Emergence of Order from Turbulence in an Isolated Planar Superfluid

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We study the relaxation dynamics of an isolated zero temperature quasi-two-dimensional superfluid Bose-Einstein condensate that is imprinted with a spatially random distribution of quantum vortices. Following a period of vortex annihilation the remaining vortices self-organize into two macroscopic coherent "Onsager vortex" clusters that are stable indefinitely—despite the absence of driving or external dissipation in the dynamics. We demonstrate that this occurs due to a novel physical mechanism—the evaporative heating of the vortices—that results in a negative-temperature phase transition in the vortex degrees of freedom. At the end of our simulations the system is trapped in a nonthermal state. Our computational results provide a pathway to observing Onsager vortex states in a superfluid Bose gas.

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The question of how thermodynamics arises from unitary quantum evolution [1] has been debated since the early days of quantum mechanics. The closest laboratory realization of an isolated quantum system of many particles is perhaps the ultracold quantum gas, and recent progress in the control and manipulation of these systems means that the question is no longer simply an academic one [2]. A particular focus has been one-dimensional Bose gases as described by the Lieb-Liniger model [3], as the integrability of the model suggests that it may be prevented from attaining thermal equilibrium following a quench [4]. Indeed, two ground-breaking experiments on the dynamics of one-dimensional Bose gases [5,6] sparked a rush of further activity due to their seemingly contradictory results on whether the experiments returned to standard thermal equilibrium. These experiments have generated significant recent theoretical interest in the nonequilibrium dynamics and relaxation of idealized isolated quantum systems following a disturbance [2,7–9].

The quantum relaxation of higher-dimensional isolated quantum systems is difficult to address computationally due to their exponential complexity. Recent experiments have quenched the interatomic interaction strength of 3D Bose-Einstein condensates (BECs) in oblate and spherical harmonic traps and followed the subsequent dynamics [10,11], including apparent saturation of the momentum distribution [10]. However the relevance of quantum dynamics in their relaxation is not clear. In some situations the classical field approximation [12,13] can provide insight into quench dynamics. Indeed, recent theoretical work has considered the classical equilibration dynamics of 2D superfluids following a quench from the perspectives of turbulence and nonthermal fixed points [14–19].

The thermalization of isolated classical systems is generally understood in terms of ergodicity and chaotic dynamics [20]. Even though the equations of motion of a physical system are entirely reversible, if a system with a sufficiently large number of degrees of freedom begins in an "atypical" state—such as a gas with all particles in one half of the container—it will quickly relax to a more "typical" state consistent with thermal equilibrium as predicted by statistical mechanics. This agrees with our everyday experience of the arrow of time that makes it abundantly clear that time only proceeds in one direction [21], and is encapsulated by the second law of thermodynamics—that isolated systems only become more disordered with time [22]. Here we present results on an isolated, quasi-two-dimensional superfluid Bose gas in which order appears with time.

Typically quenches in isolated systems increase the energy per particle, leading to more disorder and increased entropy at equilibrium. However, for systems with a limited phase space, continuing to add energy will eventually render it more ordered. This decrease in entropy with increasing energy is the definition of a state with negative absolute temperature [23,24]. Such states are actually "hotter" than those at positive temperature, as energy will spontaneously flow from negative to positive temperature systems when in contact. Thus, for negative-temperature thermodynamic states to be realized in practice, they need to be isolated from their environment. Negative temperatures have been realized in spin systems [25], and more recently with ultracold atoms in optical lattices [26].

Onsager predicted that negative-temperature states may be relevant for 2D fluids by applying statistical mechanics to a 2D model of point vortices [23]. The point-vortex model represents the full velocity field of the fluid as the superposition of the circular velocity fields generated by the individual vortices. Hence the point vortices themselves have no inertial kinetic energy—resulting in a phase space determined entirely by the finite area available to the point vortices, thus allowing negative temperatures. While intended as a model of 2D fluids in general, Onsager noted that the model

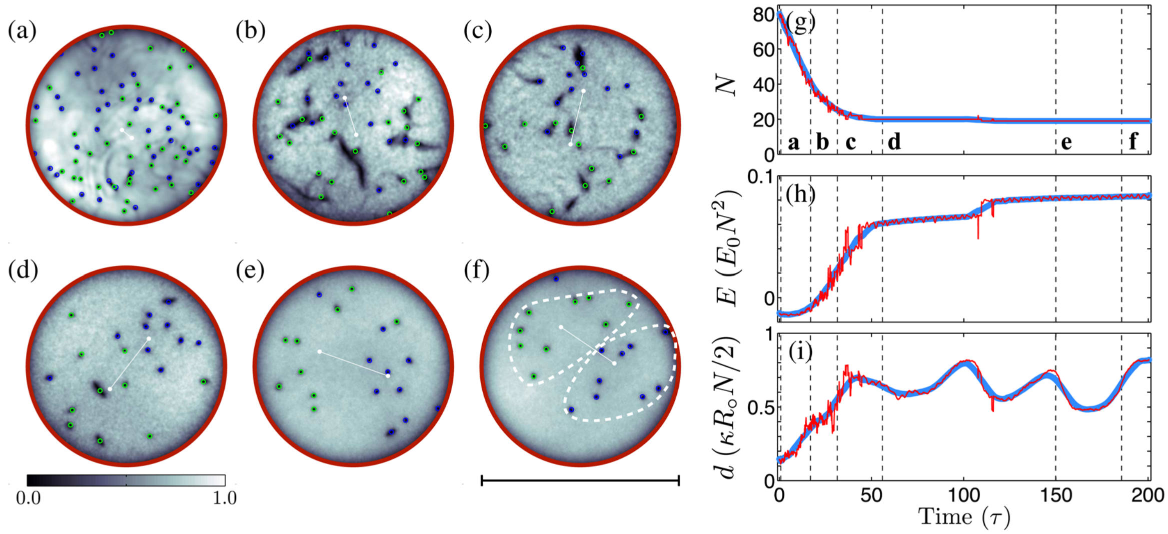

FIG. 1 (color online). Onsager vortex formation in the Gross-Pitaevskii model. (a)–(f) Time series of the condensate column density. The location of vortices with positive and negative circulations are shown using blue (dark) and green (light) circles, respectively, and the white line indicates the vortex dipole moment vector **d** as defined in the text. The initial random vortex configuration in (a) evolves to the Onsager vortices configuration in (f) characterized by two large-scale, coherent clusters of vortices in a background field of sound. (g) The total number of vortices, N, as a function of time, with the dashed vertical lines indicating the times at which we plot the condensate density in (a)–(f). (h) Kinetic energy, E, of the system. (i) Dipole moment, d, of the vortex configuration as a function of time. In (g)–(i) the thin (red) curves correspond to instantaneous values and the thick smooth (blue) curves are sliding averages over a window of 15τ to smooth out rapid fluctuations. The bar under (d) shows the scale of condensate density and the length of the scale bar under (f) is $56a_{\mathrm{osc}}$, the diameter of the trap.

was potentially particularly relevant for 2D superfluids, whose vortices have quantized circulation [23,27].

Experiments on quasi-two-dimensional ultracold quantum gases are now routine [28–30], and offer tantalizing prospects for the study of 2D turbulence in superfluids. Recently, Neely *et al.* have demonstrated that coherently stirring a quasi-2D Bose-Einstein condensate can lead to a proliferation of vortices, with evidence of transient, local clusters of like-signed vortices [31].

Recent simulations [32,33] showed that by starting with a periodic array of vortex clusters—an atypical state of the microcanonical ensemble—the dynamics evolved the vortex configuration to one that was more typical for the given energy: an Onsager vortex state. However, to date, a mechanism that enables a transition from random vortex configurations at infinite temperature to Onsager's negative absolute temperature states corresponding to long-lived, giant vortex clusters in inviscid quantum fluids has not been discovered. Indeed, while the formation of giant vortex structures in generic classical two-dimensional fluids in the limit of vanishingly small viscosity has been proven [34], Onsager's prediction of vortex clustering in inviscid flows remains unexplained [35].

In this Letter we study the relaxation dynamics of an isolated quasi-two-dimensional superfluid Bose-Einstein condensate that is "quenched" by imprinting a spatially random arrangement of vortices. We subsequently simulate the classical Hamiltonian dynamics—conserving both the energy and number of particles—and find that two Onsager vortices corresponding to negative-temperature states can arise out of the initially turbulent flow. This emergence of order in an isolated system occurs due to evaporation of vortices that enable a redistribution of the energy among the degrees of freedom, and leaves the BEC in a nonthermal state.

Our simulations begin with a BEC at zero temperature, in which we prepare an equal number of vortices and antivortices (80 in total) at stochastically sampled locations. We subsequently simulate the dynamics of the BEC using the Gross-Pitaevskii equation (GPE), a nonlinear Schrödinger equation providing a realistic mean-field description of BECs [36]. The conservative Hamiltonian evolution of the GPE preserves the total energy E_{GP}, atom number N_a, and angular momentum of the system. While the BEC itself is three dimensional, the dynamics of the vortices are essentially two dimensional [31,37]. In contrast to theoretical studies of quasi-two-dimensional quantum turbulent systems in harmonic traps [38–40] or in uniform doubly periodic domains [16,32,33,41,42], we consider a BEC confined in a disk trap [43]. Full details are provided in the Supplemental Material [43].

The results of our simulation for a typical initial condition can be seen in Movie S1 [43]. In Figs. 1(a)–1(f) we show the condensate column densities for a range of evolution times from the movie, and identify the location of the vortices and antivortices. The dynamics initially leads to the annihilation of several vortex-antivortex pairs, with the subsequent emission of sound waves in the bulk BEC. Occasionally this process is reversed. The vortex pair annihilation results in the rapid decay of the total vortex number until ~20 remain [Fig. 1(g)], while the GPE kinetic energy per vortex grows [Fig. 1(h)]. During this annihilation period, like-signed vortices exhibit a tendency to form transient clusters that grow larger with time. A remarkable feature of this system is the eventual emergence of two counterrotating clusters of like-signed vortices on opposite sides of the condensate, which we identify as Onsager vortices (OVs). These are characterized in Fig. 1(i) by the growth in the vortex dipole moment $d = |\mathbf{d}| = |\sum_i q_i \mathbf{r}_i|$, where $\mathbf{r}_i$ is the position of the ith vortex. The vortex charge is $q_i = s_i h/m$, where $s_i = \pm 1$ for vortices and antivortices, respectively, h is Planck's constant and m is the mass of an atom. After the formation of OVs, the vortex annihilation mostly ceases and the OVs remain a robust and long-lived feature of the system. Analyses of the GPE energetics and spectral features are provided in the Supplemental Material [43].

The abrupt end to the annihilation of vortices and the spontaneous emergence and persistence of the OVs in Fig. 1 is a surprising result. How has order spontaneously arisen from an initially chaotic state? The vortex gas is apparently trapped in a negative-temperature state, whereas the sound waves on top of the BEC have a positive temperature, suggesting that ergodicity has been broken in this system.

In order to develop a microscopic understanding of our results, we utilize Onsager's point-vortex model for a two-dimensional fluid, which can be derived as an approximation to the GPE in the incompressible limit [44]. We consider N singly quantized point vortices with equal numbers of each circulation confined in a disk geometry [45–47]. In the Supplemental Material [43] we present the results of Monte Carlo calculations for the equilibrium thermodynamics of this system [46]. This allows the construction of the schematic phase diagram depicted in Fig. 2.

At positive zero temperature ($T = 0^+$) in Fig. 2, a zero-entropy, zero-momentum BEC exists. In the PC phase at low positive temperatures, all vortices and antivortices are paired, and the velocity fields generated by each vortex cancel [46,49,50]. On increasing the temperature there is a critical point where the vortex pairs unbind and the system transitions to the NS. This is the entropy-dominated regime where correlations between the positions of vortices and antivortices are negligible.

The point-vortex model has a finite phase space, and the entropy S is not a monotonically increasing function of energy E. This allows $\beta = 1/k_B T = (\partial S/\partial E) < 0$ (where k_B is the Boltzmann constant) and, hence, negative-temperature states [23,24]. Continuing to increase the energy results in another abrupt change in the configuration of the vortices at a critical negative temperature T_{OV}, where well-defined clusters of like-signed vortices emerge—the Onsager vortices. Eventually, at $T = 0^-$ (see Fig. 2) the OV state corresponds to a pure, zero-entropy EBC—a BEC in a nonzero momentum state [48,50].

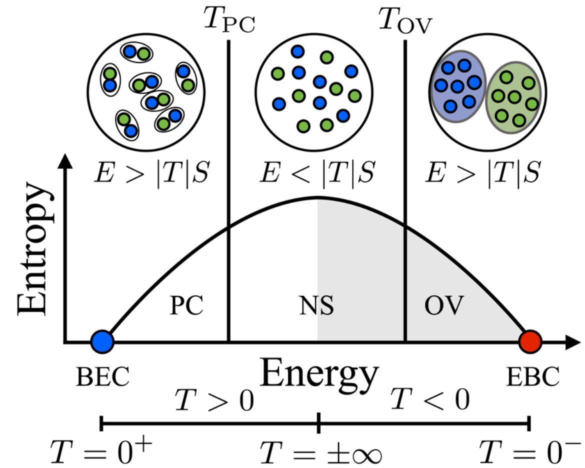

FIG. 2 (color online). A schematic plot of entropy versus energy for the point-vortex model. A zero-entropy Bose-Einstein condensate forms at $T = 0^+$ with its negative-temperature counterpart, an Einstein-Bose condensate (EBC) [48], emerging at $T = 0^-$. Entropy is maximized at $T = \pm\infty$ in the entropy-dominated normal state (NS), which has a stochastic distribution of vortices. The vortex binding-unbinding phase transition separates the normal state from the pair-collapse (PC) state at positive temperature, whereas there is a transition to the coherent Onsager vortex state at a vortex-number-dependent negative temperature.

Our GPE simulation exhibits counterintuitive collective behavior—an apparent dynamical transition in the vortex gas from an initially disordered NS phase to an ordered OV phase at negative temperature. This result raises the question—what is the physical mechanism underlying this emergent phenomenon? In short, the answer is the evaporative heating of the vortex gas.

The kinetic energy of the system can be divided into three components: an incompressible part due to the rotational vortex velocity field, a compressible part due to sound (phonon excitations), and a quantum pressure term [36]. The incompressible kinetic energy in the GPE initial state is significant, while the compressible kinetic energy is small. The GPE dynamics convert the rotational energy of vortices to sound energy via vortex-antivortex annihilation. Once the Onsager vortices form, the energy of these two components are approximately constant as shown in Fig. S3 [43].

The vortex pair annihilation, however, leads to an increase in the mean energy per vortex. The annihilation always occurs at the length scale of the vortex core size, which is much smaller than the mean distance between vortices. Since the energy of a vortex-antivortex pair decreases with separation distance [36], pair annihilation causes a decrease in the total energy of the vortex gas that is small in comparison to the mean energy per vortex, while also reducing its entropy. The subsequent dynamics of the vortex gas lead to rethermalization with a larger mean energy per vortex.

This process is evaporative heating—the reverse analog of forced evaporative cooling used to achieve quantum degeneracy in ultracold atomic gases. Evaporative cooling removes the "hottest" atoms from the system, leaving the remaining atoms to rethermalize to a lower positive temperature [51]. Instead, in our simulation the "coldest" vortices, corresponding to vortex-antivortex pairs, are removed, leaving the remaining vortices to equilibrate at a higher average energy, and hence a hotter negative temperature. The evaporative heating of the vortex gas becomes ineffective once the Onsager vortices have formed, and encounters between vortices and antivortices become rare.

The evaporative heating mechanism is strikingly confirmed by dynamical simulations of Onsager's point-vortex model with the addition of vortex-antivortex annihilation to represent the conversion of energy from the incompressible to compressible kinetic energy in the GPE. Whenever a pair of vortices of opposite sign comes within the "annihilation distance" d_a of one another, they are removed from the simulation as described in the Supplemental Material [43]. Schwarz used a similar dynamical rule to account for reconnection events in a vortex filament model of three-dimensional superfluid turbulence [52]. Campbell and O'Neil used forced vortex annihilation to describe viscosity in a two-dimensional vortex random walk model [53]. However, such effective vortex viscosity vanishes for systems where vortex annihilation is prohibited, such as Onsager vortex states or configurations consisting of only one sign of vortices. Therefore, it seems that this effective vortex viscosity behaves quite differently from the usual kinematic viscosity of classical fluids.

We have performed simulations for an ensemble of 500 stochastically sampled vortex configurations matched to the GPE initial state, and a sample trajectory is shown in Movie S2 [43]. The results for the vortex number, energy, and dipole moment are shown in Figs. 3(a)–3(c), and should be compared with the corresponding quantities for the GPE simulation in Figs. 1(g)–1(i). These results confirm that the energy per vortex increases as the vortex pairs evaporate; this is accompanied by an increase in the net vortex dipole moment, for which our Monte Carlo calculations indicate the OV transition occurs near $d \sim 0.5$.

We hence conclude that the GPE dynamics describe a transition to a coherent negative-temperature Onsager vortex state in a closed system—order emerges from chaos. This may leave the impression that the arrow of time for the vortices has been reversed. However, the heating of the background BEC, through the transfer of the fluid motion associated with vortices into sound waves, creates sufficient entropy that the results are in full compliance with the second law of thermodynamics. Nevertheless, the BEC remains trapped indefinitely in a nonthermal state, with lower entropy than that expected at full thermal equilibrium. Whether the Onsager vortex state is an example of a nonthermal fixed point [54], and whether this behavior persists in the full quantum dynamics of the system, are intriguing questions.

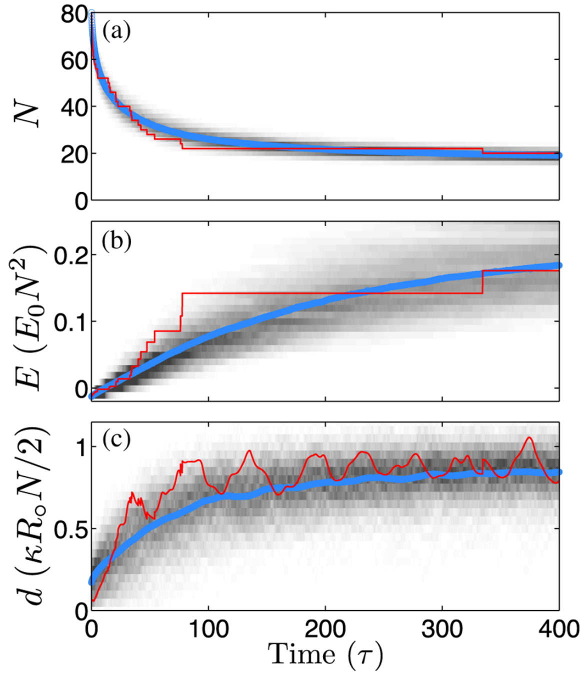

FIG. 3 (color online). Point-vortex dynamics with the addition of vortex pair annihilation. (a) Vortex number, N, (b) energy, E, and (c) dipole moment, d, as functions of time for an ensemble of 500 simulations with an initial energy corresponding to that of the GPE simulation. Thick, smooth (blue) lines are ensemble averages, thin (red) curves are results from a typical single trajectory, and background (grey scale) densities are normalized histograms of the simulation ensemble. The slow undulations of the dipole moment in (c) are due to the large-scale orbital motion of the Onsager vortices.

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