Decomposing graphs of high minimum degree into 4-cycles

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Abstract

If a graph G decomposes into edge-disjoint 4-cycles, then each vertex of G has even degree and 4 divides the number of edges in G. It is shown that these obvious necessary conditions are also sufficient when G is any simple graph having minimum degree at least $(\frac{31}{32}+o_n(1))n$, where n is the number of vertices in G. This improves the bound given by Gustavsson (1991), who showed (as part of a more general result) sufficiency for simple graphs with minimum degree at least $(1-10^{-94}+o_n(1))n$. On the other hand, it is known that for arbitrarily large values of n there exist simple graphs satisfying the obvious necessary conditions, having n vertices and minimum degree $\frac{3}{5}n-1$, but having no decomposition into edge-disjoint 4-cycles. We also show that if G is a bipartite simple graph with n vertices in each part, then the obvious necessary conditions for G to decompose into 4-cycles are sufficient when G has minimum degree at least $(\frac{31}{32}+o_n(1))n$.

1 Introduction

There are several long-standing open conjectures concerning the existence of decompositions of graphs with sufficiently high minimum degree into edge-disjoint subgraphs isomorphic to a given graph. These include the 1-*factorisation conjecture* and two conjectures of Nash-Williams concerning decompositions into Hamilton cycles and decompositions into 3-cycles. This paper addresses the problem of decomposing graphs with high minimum degree into 4-cycles, see Theorem 1.

A *decomposition* of a graph G is a set $\{G_1, G_2, \ldots, G_t\}$ of subgraphs of G such that $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$ and $E(G_1) \cup E(G_2) \cup \cdots E(G_t) = E(G)$. If each subgraph in a decomposition of G is isomorphic to a fixed graph H, then we say that G *decomposes into* H. Obvious necessary conditions for G to decompose into H are $E(H)$ divides $E(G)$ and the greatest common divisor of the degrees of the vertices in H divides the degree of each

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vertex in G. If these two conditions hold, then we say that G is *H-admissible*. Determining whether a graph G decomposes into a graph H is known to be NP-complete in general [9].

All graphs considered here are simple and without loops. The number of vertices in a graph is called its *order* and the number of edges in a graph is called its *size*. The degree in G of a vertex $x \in V(G)$ is denoted by $\deg_G(x)$. The minimum and maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A graph is *even* if each vertex has even degree. A cycle of order m is called an m-cycle and denoted by C_m, and a complete graph of order n is denoted by K_n.

Wilson [15] has shown that for any given graph H, there is a constant $N(H)$ such that every H-admissible complete graph of order greater than $N(H)$ decomposes into H. Gustavsson [13] has generalised Wilson's result by showing that for any given graph H, there is a constant $N(H)$ and a positive constant $\gamma(H)$ such that every H-admissible graph of order greater than $N(H)$ and with minimum degree at least $(1-\gamma)n$ decomposes into H. In Gustavsson's result, the constant $N(H)$ is extremely large and the constant $\gamma(H)$ is extremely small. For example, $\gamma(K_3) \approx 10^{-24}$. Gustavsson's result has been used to prove several results on decompositions of graphs with high minimum degree. These include decompositions into closed trails of arbitrary specified lengths [1] and decompositions into subgraphs isomorphic to a given list of graphs [4].

A remarkable result by Yuster [16] gives an asymptotically sharp lower bound on $\delta(G)$ in the case H is bipartite graph with $\delta(H) = 1$. Specifically, it is shown in [16] that if H is any bipartite graph with $\delta(H) = 1$, then every H-admissible graph of order n with minimum degree at least $(\frac{1}{2}+o_n(1))n$ decomposes into H. It is shown in [17] that if H is any connected graph with at least 3 vertices, then there exist H-admissible graphs of arbitrarily large order n with minimum degree $\frac{n}{2} - 2$ which do not decompose into H, and thus the bound of $(\frac{1}{2} + o_n(1))n$ is asymptotically sharp. Yuster's result does not hold if the requirement that $\delta(H) = 1$ is removed. Yuster [16] gives examples, which are attributed to Winkler and Kahn, of C_4-admissible graphs of arbitrarily large order n and minimum degree $\frac{3}{5}n - 1$ that have no decomposition into C_4.

For decompositions into C_3, a similar construction to that mentioned in the previous paragraph shows that minimum degree of at least $\frac{3}{4}n$ is necessary, and a conjecture of Nash-Williams states that every C_3-admissible graph of order n with this minimum degree decomposes into C_3. Apart from Gustavsson's result, very little is known on this conjecture. Colbourn and Rosa [7] have shown that C_3-admissible graphs of order $n > 9$ with minimum degree at least $n - 3$ decompose into C_3. Except for extremely large values of n covered by Gustavsson's result, the problem is still open even for minimum degree $n - 4$.

Given that so little is known on the Nash-Williams conjecture concerning decompositions into C_3, and given that C_4 is the smallest graph not covered by Yuster's result, it is natural to ask about decompositions into C_4. This brings us to our main result.

Theorem 1. *If G is a simple C_4-admissible graph with n vertices and minimum degree at least $(\frac{31}{32} + o_n(1))n$, then G decomposes into C_4.*

Our precise lower bound on minimum degree is given in the proof of Theorem 1, see (1) and (13), and is $\frac{31}{32}n + O(n^{\frac{3}{4}})$. In Section 3 we show that our result can be strengthened in the case where G is bipartite with the same number of vertices in each part, see Theorem 2. In [5] it is shown that if G is a simple even bipartite graph with parts X and Y such that $|Y|$ is even, the size of G is divisible by 4, and the degree of each vertex in Y is at least $\frac{95}{96}|X|$, then G decomposes into 4-cycles. Theorem 2 gives a similar result with an improved lower bound on the minimum degree, but only in the special case where the parts of G have the same cardinality.

For decompositions into C_4, Theorem 1 improves on the above-mentioned result of Gustavsson in two ways. Firstly, the bound on the minimum degree is dramatically reduced. In Gustavsson's result, the constant $\gamma(C_4) \approx 10^{-94}$, giving a bound of $(1 - 10^{-94})n$ on the minimum degree. The bound on minimum degree in Theorem 1 is asymptotically $\frac{31}{32}n$ which is not significantly far from optimal, given that we know it cannot be reduced beyond $\frac{3}{5}n$. Secondly, Gustavsson's proof is non-constructive; it does not provide an algorithm which will determine the decomposition. The proof of Theorem 1 contains a straightforward algorithm which produces the required decomposition. Fu et al [10, 11] have shown that all C_4-admissible graphs of order $n \geqslant 9$ having minimum degree at least $n - 4$ decompose into 4-cycles.

We mention two well-known conjectures concerning decompositions of graphs with sufficiently high minimum degree into spanning subgraphs, rather than into fixed subgraphs as discussed thus far. The *1-factorisation conjecture* states that every regular graph of even order n with degree at least $\frac{n}{2}$ decomposes into 1-factors. By considering the union of two vertex-disjoint complete graphs of odd order, it is easy to see that the lower bound on degree cannot be reduced. The 1-factorisation conjecture is known to hold for graphs with minimum degree at least $\frac{1}{2}(\sqrt{7}-1) \approx 0.823$ [6]. There is a similar conjecture, due to Nash-Williams, on decompositions into Hamilton cycles. It states that every regular graph of order n with even degree at least $\frac{n}{2}$ decomposes into Hamilton cycles. Buchanan [3] has shown that regular graphs of odd order n with degree $n-3$ decompose into Hamilton cycles, but the conjecture is unresolved even for degree $n - 4$.

2 Proof of Theorem 1

The condition that G is even with size divisible by 4 is clearly necessary if G decomposes into 4-cycles. For the proof of the converse, let G be a simple even graph of order n with size divisible 4. We first prove that if n is even and G has minimum degree $\delta(G) \geqslant (\frac{31}{32} + g(n))n$, where

$$g(n) = \frac{1}{2}n^{-\frac{1}{4}} + \frac{5}{8}n^{-\frac{3}{4}} + \frac{55}{16}n^{-1}, \tag{1}$$

then G decomposes into 4-cycles. Observe that the complement $\overline{G}$ of G has maximum degree

$$\Delta(\overline{G}) \leqslant h(n) - 1 \tag{2}$$

where

$$h(n) = (\frac{1}{32} - g(n))n. \tag{3}$$

Since $\delta(G) > \frac{1}{2}n$, G has a Hamilton cycle H by Dirac's Theorem [8]. Let $G - H$ be the graph obtained from G be removing the edges of H, let $\mathcal{P}$ be a partition of $V(G)$ into pairs, and let $\mathcal{C}_0$ be the set of all 4-cycles in $G - H$ of the form (x_1, y_1, x_2, y_2) where $\{x_1, x_2\}, \{y_1, y_2\} \in \mathcal{P}$. Then let G' be the graph obtained from $G - H$ by removing the edges of the 4-cycles of $\mathcal{C}_0$, let $\mathcal{D}_0$ be a (greedy) maximal set of edge-disjoint 4-cycles in G', let G'' be the graph obtained from G' by deleting the edges of the 4-cycles of $\mathcal{D}_0$, and define F_0 to be the graph obtained by adding the edges of the Hamilton cycle H to G''. Observe that F_0 is an even connected graph with size divisible by 4 (F_0 is obtained by deleting the edges of edge-disjoint 4-cycles from G). Thus, F_0 has an Euler tour $T_0 = [v_1, v_2, \ldots v_{4k}]$.

At this stage, we have a decomposition $\mathcal{C}_0 \cup \mathcal{D}_0 \cup \{F_0\}$ of G where $\mathcal{C}_0$ is a set of 4-cycles of the form (x_1, y_1, x_2, y_2) where $\{x_1, x_2\}, \{y_1, y_2\} \in \mathcal{P}$, $\mathcal{D}_0$ is a set of 4-cycles, and F_0 has an Euler tour $T_0 = [v_1, v_2, \ldots, v_{4k}]$. For convenience, we define the graph Y_0 by $V(Y_0) = \mathcal{P}$ and $E(Y_0) = \{\{P, P'\} : P, P' \in \mathcal{P}, P \cup P' = V(C), C \in \mathcal{C}_0\}$. Thus, $\{P, P'\} \in E(Y_0)$ if and only if there is a 4-cycle in $\mathcal{C}_0$ with vertex set $P \cup P'$.

We now describe an iterative process that can be applied to complete the decomposition. Set $T = T_0$, $F = F_0$, $\mathcal{C} = \mathcal{C}_0$, $\mathcal{D} = \mathcal{D}_0$ and $Y = Y_0$. As we proceed, T, F, $\mathcal{C}$, $\mathcal{D}$ and Y shall be modified, with Y always being defined by $V(Y) = \mathcal{P}$ and $E(Y) = \{\{P, P'\} : P, P' \in \mathcal{P}, P \cup P' = V(C), C \in \mathcal{C}\}$. Each major step in the process involves removing a number, t say, of 4-cycles from $\mathcal{C}$, adding $t + 1$ 4-cycles to $\mathcal{D}$, and reducing the number of edges in F by 4. Eventually, F contains no edges and $\mathcal{C} \cup \mathcal{D}$ is the required decomposition of G into 4-cycles. As described below, other minor steps are also involved in the process. First we describe how the major step works.

The Major Step: We find a subtrail $[u_0, u_1, \ldots, u_6]$ in T (this means $u_0, u_1, \ldots, u_6$ are consecutive vertices in T) such that u_0, u_2, u_4 and u_6 are pairwise distinct. We shall show later that we can guarantee the existence of such a subtrail, and we shall also add a further constraint on how the subtrail is chosen. Let $\mathcal{R}$ be the minimal set of pairs of $\mathcal{P}$ such that $\{u_0, u_2, u_4, u_6\} \subseteq \bigcup_{P \in \mathcal{R}} P$ (we note that clearly $|\mathcal{R}| \in \{2, 3, 4\}$). Define $R = \bigcup_{P \in \mathcal{R}} P$. We next find a pair $Q = \{q_1, q_2\} \in \mathcal{P}$ such that $\{Q, P\} \in E(Y)$ for each $P \in \mathcal{R}$. Again, we shall show later that such a Q can be found at each stage of the process we are describing, and we shall impose extra constraints on our choice of Q. Consider the subgraph X of G with edge set

$$E(X) = \{\{u_i, u_{i+1}\} : i \in \{0, 1, \ldots, 5\}\} \cup \{\{x, y\} : x \in Q, y \in R\}.$$

We now modify T, F, $\mathcal{C}$, $\mathcal{D}$ and Y as follows. Firstly, for each $P \in \mathcal{R}$ we remove the edge $\{Q, P\}$ from Y, and we remove the corresponding 4-cycle from $\mathcal{C}$. Secondly, in F we replace the edges $\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_5, u_6\}$ with $\{u_0, q_1\}$ and $\{u_6, q_1\}$, and we modify T as follows, which ensures that T remains an Euler tour in F.

$$[\ldots, u_0, u_1, u_2, u_3, u_4, u_5, u_6, \ldots] \to [\ldots, u_0, q_1, u_6, \ldots]$$

Thirdly, we add the following three 4-cycles to $\mathcal{D}$ (note that $\{q_1, q_2\} \cap \{u_1, u_3, u_5\} = \emptyset$ because the edges of the cycles of $\mathcal{C}$ are disjoint from the edges of T).

$$(q_2, u_0, u_1, u_2) \qquad (q_1, u_2, u_3, u_4) \qquad (q_2, u_4, u_5, u_6)$$

Finally, the edges of X which occur in neither the modified F nor these three 4-cycles induce a complete bipartite graph with parts Q and $R \setminus \{u_0, u_2, u_4, u_6\}$. This graph has a decomposition into 4-cycles and we also add the 4-cycles of this decomposition to $\mathcal{D}$. This completes the major step.

It is clear that repeated applications of the major step will produce a decomposition of G into 4-cycles. It remains to show that we can indeed repeatedly apply the major step until the desired decomposition is obtained. We now proceed to describe how this is achieved. We need to take care to ensure that F remains connected throughout. After each application of the major step, and before the next, we shall apply the following procedure which we call the *minor step*.

The Minor Step: If at any stage $v_i = v_{i+4}$ for some $i \in \{1, 2, \ldots, 4k\}$ (where $T = [v_1, v_2, \ldots, v_{4k}]$ is the Euler tour at the current stage and the subscripts are taken modulo $4k$), then $(v_i, v_{i+1}, v_{i+2}, v_{i+3})$ is a 4-cycle in F and we add this 4-cycle to $\mathcal{D}$, remove its edges from F, and redefine T as

$$T = [\ldots, v_{i-1}, v_i, v_{i+5}, \ldots].$$

Thus, T remains an Euler tour in F. We repeat this until v_i and v_{i+4} are distinct for all i, and then proceed as follows.

If $v_i = v_{i+6}$ and $v_{i+2} = v_{i+8}$, then $(v_i, v_{i+1}, v_{i+2}, v_{i+7})$ is a 4-cycle in F and we add this 4-cycle to $\mathcal{D}$, remove its edges from F, and redefine T as

$$T = [\ldots, v_{i-1}, v_i, v_{i+5}, v_{i+4}, v_{i+3}, v_{i+2}, v_{i+9}, \ldots].$$

Again, T remains an Euler tour in F. We repeat this until there is no i such that $v_i = v_{i+6}$ and $v_{i+2} = v_{i+8}$ both hold. The process described in this paragraph may mean that the process of the preceding paragraph needs to be repeated to ensure that v_i and v_{i+4} are distinct, but it is clear that we can repeat the two processes until for each $i \in \{1, 2, \ldots, 4k\}$, either v_i, v_{i+2}, v_{i+4} and v_{i+6} are pairwise distinct or v_{i+2}, v_{i+4}, v_{i+6} and v_{i+8} are pairwise distinct (or until T is a 4-cycle in which case our decomposition is complete). This completes the minor step.

The minor step is applied before and after each application of the major step. It guarantees that if $T = [v_1, v_2, \ldots, v_{4k}]$, then for any $i \in \{1, 2, \ldots, 4k\}$ at least one of the choices

$$[u_0, u_1, \ldots, u_6] = [v_i, v_{i+1}, \ldots, v_{i+6}] \qquad \text{or} \qquad [u_0, u_1, \ldots, u_6] = [v_{i+2}, v_{i+3}, \ldots, v_{i+8}] \tag{4}$$

is available for the next application of the major step (recall that we require u_0, u_2, u_4 and u_6 to be distinct). The application of the minor step which occurs immediately after the j-th application of the major step will be called the j-th application of the minor step.

In each application of the major step, we remove $|\mathcal{R}|$ edges incident with Q from Y, and we call each of these incidences a *$\mathcal{Q}$-type incidence on* Q. Similarly, for each $P \in \mathcal{R}$ we remove an edge incident with P from Y, and we call each of these incidences an *$\mathcal{R}$-type incidence on* P. Thus, each application of the major step involves $|\mathcal{R}|$ $\mathcal{Q}$-type incidences on Q and one $\mathcal{R}$-type incidence on P for each $P \in \mathcal{R}$. At any stage and for each $P \in \mathcal{P}$ we define $r_{\mathcal{Q}}(P)$ to be the number of $\mathcal{Q}$-type incidences on P that have occurred thus far, and we define $r_{\mathcal{R}}(P)$ to be the number of $\mathcal{R}$-type incidences on P that have occurred thus far. Thus, at any stage for all $P \in \mathcal{P}$ we have

$$\deg_Y(P) = \deg_{Y_0}(P) - r_{\mathcal{Q}}(P) - r_{\mathcal{R}}(P). \tag{5}$$

Now, we mentioned above that in the major step we impose an additional restriction on our choice of Q. Our restriction is that for any $P \in \mathcal{P}$, $r_{\mathcal{Q}}(P)$ must never exceed $\frac{1}{2}n^{\frac{3}{4}} + 3$.

$$r_{\mathcal{Q}}(P) \leqslant \frac{1}{2}n^{\frac{3}{4}} + 3 \tag{6}$$

Thus, we say that a pair P is *full* if $r_{\mathcal{Q}}(P) \in \{\frac{1}{2}n^{\frac{3}{4}}, \frac{1}{2}n^{\frac{3}{4}}+1, \frac{1}{2}n^{\frac{3}{4}}+2, \frac{1}{2}n^{\frac{3}{4}}+3\}$, and we demand that no full pair is ever chosen as Q.

We now proceed to obtain an upper bound for $r_{\mathcal{R}}(P)$. We must be careful because in each application of the major step the subtrail $[u_0, q_1, u_6]$ is inserted into the Euler tour T. With this in mind, we next describe an additional constraint on the selection of our subtrail $[u_0, \dots, u_6]$ for the major step. We seek to ensure that for each $\mathcal{R}$-type incidence on P, there is at least one corresponding reduction by 2 of $\deg_F(P)$, where $\deg_F(P)$ is defined by

$$\deg_F(P) = \deg_F(x) + \deg_F(y)$$

when $P = \{x, y\}$.

In the j-th application of the major step, there is an $\mathcal{R}$-type incidence on each $P \in \mathcal{R}$. That is, each pair containing u_0, u_2, u_4 or u_6. For $i \in \{0, 2, 4, 6\}$, let P_i be the pair containing u_i. It is possible that $\{u_i, u_j\} \in \mathcal{P}$ for distinct $i, j \in \{0, 2, 4, 6\}$ in which case $P_i = P_j$, but the following argument still works if this is the case. We seek reductions by 2 of $\deg_F(P_0)$, $\deg_F(P_2)$, $\deg_F(P_4)$ and $\deg_F(P_6)$. The desired reductions of $\deg_F(P_2)$ and $\deg_F(P_4)$ are achieved during the j-th application of the major step (because $\deg_F(u_2)$ and $\deg_F(u_4)$ are each reduced by 2 in the major step). So we are concerned with reducing $\deg_F(P_0)$ and $\deg_F(P_6)$.

During the application of the major step, if a pair P contains u_1, u_3 or u_5, then $\deg_F(P)$ is reduced by 2, or by an additional 2 if the other element of P is in $\{u_1, u_2, u_3, u_4, u_5\}$. Reductions of $\deg_F(P)$ may also occur during applications of the minor step. The reductions of $\deg_F(P)$ described in this paragraph will be called *bonus reductions*. We need to ensure that there is a corresponding bonus reduction of $\deg_F(P_0)$ and $\deg_F(P_6)$ for the $\mathcal{R}$-type incidences on P_0 and P_6 in the j-th application of the major process.

If there are bonus reductions of both $\deg_F(P_0)$ and $\deg_F(P_6)$ during the j-th application of the minor step then we are done. If there is no bonus reduction of $\deg_F(P_0)$ nor $\deg_F(P_6)$ during the j-th application of the minor step, then either $[u_0, q_1, u_6]$ or $[u_6, q_1, u_0]$ is a subtrail of our Euler tour when we perform the $(j+1)$-th application of the major step. Thus, by (4) we can ensure that a bonus reduction of both $\deg_F(P_0)$ and $\deg_F(P_6)$ occurs in the $(j+1)$-th application of the major step (by choosing our subtrail $[u'_0, u'_1, \dots, u'_6]$ for the $(j+1)$-th application of the major step so that either $\{u_0, u_6\} = \{u'_1, u'_3\}$ or $\{u_0, u_6\} = \{u'_3, u'_5\}$). Finally, if there is a bonus reduction of exactly one of $\deg_F(P_0)$ and $\deg_F(P_6)$ in the j-th application of the minor step, then again by (4) we can ensure that a bonus reduction of the other occurs in the $(j+1)$-th application of the major step.

We have shown that we can ensure that for each $\mathcal{R}$-type incidence on P, there is at least one corresponding reduction by 2 of $\deg_F(P)$. We now obtain an upper bound on $\deg_{F_0}(P)$ for any pair $P \in \mathcal{P}$. Let $P = \{a,b\}$ be an arbitrary pair in $\mathcal{P}$. We partition the pairs of $\mathcal{P} \setminus \{a,b\}$ into three sets $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ as follows, and for $i = 1,2,3$ we let $|\mathcal{T}_i| = t_i$.

- $\{x,y\}$ is in $\mathcal{T}_1$ if $\{a,x\}$, $\{a,y\}$, $\{b,x\}$ or $\{b,y\}$ is in $E(H)$.
- $\{x,y\}$ is in $\mathcal{T}_2$ if it is not in $\mathcal{T}_1$ and $|\{\{a,x\},\{a,y\},\{b,x\},\{b,y\}\} \cap E(G)| \leqslant 3$.
- $\{x,y\}$ is in $\mathcal{T}_3$ if it is not in $\mathcal{T}_1$ and $|\{\{a,x\},\{a,y\},\{b,x\},\{b,y\}\} \cap E(G)| = 4$.

There are no edges of F_0 joining a or b to any vertex in the pairs of $\mathcal{T}_3$. Since G'' contains no 4-cycles, there is at most one pair $\{x,y\}$ in $\mathcal{T}_1 \cup \mathcal{T}_2$ such that $|\{\{a,x\},\{a,y\},\{b,x\},\{b,y\}\} \cap E(G'')| = 3$. Hence, $\deg_{F_0}(P) \leqslant 4 + 2(t_1+t_2)+1$ (the term 4 is present because there are up to 4 edges of H incident with a or b). Since $t_1 \leqslant 4$, we have $\deg F_0(P) \leqslant 13 + 2t_2$. Now, if $\{x,y\} \in \mathcal{T}_2$, then at least one of the edges $\{a,x\}$, $\{a,y\}$, $\{b,x\}$ or $\{b,y\}$ is in the complement $\overline{G}$ of G. Thus, $t_2 \leqslant 2\Delta(\overline{G}) \leqslant 2(h(n)-1)$ (see (2)), and we have

$$\deg_{F_0}(P) \leqslant 13 + 4(h(n)-1) = 4h(n) + 9. \tag{7}$$

Since we ensure that for each $\mathcal{R}$-type incidence on P, there is at least one corresponding reduction by 2 of $\deg_F(P)$, and since the only time $\deg_F(P)$ increases is when there is a Q-type incidence on P (that is, $q_1 \in P$ in an application of the major step), by (7) we can conclude that $r_{\mathcal{R}}(P) \leqslant \frac{1}{2}(4h(n)+9) + r_Q(P)$, and hence using (6) that

$$r_{\mathcal{R}}(P) \leqslant 2h(n) + \tfrac{1}{2}n^{\frac{3}{4}} + \tfrac{15}{2}. \tag{8}$$

Now consider an arbitrary fixed pair $P = \{a,b\} \in \mathcal{P}$. If $P' = \{x,y\} \in \mathcal{P} \setminus \{P\}$ and $\{P,P'\} \notin E(Y_0)$, then at least one of the edges $\{a,x\}$, $\{a,y\}$, $\{b,x\}$, $\{b,y\}$ is in $\overline{G-H}$. Since there are at most $2\Delta(\overline{G-H})$ edges of $\overline{G-H}$ incident with a or b, we have $\deg_{Y_0}(P) \geqslant \frac{n}{2} - 1 - 2\Delta(\overline{G-H})$. Since $\Delta(\overline{G}) \leqslant h(n)-1$ (see (2)), we have $\Delta(\overline{G-H}) \leqslant h(n)+1$, and it follows that

$$\deg_{Y_0}(P) \geqslant \tfrac{n}{2} - 2h(n) - 3. \tag{9}$$

Combining the lower bound on $\deg_{Y_0}(P)$ from (9) with the upper bounds on $r_{\mathcal{Q}}(P)$ and $r_{\mathcal{R}}(P)$ from (6) and (8) respectively, (5) gives us

$$\deg_Y(P) \geqslant \tfrac{n}{2} - 4h(n) - n^{\frac{3}{4}} - \tfrac{27}{2}. \tag{10}$$

Now, since each application of the major step reduces the number of edges in F by 4, and since the number of Q-type incidences in each application of the major step is at most 4, the total number of Q-type incidences never exceeds $|E(F_0)|$. Since G'' has no 4-cycles, since F_0 is obtained from G'' by adding the edges of the Hamilton cycle H, and since Reiman [14] has shown that a graph of order n with no 4-cycles has at most $\frac{1}{4}n+\frac{1}{2}n^{\frac{3}{2}}$ edges (see [12]), we have $|E(F_0)| \leqslant \frac{5}{4}n+\frac{1}{2}n^{\frac{3}{2}}$. Thus, the total number of Q-type incidences never exceeds

$$\frac{5}{4}n+\frac{1}{2}n^{\frac{3}{2}}. \tag{11}$$

Let m be the number of full pairs at a particular stage. Since each full pair P satisfies $r_Q(P) \geqslant \frac{1}{2}n^{\frac{3}{4}}$, by (11) we have $\frac{1}{2}n^{\frac{3}{4}}m \leqslant \frac{5}{4}n+\frac{1}{2}n^{\frac{3}{2}}$. That is,

$$m \leqslant \frac{5}{2}n^{\frac{1}{4}}+n^{\frac{3}{4}}. \tag{12}$$

We are now ready to show that we can always find a suitable choice for Q when applying the major step. We require a Q in $\mathcal{P}\setminus\mathcal{R}$ such that Q is adjacent in Y to each pair $P\in\mathcal{R}$, and such that Q is not already full. Let $k=|\mathcal{R}|$ and let $P\in\mathcal{R}$. There are $k-1$ other pairs in $\mathcal{R}$, none of which can be chosen as Q, and there are at most m full pairs, none of which can be chosen as Q. Thus, for each pair $P\in\mathcal{R}$, we have $\deg_Y(P)-(k-1)-m$ suitable choices for Q. Since $|\mathcal{R}|\in\{2,3,4\}$, it is sufficient to show that for each $k\in\{2,3,4\}$

$$k(\deg_Y(P)-(k-1)-m) \geqslant (k-1)(\tfrac{n}{2}-k)+1.$$

Showing this is routine using (10) and (12) (and the values of $g(n)$ and $h(n)$ given in (1) and (3)). This completes the proof for the case n is even.

Now assume that n is odd and that G has minimum degree $\delta(G)=(\frac{31}{32}+f(n))n$ where

$$f(n)=g(n)+\frac{97}{32}n^{-1}. \tag{13}$$

Let $\infty\in V(G)$ and let $\mathcal{P}$ be a partition of $V(G)\setminus\{\infty\}$ into pairs such that for each $P\in\mathcal{P}$ the number of vertices in P that are adjacent to ∞ is either 0 or 2. Let Z be the graph with vertex set $V(Z)=\mathcal{P}$ and an edge joining $\{x_1,y_1\}\in\mathcal{P}$ to $\{x_2,y_2\}\in\mathcal{P}$ if and only if (x_1,x_2,y_1,y_2) is a 4-cycle in G.

It follows easily from $\delta(G)\geqslant(\frac{31}{32}+f(n))n$, that $\delta(Z)\geqslant\frac{1}{2}|V(Z)|$, and so Z has a Hamilton cycle by Dirac's Theorem. Thus, there exists a derangement π of $\mathcal{P}$ such that $\{P,\pi(P)\}\in E(Z)$ for each $P\in\mathcal{P}$. For each $P=\{x,y\}\in\mathcal{P}$ such that $\{\infty,x\},\{\infty,y\}\in E(G)$, there is a 4-cycle $C_P=(\infty,x,w,y)$ in G where $w\in\pi(P)$. Let $\mathcal{B}=\{C_P : P=\{x,y\}\in\mathcal{P},\{\infty,x\},\{\infty,y\}\in E(G)\}$. The 4-cycles in $\mathcal{B}$ are edge disjoint and contain every edge of G that is incident with ∞. Moreover, any other vertex of G is in at most two 4-cycles of $\mathcal{B}$. Thus, if G^* is the graph obtained from G by deleting the edges of the 4-cycles in $\mathcal{B}$, then

$$\delta(G^*)\geqslant(\frac{31}{32}+f(n))n-4$$

and G^* is an even graph with size divisible by 4. It is routine to check that

$$(\frac{31}{32}+f(n))n-4=\frac{31}{32}(n-1)+ng(n)\geqslant(\frac{31}{32}+g(n-1))(n-1).$$

Thus, G^* is a simple even graph of even order $n'=n-1$ with size divisible by 4 and minimum degree $\delta(G^*)\geqslant(\frac{31}{32}+g(n'))n'$. We have already shown that such graphs have decompositions into 4-cycles. Thus, the union of $\mathcal{B}$ and a decomposition of G^* into 4-cycles is the required decomposition of G into 4-cycles. This completes the proof. $\square$

3 Decompositions of bipartite graphs

In this section we observe that essentially the same proof gives us a stronger result in the case where G is bipartite with the same number of vertices in each part. Our aim is to prove Theorem 2 below. We employ the terminology and method of the previous section, mentioning only the differences.

Let G be a simple even bipartite graph with parts of cardinality n and size divisible by 4. We first prove that if n is even and G has minimum degree $\delta(G) \geqslant (\frac{31}{32} + g(n))n$, where

$$g(n) = \tfrac{(1+2\sqrt{2})}{4} n^{-\frac{1}{4}} + \tfrac{5}{4} n^{-\frac{3}{4}} + \tfrac{67}{16} n^{-1}, \tag{14}$$

then G decomposes into 4-cycles. Observe that the complement $\overline{G}$ of G has maximum degree

$$\Delta(\overline{G}) \leqslant h(n) \tag{15}$$

where $h(n)$ is given in (3). Since $\delta(G) \geqslant (n+1)/2$, G has a Hamilton cycle H by the bipartite version of the Bondy-Chvátal Theorem (Theorem 6.2 in [2]). We define $\mathcal{P}$ as in the previous section with the extra condition that any pair from $\mathcal{P}$ contains two vertices from the same part of G.

The bound in (15) above effects the following changes in the expressions (7), (8), (9) and (10) from the previous proof (with change only to the constant terms):

$$\deg_{F_0}(P) \leqslant 4h(n) + 13; \tag{16}$$

$$r_{\mathcal{R}}(P) \leqslant 2h(n) + \tfrac{1}{2} n^{\frac{3}{4}} + \tfrac{19}{2}; \tag{17}$$

$$\deg_{Y_0}(P) \geqslant \tfrac{n}{2} - 2h(n) - 4 \tag{18}$$

and

$$\deg_{Y}(P) \geqslant \tfrac{n}{2} - 4h(n) - n^{\frac{3}{4}} - \tfrac{33}{2}. \tag{19}$$

Since G has $2n$ vertices and H has the same number of edges, $|E(F_0)| \leqslant \frac{5}{2}n + \sqrt{2}n^{\frac{3}{2}}$. In turn, (12) changes to:

$$m \leqslant 5n^{\frac{1}{4}} + 2\sqrt{2}n^{\frac{3}{4}}. \tag{20}$$

Next, when choosing Q, we choose a pair from the opposite part of G to that containing the pairs in $\mathcal{R}$. Thus, for each pair $P \in \mathcal{R}$, we have at least $\deg_Y(P) - m$ suitable choices for Q (in the worst case scenario, all full pairs belongs to the same part of G). Thus, it is sufficient to show for each pair $k \in \{2, 3, 4\}$

$$k(\deg_Y(P) - m) \geqslant (k-1)(n/2) + 1.$$

This follows from (14), (19) and (20) above.

Now assume that n is odd and that G has minimum degree $\delta(G) = (\frac{31}{32} + f(n))n$ where

$$f(n) = g(n) + \tfrac{97}{32} n^{-1}. \tag{21}$$

Let $\infty_1, \infty_2 \in V(G)$ be from distinct parts of G and let $\mathcal{P}$ be a partition of $V(G) \setminus \{\infty_1, \infty_2\}$ into pairs such that for $i \in \{1, 2\}$ and for each $P \in \mathcal{P}$, the number of vertices in P that are adjacent to ∞_i is either 0 or 2. Moreover, let $\mathcal{P}$ be such that the two vertices from any pair in $\mathcal{P}$ are from the same part of G. Let Z be the graph with vertex set $V(Z) = \mathcal{P}$ and an edge joining $\{x_1, y_1\} \in \mathcal{P}$ to $\{x_2, y_2\} \in \mathcal{P}$ if and only if (x_1, x_2, y_1, y_2) is a 4-cycle in G.

It follows easily from $\delta(G) \geqslant (\frac{31}{32}+f(n))n$, that $\delta(Z) \geqslant \frac{1}{2}|V(Z)|$, and so Z has a Hamilton cycle by the bipartite version of the Bondy-Chvátal Theorem (Theorem 6.2 in [2]). Thus, there exists a derangement π of $\mathcal{P}$ such that $\{P, \pi(P)\} \in E(Z)$ for each $P \in \mathcal{P}$. For each $P = \{x, y\} \in \mathcal{P}$ and $\infty \in \{\infty_1, \infty_2\}$ such that $\{\infty, x\}, \{\infty, y\} \in E(G)$, there is a 4-cycle $C_P = (\infty, x, w, y)$ in G where $w \in \pi(P)$. Let $\mathcal{B} = \{C_P : P = \{x, y\} \in \mathcal{P}, \{\infty, x\}, \{\infty, y\} \in E(G), \infty \in \{\infty_1, \infty_2\}\}$. The 4-cycles in $\mathcal{B}$ are edge disjoint and contain every edge of G that is incident with ∞_1 or ∞_2. Moreover, any other vertex of G is in at most two 4-cycles of $\mathcal{B}$. Thus, if G^* is the graph obtained from G by deleting the edges of the 4-cycles in $\mathcal{B}$, then

$$\delta(G^*) \geqslant (\frac{31}{32} + f(n))n - 4$$

and G^* is an even bipartite graph with parts of cardinality $n-1$ and size divisible by 4. The remainder of the proof follows as in the previous section. □

Theorem 2. *If G is a simple bipartite C_4-admissible graph with n vertices in each part and minimum degree at least $(\frac{31}{32} + o_n(1))n$, then G decomposes into C_4.*

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