_t]$ possesses the following properties:

1. Time-consistency: $\mathcal{E}_g[\mathcal{E}_g[Q|\mathcal{F}_t]|\mathcal{F}_s]=\mathcal{E}_g[Q|\mathcal{F}_s]$, $s\in[0,t]$, $Q\in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$.
2. Constant-preserving: $\mathcal{E}_g[Q|\mathcal{F}_t]=Q$, $Q\in L^2(\Omega,\mathcal{F}_t,\mathbb{P})$.
3. Zero-one law: If $g(\cdot,0,0)=0$, then $\mathbf{1}_A\mathcal{E}_g[Q|\mathcal{F}_t]=\mathcal{E}_g[\mathbf{1}_AQ|\mathcal{F}_t]$, $Q\in L^2(\Omega,\mathcal{F}_T,\mathbb{P}), A\in\mathcal{F}_t$.

(4) Translation invariance: If g is independent of Y, then $\mathcal{E}_g[Q+q|\mathcal{F}_t] = \mathcal{E}_g[Q|\mathcal{F}_t]+q, Q \in L^2(\Omega,\mathcal{F}_T,\mathbb{P}), q \in L^2(\Omega,\mathcal{F}_t,\mathbb{P})$.

(5) Strict monotonicity: For $Q, Q' \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ with $Q' \leq Q$, $\mathcal{E}_g[Q'|\mathcal{F}_t] \leq \mathcal{E}_g[Q|\mathcal{F}_t]$ with equality only if $Q = Q'$.

(6) If g is convex, then $\mathcal{E}_g[\,\cdot\,|\mathcal{F}_t]$ is a convex operator on $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ for any $t \in [0,T]$.

(7) If g is positively homogeneous, then $\mathcal{E}_g[\,\cdot\,|\mathcal{F}_t]$ is a positively homogeneous operator on $L^2(\Omega,\mathcal{F}_T,\mathbb{P})$ for any $t \in [0,T]$.

These results follow from the uniqueness of solutions of BSDE (2.1) ([16], [17], [18]).

3. g-variance and the p-confidence Interval

From Definition 2.1, for any $Q \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$, the g-expectation $\mathcal{E}_g[Q]$ of Q is a constant. Suppose $(Q-\mathcal{E}_g[Q])^2 \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. Similarly, for $t \in [0,T]$, the BSDE

$$Y'_t - \int_t^T g(\omega,s,Y'_s,Z'_s)ds + \int_t^T Z'_s dW_s = (Q-\mathcal{E}_g[Q])^2 \tag{3.1}$$

has a unique solution pair $(Y'_t, Z'_t) \in L^2(T;\mathbb{R}) \times L^2(T;\mathbb{R}^d)$.

Definition 3.1. Suppose (Y'_t, Z'_t) is the solution of Equation 3.1 with terminal condition $(Q-\mathcal{E}_g[Q])^2 \in L^2(\Omega,\mathcal{F}_T,\mathbb{P})$. The *$g$-variance* $\mathcal{V}_g$ of Q is defined as

$$\mathcal{V}_g[Q] = Y'_0.$$

By the Comparison Theorem in [17], for any terminal condition Q, its g-variance $\mathcal{V}_g[Q]$ is always non-negative.

Definition 3.2. Given a constant p $(0 < p < 1)$, if there exists $\epsilon > 0$, such that $\mathbb{P}(|Q-\mathcal{E}_g[Q]| \leq \epsilon) \geq p$, then the interval $[\mathcal{E}_g[Q]-\epsilon, \mathcal{E}_g[Q]+\epsilon]$ is called the *p-confidence interval* of Q with respect to its g-expectation $\mathcal{E}_g[Q]$.

In the sequel, for different assumptions on g, we shall obtain the corresponding p-confidence interval of Q, which is related to the driver g and the g-covariance.

3.1. General Assumptions on g.

Theorem 3.3. *Suppose (Y'_t, Z'_t) is the solution of BSDE* (3.1)*, $\mathcal{V}_g[Q]$ is the g-variance of Q. Given a constant p $(0 < p < 1)$, for general assumptions on g, there exists*

$$\epsilon = \sqrt{\frac{\mathcal{V}_g[Q] + C\int_0^T \mathbb{E}[|Z'_s|]ds}{1-p}},$$

where $C(>0)$ is the Lipschitz constant of the function g, such that $\mathbb{P}(|Q-\mathcal{E}_g[Q]| \leq \epsilon) \geq p$.

Proof. We shall prove $\mathbb{P}(|Q-\mathcal{E}_g[Q]| \leq \epsilon) \geq p$ as long as we prove

$$\mathbb{P}(|Q-\mathcal{E}_g[Q]| > \epsilon) \leq 1-p.$$

By Chebyshev's inequality,

$$\mathbb{P}(|Q-\mathcal{E}_g[Q]| > \epsilon) \leq \frac{\mathbb{E}[(Q-\mathcal{E}_g[Q])^2]}{\epsilon^2}.$$

When $t = 0$, taking expectation on both sides of Equation (3.1), we obtain

$$\mathbb{E}[(Q - \mathcal{E}_g[Q])^2] = \mathcal{V}_g[Q] - \int_0^T \mathbb{E}[g(\omega, s, Y'_s, Z'_s)]ds. \tag{3.2}$$

Therefore,

$$\frac{\mathbb{E}[(Q - \mathcal{E}_g[Q])^2]}{\epsilon^2} = (1-p)\frac{\mathcal{V}_g[Q] - \int_0^T \mathbb{E}[g(\omega, s, Y'_s, Z'_s)]ds}{\mathcal{V}_g[Q] + C\int_0^T \mathbb{E}[|Z'_s|]ds}.$$

From Assumptions (A) and (C) on g, we have

$$|g(t, y, z)| \leq C|z|.$$

Therefore,

$$\mathcal{V}_g[Q] - \int_0^T \mathbb{E}[g(\omega, s, Y'_s, Z'_s)]ds \leq \mathcal{V}_g[Q] + C\int_0^T \mathbb{E}[|Z'_s|]ds.$$

Since $\mathcal{V}_g[Q]$ is non-negative, then

$$\frac{\mathcal{V}_g[Q] - \int_0^T \mathbb{E}[g(\omega, s, Y'_s, Z'_s)]ds}{\mathcal{V}_g[Q] + C\int_0^T \mathbb{E}[|Z'_s|]ds} \leq 1.$$

Finally, we have

$$\mathbb{P}(|Q - \mathcal{E}_g[Q]| > \epsilon) \leq \frac{\mathbb{E}[(Q - \mathcal{E}_g[Q])^2]}{\epsilon^2} \leq 1 - p.$$

□

Given a constant p $(0 < p < 1)$, for general assumptions on g, the interval

$$\left[\mathcal{E}_g[Q] - \sqrt{\frac{\mathcal{V}_g[Q] + C\int_0^T \mathbb{E}[|Z'_s|]ds}{1-p}},\ \mathcal{E}_g[Q] + \sqrt{\frac{\mathcal{V}_g[Q] + C\int_0^T \mathbb{E}[|Z'_s|]ds}{1-p}}\right]$$

is the p-confidence interval of Q with respect to its g-expectation $\mathcal{E}_g[Q]$.

3.2. Positively Homogeneous Assumption on g. (D) Suppose now g is positively homogeneous in (y, z), that is, for all $t \in [0, T], (y, z) \in \mathbb{R} \times \mathbb{R}^d, \lambda \geq 0$,

$$\lambda g(t, y, z) = g(t, \lambda y, \lambda z), \mathbb{P} - a.s.$$

Then given the terminal condition $\frac{(Q-\mathcal{E}_g[Q])^2}{1-p} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, the BSDE

$$Y_t^* - \int_t^T g(\omega, s, Y_s^*, Z_s^*)ds + \int_t^T Z_s^* dW_s = \frac{(Q - \mathcal{E}_g[Q])^2}{1-p} \tag{3.3}$$

has a unique solution pair $(Y_t^*, Z_t^*) \in L^2(T; \mathbb{R}) \times L^2(T; \mathbb{R}^d)$.

Since g is positively homogeneous in (y, z) and $1 - p > 0$, we can rearrange the above BSDE (3.3) as

$$(1-p)Y_t^* - \int_t^T g(\omega, s, (1-p)Y_s^*, (1-p)Z_s^*)ds + \int_t^T (1-p)Z_s^* dW_s = (Q - \mathcal{E}_g[Q])^2.$$

Hence the pair $((1-p)Y_t^*, (1-p)Z_t^*)$ is the solution of BSDE (3.1). Since the solution of BSDE (3.1) is unique, it follows that $Y_t' = (1-p)Y_t^*, Z_t' = (1-p)Z_t^*$ for any $t \in [0,T]$.

Theorem 3.4. *Given a constant p $(0 < p < 1)$, for positively homogeneous assumption on g, there exists*

$$\epsilon = \sqrt{\frac{\mathcal{V}_g[Q]}{1-p} - \int_0^T \mathbb{E}\left[g\left(\omega, s, \frac{Y_s'}{1-p}, \frac{Z_s'}{1-p}\right)\right] ds},$$

such that $\mathbb{P}(|Q - \mathcal{E}_g[Q]| \leq \epsilon) \geq p$.

Proof. By Chebyshev's inequality, we shall prove $\mathbb{P}(|Q - \mathcal{E}_g[Q]| > \epsilon) \leq 1-p$ as long as we prove

$$\frac{\mathbb{E}[(Q - \mathcal{E}_g[Q])^2]}{1-p} = \epsilon^2.$$

When $t = 0$, taking expectation on both sides of Equation (3.3), we obtain

$$\mathbb{E}\left[\frac{(Q - \mathcal{E}_g[Q])^2}{1-p}\right] = Y_0^* - \int_0^T \mathbb{E}[g(\omega, s, Y_s^*, Z_s^*)] ds. \tag{3.4}$$

Since $Y_t' = (1-p)Y_t^*, Z_t' = (1-p)Z_t^*$ for any $t \in [0,T]$, then

$$\mathbb{E}\left[\frac{(Q - \mathcal{E}_g[Q])^2}{1-p}\right] = \frac{\mathcal{V}_g[Q]}{1-p} - \int_0^T \mathbb{E}\left[g\left(\omega, s, \frac{Y_s'}{1-p}, \frac{Z_s'}{1-p}\right)\right] ds.$$

Therefore,

$$\frac{\mathbb{E}[(Q - \mathcal{E}_g[Q])^2]}{1-p} = \epsilon^2.$$

□

Given a constant p $(0 < p < 1)$, for positively homogeneous assumption on g, the interval $[\mathcal{E}_g[Q] - \epsilon, \mathcal{E}_g[Q] + \epsilon]$ is the p-confidence interval of Q with respect to its g-expectation $\mathcal{E}_g[Q]$, where

$$\epsilon = \sqrt{\frac{\mathcal{V}_g[Q]}{1-p} - \int_0^T \mathbb{E}\left[g\left(\omega, s, \frac{Y_s'}{1-p}, \frac{Z_s'}{1-p}\right)\right] ds}.$$

3.3. Subadditive and Convex Assumptions on g. (E) Suppose now g is subadditive in (y, z), that is, for all $t \in [0,T], (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$g(t, y_1 + y_2, z_1 + z_2) \leq g(t, y_1, z_1) + g(t, y_2, z_2), \mathbb{P} - a.s.$$

(F) Further, suppose g is convex in (y, z), that is, for all $t \in [0,T], (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d, \alpha \in (0,1)$,

$$g(t, \alpha y_1 + (1-\alpha)y_2, \alpha z_1 + (1-\alpha)z_2) \leq \alpha g(t, y_1, z_1) + (1-\alpha)g(t, y_2, z_2), \mathbb{P} - a.s.$$

Consequently, if g is subadditive and convex, then $\mathcal{E}_g[\cdot]$ is also subadditive and convex (see [19]).

Theorem 3.5. *Given a constant p $(0 < p < 1)$, for subadditive and convex assumptions on g, there exists*

$$\epsilon = \sqrt{\frac{1+p}{1-p}\mathcal{V}_g[Q] - \frac{1}{1-p}\int_0^T \mathbb{E}[g(\omega, s, Y'_s, Z'_s)]ds},$$

such that $\mathbb{P}(|Q - \mathcal{E}_g[Q]| \le \epsilon) \ge p$.

Proof. Again, by Chebyshev's inequality, we shall prove $\mathbb{P}(|Q-\mathcal{E}_g[Q]| > \epsilon) \le 1-p$ if we prove

$$\frac{\mathbb{E}[(Q-\mathcal{E}_g[Q])^2]}{1-p} \le \epsilon^2.$$

Given the terminal condition $\frac{(Q-\mathcal{E}_g[Q])^2}{1-p} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, the BSDE

$$\widehat{Y}_t - \int_t^T g(\omega, s, \widehat{Y}_s, \widehat{Z}_s)ds + \int_t^T \widehat{Z}_s dW_s = \frac{(Q-\mathcal{E}_g[Q])^2}{1-p} \tag{3.5}$$

has a unique solution pair $(\widehat{Y}_t, \widehat{Z}_t) \in L^2(T;\mathbb{R}) \times L^2(T;\mathbb{R}^d)$.

When $t = 0$, taking expectation on both sides of Equation (3.5), we obtain

$$\mathbb{E}\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] = \mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] - \int_0^T \mathbb{E}[g(\omega, s, \widehat{Y}_s, \widehat{Z}_s)]ds. \tag{3.6}$$

Since

$$\frac{(Q-\mathcal{E}_g[Q])^2}{1-p} = (1-p)(Q-\mathcal{E}_g[Q])^2 + p\left(\frac{2-p}{1-p}(Q-\mathcal{E}_g[Q])^2\right),$$

then by the convexity of $\mathcal{E}_g[\cdot]$, we have

$$\begin{aligned}\mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] &= \mathcal{E}_g\left[(1-p)(Q-\mathcal{E}_g[Q])^2 + p\left(\frac{2-p}{1-p}(Q-\mathcal{E}_g[Q])^2\right)\right] \\ &\le (1-p)\mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2] + p\mathcal{E}_g\left[\frac{2-p}{1-p}(Q-\mathcal{E}_g[Q])^2\right]. \end{aligned} \tag{3.7}$$

Since

$$\mathcal{E}_g\left[\frac{2-p}{1-p}(Q-\mathcal{E}_g[Q])^2\right] = \mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p} + (Q-\mathcal{E}_g[Q])^2\right],$$

then by the subaddtivity of $\mathcal{E}_g[\cdot]$, we have

$$\mathcal{E}_g\left[\frac{2-p}{1-p}(Q-\mathcal{E}_g[Q])^2\right] \le \mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] + \mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2], \tag{3.8}$$

therefore, by (3.7) and (3.8),

$$\begin{aligned}\mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] &\le (1-p)\mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2] \\ &\quad + p\left\{\mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] + \mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2]\right\}. \end{aligned} \tag{3.9}$$

We can rearrange the above inequality (3.9) as

$$\mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] \le \frac{1}{1-p}\mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2].$$

By (3.6),

$$\mathbb{E}\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] \le \frac{1}{1-p}\mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2] - \int_0^T \mathbb{E}[g(\omega,s,\widehat{Y}_s,\widehat{Z}_s)]ds. \qquad (3.10)$$

Since

$$\frac{(Q-\mathcal{E}_g[Q])^2}{1-p} \ge (Q-\mathcal{E}_g[Q])^2, \mathbb{P}-a.s.,$$

by the Comparison Theorem in [17], we have

$$\mathcal{E}_g\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] \ge \mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2].$$

From (3.2) and (3.6), we have

$$\begin{aligned}\mathbb{E}\left[\frac{(Q-\mathcal{E}_g[Q])^2}{1-p}\right] &+ \int_0^T \mathbb{E}[g(\omega,s,\widehat{Y}_s,\widehat{Z}_s)]ds \\ &\ge \mathbb{E}[(Q-\mathcal{E}_g[Q])^2] + \int_0^T \mathbb{E}[g(\omega,s,Y'_s,Z'_s)]ds.\end{aligned}$$

Then

$$\int_0^T \mathbb{E}[g(\omega,s,\widehat{Y}_s,\widehat{Z}_s)]ds \ge \int_0^T \mathbb{E}[g(\omega,s,Y'_s,Z'_s)]ds - \frac{p}{1-p}\mathbb{E}[(Q-\mathcal{E}_g[Q])^2].$$

From (3.2), we have

$$\int_0^T \mathbb{E}[g(\omega,s,\widehat{Y}_s,\widehat{Z}_s)]ds \ge \frac{1}{1-p}\int_0^T \mathbb{E}[g(\omega,s,Y'_s,Z'_s)]ds - \frac{p}{1-p}\mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2].$$

By (3.10), we see

$$\frac{\mathbb{E}[(Q-\mathcal{E}_g[Q])^2]}{1-p} \le \frac{1+p}{1-p}\mathcal{E}_g[(Q-\mathcal{E}_g[Q])^2] - \frac{1}{1-p}\int_0^T \mathbb{E}[g(\omega,s,Y'_s,Z'_s)]ds.$$

Therefore,

$$\frac{\mathbb{E}[(Q-\mathcal{E}_g[Q])^2]}{1-p} \le \epsilon^2.$$

□

Given a constant p $(0<p<1)$, for subadditive and convex assumptions on g, the interval $[\mathcal{E}_g[Q]-\epsilon, \mathcal{E}_g[Q]+\epsilon]$ is the p-confidence interval of Q with respect to its g-expectation $\mathcal{E}_g[Q]$, where

$$\epsilon = \sqrt{\frac{1+p}{1-p}\mathcal{V}_g[Q] - \frac{1}{1-p}\int_0^T \mathbb{E}[g(\omega,s,Y'_s,Z'_s)]ds}.$$

4. Conclusion

In this paper, we have defined g-variance and p-confidence intervals for the g-expectation. The p-confidence interval for some forms of the driver g has been found. From Theorem 3.4 and Theorem 3.5, it is clear that the confidence interval under the subadditive and convex assumptions is wider than the one under the positively homogeneous assumption.

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Leo Shen: 12 Narona St, Middle Park, QLD 4074, AUSTRALIA
E-mail address: leoshen.au@gmail.com

Rodney C. Wolff: WH Bryan Mining and Geology Research Centre, Sustainable Minerals Institute, The University of Queensland, Brisbane, QLD 4072, AUSTRALIA
E-mail address: rodney.wolff@uq.edu.au