Accepted Manuscript

Decomposing Profit Efficiency using a Slack-based Directional Distance Function

Rolf Färe, Hirofumi Fukuyama, Shawna Grosskopf, Valentin Zelenyuk

PII: S0377-2217(15)00437-3
DOI: 10.1016/j.ejor.2015.05.047
Reference: EOR 12971

To appear in: *European Journal of Operational Research*

Received date: 30 September 2014
Revised date: 14 May 2015
Accepted date: 18 May 2015

Please cite this article as: Rolf Färe, Hirofumi Fukuyama, Shawna Grosskopf, Valentin Zelenyuk, Decomposing Profit Efficiency using a Slack-based Directional Distance Function, *European Journal of Operational Research* (2015), doi: 10.1016/j.ejor.2015.05.047

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

ACCEPTED MANUSCRIPT

Highlights

- We follow up on recent work of Cooper, Pastor, Aparicio and Borras (2011, EJOR).
- We operate with slack-based directional distance function.
- We develop a slack-based decomposition of profit efficiency

ACCEPTED MANUSCRIPT

ACCEPTED MANUSCRIPT

Decomposing Profit Efficiency using a Slack-based Directional Distance Function

Rolf Färe[*], Hirofumi Fukuyama[†], Shawna Grosskopf[‡], Valentin Zelenyuk[§]

May 22, 2015

Abstract

This paper develops a slack-based decomposition of profit efficiency based on a directional distance function. It is an alternative to Cooper, Pastor, Aparicio and Borras (2011).

Keywords: Directional distance function, Slacks, Data envelopment analysis.

JEL Classification: D24.

ACCEPTED MANUSCRIPT

[*]Department of Economics, Oregon State University and Department of Agricultural and Resource Economics, University of Maryland, USA

[†]Faculty of Commerce, Fukuoka University, Japan

[‡]Department of Economics, Oregon State University, USA and Center for Environmental and Resource Economics (CERE), Sweden

[§]School of Economics and Centre for Efficiency and Productivity Analysis (CEPA) at The University of Queensland, Australia; *Address of the corresponding author*: 530, Colin Clark Building (39), St Lucia, Brisbane, Qld 4072, AUSTRALIA; e-mail: v.zelenyuk@uq.edu.au; tel: + 61 7 3346 7054;

ACCEPTED MANUSCRIPT

1 Introduction

Chambers, Chung and Färe (1998) introduced an additive decomposition of profit efficiency based on the directional technology distance function.[1] Under suitable regularity conditions on technology, this distance function has the indication property that it equals zero if and only if the input-output vector belongs to the g-isoquant (see Färe and Grosskopf, 2010). A Leontief technology is an example for which the isoquant may be larger than the Pareto-Koopmans efficiency set. Thus the Chambers et al decomposition does not account for slacks.

This observation has prompted Cooper, Pastor, Aparicio and Borras (2011) to "... show that the usual measure of profit inefficiency can be decomposed into its technical component, measured through a weighted additive model, and its allocative component derived as a residual. Our findings are all based on Fenchel-Mahler inequality using duality theory (see Färe and Grosskopf, 2000)" p.411.

In this paper we introduce an alternative model to Cooper et al. (2011) using a slack-based directional technology distance function introduced by Färe and Grosskopf (2010). We combine this function with the fact that the profit maximum is at least as large as the profit from any feasible input-output vector.

In the next section, after discussing profit-maximization, we recall the Chambers et al. (1998) decomposition. Then we introduce our new decomposition of profit efficiency using a slack-based directional technology distance function.

2 The Main Results

Denote inputs by a column vector $x = (x_1, \ldots, x_N)' \in \mathbb{R}^N_+$ and outputs by a column vector $y = (y_1, \ldots, y_M)' \in \mathbb{R}^M_+$; the production technology is given by

$$T = \{(x, y) : x \text{ can produce } y\} \tag{1}$$

[1] Luenberger introduced this function as a "shortage function", see e.g., Luenberger (1995).

ACCEPTED MANUSCRIPT

ACCEPTED MANUSCRIPT

which we assume satisfies standard regularity conditions, and strong disposability of inputs and outputs in particular.[2]

Let $w = (w_1, \ldots, w_N) \in \mathbb{R}^N_+$ be a row vector of input prices and $p = (p_1, \ldots, p_M) \in \mathbb{R}^M_+$, be a row vector of output prices. The profit function is defined as[3]

$$\pi(p, w) \quad = \quad \max_{(x,y)}\{py - wx \ : \ (x, y) \in T\}. \tag{2}$$

This definition implies that

$$\pi(p, w) \quad \geqq \quad py - wx, \ \forall (x, y) \in T \tag{3}$$

which is the first of our building blocks.

Next, introduce the directional vector

$$g \quad = \quad (g_x, g_y) \in \mathbb{R}^{N+M}_+ \ : \ g \neq 0.$$

This vector determines the direction in which the input-output vector is projected onto the boundary of T.

To recall the Chambers et al. (1998) decomposition, we define the directional technology distance function

$$\overrightarrow{D_T}(x, y; g) \quad = \quad \max_{\beta}\{\beta \in \mathbb{R}_+ \ : \ (x - \beta g_x, y + \beta g_y) \in T\}. \tag{4}$$

This function is defined to contract inputs and expand outputs, as in profit maximization, except that the optimization for it is done along the chosen direction $g = (g_x, g_y)$. In

[2]See Färe and Primont (1995) for an axiomatic discussion of T.
[3]See Färe and Primont (1995) for details about existence and other properties of this function.

ACCEPTED MANUSCRIPT

particular, note that

$$(x - \beta^* g_x, y + \beta^* g_y) \quad \in \quad T$$

where β^* is the optimizer in (4), i.e., $\overrightarrow{D_T}(x, y; g)$. It is important to note that the directional vector has units of measurement, which makes the scalar β and the directional distance function independent of units of measurement.

Combining this with the profit inequality yields

$$\begin{aligned} \pi(p, w) &\geqq p(y + \beta^* g_y) - w(x - \beta^* g_x),\ \forall (x, y) \in T \\ &= (py - wx) + \beta^*(pg_y + wg_x) \end{aligned} \tag{5}$$

or, for $(pg_y + wg_x) \neq 0$, we have

$$\frac{\pi(p, w) - (py - wx)}{(pg_y + wg_x)} \geqq \overrightarrow{D_T}(x, y; g),\ \forall (x, y) \in T \tag{6}$$

and adding allocative efficiency to the right hand side closes the inequality. This is the Chambers et al. decomposition.

Note that if all prices (on the left hand side of (6)) are scaled by $\lambda > 0$, the efficiency score does not change, thus we may use US Dollar or Swedish kroner as the unit and the results will be the same. Again, this decomposition would result in slacks being captured by the residual rather than by technical inefficiency.

Following Färe and Grosskopf (2010), the slack-based directional technology distance function, when the direction vector is chosen to be the unit vector, i.e., $g = \mathbf{1} = (1, ..., 1)$, is

$$\begin{aligned} \overrightarrow{SBD_T}(x, y; \mathbf{1}) = \max\{&\sum_{n=1}^{N} \beta_n + \sum_{m=1}^{M} \gamma_m : \\ &(x_1 - \beta_1 \cdot 1, ..., x_N - \beta_N \cdot 1,\ y_1 + \gamma_1 \cdot 1, ..., y_M + \gamma_M \cdot 1) \in T\} \end{aligned} \tag{7}$$

where the optimization is done over $B = (\beta_1, ..., \beta_N) \geqq 0$, $\Gamma = (\gamma_1, ..., \gamma_M) \geqq 0$.[4] It is worth noting that the g vector here equals one unit for each element,[5] and so has a unit measure, making β's and γ's independent of units of measurement and hence they can be summed.[6]

Let $B^* = (\beta_1^*, ..., \beta_N^*)$, $\Gamma^* = (\gamma_1^*, ..., \gamma_M^*)$ be the optimizers in (7), then under standard regularity conditions on technology set T with strong disposability of inputs and outputs, $\overrightarrow{SBD_T}(x, y; \mathbf{1}) = 0$ if and only if $(\beta_1^*, ..., \beta_N^*, \gamma_1^*, ..., \gamma_M^*) = \mathbf{0}$. Thus, if $\overrightarrow{SBD_T}(x, y; \mathbf{1}) = 0$ we say that each x_n and y_m is technically efficient, i.e., (x, y) is on the (Pareto-Koopmans) efficient frontier of the technology set T.

In the appendix we show that $\overrightarrow{SBD_T}(x, y; \mathbf{1})$ is independent of units of measurement. The proof also shows that each component in$(\beta_1^*, ..., \beta_N^*, \gamma_1^*, ..., \gamma_M^*)$ is unit independent.

Combining $\overrightarrow{SBD_T}(x, y; \mathbf{1})$ with the profit inefficiency, yields

$$\pi(p, w) \geqq \sum_{m=1}^{M} p_m y_m + \sum_{m=1}^{M} p_m \gamma_m^* \cdot 1 - \sum_{n=1}^{N} w_n x_n + \sum_{n=1}^{N} w_n \beta_n^* \cdot 1, \forall (x, y) \in T \tag{8}$$

From equation (8) it follows that

$$\pi(p, w) - (py - wx) \geqq \sum_{m=1}^{M} p_m \gamma_m^* \cdot 1 + \sum_{n=1}^{N} w_n \beta_n^* \cdot 1, \ \forall (x, y) \in T \tag{9}$$

As in the Chambers et al. case, we can normalize with the value of the directional vector, which is $g = \mathbf{1}$ here, i.e., $(\sum_{m=1}^{M} p_m \cdot 1 + \sum_{n=1}^{N} w_n \cdot 1)$, which yields

---

[4]In this paper we focus on efficiency measurement, so we consider only points that belong to the technology set and this requires the non-negativity constraint, to ensure that the reference point on the frontier is obtained by not increasing any of the inputs (i.e., contracting or keeping constant) and by not decreasing any outputs (i.e., expanding or keeping constant). It is possible to extend this measure to the case when a point does not belong to the technology set (e.g., for measuring productivity changes), and for such points the non-negativity constraints should be changed to a non-positivity constraint for analogous reasons, in addition to feasibility of the solution assumption. We thank an anonymous referee for this remark and leave further details for future work.

[5]To clarify the notation further, note that we explicitly write the multiplication by 1 to emphasize that in each instance it is multiplication by the unit of measurement corresponding to each different input or output, e.g., the first element in **1** could stand for 1 litre of petrol, the second could stand for 1 hour of labour, the last two could stand for 1 kg of apples and 1 kg of oranges, respectively, etc. Also, see appendix for the proof of independence from units of measurement of this measure.

[6]Note that if $g_x = x$ and $g_y = y$ then we have a Russell-type efficiency measure defined on T. See Färe, Fukuyama, Grosskopf and Zelenyuk (2015) for a related discussion.

ACCEPTED MANUSCRIPT

$$\begin{aligned}\frac{\pi(p,w)-(py-wx)}{(\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1)} &\geqq \sum_{m=1}^{M}\frac{p_m\gamma_m^*\cdot 1}{\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1} \\ &+ \sum_{n=1}^{N}\frac{w_n\beta_n^*\cdot 1}{\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1}, \forall(x,y)\in T\end{aligned} \tag{10}$$

Define the price-share weights $s_m = \frac{p_m\cdot 1}{\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1}$, $m = 1, ..., M$ and $s_n = \frac{w_n\cdot 1}{\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1}$, $n = 1, ..., N$, then $s_m \geqq 0$ and $s_n \geqq 0$ for all $m = 1, ..., M$ and $n = 1, ..., N$ and $\sum_{m=1}^{M} s_m + \sum_{n=1}^{N} s_n = 1$. With these weights, we have

$$\frac{\pi(p,w)-(py-wx)}{(\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1)} \geqq \sum_{m=1}^{M} s_m\gamma_m^* + \sum_{n=1}^{N} s_n\beta_n^*,\ \forall(x,y)\in T \tag{11}$$

as the basis of our decomposition, with allocative inefficiency defined as the residual.

Next, note that we should have $\sum_{m=1}^{M} p_m\gamma_m^* + \sum_{n=1}^{N} w_n\beta_n^* > 0$ if and only if $\overrightarrow{SBD_T}(x,y;\mathbf{1}) = \sum_{m=1}^{M}\gamma_m^* + \sum_{n=1}^{N}\beta_n^* > 0$, because $(\beta_1^*, ..., \beta_N^*, \gamma_1^*, ..., \gamma_M^*) \geqq 0$. So, for the case when $\overrightarrow{SBD_T}(x,y;\mathbf{1}) > 0$ we can multiply and divide the right hand side of (11) with $\overrightarrow{SBD_T}(x,y;\mathbf{1})$, to get

$$\begin{aligned}\frac{\pi(p,w)-(py-wx)}{(\sum_{m=1}^{M}p_m\cdot 1+\sum_{n=1}^{N}w_n\cdot 1)} &\geqq \overrightarrow{SBD_T}(x,y;\mathbf{1}) \\ &\times \Big(\sum_{m=1}^{M} s_m\frac{\gamma_m^*}{\sum_{m=1}^{M}\gamma_m^*+\sum_{n=1}^{N}\beta_n^*} \\ &+ \sum_{n=1}^{N} s_n\frac{\beta_n^*}{\sum_{m=1}^{M}\gamma_m^*+\sum_{n=1}^{N}\beta_n^*}\Big), \forall(x,y)\in T\end{aligned} \tag{12}$$

where the right hand side is a measure of technical efficiency. The new decomposition follows from adding an allocative inefficiency term.[7]

Obviousely, (10), (11) and (12) are equivalent, however, (12) allows us to identify the contribution of each input and output as a share of the total technical inefficiency.[8]

[7]Note that the left hand side of (12) is the ratio of profit differences and the value of the directional vector, hence it is independent of the units of measurement.

[8]Also note that the same function ($\overrightarrow{SBD_T}$) appears both inside and outside the parentheses of the decomposition on the right hand side of (12). Such mathematical rearrangement is commonly used in other

ACCEPTED MANUSCRIPT

ACCEPTED MANUSCRIPT

To take a closer look at the results, note that the right hand side of (12) is a product of two parts, the $\overrightarrow{SBD_T}(x, y; \mathbf{1})$ itself and a share weighted sum of various optimal values from (7). Let us concentrate on the m^{th} term of the second part of this formula. This term can be seen as: (i) the m^{th} output inefficiency $\frac{\gamma_m^*}{\sum_{m=1}^{M}\gamma_m^* + \sum_{n=1}^{N}\beta_n^*}$ and (ii) weights $s_m = \frac{p_m \cdot 1}{(\sum_{m=1}^{M} p_m \cdot 1 + \sum_{n=1}^{N} w_n \cdot 1)}$. That is, the expression in the parentheses consists of the share weighted sum (share of the value of the directional vector) of the individual input and output inefficiencies. Note that we have $\sum_{i=1}^{N+M} s_i = 1$ and $s_i \geqq 0$. Incidentally, also note that if $\gamma_m = \beta_n = \beta$ for $n = 1, ..., N$ and $m = 1, ..., M$, then the Chambers et al. decomposition is obtained as a special case. Also note that this decomposition is new to the literature and it is possible to adapt it to produce a similar decomposition for the Russell-based directional distance function (see Fukuyama and Weber (2009)).

Finally, a natural approach to estimate these measures in practice from data is to adapt the data envelopment analysis approach (see Farrell (1957), Charnes, Cooper and Rhodes (1978), Banker, Charnes and Cooper (1984) for the origins) in a similar fashion as in Cooper, Pastor, Aparicio and Borras (2011) and Färe, Fukuyama, Grosskopf and Zelenyuk (2015).

instances in performance measurement. For example, the same distance functions appear inside and outside of parentheses to arrive at decompositions of Malmquist productivity indexes (e.g., see Färe, Grosskopf, Norris and Zhang (1994)).

ACCEPTED MANUSCRIPT

Appendix

To show that $\overrightarrow{SBD_T}(x, y; \mathbf{1})$ is independent of the units of measurement, let Ω_x and Ω_y be arbitrary strictly positive diagonal matrices that transform the units of measurement of, respectively, inputs and outputs (x and y), into $\tilde{x} = \Omega_x x$ and $\tilde{y} = \Omega_y y$, so that

$$(x, y) \in T \iff (\tilde{x}, \tilde{y}) \in \tilde{T},$$

where $\tilde{T}$ is T expressed in the new units of measurement, i.e.,

$$\tilde{T} = \{(\tilde{x}, \tilde{y}) : \tilde{x} = \Omega_x x \text{ can produce } \tilde{y} = \Omega_y y\}$$

Furthermore, let $\tilde{\mathbf{1}} = (1 \cdot u_1, ..., 1 \cdot u_N; 1 \cdot v_1, ..., 1 \cdot v_M)$ be the direction vector defining the units corresponding to $(\tilde{x}; \tilde{y})$. Now, note that

$$\begin{aligned}
\overrightarrow{SBD_{\tilde{T}}}(\tilde{x}, \tilde{y}; \tilde{\mathbf{1}}) &= \max\{\sum_{n=1}^{N} \beta_n + \sum_{m=1}^{M} \gamma_m : \\
&\quad (\tilde{x}_1, -\beta_1 \cdot \tilde{1}, ..., \tilde{x}_N - \beta_N \cdot \tilde{1},\ \tilde{y}_1 + \gamma_1 \cdot \tilde{1}, ..., \tilde{y}_M + \gamma_M \cdot \tilde{1}) \in \tilde{T}\} \\
&= \max\{\sum_{n=1}^{N} \beta_n + \sum_{m=1}^{M} \gamma_m : \\
&\quad (\tilde{x}_1, -\beta_1 \cdot 1 \cdot u_1, ..., \tilde{x}_N - \beta_N \cdot 1 \cdot u_N,\ \tilde{y}_1 + \gamma_1 \cdot 1 \cdot v_1, ..., \tilde{y}_M + \gamma_M \cdot 1 \cdot v_M) \in \tilde{T}\} \\
&= \max\{\sum_{n=1}^{N} \beta_n + \sum_{m=1}^{M} \gamma_m : \\
&\quad (u_1(x_1 - \beta_1 \cdot 1), ..., u_N(x_N - \beta_N \cdot 1),\ v_1(y_1 + \gamma_1 \cdot 1), ..., v_M(y_M + \gamma_M \cdot 1)) \in \tilde{T}\} \\
&= \max\{\sum_{n=1}^{N} \beta_n + \sum_{m=1}^{M} \gamma_m : \\
&\quad (\Omega_x(x_1 - \beta_1 \cdot 1, ..., x_N - \beta_N \cdot 1),\ \Omega_y(y_1 + \gamma_1 \cdot 1, ..., y_M + \gamma_M \cdot 1)) \in \tilde{T}\} \\
&= \max\{\sum_{n=1}^{N} \beta_n + \sum_{m=1}^{M} \gamma_m : \\
&\quad (x_1 - \beta_1 \cdot 1, ..., x_N - \beta_N \cdot 1,\ y_1 + \gamma_1 \cdot 1, ..., y_M + \gamma_M \cdot 1) \in T\} \\
&= \overrightarrow{SBD_T}(x, y; \mathbf{1}).
\end{aligned}$$

ACCEPTED MANUSCRIPT

Thus, for any scalar-type transformation of units of measurement given by $\tilde{x} = \Omega_x x$ and $\tilde{y} = \Omega_y y$, we have $\overrightarrow{SBD}_{\tilde{T}}(\tilde{x}, \tilde{y}; \tilde{\mathbf{1}}) = \overrightarrow{SBD}_T(x, y; \mathbf{1})$.

ACCEPTED MANUSCRIPT

ACCEPTED MANUSCRIPT

References

[1] Banker, R.D., A.Charnes, W.W. Cooper (1984) "Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis," *Management Science* 30:9, 1078-92.

[2] Chambers, R.G., Y. Chung and R. Färe (1998) "Profit, directional distance functions, and Nerlovian efficiency," *Journal of Optimization Theory and Applications* 98, 351–364.

[3] Charnes, A., W.W. Cooper and E. Rhodes (1978) "Measuring the Efficiency of Decision Making Units," *European Journal of Operational Research* 2:6, 429-444.

[4] Cooper, W., J. Pastor, J. Aparicio, and F. Borras (2011) "Decomposing profit inefficiency in DEA through the weighted additive model," *European Journal of Operational Research* 212, 411-416.

[5] Färe, R., S. Grosskopf, M. Norris, and Z. Zhang. Productivity growth, technical progress, and efficiency change in industrialized countries. American Economic Review 1994; 84: 66-83.

[6] Fukuyama, H. and W.L. Weber (2009) "A Directional Slacks-based Measure of Technical Inefficiency," *Socio-Economic Planning Sciences* 43:4, 274-287.

[7] Luenberger, D.G. (1995). Microeconomic Theory. McGraw-Hill, Boston.

[8] Färe, R. and S. Grosskopf (2000) "Notes on Some Inequalities in Economics," *Economic Theory* 15, 227-223.

[9] Färe, R., and S. Grosskopf (2010) "Directional distance functions and slacks-based measures of efficiency," *European Journal of Operational Research* 200, 320–322.

[10] Färe, R., H. Fukuyama, S. Grosskopf and V. Zelenyuk (2015) "Cost Decompositions and the Efficient Subset," CEPA Working Papers.

ACCEPTED MANUSCRIPT

ACCEPTED MANUSCRIPT

[11] Färe, R., and D. Primont (1995). Multi-Output Production and Duality: Theory and Applications. Kluwer Academic Publishers, Boston.

ACCEPTED MANUSCRIPT