THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

Computational Graph Theory

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B. Science (Honours)

A thesis submitted for the degree of Doctor of Philosophy at
The University of Queensland in 2014
School of Mathematics and Physics

Abstract

This thesis involves the application of computational techniques to various problems in graph theory and low dimensional topology. The first two chapters of this thesis focus on problems in graph theory itself; in particular on graph decomposition problems. The last three chapters look at applications of graph theory to combinatorial topology, focusing on the exhaustive generation of certain families of 3-manifold triangulations.

Chapter 1 shows that the obvious necessary conditions are sufficient for the existence of a decomposition of the complete graph into cycles of arbitrary specified lengths. This problem was formally posed in 1981 by Brian Alspach, but has its origins in the mid 1800s. A complete discussion of problem, as well as a full solution, is presented in Chapter 1. This work has been published, see [34].

Chapter 2 solves a problem closely related to the Oberwolfach Problem, which was originally posed by Gerhard Ringel at a graph theory conference in Oberwolfach in 1967. We show that if a complete multipartite graph K has even degree, and F is a bipartite two factor of K, then there exists a factorisation of K into F (with the exception that there is no factorisation of the 6-regular complete bipartite graph into the 2-factor consisting of two 6-cycles). This work has been published, see [27].

The latter chapters of this thesis deal with the use of graph theory in combinatorial topology; in particular combinatorial 3-manifold topology. Chapter 3 gives an introduction to the field of combinatorial topology, especially to the graph theoretic structures required for this thesis. Chapter 3 also gives an overview of the census enumeration problem, which we focus on for the last two chapters, and outlines an existing state-of-the-art algorithm for this problem.

In Chapter 4 we look at face pairing graphs of 3-manifold triangulations. When enumerating a census of triangulations, one often starts with a potential face pairing graph and attempts to flesh it out into a full 3-manifold triangulation. Computationally however, much time is spent on potential graphs which do not lead to any interesting triangulations. We show that determining whether such a graph will lead to a 3-manifold triangulation is fixed parameter tractable in the tree width of the graph. This work has been published, see [47].

In Chapter 5 we give a new census enumeration algorithm for 3-manifold triangulations. We use graph decompositions as the basis for this algorithm, which topologically is equivalent to identifying edges of tetrahedra together (as opposed to identifying faces together). We show that this algorithm complements existing state-of-the-art algorithms, potentially reducing census enumeration running times by a factor of two or more.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

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Publications during candidature

Cycle decompositions V: Complete graphs into cycles of arbitrary lengths
Darryn Bryant; Daniel Horsley; William Pettersson
Proceedings of the London Mathematical Society 2013; doi: 10.1112/plms/pdt051

Bipartite 2-Factorizations of Complete Multipartite Graphs
Darryn Bryant; Peter Danziger; William Pettersson
Journal of Graph Theory 2014; doi: 10.1002/jgt.21806

Fixed parameter tractable algorithms in combinatorial topology
Benjamin A. Burton; William Pettersson
20th International Computing and Combinatorics Conference 2014.

Publications included in this thesis

The work from the following publications is included in the respective chapters. In all cases, the candidate contributed substantial technical and intuitive knowledge and contributed significantly to the writing of the publication. The percentages given in the table are rough guidelines; as is normal in this field all authors contribute equally to each publication.

Incorporated as Chapter 1:
Cycle decompositions V: Complete graphs into cycles of arbitrary lengths
Darryn Bryant; Daniel Horsley; William Pettersson
Proceedings of the London Mathematical Society 2013; doi: 10.1112/plms/pdt051

Contributor	Statement of contribution
William Pettersson (Candidate)	Research (30%) Computational work (90%) Writing and editing (40%)
Darryn Bryant	Research (35%) Computational work (5%) Writing and editing (30%)
Daniel Horsley	Research (35%) Computational work (5%) Writing and editing (30%)

Incorporated as Chapter 2:
Bipartite 2-Factorizations of Complete Multipartite Graphs
Darryn Bryant; Peter Danziger; William Pettersson
Journal of Graph Theory 2014; doi: 10.1002/jgt.21806

Contributor	Statement of contribution
William Pettersson (Candidate)	Research (30%) Computational work (90%) Writing and editing (25%)
Darryn Bryant	Research (35%) Computational work (5%) Writing and editing (37.5%)
Peter Danziger	Research (35%) Computational work (5%) Writing and editing (37.5%)

Incorporated as Chapter 4:
Fixed parameter tractable algorithms in combinatorial topology
Benjamin A. Burton; William Pettersson
20th International Conference on Combinatorial Computing (COCOON) 2014.

Contributor	Statement of contribution
William Pettersson (Candidate)	Research (50 %) Writing and editing (50 %)
Benjamin A. Burton	Research (50 %) Writing and editing (50 %)

Contributions by others to the thesis

No contributions by others.

Statement of parts of the thesis submitted to qualify for the award of another degree

None.

Acknowledgements

Huge thanks first and foremost go to my supervisors Dr Darryn Bryant, Dr Barbara Maenhaut and Dr Benjamin A. Burton for their support, both academic and otherwise. Without them, reading this thesis would be a much harder undertaking. I would also like to extend these thanks to the wider community at the University of Queensland; from academic staff to the brave souls who keep the bush turkeys at bay.

I also wish to thank all of my family and friends for believing in me. Without their help and generosity I would not have achieved this goal.

Keywords

graph theory, decompositions, factorisations, topology, triangulations, algorithms, parameterised complexity

Australian and New Zealand Standard Research Classifications (ANZSRC)

ANZSRC code: 010104, Combinatorics and Discrete Mathematics, 66%
ANZSRC code: 010112, Topology, 34%

Fields of Research Classification

FoR code: 0101, Pure Mathematics, 100%

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Chapter 1

Cycle decompositions of complete graphs

1.1 Introduction

A decomposition of a graph K is a set of subgraphs of K whose edge sets partition the edge set of K. In 1981, Alspach [3] asked whether it is possible to decompose the complete graph on n vertices, denoted K_n, into t cycles of specified lengths $m_1, \ldots, m_t$ whenever the obvious necessary conditions are satisfied; namely that n is odd, $3 \le m_i \le n$, and $m_1 + \cdots + m_t = \binom{n}{2}$. He also asked whether it is possible to decompose K_n into a perfect matching and t cycles of specified lengths $m_1, \ldots, m_t$ whenever n is even, $3 \le m_i \le n$, and $m_1 + \cdots + m_t = \binom{n}{2} - \frac{n}{2}$. Again, these conditions are obviously necessary.

In this chapter we solve Alspach's problem by proving the following theorem.

Theorem 1.1.1. *There is a decomposition $\{G_1, \ldots, G_t\}$ of K_n in which G_i is an m_i-cycle for $i = 1, \ldots, t$ if and only if n is odd, $3 \le m_i \le n$ for $i = 1, \ldots, t$, and $m_1 + \cdots + m_t = \frac{n(n-1)}{2}$. There is a decomposition $\{G_1, \ldots, G_t, I\}$ of K_n in which G_i is an m_i-cycle for $i = 1, \ldots, t$ and I is a perfect matching if and only if n is even, $3 \le m_i \le n$ for $i = 1, \ldots, t$, and $m_1 + \cdots + m_t = \frac{n(n-2)}{2}$.*

Let K be a graph and let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \ge 3$ for $i = 1, \ldots, t$. If each vertex of K has even degree, then an (M)-decomposition of K is a decomposition $\{G_1, \ldots, G_t\}$ such that G_i is an m_i-cycle for $i = 1, \ldots, t$. If each vertex of K has odd degree, then an (M)-decomposition of K is a decomposition $\{G_1, \ldots, G_t, I\}$ such that G_i is an m_i-cycle for $i = 1, \ldots, t$ and I is a perfect matching in K.

We say that a list $(m_1, \ldots, m_t)$ of integers is *n-admissible* if $3 \le m_1, \ldots, m_t \le n$ and $m_1 + \cdots + m_t = n\lfloor \frac{n-1}{2} \rfloor$. Note that $n\lfloor \frac{n-1}{2} \rfloor = \binom{n}{2}$ if n is odd, and $n\lfloor \frac{n-1}{2} \rfloor = \binom{n}{2} - \frac{n}{2}$ if n is even. Thus, we can rephrase Alspach's question as follows. *Prove that for each n-admissible list M, there exists an (M)-decomposition of K_n.*

A decomposition of K_n into 3-cycles is equivalent to a *Steiner triple system of order n*, and a decomposition of K_n into n-cycles is a *Hamilton decomposition*. Thus, the work of Kirkman [67] and Walecki (see [6, 74]) from the 1800s addresses Alspach's problem in the cases where M is of the form $(3, 3, \ldots, 3)$ or $(n, n, \ldots, n)$. The next results on Alspach's problem appeared in the 1960s [69, 85, 86], and a multitude of results have appeared since then. Many of these focused on the case of decompositions into cycles of uniform length [7, 9, 16, 19, 58, 62, 63, 87], and a complete solution in this case was eventually obtained [5, 88].

There have also been many papers on the case where the lengths of the cycles in the decomposition may vary. In recent work [28, 29, 31], the first two authors have made progress by developing methods introduced in [30] and [32]. In [29], Alspach's problem is settled in the case where all the cycle lengths are greater than about $\frac{n}{2}$, and in [31] the problem is completely settled for sufficiently large odd n. Earlier results for the case of cycles of varying lengths can be found in [1, 2, 14, 35, 36, 58, 60, 66]. See [23] for a survey on Alspach's problem, and see [39] for a survey on cycle decompositions generally.

The analogous problems on decompositions of complete graphs into matchings, stars or paths have all been completely solved, see [15], [71] and [24] respectively. It is also worth mentioning that the easier problems in which each G_i is required only to be a closed trail of length m_i or each G_i is required only to be a 2-regular graph of order m_i have been solved in [13], [32] and [37]. Decompositions of complete multigraphs into cycles are considered in [33].

Balister [14] has verified by computer that Theorem 1.1.1 holds for $n \leq 14$, and we include this result as a lemma for later reference.

Lemma 1.1.2 ([14])**.** *Theorem 1.1.1 holds for $n \leq 14$.*

Our proof of Theorem 1.1.1 relies heavily on the reduction of Alspach's problem obtained in [31], see Theorem 1.1.3 below. Throughout this chapter, we use the notation $\nu_i(M)$ to denote the number of occurrences of i in a given list M.

Definition A list M is an *n-ancestor* list if it is n-admissible and satisfies

(1) $\nu_6(M) + \nu_7(M) + \cdots + \nu_{n-1}(M) \in \{0, 1\}$;

(2) if $\nu_5(M) \geq 3$, then $2\nu_4(M) \leq n - 6$;

(3) if $\nu_5(M) \geq 2$, then $3\nu_3(M) \leq n - 10$;

(4) if $\nu_4(M) \geq 1$ and $\nu_5(M) \geq 1$, then $3\nu_3(M) \leq n - 9$;

(5) if $\nu_4(M) \geq 1$, then $\nu_i(M) = 0$ for $i \in \{n-2, n-1\}$; and

(6) if $\nu_5(M) \geq 1$, then $\nu_i(M) = 0$ for $i \in \{n-4, n-3, n-2, n-1\}$.

Thus, an n-ancestor list is of the form

$$(3, 3, \ldots, 3, 4, 4, \ldots, 4, 5, 5, \ldots, 5, k, n, n, \ldots, n)$$

where k is either absent or in the range $6 \leq k \leq n-1$, and there are additional constraints involving the number of occurrences of cycle lengths in the list. The following theorem was proved in [31].

Theorem 1.1.3. ([31], Theorem 4.1) *For each positive integer n, if there exists an (M)-decomposition of K_n for each n-ancestor list M, then there exists an (M)-decomposition of K_n for each n-admissible list M.*

Our goal is to construct an (M)-decomposition of K_n for each n-ancestor list M. We split this problem into two cases: the case where $\nu_n(M) \geq 2$ and the case where $\nu_n(M) \leq 1$. In particular, we prove the following two results.

Lemma 1.1.4. *If M is an n-ancestor list with $\nu_n(M) \geq 2$, then there is an (M)-decomposition of K_n.*

Proof See Section 1.3. □

Lemma 1.1.5. *If Theorem 1.1.1 holds for K_{n-3}, K_{n-2} and K_{n-1}, then there is an (M)-decomposition of K_n for each n-ancestor list M satisfying $\nu_n(M) \leq 1$.*

Proof The case $\nu_n(M) = 0$ is proved in Section 1.4 (see Lemma 1.4.8) and the case $\nu_n(M) = 1$ is proved in Section 1.5 (see Lemma 1.5.22). □

Lemmas 1.1.4 and 1.1.5 allow us to prove our main result using induction on n.

Proof of Theorem 1.1.1 The proof is by induction on n. By Lemma 1.1.2, Theorem 1.1.1 holds for $n \leq 14$. So let $n \geq 15$ and assume Theorem 1.1.1 holds for complete graphs having fewer than n vertices. By Theorem 1.1.3, it suffices to prove the existence of an (M)-decomposition of K_n for each n-ancestor list M. Lemma 1.1.4 covers each n-ancestor list M with $\nu_n(M) \geq 2$, and using the inductive hypothesis, Lemma 1.1.5 covers those with $\nu_n(M) \leq 1$. □

1.2 Notation

We shall sometimes use superscripts to specify the number of occurrences of a particular integer in a list. That is, we define $(m_1^{\alpha_1}, \ldots, m_t^{\alpha_t})$ to be the list comprised of α_i occurrences of m_i for $i = 1, \ldots, t$. Let $M = (m_1^{\alpha_1}, \ldots, m_t^{\alpha_t})$ and let $M' = (m_1^{\beta_1}, \ldots, m_t^{\beta_t})$, where $m_1, \ldots, m_t$ are distinct. Then (M, M') is the list $(m_1^{\alpha_1+\beta_1}, \ldots, m_t^{\alpha_t+\beta_t})$ and, if $0 \leq \beta_i \leq \alpha_i$ for $i = 1, \ldots, t$, $M - M'$ is the list $(m_1^{\alpha_1-\beta_1}, \ldots, m_t^{\alpha_t-\beta_t})$.

Let Γ be a finite group and let S be a subset of Γ such that the identity of Γ is not in S and such that the inverse of any element of S is also in S. The *Cayley graph* on Γ with *connection set* S, denoted $\mathrm{Cay}(\Gamma, S)$, has the elements of Γ as its vertices and there is an edge between vertices g and h if and only if $g = hs$ for some $s \in S$.

A Cayley graph on a cyclic group is called a *circulant graph*. For any graph with vertex set $\mathbb{Z}_n$, we define the *length* of an edge xy to be $x - y$ or $y - x$, whichever is in $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. It is convenient to be able to describe the connection set of a circulant graph on $\mathbb{Z}_n$ by listing only one of s and $n - s$. Thus, we use the following notation. For any subset S of $\mathbb{Z}_n \setminus \{0\}$ such that $s \in S$ and $n - s \in S$ implies $n = 2s$, we define $\langle S \rangle_n$ to be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S \cup -S)$.

Let $m \in \{3, 4, 5\}$ and let $D = \{a_1, \ldots, a_m\}$ where $a_1, \ldots, a_m$ are positive integers. If there is a partition $\{D_1, D_2\}$ of D such that $\sum D_1 - \sum D_2 = 0$, then D is called a *difference m-tuple*. If there is a partition $\{D_1, D_2\}$ of D such that $\sum D_1 - \sum D_2 = 0 \pmod{n}$, then D is called a *modulo n difference m-tuple*. Clearly, any difference m-tuple is also a modulo n difference m-tuple for all n. We may use the terms difference triple, quadruple and quintuple respectively rather than 3-tuple, 4-tuple and 5-tuple. For $m \in \{3, 4, 5\}$, it is clear that if D is a difference m-tuple, then there is an (m^n)-decomposition of $\langle D \rangle_n$ for all $n \geq 2\max(D) + 1$, and that if D is a modulo n difference m-tuple, then there is an (m^n)-decomposition of $\langle D \rangle_n$.

We denote the complete graph with vertex set V by K_V and the complete bipartite graph with parts U and V by $K_{U,V}$. If G and H are graphs then $G - H$ is the graph with vertex set

$V(G) \cup V(H)$ and edge set $E(G) \setminus E(H)$. If G and H are graphs whose vertex sets are disjoint then $G \vee H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. A cycle with m edges is called an *m-cycle* and is denoted $(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m$ are the vertices of the cycle and $x_1x_2, \ldots, x_{m-1}x_m, x_mx_1$ are the edges. A path with m edges is called an *m-path* and is denoted $[x_0, \ldots, x_m]$, where $x_0, \ldots, x_m$ are the vertices of the path and $x_0x_1, \ldots, x_{m-1}x_m$ are the edges. A graph is said to be *even* if every vertex of the graph has even degree and is said to be *odd* if every vertex of the graph has odd degree.

A *packing* of a graph K is a decomposition of some subgraph G of K, and the graph $K - G$ is called the *leave* of the packing. An (M)-*packing* of K_n is an (M)-decomposition of some subgraph G of K_n such that G is an even graph if n is odd and G is an odd graph if n is even (recall that an (M)-decomposition of an odd graph contains a perfect matching). Thus, the leave of an (M)-packing of K_n is an even graph and, like an (M)-decomposition of K_n, an (M)-packing of K_n contains a perfect matching if and only if n is even. A decomposition of a graph into Hamilton cycles is called a *Hamilton decomposition*.

1.3 The case of at least two Hamilton cycles

The purpose of this section is to prove Lemma 1.1.4 which states that there is an (M)-decomposition of K_n for each n-ancestor list M with $\nu_n(M) \geq 2$. We first give a general outline of this proof. Theorem 1.1.1 has been proved in the case where $M = (3^a, n^b)$ for some $a, b \geq 0$ [36], so we will restrict our attention to ancestor lists which are not of this form. The basic construction involves decomposing K_n into $\langle S \rangle_n$ and $K_n - \langle S \rangle_n$ where, for some $x \leq 8$, the connection set S is either $\{1, \ldots, x\}$ or $\{1, \ldots, x-1\} \cup \{x+1\}$ so that $\sum S$ is even. We partition any given n-ancestor list M into two lists M_s and $\overline{M_s} = M - M_s$, and construct an (M_s)-decomposition of $\langle S \rangle_n$ and an $(\overline{M_s})$-decomposition of $K_n - \langle S \rangle_n$. This yields the desired (M)-decomposition of K_n. Taking $S = \{1, \ldots, x-1\} \cup \{x+1\}$, rather than $S = \{1, \ldots, x\}$, is necessary when $1 + \cdots + x$ is odd as many desired cycle decompositions of $\langle \{1, \ldots, x\} \rangle_n$ do not exist when $1 + \cdots + x$ is odd, see [38].

If $M = (3^{\alpha_3 n + \beta_3}, 4^{\alpha_4 n + \beta_4}, 5^{\alpha_5 n + \beta_5}, k^\gamma, n^\delta)$ where $\alpha_i \geq 0$ and $0 \leq \beta_i \leq n-1$ for $i \in \{3, 4, 5\}$, $6 \leq k \leq n-1$, $\gamma \in \{0, 1\}$, and $\delta \geq 2$, then we usually choose $M_s = (3^{\beta_3}, 4^{\beta_4}, 5^{\beta_5}, k^\gamma)$. However, if this would result in $\sum M_s$ being less than $4n$, then we sometimes adjust this definition slightly. We always choose M_s such that $\sum M_s$ is at most $8n$, which explains why we have $|S| \leq 8$.

Our (M_s)-decompositions of $\langle S \rangle_n$ will be constructed using adaptations of techniques used in [38] and [40]. We construct our $(\overline{M_s})$-decompositions of $K_n - \langle S \rangle_n$ using a combination of difference methods and results on Hamilton decompositions of circulant graphs. In general, we split the problem into the case $\nu_5(M) \leq 2$ and the case $\nu_5(M) \geq 3$. In the former case it will follow from our choice of M_s that $\overline{M_s} = (3^{tn}, 4^{qn}, n^h)$ for some $t, q, h \geq 0$ and in the latter case it will follow from our choice of M_s that $\overline{M_s} = (5^{rn}, n^h)$ for some $r, h \geq 0$.

The precise definition of M_s is given in Lemma 1.3.1, which details the properties that we require of our partition of M into M_s and $\overline{M_s}$, and establishes its existence. The definition includes several minor technicalities in order to deal with complications and exceptions that arise in the above-described approach. Throughout the remainder of this section, for a given n-ancestor list M such that $\nu_n(M) \geq 2$ and $M \neq (3^a, n^b)$ for any $a, b \geq 0$, we shall use the notation M_s and $\overline{M_s}$ to denote the lists constructed in the proof of Lemma 1.3.1. If $\nu_n(M) \leq 1$ or $M = (3^a, n^b)$ for some $a, b \geq 0$, then M_s and $\overline{M_s}$ are not defined.

Lemma 1.3.1. *If M is any n-ancestor list such that $\nu_n(M) \geq 2$ and $M \neq (3^a, n^b)$ for any*

$a, b \geq 0$, then there exists a partition of M into sublists M_s and $\overline{M_s}$ such that

(1) $\sum M_s \in \{2n, 3n, \ldots, 8n\}$ and $\sum M_s \neq 8n$ when $\nu_5(M) \leq 2$;

(2) if $\sum M_s = 2n$, then $\nu_n(M_s) = 1$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$;

(3) if $\sum M_s = 3n$, then $\nu_n(M_s) \in \{0, 1\}$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$;

(4) if $\sum M_s \in \{4n, 5n, \ldots, 8n\}$ and $\nu_5(M) \geq 3$, then $\nu_n(M_s) = 0$ and $\overline{M_s} = (5^{rn}, n^h)$ for some $r \geq 0$, $h \geq 2$;

(5) if $\sum M_s \in \{4n, 5n, \ldots, 7n\}$ and $\nu_5(M) \leq 2$, then $\nu_n(M_s) = 0$ and $\overline{M_s} = (3^{tn}, 4^{qn}, n^h)$ for some $t, q \geq 0$, $h \geq 2$; and

(6) $M_s \neq (3^{\frac{5n}{3}})$.

Proof Let M be an n-ancestor list. The conditions of the lemma imply $n \geq 7$. We will first define a list M_e which in many cases will serve as M_s, but will sometimes need to be adjusted slightly.

If

$$M = (3^{\alpha_3 n + \beta_3}, 4^{\alpha_4 n + \beta_4}, 5^{\alpha_5 n + \beta_5}, k^\gamma, n^\delta)$$

where $\alpha_i \geq 0$ and $0 \leq \beta_i \leq n-1$ for $i \in \{3, 4, 5\}$, $6 \leq k \leq n-1$, $\gamma \in \{0, 1\}$, and $\delta \geq 2$, then

$$M_e = (3^{\beta_3}, 4^{\beta_4}, 5^{\beta_5}, k^\gamma).$$

It is clear from the definition of n-ancestor list that if we take $M_s = M_e$, then (4) and (5) are satisfied.

We now show that $\sum M_e \in \{0, n, 2n, \ldots, 8n\}$, and that $\sum M_e \neq 8n$ when $\nu_5(M) \leq 2$. Noting that $\sum M_e \leq 3\beta_3 + 4\beta_4 + 5\beta_5 + (n-1)$ and separately considering the cases $\nu_5(M) \geq 3$, $\nu_5(M) \in \{1, 2\}$ and $\nu_5(M) = 0$, it is routine to use the definition of (M)-ancestor lists to show that $\sum M_e < 9n$, and that $\sum M_e < 8n$ when $\nu_5(M) \leq 2$. Thus, because it follows from $\sum M = n\lfloor \frac{n-1}{2} \rfloor$ and the definition of M_e that n divides $\sum M_e$, we have that $\sum M_e \in \{0, n, 2n, \ldots, 8n\}$, and that $\sum M_e \neq 8n$ when $\nu_5(M) \leq 2$.

If $\sum M_e \in \{4n, 5n, 6n, 7n, 8n\}$, then we let $M_s = M_e$. If $\sum M_e \in \{0, n, 2n, 3n\}$, then we define M_s by

$$M_s = \begin{cases} (M_e, 4^n) & \text{if } \alpha_4 > 0; \\ (M_e, 5^n) & \text{if } \alpha_4 = 0 \text{ and } \alpha_5 > 0; \\ (M_e, 3^n) & \text{if } \alpha_4 = \alpha_5 = 0 \text{ and } \alpha_3 > 0; \\ (M_e, n) & \text{if } \alpha_3 = \alpha_4 = \alpha_5 = 0 \text{ and } \sum M_e \in \{n, 2n\}; \\ M_e & \text{otherwise.} \end{cases}$$

Using the definition of M_s and the fact that M is an n-ancestor list with $M \neq (3^a, n^b)$ for any $a, b \geq 0$, it is routine to check that M_s satisfies (1)–(6). □

Before proving Lemma 1.1.4, we need a number of preliminary lemmas. The first three give us the necessary decompositions of $\langle S \rangle_n$ where $S = \{1, \ldots, x\}$ or $S = \{1, \ldots, x-1\} \cup \{x+1\}$ for some $x \leq 8$. Lemma 1.3.3 was proven independently in [22] and [84], and is a special case of Theorem 5 in [38]. Lemmas 1.3.2 and 1.3.4 will be proved in Section 1.6.

Lemma 1.3.2. *If*

$$S \in \{\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\},$$

$n \geq 2\max(S)+1$, *and* $M = (m_1,\ldots,m_t,k)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i = 1,\ldots,t$, $3 \leq k \leq n$, *and* $\sum M = |S|n$, *then there is an* (M)*-decomposition of* $\langle S\rangle_n$, *except possibly when*

- $S = \{1,2,3,4,6\}$, $n \equiv 3 \pmod 6$ *and* $M = (3^{\frac{5n}{3}})$*; or*
- $S = \{1,2,3,4,6\}$, $n \equiv 4 \pmod 6$ *and* $M = (3^{\frac{5n-5}{3}},5)$.

Proof See Section 1.6. □

Lemma 1.3.3. ([22, 84]) *If* $n \geq 5$ *and* $M = (m_1,\ldots,m_t,n)$ *is any list satisfying* $m_i \in \{3,\ldots,n\}$ *for* $i = 1,\ldots,t$, *and* $\sum M = 2n$, *then there is an* (M)*-decomposition of* $\langle\{1,2\}\rangle_n$.

Lemma 1.3.4. *If* $n \geq 7$ *and* $M = (m_1,\ldots,m_t,k,n)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i = 1,\ldots,t$, $3 \leq k \leq n$, *and* $\sum M = 3n$, *then there is an* (M)*-decomposition of* $\langle\{1,2,3\}\rangle_n$.

Proof See Section 1.6. □

We now present the lemmas which give us the necessary decompositions of $K_n - \langle S\rangle_n$. Lemma 1.3.5 was proved in [36] where it was used to prove Theorem 1.1.1 in the case where $M = (3^a, n^b)$ for some $a, b \geq 0$. Lemmas 1.3.6 and 1.3.7 give our main results on decompositions of $K_n - \langle S\rangle_n$. Lemma 1.3.6 is for the case $\nu_5(M) \leq 2$ and Lemma 1.3.7 is for the case $\nu_5(M) \geq 3$.

Lemma 1.3.5. ([36], Lemma 3.1) *If* $1 \leq h \leq \lfloor\frac{n-1}{2}\rfloor$, *then there is an* (n^h)*-decomposition of* $K_n - \langle\{1,\ldots,\lfloor\frac{n-1}{2}\rfloor - h\}\rangle_n$.

Lemma 1.3.6. *If* $S \in \{\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\}\}$ *and* $n \geq 2\max(S)+1$, $t \geq 0$, $q \geq 0$ *and* $h \geq 2$ *are integers satisfying* $3t+4q+h = \lfloor\frac{n-1}{2}\rfloor - |S|$, *then there is a* $(3^{tn}, 4^{qn}, n^h)$*-decomposition of* $K_n - \langle S\rangle_n$, *except possibly when* $h = 2$, $S = \{1,2,3,4,5,6,7\}$ *and*

- $n \in \{25, 26\}$ *and* $t = 1$*; or*
- $n = 31$ *and* $t = 2$.

Proof See Section 1.7. □

Lemma 1.3.7. *If* $S \in \{\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\}$ *and* $n \geq 2\max(S)+1$, $r \geq 0$ *and* $h \geq 2$ *are integers satisfying* $5r+h = \lfloor\frac{n-1}{2}\rfloor - |S|$, *then there is a* $(5^{rn}, n^h)$*-decomposition of* $K_n - \langle S\rangle_n$.

Proof See Section 1.7. □

We also need Lemmas 1.3.9 and 1.3.10 below to deal with cases arising from the possible exceptions in Lemmas 1.3.2 and 1.3.6 respectively. To prove Lemma 1.3.9 we use the following special case of Lemma 2.8 in [31].

Lemma 1.3.8. *If there exists an $(M, 4^2)$-decomposition of K_n in which there are two 4-cycles intersecting in exactly one vertex, then there exists an $(M, 3, 5)$-decomposition of K_n.*

Lemma 1.3.9. *If M is an n-ancestor list such that $\nu_n(M) \geq 2$, $M_s = (3^{\frac{5n-5}{3}}, 5)$ and $n \equiv 4 \pmod 6$ then there is an (M)-decomposition of K_n.*

Proof We will construct an $(\overline{M_s}, 3^{\frac{5n-8}{3}}, 4^2)$-decomposition of K_n in which two 4-cycles intersect in exactly one vertex. The required (M)-decomposition of K_n can then be obtained by applying Lemma 1.3.8.

By Lemma 1.3.6 there is an $(\overline{M_s})$-decomposition of $K_n - \langle\{1, 2, 3, 4, 6\}\rangle_n$, so it suffices to construct a $(3^{\frac{5n-8}{3}}, 4^2)$-decomposition of $\langle\{1, 2, 3, 4, 6\}\rangle_n$ in which the two 4-cycles intersect in exactly one vertex for all $n \equiv 4 \pmod 6$ with $n \geq 16$ (note that the conditions of the lemma imply $n \geq 16$). The union of the following two sets of cycles gives such a decomposition.

$$\{\ (0, 4, 2, 6), (2, 3, 5, 8), (1, 5, 7), (3, 4, 7), (3, 6, 9), (4, 5, 6)\ \}$$

$$\{\ (x + 6i, y + 6i, z + y6)\ : i \in \{0, \ldots, \tfrac{n-10}{6}\},\ (x, y, z) \in \{(4, 8, 10), (5, 9, 11), (6, 8, 12), (6, 7, 10), (7, 11, 13), (7, 8, 9), (9, 12, 15), (9, 10, 13), (10, 11, 12), (8, 11, 14)\ \}$$

□

Lemma 1.3.10. *If M is an n-ancestor list such that $\nu_5(M) \leq 2$, $\nu_n(M) = 2$, $\sum M_s = 7n$, and*

- *$n = 25$ and $\nu_3(\overline{M_s}) = 25$;*
- *$n = 26$ and $\nu_3(\overline{M_s}) = 26$; or*
- *$n = 31$ and $\nu_3(\overline{M_s}) = 62$;*

then there is an (M)-decomposition of K_n.

Proof We begin by showing that it is possible to partition M_s into two lists M_s^1 and M_s^2 such that $\sum M_s^1 = 3n$ and $\sum M_s^2 = 4n$. If $\nu_3(M_s) \geq n$ or $\nu_4(M_s) \geq n$, then clearly such a partition exists. Otherwise, $\nu_n(M_s) = 0$ by Property (5) of Lemma 1.3.1, and so by the definition of n-ancestor list and the hypotheses of this lemma, we have that

$$7n = \sum M_s \leq 3\nu_3(M_s) + 4\nu_4(M_s) + 10 + (n - 1).$$

It is routine to check, using $3\nu_3(M_s) \leq 3n - 3$ and $4\nu_4(M_s) \leq 4n - 4$, that $\nu_4(M_s) \geq \frac{3n-6}{4}$ and $\nu_3(M_s) \geq \frac{2n-5}{3}$. Thus for $n = 25$, $n = 26$ and $n = 31$, we can choose $M_s^1 = (3, 4^{18})$, $M_s^1 = (3^2, 4^{18})$, and $M_s^1 = (3^3, 4^{21})$ respectively. This yields the desired partition of M_s.

For $n = 25$ we note that $\langle\{1, 2, 3\}\rangle_n \cong \langle\{2, 4, 6\}\rangle_n$ (with $x \mapsto 2x$ being an isomorphism) and $\langle\{1, 2, 3, 4\}\rangle_n \cong \langle\{1, 7, 8, 9\}\rangle_n$ (with $x \mapsto 8x$ being an isomorphism). Since $\{3, 10, 12\}$ is a modulo 25 difference triple and $\langle\{5, 11\}\rangle_{25}$ has a Hamilton decomposition (by a result of Bermond et al [18], see Lemma 1.7.1), this gives us a decomposition of K_{25} into a copy of $\langle\{1, 2, 3\}\rangle_{25}$, a copy of $\langle\{1, 2, 3, 4\}\rangle_{25}$, twenty-five 3-cycles and two Hamilton cycles. By Lemma 1.3.2, there is an (M_s^1)-decomposition of $\langle\{1, 2, 3\}\rangle_{25}$ and an (M_s^2)-decomposition of $\langle\{1, 2, 3, 4\}\rangle_{25}$, and this gives us the required (M)-decomposition of K_{25}.

For $n = 26$ we note that $\langle\{1, 2, 3, 4\}\rangle_n \cong \langle\{5, 6, 10, 11\}\rangle_n$ (with $x \mapsto 5x$ being an isomorphism). Since $\{4, 8, 12\}$ is a difference triple and $\langle\{7, 9\}\rangle_{26}$ has a Hamilton decomposition (by a result of Bermond et al [18], see Lemma 1.7.1), this gives us a decomposition of K_{26} into a copy of $\langle\{1,2,3\}\rangle_{26}$, a copy of $\langle\{1,2,3,4\}\rangle_{26}$, twenty-six 3-cycles and two Hamilton cycles. By Lemma 1.3.2, there is an (M_s^1)-decomposition of $\langle\{1,2,3\}\rangle_{26}$ and an (M_s^2)-decomposition of $\langle\{1,2,3,4\}\rangle_{26}$, and this gives us the required (M)-decomposition of K_{26}.

For $n = 31$ we note that $\langle\{1,2,3,4\}\rangle_n \cong \langle\{4,8,12,15\}\rangle_n$ (with $x \mapsto 4x$ being an isomorphism). Since $\{5,6,11\}$ is a difference triple, $\{7,10,14\}$ is a modulo 31 difference triple, and $\langle\{9,13\}\rangle_{31}$ has a Hamilton decomposition (by a result of Bermond et al [18], see Lemma 1.7.1), this gives us a decomposition of K_{31} into a copy of $\langle\{1,2,3\}\rangle_{31}$, a copy of $\langle\{1,2,3,4\}\rangle_{31}$, sixty-two 3-cycles and two Hamilton cycles. By Lemma 1.3.2, there is an (M_s^1)-decomposition of $\langle\{1,2,3\}\rangle_{31}$ and an (M_s^2)-decomposition of $\langle\{1,2,3,4\}\rangle_{31}$, which yields required (M)-decomposition of K_{31}. $\square$

We can now prove Lemma 1.1.4 which states that if M is an n-ancestor list with $\nu_n(M) \geq 2$, then there is an (M)-decomposition of K_n.

Proof of Lemma 1.1.4 If $M = (3^a, n^b)$ for some $a, b \geq 0$, then we can use the main result from [36] to obtain an (M)-decomposition of K_n, so we can assume that $M \neq (3^a, n^b)$ for any $a, b \geq 0$. By Lemma 1.1.2 we can assume that $n \geq 15$. Partition M into M_s and $\overline{M_s}$. The proof splits into cases according to the value of $\sum M_s$, which by Lemma 1.3.1 is in $\{2n, 3n, \ldots, 8n\}$.

Case 1 Suppose that $\sum M_s = 2n$. In this case, from Property (2) of Lemma 1.3.1 we have $\nu_n(M_s) = 1$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$. The required decomposition of K_n can be obtained by combining an (M_s)-decomposition of $\langle\{1,2\}\rangle_n$ (which exists by Lemma 1.3.3) with a Hamilton decomposition of $K_n - \langle\{1,2\}\rangle_n$ (which exists by Lemma 1.3.5).

Case 2 Suppose that $\sum M_s = 3n$. In this case, from Property (3) of Lemma 1.3.1 we have $\nu_n(M_s) \in \{0,1\}$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$. The required decomposition of K_n can be obtained by combining an (M_s)-decomposition of $\langle\{1,2,3\}\rangle_n$ (which exists by Lemma 1.3.2 or 1.3.4) with a Hamilton decomposition of $K_n - \langle\{1,2,3\}\rangle_n$ (which exists by Lemma 1.3.5).

Case 3 Suppose that $\sum M_s \in \{4n, 5n, 6n, 7n, 8n\}$ and $\nu_5(M) \geq 3$. In this case, from Property (4) of Lemma 1.3.1 we have $\nu_n(M_s) = 0$ and $\overline{M_s} = (5^{rn}, n^h)$ for some $r \geq 0$, $h \geq 2$, and we also have $3\nu_3(M) \leq n - 10$ from the definition of n-ancestor list. We let

$$S \in \{\{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}, \{1,2,3,4,5,6,7,8\}\}$$

such that $|S| = \frac{1}{n}\sum M_s$ and obtain the required decomposition of K_n by combining an (M_s)-decomposition of $\langle S\rangle_n$ (which exists by Lemma 1.3.2), with an $(\overline{M_s})$-decomposition of $K_n - \langle S\rangle_n$ (which exists by Lemma 1.3.7). Note that the condition $3\nu_3(M) \leq n - 10$ implies that the required (M_s)-decomposition of $\langle S\rangle_n$ is not among the listed possible exceptions in Lemma 1.3.2. Note also that the condition $n \geq 2\max(S) + 1$ required in Lemmas 1.3.2 and 1.3.7 is easily seen to be satisfied because $n \geq 15$ and $\sum M_s \leq n\lfloor\frac{n-1}{2}\rfloor$.

Case 4 Suppose that $\sum M_s \in \{4n, 5n, 6n, 7n, 8n\}$ and $\nu_5(M) \leq 2$. In this case we have $\nu_n(M_s) = 0$ and $\overline{M_s} = (3^{tn}, 4^{qn}, n^h)$ for some $t, q \geq 0$, $h \geq 2$ (see Property (5) in Lemma 1.3.1), and $\sum M_s \neq 8n$ (see Property (1) in Lemma 1.3.1). We let

$$S \in \{\{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}\}$$

such that $|S| = \frac{1}{n}\sum M_s$. If Lemma 1.3.2 gives us an (M_s)-decomposition of $\langle S\rangle_n$ and Lemma 1.3.6 gives us an $(\overline{M_s})$-decomposition of $K_n - \langle S\rangle_n$, then we have the required decomposition of K_n. The condition $n \geq 2\max(S) + 1$ required in Lemmas 1.3.2 and 1.3.6 is satisfied because

$n \geq 15$. This leaves only the cases arising from the possible exceptions in Lemma 1.3.2 and Lemma 1.3.6, and these are covered by Lemmas 1.3.9 and 1.3.10 respectively. $\square$

1.4 The case of no Hamilton cycles

In this section we prove that Lemma 1.1.5 holds in the case $\nu_n(M) = 0$. In this case, for $n \geq 15$, one of $\nu_3(M)$, $\nu_4(M)$ and $\nu_5(M)$ must be sizable, and the proof splits into three cases accordingly. Each of these three cases splits into subcases according to whether n is even or odd. In each case we construct the required decomposition of K_n from a suitable decomposition of K_{n-1} or K_{n-2}.

1.4.1 Many 3-cycles and no Hamilton cycles

In Lemma 1.4.1 we construct the required decompositions of complete graphs of odd order and in Lemma 1.4.2 we construct the required decompositions of complete graphs of even order.

Lemma 1.4.1. *If n is odd, Theorem 1.1.1 holds for K_{n-1}, and $(M, 3^{\frac{n-1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-1}{2}})$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n - 1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that M is $(n-1)$-admissible. Thus, by assumption there is an (M)-decomposition $\mathcal{D}$ of K_U. Let I be the perfect matching in $\mathcal{D}$. Then

$$\mathcal{D} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-1}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(3^{\frac{n-1}{2}})$-decomposition of $K_{\{\infty\}} \vee I$. $\square$

Lemma 1.4.2. *If n is even, Theorem 1.1.1 holds for K_{n-1}, and $(M, 3^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-2}{2}})$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n - 1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-2)$ is $(n-1)$-admissible and so by assumption there is an $(M, n-2)$-decomposition $\mathcal{D}$ of K_U. Let C be an $(n-2)$-cycle in $\mathcal{D}$, let $\{I, I_1\}$ be a decomposition of C into two matchings, and let x be the vertex in $U \setminus V(C)$. Then

$$(\mathcal{D} \setminus \{C\}) \cup \{I + \infty x\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-2}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(3^{\frac{n-2}{2}})$-decomposition of $K_{\{\infty\}} \vee I_1$. $\square$

1.4.2 Many 4-cycles and no Hamilton cycles

Lemma 1.4.3. *If n is odd, Theorem 1.1.1 holds for K_{n-2}, and $(M, 4^{\frac{n+1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n+1}{2}})$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n-2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n+1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n-3$. Thus, it is easily seen that $(M, 5)$ is $(n-2)$-admissible and by assumption there is an $(M, 5)$-decomposition $\mathcal{D}$ of K_U.

Let C be a 5-cycle in $\mathcal{D}$ and let x, y and z be vertices of C such that x and y are adjacent in C and z is not adjacent to either x or y in C. Then

$$(\mathcal{D} \setminus \{C\}) \cup \mathcal{D}_1 \cup \mathcal{D}_2$$

is an $(M, 4^{\frac{n+1}{2}})$-decomposition of K_V, where

- $\mathcal{D}_1$ is a $(4^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U \setminus \{x,y,z\}}$; and
- $\mathcal{D}_2$ is a (4^3)-decomposition of $K_{\{\infty_1,\infty_2\},\{x,y,z\}} \cup [\infty_1, \infty_2] \cup C$.

These decompositions are straightforward to construct. □

Lemma 1.4.4. *If n is even, Theorem 1.1.1 holds for K_{n-2}, and $(M, 4^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n-2}{2}})$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n-2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n-3$. Thus, it is easily seen that M is $(n-2)$-admissible and by assumption there is an (M)-decomposition $\mathcal{D}$ of K_U. Let I be the perfect matching in $\mathcal{D}$. Then

$$(\mathcal{D} \setminus \{I\}) \cup \{I + \infty_1\infty_2\} \cup \mathcal{D}_1$$

is an $(M, 4^{\frac{n-2}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(4^{\frac{n-2}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U}$. □

1.4.3 Many 5-cycles and no Hamilton cycles

We will make use of the following lemma in this subsection and in Subsection 1.5.5.

Lemma 1.4.5. *If G is a 3-regular graph which contains a perfect matching and ∞ is a vertex not in $V(G)$, then there is a decomposition of $K_{\{\infty\}} \vee G$ into $\frac{1}{2}|V(G)|$ 5-cycles.*

Proof Let I be a perfect matching in G. Then $G - I$ is a 2-regular graph on the vertex set $V(G)$ and hence it can be given a coherent orientation O. Let

$$\mathcal{D} = \{(\infty, a, b, c, d) : bc \in E(I) \text{ and } (b, a), (c, d) \in E(O)\}$$

be a set of (undirected) 5-cycles. Because O contains exactly one arc directed from each vertex of $V(G)$, $|\mathcal{D}| = |E(I)| = \frac{1}{2}|V(G)|$ and each edge of G appears in exactly one cycle in $\mathcal{D}$. Further, because O contains exactly one arc directed to each vertex of $V(G)$, each edge of $K_{\{\infty\},V}$ appears in exactly one cycle in $\mathcal{D}$. Thus $\mathcal{D}$ is a decomposition of $K_{\{\infty\}} \vee G$ into $\frac{1}{2}|V(G)|$ 5-cycles. □

Lemma 1.4.6. *If n is odd, Theorem 1.1.1 holds for K_{n-1}, and $(M, 5^{\frac{n-1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{\frac{n-1}{2}})$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since the list $(M, n-1)$ is easily seen to be $(n-1)$-admissible, by assumption there is an $(M, n-1)$-decomposition $\mathcal{D}$ of K_U. Let C be an $(n-1)$-cycle in $\mathcal{D}$ and let I be the perfect matching in $\mathcal{D}$. Then

$$(\mathcal{D} \setminus \{C, I\}) \cup \mathcal{D}_1$$

is an $(M, 5^{\frac{n-1}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(5^{\frac{n-1}{2}})$-decomposition of $K_{\{\infty\}} \vee (C \cup I)$ (this exists by Lemma 1.4.5). □

Lemma 1.4.7. *If n is even, Theorem 1.1.1 holds for K_{n-2}, and $(M, 5^{n-2})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{n-2})$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n-2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 5^{n-2})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (6) in the definition of ancestor lists that any cycle length in M is at most $n-5$. Thus, it is easily seen that the list $(M, (n-2)^3)$ is $(n-2)$-admissible and by assumption there is an $(M, (n-2)^3)$-decomposition $\mathcal{D}$ of K_U. Let C_1, C_2 and C_3 be distinct $(n-2)$-cycles in $\mathcal{D}$ and let I be the perfect matching in $\mathcal{D}$. Let $\{I_1, I_2\}$ be a decomposition of C_3 into two perfect matchings. Then

$$(\mathcal{D} \setminus \{C_1, C_2, C_3, I\}) \cup \{I + \infty_1\infty_2\} \cup \mathcal{D}_1 \cup \mathcal{D}_2$$

is an $(M, 5^{n-2})$-decomposition of K_V, where for $i = 1, 2$, $\mathcal{D}_i$ is a $(5^{\frac{n-2}{2}})$-decomposition of $K_{\{\infty_i\}} \vee (C_i \cup I_i)$ (these exist by Lemma 1.4.5). □

1.4.4 Proof of Lemma 1.1.5 in the case of no Hamilton cycles

Lemma 1.4.8. *If Theorem 1.1.1 holds for K_{n-1} and K_{n-2}, then there is an (M)-decomposition of K_n for each n-ancestor list M satisfying $\nu_n(M) = 0$.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. If there is a cycle length in M which is at least 6 and at most $n-1$, then let k be this cycle length. Otherwise let $k = 0$. We deal separately with the case n is odd and the case n is even.

Case 1 Suppose that n is odd. Since $n \geq 15$ and $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k = \frac{n(n-1)}{2}$, it can be seen that either $\nu_3(M) \geq \frac{n-1}{2}$, $\nu_4(M) \geq \frac{n+1}{2}$ or $\nu_5(M) \geq \frac{n-1}{2}$. If $\nu_3(M) \geq \frac{n-1}{2}$, then the result follows by Lemma 1.4.1. If $\nu_4(M) \geq \frac{n+1}{2}$, then the result follows by Lemma 1.4.3. If $\nu_5(M) \geq \frac{n-1}{2}$, then the result follows by Lemma 1.4.6.

Case 2 Suppose that n is even. Since $n \geq 16$, $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k = \frac{n(n-2)}{2}$ and $k \leq n-1$, it can be seen that either $\nu_3(M) \geq \frac{n-2}{2}$, $\nu_4(M) \geq \frac{n-2}{2}$ or $\nu_5(M) \geq n-2$. (To see this consider the cases $\nu_5(M) \geq 3$ and $\nu_5(M) \leq 2$ separately and use the definition of n-ancestor list.) If $\nu_3(M) \geq \frac{n-2}{2}$, then the result follows by Lemma 1.4.2. If $\nu_4(M) \geq \frac{n-2}{2}$, then the result follows by Lemma 1.4.4. If $\nu_5(M) \geq n-2$, then the result follows by Lemma 1.4.7. □

1.5 The case of exactly one Hamilton cycle

In this section we prove that Lemma 1.1.5 holds in the case $\nu_n(M) = 1$. Again in this case, for $n \geq 15$, one of $\nu_3(M)$, $\nu_4(M)$ and $\nu_5(M)$ must be sizable, and the proof splits into cases accordingly. The case in which $\nu_3(M)$ is sizable further splits according to whether $\nu_4(M) \geq 1$, $\nu_5(M) \geq 1$, or $\nu_4(M) = \nu_5(M) = 0$. We first require some preliminary definitions and results.

1.5.1 Preliminaries

Let $\mathcal{P}$ be an (M)-packing of K_n, let $\mathcal{P}'$ be an (M')-packing of K_n and let S be a subset of $V(K_n)$. We say that $\mathcal{P}$ and $\mathcal{P}'$ are *equivalent on* S if we can write $\{G \in \mathcal{P} : V(G) \cap S \neq \emptyset\} = \{G_1, \ldots, G_t\}$ and $\{G \in \mathcal{P}' : V(G) \cap S \neq \emptyset\} = \{G'_1, \ldots, G'_t\}$ such that

- for $i \in \{1, \ldots, t\}$, G_i is isomorphic to G'_i;
- for each $x \in S$ and for $i \in \{1, \ldots, t\}$, $x \in V(G_i)$ if and only if $x \in V(G'_i)$; and
- for all distinct $x, y \in S$ and for $i \in \{1, \ldots, t\}$, $xy \in E(G_i)$ if and only if $xy \in E(G'_i)$.

The following lemma is from [29]. It encapsulates a key *edge swapping* technique which was used in many of the proofs in [31], and which we shall make use of in this section.

Lemma 1.5.1. ([29]), Lemma 2.1) *Let n be a positive integer, let M be a list of integers, let $\mathcal{P}$ be an (M)-packing of K_n with a leave, L say, let α and β be vertices of L, let π be the transposition $(\alpha\beta)$, and let $Z = Z(\mathcal{P}, \alpha, \beta) = (\mathrm{Nbd}_L(\alpha) \cup \mathrm{Nbd}_L(\beta)) \setminus ((\mathrm{Nbd}_L(\alpha) \cap \mathrm{Nbd}_L(\beta)) \cup \{\alpha, \beta\})$. Then there exists a partition of the set Z into pairs such that for each pair $\{u, v\}$ of the partition, there exists an (M)-packing of K_n, $\mathcal{P}'$ say, with a leave, L' say, which differs from L only in that αu, αv, βu and βv are edges in L' if and only if they are not edges in L. Furthermore, if $\mathcal{P} = \{C_1, \ldots, C_t\}$ (n odd) or $\mathcal{P} = \{I, C_1, \ldots, C_t\}$ (n even) where $C_1, \ldots, C_t$ are cycles and I is a perfect matching, then $\mathcal{P}' = \{C'_1, \ldots, C'_t\}$ (n odd) or $\mathcal{P}' = \{I', C'_1, \ldots, C'_t\}$ (n even) where for $i = 1, \ldots, t$, C'_i is a cycle of the same length as C_i and I' is a perfect matching such that*

- *either $I' = I$ or $I' = \pi(I)$;*
- *for $i = 1, \ldots, t$ if neither α nor β is in $V(C_i)$ then $C'_i = C_i$;*
- *for $i = 1, \ldots, t$ if exactly one of α and β is in $V(C_i)$ then either $C'_i = C_i$ or $C'_i = \pi(C_i)$; and*
- *for $i = 1, \ldots, t$ if both α and β are in $V(C_i)$ then $C'_i \in \{C_i, \pi(C_i), \pi(P_i) \cup P_i^\dagger, P_i \cup \pi(P_i^\dagger)\}$ where P_i and $P_i^\dagger$ are the two paths in C_i which have endpoints α and β.*

We say that $\mathcal{P}'$ is the (M)-packing obtained from $\mathcal{P}$ by performing the (α, β)-switch with origin u and terminus v (we could equally call v the origin and u the terminus). For our purposes here, it is important to note that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V(L) \setminus \{\alpha, \beta\}$.

We will also make use of three lemmas from [31]. The original version of Lemma 1.5.2 (Lemma 2.15 in [31]) does not include the claim that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V(L) \setminus \{a, b\}$, this follows directly from the fact that the proof uses only (a, b)-switches.

Lemma 1.5.2. *Let n be a positive integer and let M be a list of integers. Suppose that there exists an (M)-packing $\mathcal{P}$ of K_n with a leave L which contains two vertices a and b such that $\deg_L(a) + 2 \leq \deg_L(b)$. Then there exists an (M)-packing $\mathcal{P}'$ of K_n, which is equivalent to $\mathcal{P}$ on $V(L) \setminus \{a, b\}$, and which has a leave L' such that $\deg_{L'}(a) = \deg_L(a) + 2$, $\deg_{L'}(b) = \deg_L(b) - 2$ and $\deg_{L'}(x) = \deg_L(x)$ for all $x \in V(L) \setminus \{a, b\}$. Furthermore,* *(i) if a and b are adjacent in L, then L' has the same number of non-trivial components as L;*

(ii) if $\deg_L(a) = 0$ and b is not a cut-vertex of L, then L' has the same number of non-trivial components as L; and

(iii) if $\deg_L(a) = 0$, then either L' has the same number of non-trivial components as L, or L' has one more non-trivial component than L.

Similarly, the original versions of Lemmas 1.5.3 and 1.5.4 (Lemmas 2.14 and 2.11 respectively in [31]) did not include the claims that the final decompositions are equivalent to the initial packings on $V \setminus U$. However, these claims can be seen to hold as the proofs of the lemmas given in [31] require switching only on vertices of positive degree in the leave, with one exception which we discuss shortly. The lemmas below each contain the additional hypothesis that $\deg_L(x) = 0$ for all $x \in V \setminus U$, and this ensures all the switches are on vertices of U and hence that the final decomposition is equivalent to the initial packing on $V \setminus U$.

The exception mentioned above occurs in the proof of the original version of Lemma 1.5.4 where a switch on a vertex of degree 0 in the leave is required when $3 \in \{m_1, m_2\}$. We can ensure this switch is on a vertex in U because we have the additional hypothesis that $\deg_L(x) = 0$ for some $x \in U$ when $3 \in \{m_1, m_2\}$. This additional hypothesis also allows us to omit the hypothesis, included in the original version of Lemma 1.5.4, that the size of the leave be at most $n+1$, because in the proof this was used only to ensure the existence of a vertex of degree 0 in the leave when $3 \in \{m_1, m_2\}$. Thus the modified versions stated below hold by the proofs presented in [31].

Lemma 1.5.3. *Let V be a vertex set and let U be a subset of V. Let M be a list of integers and let k, m_1 and m_2 be positive integers such that $m_1, m_2 \geq \max(\{3, k+1\})$. Suppose that there exists an (M)-packing $\mathcal{P}$ of K_V with a leave L of size $m_1 + m_2$ such that $\Delta(L) = 4$, exactly one vertex of L has degree 4, L has exactly k non-trivial components, L does not have a decomposition into two odd cycles if m_1 and m_2 are both even, and $\deg_L(x) = 0$ for all $x \in V \setminus U$. Then there exists an (M, m_1, m_2)-decomposition of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.*

Lemma 1.5.4. *Let V be a vertex set and let U be a subset of V. Let M be a list of integers and let m_1 and m_2 be integers such that $m_1, m_2 \geq 3$. Suppose that there exists (M)-packing $\mathcal{P}$ of K_V with a leave L of size $m_1 + m_2$ such that $\Delta(L) = 4$, exactly two vertices of L have degree 4, L has exactly one non-trivial component, $\deg_L(x) = 0$ for all $x \in V \setminus U$, and $\deg_L(x) = 0$ for some $x \in U$ if $3 \in \{m_1, m_2\}$. Then there exists an (M, m_1, m_2)-decomposition of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.*

We also require Lemma 1.5.5, which deals with some small order cases.

Lemma 1.5.5. *Let n be an integer such that $n \in \{15, 16, 17, 18, 19, 20, 22, 24, 26\}$ and let M be an n-ancestor list such that $\nu_n(M) = 1$, $\nu_5(M) \geq 3$, and $\nu_4(M) \geq 2$ if $n = 24$. Then there is an (M)-decomposition of K_n.*

Proof If there is a cycle length in M which is at least 6 and at most $n-1$, then let k be this cycle length. Otherwise let $k = 0$. Note that $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k + n = n\lfloor \frac{n-1}{2} \rfloor$ and that, because M is an n-ancestor list with $\nu_5(M) \geq 3$, it follows that $3\nu_3(M) \leq n - 10$, $2\nu_4(M) \leq n - 6$, and $k \leq n - 5$.

Using this, it is routine to check that if $n = 15$ then M must be one of 12 possible lists and if $n = 16$ then M must be one of 26 possible lists. In each of these cases we have constructed an (M)-decomposition of K_n by computer search.

If $n \in \{17, 18, 19, 20, 22, 24, 26\}$, then we partition $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into sets S_1, S_2 and S_3 according to the following table.

n	S_1	S_2	S_3
17	$\{1,2,3,4,5,6,7\}$	$\emptyset$	$\{8\}$
18	$\{1,2,3,4,5,6,7\}$	$\emptyset$	$\{8,9\}$
19	$\{1,2,3,4,5,6,7,8\}$	$\emptyset$	$\{9\}$
20	$\{1,2,3,4,5,6,7,8\}$	$\emptyset$	$\{9,10\}$
22	$\{1,2,3,4\}$	$\{6,7,8,9,10\}$	$\{5,11\}$
24	$\{1,2,3\}$	$\{4,6,7,8,11\}$	$\{5,9,10,12\}$
26	$\{1,2,3,4,5,7\}$	$\{6,8,9,10,11\}$	$\{12,13\}$

Using $3\nu_3(M) \leq n - 10$, $2\nu_4(M) \leq n - 6$ and $k \leq n - 5$, it is routine to check that $\nu_5(M) \geq n$ when $n \in \{22, 26\}$, and that $\nu_5(M) \geq n + 8$ when $n = 24$. By Lemma 1.3.2, there is an (M')-decomposition of $\langle S_1 \rangle_n$, where $M = (M', n)$ when $n \in \{17, 18, 19, 20\}$, $M = (M', 5^n, n)$ when $n \in \{22, 26\}$, and $M = (M', 4^2, 5^{n+8}, n)$ when $n = 24$. For $n \in \{22, 24, 26\}$, it is easy to see that S_2 is a modulo n difference 5-tuple, and so there is a (5^n)-decomposition of $\langle S_2 \rangle_n$. For $n \in \{17, 18, 19, 20, 22, 26\}$, there is an (n)-decomposition of the graph $\langle S_3 \rangle_n$, as it is either an n-cycle or a connected 3-regular Cayley graph on a cyclic group, and the latter are well known to contain a Hamilton cycle, see [50]. For $n = 24$, $\langle \{5, 9, 10\} \rangle_n \cong \langle \{1, 2, 3\} \rangle_n$ (with $x \mapsto 5x$ being an isomorphism). Thus, by Lemma 1.3.4 there is a $(4^2, 5^8, 24)$-decomposition of $\langle \{5, 9, 10, 12\} \rangle_{24}$ (as $\langle \{12\} \rangle_{24}$ is a perfect matching). Combining these decompositions of $\langle S_1 \rangle_n$, $\langle S_2 \rangle_n$ and $\langle S_3 \rangle_n$ gives us the required (M)-decomposition of K_n. □

1.5.2 Many 3-cycles, one Hamilton cycle, and at least one 4- or 5-cycle

In Lemmas 1.5.6 and 1.5.7 we construct the required decompositions of complete graphs of odd order in the cases where the decomposition contains at least one 4-cycle or at least one 5-cycle, respectively. In Lemmas 1.5.8 and 1.5.9 we construct the required decompositions of complete graphs of even order in the cases where the decomposition contains at least one 4-cycle or at least one 5-cycle, respectively. These results are proved by constructing the required decomposition of K_n from a suitable decomposition of K_{n-1}, K_{n-2} or K_{n-3}.

Lemma 1.5.6. *If n is odd, Theorem 1.1.1 holds for K_{n-1}, and $(M, 3^{\frac{n-5}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-5}{2}}, 4, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n - 1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-5}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n - 2)$ is $(n - 1)$-admissible and by assumption there is an $(M, n - 2)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-2)$-cycle in $\mathcal{D}$, let I be the perfect matching in $\mathcal{D}$, and let $[w, x, y, z]$ be a path in $I \cup H$ such that $w \notin V(H)$, $wx, yz \in E(I)$ and $xy \in E(H)$. Then

$$(\mathcal{D} \setminus \{I, H\}) \cup \{H', (\infty, x, y, z)\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-5}{2}}, 4, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, w, \infty, y]$; and
- $\mathcal{D}_1$ is a $(3^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty\},U\setminus\{w,x,y,z\}} \cup (I - \{wx, yz\})$. $\square$

Lemma 1.5.7. *If n is odd, Theorem 1.1.1 holds for K_{n-1}, and $(M, 3^{\frac{n-5}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-5}{2}}, 5, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n - 1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-5}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-1)$ is $(n-1)$-admissible and so by assumption there is an $(M, n-1)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-1)$-cycle in $\mathcal{D}$, let I be the perfect matching in $\mathcal{D}$, and let $[w, x, y, z]$ be a path in $I \cup H$ such that $wx, yz \in E(I)$ and $xy \in E(H)$. Then

$$(\mathcal{D} \setminus \{I, H\}) \cup \{H', (\infty, w, x, y, z)\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-5}{2}}, 5, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty, y]$; and
- $\mathcal{D}_1$ is a $(3^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty\},U\setminus\{w,x,y,z\}} \cup (I - \{wx, yz\})$. $\square$

Lemma 1.5.8. *If n is even, Theorem 1.1.1 holds for K_{n-3}, and $(M, 3^{\frac{3n-14}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{3n-14}{2}}, 4, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n - 3$, let ∞_1, ∞_2 and ∞_3 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2, \infty_3\}$. Since $(M, 3^{\frac{3n-14}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n - 3$. Thus, it is easily seen that $(M, (n-4)^2, n-3)$ is $(n-3)$-admissible and so by assumption there is an $(M, (n-4)^2, n-3)$-decomposition $\mathcal{D}$ of K_U.

Let C_1 and C_2 be distinct $(n-4)$-cycles in $\mathcal{D}$, let H be an $(n-3)$-cycle in $\mathcal{D}$, let $\{I, I_1\}$ be a decomposition of C_1 into two matchings, let $\{I_2, I_3\}$ be a decomposition of C_2 into two matchings, let w be the vertex in $U \setminus V(C_1)$, and let $[x, y, z]$ be a path in $H \cup I_3$ such that $x \notin V(C_2)$, $xy \in E(H)$ and $yz \in E(I_3)$ (possibly $w \in \{x, y, z\}$). Then

$$(\mathcal{D} \setminus \{H, C_1, C_2\}) \cup \{I + \{\infty_1 w, \infty_2\infty_3\}, H', (\infty_3, x, y, z)\} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

is an $(M, 3^{\frac{3n-14}{2}}, 4, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty_2, \infty_1, \infty_3, y]$;
- for $i = 1, 2$, $\mathcal{D}_i$ is a $(3^{\frac{n-4}{2}})$-decomposition of $K_{\{\infty_i\}} \vee I_i$; and
- $\mathcal{D}_3$ is a $(3^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty_3\},U\setminus\{x,y,z\}} \cup (I_3 - yz)$. $\square$

Lemma 1.5.9. *If n is even, Theorem 1.1.1 holds for K_{n-1}, and $(M, 3^{\frac{n-6}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-6}{2}}, 5, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-6}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-2, n-1)$ is $(n-1)$-admissible and so by assumption there is an $(M, n-2, n-1)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-1)$-cycle in $\mathcal{D}$, let C be an $(n-2)$-cycle in $\mathcal{D}$, let $\{I, I_1\}$ be a decomposition of C into two matchings, let $[w, x, y, z]$ be a path in $I_1 \cup H$ such that $wx, yz \in E(I_1)$ and $xy \in E(H)$, and let v be the vertex in $U \setminus V(C)$. Then

$$(\mathcal{D} \setminus \{C, H\}) \cup \{I + v\infty, H', (\infty, w, x, y, z)\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-6}{2}}, 5, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty, y]$; and
- $\mathcal{D}_1$ is a $(3^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty\}, U \setminus \{v,w,x,y,z\}} \cup (I - \{wx, yz\})$. □

1.5.3 Many 3-cycles, one Hamilton cycle and no 4- or 5-cycles

We show that the required decompositions exist in Lemma 1.5.13. We first require three preliminary lemmas. These results are proved using the edge swapping techniques mentioned previously.

Lemma 1.5.10. *Let n and k be positive integers, and let M be a list of integers. If there exists an (M)-packing of K_n whose leave has a decomposition into two 3-cycles, T_1 and T_2, and a k-cycle C such that $|V(T_1) \cap V(T_2)| = 1$, $|V(T_2) \cap V(C)| = 1$ and $V(T_1) \cap V(C) = \emptyset$, then there exists an $(M, 3, k+3)$-decomposition of K_n.*

Proof Let $\mathcal{P}$ be an (M)-packing of K_n which satisfies the conditions of the lemma and let L be its leave. Let $[w, x, y, z]$ be a path in L such that $w \in V(T_1) \setminus V(T_2)$, $V(T_1) \cap V(T_2) = \{x\}$, $V(T_2) \cap V(C) = \{y\}$, and $z \in V(C) \setminus V(T_2)$. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (w, z)-switch S with origin x. If the terminus of S is y, then the leave of $\mathcal{P}'$ has a decomposition into the 3-cycle T_2 and a $(k+3)$-cycle. Otherwise the terminus of S is not y and the leave of $\mathcal{P}'$ has a decomposition into the 3-cycle (x, y, z) and a $(k+3)$-cycle. In either case we complete the proof by adding these cycles to $\mathcal{P}'$. □

Lemma 1.5.11. *Let n and k be positive integers such that $k \leq n-4$, and let M be a list of integers. Suppose that there exists an (M)-packing of K_n whose leave has a decomposition into two 3-cycles, T_1 and T_2, and a k-cycle C such that $|V(T_1) \cap V(C)| \leq 1$, $|V(T_2) \cap V(C)| \leq 1$, and $|V(T_1) \cap V(T_2)| = 1$ if $k = n-4$. Then there exists an $(M, 3, k+3)$-decomposition of K_n.*

Proof Let $\mathcal{P}$ be an (M)-packing of K_n which satisfies the conditions of the lemma and let L be its leave.

Case 1 Suppose that $\Delta(L) = 2$. Then T_1, T_2 and C are pairwise vertex disjoint. Let $x \in V(T_1)$ and $y \in V(C)$ and let z be a neighbour in T_1 of x. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (x, y)-switch S with origin z, and let L' be the leave of $\mathcal{P}'$. Then either the non-trivial components of L' are a 3-cycle and a $(k+3)$-cycle or $\Delta(L') = 4$, exactly one vertex of L' has degree 4, and L' has exactly two nontrivial components. In the former case we can add these cycles to $\mathcal{P}'$ to complete the proof. In the latter case we can apply Lemma 1.5.3 to complete the proof.

Case 2 Suppose that $\Delta(L) = 4$. If exactly one vertex of L has degree 4, then L has exactly two nontrivial components and we can complete the proof by applying Lemma 1.5.3. Thus we can assume that L has at least two vertices of degree 4. We can further assume that $\mathcal{P}$ does not satisfy the conditions of Lemma 1.5.10, for otherwise we can complete the proof by applying Lemma 1.5.10. Noting that $|V(T_1) \cap V(T_2)| \in \{0, 1\}$, it follows that $|V(T_1) \cap V(C)| = 1$ and $|V(T_2) \cap V(C)| = 1$. Because $k \leq n - 4$ and $|V(T_1) \cap V(T_2)| = 1$ if $k = n - 4$, there is an isolated vertex z in L. Let w be the vertex in $V(T_1) \cap V(C)$, let x and y be the neighbours in T_1 of w. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (w, z)-switch S with origin x, and let L' be the leave of $\mathcal{P}'$. If the terminus of S is not y, then L' has a decomposition into a $(k + 3)$-cycle and a 3-cycle, and we complete the proof by adding these cycles to $\mathcal{P}'$. Otherwise the terminus of S is y and either $\Delta(L') = 4$, exactly one vertex of L' has degree 4, and L' has exactly two nontrivial components (this occurs when $|V(T_1) \cap V(T_2)| = 0$) or $\mathcal{P}'$ satisfies the conditions of Lemma 1.5.10 (this occurs when $|V(T_1) \cap V(T_2)| = 1$). Thus we can complete the proof by applying Lemma 1.5.3 or Lemma 1.5.10.

Case 3 Suppose that $\Delta(L) \geq 6$. In this case, exactly one vertex of L has degree 6 and every other vertex of L has degree at most 2. Let w be the vertex of degree 6 in L, let x be a neighbour in T_2 of w, and let y and z be the neighbours in T_1 of w. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (w, x)-switch S with origin y, and let L' be the leave of $\mathcal{P}'$. If the terminus of S is z, then $\mathcal{P}'$ satisfies the conditions of Lemma 1.5.10 and we complete the proof by applying Lemma 1.5.10. Otherwise the terminus of S is not z and L' has a decomposition into the 3-cycle T_2 and a $(k + 3)$-cycle, and we complete the proof by adding these cycles to $\mathcal{P}'$. □

Lemma 1.5.12. *Let n, k and t be positive integers such that $3 \leq k \leq n - 4$. If there exists a $(3^t, k, n)$-decomposition of K_n, then there exists a $(3^{t-1}, k + 3, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. Let V be a vertex set with $|V| = n$, let $\mathcal{D}$ be a $(3^t, k, n)$-decomposition of K_V, and let C be a k-cycle in $\mathcal{D}$. Let $U = V \setminus V(C)$. The n-cycle in $\mathcal{D}$ contains at most $|U| - 1$ edges of K_U (as the subgraph of the n-cycle induced by U is a forest). Also, if n is even, then the perfect matching in $\mathcal{D}$ contains at most $\lfloor \frac{1}{2}|U| \rfloor$ edges of K_U. The proof now splits into two cases depending on whether $k = n - 4$.

Case 1 Suppose that $k \leq n - 5$. Then $|U| \geq 5$ and, by the comments in the preceding paragraph, the 3-cycles in $\mathcal{D}$ contain at least four edges of K_U. Thus there are distinct 3-cycles $T_1, T_2 \in \mathcal{D}$ such that each contains at least one edge of K_U. We can remove C, T_1 and T_2 from $\mathcal{D}$ and apply Lemma 1.5.11 to the resulting packing to complete the proof.

Case 2 Suppose that $k = n - 4$. Then $|U| = 4$, the n-cycle in $\mathcal{D}$ contains at most three edges of K_U and the perfect matching in $\mathcal{D}$ contains at most two edges of K_U. This leaves at least one edge of K_U which occurs in a 3-cycle $T_1 \in \mathcal{D}$. Let $U = \{u_1, u_2, u_3, u_4\}$ and let H be the n-cycle in $\mathcal{D}$. This case now splits into two subcases depending on whether $V(T_1) \cap V(C) = \emptyset$.

Case 2a Suppose that $V(T_1) \cap V(C) = \emptyset$. Then we can assume without loss of generality that $T_1 = (u_1, u_2, u_3)$. If any of the three edges u_1u_4, u_2u_4, u_3u_4 is in a 3-cycle $T_2 \in \mathcal{D}$, then we can remove C, T_1 and T_2 from $\mathcal{D}$ and apply Lemma 1.5.11 to the resulting packing to complete the proof. Thus, we assume there is no such 3-cycle in $\mathcal{D}$. Without loss of generality, it follows that n is even, that u_1u_4 is an edge of the perfect matching in $\mathcal{D}$, and that $u_2u_4, u_3u_4 \in E(H)$. Let z be a vertex in C which is not adjacent in H to a vertex in U (such a vertex exists as $n \geq 15$ implies $|V(C)| \geq 11$ and there are only at most four vertices of C which are adjacent in H to vertices in U).

Now let $\mathcal{P}'$ be the $(3^{t-1}, n)$-packing of K_V obtained from $\mathcal{D} \setminus \{C, T_1\}$ by performing the (u_1, z)-

switch S with origin u_2. If the terminus of S is not u_3, then the only non-trivial component in the leave of $\mathcal{P}'$ is a $(k+3)$-cycle and we can complete the proof by adding this cycle to $\mathcal{P}'$. Otherwise, the terminus of S is u_3 and the only non-trivial component in the leave of $\mathcal{P}'$ is $(u_2, u_3, z) \cup C$. Furthermore, since z is not adjacent in H to a vertex in U, the final dot point in Lemma 1.5.1 guarantees that neither u_1u_2 nor u_1u_3 is an edge of the n-cycle in $\mathcal{P}'$. Since u_1u_2 and u_1u_3 cannot both be edges of the perfect matching in $\mathcal{P}'$, this means that one of them must be in a 3-cycle $T_2' \in \mathcal{P}'$. Thus, we can remove T_2' from $\mathcal{P}'$ and apply Lemma 1.5.11 to the resulting packing to complete the proof.

Case 2b Suppose that $|V(T_1) \cap V(C)| = 1$. Let $T_1 = (x, y, z)$ with $x \in V(C)$ and $y, z \in U$, and let $w \in U \setminus \{y, z\}$. Let $\mathcal{P}'$ be the $(3^{t-1}, n)$-packing of K_V obtained from $\mathcal{D} \setminus \{C, T_1\}$ by performing the (w, x)-switch S with origin y, and let L' be the leave of $\mathcal{P}'$. If the terminus of S is not z, then the only non-trivial component in the leave of $\mathcal{P}'$ is a $(k+3)$-cycle and we can complete the proof by adding this cycle to $\mathcal{P}'$. Otherwise the terminus of S is z, and the only non-trivial components in the leave of $\mathcal{P}'$ are C and (w, y, z). By adding these cycles to $\mathcal{P}'$ we obtain a $(3^t, k, n)$-decomposition of K_V which contains a 3-cycle and an $(n-4)$-cycle which are vertex disjoint, and we can proceed as we did in Case 2a. $\square$

We are now ready to prove the main result of this subsection.

Lemma 1.5.13. *If n, k and t are positive integers such that $3 \le k \le n-1$, Theorem 1.1.1 holds for K_{n-3}, K_{n-2} and K_{n-1}, and $(3^t, k, n)$ is an n-ancestor list, then there is a $(3^t, k, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \ge 15$. Let $r \in \{3, 4, 5\}$ such that $r \equiv k \pmod 3$. It suffices to find a $(3^{t+\frac{k-r}{3}}, r, n)$-decomposition of K_n, since we can then obtain a $(3^t, k, n)$-decomposition of K_n by repeatedly applying Lemma 1.5.12 ($\frac{k-r}{3}$ times). If $r = 3$ then the existence of a $(3^{t+\frac{k-r}{3}}, r, n)$-decomposition of K_n follows from the main result of [36], so we may assume $r \in \{4, 5\}$. Thus, the existence of the required $(3^{t+\frac{k-r}{3}}, r, n)$-decomposition of K_n is given by one of Lemmas 1.5.6, 1.5.7, 1.5.8 and 1.5.9, provided that $t + \frac{k-r}{3} \ge \frac{3n-14}{2}$ (the number of 3-cycles in the decompositions given by Lemma 1.5.8 is at least $\frac{3n-14}{2}$ and the number is smaller for the other three lemmas for $n \ge 15$). However, it follows from $3t + k + n = n\lfloor \frac{n-1}{2} \rfloor$, $k \le n-1$ and $n \ge 15$ that $t \ge \frac{3n-14}{2}$. $\square$

1.5.4 Many 4-cycles and one Hamilton cycle

In Lemma 1.5.14 we construct the required decompositions of complete graphs of odd order and in Lemma 1.5.15 we construct the required decompositions of complete graphs of even order. In each case we construct the required decomposition of K_n from a suitable decomposition of K_{n-2}.

Lemma 1.5.14. *If n is odd, Theorem 1.1.1 holds for K_{n-2}, and $(M, 4^{\frac{n-3}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n-3}{2}}, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \ge 15$. Let U be a vertex set with $|U| = n-2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n-3}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n-3$. Thus, it is easily seen that $(M, n-3)$ is $(n-2)$-admissible and so by assumption there is an $(M, n-3)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-3)$-cycle in $\mathcal{D}$, let z be the vertex in $U \setminus V(H)$, and let x and y be adjacent vertices in H. Then

$$(\mathcal{D} \setminus \{H\}) \cup \{H', (\infty_1, y, x, \infty_2)\} \cup \mathcal{D}_1$$

is an $(M, 4^{\frac{n-3}{2}}, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty_1, z, \infty_2, y]$; and
- $\mathcal{D}_1$ is a $(4^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U\setminus\{x,y,z\}}$. □

Lemma 1.5.15. *If n is even, Theorem 1.1.1 holds for K_{n-2}, and $(M, 4^{\frac{n-2}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n-2}{2}}, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n - 2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n-2}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n-3$. Thus, it is easily seen that $(M, 3, n-3)$ is $(n-2)$-admissible and so by assumption there is an $(M, 3, n-3)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-3)$-cycle in $\mathcal{D}$, let C be a 3-cycle in $\mathcal{D}$, and let I be the perfect matching in $\mathcal{D}$. Let z be the vertex in $U \setminus V(H)$, let w and x be distinct vertices in $V(C) \cap V(H)$, let u be the vertex in $V(C) \setminus \{w, x\}$ (possibly $u = z$), and let y be a vertex adjacent to x in H. Then

$$(\mathcal{D} \setminus \{I, C, H\}) \cup \{I + \infty_1\infty_2, H', (\infty_1, y, x, w), (\infty_2, x, u, w)\} \cup \mathcal{D}_1$$

is an $(M, 4^{\frac{n-2}{2}}, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty_1, z, \infty_2, y]$; and
- $\mathcal{D}_1$ is a $(4^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U\setminus\{w,x,y,z\}}$. □

1.5.5 Many 5-cycles and one Hamilton cycle

In Lemma 1.5.20 we construct the required decompositions of complete graphs of odd order and in Lemma 1.5.21 we construct the required decompositions of complete graphs of even order. We first require four preliminary lemmas.

Lemma 1.5.16. *Every even graph has a decomposition into cycles such that any two cycles in the decomposition share at most two vertices.*

Proof It is well known that every even graph has a decomposition into cycles. Let G be an even graph. Amongst all decompositions of G into cycles, let $\mathcal{D}$ be one with a maximum number of cycles. We claim that any pair of cycles in $\mathcal{D}$ shares at most two vertices. Suppose otherwise. That is, there are distinct cycles A and B in $\mathcal{D}$ and distinct vertices x, y and z of G such that $\{x, y, z\} \subseteq V(A) \cap V(B)$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where A_1, A_2, B_1 and B_2 are paths from x to y such that $z \in V(A_2)$ and $z \in V(B_2)$. Then it is easy to see that $A_1 \cup B_1$ and $A_2 \cup B_2$ are both nonempty even graphs. For $i = 1, 2$, let $\mathcal{D}_i$ be a decomposition of $A_i \cup B_i$ into cycles, and note that $|\mathcal{D}_2| \geq 2$ because $\deg_{A_2 \cup B_2}(z) = 4$. Then $(\mathcal{D} \setminus \{A, B\}) \cup \mathcal{D}_1 \cup \mathcal{D}_2$ is a decomposition of G into cycles which contains more cycles than $\mathcal{D}$, contradicting our definition of $\mathcal{D}$. □

Lemma 1.5.17. *Let V be a vertex set and let U be a subset of V such that $|U| \geq 10$. Let M be a list of integers, let $m \in \{3,4,5\}$, and let $\mathcal{P}$ be an (M)-packing of K_V with a leave L such that $\deg_L(x) = 0$ for all $x \in V \setminus U$. If there exists a spanning even subgraph G of L such that at least one of the following holds,*

(i) *$\Delta(G) = 4$, exactly one vertex of G has degree 4, G has at most two nontrivial components, $|E(G)| \geq m+3$, and G does not have a decomposition into two odd cycles if $m = 4$;*

(ii) *$\Delta(G) = 4$, exactly two vertices of G have degree 4, G has exactly one nontrivial component, $|E(G)| \geq m+3$, and $\deg_G(x) = 0$ for some $x \in U$ if $m = 3$;*

(iii) *$m = 4$, G has exactly one nontrivial component, G has a decomposition into three cycles each pair of which intersect in exactly one vertex, and $\deg_G(x) = 0$ for some $x \in U$; or*

(iv) *$m = 5$, $\Delta(G) \geq 4$, and G has a decomposition into three cycles such that any two intersect in at most two vertices, and such that any two which intersect have lengths adding to 6 or 7;*

then there exists (M,m)-packing $\mathcal{P}'$ of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.

Proof Suppose that there is a spanning even subgraph G of L which satisfies one of (i), (ii), (iii) or (iv). Because L and G are both even graphs, it follows that $L - G$ is an even graph and hence has a decomposition $\mathcal{A} = \{A_1, \ldots, A_t\}$ into cycles. Let $a_i = |V(A_i)|$ for $i = 1, \ldots, t$ and let $M^\dagger = (a_1, \ldots, a_t)$. So $\mathcal{P} \cup \mathcal{A}$ is an $(M, M^\dagger)$-packing of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$. The leave of $\mathcal{P} \cup \mathcal{A}$ is G. Let $e = |E(G)|$.

If we can produce an $(M, M^\dagger, m, e-m)$-decomposition $\mathcal{D}$ of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$, then there will be cycles in $\mathcal{D}$ with lengths $a_1, \ldots, a_t, e-m$ whose vertex sets are subsets of U, and we can complete the proof by removing these cycles from $\mathcal{D}$. So it suffices to find such a decomposition. The proof now splits into cases.

Case 1 Suppose that G satisfies (i). Then we can apply Lemma 1.5.3 to obtain the required decomposition.

Case 2 Suppose that G satisfies (ii). Then we can apply Lemma 1.5.4 to obtain the required decomposition. The only non-trivial thing to check is that there is an $x \in U$ with $\deg_G(x) = 0$ when $m \in \{4,5\}$ and $e - m = 3$. In this case we have $e \in \{7,8\}$ and, because G is even and $|U| \geq 10$, there is indeed an $x \in U$ with $\deg_G(x) = 0$.

Case 3 Suppose that G satisfies (iii). Either exactly one vertex of G has degree 6 and every other vertex of G has degree at most 2, or exactly three vertices of G have degree 4 and every other vertex of G has degree at most 2. In the former case we apply Lemma 1.5.2, choosing b to be the vertex of degree 6 in G and a to be a neighbour in G of b. In the latter case we apply Lemma 1.5.2, choosing b to be a vertex of degree 4 in G and a to be a vertex in U which has degree 0 in G. In either case we obtain an $(M, M^\dagger)$-packing $\mathcal{P}'$ of K_V, which is equivalent to $\mathcal{P}$ on $V \setminus U$, with a leave G' such that $\Delta(G') = 4$, exactly two vertices of G' have degree 4, G' has exactly one nontrivial component, and $|E(G')| \geq 9$. Thus we can apply Lemma 1.5.4 to obtain the required decomposition.

Case 4 Suppose that G satisfies (iv). Since $\Delta(G) \geq 4$ there is at least one pair of intersecting cycles in any cycle decomposition of G. Thus, there exists a decomposition $\{B_1, B_2, B_3\}$ of G into three cycles such that $|V(B_1) \cap V(B_2)| \in \{1,2\}$ and $|E(B_1)| + |E(B_2)| \in \{6,7\}$.

Case 4a Suppose that B_3 is a component of G. If $|V(B_1) \cap V(B_2)| = 1$, then we can apply Lemma 1.5.3 to obtain the required decomposition, so we may assume that $|V(B_1) \cap V(B_2)| = 2$. Let $x \in V(B_1) \cap V(B_2)$, let $y \in V(B_3)$, and let z be a neighbour in G of x. Let $\mathcal{P}'$ be the $(M, M^\dagger)$-packing of K_n obtained from $\mathcal{P}$ by performing the (x, y)-switch S with origin z and let G' be its leave. Then $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V \setminus U$, G' has exactly one nontrivial component, $\Delta(G') = 4$, exactly two vertices of G' have degree 4, and $|E(G')| \geq 9$. Thus we can apply Lemma 1.5.4 to obtain the required decomposition.

Case 4b Suppose that B_3 is not a component of G. Then B_3 intersects with B_1 or B_2 and so $|E(B_3)| \in \{3, 4\}$. Thus, $e \in \{9, 10, 11\}$ as we have $|E(B_1)| + |E(B_2)| \in \{6, 7\}$. Let $\mathcal{P}'$ be the $(M, M^\dagger)$-packing of K_n obtained from $\mathcal{P}$ by repeatedly applying Lemma 1.5.2, each time choosing b to be a vertex maximum degree in the leave and a to be a vertex in U of degree 0 in the leave, until the leave has maximum degree 4 and has exactly one vertex of degree 4 (a suitable choice for a will exist each time since $e \leq 11$ and $|U| \geq 10$). Let G' be the leave of $\mathcal{P}'$. Then $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V \setminus U$, $\Delta(G') = 4$, exactly one vertex of G' has degree 4, $|E(G')| \in \{9, 10, 11\}$, and G' has at most two components (because $|E(G')| \leq 11$). Thus we can apply Lemma 1.5.3 to obtain the required decomposition. □

Lemma 1.5.18. *Let V be a vertex set and let U be a subset of V such that $|U| \geq 10$. Let $m \in \{3, 4, 5\}$, let M be a list of integers, and let $\mathcal{P}$ be an (M)-packing of K_V with a leave L such that $|E(L)| \geq |U| + m$ and $\deg_L(x) = 0$ for all $x \in V \setminus U$. Then there exists an (M, m)-packing of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.*

Proof Since L is an even graph, Lemma 1.5.16 guarantees that there is a decomposition $\mathcal{D}$ of L such that any pair of cycles in $\mathcal{D}$ intersect in at most two vertices. Let $e = |E(L)|$. Since $e \geq |U| + m \geq 13$ it follows that $\mathcal{D}$ contains at least three cycles. Also, since $e > |U|$, there is at least one pair of intersecting cycles in $\mathcal{D}$. We now consider separately the cases $m = 3$, $m = 4$ and $m = 5$.

Case 1 Suppose that $m = 3$. We can assume that there are no 3-cycles in $\mathcal{D}$ (otherwise we can simply add one to $\mathcal{P}$ to complete the proof). Let C_1, C_2 and C_3 be distinct cycles in $\mathcal{D}$ such that C_1 and C_2 intersect. If $|V(C_1) \cap V(C_2)| = 1$, then we can apply Lemma 1.5.17 (i) (with $E(G) = E(C_1 \cup C_2)$) to complete the proof, so we may assume that $|V(C_1) \cap V(C_2)| = 2$. If $|E(C_1)| + |E(C_2)| \leq |U| + 1$, then there is at least one vertex of U that is not in $V(C_1) \cup V(C_2)$ and we can apply Lemma 1.5.17 (ii) (with $E(G) = E(C_1 \cup C_2)$) to complete the proof. Thus, we may assume $|E(C_1)| + |E(C_2)| = |U| + 2$, and it follows from this that $V(C_1) \cup V(C_2) = U$. This means that $V(C_3) \subseteq V(C_1) \cup V(C_2)$. Thus, since we have $V(C_3) \geq 4$, $V(C_1) \cap V(C_3) \leq 2$ and $V(C_2) \cap V(C_3) \leq 2$, it follows that $V(C_3) = 4$, $V(C_1) \cap V(C_3) = 2$ and $V(C_2) \cap V(C_3) = 2$. We can assume without loss of generality that $|E(C_1)| \leq |E(C_2)|$ and hence that $|E(C_2)| \geq 5$ (since $|U| \geq 10$). This means that there is at least one vertex of U that is in neither C_1 nor C_3, and so we can apply Lemma 1.5.17 (ii) (with $E(G) = E(C_1 \cup C_3)$) to complete the proof.

Case 2 Suppose that $m = 4$. If two cycles in $\mathcal{D}$ intersect in exactly two vertices, then we can apply Lemma 1.5.17 (ii) (with the edges of G being the edges of two such cycles) to complete the proof. So we may assume that any two cycles in $\mathcal{D}$ intersect in at most one vertex. Let $\{C_1, C_2\}$ be a pair of intersecting cycles in $\mathcal{D}$ such that $|E(C_1 \cup C_2)| \leq |E(C_i \cup C_j)|$ for any pair $\{C_i, C_j\}$ of intersecting cycles in $\mathcal{D}$. If there is a cycle in $\mathcal{D}$ which is vertex disjoint from $C_1 \cup C_2$, then we can apply Lemma 1.5.17 (i) (with the edges of G being the edges of C_1, C_2 and this cycle) to complete the proof. If there is a cycle in $\mathcal{D}$ which intersects with exactly one of C_1 and C_2, then we can apply Lemma 1.5.17 (ii) (with the edges of G being the edges of C_1, C_2 and this cycle) to complete the proof. So we may assume that every cycle in $\mathcal{D} \setminus \{C_1, C_2\}$ intersects (in exactly one vertex) with C_1 and with C_2. Let C_3 be a shortest cycle in $\mathcal{D} \setminus \{C_1, C_2\}$ and note that

$|V(C_i)| \le |V(C_3)|$ for $i = 1,2$ by our definition of C_1 and C_2. If $V(C_1 \cup C_2 \cup C_3) \ne U$, then we can apply Lemma 1.5.17 (iii) (with $E(G) = E(C_1 \cup C_2 \cup C_3)$) to complete the proof. Otherwise $V(C_1 \cup C_2 \cup C_3) = U$ which means that $|V(C_1)| + |V(C_2)| + |V(C_3)| \in \{|U|+2, |U|+3\}$. However, we have $e \ge |U|+4$ and so there is a cycle $C_4 \in \mathcal{D} \setminus \{C_1, C_2, C_3\}$. Thus C_4 is a 3-cycle (as $V(C_1 \cup C_2 \cup C_3) = U$ and C_4 intersects each of C_1, C_2 and C_3 in exactly one vertex). It then follows from the minimality of C_3 and from $|V(C_i)| \le |V(C_3)|$ for $i = 1,2$ that C_1, C_2 and C_3 are also 3-cycles. Since $|U| \ge 10$, this is a contradiction and the result is proved.

Case 3 Suppose that $m = 5$. Let C_1, C_2 and C_3 be three cycles in $\mathcal{D}$ such that C_1 and C_2 intersect. If there are a pair of cycles in $\{C_1, C_2, C_3\}$ which intersect and whose lengths add to at least 8, then the union of this pair of cycles has at least $m+3$ edges and we can apply Lemma 1.5.17 (i) or Lemma 1.5.17 (ii) (with the edges of G being the edges of this pair of cycles) to complete the proof. Otherwise we can apply Lemma 1.5.17 (iv) to complete the proof. $\square$

Lemma 1.5.19. *Let u and k be integers such that u is even, $u \ge 16$ and $6 \le k \le u-1$, let U be a vertex set such that $|U| = u$, and let x and y be distinct vertices in U. Then there exists a packing of K_U with a perfect matching, a u-cycle, a $(u-1)$-path from x to y, a k-cycle, three $(u-2)$-cycles each having vertex set $U \setminus \{x,y\}$, and a 2-path from x to y.*

Proof Let $U = \mathbb{Z}_{u-3} \cup \{\infty, x, y\}$. For $i = 0, \ldots, 5$, let

$$H_i = (\infty, i, i+1, i+(u-4), i+2, i+(u-5), \ldots, i+\tfrac{u-6}{2}, i+\tfrac{u}{2}, i+\tfrac{u-4}{2}, i+\tfrac{u-2}{2}),$$

and let

$$I = [\tfrac{u-6}{2}, \tfrac{u-2}{2}] \cup [\tfrac{u-8}{2}, \tfrac{u}{2}] \cup \cdots \cup [0, u-4] \cup [\infty, \tfrac{u-4}{2}],$$

so that $\{I, H_0, \ldots, H_5\}$ is a packing of $\mathbb{Z}_{u-3} \cup \{\infty\}$ with one perfect matching and six $(u-2)$-cycles (recall that $u \ge 16$). Then

$$\{I+xy, (H_0-[\infty,0,1]) \cup [\infty,x,0,y,1], (H_1-[1,2]) \cup [1,x] \cup [2,y], P \cup [a,x,b], H_3, H_4, H_5, [x,c,y]\}$$

is the required packing, where P is a $(k-2)$-path in H_2 with endpoints a and b such that $a, b \in \mathbb{Z}_{u-3} \setminus \{0,1\}$ (P exists as there are $u-2$ distinct paths of length $k-2$ in H_2, and at most six having ∞, 0 or 1 as an endpoint), and c is any vertex in $\mathbb{Z}_{u-3} \setminus \{0,1,2,a,b\}$. $\square$

Lemma 1.5.20. *If n is odd and $(M, 5^{\frac{3n-11}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{\frac{3n-11}{2}}, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 (for $n \le 13$) and Lemma 1.5.5 (for $n \in \{15, 17\}$) we may assume that $n \ge 19$ (Lemma 1.5.5 can indeed be applied as $\nu_5(M, 5^{\frac{3n-11}{2}}, n) \ge 3$ when $n \in \{15, 17\}$). Let U be a vertex set with $|U| = n-3$, let x and y be distinct vertices in U, let $\infty^\dagger$, ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty^\dagger, \infty_1, \infty_2\}$.

Since $(M, 5^{\frac{3n-11}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (6) in the definition of ancestor lists that any cycle length in M is at most $n-5$. If there is a cycle length in M which is at least 6, then let k be this cycle length. Otherwise let $k = 0$ (so $k \in \{0\} \cup \{6, \ldots, n-5\}$). By Lemma 1.5.19, there exists a packing $\mathcal{P}$ of K_U with a perfect matching I, an $(n-4)$-path P_1 from x to y, a k-cycle (if $k \ne 0$), three $(n-5)$-cycles C_1, C_2 and C_3 each having vertex set $U \setminus \{x,y\}$, and a 2-path P_2 from x to y. Let $\{I_1, I_2\}$ be a decomposition of C_3 into two matchings.

Let

$$\mathcal{P}' = (\mathcal{P} \setminus \{I, P_1, P_2, C_1, C_2, C_3\}) \cup \{P_1 \cup [x, \infty_1, \infty^\dagger, \infty_2, y], P_2 \cup [x, \infty_2, \infty_1, y]\} \cup \mathcal{D} \cup \mathcal{D}_1 \cup \mathcal{D}_2,$$

where

- $\mathcal{D}$ is a $(3^{\frac{n-3}{2}})$-decomposition of $K_{\{\infty^\dagger\}} \vee I$;

- for $i = 1, 2$, $\mathcal{D}_i$ is a $(5^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty_i\}} \vee (C_i \cup I_i)$ (this exists by Lemma 1.4.5).

Then $\mathcal{P}'$ is a $(3^{\frac{n-3}{2}}, 5^{n-4}, k, n)$-packing of K_V (a $(3^{\frac{n-3}{2}}, 5^{n-4}, n)$-packing of K_V if $k = 0$) such that

(i) $\frac{n-3}{2}$ 3-cycles in $\mathcal{P}'$ contain the vertex $\infty^\dagger$;

(ii) $\infty^\dagger\infty_1$ and $\infty^\dagger\infty_2$ are edges of the n-cycle in $\mathcal{P}'$; and

(iii) $\infty^\dagger$, ∞_1 and ∞_2 all have degree 0 in the leave of $\mathcal{P}'$.

Since $(M, 5^{\frac{3n-11}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it can be seen that by beginning with $\mathcal{P}'$ and repeatedly applying Lemma 1.5.18 we can obtain an $(M, 3^{\frac{n-3}{2}}, 5^{n-4}, n)$-packing of K_V which is equivalent to $\mathcal{P}'$ on $V \setminus U$. Note that the leave of this packing has $n - 3$ edges. Thus, by then repeatedly applying Lemma 1.5.2 we can obtain an $(M, 3^{\frac{n-3}{2}}, 5^{n-4}, n)$-packing $\mathcal{P}''$ of K_V which is equivalent to $\mathcal{P}'$ on $V \setminus U$ and whose leave L'' has the property that $\deg_{L''}(x) = 0$ for each $x \in \{\infty^\dagger, \infty_1, \infty_2\}$ and $\deg_{L''}(x) = 2$ for each $x \in U$. Because $\mathcal{P}''$ is equivalent to $\mathcal{P}'$ on $V \setminus U$, it follows from (i) and (ii) that there is a set $\mathcal{T}$ of $\frac{n-3}{2}$ 3-cycles in $\mathcal{P}''$ each of which contains the vertex $\infty^\dagger$ and two vertices in U. Let T be the union of the 3-cycles in $\mathcal{T}$. Then

$$(\mathcal{P}'' \setminus \mathcal{T}) \cup \mathcal{D}''$$

is an $(M, 5^{\frac{3n-11}{2}}, n)$-decomposition of K_V where $\mathcal{D}''$ is a $(5^{\frac{n-3}{2}})$-decomposition of $T \cup L''$ (this exists by Lemma 1.4.5, noting that $E(T \cup L'') = E(K_{\{\infty^\dagger\}} \vee G)$ for some 3-regular graph G on vertex set U which contains a perfect matching). □

Lemma 1.5.21. *If n is even and $(M, 5^{2n-9}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{2n-9}, n)$-decomposition of K_n.*

Proof By Lemma 1.1.2 (for $n \leq 14$) and Lemma 1.5.5 (for $n \in \{16, 18\}$) we may assume that $n \geq 20$ (Lemma 1.5.5 can indeed be applied as $\nu_5(M, 5^{2n-9}, n) \geq 3$ when $n \in \{16, 18\}$). Let U be a vertex set with $|U| = n - 4$, let x and y be distinct vertices in U, let $\infty_1^\dagger$, $\infty_2^\dagger$, ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1^\dagger, \infty_2^\dagger, \infty_1, \infty_2\}$.

Since $(M, 5^{2n-9}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (6) in the definition of ancestor lists that any cycle length in M is at most $n - 5$. If there is a cycle length in M which is at least 6 then let k be this cycle length. Otherwise let $k = 0$ (so $k \in \{0\} \cup \{6, \ldots, n - 5\}$). By Lemma 1.5.19, there exists a packing $\mathcal{P}$ of K_U with a perfect matching I, an $(n - 4)$-cycle B, an $(n - 5)$-path P_1 from x to y, a k-cycle (if $k \neq 0$), and three $(n - 6)$-cycles C_1, C_2 and C_3 each having vertex set $U \setminus \{x, y\}$, and a 2-path P_2 from x to y. Let $\{I_1^\dagger, I_2^\dagger\}$ be a decomposition of B into two matchings and $\{I_1, I_2\}$ be a decomposition of C_3 into two matchings.

Let

$$\begin{aligned}\mathcal{P}' = \;& (\mathcal{P} \setminus \{B, P_1, P_2, C_1, C_2, C_3\}) \cup \\ & \{I + \{\infty_1\infty_2^\dagger, \infty_2\infty_1^\dagger\}, P_1 \cup [x, \infty_1, \infty_1^\dagger, \infty_2^\dagger, \infty_2, y], P_2 \cup [x, \infty_2, \infty_1, y]\} \cup \mathcal{D}_1^\dagger \cup \mathcal{D}_2^\dagger \cup \mathcal{D}_1 \cup \mathcal{D}_2,\end{aligned}$$

where

- for $i = 1, 2$, $\mathcal{D}_i^\dagger$ is a $(3^{\frac{n-4}{2}})$-decomposition of $K_{\{\infty_i^\dagger\}} \vee I_i^\dagger$;

- for $i = 1, 2$, $\mathcal{D}_i$ is a $(5^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty_i\}} \vee (C_i \cup I_i)$ (this exists by Lemma 1.4.5).

Then $\mathcal{P}'$ is a $(3^{n-4}, 5^{n-5}, k, n)$-packing of K_V (a $(3^{n-4}, 5^{n-5}, n)$-packing of K_V if $k = 0$) such that

(i) for $i = 1, 2$, $\frac{n-4}{2}$ 3-cycles in $\mathcal{P}'$ contain the vertex $\infty_i^\dagger$;

(ii) $\infty_1^\dagger\infty_1$, $\infty_1^\dagger\infty_2^\dagger$ and $\infty_2^\dagger\infty_2$ are edges of the n-cycle in $\mathcal{P}'$;

(iii) $\infty_1^\dagger\infty_2$ and $\infty_2^\dagger\infty_1$ are edges of the perfect matching in $\mathcal{P}'$; and

(iv) $\infty_1^\dagger$, $\infty_2^\dagger$, ∞_1 and ∞_2 all have degree 0 in the leave of $\mathcal{P}'$.

Since $(M, 5^{2n-9}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, by beginning with $\mathcal{P}'$ and repeatedly applying Lemma 1.5.18 we can obtain an $(M, 3^{n-4}, 5^{n-5}, n)$-packing of K_V, which is equivalent to $\mathcal{P}'$ on $V \setminus U$. Note that the leave of this packing has $2n - 8$ edges. Thus, by then repeatedly applying Lemma 1.5.2 we can obtain an $(M, 3^{n-4}, 5^{n-5}, n)$-packing $\mathcal{P}''$ of K_V which is equivalent to $\mathcal{P}'$ on $V \setminus U$ and whose leave L'' has the property that $\deg_{L''}(x) = 0$ for $x \in \{\infty_1^\dagger, \infty_2^\dagger, \infty_1, \infty_2\}$ and $\deg_{L''}(x) = 4$ for all $x \in U$. By Petersen's Theorem [81], L'' has a decomposition $\{H_1, H_2\}$ into two 2-regular graphs, each with vertex set U. Because $\mathcal{P}''$ is equivalent to $\mathcal{P}'$ on $V \setminus U$, it follows from (i), (ii) and (iii) that, for $i = 1, 2$ there is a set $\mathcal{T}_i$ of $\frac{n-4}{2}$ 3-cycles in $\mathcal{P}''$ each of which contains the vertex $\infty_i^\dagger$ and two vertices in U. For $i = 1, 2$, let T_i be the union of the 3-cycles in $\mathcal{T}_i$. Then

$$(\mathcal{P}'' \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)) \cup \mathcal{D}_1'' \cup \mathcal{D}_2''$$

is an $(M, 5^{2n-9}, n)$-decomposition of K_V where, for $i = 1, 2$, $\mathcal{D}_i''$ is a $(5^{\frac{n-4}{2}})$-decomposition of $T_i \cup H_i$ (these decompositions exist by Lemma 1.4.5, noting that for $i = 1, 2$, $E(T_i \cup H_i) = E(K_{\{\infty_i^\dagger\}} \vee G)$ for some 3-regular graph G with vertex set U that contains a perfect matching). $\square$

1.5.6 Proof of Lemma 1.1.5 in the case of one Hamilton cycle

Lemma 1.5.22. *If Theorem 1.1.1 holds for K_{n-1}, K_{n-2} and K_{n-3}, then there is an (M)-decomposition of K_n for each n-ancestor list M satisfying $\nu_n(M) = 1$.*

Proof By Lemma 1.1.2 we can assume that $n \geq 15$. If there is a cycle length in M which is at least 6 and at most $n - 1$ then let k be this cycle length. Otherwise let $k = 0$. We deal separately with the case n is odd and the case n is even.

Case 1 Suppose that n is odd. Since $n \geq 15$ and $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k + n = \frac{n(n-1)}{2}$, it can be seen that either

(i) $n \in \{15, 17, 19\}$ and $\nu_5(M) \geq 3$;

(ii) $\nu_3(M) \geq \frac{n-5}{2}$;

(iii) $\nu_4(M) \geq \frac{n-3}{2}$; or

(iv) $\nu_5(M) \geq \frac{3n-11}{2}$.

(To see this consider the cases $\nu_5(M) \geq 3$ and $\nu_5(M) \leq 2$ separately and use the definition of n-ancestor list.) If (i) holds, then the result follows by Lemma 1.5.5. If (ii) holds, then the result follows by one of Lemmas 1.5.6, 1.5.7 or 1.5.13. If (iii) holds, then the result follows by Lemma 1.5.14. If (iv) holds, then the result follows by Lemma 1.5.20.

Case 2 Suppose that n is even. Since $n \geq 16$ and $3\nu_3(M)+4\nu_4(M)+5\nu_5(M)+k+n = \frac{n(n-2)}{2}$, it can be seen that either

(i) $n \in \{16, 18, 20, 22, 24, 26\}$, $\nu_5(M) \geq 3$, and $\nu_4(M) \geq 2$ if $n = 24$;

(ii) $\nu_3(M) \geq \frac{3n-14}{2}$;

(iii) $\nu_4(M) \geq \frac{n-2}{2}$; or

(iv) $\nu_5(M) \geq 2n - 9$.

(To see this consider the cases $\nu_5(M) \geq 3$ and $\nu_5(M) \leq 2$ separately and use the definition of n-ancestor list.) If (i) holds, then the result follows by Lemma 1.5.5. If (ii) holds, then the result follows by one of Lemmas 1.5.8, 1.5.9 or 1.5.13 (note that $\frac{3n-14}{2} \geq \frac{n-6}{2}$ for $n \geq 16$). If (iii) holds, then the result follows by Lemma 1.5.15. If (iv) holds, then the result follows by Lemma 1.5.21. □

1.6 Decompositions of $\langle S \rangle_n$

Our general approach to constructing decompositions of $\langle S \rangle_n$ follows the approach used in [38] and [40]. For each connection set S in which we are interested, we define a graph J_n for each positive integer n such that there is a natural bijection between $E(J_n)$ and $E(\langle S \rangle_n)$, and such that $\langle S \rangle_n$ can be obtained from J_n by identifying a small number (approximately $|S|$) of pairs of vertices. Thus, decompositions of J_n yield decompositions of $\langle S \rangle_n$.

The key property of the graph J_n is that it can be decomposed into a copy of J_{n-y} and a copy of J_y for any positive integer y such that $1 \leq y < n$, and this facilitates the construction of desired decompositions of J_n for arbitrarily large n from decompositions of J_i for various small values of i. For example, in the case $S = \{1, 2, 3\}$ we define J_n by $V(J_n) = \{0, \ldots, n+2\}$ and $E(J_n) = \{\{i, i+1\}, \{i+1, i+3\}, \{i, i+3\}\} : i = 0, \ldots n-1\}$. It is straightforward to construct a (3)-decomposition of J_1, a (4, 5)-decomposition of J_3, a (4^3)-decomposition of J_4 and a (5^3)-decomposition of J_5. Moreover, since J_n decomposes into J_{n-y} and J_y, it is easy to see that these decompositions can be combined to produce an (M)-decomposition of J_n for any list $M = (m_1, \ldots, m_t)$ satisfying $\sum M = 3n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$. For all $n \geq 7$, an (M)-decomposition of $\langle \{1, 2, 3\} \rangle_n$ can be obtained from an (M)-decomposition of J_n by identifying vertex i with vertex $i + n$ for $i = 0, 1, 2$.

In what follows, this general approach is modified to allow for the construction of decompositions which, in addition to cycles of lengths 3, 4 and 5, contain one arbitrarily long cycle or, in the case $S = \{1, 2, 3\}$, one arbitrarily long cycle and one Hamilton cycle. The constructions used to prove Lemma 1.3.2 proceed in a similar fashion for each connection set S.

1.6.1 Proof of Lemma 1.3.2

In this section we prove Lemma 1.3.2, which we restate here for convenience.

Lemma 1.3.2 *If*

$$S \in \{\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\},$$

$n \geq 2\max(S)+1$, and $M = (m_1,\ldots,m_t,k)$ is any list satisfying $m_i \in \{3,4,5\}$ for $i = 1,\ldots,t$, $3 \leq k \leq n$, and $\sum M = |S|n$, then there is an (M)-decomposition of $\langle S\rangle_n$, except possibly when

- *$S = \{1,2,3,4,6\}$, $n \equiv 3 \pmod 6$ and $M = (3^{\frac{5n}{3}})$; or*
- *$S = \{1,2,3,4,6\}$, $n \equiv 4 \pmod 6$ and $M = (3^{\frac{5n-5}{3}},5)$.*

Let

$$\mathcal{S} = \{\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\}.$$

We shall show that the required decompositions exist for each $S \in \mathcal{S}$ separately.

Our proof is essentially inductive and requires a large number of specific base decompositions, and these are given in the appendix. Some of the constructions could possibly have been completed using a smaller number of base decompositions, but since these were found using a computer search, we decided to keep the inductive steps themselves as simple as possible, at the cost of requiring a larger number of base decompositions.

$S = \{1,2,3\}$

In this section we show the existence of required decompositions for the case $S = \{1,2,3\}$ in Lemma 1.3.2. We first define J_n by

$$E(J_n) = \{\{i,i+1\},\{i+1,i+3\},\{i,i+3\} : i = 0,\ldots,n-1\}\}$$

and $V(J_n) = \{0,\ldots,n+2\}$. We note the following basic properties of J_n. For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$.

- For $n \geq 7$, if for each $i \in \{0,1,2\}$ we identify vertex i of J_n with vertex $i+n$ of J_n then the resulting graph is $\langle\{1,2,3\}\rangle_n$. This means that for $n \geq 7$, we can obtain an (M)-decomposition of $\langle\{1,2,3\}\rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0,1,2\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i+n$.
- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x+(n-y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations*.

Lemma 1.6.1. *If n is a positive integer and $M = (m_1,\ldots,m_t)$ is a list such that $\sum M = 3n$, $m_i \in \{3,4,5\}$ for $i = 1,\ldots,t$, then there is a decomposition $J_n \to M$.*

Proof Since J_1 is a 3-cycle, the result holds trivially for $n = 1$, so let $n \geq 2$ and suppose by induction that the result holds for each integer n' in the range $1 \leq n' < n$. The following decompositions are given in Table A.3 in the appendix.

$$J_1 \to 3 \qquad J_3 \to 4,5 \qquad J_4 \to 4^3 \qquad J_5 \to 5^3$$

It is routine to check that if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some (possibly empty) list. If X is empty, then we are finished immediately. If X is nonempty then we can obtain a decomposition $J_n \to M$ by concatenating a decomposition $J_{n-y} \to X$ (which exists by our inductive hypothesis) with a decomposition $J_y \to Y$. □

Lemma 1.6.2. *For $6 \le k \le 10$, if $n \ge k$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 3n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$ such that the k-cycle is incident upon vertices $\{0, 1, \ldots, k-1\}$.*

Proof First we note that Table A.1 in the appendix lists a number of decompositions containing a k-cycle C_k for some $6 \le k \le 10$ such that $V(C_k) \subseteq \{0, \ldots, k-1\}$. For each k, it is easy to use the value of $k \pmod 3$ to check that for $n \ge k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k$ is one of the decompositions in Table A.1, and Y is some (possibly empty) list with the property $\sum Y = 3y$ for some integer y. If Y is empty we are done, else Lemma 1.6.1 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_x \to X, k$ with $J_y \to Y$.

The k-cycle in the resulting decomposition will be incident on vertices $\{0, 1, \ldots, k-1\}$ as this property held in the decomposition $J_x \to X, k$ and thus the resulting decomposition satisfies the condition in the lemma. □ Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \ge 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- C is a k-cycle such that $V(C) = \{n-k+3, \ldots, n+2\}$ and $\{n, n+2\} \in E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 1.6.4 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{3, \ldots, 8\}$, the graph obtained from J_y by adding the edge $\{0, 2\}$ will be denoted J_y^+. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \ge 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A is a path from 0 to 2 such that $1 \notin V(A)$; and
- $|E(A)| = l + 1$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{n, n+2\} \in E(A)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{3, \ldots, 8\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edge $\{n-y, n-y+2\}$ and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k+l$. The edge $\{n, n+2\}$ of the k-cycle in the decomposition of J_{n-y} is replaced by the path in the decomposition of J_y^+ to form the $(k+l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k+y)^*$.

Lemma 1.6.3. *For $11 \leq k \leq 16$, if $n \geq k$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 3n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k^*$.*

Proof First we note the existence of decompositions of the form $J_x \to X, k^*$ listed in Table A.2 in the appendix. For each k, it is routine to use the value of $k \pmod 3$ to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions in Table A.2, and Y is some (possibly empty) list with the property $\sum Y = 3y$ for some integer y. If Y is empty we are done, else Lemma 1.6.1 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$.

□

Lemma 1.6.4. *If $n \geq 11$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 2n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof Lemma 1.6.3 shows that the result holds for $11 \leq n \leq 16$. So let $n \geq 17$ and suppose by induction that the result holds for each integer n' in the range $11 \leq n' < n$. The following decompositions can be seen in Table A.3 in the appendix.

$$J_5^+ \to 5^2, 5^{+*} \qquad J_6^+ \to 4^3, 6^{+*} \qquad J_6^+ \to 3^4, 6^{+*}$$

It is routine to check, using $\sum M = 2n \geq 34$, that if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n - y \geq n - 6 \geq 11$) with a decomposition $J_y^+ \to Y, y^{+*}$. □

Lemma 1.6.5. *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 3n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 7$ all cycles in this decomposition have the property that for $i \in \{0, 1, 2\}$ no cycle is incident upon both vertex i and vertex $n + i$.*

Proof We first note that if $n \geq 7$ it is clear that any 3-, 4- or 5-cycle in such a decomposition cannot be incident on two vertices i and $i+n$ for any $i \in \{0, 1, 2\}$. As such, Lemma 1.6.2 shows that the result holds for all n with $6 \leq k \leq 10$, so in the following we deal only with $k \geq 11$.

Lemma 1.6.3 shows that the result holds for all n with $11 \leq k \leq 16$ with the additional property that the k-cycle is not incident upon any vertex in $\{0, 1, 2\}$, and Lemma 1.6.4 shows that the result holds for all $n = k$ with the same property on the k-cycle. We can therefore assume that $17 \leq k \leq n - 1$, so let $n \geq 18$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$ with the additional property that the k-cycle is not incident upon any vertex in $\{0, 1, 2\}$.

The following decompositions exist by Lemma 1.6.1.

$$J_1 \to 3 \qquad J_3 \to 4, 5 \qquad J_4 \to 4^3 \qquad J_5 \to 5^3$$

Case 1 Suppose that $k \leq n-5$. Then it is routine to check, using $\sum M = 3n-k \geq 2n+5 \geq 41$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$ (which exists by our inductive hypothesis, since $k \le n-6 \le n-y$) with a decomposition $J_y \to Y$. Since $n \ge 17$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0,1,2\}$ has no vertex in $\{n, n+1, n+2\}$, and by our inductive hypothesis the same holds for the k-cycle.

Case 2 Suppose that $n-4 \le k \le n-1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, 3)$ for some list X, if $k \in \{n-4, n-3\}$ and $M = (X, 4, 5)$ for some list X, and if $k = n-4$ and $M = (X, 4^3)$ for some list X. So we may assume that none of these hold. Additionally, we can construct the decomposition $J_{18} \to 5^8, 14$ as the concatenation of $J_{13} \to 5^5, 14^*$ with $J_5 \to 5^3$ (both given in Table A.3 in the appendix) as it cannot be constructed by the method shown below, so in the following also note that we do not consider this decomposition.

Given this, using $\sum M = 3n - k \ge 2n + 1 \ge 37$, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below (note that, since $\nu_3(M) = 0$, in each case we can deduce the given value of $\nu_5(M) \pmod 2$ from $\sum M = 3n - k$). The decompositions in the third column exist by Lemma 1.6.4 (since $k \ge 17$), and the decompositions listed in the last column are shown in Table A.3 in the appendix.

k	$\nu_5(M) \pmod 2$	first decomposition	second decomposition
$n-4$	0	$J_{n-8} \to (M - (5^4)), (n-8)^*$	$J_8^+ \to 5^4, 4^+$
$n-3$	1	$J_{n-6} \to (M - (5^3)), (n-6)^*$	$J_6^+ \to 5^3, 3^+$
$n-2$	0	$J_{n-3} \to (M - (4^2)), (n-3)^*$ $J_{n-4} \to (M - (5^2)), (n-4)^*$	$J_3^+ \to 4^2, 1^+$ $J_4^+ \to 5^2, 2^+$
$n-1$	1	$J_{n-4} \to (M - (4,5)), (n-4)^*$ $J_{n-7} \to (M - (5^3)), (n-7)^*$	$J_4^+ \to 4, 5, 3^+$ $J_7^+ \to 5^3, 6^+$

Since $n \ge 18$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0,1,2\}$ has no vertex in $\{n, \dots, n+2\}$, and by the definition of the decompositions given in the third column the k-cycle has no vertex in $\{0,1,2\}$, so these decompositions do have the required properties.

□

Lemma 1.6.6. *If $S = \{1,2,3\}$, $n \ge 7$, and $M = (m_1, \dots, m_t, k)$ is any list satisfying $m_i \in \{3,4,5\}$ for $i = 1, \dots, t$, $3 \le k \le n$, and $\sum M = 3n$, then there is an (M)-decomposition of $\langle S \rangle_n$.*

Proof As noted above, for $n \ge 7$ we can obtain an (M)-decomposition of $\langle \{1,2,3\} \rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0,1,2\}$, no cycle contains both vertex i and vertex $i+n$. Thus, for $S = \{1,2,3\}$, the required result follows by Lemma 1.6.1 for $k \in \{3,4,5\}$ and by Lemma 1.6.5 for $6 \le k \le n$. □

$S = \{1,2,3,4\}$

In this section we show the existence of required decompositions for the case $S = \{1,2,3,4\}$ in Lemma 1.3.2. We first define J_n by

$$E(J_n) = \{\{i+2, i+3\}, \{i+2, i+4\}, \{i, i+3\}, \{i, i+4\} : i = 0, \dots, n-1\}\}$$

and $V(J_n) = \{0, \ldots, n+3\}$. We note the following basic properties of J_n. For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$.

- For $n \geq 9$, if for each $i \in \{0, 1, 2, 3\}$ we identify vertex i of J_n with vertex $i + n$ of J_n then the resulting graph is $\langle\{1, 2, 3, 4\}\rangle_n$. This means that for $n \geq 9$, we can obtain an (M)-decomposition of $\langle\{1, 2, 3, 4\}\rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0, 1, 2, 3\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i + n$.

- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x + (n - y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations*.

Lemma 1.6.7. *If n is a positive integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 4n$, $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M$.*

Proof Since J_1 is a 4-cycle, the result holds trivially for $n = 1$, so let $n \geq 2$ and suppose by induction that the result holds for each integer n' in the range $1 \leq n' < n$. The following decompositions can be seen in Table A.6 in the appendix.

$$J_1 \to 4 \qquad J_2 \to 3, 5 \qquad J_3 \to 3^4 \qquad J_5 \to 5^4$$

It is routine to check that if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some (possibly empty) list. If X is empty, then we are finished immediately. If X is nonempty then we can obtain a decomposition $J_n \to M$ by concatenating a decomposition $J_{n-y} \to X$ (which exists by our inductive hypothesis) with a decomposition $J_y \to Y$. □

Lemma 1.6.8. *For $6 \leq k \leq 8$, if $n \geq k$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 4n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$ such that the k-cycle is incident upon vertices $\{0, 1, \ldots, k-1\}$.*

Proof First we note that Table A.4 in the appendix lists a number of decompositions containing a k-cycle C_k for some $6 \leq k \leq 8$ such that $V(C_k) \subseteq \{0, \ldots, k-1\}$.

For each k, it is easy to use the value of $k \pmod 4$ to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k$ is one of the decompositions in Table A.4 and Y is some (possibly empty) list with the property $\sum Y = 4y$ for some integer y. If Y is empty we are done, else Lemma 1.6.7 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_x \to X, k$ with $J_y \to Y$.

The k-cycle in the resulting decomposition will be incident on vertices $\{0, 1, \ldots, k\}$ as this property held in the decomposition $J_x \to X, k$ and thus the resulting decomposition satisfies the condition in the lemma. □

Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and

- C is a k-cycle such that $V(C) = \{n-k+4, \ldots, n+3\}$ and $\{\{n, n+2\}, \{n+1, n+3\}\} \subseteq E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 1.6.10 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{1, \ldots, 7\}$, the graph obtained from J_y by adding the edges $\{0, 2\}$ and $\{1, 3\}$ will be denoted J_y^+. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A_1, A_2\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A_1 and A_2 are vertex-disjoint paths, one from 0 to 2 and one from 1 to 3; and
- $|E(A_1)| + |E(A_2)| = l + 2$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{\{n, n+2\}, \{n+1, n+3\}\} \subseteq E(A)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{1, \ldots, 7\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edges $\{n-y, n-y+2\}$ and $\{n-y+1, n-y+3\}$, and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k+l$. The edges $\{n, n+2\}$ and $\{n+1, n+3\}$ of the k-cycle in the decomposition of J_{n-y} are replaced by the two paths in the decomposition of J_y^+ to form the $(k+l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k+y)^*$.

Lemma 1.6.9. *For $9 \leq k \leq 13$, if $n \geq k$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 4n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k^*$.*

Proof First we note the existence of decompositions of the form $J_x \to X, k^*$ listed in Table A.5 in the appendix. For each k, it is routine to use the value of $k \pmod 4$ to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions in Table A.5, and Y is some (possibly empty) list with the property $\sum Y = 4y$ for some integer y. If Y is empty we are done, else Lemma 1.6.7 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$.

□

Lemma 1.6.10. *If $n \geq 9$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 3n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof Lemma 1.6.9 shows that the result holds for $9 \leq n \leq 13$. So let $n \geq 14$ and suppose by induction that the result holds for each integer n' in the range $9 \leq n' < n$. The following decompositions are given in Table A.6 in the appendix.

$$J_1^+ \to 3, 1^{+*} \qquad J_3^+ \to 4, 5, 3^{+*} \qquad J_4^+ \to 4^3, 4^{+*} \qquad J_5^+ \to 5^3, 5^{+*}$$

It is routine to check, using $\sum M = 3n \geq 42$, that M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n-y \geq n-5 \geq 9$) with a decomposition $J_y^+ \to Y, y^{+*}$. $\square$

Lemma 1.6.11. *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 4n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 9$ all cycles in this decomposition have the property that for $i \in \{0, 1, 2, 3\}$ no cycle is incident upon both vertex i and vertex $n+i$.*

Proof We first note that if $n \geq 9$ it is clear that any 3-, 4- or 5-cycle in such a decomposition cannot be incident on two vertices i and $i+n$ for any $i \in \{0, 1, 2, 3\}$. As such, Lemma 1.6.8 shows that the result holds for all n with $6 \leq k \leq 8$, so in the following we deal only with $k \geq 9$.

Lemma 1.6.9 shows that the result holds for all n with $9 \leq k \leq 13$ with the additional property that the k-cycle is not incident upon any vertex in $\{0, 1, 2, 3\}$, and Lemma 1.6.10 shows that the result holds for all $n = k$ with the same property on the k-cycle. We can therefore assume that $14 \leq k \leq n-1$, so let $n \geq 15$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$ with the additional property that the k-cycle is not incident upon any vertex in $\{0, 1, 2, 3\}$.

The following decompositions exist by Lemma 1.6.7.

$$J_1 \to 4 \qquad J_2 \to 3, 5 \qquad J_3 \to 3^4 \qquad J_5 \to 5^4$$

Case 1 Suppose that $k \leq n-5$. Then it is routine to check, using $\sum M = 4n-k \geq 3n+5 \geq 50$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$ (which exists by our inductive hypothesis, since $k \leq n-6 \leq n-y$) with a decomposition $J_y \to Y$. Since $n \geq 15$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, 2, 3\}$ has no vertex in $\{n, n+1, n+2, n+3\}$, and by our inductive hypothesis the same holds for the k-cycle.

Case 2 Suppose that $n-4 \leq k \leq n-1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, 4)$ for some list X, if $k \in \{n-4, n-3, n-2\}$ and $M = (X, 3, 5)$ for some list X, and if $k \in \{n-4, n-3\}$ and $M = (X, 3^4)$ for some list X. So we may assume that none of these hold.

Given this, using $\sum M = 4n - k \geq 3n + 1 \geq 46$, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below (note that, since $\nu_4(M) = 0$, in each case we can deduce the given value of $\nu_5(M) \pmod 3$ from $\sum M = 3n - k$). The decompositions in the third column exist by Lemma 1.6.10 (since $k \geq 14$), and the decompositions listed in the last column are shown in Table A.6 in the appendix.

k	$\nu_5(M) \pmod 3$	first decomposition	second decomposition
$n-4$	2	$J_{n-7} \to (M - (5^5)), (n-7)^*$	$J_7^+ \to 5^5, 3^+$
$n-3$	0	$J_{n-4} \to (M - (5^3)), (n-4)^*$	$J_4^+ \to 5^3, 1^+$
$n-2$	1	$J_{n-6} \to (M - (5^4)), (n-6)^*$	$J_6^+ \to 5^4, 4^+$
$n-1$	2	$J_{n-3} \to (M - (5^2)), (n-3)^*$	$J_2^+ \to 5^2, 2^+$

For $n \geq 9$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, 2, 3\}$ has no vertex in $\{n, \ldots, n+3\}$, and by the definition of the decompositions given in the third column the k-cycle has no vertex in $\{0,1,2,3\}$, so these decompositions do have the required properties.

$\square$

Lemma 1.6.12. *If $S = \{1,2,3,4\}$, $n \geq 9$, and $M = (m_1,\dots,m_t,k)$ is any list satisfying $m_i \in \{3,4,5\}$ for $i = 1,\dots,t$, $3 \leq k \leq n$, and $\sum M = 4n$, then there is an (M)-decomposition of $\langle S\rangle_n$.*

Proof As noted earlier, for $n \geq 9$ we can obtain an (M)-decomposition of $\langle\{1,2,3,4\}\rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0,1,2,3\}$, no cycle contains both vertex i and vertex $i+n$. Thus, for $S = \{1,2,3,4\}$, the required result follows by Lemma 1.6.7 for $k \in \{3,4,5\}$ and by Lemma 1.6.11 for $6 \leq k \leq n$. $\square$

$S = \{1,2,3,4,6\}$

In this section we show the existence of required decompositions for the case $S = \{1,2,3,4,6\}$ in Lemma 1.3.2. We first define J_n by

$$E(J_n) = \{\{i+2,i+3\},\{i+2,i+4\},\{i+3,i+6\},\{i,i+4\},\{i,i+6\} : i = 0,\dots,n-1\}$$

and $V(J_n) = \{0,\dots,n+5\}$. We note the following basic properties of J_n.

For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$. We note the following basic properties of J_n.

- For $n \geq 13$, if for each $i \in \{0,1,2,3,4,5\}$ we identify vertex i of J_n with vertex $i+n$ of J_n then the resulting graph is $\langle\{1,2,3,4,6\}\rangle_n$. This means that for $n \geq 13$, we can obtain an (M)-decomposition of $\langle\{1,2,3,4,6\}\rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0,1,2,3,4,5\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i+n$.
- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x+(n-y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations.*

Lemma 1.6.13. *If n is a positive integer and $M = (m_1,\dots,m_t)$ is a list such that $\sum M = 5n$, $m_i \in \{3,4,5\}$ for $i = 1,\dots,t$, and $M \notin \mathcal{E}$ where*

$$\mathcal{E} = \{(3^2,4)\} \cup \{(3^{5i}) : i \geq 1 \text{ is odd}\} \cup \{(3^{5i},5) : i \geq 1 \text{ is odd}\},$$

then there is a decomposition $J_n \to M$.

Proof We have verified by computer search and concatenation that the result holds for $n \leq 10$. So assume $n \geq 11$ and suppose by induction that the result holds for each integer n' in the range $1 \leq n' < n$. The following decompositions are given in Table A.9 in the appendix.

$$J_1 \to 5 \qquad J_4 \to 4^5 \qquad J_6 \to 3^{10} \qquad J_5 \to 3^7,4 \qquad J_4 \to 3^4,4^2 \qquad J_3 \to 3,4^3$$

It is routine to check that if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some (possibly empty) list. If X is empty, then we are finished immediately. If X is nonempty and $X \notin \mathcal{E}$, then we can obtain a decomposition $J_n \to M$ by concatenating a decomposition $J_{n-y} \to X$ (which exists by our inductive hypothesis) with a decomposition $J_y \to Y$. Thus, we can assume $X \in \mathcal{E}$. But since $n \geq 11$ and $\sum Y \leq 30$, we have $\sum X \geq 25$ which implies $X \in \{(3^{5i}) : i \geq 3 \text{ is odd}\} \cup \{(3^{5i}, 5) : i \geq 3 \text{ is odd}\}$.

It follows that $M = (3^{10}, X')$ for some nonempty list $X' \notin \mathcal{E}$ (because $M \notin \mathcal{E}$) and we can obtain a decomposition $J_n \to M$ by concatenating a decomposition $J_{n-6} \to X'$ (which exists by our inductive hypothesis) with a decomposition $J_6 \to 3^{10}$. □

Lemma 1.6.14. *For $6 \leq k \leq 10$, if $n \geq k$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 5n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 13$ any cycle in this decomposition having a vertex in $\{0, \ldots, 5\}$ has no vertex in $\{n, \ldots, n+5\}$.*

Proof First we note that Table A.7 in the appendix lists a number of decompositions containing a k-cycle C_k for some $6 \leq k \leq 10$ such that $V(C_k) \subseteq \{0, \ldots, 12\}$. For each k, it is routine to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k$ is one of the decompositions in Table A.7, and Y is some (possibly empty) list with the property $\sum Y = 5y$ for some integer y. If Y is empty we are done, else Lemma 1.6.13 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k$. All decompositions in the list have the property that the k-cycle is only incident upon some subset of the vertices $\{0, \ldots, 12\}$, and the resulting decomposition after concatenation will still have this property. Additionally it is simple to check that every decomposition used has the property that no 3-, 4- or 5-cycle contains two vertices v_a and v_b such that $|v_a - v_b| \geq 8$, so this gives the required decompositions. □

Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- C is a k-cycle such that $V(C) = \{n-k+6, \ldots, n+5\}$ and $\{\{n, n+3\}, \{n+1, n+4\}, \{n+2, n+5\}\} \subseteq E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 1.6.16 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{4, \ldots, 11\}$, the graph obtained from J_y by adding the three edges $\{0, 3\}$, $\{1, 4\}$ and $\{2, 5\}$ will be denoted J_y^+. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A_1, A_2, A_3\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A_1, A_2 and A_3 are vertex-disjoint paths, one from 0 to 3, one from 1 to 4, and one from 2 to 5; and

- $|E(A_1)| + |E(A_2)| + |E(A_3)| = l + 3$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{\{n, n+3\}, \{n+1, n+4\}, \{n+2, n+5\}\} \subseteq E(A_1) \cup E(A_2) \cup E(A_3)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{4, \ldots, 11\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edges in $\{\{n-y, n-y+3\}, \{n-y+1, n-y+4\}, \{n-y+2, n-y+5\}\}$ and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k+l$. The three edges $\{n, n+3\}$, $\{n+1, n+4\}$ and $\{n+2, n+5\}$ of the k-cycle in the decomposition of J_{n-y} are replaced by the three paths in the decomposition of J_y^+ to form the $(k+l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k+y)^*$.

Lemma 1.6.15. *For $11 \leq k \leq 16$, if $n \geq k$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 3n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k^*$.*

Proof First we note the existence of the decompositions given in Table A.8 in the appendix. For $11 \leq k \leq 14$ it is routine to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions in Table A.8, and Y is some (possibly empty) list with the properties that $\sum Y = 5y$ for some integer y and Y is not one of the exceptions to Lemma 1.6.13. If Y is empty, we are done, else Lemma 1.6.13 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$.

For $15 \leq k \leq 16$ we note the existence of the following decompositions, given in Table A.9 in the appendix.

$$J_4^+ \to 4^4, +4^* \qquad J_4^+ \to 3^2, 5^2, +4^* \qquad J_4^+ \to 3, 4^2, 5, +4^* \qquad J_4^+ \to 3^4, 4, +4^*$$
$$J_5^+ \to 5^4, +5^* \qquad J_5^+ \to 3^5, +5^*$$

For $n \geq k$, if M can be written as (X, Y) such that $J_y^+ \to Y, +y$ is a decomposition in the above list and $k - y \geq 11$, then the required decomposition can be obtained by concatenation of a decomposition of $J_x \to X, (k-y)^*$ (which exists by the argument above) with the decomposition $J_y^+ \to Y, +y$.

For $k \geq 15$ we therefore assume that $Y \notin M$ for any $Y \in \{(4^4), (3^2, 5^2), (3, 4^2, 5), (3^4, 4)\}$ and for $k = 16$ we add the additional assumptions that $5^4 \notin M$ and $3^5, 5 \notin M$.

Given $n \geq k$ and a list M that satisfies these assumptions (where applicable) and the conditions of the lemma, it is routine to check that we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions listed in Table A.8 in the appendix, and Y is some (possibly empty) list with the property $\sum Y = 5y$ for some integer y. If Y is empty we are done, else Lemma 1.6.13 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$. □

Lemma 1.6.16. *If $n \geq 11$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 4n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof Lemma 1.6.15 shows that the result holds for $11 \leq n \leq 16$. So let $n \geq 17$ and suppose by induction that the result holds for each integer n' in the range $11 \leq n' < n$. The following decompositions are given in Table A.9 in the appendix.

$$J_4^+ \to 4^4, 4^{+*} \qquad J_5^+ \to 5^4, 5^{+*} \qquad J_6^+ \to 3^8, 6^{+*}$$

It is routine to check, using $\sum M = 4n \geq 68$, that M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n - y \geq n - 6 \geq 11$) with a decomposition $J_y^+ \to Y, y^{+*}$. $\square$

Lemma 1.6.17. *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 5n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 13$ any cycle in this decomposition having a vertex in $\{0, \ldots, 5\}$ has no vertex in $\{n, \ldots, n+5\}$.*

Proof Lemmas 1.6.14 and 1.6.15 show that the result holds for $6 \leq k \leq 16$, and Lemma 1.6.16 shows that it holds for $k = n$, so we can also assume that $17 \leq k \leq n - 1$. Let $n \geq 18$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$. The following decompositions exist by Lemma 1.6.13.

$$J_1 \to 5^1 \qquad J_3 \to 3^1 4^3 \qquad J_4 \to 4^5 \qquad J_6 \to 3^{10}$$

Case 1 Suppose that $k \leq n-6$. Then it is routine to check, using $\sum M = 5n-k \geq 4n+6 \geq 78$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$ (which exists by our inductive hypothesis, since $k \leq n - 6 \leq n - y$) with a decomposition $J_y \to Y$. As this concatenation does not change the k-cycle, this decomposition has the desired properties.

Case 2 Suppose that $n - 5 \leq k \leq n - 1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, 5)$ for some list X, if $k \in \{n-5, n-4, n-3\}$ and $M = (X, 3, 4^3)$ for some list X, and if $k \in \{n-5, n-4\}$ and $M = (X, 4^5)$ for some list Y. So we may assume that none of these hold. Additionally, the following two decompositions are noted here.

$$J_{18} \to 3^{23}, 4, 17 \quad J_{21} \to 3^{28}, 4, 17$$

The first of these decompositions is given in Table A.9 in the appendix. The decomposition $J_{21} \to 3^{28}, 4, 17$ can be obtained by concatenation of $J_{12} \to 3^{16}, 12^*$ (which exists by Lemma 1.6.16) with $J_9^+ \to 3^{12}, 4, 5^+$ (also given in Table A.9). As such, we do not consider these decompositions in what follows.

Given the exceptions above, using $\sum M = 5n - k \geq 4n + 1 \geq 73$, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below (note that, since $\nu_5(M) = 0$, in each case we can deduce the given value of $\nu_3(M) \pmod 4$ from $\sum M = 5n - k$). The decompositions in the third column exist by Lemma 1.6.16 (since $k \geq 17$), and the decompositions listed in the last column are given in Table A.9.

k	$\nu_3(M) \pmod 4$	first decomposition	second decomposition
$n-5$	3	$J_{n-10} \to (M - (3^{15})), (n-10)^*$	$J_{10}^+ \to 3^{15}, 5^+$
$n-4$	0	$J_{n-11} \to (M - (3^{16})), (n-11)^*$	$J_{11}^+ \to 3^{16}, 7^+$
$n-3$	1	$J_{n-9} \to (M - (3^{13})), (n-9)^*$	$J_9^+ \to 3^{13}, 6^+$
$n-2$	2	$J_{n-7} \to (M - (3^{10})), (n-7)^*$	$J_7^+ \to 3^{10}, 5^+$
		$J_{n-5} \to (M - (3^2, 4^4)), (n-5)^*$	$J_5^+ \to 3^2, 4^4, 3^+$
$n-1$	3	$J_{n-8} \to (M - (3^{11})), (n-8)^*$	$J_8^+ \to 3^{11}, 7^+$
		$J_{n-4} \to (M - (3^3, 4^2)), (n-4)^*$	$J_4^+ \to 3^3, 4^2, 3^+$

By the definition of the decompositions given in the third column the vertex set of the k-cycle is some subset of $\{6, \ldots, n+5\}$. $\square$

Lemma 1.6.18. *If $S = \{1, 2, 3, 4, 6\}$, $n \geq 13$, and $M = (m_1, \ldots, m_t, k)$ is any list satisfying $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, $3 \leq k \leq n$, and $\sum M = 5n$, then there is an (M)-decomposition of $\langle S \rangle_n$, except possibly when*

- $n \equiv 3 \pmod 6$ *and* $M = (3^{\frac{5n}{3}})$*; or*
- $n \equiv 4 \pmod 6$ *and* $M = (3^{\frac{5n-5}{3}}, 5)$.

Proof As noted above, for $n \geq 13$ we can obtain an (M)-decomposition of $\langle \{1, 2, 3, 4, 6\} \rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0, \ldots, 5\}$, no cycle contains both vertex i and vertex $i + n$. Thus, for $S = \{1, 2, 3, 4, 6\}$, Lemma 1.3.2 follows by Lemma 1.6.13 for $k \in \{3, 4, 5\}$ and by Lemma 1.6.17 for $6 \leq k \leq n$. $\square$

$S = \{1, 2, 3, 4, 5, 7\}$

In this section we show the existence of required decompositions for the case $S = \{1, 2, 3, 4, 5, 7\}$ in Lemma 1.3.2. We first define J_n by

$$E(J_n) = \{\{i+6, i+7\}, \{i+5, i+7\}, \{i+3, i+6\}, \{i+3, i+7\}, \{i, i+5\}, \{i, i+7\} : i = 0, \ldots, n-1\}\}$$

and $V(J_n) = \{0, \ldots, n+6\}$. We note the following basic properties of J_n. For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$.

- For $n \geq 15$, if for each $i \in \{0, 1, 2, 3, 4, 5, 6\}$ we identify vertex i of J_n with vertex $i + n$ of J_n then the resulting graph is $\langle \{1, 2, 3, 4, 5, 7\} \rangle_n$. This means that for $n \geq 15$, we can obtain an (M)-decomposition of $\langle \{1, 2, 3, 4, 5, 7\} \rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0, 1, \ldots, 6\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i + n$.
- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x + (n - y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations*.

Lemma 1.6.19. *If n is a positive integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 6n$, $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, and $M \neq (4^3)$ then there is a decomposition $J_n \to M$.*

Proof We first note the existence of the following decompositions, given in Table A.12 in the appendix.

$J_1 \to 3^2$	$J_2 \to 3, 4, 5$	$J_3 \to 4^2, 5^2$	$J_3 \to 3, 5^3$
$J_3 \to 3^2, 4^3$	$J_4 \to 4, 5^4$	$J_4 \to 4^6$	$J_4 \to 3, 4^4, 5$
$J_5 \to 5^6$	$J_5 \to 4^5, 5^2$	$J_6 \to 4^9$	

The only required decomposition of J_1 is shown in the table above, so we may assume $n \geq 2$ and assume by induction that the result holds for any positive integer n' in the range $1 \leq n' < n$.

It is routine to check that for $n \geq 2$ if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ for some (possibly empty) list $Y \neq (4^3)$ where $J_x \to X$ is one of the decompositions listed above. If Y is empty, we are done, else we can obtain the required decomposition by concatenation of $J_y \to Y$ (which exists by our inductive hypothesis since $Y \neq (4^3)$) with the decomposition $J_x \to X$. □

Lemma 1.6.20. *For $n \geq 4$, if $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + 6 = 6n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, 6$ such that the 6-cycle is incident upon vertices $\{i, i+1, \ldots, i+5\}$ for some integer i.*

Proof First we note that Table A.10 in the appendix lists a number of decompositions containing a 6-cycle C_6 such that $V(C_6) \subseteq \{4, \ldots, 9\}$.

It is routine to check that for $n \geq 4$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k$ is one of the decompositions in Table A.10, and $Y \neq (4^3)$ is some (possibly empty) list. If Y is empty we are done, else Lemma 1.6.19 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_x \to X, k$ with $J_y \to Y$. □

Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- C is a k-cycle such that $V(C) = \{n - k + 7, \ldots, n + 6\}$ and $\{n + 4, n + 6\} \in E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 1.6.22 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{2, \ldots, 7\}$, the graph obtained from J_y by adding the edge $\{4, 6\}$ will be denoted J_y^+. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A is a path from 4 to 6 such that $\{0, 1, 2, 3, 5\} \cap V(A) = \emptyset$; and
- $|E(A)| = l + 1$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{n + 4, n + 6\} \in E(A)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{2, \ldots, 7\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edge $\{n - y + 4, n - y + 6\}$, and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n - y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k + l$. The edge $\{n + 4, n + 6\}$ of the k-cycle in the decomposition of J_{n-y} are replaced by the two paths in the decomposition of J_y^+ to form the $(k + l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k + y)^*$.

Lemma 1.6.21. *For $7 \leq k \leq 12$, if n is an integer with $n \geq k$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 6n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k^*$.*

Proof

For each k, it is routine to use the value of $k \pmod 6$ to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions in Table A.11 in the appendix, and $Y \neq (4^3)$ is some (possibly empty) list. If Y is empty, then we are done, else Lemma 1.6.19 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$. □

Lemma 1.6.22. *Given an integer $n \geq 7$, if $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 5n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof Lemma 1.6.21 shows that the result holds for $7 \leq n \leq 12$. So let $n \geq 13$ and suppose by induction that the result holds for each integer n' in the range $7 \leq n' < n$. The following decompositions are given in Table A.12 in the appendix.

$$J_4^+ \to 5^4, 4^* \qquad J_4^+ \to 4^5, 4^* \qquad J_6^+ \to 3^{10}, 6^*$$

It is routine to check, using $\sum M = 5n \geq 65$, that M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n - y \geq n - 6 \geq 7$) with a decomposition $J_y^+ \to Y, y^{+*}$. □

Lemma 1.6.23. *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 6n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 15$ all cycles in this decomposition have the property that for $i \in \{0, 1, \ldots, 6\}$ no cycle is incident upon both vertex i and vertex $n + i$.*

Proof We first note that if $n \geq 15$ it is clear that any 3-, 4- or 5-cycle in such a decomposition cannot be incident on two vertices i and $i + n$ for any $i \in \{0, 1, \ldots, 6\}$. As such, Lemma 1.6.20 shows that the result holds for all n with $k = 6$, so in the following we deal only with $k \geq 7$.

Lemma 1.6.21 shows that the result holds for all $n \geq k$ with $7 \leq k \leq 12$ with the property that the k-cycle is not incident upon any vertex in $\{0, 1, \ldots, 6\}$, and Lemma 1.6.22 shows that the result holds for all $n = k$ with the same property on the k-cycle. We can therefore assume that $13 \leq k \leq n - 1$, so let $n \geq 14$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$ with the property that the k-cycle is not incident upon any vertex in $\{0, 1, \ldots, 6\}$.

The following decompositions exist by Lemma 1.6.19.

$$\begin{array}{lll} J_1 \to 3^2 & J_2 \to 3, 4, 5 & J_3 \to 4^2, 5^2 \\ J_4 \to 4^6 & J_4 \to 4, 5^4 & J_5 \to 5^6 \end{array}$$

Case 1 Suppose that $k \leq n - 5$. Then it is routine to check, using $\sum M = 6n - k \geq 5n + 5 \geq 75$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$

(which exists by our inductive hypothesis, since $k \leq n-5 \leq n-y$) with a decomposition $J_y \to Y$. For $n \geq 15$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, \ldots, 6\}$ has no vertex in $\{n, n+1, \ldots, n+6\}$, and by our inductive hypothesis the same holds for the k-cycle.

Case 2 Suppose that $n-4 \leq k \leq n-1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, 3^2)$ for some list X, if $k \in \{n-4, n-3, n-2\}$ and $M = (X, 3, 4, 5)$ for some list X, if $k \in \{n-4, n-3\}$ and $M = (X, 4^2, 5^2)$ for some list X, and if $k = n-4$ and $M = (X, 4^6)$ or $M = (X, 4, 5^4)$ for some list X. So we may assume that none of these hold.

Given this, using $\sum M = 6n - k \geq 5n + 1 \geq 71$, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below. Note that, since $\nu_3(M) < 2$, there are only two cases to take for $\nu_3(M)$ and in either case we can deduce the given value of $\nu_4(M) \pmod 5$ from $\sum M = 6n - k$. For $k = n-4$ this shows that $\nu_4(M) \geq 1$ so it is routine to see that all required decompositions for $k = n-4$ can be constructed in the manner described in the previous paragraph. The decompositions in the fourth column exist by Lemma 1.6.22, and the decompositions listed in the last column are shown in Table A.12 in the appendix.

k	$\nu_3(M)$	$\nu_4(M) \pmod 5$	first decomposition	second decomposition
$n-3$	1	0	$J_{n-4} \to (M - (3, 4^5)), (n-4)^*$	$J_4^+ \to 3, 4^5, 1^+$
			$J_{n-4} \to (M - (3, 5^4)), (n-4)^*$	$J_4^+ \to 3, 5^4, 1^+$
$n-3$	0	2	$J_{n-5} \to (M - (4^7)), (n-5)^*$	$J_5^+ \to 4^7, 2^+$
$n-2$	1	1	$J_{n-5} \to (M - (3, 4^6)), (n-5)^*$	$J_5^+ \to 3, 4^6, 3^+$
$n-2$	0	3	$J_{n-3} \to (M - (4^3, 5)), (n-3)^*$	$J_3^+ \to 4^3, 5, 1^+$
			$J_{n-6} \to (M - (4^8)), (n-6)^*$	$J_6^+ \to 4^8, 4^+$
$n-1$	1	2	$J_{n-2} \to (M - (3, 4^2)), (n-2)^*$	$J_2^+ \to 3, 4^2, 1^+$
$n-1$	0	4	$J_{n-3} \to (M - (4^3, 5)), (n-3)^*$	$J_3^+ \to 4^3, 5, 1^+$
			$J_{n-7} \to (M - (4^9)), (n-7)^*$	$J_7^+ \to 4^9, 6^+$

For $n \geq 15$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, \ldots, 6\}$ has no vertex in $\{n, \ldots, n+6\}$, and by the definition of the decompositions given in the fourth column the k-cycle has no vertex in $\{0, 1, \ldots, 6\}$, so these decompositions do have the required properties. $\square$

Lemma 1.6.24. *If $S = \{1, 2, 3, 4, 5, 7\}$, $n \geq 15$, and $M = (m_1, \ldots, m_t, k)$ is any list satisfying $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, $3 \leq k \leq n$, and $\sum M = 6n$, then there is an (M)-decomposition of $\langle S \rangle_n$.*

Proof As noted earlier, for $n \geq 15$ we can obtain an (M)-decomposition of $\langle \{1, 2, 3, 4, 5, 7\} \rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0, 1, \ldots, 6\}$, no cycle contains both vertex i and vertex $i+n$. Thus, for $S = \{1, 2, 3, 4, 5, 7\}$, the required result follows by Lemma 1.6.19 for $k \in \{3, 4, 5\}$ and by Lemma 1.6.23 for $6 \leq k \leq n$. $\square$

$S = \{1, 2, 3, 4, 5, 6, 7\}$

In this section we show the existence of required decompositions exist for the case $S = \{1, 2, 3, 4, 5, 6, 7\}$ in Lemma 1.3.2. We first define J_n by

$$E(J_n) = \{\{i+3, i+4\}, \{i+5, i+7\}, \{i+3, i+6\}, \{i, i+4\}, \{i, i+5\}, \{i, i+6\}, \{i, i+7\} : i = 0, \ldots, n-1\}\}$$

and $V(J_n) = \{0, \ldots, n+6\}$. We note the following basic properties of J_n. For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$.

- For $n \geq 15$, if for each $i \in \{0, 1, 2, 3, 4, 5, 6\}$ we identify vertex i of J_n with vertex $i + n$ of J_n then the resulting graph is $\langle\{1, 2, 3, 4, 5, 6, 7\}\rangle_n$. This means that for $n \geq 15$, we can obtain an (M)-decomposition of $\langle\{1, 2, 3, 4, 5, 6, 7\}\rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0, 1, \ldots, 6\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i + n$.
- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x + (n - y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations*.

Lemma 1.6.25. *If n is a positive integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 7n$, $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, and $M \neq (3^7)$ then there is a decomposition $J_n \to M$.*

Proof The following decompositions are given in Table A.15 in the appendix.

$J_1 \to 3, 4$	$J_2 \to 4, 5^2$	$J_2 \to 3^3, 5$	$J_3 \to 4^4, 5$
$J_3 \to 3^2, 5^3$	$J_4 \to 4^7$	$J_4 \to 3, 5^5$	$J_5 \to 5^7$
$J_6 \to 3^{14}$	$J_9 \to 3^{21}$		

The only required decomposition of J_1 is shown in the table above, so we may assume $n \geq 2$ and assume by induction that the result holds for any positive integer n' in the range $1 \leq n' < n$.

It is routine to check that for $n \geq 2$ if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ for some (possibly empty) list $Y \neq (3^7)$ where $J_x \to X$ is one of the decompositions listed above. If Y is empty, we are done, else we can obtain the required decomposition by concatenation of $J_y \to Y$ (which exists by our inductive hypothesis since $Y \neq (3^7)$) with the decomposition $J_x \to X$. $\square$

Lemma 1.6.26. *For $k \in \{6, 7\}$ and $n \geq k + 1$, if $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 7n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$ such that the k-cycle is incident upon vertices $\{i, i + 1, \ldots, i + 5\}$ for some integer i.*

Proof

First we note that Table A.13 in the appendix lists a number of decompositions required for this lemma. All of these decompositions contain a k-cycle for some $k \in \{6, 7\}$ where the k-cycle is incident on some subset of the vertices $\{4, \ldots, k + 3\}$.

It is routine to check that for $n \geq 4$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k$ is one of the decompositions in Table A.13 in the appendix, and $Y \neq (3^7)$ is some (possibly empty) list. If Y is empty we are done, else Lemma 1.6.25 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_x \to X, k$ with $J_y \to Y$. $\square$

Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \dots, t$; and
- C is a k-cycle such that $V(C) = \{n-k+7, \dots, n+6\}$ and $\{n+4, n+6\} \in E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 1.6.28 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{5, \dots, 10\}$, the graph obtained from J_y by adding the edge $\{4, 6\}$ will be denoted J_y^+. Let $M = (m_1, \dots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \dots, t$. A decomposition $\{G_1, \dots, G_t, A\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \dots, t$;
- A is a path from 4 to 6 such that $\{0, 1, 2, 3, 5\} \cap V(A) = \emptyset$; and
- $|E(A)| = l + 1$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{n+4, n+6\} \in E(A)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{5, \dots, 10\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edge $\{n-y+4, n-y+6\}$, and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k+l$. The edge $\{n+4, n+6\}$ of the k-cycle in the decomposition of J_{n-y} are replaced by the two paths in the decomposition of J_y^+ to form the $(k+l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k+y)^*$.

Lemma 1.6.27. *For $8 \leq k \leq 17$, if n is an integer with $n \geq k$ and $M = (m_1, \dots, m_t)$ is a list such that $\sum M + k = 7n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \dots, t$, then there is a decomposition $J_n \to M, k^*$.*

Proof

For each k, it is routine to use the value of $k \pmod 7$ to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions in Table A.14 in the appendix, and $Y \neq (3^7)$ is some (possibly empty) list. If Y is empty, then we are done, else Lemma 1.6.25 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$. □

Lemma 1.6.28. *Given an integer $n \geq 7$, if $M = (m_1, \dots, m_t)$ is a list such that $\sum M = 6n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \dots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof Lemma 1.6.27 shows that the result holds for $8 \leq n \leq 17$. So let $n \geq 18$ and suppose by induction that the result holds for each integer n' in the range $8 \leq n' < n$. The following decompositions are given in Table A.15 in the appendix.

$J_7^+ \to 4^3, 5^6, 7^{+*}$ $\quad$ $J_7^+ \to 3^4, 5^6, 7^{+*}$ $\quad$ $J_7^+ \to 3^6, 4^6, 7^{+*}$ $\quad$ $J_7^+ \to 3^{14}, 7^{+*}$
$J_8^+ \to 4^{12}, 8^{+*}$ $\quad$ $J_{10}^+ \to 5^{12}, 10^{+*}$

It is routine to check, using $\sum M = 6n \geq 108$, that M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n - y \geq n - 10 \geq 8$) with a decomposition $J_y^+ \to Y, y^{+*}$. $\square$

Lemma 1.6.29. *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 7n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 15$ all cycles in this decomposition have the property that for $i \in \{0, 1, \ldots, 6\}$ no cycle is incident upon both vertex i and vertex $n + i$.*

Proof We first note that if $n \geq 15$ it is clear that any 3-, 4- or 5-cycle in such a decomposition cannot be incident on two vertices i and $i + n$ for any $i \in \{0, 1, \ldots, 6\}$. As such, Lemma 1.6.26 shows that the result holds for all n with $k \in \{6, 7\}$, so in the following we deal only with $k \geq 8$.

Lemma 1.6.27 shows that the result holds for all $n \geq k$ with $8 \leq k \leq 17$ with the property that the k-cycle is not incident upon any vertex in $\{0, 1, \ldots, 6\}$, and Lemma 1.6.28 shows that the result holds for all $n = k$ with the same property on the k-cycle. We can therefore assume that $18 \leq k \leq n - 1$, so let $n \geq 19$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$ with the property that the k-cycle is not incident upon any vertex in $\{0, 1, \ldots, 6\}$.

The following decompositions exist by Lemma 1.6.25.

$$\begin{array}{llll} J_1 \to 3, 4 & J_2 \to 4, 5^2 & J_2 \to 3^3, 5 & J_3 \to 4^4, 5 \\ J_3 \to 3^2, 5^3 & J_4 \to 4^7 & J_4 \to 3, 5^5 & J_5 \to 5^7 \\ J_6 \to 3^{14} & & & \end{array}$$

Case 1 Suppose that $k \leq n - 6$. Then it is routine to check, using $\sum M = 7n - k \geq 6n + 6 \geq 120$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$ (which exists by our inductive hypothesis, since $k \leq n - 6 \leq n - y$) with a decomposition $J_y \to Y$. Since $n \geq 18$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, \ldots, 6\}$ has no vertex in $\{n, n + 1, \ldots, n + 6\}$, and by our inductive hypothesis the same holds for the k-cycle.

Case 2 Suppose that $n - 5 \leq k \leq n - 1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, Y)$ for some list X where $J_x \to X$ is one of the decompositions shown above and $k + x \leq n$. In what follows we take deal with each case of $k \in \{n - 5, n - 4, \ldots, n - 1\}$ separately, and in each case we assume that M cannot be written as (X, Y) for any such X.

Given this, using $\sum M = 7n - k \geq 6n + 1 \geq 115$, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below. We can use the fact that $\sum M = 7n - k$ to determine that $\nu_4(M) + 2\nu_5(M) \equiv (n - k) \pmod 3$, and this is also shown in the table (Note that by this, it is routine to see that for $k = n - 5$, $\nu_4(M) + 2\nu_5(M) \equiv 2 \pmod 3$ and thus all required decompositions can be constructed in the manner described in the previous paragraph). The decompositions in the third column exist by Lemma 1.6.28, and the decompositions listed in the last column are shown in Table A.15 in the appendix.

k	$\nu_4(M)+2\nu_5(M) \pmod 3$	first decomposition	second decomposition
$n-4$	1	$J_{n-6} \to (M-(5^8)),(n-6)^*$	$J_4^+ \to 5^8, 2^+$
$n-3$	0	$J_{n-5} \to (M-(3^{11})),(n-5)^*$ $J_{n-7} \to (M-(5^9)),(n-7)^*$	$J_5^+ \to 3^{11}, 2^+$ $J_7^+ \to 5^9, 4^+$
$n-2$	2	$J_{n-5} \to (M-(4^8)),(n-5)^*$ $J_{n-8} \to (M-(5^{10})),(n-8)^*$	$J_5^+ \to 4^8, 3^+$ $J_8^+ \to 5^{10}, 6^+$
$n-1$	1	$J_{n-5} \to (M-(3^2,5^5)),(n-5)^*$ $J_{n-5} \to (M-(3^7,5^2)),(n-5)^*$ $J_{n-6} \to (M-(4^8,5)),(n-6)^*$ $J_{n-8} \to (M-(4,5^9)),(n-8)^*$ $J_{n-9} \to (M-(5^{11})),(n-9)^*$	$J_5^+ \to 3^2, 5^5, 4^+$ $J_5^+ \to 3^7, 5^2, 4^+$ $J_6^+ \to 4^8, 5, 5^+$ $J_8^+ \to 4, 5^9, 7^+$ $J_9^+ \to 5^{11}, 8^+$

Since $n \geq 18$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0,1,\ldots,6\}$ has no vertex in $\{n,\ldots,n+6\}$, and by the definition of the decompositions given in the fourth column the k-cycle has no vertex in $\{0,1,\ldots,6\}$, so these decompositions do have the required properties. □

Lemma 1.6.30. *If $S=\{1,2,3\}$, $n \geq 15$, and $M=(m_1,\ldots,m_t,k)$ is any list satisfying $m_i \in \{3,4,5\}$ for $i=1,\ldots,t$, $3 \leq k \leq n$, and $\sum M = 7n$, then there is an (M)-decomposition of $\langle S\rangle_n$.*

Proof As noted earlier, for $n \geq 15$ we can obtain an (M)-decomposition of $\langle\{1,2,3,4,5,6,7\}\rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0,1,\ldots,6\}$, no cycle contains both vertex i and vertex $i+n$. Thus, for $S=\{1,2,3,4,5,6,7\}$, the required result follows by Lemma 1.6.25 for $k \in \{3,4,5\}$ and by Lemma 1.6.29 for $6 \leq k \leq n$. □

$S=\{1,2,3,4,5,6,7,8\}$

In this section we show the existence of required decompositions for the case $S=\{1,2,3,4,5,6,7,8\}$ in Lemma 1.3.2. We first define J_n by

$$\begin{aligned}E(J_n)=\{&\{i+7,i+8\},\{i+5,i+7\},\{i+5,i+8\},\{i+5,i+9\},\{i,i+5\},\\&\{i+1,i+7\},\{i,i+7\},\{i+1,i+9\} : i=0,\ldots,n-1\}\}\end{aligned}$$

and $V(J_n)=\{0,\ldots,n+8\}$. We note the following basic properties of J_n. For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$.

- For $n \geq 17$, if for each $i \in \{0,1,\ldots,8\}$ we identify vertex i of J_n with vertex $i+n$ of J_n then the resulting graph is $\langle\{1,2,3,4,5,6,7,8\}\rangle_n$. This means that for $n \geq 17$, we can obtain an (M)-decomposition of $\langle\{1,2,3,4,5,6,7,8\}\rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0,1,\ldots,9\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i+n$.
- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x+(n-y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations*.

Lemma 1.6.31. *If n is a positive integer and $M=(m_1,\ldots,m_t)$ is a list such that $\sum M = 8n$, $m_i \in \{3,4,5\}$ for $i=1,\ldots,t$, and $M \neq (3^8)$ then there is a decomposition $J_n \to M$.*

Proof We first note the existence of the following decompositions, given in Table A.18 in the appendix.

$$\begin{array}{llll} J_1 \to 4^2 & J_1 \to 3,5 & J_2 \to 3^4,4 & J_3 \to 4,5^4 \\ J_4 \to 3^9,5 & J_5 \to 5^8 & J_5 \to 3^{12},4 & J_6 \to 3^{16} \\ J_9 \to 3^{24} & & & \end{array}$$

The only required decompositions of J_1 are shown in the table above, so we may assume $n \geq 2$ and assume by induction that the result holds for any positive integer n' in the range $1 \leq n' < n$.

It is routine to check that for $n \geq 2$ if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ for some (possibly empty) list $Y \neq (3^8)$ where $J_x \to X$ is one of the decompositions listed above. If Y is empty, we are done, else we can obtain the required decomposition by concatenation of $J_y \to Y$ (which exists by our inductive hypothesis since $Y \neq (3^8)$) with the decomposition $J_x \to X$. □

Lemma 1.6.32. *For $k \in \{6,7,8\}$ and $n \geq 3$, if $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 8n$ and $m_i \in \{3,4,5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$ such that $V(C_k) \subseteq \{n, n+1, \ldots, n+9\}$.*

Proof First we note that Table A.16 in the appendix lists a number of decompositions required for this lemma. All of these decompositions contain a k-cycle for some $6 \leq k \leq 8$ where the k-cycle is incident on some subset of the vertices $\{n, \ldots, n+9\}$. some subset of the It is routine to check that for $n \geq 3$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k$ is one of the decompositions in Table A.16, and $Y \neq (3^8)$ is some (possibly empty) list. If Y is empty we are done, else Lemma 1.6.31 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k$. Since we concatenate with the k-cycle on the right, it is clear that the k-cycle is still incident upon some subset of $\{n, n+1, \ldots, n+9\}$. □

Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- C is a k-cycle such that $V(C) = \{n-k+9, \ldots, n+8\}$ and $\{\{n+1, n+5\}, \{n+2, n+6\}, \{n+3, n+7\}, \{n+4, n+8\}\} \subseteq E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 1.6.34 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{2, \ldots, 8\}$, the graph obtained from J_y by adding the edges $\{1,5\}, \{2,6\}, \{3,7\}, \{4,8\}$ will be denoted J_y^+. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A_1, A_2, A_3, A_4\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A_i is a path from i to $i+4$ with $0 \notin V(A_i)$ for $i = 1,2,3,4$;
- $V(A_i) \cap V(A_j) = \emptyset$ for $i \neq j$; and

- $|E(A_1)| + |E(A_2)| + |E(A_3)| + |E(A_4)| = l + 4$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{\{n+1, n+5\}, \{n+2, n+6\}, \{n+3, n+7\}, \{n+4, n+8\}\} \subseteq E(A_1) \cup E(A_2) \cup E(A_3) \cup E(A_4)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{2, \ldots, 8\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edges $\{n-y+1, n-y+5\}, \{n-y+2, n-y+6\}, \{n-y+3, n-y+7\}, \{n-y+4, n-y+8\}$, and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k+l$. The removed edges of the k-cycle in the decomposition of J_{n-y} are replaced by the four paths in the decomposition of J_y^+ to form the $(k+l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k+y)^*$.

Lemma 1.6.33. *For $9 \leq k \leq 14$, if n is an integer with $n \geq k$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M + k = 8n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k^*$.*

Proof

For each k, it is routine to use the value of $k \pmod 8$ to check that for $n \geq k$ and any M that satisfies the hypotheses of the lemma we can write M as (X, Y) where $J_x \to X, k^*$ is one of the decompositions given in Table A.17 in the appendix, and $Y \neq (3^8)$ is some (possibly empty) list. If Y is empty, then we are done, else Lemma 1.6.31 gives us the existence of a decomposition $J_y \to Y$ and the required decomposition can be obtained by concatenation of $J_y \to Y$ with $J_x \to X, k^*$. □

Lemma 1.6.34. *Given an integer $n \geq 9$, if $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 7n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof Lemma 1.6.33 shows that the result holds for $9 \leq n \leq 14$. So let $n \geq 15$ and suppose by induction that the result holds for each integer n' in the range $9 \leq n' < n$. The following decompositions are given in Table A.18 in the appendix.

$$J_4^+ \to 4^7, 4^{+*} \qquad J_5^+ \to 5^7, 5^{+*} \qquad J_6^+ \to 3^{14}, 6^{+*}$$

It is routine to check, using $\sum M = 7n \geq 105$, that M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n - y \geq n - 6 \geq 9$) with a decomposition $J_y^+ \to Y, y^{+*}$. □

Lemma 1.6.35. *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 8n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 17$ all cycles in this decomposition have the property that for $i \in \{0, 1, \ldots, 9\}$ no cycle is incident upon both vertex i and vertex $n + i$.*

Proof We first note that if $n \geq 17$ it is clear that any 3-, 4- or 5-cycle in such a decomposition cannot be incident on two vertices i and $i + n$ for any $i \in \{0, 1, \ldots, 9\}$. As such, Lemma 1.6.32 shows that the result holds for all n with $k \in \{6, 7, 8\}$, so in the following we deal only with $k \geq 9$.

Lemma 1.6.33 shows that the result holds for all $n \geq k$ with $9 \leq k \leq 14$ with the property that the k-cycle is not incident upon any vertex in $\{0, 1, \ldots, 9\}$, and Lemma 1.6.34 shows that the result holds for all $n = k$ with the same property on the k-cycle. We can therefore assume that $15 \leq k \leq n-1$, so let $n \geq 16$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$ with the property that the k-cycle is not incident upon any vertex in $\{0, 1, \ldots, 9\}$.

The following decompositions exist by Lemma 1.6.31.

$$\begin{array}{llll} J_1 \to 4^2 & J_1 \to 3, 5 & J_2 \to 3^4, 4 & J_3 \to 4, 5^4 \\ J_4 \to 3^9, 5 & J_5 \to 5^8 & J_5 \to 3^{12}, 4 & J_6 \to 3^{16} \end{array}$$

Case 1 Suppose that $k \leq n-6$. Then it is routine to check, using $\sum M = 8n - k \geq 7n + 6 \geq 118$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$ (which exists by our inductive hypothesis, since $k \leq n - 6 \leq n - y$) with a decomposition $J_y \to Y$. Since $n \geq 15$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, \ldots, 9\}$ has no vertex in $\{n, n+1, \ldots, n+9\}$, and by our inductive hypothesis the same holds for the k-cycle.

Case 2 Suppose that $n - 5 \leq k \leq n - 1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, Y)$ for some list X where $J_x \to X$ is one of the decompositions shown above and $k + x \leq n$. We can therefore assume that M cannot be written as (X, Y) for any such list X. In particular, since $J_1 \to 4^2$ exists we can assume $\nu_4(M) \leq 1$, and since $J_1 \to 3, 5$ exists we can assume either $\nu_3(M) = 0$ or $\nu_5(M) = 0$. As a result, using $\sum M = 8n - k \geq 7n + 1 \geq 113$ we have either $\nu_3(M) \geq 33$ or $\nu_5(M) \geq 20$.

Given this, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below.

The decompositions in the second column exist by Lemma 1.6.34 (since $k \geq 15$), and the decompositions listed in the last column are shown in Table A.18 in the appendix.

k	first decomposition	second decomposition
$n-5$	$J_{n-7} \to (M - (3^{18})), (n-7)^*$	$J_7^+ \to 3^{14}, 2^+$
$n-4$	$J_{n-8} \to (M - (5^{12})), (n-8)^*$	$J_8^+ \to 5^{12}, 4^+$
	$J_{n-5} \to (M - (3^{13})), (n-5)^*$	$J_5^+ \to 3^{13}, 1^+$
$n-3$	$J_{n-6} \to (M - (5^9)), (n-6)^*$	$J_6^+ \to 5^9, 3^+$
	$J_{n-6} \to (M - (3^{15})), (n-6)^*$	$J_6^+ \to 3^{15}, 3^+$
$n-2$	$J_{n-4} \to (M - (5^6)), (n-4)^*$	$J_4^+ \to 5^6, 2^+$
	$J_{n-4} \to (M - (3^{10})), (n-4)^*$	$J_4^+ \to 3^{10}, 2^+$
$n-1$	$J_{n-2} \to (M - (5^3)), (n-2)^*$	$J_2^+ \to 5^3, 1^+$
	$J_{n-2} \to (M - (3^5)), (n-2)^*$	$J_2^+ \to 3^5, 1^+$

If $n \geq 17$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, 1, \ldots, 9\}$ has no vertex in $\{n, \ldots, n+9\}$, and by the definition of the decompositions given in the second column the k-cycle has no vertex in $\{0, 1, \ldots, 9\}$, so these decompositions do have the required properties. □

Lemma 1.6.36. *If $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $n \geq 17$, and $M = (m_1, \ldots, m_t, k)$ is any list satisfying $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, $3 \leq k \leq n$, and $\sum M = 8n$, then there is an (M)-decomposition of $\langle S \rangle_n$.*

Proof As noted earlier, for $n \geq 17$ we can obtain an (M)-decomposition of $\langle\{1,2,3,4,5,6,7,8\}\rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0,1,\ldots,8\}$, no cycle contains both vertex i and vertex $i+n$. Thus, for $S=\{1,2,3,4,5,6,7,8\}$, the required result follows by Lemma 1.6.31 for $k \in \{3,4,5\}$ and by Lemma 1.6.35 for $6 \leq k \leq n$. $\square$

We now prove Lemma 1.3.2.

Lemma 1.3.2 *If*

$$S \in \{\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\},$$

$n \geq 2\max(S)+1$*, and* $M=(m_1,\ldots,m_t,k)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i=1,\ldots,t$*,* $3 \leq k \leq n$*, and* $\sum M = |S|n$*, then there is an* (M)*-decomposition of* $\langle S\rangle_n$*, except possibly when*

- $S=\{1,2,3,4,6\}$*,* $n \equiv 3 \pmod 6$ *and* $M=(3^{\frac{5n}{3}})$*; or*
- $S=\{1,2,3,4,6\}$*,* $n \equiv 4 \pmod 6$ *and* $M=(3^{\frac{5n-5}{3}},5)$*.*

Proof The required decompositions exist by Lemma 1.6.6 for $S=\{1,2,3\}$, by Lemma 1.6.12 for $S=\{1,2,3,4\}$, by Lemma 1.6.18 for $S=\{1,2,3,4,6\}$, by Lemma 1.6.24 for $S=\{1,2,3,4,5,7\}$, by Lemma 1.6.30 for $S=\{1,2,3,4,5,6,7\}$, and by Lemma 1.6.36 for $S=\{1,2,3,4,5,6,7,8\}$. $\square$

1.6.2 Proof of Lemma 1.3.4

In this section we prove Lemma 1.3.4, which we restate here for convenience.

Lemma 1.3.4 *If* $n \geq 7$ *and* $M=(m_1,\ldots,m_t,k,n)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i=1,\ldots,t$*,* $3 \leq k \leq n$*, and* $\sum M = 3n$*, then there is an* (M)*-decomposition of* $\langle\{1,2,3\}\rangle_n$*.*

The proof of Lemma 1.3.4 proceeds along similar lines to the proof of Lemma 1.3.2. We make use of the graphs $J_n^{\{1,2,3\}}$ defined in the proof of Lemma 1.3.2, which in this subsection we denote by just J_n. Recall that $J_n^{\{1,2,3\}}$ is the graph with vertex set $\{0,\ldots,n+2\}$ and edge set

$$\{\{i,i+1\},\{i+1,i+3\},\{i,i+3\} : i=0,\ldots,n-1\}.$$

We first construct decompositions of graphs which are related to the graphs J_n, then concatenate these decompositions to produce decompositions of the graphs J_n, and finally identify pairs of vertices to produce the required decompositions of $\langle\{1,2,3\}\rangle_n$.

For $n \geq 1$, the graph obtained from J_n by adding the edges $\{n,n+1\}$ and $\{n+1,n+2\}$ will be denoted L_n. Let $M=(m_1,\ldots,m_t)$ be a list of integers with $m_i \geq 3$ for $i=1,\ldots,t$. A decomposition $\{G_1,\ldots,G_t,A,B\}$ of L_n such that

- G_i is an m_i-cycle for $i=1,\ldots,t$;
- A is a path of length k from n to $n+1$; and
- B is a path of length $n-1$ from $n+1$ to $n+2$ such that $0,1,2 \notin V(B)$;

will be denoted $L_n \to M, k^+, (n-1)^H$.

In Lemmas 1.6.37 and 1.6.38 we form new decompositions of graphs L_n by concatenating decompositions of L_{n-y} with decompositions of graphs P_y which we will now define. For $y \in \{3,4,5,6\}$, the graph obtained from J_y by deleting the edges in $\{\{0,1\},\{1,2\}\}$ and adding the edges in $\{\{y,y+1\},\{y+1,y+2\}\}$ will be denoted P_y. Let $M = (m_1,\ldots,m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1,\ldots,t$. A decomposition $\{G_1,\ldots,G_t,A_1,A_2,B_1,B_2\}$ of P_y such that

- G_i is an m_i-cycle for $i = 1,2,\ldots,t$;
- A_1 and A_2 are vertex-disjoint paths with endpoints 0, 1, y and $y+1$, such that A_1 has endpoints 0 and y or 0 and $y+1$;
- $|E(A_1)| + |E(A_2)| = k'$ and $2 \notin V(A_1) \cup V(A_2)$;
- B_1 and B_2 are vertex-disjoint paths with endpoints 1, 2, $y+1$ and $y+2$, such that B_1 has endpoints 1 and $y+1$ or 1 and $y+2$; and
- $|E(B_1)| + |E(B_2)| = y$, and $0 \notin V(B_1) \cup V(B_2)$;

will be denoted $P_y \to M, k'^+, y^H$.

For $y \in \{3,4,5,6\}$ and $n > y$, the graph L_n is the union of the graph L_{n-y} and the graph obtained from P_y by applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $L_{n-y} \to M, k^+, (n-y-1)^H$ and a decomposition $P_y \to M', k'^+, y^H$, then there is a decomposition $L_n \to M, M', (k+k')^+, (n-1)^H$.

Lemma 1.6.37. *If $n \geq 2$ is an integer and $M = (m_1,\ldots,m_t)$ is a list such that $\sum M = n+1$, $M \notin \{(3^i) : i \text{ is even}\}$ and $m_i \in \{3,4,5\}$ for $i = 1,\ldots,t$, then there is a decomposition $L_n \to M, (n+2)^+, (n-1)^H$.*

Proof The following decompositions are given in full detail in Table A.19 in the appendix, thus verifying the lemma for $n \in \{2,3,4\}$.

$$\begin{array}{lll} L_2 \to 3, 4^+, 1^H & L_3 \to 4, 5^+, 2^H & L_4 \to 5, 6^+, 3^H \\ P_4 \to 4, 4^+, 4^H & P_5 \to 5, 5^+, 5^H & P_6 \to 3^2, 6^+, 6^H \end{array}$$

So let $n \geq 5$ and assume by induction that the result holds for each integer n' in the range $2 \leq n' < n$. It is routine to check that for $n \geq 5$, if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $n - y \geq 2$, $X \notin \{(3^i) : i \text{ is even}\}$, and $P_y \to Y, y^+, y^H$ is one of the decompositions above. We can obtain the required decomposition $L_n \to M, (n+2)^+, (n-1)^H$ by concatenating a decomposition $L_{n-y} \to X, (n-y+2)^+, (n-y-1)^H$ (which exists by our inductive hypothesis) with a decomposition $P_y \to Y, y^+, y^H$. □

Lemma 1.6.38. *If n and k are positive integers with $\frac{4n+12}{5} \leq k \leq n+2$, and $M = (m_1,\ldots,m_t)$ is a list such that $\sum M = 2n-k+3$, $M \notin \{(3^i) : i \text{ is even}\}$ and $m_i \in \{3,4,5\}$ for $i = 1,\ldots,t$, then there is a decomposition $L_n \to M, k^+, (n-1)^H$.*

Proof The proof will be by induction on $j = n-k+2$. For a given n we need to prove the result for each integer j in the range $0 \leq j \leq \frac{n-2}{5}$. The case $j = 0$ is covered in Lemma 1.6.37, so assume $1 \le j \le \frac{n-2}{5}$ and that the result holds for each integer j' in the range $0 \le j' < j$. Note that, since $\frac{4n+12}{5} \le k$ and $j \ge 1$, we have $n \ge 7$. The following decompositions are given in Table A.19 in the appendix.

$$P_3 \to 4, 2^+, 3^H \qquad P_4 \to 5, 3^+, 4^H \qquad P_5 \to 3^2, 4^+, 5^H$$

It is routine to check that for $j \le \frac{n-2}{5}$, if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $X \notin \{(3^i) : i \text{ is even}\}$ and $P_y \to Y, (y-1)^+, y^H$ is one of the decompositions listed above. A decomposition $L_{n-y} \to X, (k-y+1)^+, (n-y-1)^H$ will exist by our inductive hypothesis provided that

$$\tfrac{4(n-y)+12}{5} \le k - y + 1 \le n - y + 2$$

and it is routine to check that this holds using $\frac{4n+12}{5} \le k$, $j \ge 1$ and $y \in \{3, 4, 5\}$. Thus, the required decomposition $L_n \to M, k^+, (n-1)^H$ can be obtained by concatenating a decomposition $L_{n-y} \to X, (k-y+1)^+, (n-y-1)^H$ with a decomposition $P_y \to Y, (y-1)^+, y^H$. □

Let $(m_1, \ldots, m_t)$ be a list of integers with $m_i \ge 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, H\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- H is an n-cycle such that $0, 1, 2 \notin V(H)$ and $\{n, n+2\} \in E(H)$;

will be denoted $J_n \to m_1, \ldots, m_t, n^H$.

In Lemma 1.6.40 we will form decompositions of graphs J_n by concatenating decompositions of graphs L_{n-y} obtained from Lemma 1.6.38 with decompositions of graphs Q_y which we will now define. For each $y \in \{4, 5, 6\}$, the graph obtained from J_y by deleting the edges $\{0, 1\}$ and $\{1, 2\}$ will be denoted by Q_y. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \ge 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A, B\}$ of Q_y such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A is a path of length k' from 0 to 1 such that $\{2, y, y+1, y+2\} \notin V(A)$; and
- B is a path of length $y+1$ from 1 to 2 such that $0 \notin V(B)$ and $\{y, y+2\} \in E(B)$;

will be denoted $Q_y \to M, k'^+, (y+1)^H$.

For $y \in \{4, 5, 6\}$ and $n > y$, the graph J_n is the union of the graph L_{n-y} and the graph obtained from Q_y by applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $L_{n-y} \to M, k^+, (n-y-1)^H$ and a decomposition $Q_y \to M', k'^+, (y+1)^H$, then there is a decomposition $J_n \to M, M', k+k', n^H$. Note that, for $y \in \{4, 5, 6\}$ and $n - y \ge 3$, no cycle of this decomposition contains both vertex i and vertex $i + n$ for $i \in \{0, 1, 2\}$.

Lemma 1.6.39. *If n and k are integers with $6 \le n \le 32$, $k \ge 6$ and $n - 5 \le k \le n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 2n - k$, $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k, n^H$ such that for $i \in \{0, 1, 2\}$ no cycle of the decomposition contains both vertex i and vertex $i + n$.*

Proof First note the existence of the following decompositions, given in full detail in Table A.19 in the appendix.

$$Q_4 \to 3, 2^+, 5^H \qquad Q_5 \to 4, 3^+, 6^H \qquad Q_6 \to 5, 4^+, 7^H$$

Let $j = n - k$, then by the conditions of the lemma we have $0 \leq j \leq 5$. Note that for any M that satisfies the conditions of the lemma, if M can be written as $M = (X, y-1)$ where $L_{n-y} \to X, (k-y+2)^+, (n-y-1)^H$ exists and $Q_y \to (y-1), (y-2)^+, (y+1)^H$ is one of the decompositions listed above, then we can construct the required decomposition by concatenating $L_{n-y} \to X, (k-y+2)^+, (n-y-1)^H$ with $Q_y \to (y-1), (y-2)^+, (y+1)^H$. We can write $L_{n-y} \to X, (k-y+2)^+, (n-y-1)^H$ as $L_{n-y} \to X, (n-j-y+2)^+, (n-y-1)^H$. For $j = 0$, such decompositions exist by Lemma 1.6.37 and for $1 \leq j \leq 5$ such decompositions exist for $n \geq 12 + 5j$ by Lemma 1.6.38. Therefore, in the following we assume $1 \leq j \leq 5$ and $n < 12 + 5i$.

In tables A.20, A.21, A.22, A.23 and A.24, we list all required decompositions for $j = 1, 2, 3, 4$ and 5 respectively. That is, each table lists all required decompositions of the form $J_n \to M, k, n$ where $6 \leq k \leq n < 12 + 5j$ for the given value of $j = n - k$. □

Lemma 1.6.40. *If n and k are integers with $n \geq 6$, $k \geq 3$ and $n - 5 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 2n - k$, $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k, n^H$ such that for $i \in \{0, 1, 2\}$ no cycle of the decomposition contains both vertex i and vertex $i + n$.*

Proof For $k \in \{3, 4, 5\}$ the result holds by Lemma 1.6.5 (by letting n in this theorem be k in Lemma 1.6.4) We therefore assume $6 \leq k \leq n$.

For $6 \leq n \leq 32$ the required result holds by Lemma 1.6.39. Given this, we may assume $n \geq 33$. The special case where $M \in \{(3^i) : i \text{ is odd}\}$ will be dealt with separately in a moment.

Case 1 Suppose that $M \notin \{(3^i) : i \text{ is odd}\}$. The following decompositions are given in full detail in Table A.19 in the appendix.

$$Q_4 \to 3, 2^+, 5^H \qquad Q_5 \to 4, 3^+, 6^H \qquad Q_6 \to 5, 4^+, 7^H$$

It is routine to check for $n \geq 33$, if M satisfies the hypotheses of the lemma (and $M \notin \{(3^i) : i \text{ is odd}\}$), then M can be written as $M = (X, y-1)$ where $X \notin \{(3^i) : i \text{ is even}\}$ and $Q_y \to (y-1), (y-2)^+, (y+1)^H$ is one of the decompositions listed in Lemma 1.6.39 above. Using $n \geq 33$ and $y \in \{4, 5, 6\}$, it can be verified that a decomposition $L_{n-y} \to X, (k-y+2)^+, (n-y-1)^H$ exists by Lemma 1.6.38. Concatenation of this decomposition with $Q_y \to (y-1), (y-2)^+, (y+1)^H$ yields the required decomposition $J_n \to M, k, n^H$.

Case 2 Suppose that $M \in \{(3^i) : i \text{ is odd}\}$. Let $p = \frac{i-3}{2} - (n-k)$. We deal separately with the case $n = k$ and the case $n \in \{k+1, k+2, k+3, k+4, k+5\}$.

Case 2a Suppose that $n = k$. Note that since $n \geq 33$ and $\sum M = 3i = 2n - k$, we have $p \geq 4$ when $n = k$. The set of 3-cycles in the decomposition is the union of the following two sets.

$$\{(0,1,3), (2,4,5), (n-3, n-2, n-1)\}$$
$$\{(6j+6, 6j+7, 6j+8), (6j+9, 6j+10, 6j+11) : j \in \{0, \ldots, p-1\}\}$$

The edge set of one n-cycle is $E_1 \cup E_2 \cup E_3$ where

$$\begin{aligned}
E_1 =\ & \{\{5,3\}, \{3,4\}, \{4,6\}\},\\
E_2 =\ & \{\{n-4, n-2\}, \{n-2, n+1\}, \{n+1, n-1\}, \{n-1, n+2\}, \{n+2, n\}, \{n, n-3\}\},\\
E_3 =\ & \{\{6j+6, 6j+9\}, \{6j+9, 6j+8\}, \{6j+8, 6j+11\},\\
& \{6j+5, 6j+7\}, \{6j+7, 6j+10\}, \{6j+10, 6j+12\} : j \in \{0, \ldots, p-1\}\}.
\end{aligned}$$

Note that this n-cycle contains the edge $\{n, n+2\}$ and does not contain any of the vertices in $\{0, 1, 2\}$. The remaining edges form the edge set of the other n-cycle (here $n = k$).

Case 2b Suppose that $n \in \{k+1, k+2, k+3, k+4, k+5\}$. Since $n \geq 33$, it is routine to verify that for any integers n and k and list $M \in \{(3^i) : i \text{ is odd}\}$ which satisfy the conditions of the lemma we have $p \geq 1$, except in the case where $M = (3^{13})$ and $n = k+5$. In this special case we have $(n, k) = (34, 29)$ and we have constructed the decomposition required in this case explicitly and it is listed in Table A.24 in the appendix. Thus we can assume $p \geq 1$. Let $l = 5(n-k)$. The set of 3-cycles in the decomposition is the union of the following three sets.

$$\{(0,1,3),(2,4,5),(n-3,n-2,n-1)\}$$
$$\{(5j+6,5j+7,5j+8),(5j+9,5j+10,5j+11) : j \in \{0,\dots,(n-k)-1\}\}$$
$$\{(6j+l+6,6j+l+7,6j+l+8),(6j+l+9,6j+l+10,6j+l+11) : j \in \{0,\dots,p-1\}\}$$

The edge set of the n-cycle is $E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$\begin{aligned}
E_1 = \;& \{\{5,3\},\{3,4\},\{4,6\}\}, \\
E_2 = \;& \{\{n-4,n-2\},\{n-2,n+1\},\{n+1,n-1\},\{n-1,n+2\},\{n+2,n\},\{n,n-3\}\}, \\
E_3 = \;& \{\{5j+6,5j+9\},\{5j+9,5j+8\},\{5j+8,5j+11\}, \\
& \{5j+5,5j+7\},\{5j+7,5j+10\} : j \in \{0,\dots,(n-k)-1\}\}, \\
E_4 = \;& \{\{6j+l+6,6j+l+9\},\{6j+l+9,6j+l+8\}, \\
& \{6j+l+8,6j+l+11\},\{6j+l+5,6j+l+7\}, \\
& \{6j+l+7,6j+l+10\},\{6j+l+10,6j+l+12\} : j \in \{0,\dots,p-1\}\}.
\end{aligned}$$

Note that this n-cycle contains the edge $\{n, n+2\}$ and does not contain any of the vertices in $\{0, 1, 2\}$. The remaining edges form the edge set of the cycle of length k. □

In Lemma 1.6.41 we will form decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs R_y which we will now define. For $y \in \{5, 6\}$, the graph obtained from J_y by adding the edge $\{0, 2\}$ will be denoted by R_y. Let $M = (m_1, \dots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \dots, t$. A decomposition $\{G_1, \dots, G_t, A\}$ of R_y such that

- G_i is an m_i-cycle for $i = 1, \dots, t$; and
- A is a path of length $y+1$ from 0 to 2 such that $1 \notin V(A)$ and $\{y, y+2\} \in E(A)$;

will be denoted $R_y \to M, y^H$.

For $y \in \{5, 6\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by removing the edge $\{n-y, n-y+2\}$ and the graph obtained from R_y by applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k, (n-y)^H$ and a decomposition $R_y \to M', y^H$, then there is a decomposition $J_n \to M, M', k, n^H$. In this construction the edge $\{n-y, n-y+2\}$ in the $(n-y)$-cycle of the decomposition of J_{n-y} is replaced with the path from the decomposition of R_y to form the n-cycle in the decomposition of J_n. Note that, for $y \in \{5, 6\}$ and $n - y \geq 3$, no cycle of the decomposition contains both vertex i and vertex $i+n$ for $i \in \{0, 1, 2\}$.

Lemma 1.6.41. *Let n and k be integers with $6 \leq k \leq n$. If $M = (m_1, \dots, m_t)$ is a list such that $\sum M = 2n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \dots, t$, then there is a decomposition $J_n \to M, k, n^H$ such that for $i \in \{0, 1, 2\}$ no cycle of the decomposition contains both vertex i and vertex $i + n$.*

Proof If $k \geq n-5$, then the result follows by Lemma 1.6.40, which means the result holds for $n \leq 11$. We can therefore assume that $k \leq n-6$, $n \geq 12$ and, by induction, that the result holds for each integer n' in the range $6 \leq n' < n$.

The following decompositions are given in full detail in Table A.19 in the appendix.

$$R_5 \to 5^2, 5^H \qquad R_6 \to 3, 4, 5, 6^H \qquad R_6 \to 4^3, 6^H \qquad R_6 \to 3^4, 6^H$$

It is routine to check that if M satisfies the conditions of the lemma, then M can be written as $M = (X, Y)$ where $R_y \to Y, y^H$ is one of the decompositions above. The required decomposition can be obtained by concatenating a decomposition $J_{n-y} \to X, k, (n-y)^H$ (which exists by our inductive hypothesis since $k \leq n-6 \leq n-y$) with a decomposition $R_y \to Y, y^H$. □

Proof of Lemma 1.3.4 If $k \in \{3, 4, 5\}$, then the result follows by Lemma 1.3.2. So we can assume $k \geq 6$. Since $n \geq 7$, we can obtain an (M)-decomposition of $\langle\{1, 2, 3\}\rangle_n$ from an (M)-decomposition of J_n by identifying vertex i with vertex $i+n$ for each $i \in \{0, 1, 2\}$, provided that for each $i \in \{0, 1, 2\}$, no cycle of our decomposition contains both vertex i and vertex $i+n$. Thus, Lemma 1.3.4 follows immediately from Lemma 1.6.41. □

1.7 Decompositions of $K_n - \langle S\rangle_n$

The purpose of this section is to prove Lemmas 1.3.6 and 1.3.7, and these proofs are given in Subsections 1.7.2 and 1.7.3 respectively. In Subsection 1.7.1 we present results on Hamilton decompositions of circulant graphs that we will require.

To prove Lemma 1.3.6, we require a $(3^{tn}, 4^{qn}, n^h)$-decomposition of $\langle\overline{S}\rangle_n$, where $\overline{S} = \{1, \ldots, \lfloor\frac{n}{2}\rfloor\} \setminus S$, for almost all n, t, q and h satisfying $h \geq 2$, $n \geq 2\max(S) + 1$ and $3t + 4q + h = \lfloor\frac{n-1}{2}\rfloor - |S|$. To construct this, $\overline{S}$ will be partitioned into three subsets S_1, S_2 and S_3 such that there is a (3^{tn})-decomposition of $\langle S_1\rangle_n$, a (4^{qn})-decomposition of $\langle S_2\rangle_n$, and an (n^h)-decomposition of $\langle S_3\rangle_n$. Our (3^{tn})-decompositions of $\langle S_1\rangle_n$ are constructed by partitioning S_1 into modulo n difference triples, our (4^{qn})-decompositions of $\langle S_2\rangle_n$ are constructed by partitioning S_2 into modulo n difference quadruples, and our (n^h)-decompositions of $\langle S_3\rangle_n$ are constructed by partitioning S_3 into sets of size at most 3 to yield connected circulant graphs of degree at most 6 that are known to have Hamilton decompositions. Lemma 1.3.7 is proved in an analogous manner.

1.7.1 Decompositions of circulant graphs into Hamilton cycles

Theorems 1.7.1–1.7.3 address the open problem of whether every connected Cayley graph on a finite abelian group has a Hamilton decomposition [4]. Note that $\langle S\rangle_n$ is connected if and only if $\gcd(S \cup \{n\}) = 1$.

Theorem 1.7.1. ([18]) *Every connected 4-regular Cayley graph on a finite abelian group has a decomposition into two Hamilton cycles.*

The following theorem is an easy corollary of Theorem 1.7.1.

Theorem 1.7.2. *Every connected 5-regular Cayley graph on a finite abelian group has a decomposition into two Hamilton cycles and a perfect matching.*

Proof Let the graph be $X = \mathrm{Cay}(\Gamma, S)$. Since each vertex of X has odd degree, S contains an element s of order 2 in Γ. Let F be the perfect matching of X generated by s. If $\mathrm{Cay}(\Gamma, S \setminus \{s\})$ is connected then, as it is also 4-regular, the result follows immediately from Theorem 1.7.1. On the other hand, if $\mathrm{Cay}(\Gamma, S \setminus \{s\})$ is not connected, then it consists of two isomorphic connected components, with $x \mapsto sx$ being an isomorphism. These components are 4-regular and so by Theorem 1.7.1, each can be decomposed into two Hamilton cycles. Moreover, since $x \mapsto sx$ is an isomorphism, there exists a Hamilton decomposition $\{H_1, H_1'\}$ of the first and a Hamilton decomposition $\{H_2, H_2'\}$ of the second such that there is a pair of vertex-disjoint 4-cycles (x_1, y_1, y_2, x_2) and (x_1', y_1', y_2', x_2') in X with $x_1y_1 \in E(H_1)$, $x_1'y_1' \in E(H_1')$, $x_2y_2 \in E(H_2)$, $x_2'y_2' \in E(H_2')$, and $x_1x_2, y_1y_2, x_1'x_2', y_1'y_2' \in E(F)$. It follows that if we let G be the graph with edge set

$$(E(H_1) \cup E(H_2) \cup \{x_1x_2, y_1y_2\}) \setminus \{x_1y_1, x_2y_2\}$$

and let G' be the graph with edge set

$$(E(H_1') \cup E(H_2') \cup \{x_1'x_2', y_1'y_2'\}) \setminus \{x_1'y_1', x_2'y_2'\},$$

then G and G' are edge-disjoint Hamilton cycles in X. This proves the result. $\square$

Theorem 1.7.3. ([54]) *Every 6-regular Cayley graph on a group which is generated by an element of the connection set has a decomposition into three Hamilton cycles.*

This theorem implies that, for distinct $a, b, c \in \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$, the graph $\langle \{a, b, c\} \rangle_n$ has a decomposition into three Hamilton cycles if $\gcd(x, n) = 1$ for some $x \in \{a, b, c\}$.

In the next two lemmas we give results similar to that of Lemma 1.3.5, but for the case where the connection set is of the form $\{x-1\} \cup \{x+1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ rather than $\{x, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Lemma 1.7.4 deals with the case n is odd, and Lemma 1.7.5 deals with the case n is even.

Lemma 1.7.4. *If n is odd and $1 \leq h \leq \frac{n-3}{2}$, then there is an (n^h)-decomposition of $\langle \{\frac{n-1}{2} - h\} \cup \{\frac{n-1}{2} - h + 2, \ldots, \frac{n-1}{2}\} \rangle_n$; except when $h = 1$ and $n \equiv 3 \pmod 6$ in which case the graph is not connected.*

Proof If $h = 1$, then the graph is $\langle \frac{n-3}{2} \rangle_n$. If $n \equiv 1, 5 \pmod 6$, then $\gcd(\frac{n-3}{2}, n) = 1$ and $\langle \frac{n-3}{2} \rangle_n$ is an n-cycle. If $n \equiv 3 \pmod 6$, then $\gcd(\frac{n-3}{2}, n) = 3$ and $\langle \frac{n-3}{2} \rangle_n$ is not connected. Thus the result holds for $h = 1$. In the remainder of the proof we assume $h \geq 2$.

We first decompose $\langle \{\frac{n-1}{2} - h\} \cup \{\frac{n-1}{2} - h + 2, \ldots, \frac{n-1}{2}\} \rangle_n$ into circulant graphs by partitioning the connection set, and then decompose the resulting circulant graphs into n-cycles using Theorems 1.7.1 and 1.7.3.

If h is even, then we partition the connection set into pairs by pairing $\frac{n-1}{2} - h$ with $\frac{n-1}{2}$ and partitioning $\{\frac{n-1}{2} - h + 2, \ldots, \frac{n-3}{2}\}$ into consecutive pairs (if $h = 2$, then our partition is just $\{\{\frac{n-5}{2}, \frac{n-1}{2}\}\}$). Each of the resulting circulant graphs is 4-regular and connected and thus can be decomposed into two n-cycles by Theorem 1.7.1. If h is odd, then we partition the connection set into the triple $\{\frac{n-1}{2} - h, \frac{n-1}{2} - h + 2, \frac{n-1}{2}\}$ and consecutive pairs from $\{\frac{n-1}{2} - h + 3, \ldots, \frac{n-3}{2}\}$ (if $h = 3$, then our partition is just $\{\{\frac{n-7}{2}, \frac{n-3}{2}, \frac{n-1}{2}\}\}$). Since $\gcd(\frac{n-1}{2}, n) = 1$, the graph $\langle \{\frac{n-1}{2} - h, \frac{n-1}{2} - h + 2, \frac{n-1}{2}\} \rangle_n$ can be decomposed into three n-cycles by Theorem 1.7.3. Any other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 1.7.1. $\square$

Lemma 1.7.5. *If n is even and $1 \leq h \leq \frac{n-4}{2}$, then there is an (n^h)-decomposition of $\langle \{\frac{n}{2} - h - 1\} \cup \{\frac{n}{2} - h + 1, \ldots, \frac{n}{2}\} \rangle_n$; except when $h = 1$ and $n \equiv 0 \pmod 4$ in which case the graph is not connected.*

Proof If $h=1$, then the graph is $\langle\{\frac{n-4}{2},\frac{n}{2}\}\rangle_n$. If $n\equiv 2 \pmod 4$, then $\gcd(\frac{n-4}{2},n)=1$, $\langle\frac{n-4}{2}\rangle_n$ is an n-cycle, and $\langle\{\frac{n}{2}\}\rangle_n$ is a perfect matching. If $n\equiv 0 \pmod 4$, then $\gcd(\frac{n-4}{2},\frac{n}{2},n)=2$ and $\langle\{\frac{n-4}{2},\frac{n}{2}\}\rangle_n$ is not connected. Thus the result holds for $h=1$. In the remainder of the proof we assume $h\geq 2$.

We first decompose $\langle\{\frac{n}{2}-h-1\}\cup\{\frac{n}{2}-h+1,\ldots,\frac{n}{2}\}\rangle_n$ into circulant graphs by partitioning the connection set, and then decompose the resulting circulant graphs into n-cycles using Theorems 1.7.1, 1.7.2 and 1.7.3.

If $n\equiv 0 \pmod 4$ and h is even, then we partition the connection set into pairs and the singleton $\{\frac{n}{2}\}$ by pairing $\frac{n}{2}-h-1$ with $\frac{n-2}{2}$ and partitioning $\{\frac{n}{2}-h+1,\ldots,\frac{n-4}{2}\}$ into pairs of consecutive integers (if $h=2$, then our partition is just $\{\{\frac{n}{2}\},\{\frac{n-6}{2},\frac{n-2}{2}\}\}$). The graph $\langle\{\frac{n}{2}\}\rangle_n$ is a perfect matching. The other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 1.7.1 (note that $\gcd(\frac{n-2}{2},n)=1$).

If $n\equiv 0 \pmod 4$ and h is odd, then we partition the connection set into pairs, the triple $\{\frac{n}{2}-h-1,\frac{n}{2}-h+1,\frac{n-2}{2}\}$ and the singleton $\{\frac{n}{2}\}$ by partitioning $\{\frac{n}{2}-h+2,\ldots,\frac{n-4}{2}\}$ into pairs of consecutive integers (if $h=3$, then our partition is just $\{\{\frac{n}{2}\},\{\frac{n-8}{2},\frac{n-4}{2},\frac{n-2}{2}\}\}$). The graph $\langle\{\frac{n}{2}\}\rangle_n$ is a perfect matching and, since $\gcd(\frac{n-2}{2},n)=1$, the graph $\langle\{\frac{n}{2}-h-1,\frac{n}{2}-h+1,\frac{n-2}{2}\}\rangle_n$ can be decomposed into three n-cycles using Theorem 1.7.3. Any other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 1.7.1.

If $n\equiv 2 \pmod 4$, then we partition the connection set into pairs, the triple $\{\frac{n}{2}-h-1,\frac{n-2}{2},\frac{n}{2}\}$ and, when h is odd, the singleton $\{\frac{n-4}{2}\}$ by partitioning $\{\frac{n}{2}-h+1,\ldots,\frac{n-4}{2}\}$ into pairs of consecutive integers (when h is even) or into pairs of consecutive integers and the singleton $\{\frac{n-4}{2}\}$ (when h is odd). (Our partition is just $\{\{\frac{n-6}{2},\frac{n-2}{2},\frac{n}{2}\}\}$ if $h=2$, and just $\{\{\frac{n-8}{2},\frac{n-2}{2},\frac{n}{2}\},\{\frac{n-4}{2}\}\}$ if $h=3$.) Since $\gcd(\frac{n-2}{2},\frac{n}{2})=1$, the graph $\langle\{\frac{n}{2}-h-1,\frac{n-2}{2},\frac{n}{2}\}\rangle_n$ can be decomposed into two n-cycles and a perfect matching using Theorem 1.7.2. When h is odd, $\langle\{\frac{n-4}{2}\}\rangle_n$ is an n-cycle (note that $\gcd(\frac{n-4}{2},n)=1$). Any other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 1.7.1. □

1.7.2 Proof of Lemma 1.3.6

Before we prove Lemma 1.3.6, we require three preliminary lemmas which establish the existence of various $(4^{qn},n^h)$-decompositions of circulant graphs.

Lemma 1.7.6. *If* $S\subseteq\{1,\ldots,\lfloor\frac{n-1}{2}\rfloor\}$ *such that*

- $S=\{x+1,\ldots,x+4q\}$ *for some* x*;*
- $S=\{x\}\cup\{x+2,\ldots,x+4q-1\}\cup\{x+4q+1\}$ *for some* x*; or*
- $S=\{\frac{n-1}{2}-4q\}\cup\{\frac{n-1}{2}-4q+2,\ldots,\frac{n-1}{2}\}$ *where* n *is odd;*

then there is a (4^{qn})*-decomposition of* $\langle S\rangle_n$*.*

Proof It is sufficient to partition S into q modulo n difference quadruples. If $S=\{x+1,\ldots,x+4q\}$, then we partition S into q sets of the form $\{y,y+1,y+2,y+3\}$, each of which is a difference quadruple. If $S=\{x\}\cup\{x+2,\ldots,x+4q-1\}\cup\{x+4q+1\}$, then we partition

S into q sets of the form $\{y, y+2, y+3, y+5\}$, each of which is a difference quadruple. If $S = \{\frac{n-1}{2} - 4q\} \cup \{\frac{n-1}{2} - 4q + 2, \ldots, \frac{n-1}{2}\}$ and n is odd, then we partition S into $q-1$ sets of the form $\{y, y+2, y+3, y+5\}$, each of which is a difference quadruple, and the set $\{\frac{n-9}{2}, \frac{n-5}{2}, \frac{n-3}{2}, \frac{n-1}{2}\}$, which is a modulo n difference quadruple (note that $\frac{n-5}{2} + \frac{n-3}{2} + \frac{n-1}{2} - \frac{n-9}{2} = n$). $\square$

Lemma 1.7.7. *If h, q and n are non-negative integers with $1 \leq 4q + h \leq \lfloor \frac{n-1}{2} \rfloor$, then there is a $(4^{qn}, n^h)$-decomposition of $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 1, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$.*

Proof If $h = 0$ then the result follows immediately by Lemma 1.7.6, and if $q = 0$ then the result follows immediately by Lemma 1.3.5. For $q, h \geq 1$ we partition the connection set into the set $\{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - h\}$ and the set $\{\lfloor \frac{n-1}{2} \rfloor - h + 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - h\} \rangle_n$ has a (4^{qn})-decomposition by Lemma 1.7.6, and $\langle \{\lfloor \frac{n-1}{2} \rfloor - h + 1, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ has an (n^h)-decomposition by Lemma 1.3.5. $\square$

Lemma 1.7.8. *If h, q and n are non-negative integers with $1 \leq 4q + h \leq \lfloor \frac{n-3}{2} \rfloor$ such that n is odd when $h = 0$ and $n \equiv 1, 2, 5, 6, 7, 10, 11 \pmod{12}$ when $h = 1$, then there is a $(4^{qn}, n^h)$-decomposition of $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$.*

Proof If $h = 0$, then the result follows immediately by Lemma 1.7.6. If $q = 0$, then the result follows immediately by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). For $h, q \geq 1$ we partition the connection set into the set $\{\lfloor \frac{n-1}{2} \rfloor - h - 4q\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - h - 1\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 1\}$ and the set $\{\lfloor \frac{n-1}{2} \rfloor - h\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - h - 1\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 1\} \rangle_n$ has a (4^{qn})-decomposition by Lemma 1.7.6, and $\langle \{\lfloor \frac{n-1}{2} \rfloor - h\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ has an (n^h)-decomposition by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). $\square$

We now prove Lemma 1.3.6, which we restate here for convenience.

Lemma 1.3.6 *If $S \in \{\{1, 2, 3, 4\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$ and $n \geq 2\max(S) + 1$, $t \geq 0$, $q \geq 0$ and $h \geq 2$ are integers satisfying $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - |S|$, then there is a $(3^{tn}, 4^{qn}, n^h)$-decomposition of $K_n - \langle S \rangle_n$, except possibly when $h = 2$, $S = \{1, 2, 3, 4, 5, 6, 7\}$ and*

- *$n \in \{25, 26\}$ and $t = 1$; or*
- *$n = 31$ and $t = 2$.*

Proof We give the proof of Lemma 1.3.6 for each

$$S \in \{\{1, 2, 3, 4\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$$

separately.

Case A: ($S = \{1, 2, 3, 4\}$) The conditions $h \geq 2$ and $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 4$ imply $n \geq 6t + 13$. If $t = 0$, then the result follows immediately by Lemma 1.7.8. We deal separately with the three cases $t \in \{1, 2, 3, 4\}$, $t \in \{5, 6, 7, 8\}$, and $t \geq 9$.

Case A1: Suppose that $t \in \{1, 2, 3, 4\}$. The cases $6t + 13 \leq n \leq 6t + 17$ and the cases

$$(n, t) \in \{(30, 2), (32, 2), (36, 3)\}$$

are dealt with first. Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 5$ that in each of these cases we have $q = 0$. Thus, the value of h is uniquely determined by the values of n and t. The required decompositions are obtained by partitioning $\{5, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Suitable partitions are given in the following tables.

t=1:

n	modulo n difference triples	connection sets
19	$\{5, 6, 8\}$	$\{7, 9\}$
20	$\{5, 6, 9\}$	$\{7, 8\}$
21	$\{5, 7, 9\}$	$\{6, 8\}, \{10\}$
22	$\{5, 7, 10\}$	$\{6, 8, 11\}, \{9\}$

t=2:

n	modulo n difference triples	connection sets
25	$\{5, 9, 11\}, \{7, 8, 10\}$	$\{6, 12\}$
26	$\{5, 7, 12\}, \{6, 9, 11\}$	$\{8, 10, 13\}$
27	$\{5, 6, 11\}, \{8, 9, 10\}$	$\{7\}, \{12, 13\}$
28	$\{5, 8, 13\}, \{6, 10, 12\}$	$\{7, 9\}, \{11\}$
30	$\{5, 7, 12\}, \{6, 8, 14\}$	$\{9, 10\}, \{11, 13\}$
32	$\{5, 7, 12\}, \{6, 8, 14\}$	$\{9, 10\}, \{11, 13\}, \{15\}$

t=3:

n	modulo n difference triples	connection sets
31	$\{6, 8, 14\}, \{7, 11, 13\}, \{9, 10, 12\}$	$\{5, 15\}$
32	$\{5, 7, 12\}, \{6, 8, 14\}, \{9, 10, 13\}$	$\{11, 15\}$
33	$\{5, 7, 12\}, \{6, 8, 14\}, \{9, 11, 13\}$	$\{10\}, \{15, 16\}$
34	$\{5, 7, 12\}, \{6, 8, 14\}, \{10, 11, 13\}$	$\{9\}, \{15, 16\}$
36	$\{5, 7, 12\}, \{6, 8, 14\}, \{10, 11, 15\}$	$\{9, 13\}, \{16, 17\}$

t=4:

n	modulo n difference triples	connection sets
37	$\{5, 10, 15\}, \{6, 7, 13\}, \{8, 9, 17\}, \{11, 12, 14\}$	$\{16, 18\}$
38	$\{5, 9, 14\}, \{6, 7, 13\}, \{8, 10, 18\}, \{11, 12, 15\}$	$\{16, 17\}$
39	$\{5, 10, 15\}, \{6, 7, 13\}, \{8, 9, 17\}, \{11, 12, 16\}$	$\{14\}, \{18, 19\}$
40	$\{5, 9, 14\}, \{6, 7, 13\}, \{8, 10, 18\}, \{11, 12, 17\}$	$\{15, 16\}, \{19\}$

We now deal with $n \geq 6t + 18$ which implies $4q + h \geq 4$. Define S_t by $S_t = \{5, \ldots, 3t + 8\}$ for $t \in \{1, 4\}$, and $S_t = \{5, \ldots, 3t + 7\} \cup \{3t + 9\}$ when $t \in \{2, 3\}$. The following table gives a partition π_t of S_t into difference triples and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
1	$\{\{5,6,11\},\{7,8,9,10\}\}$
2	$\{\{5,6,11\},\{7,8,15\},\{9,10,12,13\}\}$
3	$\{\{5,6,11\},\{7,9,16\},\{8,10,18\},\{12,13,14,15\}\}$
4	$\{\{5,10,15\},\{6,11,17\},\{7,9,16\},\{8,12,20\},\{13,14,18,19\}\}$

Thus, $\langle Q_t\rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is both a $(3^{tn},4^n)$-decomposition and a $(3^{tn},n^4)$-decomposition of $\langle S_t\rangle_n$. If $q=0$, then we use the $(3^{tn},n^4)$-decomposition of $\langle S_t\rangle_n$ and if $q\geq 1$, then we use the $(3^{tn},4^n)$-decomposition of $\langle S_t\rangle_n$. This leaves us needing an (n^{h-4})-decomposition of $K_n-\langle\{1,2,3,4\}\cup S_t\rangle_n$ when $q=0$, and a $(4^{(q-1)n},n^h)$-decomposition of $K_n-\langle\{1,2,3,4\}\cup S_t\rangle_n$ when $q\geq 1$. Note that $K_n-\langle\{1,2,3,4\}\cup S_t\rangle_n$ is isomorphic to

- $\langle\{3t+9,\dots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t\in\{1,4\}$; and
- $\langle\{3t+8\}\cup\{3t+10,\dots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t\in\{2,3\}$.

When $t\in\{1,4\}$ the required decomposition exists by Lemma 1.7.7. When $t\in\{2,3\}$ and the required number of Hamilton cycles (that is, $h-4$ when $q=0$ and h when $q\geq 1$) is at least 2, the required decomposition exists by Lemma 1.7.8. So we need to consider only the cases where $q=0$, $h\in\{4,5\}$ and $t\in\{2,3\}$.

Since $3t+4q+h=\lfloor\frac{n-1}{2}\rfloor-4$, and since we have already dealt with the cases where $(n,t)\in\{(30,2),(32,2),(36,3)$, this leaves us with only the five cases where

$$(n,t,h)\in\{(29,2,4),(31,2,5),(35,3,4),(37,3,5),(38,3,5)\}.$$

In the cases $(n,t,h)\in\{(29,2,4),(35,3,4)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 0 and n is odd, and in the cases $(n,t,h)\in\{(31,2,5),(37,3,5),(38,3,5)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 1 and $n\equiv 1,2,7\pmod{12}$. So in all these cases the required decompositions exist by Lemma 1.7.8.

Case A2: Suppose that $t\in\{5,6,7,8\}$. Redefine S_t by $S_t=\{5,\dots,3t+6\}$. The following table gives a partition of S_t into difference triples and a set R_t consisting of a pair of relatively prime integers. Thus, $\langle R_t\rangle_n$ is a connected 4-regular Cayley graph, and so can be decomposed into two Hamilton cycles using Theorem 1.7.1. Thus, we have a $(3^{tn},n^2)$-decomposition of $\langle S_t\rangle_n$.

t	difference triples	R_t
5	$\{5,12,17\},\{6,13,19\},\{7,14,21\},\{8,10,18\},\{9,11,20\}$ $\{12,13,25\}$	$\{15,16\}$
6	$\{5,13,18\},\{6,14,20\},\{7,15,22\},\{8,16,24\},\{9,10,19\},\{11,12,23\}$	$\{17,21\}$
7	$\{5,15,20\},\{6,16,22\},\{7,17,24\},\{8,18,26\},\{9,12,21\},\{10,13,23\},$ $\{11,14,25\}$	$\{19,27\}$
8	$\{5,17,22\},\{6,18,24\},\{7,19,26\},\{8,13,21\},\{9,16,25\},\{10,20,30\},$ $\{11,12,23\},\{14,15,29\}$	$\{27,28\}$

Thus, we only require a $(4^{qn}, n^{h-2})$-decomposition of $K_n - \langle\{1,2,3,4\} \cup S_t\rangle_n$. But $K_n - \langle\{1,2,3,4\} \cup S_t\rangle_n$ is isomorphic to $\langle\{3t+7, \ldots, \lfloor\frac{n}{2}\rfloor\}\rangle_n$ and so this decomposition exists by Lemma 1.7.7.

Case A3: Suppose that $t \geq 9$. Redefine S_t by $S_t = \{5, \ldots, 3t+4\}$ when $t \equiv 0, 1 \pmod 4$, and $S_t = \{5, \ldots, 3t+3\} \cup \{3t+5\}$ when $t \equiv 2, 3 \pmod 4$. We now obtain a (3^{tn})-decomposition of $\langle S_t\rangle_n$.

For $t \equiv 0, 1 \pmod 4$ (respectively $t \equiv 2, 3 \pmod 4$), we can obtain a (3^{tn})-decomposition of $\langle S_t\rangle_n$ by using a Langford sequence (respectively hooked Langford sequence) of order t and defect 5, which exists since $t \geq 9$, to partition S_t into difference triples (see [90, 91]). So we have a (3^{tn})-decomposition of $\langle S_t\rangle_n$, and require a $(4^{qn}, n^h)$-decomposition of $K_n - \langle\{1,2,3,4,6\} \cup S_t\rangle_n$. Since $K_n - \langle\{1,2,3,4\} \cup S_t\rangle_n$ is isomorphic to

- $\langle\{3t+5, \ldots, \lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \equiv 0, 1 \pmod 4$; and
- $\langle\{3t+4\} \cup \{3t+6, \ldots, \lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \equiv 2, 3 \pmod 4$;

this decomposition exists by Lemma 1.7.7 or 1.7.8. □

Case B: ($S = \{1,2,3,4,6\}$) The conditions $h \geq 2$ and $3t + 4q + h = \lfloor\frac{n-1}{2}\rfloor - 5$ imply $n \geq 6t + 15$. If $t = 0$, then the result follows immediately by Lemma 1.7.8. We deal separately with the three cases $t \in \{1,2,3,4,5,6\}$, $t \in \{7,8,9,10\}$, and $t \geq 11$.

Case B1: Suppose that $t \in \{1,2,3,4,5,6\}$. The cases $6t + 15 \leq n \leq 6t + 18$ and the cases

$$(n, t) \in \{(38, 3), (39, 3), (40, 3), (44, 4), (45, 4)\}$$

are dealt with first. Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor\frac{n-1}{2}\rfloor - 5$ that in each of these cases we have $q = 0$. Thus, the value of h is uniquely determined by the values of n and t. The required decompositions are obtained by partitioning $\{5\} \cup \{7, \ldots, \lfloor\frac{n}{2}\rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Suitable partitions are given in the following tables.

t=1:

n	modulo n difference triples	connection sets
21	$\{5,7,9\}$	$\{8,10\}$
22	$\{5,7,10\}$	$\{8,9\}, \{11\}$
23	$\{5,8,10\}$	$\{7,9\}, \{11\}$
24	$\{5,9,10\}$	$\{7,8\}, \{11\}, \{12\}$

t=2:

n	modulo n difference triples	connection sets
27	$\{5,7,12\}, \{8,9,10\}$	$\{11,13\}$
28	$\{5,7,12\}, \{8,9,11\}$	$\{10,13\}, \{14\}$
29	$\{5,7,12\}, \{8,10,11\}$	$\{9\}, \{13,14\}$
30	$\{5,9,14\}, \{8,10,12\}$	$\{7\}, \{11,13\}, \{15\}$

t=3:

n	modulo n difference triples	connection sets
33	$\{5,8,13\},\{7,9,16\},\{10,11,12\}$	$\{14,15\}$
34	$\{5,10,15\},\{7,9,16\},\{8,12,14\}$	$\{11,13\},\{17\}$
35	$\{5,8,13\},\{7,9,16\},\{10,11,14\}$	$\{12,15\},\{17\}$
36	$\{5,8,13\},\{7,9,16\},\{10,12,14\}$	$\{11,15\},\{17\},\{18\}$
38	$\{5,12,17\},\{7,9,16\},\{8,10,18\}$	$\{11,13\},\{14,15\},\{19\}$
39	$\{5,12,17\},\{7,9,16\},\{8,10,18\}$	$\{11,13\},\{14,15\},\{19\}$
40	$\{5,12,17\},\{7,9,16\},\{8,10,18\}$	$\{11,13\},\{14,15\},\{19\},\{20\}$

t=4:

n	modulo n difference triples	connection sets
39	$\{5,11,16\},\{7,8,15\},\{9,10,19\},\{12,13,14\}$	$\{17,18\}$
40	$\{5,8,13\},\{7,9,16\},\{10,12,18\},\{11,14,15\}$	$\{17,19\},\{20\}$
41	$\{5,14,19\},\{7,8,15\},\{9,11,20\},\{12,13,16\}$	$\{10,17\},\{18\}$
42	$\{5,10,15\},\{7,11,18\},\{8,9,17\},\{12,14,16\}$	$\{13\},\{19,20\},\{21\}$
44	$\{5,11,16\},\{7,13,20\},\{8,10,18\},\{9,12,21\}$	$\{14,15\},\{17,19\},\{22\}$
45	$\{5,11,16\},\{7,13,20\},\{8,10,18\},\{9,12,21\}$	$\{14,15\},\{17,19\},\{22\}$

t=5:

n	modulo n difference triples	connection sets
45	$\{5,17,22\},\{7,13,20\},\{8,10,18\},\{9,12,21\},\{14,15,16\}$	$\{11,19\}$
46	$\{5,17,22\},\{7,13,20\},\{8,10,18\},\{9,12,21\},\{11,16,19\}$	$\{14,15\},\{23\}$
47	$\{5,18,23\},\{7,13,20\},\{8,11,19\},\{10,12,22\},\{14,16,17\}$	$\{9,15\},\{21\}$
48	$\{5,17,22\},\{7,13,20\},\{8,10,18\},\{9,12,21\},\{11,14,23\}$	$\{15,16\},\{19\},\{24\}$

t=6:

n	modulo n difference triples	connection sets
51	$\{5,13,18\},\{7,15,22\},\{8,16,24\},$ $\{9,10,19\},\{11,12,23\},\{14,17,20\}$	$\{21,25\}$
52	$\{5,13,18\},\{7,15,22\},\{8,16,24\},$ $\{9,10,19\},\{11,12,23\},\{14,17,21\}$	$\{20,25\},\{26\}$
53	$\{5,13,18\},\{7,12,19\},\{8,14,22\},$ $\{9,15,24\},\{10,11,21\},\{16,17,20\}$	$\{23\},\{25,26\}$
54	$\{5,17,22\},\{7,8,15\},\{9,10,19\},$ $\{11,13,24\},\{12,14,26\},\{16,18,20\}$	$\{21,23\},\{25\},\{27\}$

We now deal with $n \geq 6t+19$ which implies $4q+h \geq 4$. Define S_t by $S_t = \{5\} \cup \{7,\dots,3t+9\}$ for $t \in \{1,2,5,6\}$, and $S_t = \{5\} \cup \{7,\dots,3t+8\} \cup \{3t+10\}$ when $t \in \{3,4\}$. The following table gives a partition π_t of S_t into difference triples and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
1	$\{\{5,7,12\},\{8,9,10,11\}\}$
2	$\{\{5,9,14\},\{7,8,15\},\{10,11,12,13\}\}$
3	$\{\{5,9,14\},\{7,10,17\},\{8,11,19\},\{12,13,15,16\}\}$
4	$\{\{5,9,14\},\{7,13,20\},\{8,11,19\},\{10,12,22\},\{15,16,17,18\}\}$
5	$\{\{5,14,19\},\{7,13,20\},\{8,10,18\},\{9,15,24\},\{11,12,23\},\{16,17,21,22\}\}$
6	$\{\{5,15,20\},\{7,16,23\},\{8,14,22\},\{9,12,21\},\{10,17,27\},\{11,13,24\},$ $\{18,19,25,26\}\}$

Thus, $\langle Q_t\rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is both a $(3^{tn},4^n)$-decomposition and a $(3^{tn},n^4)$-decomposition of $\langle S_t\rangle_n$. If $q=0$, then we use the $(3^{tn},n^4)$-decomposition of $\langle S_t\rangle_n$ and if $q\geq 1$, then we use the $(3^{tn},4^n)$-decomposition of $\langle S_t\rangle_n$. This leaves us needing an (n^{h-4})-decomposition of $K_n-\langle\{1,2,3,4,6\}\cup S_t\rangle_n$ when $q=0$, and a $(4^{(q-1)n},n^h)$-decomposition of $K_n-\langle\{1,2,3,4,6\}\cup S_t\rangle_n$ when $q\geq 1$. Note that $K_n-\langle\{1,2,3,4,6\}\cup S_t\rangle_n$ is isomorphic to

- $\langle\{3t+10,\dots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t\in\{1,2,5,6\}$; and
- $\langle\{3t+9\}\cup\{3t+11,\dots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t\in\{3,4\}$.

When $t\in\{1,2,5,6\}$ the required decomposition exists by Lemma 1.7.7. When $t\in\{3,4\}$ and the required number of Hamilton cycles (that is, $h-4$ when $q=0$ and h when $q\geq 1$) is at least 2, the required decomposition exists by Lemma 1.7.8. So we need to consider only the cases where $q=0$, $h\in\{4,5\}$ and $t\in\{3,4\}$.

Since $3t+4q+h=\lfloor\frac{n-1}{2}\rfloor-5$, and since we have already dealt with the cases where $(n,t)\in\{(38,3),(39,3),(40,3),(44,4),(45,4)\}$, this leaves us with only the three cases where $(n,t,h)\in\{(37,3,4),(43,4,4),(46,4,5)\}$. In the cases $(n,t,h)\in\{(37,3,4),(43,4,4)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 0 and n is odd, and in the case $(n,t,h)=(46,4,5)$ we have that $h-4$ (the required number of Hamilton cycles) is 1 and $n\equiv 10 \pmod{12}$. So in all these cases the required decompositions exist by Lemma 1.7.8.

Case B2: Suppose that $t\in\{7,8,9,10\}$. Redefine S_t by $S_t=\{5\}\cup\{7,\dots,3t+7\}$. The following table gives a partition of S_t into difference triples and a set R_t consisting of a pair of relatively prime integers. Thus, $\langle R_t\rangle_n$ is a connected 4-regular Cayley graph, and so can be decomposed into two Hamilton cycles using Theorem 1.7.1. Thus, we have a $(3^{tn},n^2)$-decomposition of $\langle S_t\rangle_n$.

t	difference triples	R_t
7	$\{5,17,22\},\{7,16,23\},\{8,20,28\},\{9,18,27\},\{10,14,24\},\{11,15,26\},$ $\{12,13,25\}$	$\{19,21\}$
8	$\{5,17,22\},\{7,19,26\},\{8,16,24\},\{9,20,29\},\{10,21,31\},\{11,14,25\},$ $\{12,18,30\},\{13,15,28\}$	$\{23,27\}$
9	$\{5,19,24\},\{7,20,27\},\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,15,26\},$ $\{12,16,28\},\{13,17,30\},\{14,18,32\}$	$\{25,34\}$
10	$\{5,21,26\},\{7,22,29\},\{8,23,31\},\{9,24,33\},\{10,25,35\},\{11,17,28\},$ $\{12,18,30\},\{13,14,27\},\{15,19,34\},\{16,20,36\}$	$\{32,37\}$

Thus, we only require a $(4^{qn}, n^{h-2})$-decomposition of $K_n - \langle\{1,2,3,4,6\} \cup S_t\rangle_n$. But $K_n - \langle\{1,2,3,4,6\} \cup S_t\rangle_n$ is isomorphic to $\langle\{3t+8,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ and so this decomposition exists by Lemma 1.7.7.

Case B3: Suppose that $t \geq 11$. Redefine S_t by $S_t = \{5\} \cup \{7,\ldots,3t+5\}$ when $t \equiv 1,2 \pmod 4$, and $S_t = \{5\} \cup \{7,\ldots,3t+4\} \cup \{3t+6\}$ when $t \equiv 0,3 \pmod 4$. We now obtain a (3^{tn})-decomposition of $\langle S_t\rangle_n$. For $11 \leq t \leq 52$, we have found such a decomposition by partitioning S_t into difference triples with the aid of a computer. These difference triples are shown in Table A.25 in the appendix. For $t \geq 53$ and $t \equiv 1,2 \pmod 4$ (respectively $t \equiv 0,3 \pmod 4$) we set aside as one difference triple $\{5, 3t, 3t+5\}$ (respectively $\{5, 3t+1, 3t+6\}$) and form the set $S'_t = \{7,\ldots,3t-1,3t+1,\ldots,3t+4\} = \{7,\ldots,3t+4\} \setminus \{3t\}$ (respectively $S'_t = \{7,\ldots,3t,3t+2,\ldots,3t+4\} = \{7,\ldots,3t+4\} \setminus \{3t+1\}$). We can obtain a $(3^{(t-1)n})$-decomposition of $\langle S'_t\rangle_n$ by using an extended Langford sequence of order $t-1$ and defect 7 to partition S'_t into difference triples. Since $t \geq 53$, this sequence exists by Theorem 7.1 in [72] (also see [90, 91]). So we have a (3^{tn})-decomposition of $\langle S_t\rangle_n$, and require a $(4^{qn}, n^h)$-decomposition of $K_n - \langle\{1,2,3,4,6\} \cup S_t\rangle_n$. Since $K_n - \langle\{1,2,3,4,6\} \cup S_t\rangle_n$ is isomorphic to

- $\langle\{3t+6,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \equiv 1,2 \pmod 4$; and
- $\langle\{3t+5\} \cup \{3t+7,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \equiv 0,3 \pmod 4$;

this decomposition exists by Lemma 1.7.7 or 1.7.8. □

Case C: ($S = \{1,2,3,4,5,7\}$) The conditions $h \geq 2$ and $3t + 4q + h = \lfloor\frac{n-1}{2}\rfloor - 5$ imply $n \geq 6t + 17$. If $t = 0$, then the result follows immediately by Lemma 1.7.8. We deal separately with the four cases $t \in \{1,5,6,7,8\}$, $t \in \{2,3,4\}$, $t \in \{9,10,11,12\}$, and $t \geq 11$.

Case C1: Suppose that $t \in \{1,5,6,7,8\}$. The cases $6t+17 \leq n \leq 6t+20$ and the cases

$$(n,t) \in \{(28,1),(52,5),(70,8),(72,8)\}$$

are dealt with first. Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor\frac{n-1}{2}\rfloor - 6$ that in each of these cases we have $q = 0$. Thus, the value of h is uniquely determined by the values of n and t. The required decompositions are obtained by partitioning $\{6\} \cup \{8,\ldots,\lfloor\frac{n}{2}\rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Suitable partitions are given in the following tables.

t=1:

n	modulo n difference triples	connection sets
23	$\{6,8,9\}$	$\{10,11\}$
24	$\{6,8,10\}$	$\{9,11\}$
25	$\{6,8,11\}$	$\{9,10\},\{12\}$
26	$\{6,8,12\}$	$\{9,10\},\{11\}$
27	$\{6,8,13\}$	$\{9,10\},\{11,12\}$
28	$\{6,10,12\}$	$\{8,9\},\{11,13\}$

t=5:

n	modulo n difference triples	connection sets
47	$\{6,13,19\},\{8,15,23\},\{9,11,20\},\{10,12,22\},\{14,16,17\}$	$\{18,21\}$
48	$\{6,13,19\},\{8,15,23\},\{9,11,20\},\{10,12,22\},\{14,16,18\}$	$\{17,21\}$
49	$\{6,13,19\},\{8,15,23\},\{9,11,20\},\{10,12,22\},\{14,17,18\}$	$\{16,21\},\{24\}$
50	$\{6,13,19\},\{8,16,24\},\{9,11,20\},\{10,12,22\},\{15,17,18\}$	$\{14,21\},\{23\}$
52	$\{6,13,19\},\{8,16,24\},\{9,11,20\},\{10,12,22\},\{14,17,21\}$	$\{15,18\},\{23,25\}$

t=6:

n	modulo n difference triples	connection sets
53	$\{6,15,21\},\{8,17,25\},\{9,13,22\},$ $\{10,14,24\},\{11,12,23\},\{16,18,19\}$	$\{20,26\}$
54	$\{6,15,21\},\{8,17,25\},\{9,13,22\},$ $\{10,14,24\},\{11,12,23\},\{16,18,20\}$	$\{19,26\}$
55	$\{6,15,21\},\{8,17,25\},\{9,13,22\},$ $\{10,14,24\},\{11,12,23\},\{16,19,20\}$	$\{18,26\},\{27\}$
56	$\{6,15,21\},\{8,18,26\},\{9,13,22\},$ $\{10,14,24\},\{11,12,23\},\{17,19,20\}$	$\{16,25\},\{27\}$

t=7:

n	modulo n difference triples	connection sets
59	$\{6,16,22\},\{8,15,23\},\{9,19,28\},\{10,17,27\},$ $\{11,13,24\},\{12,14,26\},\{18,20,21\}$	$\{25,29\}$
60	$\{6,16,22\},\{8,17,25\},\{9,15,24\},\{10,18,28\},$ $\{11,12,23\},\{13,14,27\},\{19,20,21\}$	$\{26,29\}$
61	$\{6,15,21\},\{8,16,24\},\{9,17,26\},\{10,18,28\},$ $\{11,12,23\},\{13,14,27\},\{19,20,22\}$	$\{25,29\},\{30\}$
62	$\{6,20,26\},\{8,17,25\},\{9,15,24\},\{10,18,28\},$ $\{11,12,23\},\{13,14,27\},\{19,21,22\}$	$\{16,30,31\},\{29\}$

t=8:

n	modulo n difference triples	connection sets
65	$\{6,21,27\},\{8,18,26\},\{9,15,24\},\{10,19,29\},$ $\{11,17,28\},\{12,13,25\},\{14,16,30\},\{20,22,23\}$	$\{31,32\}$
66	$\{6,21,27\},\{8,18,26\},\{9,23,32\},\{10,19,29\},$ $\{11,17,28\},\{12,13,25\},\{14,16,30\},\{20,22,24\}$	$\{15,31\}$
67	$\{6,27,33\},\{8,18,26\},\{9,23,32\},\{10,19,29\},$ $\{11,17,28\},\{12,13,25\},\{14,16,30\},\{21,22,24\}$	$\{15,20\},\{31\}$
68	$\{6,18,24\},\{8,21,29\},\{9,23,32\},\{10,15,25\},$ $\{11,19,30\},\{12,16,28\},\{13,14,27\},\{20,22,26\}$	$\{17,31\},\{33\}$
70	$\{6,18,24\},\{9,22,31\},\{10,15,25\},\{11,19,30\},$ $\{12,16,28\},\{13,14,27\},\{17,20,33\},\{21,23,26\}$	$\{8,29\},\{32,34,35\}$
72	$\{6,16,22\},\{8,13,21\},\{9,17,26\},\{10,18,28\},$ $\{11,19,30\},\{12,15,27\},\{14,20,34\},\{23,24,25\}$	$\{29,33\},\{31,32\},\{35\}$

We now define S_t by $S_t = \{6\} \cup \{8, \ldots, 3t+9\} \cup \{3t+11\}$ for $t \equiv 0, 1 \pmod 4$, and $S_t = \{6\} \cup \{8, \ldots, 3t+9\}$ when $t \equiv 2, 3 \pmod 4$. The following table gives a partition π_t of S_t into difference triples and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
1	$\{\{6,8,14\},\{9,10,11,12\}\}$
5	$\{\{6,14,20\},\{8,15,23\},\{9,17,26\},\{10,12,22\},\{11,13,24\},\{16,18,19,21\}\}$
6	$\{\{6,13,19\},\{8,16,24\},\{9,17,26\},\{10,18,28\},$ $\{11,14,25\},\{12,15,27\},\{20,21,22,23\}\}$
7	$\{\{6,24,30\},\{8,23,31\},\{9,16,25\},\{10,17,27\},$ $\{11,18,29\},\{12,14,26\},\{13,15,28\},\{19,20,21,22\}\}$
8	$\{\{6,17,23\},\{8,16,24\},\{9,19,28\},\{10,15,25\},\{11,20,31\},$ $\{12,21,33\},\{13,22,35\},\{14,18,32\},\{26,27,29,30\}\}$

Thus, $\langle Q_t \rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is both a $(3^{tn}, 4^n)$-decomposition and a $(3^{tn}, n^4)$-decomposition of $\langle S_t \rangle_n$. If $q = 0$, then we use the $(3^{tn}, n^4)$-decomposition of $\langle S_t \rangle_n$ and if $q \geq 1$, then we use the $(3^{tn}, 4^n)$-decomposition of $\langle S_t \rangle_n$. This leaves us needing an (n^{h-4})-decomposition of $K_n - \langle \{1,2,3,4,5,7\} \cup S_t \rangle_n$ when $q = 0$, and a $(4^{(q-1)n}, n^h)$-decomposition of $K_n - \langle \{1,2,3,4,5,7\} \cup S_t \rangle_n$ when $q \geq 1$. Note that $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$ is isomorphic to

- $\langle \{3t+10, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \in \{1,5,6\}$; and
- $\langle \{3t+9\} \cup \{3t+11, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \in \{7,8\}$.

When $t \in \{6,7\}$ the required decomposition exists by Lemma 1.7.7. When $t \in \{1,5,8\}$ and the required number of Hamilton cycles (that is, $h-4$ when $q = 0$ and h when $q \geq 1$) is at least 2, the required decomposition exists by Lemma 1.7.8. So we need to consider only the cases where $q = 0$, $h \in \{4,5\}$ and $t \in \{1,5,8\}$.

Since $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 6$, and since we have already dealt with the cases where $(n,t) \in \{(27,1),(28,1),(52,5),(70,8),(72,8)\}$, this leaves us with the cases where (n,t,h) in

$$\{(27,1,4),(29,1,5),(30,1,5),(51,5,4),(53,5,5),(54,5,5),(69,8,4),(71,8,5)\}.$$

In the cases $(n,t,h) \in \{(27,1,4),(51,5,4),(69,8,4)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 0 and n is odd, and in the other cases we have that $h-4$ (the required number of Hamilton cycles) is 1 and $n \equiv 5, 6, 11 \pmod{12}$. So in all these cases the required decompositions exist by Lemma 1.7.8.

Case C2: Suppose that $t \in \{2,3,4\}$. The cases $6t+17 \leq n \leq 6t+24$ are dealt with first. Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 6$ that in each of these cases we have $q = 0$. Thus, the value of h is uniquely determined by the values of n and t. The required decompositions are obtained by partitioning $\{6\} \cup \{8, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Suitable partitions are given in the following tables.

t=2:

n	modulo n difference triples	connection sets
29	$\{6, 9, 14\}, \{8, 10, 11\}$	$\{12, 13\}$
30	$\{6, 8, 14\}, \{9, 10, 11\}$	$\{12, 13\}$
31	$\{6, 8, 14\}, \{9, 10, 12\}$	$\{11, 13\}, \{15\}$
32	$\{6, 8, 14\}, \{9, 10, 13\}$	$\{11, 12\}, \{15\}$
33	$\{6, 8, 14\}, \{10, 11, 12\}$	$\{9, 16\}, \{13, 15\}$
34	$\{6, 8, 14\}, \{10, 11, 13\}$	$\{9, 12\}, \{15, 16\}$
35	$\{6, 8, 14\}, \{10, 12, 13\}$	$\{9, 11\}, \{15, 16\}, \{17\}$
36	$\{6, 8, 14\}, \{11, 12, 13\}$	$\{9, 10\}, \{15, 16\}, \{17\}$

t=3:

n	modulo n difference triples	connection sets
35	$\{6, 8, 14\}, \{9, 11, 15\}, \{10, 12, 13\}$	$\{16, 17\}$
36	$\{6, 8, 14\}, \{9, 10, 17\}, \{11, 12, 13\}$	$\{15, 16\}$
37	$\{6, 8, 14\}, \{9, 11, 17\}, \{10, 12, 15\}$	$\{13, 16\}, \{18\}$
38	$\{6, 9, 15\}, \{8, 10, 18\}, \{11, 13, 14\}$	$\{12, 16, 19\}, \{17\}$
39	$\{6, 9, 15\}, \{8, 10, 18\}, \{12, 13, 14\}$	$\{11, 16\}, \{17, 19\}$
40	$\{6, 9, 15\}, \{8, 10, 18\}, \{11, 13, 16\}$	$\{12, 17\}, \{14, 19\}$
41	$\{6, 9, 15\}, \{8, 10, 18\}, \{12, 13, 16\}$	$\{11, 14\}, \{17, 19\}, \{20\}$
42	$\{6, 9, 15\}, \{8, 10, 18\}, \{12, 14, 16\}$	$\{11, 13\}, \{17\}, \{19, 20\}$

t=4:

n	modulo n difference triples	connection sets
41	$\{6, 8, 14\}, \{9, 15, 17\}, \{10, 11, 20\}, \{12, 13, 16\}$	$\{18, 19\}$
42	$\{6, 8, 14\}, \{9, 16, 17\}, \{10, 12, 20\}, \{11, 13, 18\}$	$\{15, 19\}$
43	$\{6, 11, 17\}, \{8, 12, 20\}, \{9, 10, 19\}, \{13, 14, 16\}$	$\{15, 18\}, \{21\}$
44	$\{6, 10, 16\}, \{8, 13, 21\}, \{9, 11, 20\}, \{12, 14, 18\}$	$\{15, 17\}, \{19\}$
45	$\{6, 11, 17\}, \{8, 12, 20\}, \{9, 10, 19\}, \{13, 14, 18\}$	$\{15, 16\}, \{21, 22\}$
46	$\{6, 11, 17\}, \{8, 12, 20\}, \{9, 10, 19\}, \{13, 15, 18\}$	$\{14, 16, 23\}, \{21, 22\}$
47	$\{6, 11, 17\}, \{8, 12, 20\}, \{9, 10, 19\}, \{13, 16, 18\}$	$\{14, 15\}, \{21, 22\}, \{23\}$
48	$\{6, 11, 17\}, \{8, 12, 20\}, \{9, 10, 19\}, \{14, 16, 18\}$	$\{13, 15\}, \{21, 22\}, \{23\}$

We now deal with $n \geq 6t + 25$ and (n, t) not covered earlier. We define S_t by $S_t = \{6\} \cup \{8, \ldots, 3t + 12\}$. The following table gives a partition π_t of S_t into difference triples, a set R_t and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers, and R_t is a pair of relatively prime integers.

t	π_t
2	$\{\{6, 12, 18\}, \{8, 9, 17\}, \{13, 14, 15, 16\}, \{10, 11\}\}$
3	$\{\{6, 14, 20\}, \{8, 13, 21\}, \{9, 10, 19\}, \{15, 16, 17, 18\}, \{11, 12\}\}$
4	$\{\{6, 11, 17\}, \{8, 12, 20\}, \{9, 13, 22\}, \{10, 14, 24\}, \{15, 16, 18, 19\}, \{21, 23\}\}$

Thus, $\langle Q_t \rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is both a $(3^{tn}, 4^n, n^2)$-decomposition and a $(3^{tn}, n^6)$-decomposition of $\langle S_t \rangle_n$. If $q = 0$, then we use the $(3^{tn}, n^6)$-decomposition of $\langle S_t \rangle_n$ and if $q \geq 1$, then we use the $(3^{tn}, 4^n, n^2)$-decomposition of $\langle S_t \rangle_n$. This leaves us needing an (n^{h-6})-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_t \rangle_n$ when $q = 0$, and a $(4^{(q-1)n}, n^{h-2})$-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_t \rangle_n$ when $q \geq 1$. Note that $K_n - \langle \{1, 2, 3, 4, 6\} \cup S_t \rangle_n$ is isomorphic to $\langle \{3t + 13, \dots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ so the required decomposition exists by Lemma 1.7.7.

Case C3: Suppose that $t \in \{9, 10, 11, 12\}$. Redefine S_t by $S_t = \{6\} \cup \{8, \dots, 3t + 8\}$. The following table gives a partition of S_t into difference triples and a set R_t consisting of a pair of relatively prime integers. Thus, $\langle R_t \rangle_n$ is a connected 4-regular Cayley graph, and so can be decomposed into two Hamilton cycles using Theorem 1.7.1. Thus, we have a $(3^{tn}, n^2)$-decomposition of $\langle S_t \rangle_n$.

t	difference triples	R_t
9	$\{6, 20, 26\}, \{8, 21, 29\}, \{9, 22, 31\}, \{10, 23, 33\}, \{11, 24, 35\},$ $\{12, 16, 28\}, \{13, 17, 30\}, \{14, 18, 32\}, \{15, 19, 34\}$	$\{25, 27\}$
10	$\{6, 23, 29\}, \{8, 20, 28\}, \{9, 18, 27\}, \{10, 22, 32\}, \{11, 25, 36\}, \{12, 26, 38\},$ $\{13, 24, 37\}, \{14, 21, 35\}, \{15, 19, 34\}, \{16, 17, 33\}$	$\{30, 31\}$
11	$\{6, 25, 31\}, \{8, 24, 32\}, \{9, 21, 30\}, \{10, 19, 29\}, \{11, 26, 37\}, \{12, 27, 39\},$ $\{13, 28, 41\}, \{14, 20, 34\}, \{15, 18, 33\}, \{16, 22, 38\}, \{17, 23, 40\}$	$\{35, 36\}$
12	$\{6, 27, 33\}, \{8, 23, 31\}, \{9, 25, 34\}, \{10, 22, 32\}, \{11, 24, 35\},$ $\{12, 28, 40\}, \{13, 29, 42\}, \{14, 30, 44\}, \{15, 26, 41\},$ $\{16, 21, 37\}, \{17, 19, 36\}, \{18, 20, 38\}$	$\{39, 43\}$

Thus, we only require a $(4^{qn}, n^{h-2})$-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_t \rangle_n$. But $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_t \rangle_n$ is isomorphic to $\langle \{3t + 9, \dots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ and so this decomposition exists by Lemma 1.7.7.

Case C4: Suppose that $t \geq 13$. Redefine S_t by $S_t = \{6\} \cup \{8, \dots, 3t + 6\}$ when $t \equiv 2, 3 \pmod 4$, and $S_t = \{6\} \cup \{8, \dots, 3t + 5\} \cup \{3t + 7\}$ when $t \equiv 0, 1 \pmod 4$. We now obtain a (3^{tn})-decomposition of $\langle S_t \rangle_n$. For $13 \leq t \leq 60$, we have found such a decomposition by partitioning S_t into difference triples with the aid of a computer. These difference triples are shown in Table A.26 in the appendix. For $t \geq 61$ and $t \equiv 2, 3 \pmod 4$ (respectively $t \equiv 0, 1 \pmod 4$) we set aside as one difference triple $\{6, 3t, 3t+6\}$ (respectively $\{6, 3t+1, 3t+7\}$) and form the set $S'_t = \{8, \dots, 3t - 1, 3t + 1, \dots, 3t + 5\} = \{8, \dots, 3t + 5\} \setminus \{3t\}$ (respectively $S'_t = \{8, \dots, 3t, 3t + 2, \dots, 3t + 5\} = \{8, \dots, 3t + 5\} \setminus \{3t + 1\}$). We can obtain a $(3^{(t-1)n})$-decomposition of $\langle S'_t \rangle_n$ by using an extended Langford sequence of order $t - 1$ and defect 8 to partition S'_t into difference triples. Since $t \geq 61$, this sequence exists by Theorem 7.1 in [72] (also see [90, 91]). So we have a (3^{tn})-decomposition of $\langle S_t \rangle_n$, and require a $(4^{qn}, n^h)$-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_t \rangle_n$. Since $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_t \rangle_n$ is isomorphic to

- $\langle \{3t + 7, \dots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \equiv 2, 3 \pmod 4$; and
- $\langle \{3t + 6\} \cup \{3t + 8, \dots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \equiv 0, 1 \pmod 4$;

this decomposition exists by Lemma 1.7.7 or 1.7.8. □

Case D: **($S = \{1, 2, 3, 4, 5, 6, 7\}$)** The conditions $h \geq 2$ and $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 7$ imply $n \geq 6t + 19$. If $t = 0$, then the result follows immediately by Lemma 1.7.8. We deal separately with the five cases $t \in \{1, 2\}$, $t \in \{3, 4, 5, 6\}$, $t \in \{7, 8, 9, 10\}$, $t \in \{11, 12, 13, 14\}$ and $t \geq 15$.

Case D1: Suppose that $t \in \{1,2\}$. The cases $6t+19 \leq n \leq 6t+30$ and the cases

$$(n,t) \in \{(38,1),(39,1),(40,1),(44,2),(45,2)\}$$

are dealt with first. Since $h \geq 2$, it follows from $3t+4q+h = \lfloor\frac{n-1}{2}\rfloor - 5$ that in each of these cases we have $q \in \{0,1\}$. The required decompositions are obtained by partitioning $\{8,\ldots,\lfloor\frac{n}{2}\rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Whenever we need to consider $q=1$, one of the given connection sets has cardinality 4 and is a difference quadruple, which means that the corresponding circulant graph has a (4^n)-decomposition. Suitable partitions are given in the following tables, noting that for $t=1$ we have $n \notin \{25,26\}$ and for $t=2$ we have $n \neq 31$..

t=1:

n	modulo n difference triples	connection sets
27	$\{8,9,10\}$	$\{11,12\},\{13\}$
28	$\{8,9,11\}$	$\{10,12,13\}$
29	$\{8,9,12\}$	$\{10,11\},\{13,14\}$
30	$\{8,9,13\}$	$\{10,11\},\{12,14,15\}$
31	$\{8,9,14\}$	$\{10,11\},\{12,13\},\{15\}$
32	$\{8,9,15\}$	$\{10,11\},\{12,13,14\}$
33	$\{10,11,12\}$	$\{8,9,13,14\},\{15,16\}$
34	$\{8,10,16\}$	$\{9,11,13,15\},\{12,14,17\}$
35	$\{10,12,13\}$	$\{8,9,14,15\},\{11\},\{16,17\}$
36	$\{10,12,14\}$	$\{8,9,15,16\},\{11,13\},\{17\}$
38	$\{10,12,16\}$	$\{8,9,14,15\},\{11,13\},\{17,18\}$
39	$\{10,13,16\}$	$\{8,9,11,12\},\{14,15\},\{17,18\},\{19\}$
40	$\{10,14,16\}$	$\{8,9,11,12\},\{13,15\},\{17,18\},\{19\}$

t=2:

n	modulo n difference triples	connection sets
32	$\{8,10,14\},\{9,11,12\}$	$\{13,15\}$
33	$\{8,10,15\},\{9,11,13\}$	$\{12,14\},\{16\}$
34	$\{8,12,14\},\{10,11,13\}$	$\{9\},\{15,16\}$
35	$\{8,11,16\},\{10,12,13\}$	$\{9,14\},\{15,17\}$
36	$\{8,13,15\},\{10,12,14\}$	$\{9,11\},\{16,17\}$
37	$\{8,10,18\},\{11,12,14\}$	$\{9,13\},\{15,16\},\{17\}$
38	$\{8,10,18\},\{11,12,15\}$	$\{9\},\{13,14\},\{16,17\}$
39	$\{8,10,18\},\{11,12,16\}$	$\{9,14\},\{13,15,17,19\}$
40	$\{8,10,18\},\{11,12,17\}$	$\{9,19\},\{13,14,15,16\}$
41	$\{8,10,18\},\{12,14,15\}$	$\{9,11\},\{13,16,17,20\},\{19\}$
42	$\{8,10,18\},\{12,14,16\}$	$\{9,11,13,15\},\{17\},\{19,20\}$
44	$\{8,9,17\},\{10,14,20\}$	$\{11,12\},\{13,15\},\{16,18,19,21\}$
45	$\{8,10,18\},\{14,15,16\}$	$\{9,11,17,19\},\{12,13\},\{20,21\},\{22\}$

We now deal with $n \geq 6t+30$ and (n,t) not covered earlier. This implies $4q+h \geq 8$. Define S_t by $S_t = \{8,\ldots,3t+14\} \cup \{3t+16\}$. The following table gives a partition π_t of S_t into difference triples and two difference quadruples Q_t' and Q_t' such that Q_t and Q_t' can each be partitioned into two pairs of relatively prime integers.

t	π_t
1	$\{\{8, 11, 19\}, \{9, 10, 12, 13\}, \{14, 15, 16, 17\}\}$
2	$\{\{8, 10, 18\}, \{9, 11, 20\}, \{12, 15, 19, 22\}, \{13, 14, 16, 17\}\}$

Thus, each of $\langle Q_t\rangle_n$ and $\langle Q_t'\rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is a $(3^{tn}, 4^{2n})$-decomposition, a $(3^{tn}, 4^n, n^4)$-decomposition and a $(3^{tn}, n^8)$-decomposition of $\langle S_t\rangle_n$. If $q = 0$, then we use the $(3^{tn}, n^8)$-decomposition of $\langle S_t\rangle_n$, if $q = 1$, then we use the $(3^{tn}, 4^n, n^4)$-decomposition of $\langle S_t\rangle_n$, and if $q \geq 2$, then we use the $(3^{tn}, 4^{2n})$-decomposition of $\langle S_t\rangle_n$.

This leaves us needing an (n^{h-8})-decomposition of $K_n - \langle\{1, 2, 3, 4, 5, 6, 7\} \cup S_t\rangle_n$ when $q = 0$, a (n^{h-4})-decomposition of $K_n - \langle\{1, 2, 3, 4, 5, 6, 7\} \cup S_t\rangle_n$ when $q = 1$, and a $(4^{(q-2)n}, n^h)$-decomposition of $K_n - \langle\{1, 2, 3, 4, 5, 6, 7\} \cup S_t\rangle_n$ when $q \geq 2$. Note that $K_n - \langle\{1, 2, 3, 4, 5, 6, 7\} \cup S_t\rangle_n$ is isomorphic to $\langle\{3t + 15\} \cup \{3t + 17, \ldots, \lfloor \frac{n}{2} \rfloor\}\rangle_n$.

When the required number of Hamilton cycles (that is, $h - 8$ when $q = 0$, $h - 4$ when $q = 1$ and h when $q \geq 2$) is at least 2, the required decomposition exists by Lemma 1.7.8. So we need to consider only the cases where

- $q = 0$ and $h \in \{8, 9\}$; and
- $q = 1$ and $h \in \{4, 5\}$.

Since $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 7$, and since we have already dealt with the cases where $(n, t) \in \{(38, 1), (39, 1), (40, 1), (44, 2), (45, 2)\}$, this leaves only

$$(n, t, h) \in \{(37, 1, 8), (43, 2, 8), (46, 2, 9)\}$$

when $q = 0$, and

$$(n, t, h) \in \{(37, 1, 4), (43, 2, 4), (46, 2, 5)$$

when $q = 1$.

In both cases ($q = 0$ and $q = 1$ respectively), when the required number of Hamilton cycles ($h - 8$ and $h - 4$ respectively) is 0 we have n odd, and when the required number of Hamilton cycles is 1 we have $n \equiv 10 \pmod{12}$. Thus, the decomposition exists by Lemma 1.7.8.

Case D2: Suppose that $t \in \{3, 4, 5, 6\}$. Redefine S_t by $S_t = \{8, \ldots, 3t + 13\}$.

The case $6t + 19 \leq n \leq 6t + 26$ is dealt with first. Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 7$ that in each of these cases we have $q = 0$. The required decompositions are obtained by partitioning $\{8, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Suitable partitions are given in the following tables.

t=3:

n	modulo n difference triples	connection sets
37	$\{8,10,18\},\{9,13,15\},\{11,12,14\}$	$\{16,17\}$
38	$\{8,10,18\},\{9,13,16\},\{11,12,15\}$	$\{14,17\}$
39	$\{8,10,18\},\{9,13,17\},\{11,12,16\}$	$\{14,15\},\{19\}$
40	$\{8,10,18\},\{9,12,19\},\{11,13,16\}$	$\{14,15\},\{17\}$
41	$\{8,10,18\},\{9,11,20\},\{12,13,16\}$	$\{14,15\},\{17,19\}$
42	$\{8,10,18\},\{9,11,20\},\{12,13,17\}$	$\{14,15\},\{16,19\}$
43	$\{8,10,18\},\{9,11,20\},\{12,14,17\}$	$\{13,15\},\{16,19\},\{21\}$
44	$\{8,10,18\},\{9,11,20\},\{13,14,17\}$	$\{12,15\},\{16,19\},\{21\}$

t=4:

n	modulo n difference triples	connection sets
43	$\{8,10,18\},\{9,12,21\},\{11,15,17\},\{13,14,16\}$	$\{19,20\}$
44	$\{8,10,18\},\{9,12,21\},\{11,14,19\},\{13,15,16\}$	$\{17,20\}$
45	$\{8,10,18\},\{9,12,21\},\{11,14,20\},\{13,15,17\}$	$\{16,19\},\{22\}$
46	$\{8,10,18\},\{9,12,21\},\{11,15,20\},\{13,14,19\}$	$\{16,17\},\{22,23\}$
47	$\{8,10,18\},\{9,12,21\},\{11,14,22\},\{13,15,19\}$	$\{16,17\},\{20,23\}$
48	$\{8,10,18\},\{9,12,21\},\{11,14,23\},\{13,15,20\}$	$\{16,17\},\{19,22\}$
49	$\{8,10,18\},\{9,12,21\},\{11,15,23\},\{13,14,22\}$	$\{16,17\},\{19,20\},\{24\}$
50	$\{8,10,18\},\{9,12,21\},\{11,17,22\},\{13,14,23\}$	$\{15,16\},\{19,20\},\{24,25\}$

t=5:

n	modulo n difference triples	connection sets
49	$\{8,12,20\},\{9,14,23\},\{10,11,21\},$ $\{13,17,19\},\{15,16,18\}$	$\{22,24\}$
50	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,18,19\},\{14,15,21\}$	$\{17,23\}$
51	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,15,23\},\{14,18,19\}$	$\{17,21\},\{25\}$
52	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,14,25\},\{15,18,19\}$	$\{17,21\},\{23\}$
53	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,15,25\},\{14,18,21\}$	$\{17,19\},\{23,26\}$
54	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,18,23\},\{14,15,25\}$	$\{17,19\},\{21,26\}$
55	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,17,25\},\{14,18,23\}$	$\{15,19\},\{21,26\},\{27\}$
56	$\{8,16,24\},\{9,11,20\},\{10,12,22\},$ $\{13,17,26\},\{14,19,23\}$	$\{15,18\},\{21,25\},\{27\}$

t=6:

n	modulo n difference triples	connection sets
55	$\{8,21,26\},\{10,20,25\},\{12,19,24\},$ $\{13,15,27\},\{14,18,23\},\{16,17,22\}$	$\{9,11\}$
56	$\{8,9,17\},\{10,19,27\},\{11,22,23\},$ $\{12,20,24\},\{13,18,25\},\{14,16,26\}$	$\{15,21\}$
57	$\{8,15,23\},\{9,16,25\},\{10,17,27\},$ $\{11,13,24\},\{12,14,26\},\{18,19,20\}$	$\{21,22\},\{28\}$
58	$\{8,15,23\},\{9,16,25\},\{10,18,28\},$ $\{11,13,24\},\{12,14,26\},\{17,19,22\}$	$\{20,21\},\{27\}$
59	$\{8,15,23\},\{9,16,25\},\{10,18,28\},$ $\{11,13,24\},\{12,14,26\},\{17,20,22\}$	$\{19,21\},\{27,29\}$
60	$\{8,15,23\},\{9,16,25\},\{10,18,28\},$ $\{11,13,24\},\{12,14,26\},\{17,21,22\}$	$\{19,20\},\{27,29\}$
61	$\{8,15,23\},\{9,16,25\},\{10,18,28\},$ $\{11,13,24\},\{12,14,26\},\{19,20,22\}$	$\{17,21\},\{27,29\},\{30\}$
62	$\{8,15,23\},\{9,16,25\},\{10,18,28\},$ $\{11,13,24\},\{12,14,26\},\{19,21,22\}$	$\{17,20\},\{27\},\{29,30\}$

We now deal with $n \geq 6t+27$. This implies $4q+h \geq 6$. Define S_t by $S_t = \{8,\dots,3t+13\}$. The following table gives a partition π_t of S_t into difference triples, a set R_t of two relatively prime integers, and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
3	$\{\{8,12,20\},\{9,13,22\},\{10,11,21\},\{14,15,16,17\},\{18,19\}\}$
4	$\{\{8,12,20\},\{9,14,23\},\{10,15,25\},\{11,13,24\},\{16,17,18,19\},\{21,22\}\}$
5	$\{\{8,14,22\},\{9,15,24\},\{10,16,26\},\{11,17,28\},\{12,13,25\},$ $\{18,19,20,21\},\{23,27\}\}$
6	$\{\{8,16,24\},\{9,17,26\},\{10,18,28\},\{11,19,30\},\{12,13,25\},\{14,15,29\}$ $\{20,21,22,23\},\{27,31\}\}$

Thus, $\langle R_t\rangle_n$ is a connected 4-regular Cayley graph, and so can be decomposed into 2 Hamilton cycles using Theorem 1.7.1. Additionally $\langle Q_t\rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is a $(3^{tn},4^n,n^2)$-decomposition, and a $(3^{tn},n^6)$-decomposition of $\langle S_t\rangle_n$. If $q=0$, then we use the $(3^{tn},n^6)$-decomposition of $\langle S_t\rangle_n$, and if $q \geq 1$, then we use the $(3^{tn},4^n,n^2)$-decomposition of $\langle S_t\rangle_n$.

This leaves us needing an (n^{h-6})-decomposition of $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$ when $q=0$, and a $(4^{(q-1)n},n^{h-2})$-decomposition of $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$ when $q \geq 2$. Note that $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$ is isomorphic to $\langle\{3t+14,\dots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$, so the decomposition exists by Lemma 1.7.7.

Case D3: Suppose that $t \in \{7,8,9,10\}$. Redefine S_t by $S_t = \{8,\dots,3t+11\}$ when $t \in \{7,8\}$ and $S_t = \{8,\dots,3t+10\} \cup \{3t+12\}$ when $t \in \{9,10\}$.

The case $6t+19 \leq n \leq 6t+22$ is dealt with first, along with the cases $(n,t) \in \{(78,9),(80,9),(84,10)$.

Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 7$ that in each of these cases we have $q = 0$. The required decompositions are obtained by partitioning $\{8, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 1.7.1. Suitable partitions are given in the following tables.

t=7:

n	modulo n difference triples	connection sets
61	$\{8,18,26\},\{9,19,28\},\{10,20,30\},\{11,16,27\},$ $\{12,13,25\},\{14,15,29\},\{17,21,23\}$	$\{22,24\}$
62	$\{8,18,26\},\{9,19,28\},\{10,20,30\},\{11,16,27\},$ $\{12,13,25\},\{14,15,29\},\{17,22,23\}$	$\{21,24\}$
63	$\{8,18,26\},\{9,19,28\},\{10,20,30\},\{11,16,27\},$ $\{12,13,25\},\{14,15,29\},\{17,22,24\}$	$\{21,23\},\{31\}$
64	$\{8,18,26\},\{9,19,28\},\{10,20,30\},\{11,16,27\},$ $\{12,13,25\},\{14,15,29\},\{17,23,24\}$	$\{21,22\},\{31\}$

t=8:

n	modulo n difference triples	connection sets
67	$\{8,18,26\},\{9,20,29\},\{10,21,31\},\{11,22,33\},$ $\{12,16,28\},\{13,14,27\},\{15,17,32\},\{19,23,25\}$	$\{24,30\}$
68	$\{8,18,26\},\{9,20,29\},\{10,21,31\},\{11,22,33\},$ $\{12,16,28\},\{13,14,27\},\{15,17,32\},\{19,24,25\}$	$\{23,30\}$
69	$\{8,18,26\},\{9,20,29\},\{10,21,31\},\{11,25,33\},$ $\{12,16,28\},\{13,14,27\},\{15,17,32\},\{22,23,24\}$	$\{19,30\},\{34\}$
70	$\{8,18,26\},\{9,19,28\},\{10,20,30\},\{11,21,32\},$ $\{12,15,27\},\{13,16,29\},\{14,17,31\},\{22,23,25\}$	$\{24,34,35\},\{33\}$

t=9:

n	modulo n difference triples	connection sets
73	$\{8,20,28\},\{9,21,30\},\{10,23,33\},\{11,24,35\},$ $\{12,15,27\},\{13,19,32\},\{14,17,31\},\{16,18,34\},$ $\{22,25,26\}$	$\{29,36\}$
74	$\{8,24,32\},\{9,22,31\},\{10,19,29\},\{11,16,27\},$ $\{12,21,33\},\{13,15,28\},\{14,20,34\},\{17,18,35\},$ $\{23,25,26\}$	$\{30,36,37\}$
75	$\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,24,35\},$ $\{12,25,37\},\{13,17,30\},\{14,18,32\},\{15,19,34\},$ $\{16,20,36\}$	$\{26,27\},\{28\}$
76	$\{8,20,28\},\{9,21,30\},\{10,23,33\},\{11,24,35\},$ $\{12,15,27\},\{13,19,32\},\{14,17,31\},\{16,18,34\},$ $\{22,25,29\}$	$\{26,36,37\}$
78	$\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,24,35\},$ $\{12,25,37\},\{13,17,30\},\{14,18,32\},\{15,19,34\},$ $\{16,20,36\}$	$\{26,27\},\{28,38,39\}$
80	$\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,24,35\},$ $\{12,25,37\},\{13,17,30\},\{14,18,32\},\{15,19,34\},$ $\{16,20,36\}$	$\{26,27\},\{28,38,39\}$

t=10:

n	modulo n difference triples	connection sets
79	$\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,26,37\},$ $\{12,20,32\},\{13,25,38\},\{14,16,30\},\{15,19,34\},$ $\{17,18,35\},\{24,27,28\}$	$\{36,39\}$
80	$\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,24,35\},$ $\{12,26,38\},\{13,17,30\},\{14,18,32\},\{15,19,34\},$ $\{16,20,36\},\{25,27,28\}$	$\{37,39\}$
81	$\{8,20,28\},\{9,22,31\},\{10,23,33\},\{11,24,35\},$ $\{12,26,38\},\{13,17,30\},\{14,18,32\},\{15,19,34\},$ $\{16,21,37\},\{25,27,29\}$	$\{36,39,40\}$
82	$\{8,22,30\},\{9,23,32\},\{10,24,34\},\{11,25,36\},$ $\{12,26,38\},\{13,27,40\},\{14,19,33\},\{15,16,31\},$ $\{17,20,37\},\{18,21,39\}$	$\{28,29\},\{35\}$
84	$\{8,22,30\},\{9,23,32\},\{10,24,34\},\{11,25,36\},$ $\{12,26,38\},\{13,27,40\},\{14,19,33\},\{15,16,31\},$ $\{17,20,37\},\{18,21,39\}$	$\{28,29\},\{35,41\}$

We now deal with $n \geq 6t+23$ and (n,t) not covered earlier. This implies $4q+h \geq 4$. Define S_t by $S_t = \{8,\ldots,3t+11\}$ when $t \in \{7,8\}$ and $S_t = \{8,\ldots,3t+10\} \cup \{3t+12\}$ when $t \in \{9,10\}$. The following table gives a partition π_t of S_t into difference triples and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
7	$\{\{8,18,26\},\{9,19,28\},\{10,20,30\},\{11,21,32\},\{12,15,27\},\{13,16,29\},$ $\{14,17,31\},\{22,23,24,25\}\}$
8	$\{\{8,16,24\},\{9,20,29\},\{10,21,31\},\{11,22,33\},\{12,23,35\},\{13,17,30\},$ $\{14,18,32\},\{15,19,34\},\{25,26,27,28\}\}$
9	$\{\{8,20,28\},\{9,21,30\},\{10,25,35\},\{11,18,29\},\{12,19,31\},\{13,24,37\},$ $\{14,22,36\},\{15,17,32\},\{16,23,39\},\{26,27,33,34\}\}$
10	$\{\{8,22,30\},\{9,23,32\},\{10,24,34\},\{11,25,36\},\{12,21,33\},\{13,18,31\},$ $\{14,26,40\},\{15,27,42\},\{16,19,35\},\{17,20,37\},\{28,29,38,39\}\}$

Thus, $\langle Q_t\rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 1.7.1. It follows that there is a $(3^{tn}, 4^n)$-decomposition, and a $(3^{tn}, n^4)$-decomposition of $\langle S_t\rangle_n$. If $q = 0$, then we use the $(3^{tn}, n^4)$-decomposition of $\langle S_t\rangle_n$, and if $q \geq 1$, then we use the $(3^{tn}, 4^n)$-decomposition of $\langle S_t\rangle_n$.

This leaves us needing an (n^{h-4})-decomposition of $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$ when $q = 0$, and a $(4^{(q-1)n}, n^h)$-decomposition of $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$ when $q \geq 1$. Note that $K_n - \langle\{1,2,3,4,6,7\} \cup S_t\rangle_n$ is isomorphic to

- $\langle\{3t+12,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \in \{7,8\}$; and

- $\langle\{3t+11\} \cup \{3t+13,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \in \{9,10\}$.

When $t \in \{7,8\}$ the required decomposition exists by Lemma 1.7.7. When $t \in \{9,10\}$ and the required number of Hamilton cycles (that is, $h-4$ when $q = 0$ and h when $q \geq 1$) is at least 2, the required decomposition exists by Lemma 1.7.8. So we need to consider only the cases where $q = 0$, $h \in \{4,5\}$ and $t \in \{9,10\}$.

Since $3t + 4q + h = \lfloor\frac{n-1}{2}\rfloor - 7$, and since we have already dealt with the cases where $(n,t) \in \{(78,t),(80,9),(84,10)\}$, this leaves us with the cases where

$$(n,t,h) \in \{(77,9,4),(79,9,5),(83,10,4),(85,10,5),(86,10,5)\}.$$

In the cases $(n,t,h) \in \{(77,9,4),(83,10,4)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 0 and n is odd, and in the case $(n,t,h) \in \{(79,9,5),(85,10,5),(86,10,5)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 1 and $n \equiv 1,2,7 \pmod{12}$. So in all these cases the required decompositions exist by Lemma 1.7.8.

Case D4: Suppose that $t \in \{11,12,13,14\}$. Define S_t by $S_t = \{8,\ldots,3t+9\}$

The following table gives a partition π_t of S_t into difference triples and a set Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
11	$\{\{8,24,32\},\{9,25,34\},\{10,26,36\},\{11,27,38\},\{12,28,40\},\{13,29,42\},$ $\{14,19,33\},\{15,20,35\},\{16,21,37\},\{17,22,39\},\{18,23,41\},\{30,31\}\}$
12	$\{\{8,26,34\},\{9,27,36\},\{10,28,38\},\{11,24,35\},\{12,29,41\},\{13,30,43\},$ $\{14,31,45\},\{15,18,33\},\{16,21,37\},\{17,23,40\},\{19,25,44\},\{20,22,42\},$ $\{32,39\}\}$
13	$\{\{8,26,34\},\{9,27,36\},\{10,28,38\},\{11,29,40\},\{12,30,42\},\{13,31,44\},$ $\{14,32,46\},\{15,33,48\},\{16,21,37\},\{17,22,39\},\{18,23,41\},\{19,24,43\},$ $\{20,25,45\},\{35,47\}\}$
14	$\{\{8,28,36\},\{9,29,38\},\{10,30,40\},\{11,31,42\},\{12,32,44\},\{13,33,46\},$ $\{14,34,48\},\{15,35,50\},\{16,23,39\},\{17,24,41\},\{18,19,37\},\{20,25,40\},$ $\{21,26,47\},\{22,27,49\},\{43,51\}\}$

Thus, $\langle Q_t\rangle_n$ is a connected 4-regular Cayley graph, which in turn can be decomposed into two Hamilton cycles using Theorem 1.7.1. It follows that there is a $(3^{tn}, n^2)$-decomposition of $\langle S_t\rangle_n$, which leaves us needing an (n^{h-2})-decomposition of $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$. Note that $K_n - \langle\{1,2,3,4,6,7\} \cup S_t\rangle_n$ is isomorphic to $\langle\{3t+10,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ so the required decomposition exists by Lemma 1.7.7.

Case D5: Suppose that $t \geq 15$. Redefine S_t by $S_t = \{8,\ldots,3t+7\}$ when $t \equiv 0,3 \pmod 4$, and $S_t = \{8,\ldots,3t+6\} \cup \{3t+8\}$ when $t \equiv 1,2 \pmod 4$. We now obtain a (3^{tn})-decomposition of $\langle S_t\rangle_n$.

For $t \equiv 0,3 \pmod 4$ (respectively $t \equiv 1,2 \pmod 4$), we can obtain a $(3^{t)n})$-decomposition of $\langle S_t\rangle_n$ by using a Langford sequence (respectively hooked Langford sequence) of order t and defect 8, which exists since $t \geq 15$, to partition S_t into difference triples (see [90, 91]). So we have a (3^{tn})-decomposition of $\langle S_t\rangle_n$, and require a $(4^{qn}, n^h)$-decomposition of $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$. Since $K_n - \langle\{1,2,3,4,5,6,7\} \cup S_t\rangle_n$ is isomorphic to

- $\langle\{3t+8,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \equiv 0,3 \pmod 4$; and
- $\langle\{3t+7\} \cup \{3t+9,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $t \equiv 1,2 \pmod 4$;

this decomposition exists by Lemma 1.7.7 or 1.7.8. □

1.7.3 Proof of Lemma 1.3.7

We now prove Lemma 1.3.7, which we restate here for convenience.

Lemma 1.3.7 *If $S \in \{\{1,2,3,4\},\{1,2,3,4,6\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\}$ and $n \geq 2\max(S)+1$, $r \geq 0$ and $h \geq 2$ are integers satisfying $5r+h = \lfloor\frac{n-1}{2}\rfloor - |S|$, then there is a $(5^{rn}, n^h)$-decomposition of $K_n - \langle S\rangle_n$.*

Proof We give the proof for each

$$S \in \{\{1,2,3,4\},\{1,2,3,4,5,7\},\{1,2,3,4,5,6,7\},\{1,2,3,4,5,6,7,8\}\}$$

separately.

Case A: ($S = \{1, 2, 3, 4\}$)

The conditions $h \geq 2$ and $5r + h = \lfloor \frac{n-1}{2} \rfloor - 4$ imply $n \geq 10r + 13$. If $r = 0$, then the result follows immediately by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). Thus, we assume $r \geq 1$.

Define S_r by $S_r = \{5, \ldots, 5r + 4\}$ when $r \equiv 0, 3 \pmod 4$, and $S_r = \{5, \ldots, 5r + 3\} \cup \{5r + 5\}$ when $r \equiv 1, 2 \pmod 4$. We now obtain a (5^{rn})-decomposition $\langle S_r \rangle_n$ by partitioning S_r into difference quintuples. We have constructed such a partition π_r of S for $1 \leq r \leq 6$, shown in Table A.27 in the appendix, and thus assume $r \geq 7$.

For $r \equiv 0, 3 \pmod 4$, we take a Langford sequence of order r and defect 4, which exists since $r \geq 7$ (see [90, 91]), and use it to partition $\{4, \ldots, 3r + 3\}$ into r difference triples. We then add 1 to each element of each of these triples to obtain a partition of $\{5, \ldots, 3r + 4\}$ into r triples of the form $\{a, b, c\}$ where $a + b = c + 1$. It is easy to construct the required partition of $\{5, \ldots, 5r + 4\}$ into difference quintuples from this by partitioning $\{3r + 5, \ldots, 5r + 4\}$ into r pairs of consecutive integers.

For $r \equiv 1, 2 \pmod 4$, we take a hooked Langford sequence of order r and defect 4, which exists since $r \geq 7$ (see [90, 91]), and use it to partition $\{4, \ldots, 3r + 2\} \cup \{3r + 4\}$ into r difference triples. We then add 1 to each element of each of these triples, except the element $3r + 4$, to obtain a partition of $\{5, \ldots, 3r + 4\}$ into $r - 1$ triples of the form $\{a, b, c\}$ where $a + b = c + 1$, and one triple of the form $\{a, b, c\}$ where $a + b = c + 2$. It is easy to construct the required partition of $\{5, \ldots, 5r + 3\} \cup \{5r + 5\}$ into difference quintuples from this by partitioning $\{3r + 5, \ldots, 5r + 3\} \cup \{5r + 5\}$ into $r - 1$ pairs of consecutive integers, and the pair $\{5r + 3, 5r + 5\}$. The pair $\{5r + 3, 5r + 5\}$ combines with the triple of the form $\{a, b, c\}$ where $a + b = c + 2$ to form a difference quintuple.

So we have a (5^{rn})-decomposition of $\langle S_r \rangle_n$, and require an (n^h)-decomposition of $K_n - \langle \{1, 2, 3, 4\} \cup S_r \rangle_n$. Note that $K_n - \langle \{1, 2, 3, 4\} \cup S_r \rangle_n$ is isomorphic to

- $\langle \{5r + 5, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 0, 3 \pmod 4$; and
- $\langle \{5r + 4\} \cup \{5r + 6, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 1, 2 \pmod 4$.

If $r \equiv 0, 3 \pmod 4$, then the decomposition exists by Lemma 1.3.5. If $r \equiv 1, 2 \pmod 4$, then the decomposition exists by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even).

Case B: ($S = \{1, 2, 3, 4, 6\}$)

The conditions $h \geq 2$ and $5r + h = \lfloor \frac{n-1}{2} \rfloor - 5$ imply $n \geq 10r + 15$. If $r = 0$, then the result follows immediately by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). Thus, we assume $r \geq 1$.

Define S_r by $S_r = \{5\} \cup \{7, \ldots, 5r + 5\}$ when $r \equiv 2, 3 \pmod 4$, and $S_r = \{5\} \cup \{7, \ldots, 5r + 4\} \cup \{5r + 6\}$ when $r \equiv 0, 1 \pmod 4$. We now obtain a (5^{rn})-decomposition $\langle S_r \rangle_n$ by partitioning S_r into difference quintuples. We have constructed such a partition of S for $1 \leq r \leq 30$ with the aid of a computer, shown in Table A.28 in the appendix, and thus assume $r \geq 31$.

For $r \geq 31$ we first partition S_r into $\{5\} \cup \{7, \ldots, 15\} = S_2$ and $S_r \setminus S_2$. We have already noted that S_2 can be partitioned into difference quintuples, so we only need to partition $S_r \setminus S_2$ into difference quintuples. Note that $S_r \setminus S_2 = \{16, \ldots, 5r + 5\}$ when $r \equiv 2, 3 \pmod 4$, and $S_r \setminus S_2 = \{16, \ldots, 5r + 4\} \cup \{5r + 6\}$ when $r \equiv 0, 1 \pmod 4$.

For $r \equiv 2, 3 \pmod 4$, we take a Langford sequence of order $r - 2$ and defect 15, which exists since $r \geq 31$ (see [90, 91]), and use it to partition $\{15, \ldots, 3r+8\}$ into $r-2$ difference triples. We then add 1 to each element of each of these triples to obtain a partition of $\{16, \ldots, 3r+9\}$ into $r-2$ triples of the form $\{a, b, c\}$ where $a+b = c+1$. It is easy to construct the required partition of $\{16, \ldots, 5r+5\}$ into difference quintuples from this by partitioning $\{3r+10, \ldots, 5r+5\}$ into $r-2$ pairs of consecutive integers.

For $r \equiv 0, 1 \pmod 4$, we take a hooked Langford sequence of order $r-2$ and defect 15, which exists since $r \geq 31$ (see [90, 91]), and use it to partition $\{15, \ldots, 3r+7\} \cup \{3r+9\}$ into $r-2$ difference triples. We then add 1 to each element of each of these triples, except the element $3r+9$, to obtain a partition of $\{16, \ldots, 3r+9\}$ into $r-3$ triples of the form $\{a, b, c\}$ where $a+b = c+1$, and one triple of the form $\{a, b, c\}$ where $a+b = c+2$. It is easy to construct the required partition of $\{16, \ldots, 5r+4\} \cup \{5r+6\}$ into difference quintuples from this by partitioning $\{3r+10, \ldots, 5r+4\} \cup \{5r+6\}$ into $r-3$ pairs of consecutive integers, and the pair $\{5r+4, 5r+6\}$. The pair $\{5r+4, 5r+6\}$ combines with the triple of the form $\{a, b, c\}$ where $a+b = c+2$ to form a difference quintuple.

So we have a (5^{rn})-decomposition of $\langle S_r \rangle_n$, and require an (n^h)-decomposition of $K_n - \langle \{1, 2, 3, 4, 6\} \cup S_r \rangle_n$. Note that $K_n - \langle \{1, 2, 3, 4, 6\} \cup S_r \rangle_n$ is isomorphic to

- $\langle \{5r+6, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 2, 3 \pmod 4$; and
- $\langle \{5r+5\} \cup \{5r+7, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 0, 1 \pmod 4$.

If $r \equiv 2, 3 \pmod 4$, then the decomposition exists by Lemma 1.3.5. If $r \equiv 0, 1 \pmod 4$, then the decomposition exists by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even).

Case C: ($S = \{1, 2, 3, 4, 5, 7\}$)

The conditions $h \geq 2$ and $5r + h = \lfloor \frac{n-1}{2} \rfloor - 6$ imply $n \geq 10r + 17$. If $r = 0$, then the result follows immediately by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). Thus, we assume $r \geq 1$.

If $r = 1$ we define $S_1 = \{6, 8, 9, 10, 11, 12, 13\}$. Since $\{6, 8, 9, 10, 13\}$ is a difference quintuple and $\langle \{11, 12\} \rangle_n$ has a decomposition into Hamilton cycles (by Lemma 1.7.1), there is a $(5^n, n^2)$-decomposition of $\langle S_1 \rangle_n$. If $h = 2$, then we are finished. Otherwise, we use Lemma 1.7.1 to obtain an (n^{h-2})-decomposition of $\langle 14, \ldots, \lfloor \frac{n}{2} \rfloor \rangle_n$ and we are finished.

If $r \geq 2$, define S_r by $S_r = \{6\} \cup \{8, \ldots, 5r+6\}$ when $r \equiv 1, 2 \pmod 4$, and $S_r = \{6\} \cup \{8, \ldots, 5r+5\} \cup \{5r+7\}$ when $r \equiv 0, 3 \pmod 4$. We now obtain a (5^{rn})-decomposition $\langle S_r \rangle_n$ by partitioning S_r into difference quintuples. We have constructed such a partition of S for $1 \leq r \leq 32$ with the aid of a computer, shown in Table A.29 in the appendix, and thus assume $r \geq 33$.

For $r \geq 33$ we first partition S_r into $\{6\} \cup \{8, \ldots, 16\} = S_2$ and $S_r \setminus S_2$. We have already noted that S_2 can be partitioned into difference quintuples, so we only need to partition $S_r \setminus S_2$ into difference quintuples. Note that $S_r \setminus S_2 = \{17, \ldots, 5r+6\}$ when $r \equiv 1, 2 \pmod 4$, and $S_r \setminus S_2 = \{17, \ldots, 5r+5\} \cup \{5r+7\}$ when $r \equiv 0, 3 \pmod 4$.

For $r \equiv 1, 2 \pmod 4$, we take a Langford sequence of order $r-2$ and defect 16, which exists since $r \geq 33$ (see [90, 91]), and use it to partition $\{16, \ldots, 3r+9\}$ into $r-2$ difference triples. We then add 1 to each element of each of these triples to obtain a partition of $\{17, \ldots, 3r+10\}$ into $r-2$ triples of the form $\{a, b, c\}$ where $a+b = c+1$. It is easy to construct the required partition of $\{17, \ldots, 5r+6\}$ into difference quintuples from this by partitioning $\{3r+11, \ldots, 5r+6\}$ into $r-2$ pairs of consecutive integers.

For $r \equiv 0, 3 \pmod 4$, we take a hooked Langford sequence of order $r-2$ and defect 16, which exists since $r \geq 33$ (see [90, 91]), and use it to partition $\{16, \ldots, 3r+8\} \cup \{3r+10\}$ into $r-2$ difference triples. We then add 1 to each element of each of these triples, except the element $3r+10$, to obtain a partition of $\{17, \ldots, 3r+10\}$ into $r-3$ triples of the form $\{a, b, c\}$ where $a+b=c+1$, and one triple of the form $\{a, b, c\}$ where $a+b=c+2$. It is easy to construct the required partition of $\{17, \ldots, 5r+5\} \cup \{5r+7\}$ into difference quintuples from this by partitioning $\{3r+11, \ldots, 5r+5\} \cup \{5r+7\}$ into $r-3$ pairs of consecutive integers, and the pair $\{5r+5, 5r+7\}$. The pair $\{5r+5, 5r+7\}$ combines with the triple of the form $\{a, b, c\}$ where $a+b=c+2$ to form a difference quintuple.

So we have a (5^{rn})-decomposition of $\langle S_r \rangle_n$, and require an (n^h)-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_r \rangle_n$. Note that $K_n - \langle \{1, 2, 3, 4, 5, 7\} \cup S_r \rangle_n$ is isomorphic to

- $\langle \{5r+7, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 1, 2 \pmod 4$; and
- $\langle \{5r+6\} \cup \{5r+8, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 0, 3 \pmod 4$.

If $r \equiv 1, 2 \pmod 4$, then the decomposition exists by Lemma 1.3.5. If $r \equiv 0, 3 \pmod 4$, then the decomposition exists by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even).

Case D: ($S = \{1, 2, 3, 4, 5, 6, 7\}$)

The conditions $h \geq 2$ and $5r + h = \lfloor \frac{n-1}{2} \rfloor - 7$ imply $n \geq 10r + 19$. If $r = 0$, then the result follows immediately by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). Thus, we assume $r \geq 1$.

If $r = 1$ we define $S_1 = \{8, 9, 10, 11, 12, 13, 14\}$. Since $\{8, 9, 10, 13, 14\}$ is a difference quintuple and $\langle \{11, 12\} \rangle_n$ has a decomposition into Hamilton cycles (by Lemma 1.7.1), there is a $(5^n, n^2)$-decomposition of $\langle S_1 \rangle_n$. If $h = 2$, then we are finished. Otherwise, we use Lemma 1.7.1 to obtain an (n^{h-2})-decomposition of $\langle 14, \ldots, \lfloor \frac{n}{2} \rfloor \rangle_n$ and we are finished.

If $r \geq 2$, define S_r by $S_r = \{8, \ldots, 5r+7\}$ when $r \equiv 0, 1 \pmod 4$, and $S_r = \{8, \ldots, 5r+6\} \cup \{5r+8\}$ when $r \equiv 2, 3 \pmod 4$. We now obtain a (5^{rn})-decomposition $\langle S_r \rangle_n$ by partitioning S_r into difference quintuples. We have constructed such a partition of S for $2 \leq r \leq 12$ with the aid of a computer, shown in Table A.30 in the appendix, and thus we assume $r \geq 13$.

For $r \equiv 0, 1 \pmod 4$, we take a Langford sequence of order r and defect 7, which exists since $r \geq 13$ (see [90, 91]), and use it to partition $\{7, \ldots, 3r+6\}$ into r difference triples. We then add 1 to each element of each of these triples to obtain a partition of $\{8, \ldots, 3r+7\}$ into r triples of the form $\{a, b, c\}$ where $a+b=c+1$. It is easy to construct the required partition of $\{8, \ldots, 5r+7\}$ into difference quintuples from this by partitioning $\{3r+8, \ldots, 5r+7\}$ into r pairs of consecutive integers.

For $r \equiv 2, 3 \pmod 4$, we take a hooked Langford sequence of order r and defect 7, which exists since $r \geq 13$ (see [90, 91]), and use it to partition $\{7, \ldots, 3r+5\} \cup \{3r+7\}$ into r difference triples. We then add 1 to each element of each of these triples, except the element $3r+7$, to obtain a partition of $\{8, \ldots, 3r+7\}$ into $r-1$ triples of the form $\{a, b, c\}$ where $a+b=c+1$, and one triple of the form $\{a, b, c\}$ where $a+b=c+2$. It is easy to construct the required partition of $\{8, \ldots, 5r+6\} \cup \{5r+8\}$ into difference quintuples from this by partitioning $\{3r+8, \ldots, 5r+6\} \cup \{5r+8\}$ into $r-1$ pairs of consecutive integers, and the pair $\{5r+6, 5r+8\}$. The pair $\{5r+6, 5r+8\}$ combines with the triple of the form $\{a, b, c\}$ where $a+b=c+2$ to form a difference quintuple.

So we have a (5^{rn})-decomposition of $\langle S_r \rangle_n$, and require an (n^h)-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 6, 7\} \cup S_r \rangle_n$. Note that $K_n - \langle \{1, 2, 3, 4, 5, 6, 7\} \cup S_r \rangle_n$ is isomorphic to

- $\langle\{5r+8,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $r \equiv 0,1 \pmod 4$; and
- $\langle\{5r+7\} \cup \{5r+9,\ldots,\lfloor\frac{n}{2}\rfloor\}\rangle_n$ when $r \equiv 2,3 \pmod 4$.

If $r \equiv 0,1 \pmod 4$, then the decomposition exists by Lemma 1.3.5. If $r \equiv 2,3 \pmod 4$, then the decomposition exists by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even).

Case E: ($S = \{1,2,3,4,5,6,7,8\}$)

The conditions $h \geq 2$ and $5r + h = \lfloor\frac{n-1}{2}\rfloor - 8$ imply $n \geq 10r + 21$. If $r = 0$, then the result follows immediately by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). Thus, we assume $r \geq 1$.

For $r = 1$ and $n \in \{31,32,33,34\}$, we firstly have the following results. For $n = 31$, we can obtain the required decomposition by noting that $\{9,11,13,14,15\}$ is a modulo 31 difference quintuple and that $\langle\{10,12\}\rangle_n$ has a Hamilton cycle decomposition by Lemma 1.7.1.

For $n = 32$, we can obtain the required decomposition by noting that $\{10,12,13,14,15\}$ is a modulo 32 difference quintuple and that $\langle\{9,11\}\rangle_n$ has a Hamilton cycle decomposition by Lemma 1.7.1.

For $n \in \{33,34\}$, we can obtain the required decomposition by noting that $\{9,10,12,15,16\}$ is a difference quintuple, that $\langle\{11,14\}\rangle_n$ has a Hamilton cycle decomposition by Lemma 1.7.1, and $\langle\{13\}\rangle_n$ is a Hamilton cycle.

For $r = 1$ and $n \geq 35$, we define $S_1 = \{9,\ldots,17\}$. Since $\{9,10,11,14,16\}$ is a difference quintuple and each of $\langle\{12,13\}\rangle_n$ and $\langle\{15,17\}\rangle_n$ has a decomposition into Hamilton cycles (by Lemma 1.7.1), there is a $(5^n, n^4)$-decomposition of $\langle S_1\rangle_n$. If $h = 4$, then we are finished. Otherwise, we use Lemma 1.7.1 to obtain an (n^{h-4})-decomposition of $\langle 18,\ldots,\lfloor\frac{n}{2}\rfloor\rangle_n$ and we are finished.

For $r = 2$, we define $S_2 = \{9,\ldots,20\}$. Since $\{9,11,14,16,18\}$ and $\{10,12,13,15,20\}$ are both difference quintuples and $\langle\{17,19\}\rangle_n$ has a decomposition into Hamilton cycles (by Lemma 1.7.1), there is a $(5^{2n}, n^2)$-decomposition of $\langle S_2\rangle_n$. If $h = 2$, then we are finished. Otherwise, we use Lemma 1.7.1 to obtain an (n^{h-2})-decomposition of $\langle 18,\ldots,\lfloor\frac{n}{2}\rfloor\rangle_n$ and we are finished.

For $r \geq 3$, define S_r by $S_r = \{9,\ldots,5r+8\}$ when $r \equiv 0,3 \pmod 4$, and $S_r = \{9,\ldots,5r+7\} \cup \{5r+9\}$ when $r \equiv 1,2 \pmod 4$. We now obtain a (5^{rn})-decomposition $\langle S_r\rangle_n$ by partitioning S_r into difference quintuples. We have constructed such a partition of S for $3 \leq r \leq 14$ with the aid of a computer, shown in Table A.31 in the appendix, and thus assume $r \geq 15$.

For $r \equiv 0,3 \pmod 4$, we take a Langford sequence of order r and defect 8, which exists since $r \geq 15$ (see [90, 91]), and use it to partition $\{8,\ldots,3r+7\}$ into r difference triples. We then add 1 to each element of each of these triples to obtain a partition of $\{9,\ldots,3r+8\}$ into r triples of the form $\{a,b,c\}$ where $a+b = c+1$. It is easy to construct the required partition of $\{9,\ldots,5r+8\}$ into difference quintuples from this by partitioning $\{3r+9,\ldots,5r+8\}$ into r pairs of consecutive integers.

For $r \equiv 1,2 \pmod 4$, we take a hooked Langford sequence of order r and defect 8, which exists since $r \geq 15$ (see [90, 91]), and use it to partition $\{8,\ldots,3r+6\} \cup \{3r+8\}$ into r difference triples. We then add 1 to each element of each of these triples, except the element $3r+8$, to obtain a partition of $\{9,\ldots,3r+8\}$ into $r-1$ triples of the form $\{a,b,c\}$ where $a+b = c+1$, and one triple of the form $\{a,b,c\}$ where $a+b = c+2$. It is easy to construct the required partition of $\{9,\ldots,5r+7\} \cup \{5r+9\}$ into difference quintuples from this by partitioning $\{3r+9,\ldots,5r+7\} \cup \{5r+8\}$ into $r-1$ pairs of consecutive integers, and the pair $\{5r+7,5r+9\}$. The pair $\{5r+7,5r+9\}$ combines with the triple of the form $\{a,b,c\}$ where

$a + b = c + 2$ to form a difference quintuple.

So we have a (5^{rn})-decomposition of $\langle S_r \rangle_n$, and require an (n^h)-decomposition of $K_n - \langle \{1, 2, 3, 4, 5, 6, 7, 8\} \cup S_r \rangle_n$. Note that $K_n - \langle \{1, 2, 3, 4, 5, 6, 7, 8\} \cup S_r \rangle_n$ is isomorphic to

- $\langle \{5r + 9, \ldots, \lfloor \frac{n}{2} \rfloor \} \rangle_n$ when $r \equiv 0, 3 \pmod 4$; and
- $\langle \{5r + 8\} \cup \{5r + 10, \ldots, \lfloor \frac{n}{2} \rfloor \} \rangle_n$ when $r \equiv 1, 2 \pmod 4$.

If $r \equiv 0, 3 \pmod 4$, then the decomposition exists by Lemma 1.3.5. If $r \equiv 1, 2 \pmod 4$, then the decomposition exists by Lemma 1.7.4 (n odd) or Lemma 1.7.5 (n even). $\square$

Chapter 2

Bipartite 2-factorizations of complete multipartite graphs

2.1 Introduction

A spanning subgraph of a graph is called a *factor*, a k-regular factor is called a *k-factor*, and a decomposition into edge-disjoint k-factors is called a *k-factorisation*. This chapter is concerned with 2-factorisations of complete multipartite graphs in which the 2-factors are all isomorphic to a given 2-factor. We shall refer to this problem as the *Oberwolfach Problem for complete multipartite graphs*, because it is a natural extension from complete graphs to complete multipartite graphs of the well-known Oberwolfach Problem, which arose out of a seating arrangement problem posed by Ringel at a graph theory meeting in Oberwolfach in 1967. The Oberwolfach Problem for complete multipartite graphs has been studied previously and we shall discuss known results shortly. The purpose of this chapter is to give a complete solution (see Theorem 2.3.5) in the case where the given 2-factor is bipartite (equivalently, where the given 2-factor is a disjoint union of cycles of even length).

The complete multipartite graph with r parts of cardinalities $s_1, s_2, \ldots, s_r$ is denoted by $K_{s_1,s_2,\ldots,s_r}$, and the notation K_{s^r} is used rather than $K_{s_1,s_2,\ldots,s_r}$ when $s_1 = s_2 = \cdots = s_r = s$. The 2-regular graph consisting of t disjoint cycles of lengths $m_1, m_2, \ldots, m_t$ will be denoted by $[m_1, m_2, \ldots, m_t]$, and exponents may be used to indicate multiple cycles of the same length. For example, $[4, 4, 6, 6, 6, 10]$ may be denoted by $[4^2, 6^3, 10]$. Throughout this chapter, the meaning of any notation involving an exponent is as defined in this paragraph.

A 2-factorisation in which each 2-factor is a single cycle is a *Hamilton decomposition*. Auerbach and Laskar [70] proved in 1976 that a complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.

Theorem 2.1.1. ([70]) *A complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.*

The complete multipartite graph with n parts each consisting of a single vertex is the complete graph on n vertices which is denoted by K_n. The problem of finding a 2-factorisation of K_n in which the 2-factors are isomorphic to a given 2-factor F is the Oberwolfach Problem. The Oberwolfach Problem has been completely settled for infinitely many values of n [40], when F consists of cycles of uniform length [8], and in many other special cases. The known results on the Oberwolfach Problem up to 2007 can be found in the survey [39], and several new results appearing after [39] was published are cited in the introduction of [26].

If n is even, then K_n has odd degree and no 2-factorisation exists. However, if F is any given 2-regular graph on n vertices where n is even, then one may ask instead for a factorisation of K_n into $\frac{n-2}{2}$ copies of F and a 1-factor. The Oberwolfach Problem is now usually considered to include this problem, and solutions are equivalent to 2-factorisations of the complete multipartite graph with $\frac{n}{2}$ parts of cardinality 2. The status of the problem is similar to that of the Oberwolfach Problem for n odd (see the survey [39] and the references cited in [26]), with a notable exception being that the problem has been completely settled in all cases where F is bipartite [25, 58]. Of course, F is never bipartite when n is odd.

Theorem 2.1.2. ([25, 58]) *If F is a bipartite 2-regular graph of order $2r$, then the complete multipartite graph K_{2^r} has a 2-factorisation into F.*

Piotrowski [82] has completely settled the Oberwolfach Problem for complete bipartite graphs. Obviously, the 2-factors are necessarily bipartite in this problem.

Theorem 2.1.3. ([82]) *If F is a bipartite 2-regular graph of order $2n$, then the complete bipartite graph $K_{n,n}$ has a 2-factorisation into F except when $n = 6$ and $F \cong [6, 6]$.*

The Oberwolfach Problem for complete multipartite graphs has also been completely settled, by Liu [73], for cases where the 2-factors consist of cycles of uniform length.

Theorem 2.1.4. ([73]) *The complete multipartite graph K_{n^r}, $r \geq 2$, has a 2-factorisation into 2-factors composed of k-cycles if and only if k divides rn, $(r-1)n$ is even, k is even when $r = 2$, and (k, r, n) is none of $(3, 3, 2)$, $(3, 6, 2)$, $(3, 3, 6)$, $(6, 2, 6)$.*

In Theorem 2.3.5, we generalise Theorems 2.1.2 and 2.1.3, completely settling the Oberwolfach Problem for complete multipartite graphs in the case of bipartite 2-factors.

2.2 Notation and preliminaries

Let Γ be a finite group. A *Cayley subset* of Γ is a subset which does not contain the identity and which is closed under taking of inverses. If S is a Cayley subset of Γ, then the *Cayley graph* on Γ with *connection set* S, denoted $\mathrm{Cay}(\Gamma, S)$, has the elements of Γ as its vertices and there is an edge between vertices g and h if and only if $g = h + s$ for some $s \in S$.

We need the following two results on Hamilton decompositions of Cayley graphs. The first was proved by Bermond et al [18], and the second by Dean [53]. Both results address the open question of whether every connected Cayley graph of even degree on a finite abelian group has a Hamilton decomposition [4].

Theorem 2.2.1. ([18]) *Every connected 4-regular Cayley graph on a finite abelian group has a Hamilton decomposition.*

Theorem 2.2.2. ([53]) *Every 6-regular Cayley graph on a cyclic group which has a generator of the group in its connection set has a Hamilton decomposition.*

A Cayley graph on a cyclic group is called a *circulant graph* and we will be using these, and certain subgraphs of them, frequently. Thus, we introduce the following notation. The *length* of an edge $\{x, y\}$ in a graph with vertex set $\mathbb{Z}_m$ is defined to be either $x - y$ or $y - x$, whichever is in $\{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$ (calculations in $\mathbb{Z}_m$). When m is even and $s \leq \frac{m-2}{2}$, we call $\{\{x, x+s\} : x = 0, 2, \ldots, m-2\}$ the *even edges of length* s and we call $\{\{x, x+s\} : x = 1, 3, \ldots, m-1\}$ the *odd edges of length* s. Note that elsewhere in the literature, the term "even (odd) edges" has sometimes been used for edges of even (odd) length.

For any $m \geq 3$ and any $S \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor\}$, we denote by $\langle S \rangle_m$ the graph with vertex set $\mathbb{Z}_m$ and edge set consisting of the edges of length s for each $s \in S$, that is, $\langle S \rangle_m = \mathrm{Cay}(\mathbb{Z}_m, S \cup -S)$. For m even, if we wish to include in our graph only the even edges of length s then we give s the superscript "e". Similarly, if we wish to include only the odd edges of length s then we give s the superscript "o". For example, the graph $\langle \{1, 2^{\mathrm{o}}, 5^{\mathrm{e}}\} \rangle_{12}$ is shown in Figure 2.1.

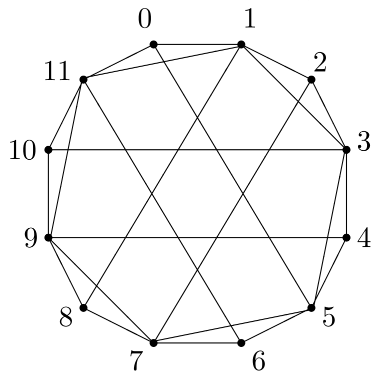


Figure 2.1: The graph $\langle \{1, 2^{\mathrm{o}}, 5^{\mathrm{e}}\} \rangle_{12}$

The *wreath product* $G \wr H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and edge set given by joining (g_1, h_1) to (g_2, h_2) precisely when g_1 is joined to g_2 in G or $g_1 = g_2$ and h_1 is joined to h_2 in H. We will be dealing frequently with the wreath product of a graph K and the empty graph with vertex set $\mathbb{Z}_2$, so we introduce the following special notation for this graph. The graph $K^{(2)}$ is defined by $V(K^{(2)}) = V(K) \times \mathbb{Z}_2$ and $E(K^{(2)}) = \{\{(x, a), (y, b)\} : \{x, y\} \in E(K),\ a, b \in \mathbb{Z}_2\}$. It is easy to see that $\mathrm{Cay}(\Gamma, S)^{(2)} \cong \mathrm{Cay}(\Gamma \times \mathbb{Z}_2, S \times \mathbb{Z}_2)$. If $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$ is a set of graphs, then we define $\mathcal{F}^{(2)} = \{F_1^{(2)}, F_2^{(2)}, \ldots, F_t^{(2)}\}$. Note that if $\mathcal{F}$ is a factorisation of K, then $\mathcal{F}^{(2)}$ is a factorisation of $K^{(2)}$.

Häggkvist [58] observed that for any bipartite 2-regular graph F on $2m$ vertices, there is a 2-factorisation of $C_m^{(2)}$ into two copies of F. The following very useful result, on which many of our constructions depend, is an immediate consequence of Häggkvist's observation and the fact that $\mathcal{F}^{(2)}$ is a factorisation of $K^{(2)}$ when $\mathcal{F}$ is a factorisation of K. If $\mathcal{F}$ is a Hamilton decomposition of K, then we obtain a 4-factorisation of $K^{(2)}$ into copies of $C_m^{(2)}$ (where m is the number of vertices in K), and we then obtain the required 2-factorisation of $K^{(2)}$ by factorising each copy of $C_m^{(2)}$ into two copies of the required bipartite 2-regular graph.

Lemma 2.2.3. ([58]) *If there is a Hamilton decomposition of K, then for each bipartite 2-regular graph F of order $|V(K^{(2)})|$, there is a 2-factorisation of $K^{(2)}$ into F.*

2.3 Main Result

We begin this section with two results on factorisations of K_{m^r} in cases where K_{m^r} has odd degree. Note that K_{m^r} has odd degree if and only if m is odd and r is even. In Lemma 2.3.1 the factorisation is into Hamilton cycles and a 3-factor isomorphic to $\langle \{1, 3^{\mathrm{e}}\} \rangle_{rm}$, and in Lemma 2.3.2 the factorisation is into Hamilton cycles and a 5-factor isomorphic to $\langle \{1, 2, 3^{\mathrm{e}}\} \rangle_{rm}$. These factorisations are used in the proof of Theorem 2.3.5.

Lemma 2.3.1. *For each even $r \geq 4$ and each odd $m \geq 1$, except $(r, m) = (4, 1)$, there is a factorisation of K_{m^r} into $\frac{(r-1)m-3}{2}$ Hamilton cycles and a copy of $\langle\{1, 3^e\}\rangle_{rm}$.*

Proof First observe that $K_{m^r} \cong \langle\{1, 2, \ldots, \frac{rm}{2}\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$. The cases $rm \equiv 0 \pmod 4$ and $rm \equiv 2 \pmod 4$ are dealt with separately. For $rm \equiv 2 \pmod 4$ it is easy to verify that the mapping $\psi : \mathbb{Z}_{rm} \mapsto \mathbb{Z}_{rm}$ given by

$$\psi(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod 4 \\ \frac{rm}{2} + \lfloor \frac{x}{2} \rfloor & \text{if } x \equiv 1, 2 \pmod 4 \\ \frac{x-1}{2} & \text{if } x \equiv 3 \pmod 4 \end{cases}$$

is an isomorphism from $\langle\{1, 3^e\}\rangle_{rm}$ to $\langle\{1, \frac{rm}{2}\}\rangle_{rm}$. So in the case $rm \equiv 2 \pmod 4$ it is sufficient to show that $\langle\{2, 3, \ldots, \frac{rm}{2} - 1\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ has a Hamilton decomposition.

Consider the sequence $S = s_1, s_2, \ldots, s_t$ (where $t = (rm - m - 3)/2$) whose terms are the elements of

$$\{2, 3, \ldots, \tfrac{rm}{2} - 1\} \setminus \{r, 2r, \ldots, \tfrac{m-1}{2}r\}$$

arranged in ascending order. Note that since r is even, consecutive terms in S are relatively prime. Thus, if a and b are consecutive terms in S, then $\langle\{a, b\}\rangle_{rm}$ is connected and thus has a Hamilton decomposition by Theorem 2.2.1. Also, since we are in the case $rm \equiv 2 \pmod 4$, we have $s_{t-1} = \frac{rm}{2} - 2$ and $\gcd(\frac{rm}{2} - 2, rm) = 1$. Thus, $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 2.2.2.

In view of the arguments in the preceding paragraph, we can obtain the required Hamilton decomposition of $\langle\{2, 3, \ldots, \frac{rm}{2} - 1\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ by factoring it into Hamilton decomposable 4-regular graphs of the form $\langle\{a, b\}\rangle_{rm}$ where a and b are consecutive terms of S, and, in the case where the number of terms of S is odd, the Hamilton decomposable 6-regular graph $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$.

Now consider the case $rm \equiv 0 \pmod 4$. It is easy to see that $\langle\{2, 3^o, \frac{rm}{2}\}\rangle_{rm} \cong \mathrm{Cay}(\mathbb{Z}_{\frac{rm}{2}} \times \mathbb{Z}_2, \{(1, 0), (\frac{rm}{4}, 0), (0, 1)\})$, and that this graph is connected. It follows that $\langle\{2, 3^o, \frac{rm}{2}\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 2.2.1. Thus, it is sufficient to show that $\langle\{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ has a Hamilton decomposition. Redefine $S = s_1, s_2, \ldots, s_t$ to be the sequence whose terms are the elements of $\{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}$ arranged in ascending order (so t is now $(rm - m - 7)/2$). As before, consecutive terms in S are relatively prime.

Since we are in the case $rm \equiv 0 \pmod 4$, we have $\gcd(\frac{rm}{2} - 1, rm) = 1$, which means that $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 2.2.2. We can thus obtain the required Hamilton decomposition of $\langle\{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ by factoring it into Hamilton decomposable 4-regular graphs of the form $\langle\{a, b\}\rangle_{rm}$ where a and b are consecutive terms of S, and, in the case where the number of terms of S is odd, the Hamilton decomposable 6-regular graph $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$. □

Lemma 2.3.2. *For each even $r \geq 4$ and each odd $m \geq 3$ such that $rm \equiv 8 \pmod{12}$, there is a factorisation of K_{m^r} into $\frac{(r-1)m-5}{2}$ Hamilton cycles and a copy of $\langle\{1, 2, 3^e\}\rangle_{rm}$.*

Proof Since $K_{m^r} \cong \langle\{1, 2, \ldots, \frac{rm}{2}\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$, it is sufficient to show that there is a Hamilton decomposition of $\langle\{3^o, 4, 5, \ldots, \frac{rm}{2}\} \setminus \{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$. Note that neither 6 nor $\frac{rm}{2}$ is in $\{r, 2r, \ldots, \frac{m-1}{2}r\}$. Now, it is easy to see that

$$\langle\{3^o, 6, \tfrac{rm}{2}\}\rangle_{rm} \cong \mathrm{Cay}(\mathbb{Z}_{\frac{rm}{2}} \times \mathbb{Z}_2, \{(3, 0), (\tfrac{rm}{4}, 0), (0, 1)\})$$

and hence that $\langle\{3^o, 6, \frac{rm}{2}\}\rangle_{rm}$ is a connected 4-regular Cayley graph (connectedness follows from $\gcd(3, \frac{rm}{2}) = 1$). Thus, $\langle\{3^o, 6, \frac{rm}{2}\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 2.2.1, and it is sufficient to show that $\langle\{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{6, r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$ has a Hamilton decomposition.

Consider the sequence $S = s_1, s_2, \ldots, s_t$ (where $t = (rm - m - 9)/2$) whose terms are the elements of

$$\{4, 5, \ldots, \tfrac{rm}{2} - 1\} \setminus \{6, r, 2r, \ldots, \tfrac{m-1}{2}r\}$$

arranged in ascending order. Note that since r is even, consecutive terms in S are relatively prime. Thus, if a and b are consecutive terms in S, then $\langle\{a, b\}\rangle_{rm}$ is connected and thus has a Hamilton decomposition by Theorem 2.2.1. Also, $\gcd(s_t, rm) = 1$ (since $s_t = \frac{rm}{2} - 1$ is odd), and so $\langle\{s_t\}\rangle_{rm}$ is an rm-cycle.

In view of the preceding paragraph, we can obtain the required Hamilton decomposition of $\langle\{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{6, r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$ by factoring it into Hamilton decomposable 4-regular graphs of the form $\langle\{a, b\}\rangle_{rm}$ where a and b are consecutive terms of S, and, in the case where the number of terms of S is odd, the cycle $\langle\{s_t\}\rangle_{rm}$. □

We also need the following result from [26].

Lemma 2.3.3. ([26]) *Let $n \equiv 0 \pmod 4$ with $n \geq 12$. For each bipartite 2-regular graph F of order n, there is a factorisation of $\langle\{1, 3^e\}\rangle^{(2)}_{n/2}$ into three copies of F; except possibly when $F \in \{[6^r], [4, 6^r] : r \equiv 2 \pmod 4\}$.*

In the proof of Theorem 2.3.5, an alternate approach is required when F is one of the possible exceptions in Lemma 2.3.3. Cases where F is of the form $[6^r]$ are covered by Theorem 2.1.4, and the following result is used together with Lemma 2.3.2 to deal with cases where F is of the form $[4, 6^r]$.

Lemma 2.3.4. *For each $k \geq 1$, there is a factorisation of $\langle\{1, 2, 3^e\}\rangle^{(2)}_{12k+8}$ into five copies of $[4, 6^{4k+2}]$.*

Proof For any subgraph F of $\langle\{1, 2, 3^e\}\rangle^{(2)}_{12k+8}$ and any $t \in \{0, 2, \ldots, 12k+6\}$, let $F+t$ denote the subgraph of $\langle\{1, 2, 3^e\}\rangle^{(2)}_{12k+8}$ obtained by applying the permutation $(x, i) \mapsto (x+t, i)$. That is, $V(F+t) = \{(x+t, i) : (x, i) \in V(F)\}$ and $E(F+t) = \{(x+t, i)(y+t, j) : (x, i)(y, j) \in E(F)\}$. For each $x \in \mathbb{Z}_{12k+8}$ and each $i \in \mathbb{Z}_2$ denote the vertex (x, i) of $\langle\{1, 2, 3^e\}\rangle^{(2)}_{12k+8}$ by x_i.

The required 2-factorisation of $\langle\{1, 2, 3^e\}\rangle^{(2)}_{12k+8}$ is given by the following five 2-factors.

(1) $(0_0, 1_0, 0_1, 1_1) \cup (2_0, 3_0, 2_1, 4_0, 5_0, 3_1) \cup (4_1, 5_1, 7_0, 6_1, 7_1, 6_0) \cup (0_0, 1_0, 0_1, 2_0, 3_0, 1_1) + t \cup (2_1, 3_1, 4_1, 5_0, 6_0, 4_0) + t \cup (5_1, 6_1, 7_1, 8_0, 9_0, 7_0) + t \cup (8_1, 9_1, 11_0, 10_1, 11_1, 10_0) + t : t = 8, 20, 32, \ldots, 12k - 4$

(2) $(0_0, 2_0, 0_1, 2_1) \cup (1_1, 3_0, 4_0, 6_1, 5_1, 3_1) \cup (4_1, 5_0, 6_0, 7_0, 9_0, 7_1) \cup (0_0, 2_0, 1_1, 0_1, 3_0, 2_1) + t \cup (3_1, 4_0, 5_1, 4_1, 7_1, 5_0) + t \cup (6_0, 7_0, 6_1, 8_1, 10_1, 9_0) + t \cup (8_0, 9_1, 11_1, 13_0, 10_0, 11_0) + t : t = 8, 20, 32, \ldots, 12k - 4$

(3) $(0_0, 3_0, 1_0, 3_1) \cup (1_1, 2_0, 4_1, 6_1, 5_0, 2_1) \cup (4_0, 6_0, 5_1, 7_1, 8_1, 7_0) \cup (0_0, 3_0, 1_0, 2_1, 1_1, 3_1) + t \cup (2_0, 5_0, 7_0, 4_1, 6_0, 5_1) + t \cup (4_0, 6_1, 8_0, 10_1, 9_1, 7_1) + t \cup (8_1, 11_0, 9_0, 10_0, 12_1, 11_1) + t : t = 8, 20, 32, \ldots, 12k - 4$

(4) $(6_0, 9_0, 6_1, 9_1) \cup (0_1, 3_0, 5_1, 2_1, 4_1, 3_1) \cup (2_0, 4_0, 7_1, 8_0, 7_0, 5_0) \cup (0_1, 2_1, 5_0, 4_0, 2_0, 3_1)+t \cup (3_0, 4_1, 6_1, 9_0, 7_1, 5_1)+t \cup (6_0, 8_1, 7_0, 8_0, 10_0, 9_1)+t \cup (10_1, 12_0, 11_1, 13_1, 11_0, 13_0)+t : t = 8, 20, 32, \ldots, 12k-4$

(5) $(6_0, 8_0, 6_1, 8_1) \cup (1_0, 2_0, 5_1, 4_0, 3_1, 2_1) \cup (3_0, 4_1, 7_0, 9_1, 7_1, 5_0) \cup (1_0, 2_0, 4_1, 2_1, 5_1, 3_1)+t \cup (3_0, 4_0, 7_0, 9_1, 6_1, 5_0)+t \cup (6_0, 7_1, 8_1, 9_0, 11_1, 8_0)+t \cup (10_0, 12_0, 11_0, 12_1, 10_1, 13_1)+t : t = 8, 20, 32, \ldots, 12k-4$

□

We are now ready to prove our main result.

Theorem 2.3.5. *If F is a bipartite 2-regular graph of order rn, then there exists a 2-factorisation of K_{n^r}, $r \geq 2$, into F if and only if n is even; except that there is no 2-factorisation of $K_{6,6}$ into $[6,6]$.*

Proof A bipartite 2-regular graph has even order, so rn is even. Since a graph having a 2-factorisation is regular of even degree, if the 2-factorisation exists, then $(r-1)n$ (the degree of K_{n^r}) is even. This together with the fact that rn is even implies that n is even when the 2-factorisation of K_{n^r} exists, and it is known that there is no 2-factorisation of $K_{6,6}$ into $[6,6]$, see [73] or [82].

Now, conversely, let n be even and let $m = n/2$ so that $K_{n^r} \cong K_{m^r}^{(2)}$. If m is even or r is odd, then K_{m^r} has even degree, and hence has a Hamilton decomposition by Theorem 2.1.1. So the result follows by Lemma 2.2.3 when m is even or r is odd. The result has been proved when $r = 2$ (see Theorem 2.1.3) and when $n = 2$ (see Theorem 2.1.2). Thus, we can assume $m \geq 3$ is odd and $r \geq 4$ is even.

By Lemma 2.3.1, there is a factorisation of K_{m^r} into $\frac{(r-1)m-3}{2}$ Hamilton cycles and a copy of $\langle\{1, 3^e\}\rangle_{rm}$, and hence a factorisation of $K_{n^r} \cong K_{m^r}^{(2)}$ into $\frac{(r-1)m-3}{2}$ copies of $C_{rm}^{(2)}$ and a copy of $\langle\{1, 3^e\}\rangle_{rm}^{(2)}$. Each copy of $C_{rm}^{(2)}$ can be factored into two copies of F by Lemma 2.2.3, and the copy of $\langle\{1, 3^e\}\rangle_{rm}^{(2)}$ can be factored into three copies of F by Lemma 2.3.3; except when $F \in \{[6^r], [4, 6^r] : r \equiv 2 \pmod 4\}$. The case $F = [6^r]$ with $r \equiv 2 \pmod 4$ is covered by Theorem 2.1.4. Thus, the proof is complete except when $r \geq 4$ is even, $m = \frac{n}{2} \geq 3$ is odd, and $F = [4, 6^{4k+2}]$ for some $k \geq 1$ (where $rm = 12k + 8$). We now deal with this special case.

By Lemma 2.3.2, there is a factorisation of K_{m^r} into $\frac{(r-1)m-5}{2}$ Hamilton cycles and a copy of $\langle\{1, 2, 3^e\}\rangle_{rm}$, and hence a factorisation of $K_{n^r} \cong K_{m^r}^{(2)}$ into $\frac{(r-1)m-5}{2}$ copies of $C_{rm}^{(2)}$ and a copy of $\langle\{1, 2, 3^e\}\rangle_{rm}^{(2)}$. Each copy of $C_{rm}^{(2)}$ can be factored into two copies of F by Lemma 2.2.3, and the copy of $\langle\{1, 2, 3^e\}\rangle_{rm}^{(2)}$ can be factored into five copies of F by Lemma 2.3.4. This completes the proof. □

We remark that the method used in the proof of Theorem 2.3.5 can also be used to obtain 2-factorisations in which the 2-factors are not all isomorphic. In the proof, distinct copies of $C_{rm}^{(2)}$, and the copy of $\langle\{1, 3^e\}\rangle_{rm}^{(2)}$ or $\langle\{1, 2, 3^e\}\rangle_{rm}^{(2)}$, can each be factored independently into specified 2-factors as described in Lemma 2.2.3, Lemma 2.3.3, and Lemma 2.3.4.

Chapter 3

Combinatorial topology

3.1 Introduction

Combinatorial topology is the study of discrete, combinatorial representations of topological spaces. In the rest of this thesis, we look at combinatorial 3-manifold topology. Chapters 4 and 5 take two different approaches to the problem of census enumeration. Many definitions are shared between the two approaches, and we introduce these here.

We cover the basic definitions of 3-manifold triangulations, as well as their related graph structures (like dual 1-skeletons). We also discuss some properties of 3-manifolds, and detail the existing state-of-the-art algorithm for enumerating a census of 3-manifolds. For a more in-depth introduction to topology see [59].

3.2 Definitions and notation

In both combinatorial topology, and graph theory, the terms edge and vertex have distinct meanings. Therefore for the rest of this thesis, the terms *edge* and *vertex* will be used to mean an edge or vertex in a triangulation or manifold and the terms *arc* and *node* will be used to mean an edge or vertex in a graph respectively.

A 3-manifold is a topological Hausdorff space that locally looks like either 3-dimensional Euclidean space (i.e., $\mathbb{R}^3$) or closed 3-dimensional Euclidean half-space (i.e., $\mathbb{R}^3_{z\geq 0}$). For the purposes of this thesis, all 3-manifolds will be connected and compact. When we refer to faces, we are explicitly talking about 2-faces (i.e., facets of a 3-manifold or 3-simplex depending on context). We represent 3-manifolds combinatorially as triangulations [80]: a collection of tetrahedra (3-simplices) with some 2-faces pairwise identified. We can also represent 3-manifolds combinatorially as spines. We briefly touch on this idea in Chapter 5; for more detail see [77]. We begin by defining a general triangulation (which may or may not represent a 3-manifold), which we later restrict to a 3-manifold triangulation.

Definition 3.2.1. *A* general triangulation *is a collection $\Delta_1, \Delta_2, \ldots, \Delta_n$ of n abstract tetrahedra, along with some affine bijections $\pi_1, \pi_2, \ldots, \pi_m$ where each bijection π_i is an affine map between two distinct triangular faces of tetrahedra, and each face of each tetrahedron is in at most one such bijection.*

We call these affine bijections *face identifications* or simply *identifications*. Note that there are

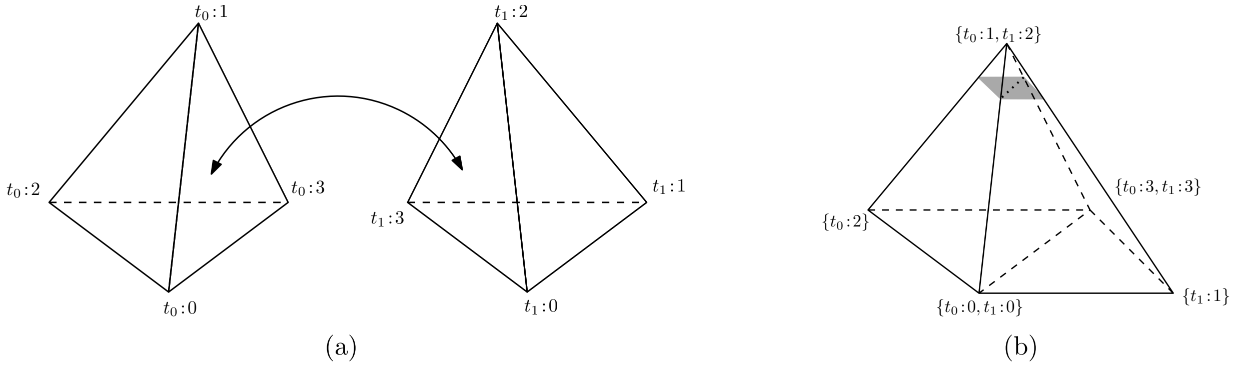


Figure 3.1: (a) A general triangulation with one face identification, shown as two distinct tetrahedra with an arrow indicating the two faces which are identified; and (b) the combined triangulation. The grey rectangle in (b) indicates the link of the vertex $\{t_0\!:\!1, t_1\!:\!2\}$.

six ways to identify two faces, given by the six symmetries of a regular triangle. Also note that this is more general than a simplicial complex (e.g., we allow an identification between two distinct faces of the same tetrahedron).

Only if the quotient space of such a triangulation is a 3-manifold will we say that the triangulation represents a 3-manifold. Later we show that we do not have to explicitly construct the quotient space to determine whether a triangulation represents a 3-manifold.

Notation 3.2.2. We use the notation $t_i : a$ to denote vertex a of tetrahedron t_i, and $t_i : abc$ to denote unique face containing the vertices a, b and c of tetrahedron t_i. Given tetrahedron vertices a and b, we will write $a \leftrightarrow b$ to denote that vertex a is identified with vertex b. Face identifications are denoted as $t_i : abc \leftrightarrow t_j : def$, which means that face abc of t_i is mapped to face def of t_j such that $a \leftrightarrow d$, $b \leftrightarrow e$ and $c \leftrightarrow f$.

Example 3.2.3. Take two tetrahedra and apply the face identification $t_1\!:\!031 \leftrightarrow t_1\!:\!032$. The resulting triangulation is topologically equivalent to a 3-ball. Figure 3.1(a) shows this triangulation, with the arrow indicating two faces being identified. The precise identification involved is not displayed in the diagram, however. Figure 3.1(b) shows the resulting triangulation.

As a result of the identification of various faces, some edges or vertices of various tetrahedra are identified together. These identifications may be the direct result of a single face identification, or may be the result of multiple face identifications.

We assign an arbitrary orientation to each edge of each tetrahedron. Given two tetrahedron edges e and e' that are identified together via the face identifications, we write $e \simeq e'$ if the orientations agree, and $e \simeq \overline{e'}$ if the orientations are reversed. In settings where we are not interested in orientation, we write $e \sim e'$ if the two edges are identified (i.e., one of $e \simeq e'$ or $e \simeq \overline{e'}$ holds).

This leads to the natural notation $[e] = \{e' : e \sim e'\}$ as an equivalence class of identified edges (ignoring orientation). We refer to $[e]$ as an *edge of the triangulation*. Likewise, we use the notation $v \sim v'$ for vertices of tetrahedra that are identified together via the face identifications, and we call an equivalence class $[v]$ of identified vertices a *vertex of the triangulation*.

We then define the degree of an edge of the triangulation, denoted $\deg(e)$, to be the number of edges of tetrahedra in the equivalence class $[e]$.

Any face of a triangulation which is not identified with any another face is called a *boundary face* of the triangulation. We then define the boundary of a triangulation T, denoted ∂T, as the union of all boundary faces of T. A *boundary edge* (respectively *boundary vertex*) of a triangulation is an edge (respectively vertex) of the triangulation whose equivalence class contains some edge (respectively vertex) which itself is contained within a boundary face. That is, the entire edge (respectively vertex) is contained within ∂T.

We can partially represent a triangulation by its face pairing graph (also known as its dual 1-skeleton), which describes *which* faces are identified together, but not *how* they are identified. In Chapter 5 we will show how to extend this definition to completely represent a triangulation.

Definition 3.2.4. *The* face pairing graph *of a triangulation $\mathcal{T}$ is the multigraph $\Gamma(\mathcal{T})$ constructed as follows. Start with an empty graph G, and insert one node for every tetrahedron in $\mathcal{T}$. For every identification π_i between a face of tetrahedron T_i and a face of tetrahedron T_j, insert one arc between the nodes corresponding to T_i and T_j into the graph G.*

Remark 3.2.5. A triangulation $\mathcal{T}$ is connected in the topological sense if and only if $\Gamma(\mathcal{T})$ is connected in the graph theoretic sense. This is not true in more general settings (i.e., simplicial complexes).

Note that a face pairing graph will have parallel arcs if there are two or more distinct face identifications between T_i and T_j, and loops if two faces of the same tetrahedron are identified together. An example of a face pairing graph is given in Figure 3.3.

We also need to define the *link* of a vertex before we can discuss 3-manifold triangulations.

Definition 3.2.6. *Given a vertex v in some triangulation, the link of v, denoted* $\mathrm{Link}(v)$*, is the (2-dimensional) frontier of a small regular neighbourhood of v.*

Figure 3.1(b) shows the link of a vertex in grey. In this case, the link is homeomorphic to a disc. We now give detail the properties a general triangulation must have to represent a 3-manifold.

Lemma 3.2.7. *A general triangulation is a* 3-manifold triangulation *if the following additional conditions hold:*

- *the triangulation is connected;*
- *the link of any vertex in the triangulation is homeomorphic to either a 2-sphere or a disc; and*
- *no edge in the triangulation is identified with itself in reverse.*

It is both well known and routine to check that these conditions are both necessary and sufficient for the underlying topological space to be a 3-manifold.

Example 3.2.8. Figure 3.2 shows how two tetrahedra may have faces identified together to form a triangulation of a 3-sphere. Each tetrahedron has two of its own faces identified together, and the other two faces identified with two faces from the other tetrahedron. The exact identifications are as follows.

$$\begin{array}{ll} t_0\!:\!103 \leftrightarrow t_1\!:\!102 & t_0\!:\!102 \leftrightarrow t_1\!:\!302 \\ t_0\!:\!132 \leftrightarrow t_0\!:\!032 & t_1\!:\!231 \leftrightarrow t_1\!:\!031 \end{array}$$

The associated face pairing graph is shown in Figure 3.3.

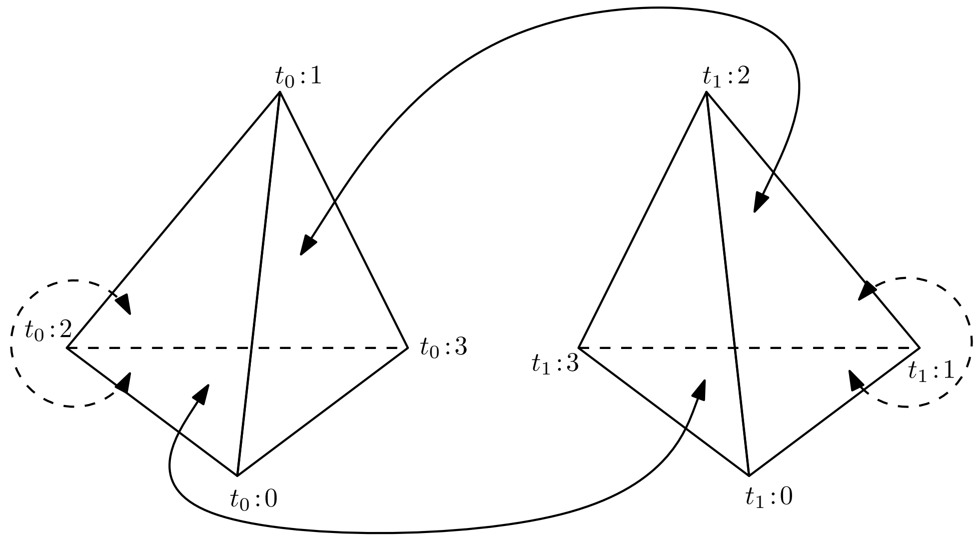


Figure 3.2: A visual representation of the 3-sphere triangulation described in Example 3.2.8. The arrows indicate which faces are identified together with dashed arrows referring to the "back" faces. Note that some of the identifications involve rotations or flips which are not shown in the figure.

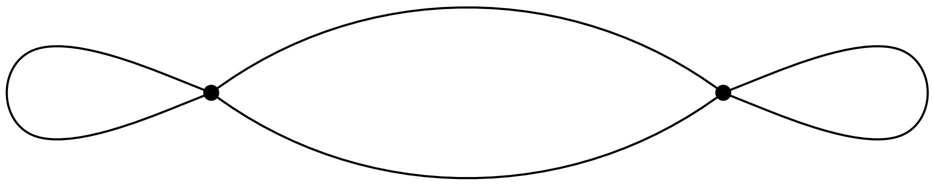


Figure 3.3: The face pairing graph of the 3-sphere triangulation from Example 3.2.8.

Note that in a 3-manifold triangulation, if a vertex has a link homeomorphic to a 2-sphere then that vertex cannot lie on the boundary of the triangulation. Similarly, if a vertex has a link homeomorphic to a disc then that vertex must lie on the boundary of a triangulation.

We will later deal with 3-manifolds with various properties which we define here. Most of these properties are properties of 3-manifolds and so we will also say that a triangulation has a given property if and only if its associated 3-manifold has said property.

Definition 3.2.9. *A 3-manifold triangulation is* closed *if it has no boundary faces.*

Definition 3.2.10. *A 3-manifold triangulation is* orientable *if every tetrahedron can be assigned an orientation (i.e., ± 1) such that if two tetrahedra of consistent orientation have faces identified then the identification is orientation preserving.*

Often one prefers to work with the "most simple" object exhibiting some particular behaviour. Since we are dealing with triangulations of 3-manifolds, we can define this simplicity both in terms of the triangulation and the manifold. Firstly, we define a notion of minimality of the triangulation.

Definition 3.2.11. *A 3-manifold triangulation of a manifold $\mathcal{M}$ is* minimal *if $\mathcal{M}$ cannot be triangulated with fewer tetrahedra.*

Minimal triangulations are well studied, both for their relevance to computation and for their applications in zero-efficient triangulations [65]. It is also known that the minimal number of tetrahedra required to triangulate a manifold is equal to the "Matveev complexity" [77] of said manifold (with exceptions for the 3-manifolds S^3, RP^3 and $L_{3,1}$).

Next we give three properties of 3-manifolds which all convey some notion of simplicity of the manifold (as opposed to the triangulation).

Definition 3.2.12. *A 3-manifold $\mathcal{M}$ is* irreducible *if every embedded 2-sphere in $\mathcal{M}$ bounds a 3-ball in $\mathcal{M}$.*

Definition 3.2.13. *A 3-manifold $\mathcal{M}$ is* prime *if it cannot be written as a connected sum of two manifolds where neither is a 3-sphere.*

Definition 3.2.14. *A 3-manifold is* $\mathbb{P}^2$-irreducible *if it is irreducible and also contains no embedded two-sided projective plane.*

Prime manifolds are the obvious manifold to work with. We note that prime 3-manifolds are either irreducible, or are one of the orientable direct product $S^2 \times S^1$ or the non-orientable twisted product $S^2 \tilde{\times} S^1$. As these are both well known and have triangulations on two tetrahedra, for any census of minimal triangulations on three or more tetrahedra we can look for irreducible 3-manifolds when we want prime manifolds. Any non-prime manifold can be constructed from a connected sum of prime manifolds, so enumerating prime manifolds is sufficient for most purposes. A similar (but more complicated) notion holds for $\mathbb{P}^2$-irreducible manifolds. As such, minimal prime $\mathbb{P}^2$-irreducible triangulations form the basic building blocks in combinatorial topology.

We now given some results on the links of vertices in various triangulations and manifolds. These results are well known, and are given for completeness. First, however, we need the following definition.

Definition 3.2.15. *The* Euler characteristic *of a triangulation is topological invariant, denoted as χ. For triangulations it can be calculated as $\chi = V - E + F - T$ where V, E, F and T are the number of vertices, edges, faces and tetrahedra in the triangulation respectively. For 2-dimensional triangulations, $\chi = V - E + F$.*

For the following proofs, we also briefly need the Euler characteristic of a cell decomposition[1]. We omit the technical details, but χ can be calculated as $\sum_i (-1)^i k_i$ where k_i is the number of i-cells in the decomposition.

It is well known by the classification of 2-manifolds ([57]) that $\chi \leq 2$ for any surface and that $\chi = 2$ if and only if the the surface is a 2-sphere. Additionally, a closed 3-manifold (that is, a compact 3-manifold with no boundary) has an Euler characteristic of zero (by Poincaré duality, see [59]).

The following lemmas will help determine the form of links in various triangulations.

Lemma 3.2.16. *Take two triangulations L and K with combinatorially equivalent boundaries, and create a new triangulation M by identifying L and K along their boundaries. Then $\chi(M) = \chi(L) + \chi(K) - \chi(\partial L)$.*

The above follows from a simple counting argument along the shared boundary.

[1] A triangulation is a cell decomposition satisfying some extra properties.

Lemma 3.2.17. *Given any connected closed triangulation T on n tetrahedra with k vertices where no edge is identified with itself in reverse, the triangulation has $n+k$ edges if and only if the link of each vertex in T is homeomorphic to a 2-sphere.*

Proof Let the triangulation have e edges. As each face of T is identified with exactly one other face, and a tetrahedron has 4 faces, we know that T must have $2n$ faces. Then $\chi(T) = k - e + 2n - n = n + k - e$ which immediately gives one direction of the proof. We must still show that if $e = n + k$ then the link of each vertex of T is homeomorphic to a 2-sphere.

Label each vertex in T by $\{v_1, \ldots, v_k\}$. Now consider the cell decomposition T' created from T by truncating T along the link of every vertex v_i. The boundary of T' is therefore the union of the links of the vertices of T. When forming T' from T, for each vertex v_i we removed one 0-cell, and then added a cell for every vertex, face and edge of $Link(v_i)$. As a result, we get that

$$\chi(T') = \chi(T) + \sum_{i=1}^{k} \left(\chi(Link(v_i)) - 1\right).$$

Furthermore, since $\chi(T) = n + k - e$ we get

$$\chi(T') = n - e + \sum_{i=1}^{k} \chi(Link(v_i)). \qquad (\star)$$

Note that T' represents a compact 3-manifold with boundary as it contains no edge identified with itself in reverse and every vertex on $\partial T'$ has a link homeomorphic to a disc.

Now take a copy of T', call the copy T'', and identify the boundary of T' with the boundary of T'' to form the triangulation $T^\dagger$. Since T' is a compact 3-manifold with boundary, $T^\dagger$ is a closed 3-manifold, giving $\chi(T^\dagger) = 0$ (even if the vertex links of T are not spheres).

If we set $L = T'$, $K = T''$ and $M = T^\dagger$ in Lemma 3.2.16, then we get the following result.

$$0 = \chi(T') + \chi(T') - \sum_{i=1}^{k} \chi(Link(v_i))$$

We then substitute in from $(\star)$ and rearrange to get

$$\sum_{i=1}^{k} \chi(Link(v_i)) = 2(e - n).$$

Since $\chi(Link(v_i)) = 2$ if and only if the link of each vertex is homeomorphic to a 2-sphere, and is less than 2 otherwise, we get that $2k = 2(e - n)$ if and only if the link of each vertex of T is homeomorphic to a 3-sphere. □

3.3 Census enumeration

A 3-manifold census is an exhaustive list of all 3-manifold triangulations on up to a certain number of tetrahedra. Such a broad census is super-exponential in size, so we instead often add limits to such a census. For example, a census of closed 3-manifolds on up to 5 tetrahedra would be a complete list of every triangulation which represents a closed 3-manifold and contains at most 5 tetrahedra. These censuses are a useful tool for making or breaking conjectures, or finding pathologically bad cases for various algorithms. However generating such a census is often time consuming.

The first such census was of cusped hyperbolic 3-manifolds on at most 5 tetrahedra by Hildebrand and Weeks [61] which was later extended to 3-manifolds on at most 9 tetrahedra [44, 49, 92]. Significant work has also been done on the census of closed orientable prime minimal triangulations, first by Matveev [76] on triangulations on at most 6 tetrahedra, which has been extended to 11 tetrahedra [41, 42, 75, 77].

One of the state of the art enumeration algorithms in use was developed for *Regina* [45], a high performance topological software suite. We give a brief explanation of this pre-existing algorithm here, and in later chapters we compare and contrast the algorithm from *Regina* with new algorithms developed in this thesis. *Regina* was chosen as it is freely available in both binary and source form, allowing for thorough comparisons. The algorithm discussed in this thesis searches for closed prime minimal triangulations of 3-manifolds. Note that *Regina* itself is very flexible and can search for other families of triangulations as well.

The algorithm in *Regina* is given, as input, a number n denoting how many tetrahedra each triangulation should contain. A list of all 4-regular multigraphs on n nodes is then generated. This is a simple and fast process, and is the first stage of many algorithms in the literature. At this point, certain graphs are discarded as they will not lead to any 3-manifold triangulations which are of interest to the census [41, 43]. Each remaining graph G is used, one at a time, as a frame to build up a triangulation, with the aim that any triangulation found will have G as its face pairing graph (or dual 1-skeleton). This second stage is by far the most time consuming.

To build a triangulation from a graph G, the algorithm starts with n tetrahedra with no faces identified and recursively attempts to identify faces. Recall that there are six symmetries of a regular triangle, and therefore six ways in which two faces may be identified. The algorithm picks an arc in the graph G denoting a pair of faces to be identified. The algorithm identifies the two faces using one of the six symmetries. Identifying two faces may result in a triangulation which can never be completed into a desired 3-manifold triangulation. In this case, the algorithm tries a different symmetry in the face identification. If all symmetries have failed, the algorithm backtracks and undoes an earlier face identification. If at some point there are no face identifications left to complete, the resulting triangulation is added to the census and the algorithm continues recursively.

Note that the selection of an arc from G is not detailed. In fact, *Regina* currently selects the arc based on the underlying data structure, which in turn is based on a lexicographical ordering of a particular representation of G. Looking at this particular problem led to new graphical representations of triangulations, which in turn formed the basis of Chapter 5.

Additionally, each census is completed without relying on any results from smaller censuses. It is well known that not all manifold triangulations on n tetrahedra can be built from smaller manifold triangulations, so the algorithm must exhaustively check the search space for triangulations regardless. This problem is also briefly discussed at the end of Chapter 4.

Chapter 4

Fixed parameter tractable algorithms in combinatorial topology

4.1 Introduction

In this chapter we will look closely at one particular problem in census enumeration algorithms. As mentioned earlier, such algorithms will first generate a list of 4-regular multigraphs and then for each such graph G the algorithm will attempt to build triangulations with G as its face pairing graph. We say a graph G is *admissible* if we find any such triangulation, and *non-admissible* if no such triangulation is found. In fact, it is often the case that a large proportion of the running time of an algorithm is spent on non-admissible graphs.

Using state-of-the-art public software [45], generating such a census on 12 tetrahedra takes 1967 CPU-days, of which over 1588 CPU-days is spent analysing non-admissible graphs. Indeed, for a typical census on ≤ 10 tetrahedra, less than 1% of 4-regular graphs are admissible [42]. Moreover, Dunfield and Thurston [56] show that the probability of a random 4-regular graph being admissible tends toward zero as the size of the graph increases. Clearly an efficient method of determining whether a given graph is admissible could have significant effect on the (often enormous) running time required to generating such a census.

We use parameterized complexity [55] to address this issue. A problem is *fixed parameter tractable* if, when some parameter of the input is fixed, the problem can be solved in polynomial time in the input size. In Corollary 4.4.5 we show that to test whether a graph G is admissible is fixed parameter tractable, where the parameter is the treewidth of G. Specifically, if the treewidth is fixed at $\leq k$ and G has size n, we can determine whether G is admissible in $O(n \cdot f(k))$ time.

Courcelle showed [51, 52] that for graphs of bounded treewidth, an entire class of problems have fixed parameter tractable algorithms. However, employing this result for our problem of testing admissibility looks to be highly non-trivial. In particular, it is not clear how the topological constraints of our problem can be expressed in monadic second-order logic, as Courcelle's theorem requires. Even if Courcelle's theorem could be used, our results here provide significantly better constants than a direct application of Courcelle's theorem would.

Following the example of Courcelle's theorem, however, we generalise our result to a larger class of problems (Theorem 4.4.4). Specifically, we introduce the concept of a *simple property*, and give a fixed parameter tractable algorithm which, for any simple property p, determines whether a graph admits a triangulated 3-manifold with property p (again the parameter is treewidth).

We show that these results are practical through an explicit implementation, and identify some simple heuristics which improve the running time and memory requirements. Lastly, we identify a clear potential for how these ideas can be extended to the more difficult enumeration problem, in those cases where a graph *is* admissible and a complete list of triangulations is required.

Parameterised complexity is very new to the field of 3-manifold topology [46, 48], and this chapter marks the first exploration of parameterised complexity in 3-manifold enumeration problems. Given that 3-manifold enumeration algorithms are often extremely slow on small graphs (and so we expect many graphs to have low treewidth), our work here highlights a growing potential for parameterised complexity to offer practical alternative algorithms in this field.

4.2 Background

Many NP-hard problems on graphs are fixed parameter tractable in the *treewidth* of the graph (e.g., [10, 12, 20, 21, 51]). Introduced by Robertson and Seymour [83], the treewidth measures precisely how "tree-like" a graph is:

Definition 4.2.1 (Tree decomposition and treewidth)**.** *Given a graph G, a tree decomposition of G is a tree H with the following additional properties:*

- *Each node of H, also called a* bag*, is associated with a set of nodes of G;*
- *For every arc a of G, some bag of H contains both endpoints of a;*
- *For any node v of G, the subforest in H of bags containing v is connected.*

If the largest bag of H contains k nodes of G, we say that the tree decomposition has width *$k+1$. The* treewidth *of G, denoted* $\mathrm{tw}(G)$*, is the minimum width of any tree decomposition of G.*

While finding the tree width of a given graph is NP-complete in general [11], Bodlaender [20] gave a linear time algorithm for determining if a graph has treewidth $\leq k$ for fixed k, and for finding such a tree decomposition, and Kloks [68] demonstrated algorithms for finding "nice" tree decompositions.

These properties are necessary and sufficient for the underlying topological space to be a 3-manifold. We say that a graph G is *admissible* if it is the face pairing graph for any closed 3-manifold triangulation $\mathcal{T}$.

Definition 4.2.2 (Partial-3-manifold triangulation)**.** *A partial-3-manifold triangulation $\mathcal{T}$ is a general triangulation for which (i) for any vertex v in $\mathcal{T}$, the link of v is homeomorphic to a 2-sphere with zero or more punctures; and (ii) no edge e in $\mathcal{T}$ is identified with itself in reverse (i.e., $e \not\simeq \bar{e}$).*

These are in essence "partially constructed" 3-manifold triangulations; the algorithms of Section 4.4 build these up into full 3-manifold triangulations. Note that the underlying space of $\mathcal{T}$ might not even be a 3-manifold with boundary: there may be "pinched vertices" whose links have many punctures.

We can make some simple observations: (i) the boundary vertices of a partial 3-manifold triangulation are precisely those whose links have at least one puncture; (ii) a connected partial-3-manifold triangulation with no boundary faces is a closed 3-manifold triangulation, and vice-versa; (iii) a partial-3-manifold triangulation with a face identification removed, or an entire tetrahedron removed, is still a partial-3-manifold triangulation.

4.3 Configurations

The algorithms in Section 4.4 build up 3-manifold triangulations one tetrahedron at a time. As we add tetrahedra, we must track what happens on the boundary of the triangulation, but we can forget about the parts of the triangulation not on the boundary—this is key to showing fixed parameter tractability. In this section we define and analyse edge and vertex configurations of general triangulations, which encode exactly those details on the boundary that we must retain.

Definition 4.3.1 (Edge configuration)**.** *The edge configuration of a triangulation $\mathcal{T}$ is a set C_e of triples detailing how the edges of the boundary faces are identified together. Each triple is of the form $((f,e),\ (f',e'),\ o)$, where: f and f' are boundary faces; e and e' are tetrahedron edges that lie in f and f' respectively; e and e' are identified in $\mathcal{T}$; and o is a boolean "orientation indicator" that is true if $e \simeq e'$ and false if $e \simeq \overline{e'}$.*

This mostly encodes the 2-dimensional triangulation of the boundary, though additional information describing "pinched vertices" is still required.

Example 4.3.2 (2-tetrahedra pinched pyramid)**.** Take two tetrahedra t_0 and t_1, each with vertices labelled $0, 1, 2, 3$, and apply the face identifications $t_0\!:\!012 \leftrightarrow t_1\!:\!012$ and $t_1\!:\!023 \leftrightarrow t_1\!:\!321$.

The resulting triangulation is a square based pyramid with one pair of opposing faces identified (see Figure 4.1(a)). The final space resembles a hockey puck with a pinch in the centre, as seen in Figure 4.1(b). Note that the vertex at top of the pyramid, which becomes the pinched centre of the puck, has a link homeomorphic to a 2-sphere with two punctures. Therefore, although this is a partial 3-manifold triangulation, the underlying space is not a 3-manifold.

The edge configuration of this triangulation is:

$$\begin{array}{ll}
\{((t_0\!:\!013,03),(t_1\!:\!013,13),f), & ((t_0\!:\!013,01),(t_1\!:\!013,01),t), \\
\ ((t_0\!:\!013,13),(t_0\!:\!123,13),t), & ((t_0\!:\!123,12),(t_0\!:\!123,23),f), \\
\ ((t_1\!:\!013,03),(t_1\!:\!023,03),t), & ((t_1\!:\!023,02),(t_1\!:\!023,23),f)\};
\end{array}$$

here t and f represent *true* and *false* respectively.

Definition 4.3.3 (Vertex configuration)**.** *The vertex configuration C_v of a triangulation $\mathcal{T}$ is a partitioning of those tetrahedron vertices that belong to boundary faces, where vertices v and v' are in the same partition if and only if $v \sim v'$.*

In partial-3-manifold triangulations, vertex links may have multiple punctures; the vertex configuration then allows us to deduce which punctures belong to the same link. In essence, the

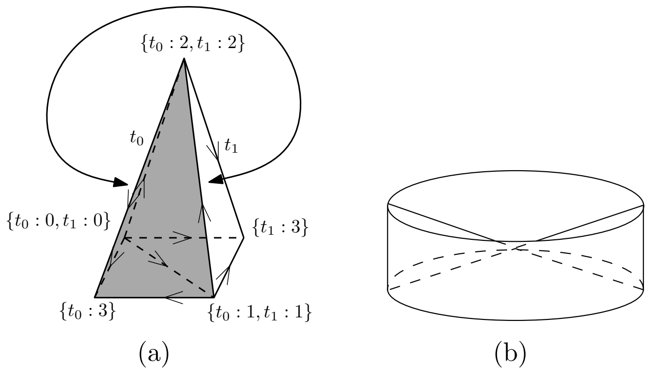


Figure 4.1: The triangulation from Example 4.3.2. The grey shaded tetrahedron is t_0. Edges are marked with their orientations, and the double-ended arrow indicates the identification of two opposing faces of the pyramid. The resulting space resembles a hockey puck with the centre pinched into a point. This pinch is the vertex $\{t_0\!:\!2, t_1\!:\!2\}$.

vertex configuration describes how the triangulation is "pinched" inside the manifold at vertices whose links have too many punctures.

For instance, the vertex configuration of Example 4.3.2 is given by

$$\{\{t_0\!:\!0, t_1\!:\!0, t_1\!:\!3\},\quad \{t_0\!:\!1, t_0\!:\!3, t_1\!:\!1\},\quad \{t_0\!:\!2, t_1\!:\!2\}\}.$$

The partition $\{t_0\!:\!2, t_1\!:\!2\}$ represents the pinch at the center of the "hockey puck". We now give the boundary configuration of a triangulated cube on 5 tetrahedra as an additional example.

Example 4.3.4 (5-tetrahedra cube)**.** Take 5 tetrahedra labelled $t_0, t_1, \ldots, t_4$, each with vertices labelled $0, 1, 2, 3$ and identify the following faces:

$$\begin{aligned} t_0\!:\!012 &\leftrightarrow t_4\!:\!012 \\ t_1\!:\!013 &\leftrightarrow t_4\!:\!013 \\ t_2\!:\!123 &\leftrightarrow t_4\!:\!123 \\ t_3\!:\!023 &\leftrightarrow t_4\!:\!023 \end{aligned}$$

The resulting triangulation is a cube, with t_4 having zero boundary faces and each other tetrahedra having three boundary faces. The edge configuration of this triangulation is:

$$\begin{aligned}
&\{((t_0\!:\!013, 01), (t_1\!:\!012, 01), t), && ((t_0\!:\!123, 12), (t_2\!:\!012, 12), t),\\
&\ ((t_0\!:\!023, 02), (t_3\!:\!012, 02), t), && ((t_1\!:\!123, 13), (t_2\!:\!013, 13), t),\\
&\ ((t_1\!:\!023, 03), (t_3\!:\!013, 03), t), && ((t_2\!:\!023, 23), (t_3\!:\!123, 23), t),\\
&\ ((t_0\!:\!013, 03), (t_0\!:\!023, 03), t), && ((t_0\!:\!013, 13), (t_0\!:\!123, 13), t),\\
&\ ((t_0\!:\!023, 23), (t_0\!:\!123, 23), t), && ((t_1\!:\!023, 02), (t_1\!:\!012, 02), t),\\
&\ ((t_1\!:\!023, 23), (t_1\!:\!123, 23), t), && ((t_1\!:\!012, 12), (t_1\!:\!123, 12), t),\\
&\ ((t_2\!:\!023, 02), (t_2\!:\!012, 02), t), && ((t_2\!:\!023, 03), (t_2\!:\!013, 03), t),\\
&\ ((t_2\!:\!012, 01), (t_2\!:\!013, 01), t), && ((t_3\!:\!012, 01), (t_3\!:\!013, 01), t),\\
&\ ((t_3\!:\!012, 12), (t_3\!:\!123, 12), t), && ((t_3\!:\!013, 13), (t_3\!:\!123, 13), t)\}.
\end{aligned}$$

If we examine the first triple in this configuration, it indicates that edge 01 on face 013 of t_0 and edge 01 on face 012 of t_1 are part of the same edge of the triangulation[1], even though no face of t_0 is identified to a face of t_1.

[1]Remember that each edge of the triangulation is an equivalence class of edges of tetrahedra.

This triangulation has no pinched vertices, but some vertices of tetrahedra are identified together, and so the vertex configuration is:

$$\{\{t_0\!:\!0, t_1\!:\!0, t_3\!:\!0\},\quad \{t_0\!:\!1, t_1\!:\!1, t_2\!:\!1\},\quad \{t_0\!:\!2, t_2\!:\!2, t_3\!:\!2\},$$
$$\{t_0\!:\!3,\},\quad \{t_1\!:\!3, t_2\!:\!3, t_3\!:\!3\},\quad \{t_1\!:\!2\},\quad \{t_2\!:\!0\},\quad \{t_3\!:\!1\}\}.$$

Definition 4.3.5 (Boundary configuration)**.** *The boundary configuration C of a triangulation $\mathcal{T}$ is the pair (C_e, C_v) where C_e is the edge configuration and C_v is the vertex configuration.*

Lemma 4.3.6. *For b boundary faces, there are $\frac{(3b)!}{(3b/2)!}$ possible edge configurations.*

Proof Note that b must be even; let $b = 2m$. Each boundary face has three edges, so there are $6m$ possible pairs (f, e) where e is an edge on a boundary face f. Each such pair must be identified with exactly one other pair, with either $e \simeq e'$ or $e \simeq \overline{e'}$, and so the number of possible edge configurations is

$$2 \cdot (6m-1) \cdot 2 \cdot (6m-3) \cdot \ldots \cdot 2 \cdot 3 \cdot 2 \cdot 1 = \frac{(6m)!}{(3m)!} = \frac{(3b)!}{(3b/2)!}.$$

□

Lemma 4.3.7. *For b boundary faces, the number of possible boundary configurations is bounded from above by*

$$\frac{(3b)!}{(3b/2)!} \cdot \left(\frac{2.376b}{\ln(3b+1)}\right)^{3b}.$$

Proof There are $3b$ tetrahedron vertices on boundary faces, and so the number of possible vertex configurations is the Bell number B_{3b}. The result now follows from Lemma 4.3.6 and the following inequality of Berend [17]:

$$B_{3b} = \frac{1}{e}\sum_{i=0}^{\infty} \frac{i^{3b}}{i!} < \left(\frac{2.376b}{\ln(3b+1)}\right)^{3b}.$$

□

Corollary 4.3.8. *The number of possible boundary configurations for a triangulation on n tetrahedra with b boundary faces depends on b, but not on n.*

The boundary configuration can be used to partially reconstruct the links of vertices on the boundary of the triangulation. In particular:

- The edge configuration allows us to follow the arcs around each puncture of a vertex link—in Figure 4.2 for instance, we can follow the sequence of arcs $a_1, a_2, \ldots$ that surround the puncture in the link of the top vertex.
- The vertex configuration tells us whether two sequences of arcs describe punctures in the *same* vertex link, versus *different* vertex links.

In this way, we can reconstruct all information about punctures in the vertex links, even though we cannot access the full (2-dimensional) triangulations of the links themselves. As the next result shows, this means that the boundary configuration retains all data required to build up a partial-3-manifold triangulation, without knowledge of the full triangulation of the underlying space.

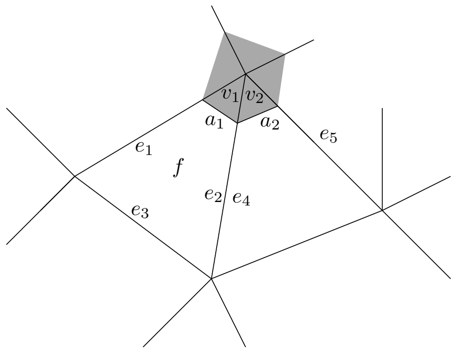


Figure 4.2: Part of the boundary of a triangulation. The link of the top vertex is shaded grey; this link does not contain the vertex, but instead cradles the vertex from below.

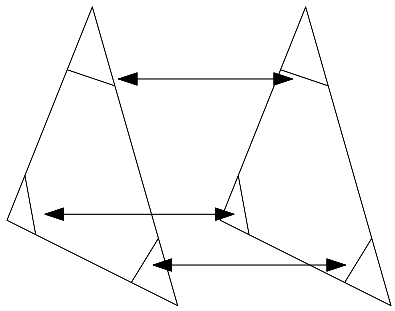


Figure 4.3: One face identification will result in three pairs of arcs on vertex links being identified, as shown by the arrows.

Lemma 4.3.9. *Let $\mathcal{T}$ be a partial 3-manifold triangulation with b boundary faces, and let $\mathcal{T}'$ be formed by introducing a new identification between two boundary faces of $\mathcal{T}$. Given the boundary configuration of $\mathcal{T}$ and the new face identification, we can test whether $\mathcal{T}'$ is also a partial-3-manifold triangulation in $O(b)$ time.*

Proof We are given the boundary configuration for $\mathcal{T}$. We need to check that the new face identification does not result in any edges being identified in reverse, and that the links of all vertices are still spheres with zero or more punctures.

It is easy to see that the face identification will identify at most three pairs of boundary edges together, and it is routine to check (using the edge configuration) whether these identifications will result in any edge identified with itself in reverse in $O(b)$ time. The rest of this proof therefore only deals with the vertex links.

To determine whether $\mathcal{T}'$ is also a partial-3-manifold triangulation we only need to determine how these new edge identifications affect the link of each vertex. Clearly any vertices that are internal to $\mathcal{T}$ must already have links homeomorphic to a 2-sphere, and cannot be changed. We noted in Section 4.3 that the cycles of arcs surrounding the punctures on the links of each boundary vertex can be determined from the edge configuration—we do not explicitly reconstruct these cycles here, but we do note that this information is accessible from the edge configuration. We note also that each link of a boundary vertex in $\mathcal{T}$ must be homeomorphic to a sphere with one or more punctures (equivalently, a disc with zero or more punctures).

The new face identification will identify three pairs of arcs on the vertex links, as shown in Figure 4.3. Each of these three arc identifications will take one of three forms (see Figure 4.4):

Type I: The two arcs being identified both bound the same puncture in the same vertex link.

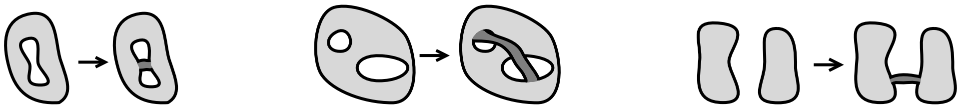


Figure 4.4: (a) Type I arc identification; (b) Type II arc identification; and (c) Type III arc identification. Light grey indicates the existing link with the white space indicating the punctures, and the dark grey indicates the new identification in the link.

Type II: The two arcs are part of the same vertex link but bound distinct punctures.

Type III: The two arcs are part of distinct vertex links.

For a type I identification, if the identification preserves the orientability of the vertex link then the new vertex link will be homeomorphic to a 2-sphere with zero or more punctures. In particular, if the puncture only contained two arcs and these are now both identified (in an orientable manner) then the puncture will be closed off (i.e., the vertex link will be a 2-sphere with one less puncture than before). If this vertex of the triangulation only had one puncture, then the vertex link will become homeomorphic to a 2-sphere and the new vertex will be an internal vertex of $\mathcal{T}'$.

If orientability is not preserved in a type I identification then we will embed a Möbius band in the vertex link, which is never allowed. Identifications of type II increase the genus of the vertex link, which is likewise not allowed (see Figure 4.4(b)), and identifications of type III simply connect two discs with zero or more punctures.

In summary: orientable identifications of type I and all identifications of type III are allowed, whereas non-orientable identifications of type I and all identifications of type II are not allowed.

Since the triangulations of all vertex links contain $3b$ boundary arcs in total, we can identify both the type and orientation of each identification in $O(b)$ time. Specifically, we use the edge configuration to determine if the identification is of type I (as well as the orientation of the identification), and we use the vertex configuration to distinguish between identifications of type II and type III. If any non-orientable type I identifications or any type II identifications are found, $\mathcal{T}'$ is not a partial-3-manifold triangulation.

Since we have only three such identifications of pairs of arcs, we can check all three in $O(b)$ time as well. Combining this with the $O(b)$ check described earlier for bad edges, we obtain the required result. □

4.4 A fixed parameter tractable admissibility algorithm

Recall that the motivating problem for our work was to quickly detect whether a given graph admits a closed 3-manifold triangulation. The algorithm we develop generalises to many other settings. For this we define a *simple property* of a partial 3-manifold triangulation (see below).

We extend boundary configurations to include an extra piece of data ϕ based on the partial triangulation that helps test our property. For instance, if p is the simple property that the triangulation contains ≤ 3 internal vertices, then ϕ might encode the number of internal vertices thus far in the partial 3-manifold triangulation (here ϕ takes one of the values $0, 1, 2, 3,$ `too_many`).

We say that a boundary configuration C is *viable* for a graph G if there exists some partial-3-manifold triangulation $\mathcal{T}$ with C as its boundary configuration and $\Gamma(\mathcal{T}) = G$. Additionally, if $\mathcal{T}$ satisfies a simple property p we say that C is *p-viable*.

Definition 4.4.1 (Simple property)**.** *A boolean property p of a partial-3-manifold triangulation is called* simple *if all of the following hold. Here all configurations have $\leq b$ boundary faces, and f, g, h are some computable functions.*

1. *The extra data ϕ in the boundary configuration satisfies $\phi \in P$ for some universal set P with $|P| \leq f(b)$.*
2. *We can determine whether a triangulation satisfies p based only on its boundary configuration (including the extra data ϕ).*
3. *Given any viable configuration and a new face identification π between two of its boundary faces, we can in $O(g(b))$ time test whether introducing this identification yields another viable configuration and, if so, calculate the corresponding value of ϕ.*
4. *Given viable configurations for two disjoint triangulations, we can in $O(h(b))$ time test whether the configuration for their union is also viable and, if so, calculate the corresponding value of ϕ.*

The four conditions above can be respectively interpreted as meaning:

1. the upper bound on the number of viable configurations (including the data ϕ) still depends on b but not the number of tetrahedra;
2. we can still test property p without examining the full triangulation;
3. new face identifications can still be checked for p-viability in $O(g(b))$ time;
4. configurations for disjoint triangulations can be combined in $O(h(b))$ time.

Example 4.4.2. Let p be the property that a triangulation contains at most x internal vertices (i.e., vertices with links homeomorphic to a 2-sphere), for some fixed integer x. Then p is simple.

Here we define $\phi \in P = \{0, 1, \ldots, x, \texttt{too\_many}\}$ to be the number of vertices in our partial 3-manifold triangulation with 2-sphere links. This clearly satisfies conditions 1 and 2. For condition 3: when identifying two faces together, a new vertex acquires a 2-sphere link if and only if the identification closes off all punctures in the link (which we can test from the edge and vertex configurations). Condition 4 is easily satisfied by summing ϕ over the disjoint configurations.

The case when $x = 1$ is highly relevant: much theoretical and computational work has gone into 1-vertex triangulations of 3-manifolds [64, 77], and these are of particular use when searching for 0-efficient triangulations [65].

We can now state the main result of this chapter:

Problem 4.4.3. p-ADMISSIBILITY(G) Let p be a simple property. Given a connected 4-regular multigraph G, determine whether there exists a closed 3-manifold triangulation $\mathcal{T}$ with property p such that $\Gamma(\mathcal{T}) = G$.

The basic idea for our algorithm is as follows. Start with an empty triangulation, and then introduce tetrahedra and face identifications in a way that essentially works from the leaves up to the root of the tree decomposition of G. For each subtree in the tree decomposition we compute which configurations are viable for the corresponding subgraph of G, and then propagate these configurations further up the tree. The running time at each node depends only on the number of boundary faces, which is bounded in terms of the bag size and thereby $\mathrm{tw}(G)$. Note that the properties of a tree decomposition will be used to ensure that two child bags of a common parent will represent disjoint triangulations.

Theorem 4.4.4. *Let p be a simple property. Given a connected 4-regular multigraph G on n nodes with treewidth $\leq k$, and a corresponding tree decomposition with $O(n)$ nodes where each bag has at most two children, we can solve p-*ADMISSIBILITY(G) *in $O(n \cdot f(k))$ time for some computable function f.*

Our requirement for such a tree decomposition is not restrictive: Bodlaender [20] gives a fixed parameter tractable algorithm to find a tree decomposition of width $\leq k$ for fixed k, and Kloks [68] gives an $O(n)$ time algorithm to transform this into a tree decomposition where each bag has at most two children. The proof itself holds as long as the number of children at each bag is independent of n, which can be shown by following the ideas used in [68]. We use the "two children" constraint to keep the proof simple, however, as it does not affect the result.

Proof We will give this proof in three sections. First, we describe the algorithm in detail. Then we show that the algorithm is correct. Lastly we demonstrate the running time of the algorithm.

We begin, however, with some preliminaries. Let the tree decomposition be H. Recall that in a tree decomposition, each node of H represents a collection of nodes of G. We will use the term *bag* to refer to a node of H, and *node* to refer to a node of G. Each node w in G will represent a corresponding tetrahedron Δ_w.

Arbitrarily choosing one bag of H and make it the root bag, so that the tree becomes a hierarchy of subtrees. For each bag ν in H, the subtree H_ν is defined as the subtree obtained by taking only the bag ν and any bag which appears below ν in H.

We now define the subgraph G_ν of G, which contains precisely those nodes of G that appear *only* in H_ν. In other words, node w is in G_ν if and only if w does *not* appear in any bag in $H \setminus H_\nu$. The subgraph G_ν contains all corresponding arcs of G, i.e., all arcs of G that link nodes of G_ν.

We first make the following observation. If two children ν_i, ν_j of some bag ν were to contain a common node w, then since H is a tree decomposition any such w must also appear in the bag ν. Therefore *no two subgraphs G_{ν_i}, G_{ν_j} may contain a common node representing a common tetrahedron.*

As a result, we can combine the boundary configurations of children of ν by simply taking the union of the configurations, as they correspond to disjoint triangulations. The same process can be used to extend some boundary configuration with the boundary configuration of a new standalone tetrahedron.

The algorithm: For each bag ν, we will construct all boundary configurations for G_ν. To achieve this, we take the following steps:

1. Take every possible combination of configurations from the children of ν, where each combination contains exactly one configuration from each child. We showed earlier that these must represent disjoint triangulations, and that for each combination we can construct the configuration of their union.

2. For each such combined configuration C:

   (a) For each element w inside the bag ν, if w does not appear anywhere in a higher bag in H, then add the boundary configuration of a single standalone tetrahedron (corresponding to tetrahedron Δ_w) to C. Again, the earlier observation shows that this is possible. Then:

      i. For each arc e in G incident to the node w, if the other endpoint w' of the arc e is also in G_ν, use Lemma 4.3.9 to try to add each of the six possible corresponding face identifications to C in turn (recall that these come from the six symmetries of a regular triangle). For each viable (but not necessarily p-viable) configuration thus created, continue by recursing to Step 2a and taking the next element w.

   (b) If all elements of ν have been successfully processed in Step 2a then a viable configuration has been found. Store this as a viable boundary configuration for G_ν.

If any bag contains no viable configurations, we immediately know that there are no closed 3-manifold triangulations $\mathcal{T}$ satisfying p-ADMISSIBILITY(G).

Once all configurations have been constructed, if the root bag contains any p-viable boundary configurations (by construction all boundary configurations at the root node have empty edge and vertex configurations), then there does exist some closed 3-manifold triangulation $\mathcal{T}$ with property p such that $\Gamma(\mathcal{T}) = G$. If, however, the root bag contains no p-viable configuration then such a triangulation does not exist.

Correctness: For each bag ν, we have a corresponding graph G_ν. If a closed 3-manifold triangulation $\mathcal{T}$ with property p and $\Gamma(\mathcal{T}) = G$ exists, then define $\mathcal{T}_\nu$ to be the partial-3-manifold triangulation constructed by removing from $\mathcal{T}$ the tetrahedra and face identifications which respectively represent nodes and arcs not present in G_ν. Each such $\mathcal{T}_\nu$ must be a partial-3-manifold triangulation, and so each bag ν must have at least one viable configuration.

If the root bag does contain some p-viable boundary configuration, then each arc in G has been through Step 2(a)i in the algorithm and by Lemma 4.3.9 we know that each such configuration must represent a partial 3-manifold triangulation with property p (or possibly many such triangulations). Since G is 4-regular, we also know that these triangulations can have no boundary faces, and so these partial 3-manifold triangulations must in fact be closed 3-manifold triangulations with property p.

Running time: We begin by showing that the number of configurations at each bag ν is bounded by a function of k, but is independent of n.

Consider a boundary face f of tetrahedron Δ in some triangulation $\mathcal{T}_\nu$ represented by some configuration C in ν. In the graph G, there must exist some arc a which represents the identification of f with some other face f' of some tetrahedron Δ'. However, since f is a boundary face of $\mathcal{T}_\nu$, this must mean that the node representing Δ' must occur in some bag "higher up" in the tree decomposition; that is, the node representing Δ' must occur in some bag in $H \setminus H_\nu$. However, the nodes representing Δ and Δ' must occur together in some bag (as they are the endpoints of arc a), so by Definition 4.2.1 the node representing Δ must occur in the bag ν itself. From this, it is easy to see that as ν has at most $k+1$ elements, configurations at ν must represent triangulations with at most $4(k+1)$ boundary faces. Therefore the number of configurations is $O(c(k) \cdot f(k))$, where $c(k)$ is the bound given in Lemma 4.3.7 and $f(k)$ is the function given in Definition 4.4.1.

We now calculate the running time of each step in the algorithm. The tree decomposition has $O(n)$ nodes, and at each node we go through all three steps. We again will use the functions $g(b)$ and $h(b)$ as given in Definition 4.4.1, and note that $O(k) = O(b)$.

Take any bag, along with its two children. Each of the three contains at most $O(k)$ elements. From the above argument, each child stores at most $O(c(k) \cdot f(k))$ viable configurations. At Step 1, we are combining configurations. As each bag has two children, Step 1 takes $O((c(k) \cdot f(k))^2 \cdot h(k))$ time per bag.

Step 2 takes each such configuration, and at Step 2a extends the configuration. Therefore Step 2a runs at most $O((c(k) \cdot f(k))^2)$ times per bag. We know that G is 4-regular, so by Lemma 4.3.9 and Definition 4.4.1 there are at most four distinct pairs of faces to identify per each introduced tetrahedron, and thus Step 2(a)i runs in $O(g(k))$ time and therefore Step 2a likewise runs in $O(k \cdot g(k))$ time. Step 2b is simply storing configurations. Step 2 can therefore be completed for each viable configuration in $O(k \cdot g(k))$ time.

Since each viable configuration is built from one of the $O((c(k) \cdot f(k))^2)$ configurations obtained in Step 1, each bag can be processed in $O((c(k) \cdot f(k))^2 \cdot (h(k) + k \cdot g(k)))$. Combining the above counts for each of the $O(n)$ bags in the tree decomposition gives a running time of $O(n \cdot ((c(k) \cdot f(k))^2 \cdot (h(k) + k \cdot g(k))))$ and the required result. □

By taking a trival property p which is always true, we obtain the original desired result. Note that for the this case, the functions f and h are trivial and therefore constant. By Lemma 4.3.9, g is $O(k)$ and so the running time in this case simply becomes $O(n \cdot k^2 \cdot c(k)^2)$.

Corollary 4.4.5. *Given a connected 4-regular multigraph G, the problem of determining whether there exists a closed 3-manifold triangulation $\mathcal{T}$ such that $\Gamma(\mathcal{T}) = G$ is fixed parameter tractable in the treewidth of G.*

4.5 Implementation and experimentation

The algorithm was implemented in Java, using the treewidth library from [93]. Although our theoretical bound on the number of configurations is extremely large (Lemma 4.3.7), we store all configurations using hash maps to exploit situations where in practice the number of viable configurations is much smaller. As seen below, we find that such a discrepancy does indeed arise (and significantly so).

We also introduce another modification that yields significant speed improvements in practice. The algorithm builds up a complete list of all viable configurations at each bag ν of the tree decomposition. However, for an affirmative answer to the problem, only a small subset of these may be required. We take advantage of this as follows.

For any bag ν with no children, configurations are computed as normal. Once a viable configuration is found, it is immediately propagated up the tree in a depth-first manner. This means that, rather than calculating every possible viable configuration for every subgraph G_ν, the improved algorithm can identify a full triangulation with property p quickly and allow early termination.

We implemented the program with p defined to be *one-vertex and possibly minimal*, using criteria on the degrees of edges from [41]. This allowed us to compare both correctness and timing with the existing software *Regina* [45]. We ran our algorithm on all 4-regular graphs on 4, 5 or 6 nodes (see Table 4.1)to verify correctness. We see that the average time to process a graph increases with treewidth, as expected. We also see that the number of viable configurations is indeed significantly lower than the upper bound of Lemma 4.3.7, as we had hoped.

Table 4.1: Results from the algorithm. From left to right, the columns denote the number of nodes in the graph, the treewidth of the graph, the number of distinct graphs with these parameters, the average running time of the algorithm on these graphs, and the largest number of configurations found at any bag, for any graph.

$\lvert(V(G))\rvert$	tw(G)	# of graphs	Avg. run time (ms)	max(|configurations|)
4	1	1	680	2
4	2	8	4036	7
4	3	1	13280	17
5	1	1	780	17
5	2	22	13446	156
5	3	4	29505	307
5	4	1	94060	39
6	1	1	890	17
6	2	68	64650	1583
6	3	25	346028	5471
6	4	3	297183	1266

Regina significantly outperforms our algorithm on all of these graphs, though these are small problems for which asymptotic behaviour plays a less important role. What matters more is performance on larger graphs, where existing software begins to break down.

We therefore ran a sample of 12-node graphs through our algorithm, selected randomly from graphs which cause significant slowdown in existing software. This “biased” sampling was deliberate—our aim is not for our algorithm to *always* outperform existing software, but instead to seek new ways of solving those difficult cases that existing software cannot handle. Here we do find success: our algorithm was at times 600% faster at identifying non-admissible graphs than *Regina*, though this improvement was not consistent across all trials.

In summary: for larger problems, our proof-of-concept code already exhibits far superior performance for some cases that *Regina* struggles with. With more careful optimisation (e.g., for dealing with combinatorial isomorphism), we believe that this algorithm would be an important tool that complements existing software for topological enumeration.

The full source code for the implementation of this algorithm is available at `http://www.github.com/WPettersson/AdmissibleFPG`.

4.6 Applications and extensions

We first note that our meta-theorem is useful: here we list several simple properties p that are important in practice, with a brief motivation for each.

1. *One-vertex triangulations* are crucial for computation: they typically use very few tetrahedra, and have desirable combinatorial properties. This is especially evident with 0-efficient triangulations [65].

2. Likewise, *minimal triangulations* (which use the fewest possible tetrahedra) are important for both combinatorics and computation [41, 42]. Although minimality is not a simple property, it has many simple necessary conditions, which are used in practical enumeration software [42, 77].

3. *Ideal triangulation of hyperbolic manifolds* play a key role in 3-manifold topology. An extension of Theorem 4.4.4 allows us to support several necessary conditions for hyperbolicity, which again are used in real software [49, 61].

Finally: a major limitation of all existing 3-manifold enumeration algorithms is that they cannot "piggyback" on prior results for fewer tetrahedra, a technique that has been remarkably successful in other areas such as graph enumeration [79]. This is not a simple oversight: it is well known that we cannot build all "larger" 3-manifold triangulations from smaller 3-manifold triangulations. The techniques presented here, however, may allow us to overcome this issue—we can modify the algorithm of Theorem 4.4.4 to store entire families of triangulations at each bag of the tree decomposition. We would lose fixed parameter tractability, but for the first time we would be able to cache and reuse partial results across different graphs and even different numbers of tetrahedra, offering a real potential to extend census data well beyond its current limitations.

Chapter 5

A new algorithm for census enumeration

5.1 Introduction

Most (if not all) census algorithms in the literature enumerate 3-manifolds on n tetrahedra in two main stages. The first of these is generating a list of all 4-regular multigraphs on n nodes. The second stage takes each such graph G, and sequentially identifies faces of tetrahedra together to form a triangulation with G as its dual 1-skeleton (see Section 3.3 for more details). In this chapter we describe a different approach to generating a census of 3-manifolds. The first stage remains the same, but in the second stage we build up the link of each edge in the triangulation sequentially. Since many algorithms identify faces of tetrahedra together (or take combinatorially equivalent steps), this is a paradigm shift in census enumeration. We achieve this result by extending each possible dual 1-skeleton graph to a "fattened face pairing graph" and then finding particular decompositions of these new graphs. We also show how various improvements to typical census algorithms (such as those in [41]) can be translated into this new setting.

We implement the new algorithm and compare its running time to that of existing algorithms. Results show that our new algorithm complements existing algorithms very well, and we predict that a heuristic combination of existing algorithms and this new algorithm can significantly speed up census enumeration.

5.2 Manifold decompositions

In this section we define a fattened face pairing graph, and then describe how a specific decomposition of such a graph is exactly equivalent to a general triangulation. We also show how the conditions for being a 3-manifold triangulation can translate into this context. Lastly we comment on how these decompositions relate to spine representations of 3-manifolds.

A *fattened face pairing graph* is an extension of a face pairing graph $F = (V, E)$ which we use in a dual representation of the corresponding triangulation. Instead of one node for each tetrahedron, a fattened face pairing graph contains one node for each face of each tetrahedron. Additionally, a face identification in the triangulation is represented by three arcs in the fattened face pairing graph; these three arcs loosely correspond to three pairs of edges which are identified

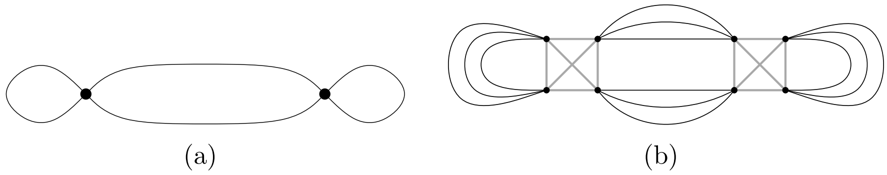


Figure 5.1: The face pairing graph (a) and fattened face pairing graph (b) of a 3-sphere triangulation. Note that the grey arcs are internal arcs, while the black arcs are external arcs.

when two faces of tetrahedra are identified.

Definition 5.2.1. *Given a face pairing graph F, a fattened face pairing graph is constructed by first tripling each arc (i.e., for each arc e in F, add two more arcs parallel to e), and then replacing each node ν of F with a copy of K_4 such that each node of the K_4 is incident with exactly one set of triple arcs that met ν.*

Example 5.2.2. Figure 5.1 shows a face pairing graph and the resulting fattened face pairing graph. The arcs shown in grey are what we call *internal* arcs. Each original node has been replaced with a copy of K_4 and in place of each original arc a set of three parallel arcs have been added.

We will refer to the arcs of each K_4 as *internal arcs*, and the arcs between distinct copies of K_4 as *external arcs*. Each such K_4 represents a tetrahedron in the associated face pairing graph, and as such we will say that a fattened face pairing graph has n tetrahedra if it contains $4n$ nodes.

Triangulations are often labelled or indexed in some manner, and changing the labels does not change the triangulation. Given any labelling of a triangulation, we label the corresponding fattened face pairing graph as follows. For each tetrahedron i with faces a, b, c and d, we label the nodes of the corresponding K_4 in the fattened face pairing graph $v_{i,a}$, $v_{i,b}$, $v_{i,c}$ and $v_{i,d}$ such that if face a of tetrahedron i is identified with face b of tetrahedron j then there are three parallel external arcs between $v_{i,a}$ and $v_{j,b}$.

In such a labelling, the node $v_{i,a}$ represents face a of tetrahedron i. Each internal arc $\{v_{i,a}, v_{i,b}\}$ represents the unique edge common to faces a and b of tetrahedron i. Each external arc $\{v_{i,a}, v_{j,b}\}$ represents one of the three pairs of edges of tetrahedra which become identified as a result of identifying face a of tetrahedron i with face b of tetrahedron j. Note that the arc only represents the pair of edges being identified; not the orientation of said identification.

We now define *ordered decompositions* of fattened face pairing graphs. Later we show that there is a one-to-one correspondence between such a decomposition and a general triangulation. Note that our definitions for general triangulations (respectively fattened face pairing graphs) do not involve labels on the vertices (respectively nodes). As such we can ignore any isomorphisms due to labellings when discussing such a bijection. Later we show exactly how the 3-manifold constraints on general triangulations (see Lemma 3.2.7) can be translated to constraints on these decompositions.

Definition 5.2.3. *An* ordered decomposition *of a fattened face pairing graph $F = (E, V)$ is a set of closed walks $\{P_1, P_2, \ldots, P_n\}$ such that:*

- *$\{P_1, P_2, \ldots, P_n\}$ partition the arc set E;*

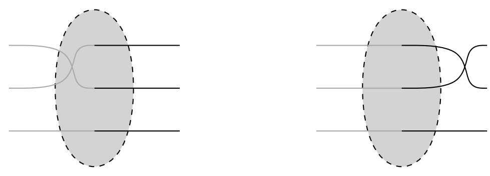


Figure 5.2: Two close up views of a node of a fattened face pairing graph with the same pairing of arcs. The node itself is represented by the grey ellipse, and all 6 arcs are incident upon this node. Note how both figures show the same pairing of edges, the only difference is where the "twist" occurs.

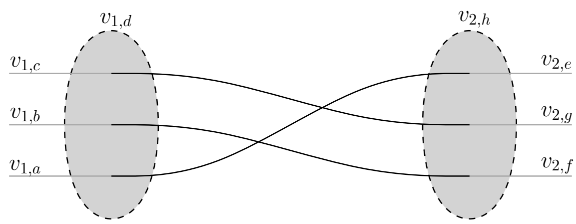


Figure 5.3: A partial drawing of a fattened face pairing graph.

- *P_i is a closed walk of even length for each i; and*
- *if arc e_{j+1} immediately follows arc e_j in one of the walks then exactly one of e_j or e_{j+1} is an internal arc.*

An ordered decomposition of a fattened face pairing graph exactly describes a general triangulation. We will outline this idea first by showing how three parallel external arcs can represent an identification of faces, and give the full proof in Theorem 5.2.4.

Since the ordered decomposition consists of closed walks of alternating internal and external arcs, the decomposition pairs up arcs at such nodes so that at each node of degree 6, one external arc is paired with exactly one internal arc. To help visualise this, we can draw such nodes as larger ellipses, with 3 external arcs and 3 internal arcs entering the ellipse, as in Figure 5.2. Each external arc meets exactly one internal arc inside this ellipse. This only represents how such arcs are pairing up in a given decomposition, the node is still incident with all 6 arcs. We also see in Figure 5.2 that the fattened face pairing graph can always be drawn such that any "crossings" of arcs only occur between external arcs. Such crossings are simply artefacts of how the fattened face pairing graph is drawn in the plane, and in no way represent any sort of underlying topological twist.

Figure 5.3 shows a partial drawing of an ordered decomposition of a fattened face pairing graph. In this, we see a set of three parallel external arcs between nodes $v_{1,d}$ and $v_{2,h}$. This tells us that face d of tetrahedron 1 is identified with face h of tetrahedron 2. Additionally, we see that one of the external arcs joins arc $\{v_{1,c}, v_{1,d}\}$ to arc $\{v_{2,g}, v_{2,h}\}$. This tells us that edge ab of tetrahedron 1 (represented by $\{v_{1,c}, v_{1,d}\}$) is identified with edge ef of tetrahedron 2 (represented by $\{v_{2,g}, v_{2,h}\}$). Since we know that face abc is somehow identified with face efg, this tells us that vertex c is identified with vertex g in this face identification. We can repeat this process for the other paired arcs to see that vertex a is identified with vertex e and vertex b is identified with vertex f. The resulting identification is therefore $abc \leftrightarrow efg$.

We now extend these ideas, detailing exactly how to construct the general triangulation from an ordered decomposition and vice-versa.

Construction 5.2.4. *Constructing a general triangulation from an ordered decomposition of a fattened face pairing graph.*
It is straight forward to see that we can simplify a ordered decomposition of a fattened face pairing graph into a regular face pairing graph, and this gives a collection of tetrahedra and shows which faces are identified. What remains is to determine the exact identification between each pair of faces.

First we label the nodes of the fattened face pairing graph such that each K_4 in the fattened face pairing graph has nodes labelled $v_{i,a}, v_{i,b}, v_{i,c}, v_{i,d}$. The choice of i here assigns the label i to the corresponding tetrahedron in the triangulation. Similarly, the assignment of $v_{i,a}$ to a node labels a face of the corresponding tetrahedron. Different labellings of nodes will therefore result in a triangulation with different labels on tetrahedra and vertices. However, up to isomorphism the actual triangulation is not changed.

For each identification of two tetrahedron faces, we have three corresponding external arcs in the fattened face pairing graph. Each arc e out of these three belongs to one walk in the ordered decomposition, and in said walk e has exactly one arc e_1 preceding it and one arc e_2 succeeding it such that the sequence of arcs (e_1, e, e_2) occurs the walk.

Since e is an external arc, e_1 and e_2 must be internal arcs and therefore of the form $\{v_{i,a}, v_{i,b}\}$ where $a \neq b$. Let $e = \{v_{i,b}, v_{j,c}\}$, $e_1 = \{v_{i,a}, v_{i,b}\}$ and $e_2 = \{v_{j,c}, v_{j,d}\}$. This tells us that this identification is between face a of tetrahedron i and face d of tetrahedron j, and in this identification the edge common to faces a and b on tetrahedra t_i is identified with the edge common to faces c and d on tetrahedra t_j. The orientation of this edge identification is not given, however it is not needed.. Each of faces a and d have three vertices, and this identification of edges also identifies two vertices from face a with two vertices from face d. This leaves one vertex from each face, which must be identified together. By repeating this process for the two external arcs parallel to e we can therefore determine the actual face identification between face a and face d.

Recall that $\deg(e)$ is the number of edges of tetrahedra identified together to form edge e in the triangulation. It is clear that in the above construction all internal arcs that belong to one walk in the ordered decomposition represent edges of tetrahedra which are all identified together, leading to the following corollary.

Corollary 5.2.5. *Given an ordered decomposition $\{P_1, \ldots, P_t\}$, each walk P_i corresponds to exactly one edge e in the corresponding general triangulation. Additionally we get $2 \cdot \deg(e) = |P_i|$.*

We have shown that we can construct a triangulation from an ordered decomposition. We now give a reverse construction of an ordered decomposition from a triangulation. Together these show the 1-to-1 correspondence between the two.

Example 5.2.6. First we give an example of how to partially build an ordered decomposition. For this example we have a triangulation edge of degree ≥ 3, depicted in Figure 5.4 as the thicker central edge. Recall that face x of a tetrahedron is the face opposite vertex x. We see that in the leftmost tetrahedron, the thickened edge is opposite vertices 1 and 2, and that face 1 is identified with face 4. We therefore have the sequence $(\{v_{1,1}, v_{1,2}\}, \{v_{1,1}, v_{2,4}\}, \ldots)$ occurring in one of the walks of the ordered decomposition.

Continuing this process shows that the sequence

$$(\{v_{1,1}, v_{1,2}\}, \{v_{1,1}, v_{2,4}\}, \{v_{2,4}, v_{2,3}\}, \{v_{2,3}, v_{3,6}\}, \{v_{3,6}, v_{3,5}\}, \ldots)$$

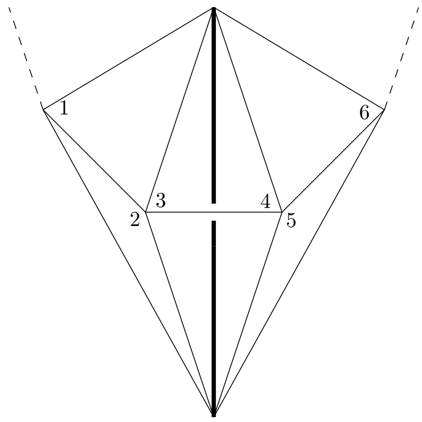


Figure 5.4: Three tetrahedra about a central edge. Note that only vertices of tetrahedra are labelled in this diagram (i.e., vertices 2 and 3 are both of tetrahedra, but in the triangulation they are identified together), and recall that vertex 1 is opposite face 1.

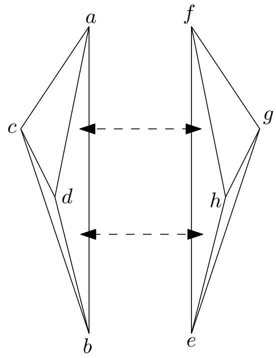


Figure 5.5: Face c of tetrahedron i is identified with face g of tetrahedron j. As a result, one of the walks of the ordered decomposition contains the three arcs $\{\{v_{i,d}, v_{i,c}\}, \{v_{i,c}, v_{j,g}\}, \{v_{j,g}, v_{j,h}\}$ in order.

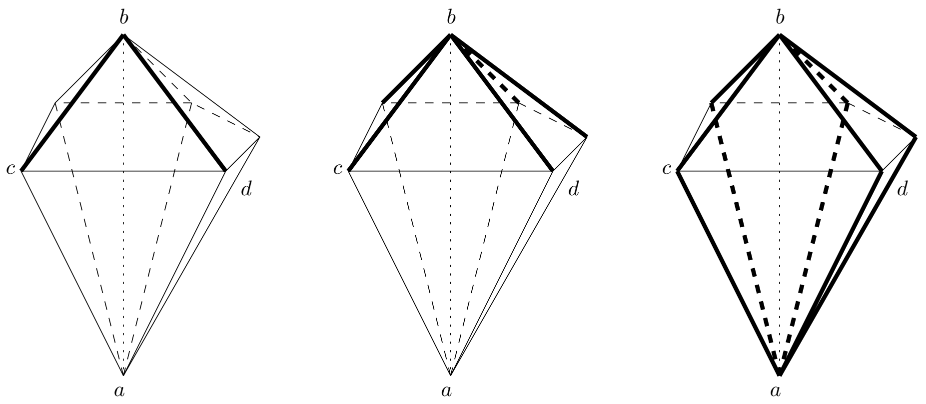
occurs in one of the walks of the ordered decomposition.

Construction 5.2.7. *Constructing an ordered decomposition from a general triangulation.* First construct the fattened face pairing graph from the face pairing graph of the triangulation. We now label the fattened face pairing graph. Begin by labelling the tetrahedra in the triangulation, and their vertices. Label the individual nodes of the fattened face pairing graph such that if face a of tetrahedron i is identified with face b of tetrahedron j then the corresponding three parallel arcs are between node $v_{i,a}$ and node $v_{j,b}$ in the fattened face pairing graph.

Recall that an edge ab is the edge between vertices a and b. Given a tetrahedron with vertices labelled a, b, c and d, the edge ab has as endpoints the two vertices a and b and thus is the intersection of face c and face d, so the edge ab in the triangulation is represented by the arc $\{v_{i,c}, v_{i,d}\}$ in the fattened face pairing graph.

Start with an edge ab on tetrahedron i in the triangulation, and add $\{v_{i,c}, v_{i,d}\}$ to the start of what will become a walk in the ordered decomposition.

Face c on this tetrahedron must be identified with some face g on tetrahedron j. For a diagram, see Figure 5.5. Through this identification, the edge ab must be identified with some edge on face g. Call this edge ef. Add one of the three arcs $\{v_{i,c}, v_{j,g}\}$ to the current walk. Since a face contains three edges, by construction we can always find such an arc which is not already in one of the walks of the ordered decomposition. If $\{v_{j,g}, v_{j,h}\}$ is already in this walk then we are finished with the walk. Otherwise, add the arc $\{v_{j,g}, v_{j,h}\}$ into the walk. The process then



(a) Marking process (b) A "good" marking (c) A "bad" marking

Figure 5.6: In these figures, the thicker edges are marked. In (a), edge bc was arbitrarily marked as being above edge ab. Edge bd is marked because bc and bd share a common face which does not contain ab. If the marking in (b) is reached, then the edge ab is not identified with itself in reverse. If, however, (c) is reached, then ab is identified with itself in reverse.

continues with the edge ef. Since each tetrahedron edge is the intersection of two faces of a tetrahedron, it is clear that this process will continue until the initial edge ab is reached and the current walk is complete.

The above procedure is then repeated until all arcs have been added to a walk. By construction, we have created an ordered decomposition with the required properties.

We now give a decomposition representation of the property that a 3-manifold triangulation has no edge that is identified with itself in reverse. In the triangulation one may consider the ring of tetrahedra $\Delta_1, \ldots, \Delta_k$ (which need not be distinct) around an edge $e = ab$, as in Figure 5.6. Start on Δ_1, and mark one edge incident to e (say bc) as being "above" e. Since bc is "above" e, the face bcd must be the "top" face of Δ_1, and thus the edge bd must also be "above" e and is marked. We can then track the edge bd through a face identification, and across the top of the next tetrahedron. At some point, we must reach Δ_1 again. If Δ_1 is reached via one of the edges ac or ad, then e is identified with itself in reverse. However, if Δ_1 is reached via the edge bc again, then we know that the edge ab is not been identified with itself in reverse.

Loosely speaking, in the decomposition setting, we look at one walk P_x of our ordered decomposition and mark arcs in the decomposition as being "above" arcs in the walk P_x. If we again consider the edge bc as "above" ab, we mark the arc[1] $\{v_{i,a}, v_{i,d}\}$. Since the ordered decomposition corresponds to exactly one triangulation, we can use the ordered decomposition to determine which edge is identified with $\{v_{i,a}, v_{i,d}\}$. We then mark the next edge, and proceed as in the previous paragraph.

Definition 5.2.8. *Given an ordered decomposition $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$, we can mark a walk P_x as follows. For a diagram of the ordered decomposition, see Figure 5.7.*

Pick an external arc e_s from P_x. Arbitrarily pick an external arc e_S parallel to e_s, and mark e_S as being above e_s. Then let $e_a = e_s$ and $e_A = e_S$ and continue as follows:

- *Let e_b be the next external arc in P_x after e_a.*

[1]Recall that the arc $\{v_{i,a}, v_{i,d}\}$ denotes the edge common to face a and face d. Since face a contains vertices b,c and d and face d contains vertices a, b and c this edge must be bc.

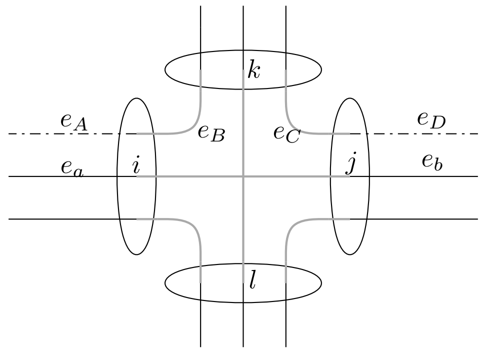


Figure 5.7: The process used to mark edges as per Definition 5.2.8. The dot-dashed arcs are the ones marked as "above". Recall that the ellipses are whole nodes, the insides of which denote how internal and external arcs are paired up in the decomposition.

- *The internal arc preceding e_b joins two nodes. Call these nodes i and j, such that e_b is incident on j.*
- *Some external arc e_A incident on i must be marked. Find the closed walk which e_A belongs to. In this closed walk there must exist some internal arc which either immediately precedes or follows e_A through node i. Call this internal arc e_B. Note that the walk containing these two arcs need not be, and often is not, P_x. Arc e_B must be incident to i, and some other node which we shall call k.*
- *Find the internal arc e_C between nodes k and j, and find the walk P_y it belongs to. In this walk, one of the arcs parallel to e_b must either immediately precede or follow e_C and be incident upon node j. Call this arc e_D.*
- *If $e_b = e_s$, and e_D is already marked as being above e_b, we are finished with this walk.*
- *Else, mark the arc e_D as being above e_b and repeat the above steps, using e_b in place of e_a, and using e_D in place of e_A.*

Note that this processing of marking specifically marks one arc as being "above" another. It does not mark arcs as being "above" in general.

To visualise this definition in terms of the decomposition, see Figure 5.7. The arcs e_a and e_b are part of a closed walk, and we are marking the edges "above" this walk. Arc e_A was arbitrarily chosen. Arc e_B follows e_C, and then we find e_C as the arc sharing one node with e_B and one with e_b. From e_C we can find and mark e_D.

In terms of triangulations, each of e_A and e_D represent edges of tetrahedra in the triangulation which are "above" the edge represented by P_x. Each arc actually represents one of the three edge identifications in a face identification, since they are external arcs. The process of translating e_A to e_B (and e_C to e_D) is following this edge of the face identification onto an edge of the tetrahedron (or in the case of e_C to e_D, the reverse). Both e_B and e_C are internal arcs of the same tetrahedron and share a common node k so we know that both these internal arcs represent edges of the tetrahedron which share a common face. This face is what we called the "top" face in the triangulated manifold.

Lemma 5.2.9. *Take an ordered decomposition containing a walk P_x with arcs marked according to Definition 5.2.8 and the corresponding triangulation. If there exists some external arc e in P_x that has two distinct external arcs marked as "above" e, then the edge of the triangulation represented by P_x is identified to itself in reverse.*

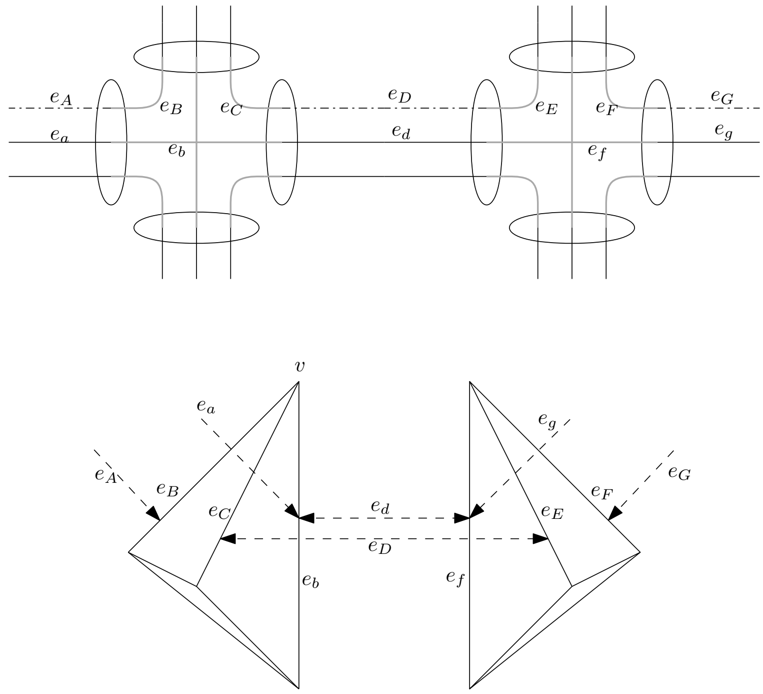


Figure 5.8: Part of an ordered decomposition, and associated tetrahedra. Identifications of edges are shown with dashed arrows.

Proof Part of an ordered decomposition is shown in Figure 5.8, and we use the notation as shown there. The part shown represents a single face identification between two (not necessarily distinct) tetrahedra. The markings on the tetrahedra denote exactly what each labelled arc in the fattened face pairing graph represents. As such, we say that an internal arc of the ordered decomposition "is" also an edge of a tetrahedron. For example, e_C is an internal arc, so it represents the edge of the tetrahedron yet we say that e_C is the edge on the tetrahedron. The external arcs all represent edges in face identifications, and are drawn with dashed lines.

We prove the result by applying an orientation onto each of the edges of tetrahedra contained in the edge of the triangulation represented by P_x. Consider first the arc e_a, which represents one edge identification in some face identification. The arc e_A (one of the two arcs parallel to e_a) is marked as being "above" e_a. This is equivalent to assigning an orientation onto each of the pair of edges represented by e_a. Since e_b is one of these, we now have an orientation on the edge e_b. We want to fix an orientation onto the edge e_f such that the orientations of e_b and e_f agree after the identification of faces. Since e_B immediately follows e_A (or vice-versa) and e_C immediately follows e_D (or vice-versa, again) in the ordered decomposition, edges e_B and e_C meet in a common tetrahedron vertex, call this vertex v. We also see that the edge e_b meets v. Since the edge e_b is identified with the edge e_f (via the edge identification represented by e_d), and the edge e_C is identified with the edge e_E (via the edge identification represented by e_D), v must be identified to the vertex common to edges e_f and e_E.

The orientation of the edge represented by e_b has been used to orient the edge represented by e_f such that the two orientations agree after the face identification. Repeating this process for all arcs in P_x in turn orients all the edges of tetrahedra that are contained in the edge of the triangulation.

If every external arc e in P_x has exactly one external arc marked as "above" e, then we have exactly one orientation for each edge of a tetrahedron. That is, the edge of the triangulation corresponding to P_x cannot be identified with itself in reverse.

If some external arc e in P_x has two distinct external arcs marked as "above" e, then every external arc must have two such other arcs marked (as the marking process can only terminate when it reaches e_s in Definition 5.2.8). This must mean that we have assigned two distinct orientations to each tetrahedron edge in the triangulation edge corresponding to P_x and therefore this triangulation edge is identified with itself in reverse. $\square$

If a walk P_x in an ordered decomposition satisfies Lemma 5.2.9, we say that this walk is *non-reversing.*

Definition 5.2.10. *A* manifold decomposition *is an ordered decomposition of a fattened face pairing graph satisfying all of the following conditions.*

- *The ordered decomposition contains* $n + 1$ *closed walks.*
- *The fattened face pairing graph contains* $4n$ *nodes.*
- *Each walk is non-reversing.*
- *The associated manifold triangulation contains exactly* 1 *vertex.*

Theorem 5.2.11. *There is a one-to-one correspondence between manifold decompositions of a connected fattened face pairing graph and 1-vertex 3-manifold triangulations.*

Proof Constructions 5.2.4 and 5.2.7 give the correspondence between general triangulations and ordered decompositions. All that remains is to show that the extra properties of a manifold decomposition force the corresponding triangulation to be a 3-manifold triangulation. Since the decomposition contains $n + 1$ walks, Corollary 5.2.5 tells us the triangulation has $n + 1$ edges. Additionally, each tetrahedron corresponds to four nodes in the fattened face pairing graph, so the triangulation has n tetrahedra and thus by Lemma 3.2.17 we see that the link of each vertex is homeomorphic to a 2-sphere. Each walk is non-reversing so Lemma 5.2.9 says that no edge in the corresponding triangulation is identified with itself in reverse, and we have the required result. $\square$

We now define the notation used to refer to specific ordered decompositions. This notation is used in Algorithm 1, in particular for some performance improvements. The notation is unintentionally defined such that it can also be interpreted as a *spine code* (as used by Matveev's Manifold Recognizer [78]), and that the spine generated from such a spine code is a dual representation of the same combinatorial object as the manifold decomposition. For more detail on spine codes, see Section 5.2.1.

Notation 5.2.12. Take an ordered decomposition of a fattened face pairing graph with $4n$ nodes, and label each set of three parallel external arcs with a distinct value taken from the set $\{1, \ldots, 2n\}$. Assign an arbitrary orientation onto each set of three parallel external arcs. For each walk in the ordered decomposition:

1. Create a corresponding empty ordered list.
2. Follow the external arcs in the walk.
   (a) For each external arc labelled i met in the walk, if the arc in the walk is traversed in a manner consistent with the applied orientation add i to the end of the corresponding ordered list.

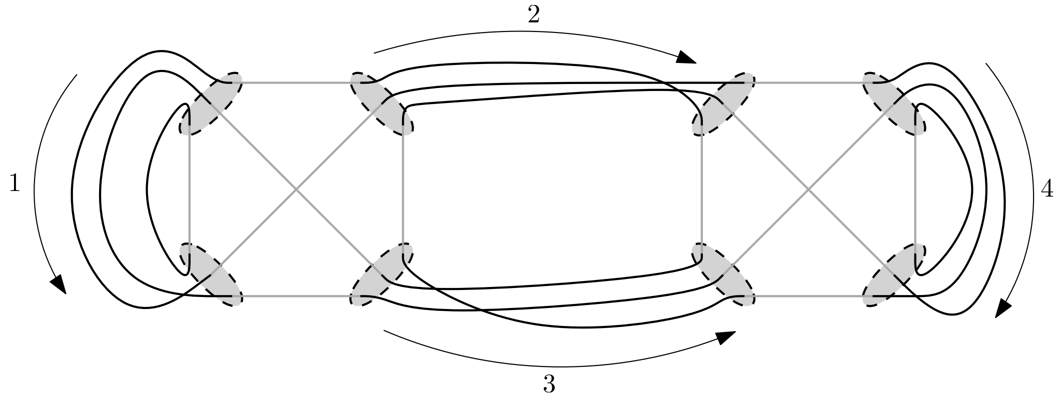


Figure 5.9: A manifold decomposition of a fattened face pairing graph that represents a 3-sphere. Recall that each grey ellipse is actually a node in the fattened face pairing graph.

(b) If instead the arc in the walk is traversed in the reverse direction, add $-i$ to the end of the list.

(c) Continue until the first external arc in the walk is reached.

See Example 5.2.13 for an example of the use of this notation. Note that this notation only records the external arcs, and does not note any internal arcs in walks.

We can also reconstruct the face pairing graph (and therefore the fattened face pairing graph) from this notation (i.e., without the internal arcs). Since each external arc represents some identification of two faces (and three parallel external arcs will represent the same identification), we can use the orientation of each arc to distinguish between the two faces in each identification and build up the face pairing graph.

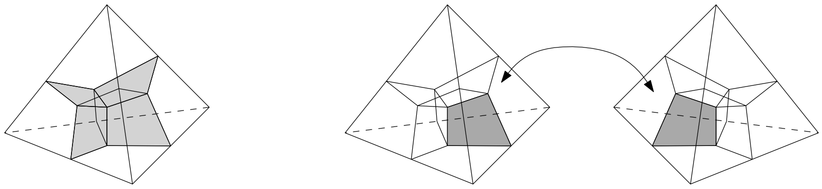
It is routine to check that if one is given a fattened face pairing graph and a partial ordered decomposition in which all the internal arcs are missing from each walk, it is still possible (and indeed trivial) to reconstruct the complete ordered decomposition. For the theoretical discussions in this section we have retained complete information regarding the ordered decompositions, but in the implementation we only store the sequential list of external arcs as in Notation 5.2.12.

Example 5.2.13. The following set of walks (remember, we omit internal arcs and instead prescribe orientations on external arcs) describes a manifold decomposition of a 3-sphere.

$$T = \{(1), (1, 2, 4, -2, 3, -4, -3, -1, 3, -2), (4)\}$$

Figure 5.9 shows this manifold decomposition of a 3-sphere. Given the appropriate vertex labellings, this represents the same triangulation as that given in Example 3.2.8.

Each integer in T represents an identification of faces, and we can also track each face in an identification individually using the sign of said integer. For example, -3 is before -1 in the second walk, so we can say that the "second" face in identification 1 belongs to the same tetrahedron as the "first" face in identification 3. Each integer (or its negative) appears exactly three times in an ordered decomposition, so we can determine exactly which faces belong to the same tetrahedron. For example, both faces involved in identification 1 belong to the same tetrahedron as the "first" face in identification 2 and the "first" face in identification 3.



(a) (b)

Figure 5.10: (a) A *butterfly* inside a tetrahedron; and (b) the identification of two faces will identify quadrilaterals together. In this case, the two grey quadrilaterals will have their edges identified.

5.2.1 Special spines of triangulations

We now briefly discuss a dual representation of 3-manifold triangulations, the *spine* representation. Since spines and manifold decompositions represent equivalent objects, we expect there to be an equivalence between them. However, despite the different construction methods involved, the representation of a manifold decomposition as defined in Notation 5.2.12 can directly be interpreted as a spine which represents the same manifold (and vice-versa). We only give a brief introduction to special spines (which we shall just call spines) to highlight the differences between spines and manifold decompositions, for full details on spines, see [77].

Figure 5.10(a) shows what is called a *butterfly* inside a tetrahedron. This butterfly is essentially the spine of the tetrahedron, and consists of six intersecting quadrilaterals. A manifold can be *collapsed* to a spine (for details, see [77]), leading to the relationship between the two. Each face of the tetrahedron is incident with three of these six quadrilaterals, and it is easy to see that one can translate a face identification between two faces into an identification of these quadrilaterals (see Figure 5.10(b)). Also it is important to note that there is exactly one quadrilateral that is incident with any two faces of the tetrahedron.

To work with spines combinatorially, we need some combinatorial representation. Each set of quadrilaterals forms a disc about an edge of the triangulation, so we can arbitrarily orient the disc and then follow a path that traces a circle around the edge. This path crosses over various quadrilaterals by transitioning between faces of tetrahedra. Since we are dealing with 3-manifold triangulations, each face is involved in exactly one face identification, so we note down which face identification is involved. Note that since the disc and circle are oriented, we also need to track whether we use each face identification in a positive or negative manner.

This notation is used by Matveev's Manifold Recognizer [78], As mentioned earlier, if a spine code and a representation of a manifold decomposition as defined in Notation 5.2.12 are equivalent (up to isomorphisms due to relabelling), then the corresponding spine and manifold decompositions are dual to each other.

5.3 Algorithm and improvements

In this section we first give a basic outline of the new algorithm to enumerate manifold decompositions (3-manifold triangulations). We then give various improvements to this algorithm based on known theoretical results in 3-manifold topology combined with suitable data structures.

Many existing algorithms in the literature [45, 77] build triangulations by identifying faces pairwise (or taking combinatorially equivalent steps). The algorithm we give here essentially identifies edges of faces together. Therefore the search tree traversed by our new algorithm is significantly different than that traversed by other algorithms. This is highlighted in the results given in Section 5.4.

5.3.1 Algorithm

The basic algorithm, detailed below, begins by labelling with an index each set of three parallel external arcs arbitrarily. It then simply tries to create suitable walks beginning with the lowest index arc. Recursion is used to ensure that every possible walk is tried, and once every external arc is in a walk we have an ordered decomposition.

Algorithm 1 Find an ordered decomposition (G)

Require: G is a fattened face pairing graph on n nodes
Index and assign orientation to each set of three parallel arcs
$e :=$ external arc with lowest index not yet in any walk
Finish walk(G,(e),{})

Algorithm 2 Finish walk (G,$\mathcal{P}$,$\mathcal{D}$)

Require: G is a fattened face pairing graph
Require: $\mathcal{P}$ is a partial walk in an ordered decomposition
Require: $\mathcal{D}$ is a partial ordered decomposition
Notation: $\mathcal{P}_1 \lhd \mathcal{P}_2$ is the walk generated by appending the walk $\mathcal{P}_2$ to the walk $\mathcal{P}_1$
$e :=$ last external arc in $\mathcal{P}$, with orientation
$n_1 :=$ last node e visits
for each internal arc i incident on n_1 **do**
  if i has not yet been placed into any walk in $\mathcal{D}$ **then**
    $n_2 :=$ node incident to i and distinct to n_1
    if n_2 is first node of first arc in $\mathcal{P}$ **then**
      $\mathcal{P}' := \mathcal{P} \lhd (i)$
      if all external arcs of G are in some walk **then**
        Store $\mathcal{D} \cup \mathcal{P}'$ as ordered decomposition
      else
        $e :=$ external arc with lowest index not yet in any walk
        Finish walk (G,(e),$\mathcal{D} \cup \mathcal{P}'$)
      end if
    else
      $e_2 :=$ external arc incident to n_2, not yet in any walk
      if e_2 will be traversed in forward direction **then**
        Finish walk (G,$\mathcal{P} \lhd (i, e_2)$,$\mathcal{D}$)
      else
        Finish walk (G,$\mathcal{P} \lhd (i, -e_2)$,$\mathcal{D}$)
      end if
    end if
  end if
end for

Algorithm 1 can be considered an initialisation algorithm and is relatively easy to understand, while Algorithm 2 is the recursive function which does the actual searching and is more complex.

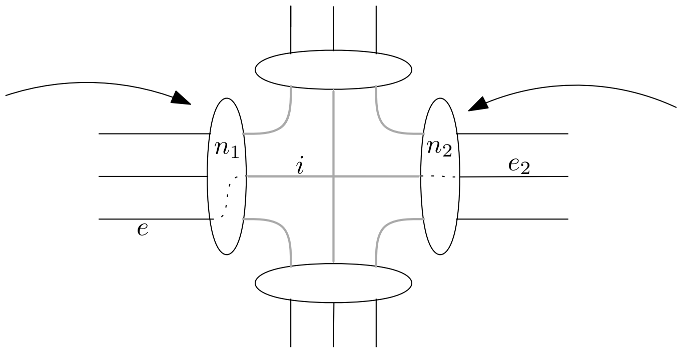


Figure 5.11: Partial walk being built as in Algorithm 2. In this diagram, e was the starting arc. The choice of i is shown. Note that all three possible choices for e_2 are equivalent. Also since the orientation on e_2 is "backwards", the new partial walk would be $\mathcal{P} \lhd (i, -e_2)$.

A diagram helping explain Algorithm 2 is given in Figure 5.11. This second algorithm starts with a partial walk (initially a single external edge), and attempts to complete said walk. The partial walk always ends with an external arc, so there are three choices for the next internal arc (Line 3). The first check is to ensure that the next internal arc i only occurs in one walk in the current ordered decomposition. Then if using i completes the current walk (Line 6), the algorithm either finishes the ordered decomposition (Line 9) or tries to find the next walk (Line 11). If using i does not complete the current walk, the algorithm continues with the current walk. Note that the algorithm needs to track whether the next external arc e_2 is used in the forward (Line 17) or reverse (Line 19) direction. However, since each of the three parallel choices for e_2 are equivalent, there is no need to distinguish between these.

5.3.2 Limiting the size of walks

Enumeration algorithms [41, 42, 45, 75, 76, 77] in 3-manifold topology often focus on closed, minimal, irreducible and $\mathbb{P}^2$-irreducible 3-manifold triangulations. We do the same here, see Chapter 3 for details on these properties.

The following results are taken from [41], but similar lemmas for orientable cases were used by the enumeration algorithms in [75, 77].

Lemma 5.3.1. (2.1 in [41]) *No closed minimal triangulation has an edge of degree three that belongs to three distinct tetrahedra.*

Lemma 5.3.2. (2.3 in [41]) *No closed minimal $\mathbb{P}^2$-irreducible triangulation with ≥ 3 tetrahedra contains an edge of degree two.*

Lemma 5.3.3. (2.4 in [41]) *No closed minimal $\mathbb{P}^2$-irreducible triangulation with ≥ 3 tetrahedra contains an edge of degree one.*

Remembering that the degree of an edge of a triangulation is the number of edges of tetrahedra which are identified together to form said edge, we can see that these lead to the following.

Corollary 5.3.4. *No closed minimal $\mathbb{P}^2$-irreducible manifold decomposition with ≥ 3 tetrahedra contains a walk which itself contains less than three external arcs.*

Corollary 5.3.5. *No closed minimal manifold decomposition contains a walk which itself contains exactly three internal arcs representing edges on distinct tetrahedra (i.e., belonging to three distinct K_4 subgraphs).*

The above results are direct corollaries, as it is simple to translate the terms involved and the results are simple enough to implement in an algorithm. In the algorithm, we can add a check on the number of arcs in $\mathcal{P}'$ after Line 7. This is implementable as a constant time check if the length of the current partial walk is stored.

Additionally, we use these results a second time. For a census on n tetrahedra, the ordered decomposition needs to contain $n+1$ walks to be a manifold decomposition. If the algorithm has completed k walks, then there are $n+1-k$ walks left to complete. Each such walk must contain at least three external arcs, so if there are less than $3(n+1-k)$ unused external arcs, the current subtree of the search space can be pruned.

Improvement 5.3.6. At Line 1 of Algorithm 2, if the number of unused external arcs is less than $3(n+1-|\mathcal{D}|)$, end the current iteration of the algorithm.

This result is extended one step further. There is only one closed walk in a fattened face pairing graph on more than one tetrahedron which contains three internal arcs that are not all from distinct tetrahedra, shown in Figure 5.12. We modify Algorithm 1 to enumerate all such closed walks first. Each such walk is either present or absent in any manifold decomposition. For each possible combination of such walks, we fix said walks and then run Algorithm 2 on the remaining arcs. All other walks must now contain at least four external arcs, so during the census on n tetrahedra if the algorithm has completed k walks and there are less than $4(n+1-k)$ unused external arcs we know that the partial decomposition can not be completed to a manifold decomposition.

Improvement 5.3.7. Use Algorithm 3 instead of Algorithm 1. At Line 1 of Algorithm 2, if the number of unused external arcs is less than $4(n+1-|\mathcal{D}|)$, end the current iteration of the algorithm.

Algorithm 3 Find an ordered decomposition (G)

Require: G is a fattened face pairing graph on n nodes

```
Index and assign orientation to each set of three parallel arcs
S := {}
for each K_4 in G do
    if two internal arcs of the K_4 can be used together in a walk containing three internal
    arcs then
        Add the walk to S
    end if
end for
for each subset s of S do
    e := external arc with lowest index not yet in any walk
    Complete walk(G,(e),s)
end for
```

5.3.3 Avoiding cone faces

For the next results, it is not clear that computationally cheap implementations are possible. Instead we look more closely at the results for triangulations to find partial results that are fast when implemented. The following was shown in [41].

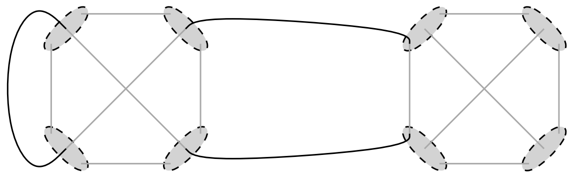


Figure 5.12: The only possible walk containing 3 internal arcs not all from distinct tetrahedra in a fattened face pairing graph on more than 1 tetrahedron. Only the external arcs used in the walk are shown, other external arcs are not shown.

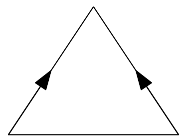


Figure 5.13: A one-face cone formed by identifying the two marked edges.

Lemma 5.3.8. (2.8 in [41]) *Let T be a closed minimal $\mathbb{P}^2$-irreducible triangulation containing ≥ 3 tetrahedra. Then no single face of T has two of its edges identified to form a cone as illustrated in Figure 5.13.*

For manifold decompositions, our translation of this result also requires the orientability of the underlying manifold. It should be noted that this is not due to some inherent flaw in manifold decompositions. This simply ensures that implementing the resulting check is viable.

Lemma 5.3.9. *Let D be a closed minimal $\mathbb{P}^2$-irreducible manifold decomposition of an orientable manifold containing ≥ 3 tetrahedra. Then no walk of D can use two parallel external arcs in opposite directions (as seen in Figure 5.14).*

Proof Recall that by our definition, if some walk P_x of a manifold decomposition contains the sequence of arcs $(\{v_{i,a}, v_{i,b}\}, \{v_{i,a}, v_{j,c}\})$ then face a of tetrahedron i is identified with face c of tetrahedron j. Assume towards a contradiction that we also have the sequence of arcs $(\{v_{i,a}, v_{j,c}\}, \{v_{i,a}, v_{i,d}\})$ in the walk P_x somewhere, such that the parallel arcs of the form $\{v_{i,a}, v_{j,c}\}$ are used in the walk in both directions.

Affix some orientation onto the edge of the manifold represented by P_x, and consider the ring of tetrahedra surrounding this edge. Since we have an orientable manifold, we can make use of a "right-hand" rule. See Figure 5.15 for a visual aid. Imagine a right hand inside tetrahedron i, gripping edge cd (represented by $\{v_{i,a}, v_{i,b}\}$) such that the thumb points towards the positive end of the edge and the fingers curl around the edge so that they leave the tetrahedron through

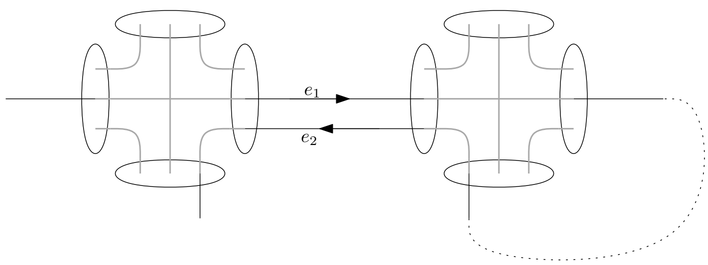


Figure 5.14: The depicted walk cannot occur in a closed minimal $\mathbb{P}^2$-irreducible orientable manifold decomposition as external arcs e_1 and e_2 are used in opposite directions. The dotted lines indicates the walk continues through parts of the fattened face pairing graph not depicted.

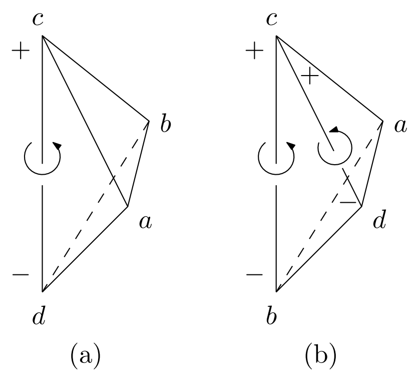


Figure 5.15: A tetrahedron in an oriented triangulation. The circular line with the arrow head indicates a "right hand" gripping the edge. In (a) the edge *cd* is given an orientation. If two parallel external arcs are used in opposite directions (as seen in Figure 5.14) then (b) must occur. Note that since the triangulation is oriented, the labelling of (b) is forced, given the labelling of (a).

face a (see Figure 5.15(a)). Since the manifold is orientable, any time this hand is back inside tetrahedron i it must have this same orientation. Now since $\{v_{i,a}, v_{i,b}\}$ preceded $\{v_{i,a}, v_{j,c}\}$ in the walk and the fingers curl "out" through face a of tetrahedron i, if some other arc $\{v_{i,a}, v_{i,d}\}$ succeeds arc $\{v_{i,a}, v_{j,c}\}$ then the fingers must curl "in" through face a of tetrahedron i as the hand grips edge *cd*. As yet, this is no contradiction, as the hand is gripping the one edge of the triangulation, but is therefore gripping many edges of tetrahedra. However, this necessarily leads to these two edges of tetrahedra having the same common vertex as their "positive" end (see Figure 5.15(b)). Then face a has two edges identified as in 5.13, contradicting Lemma 5.2.9. □

Improvement 5.3.10. After Line 15 of Algorithm 2, if e_2 is parallel to another external arc e_3 such that e_3 is already used in the reverse direction, do not use e_2 in this walk.

Note that this improvement only applies if the algorithm is searching for orientable manifold decompositions.

5.3.4 One vertex tests

Definition 5.2.10 requires that the associated manifold only have one vertex. We now show how the algorithm can determine this. More accurately, we show how the algorithm detects when a partially constructed manifold must have more than one vertex, which is used to prune the search tree. We do this by tracking how the link of each vertex may be triangulated[2]. For this, we will use the term *frontier edge* to refer to an edge on the frontier of the triangulation of a link (see Figure 5.16).

Initially, the link of each vertex may be triangulated as a single triangular face, and therefore has 3 frontier edges. Each time an external arc is used in a walk, two edges in the triangulation are identified together, and as a result two frontier edges are identified together (see Figure 5.17).

The orientation of this identification is not known, but is also not required. We only require that the triangulation only have one vertex and we do this by tracking how many frontier edges

[2]Since we are triangulating the link of a vertex, this triangulation is two-dimensional.

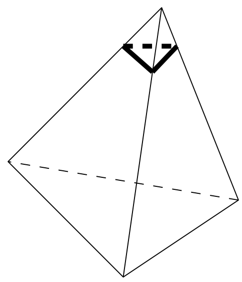


Figure 5.16: A tetrahedron, with the link of the top vertex drawn in heavier lines. This link, when triangulated, is homeomorphic to a disc. Each of the three heavier lines is a frontier edge.

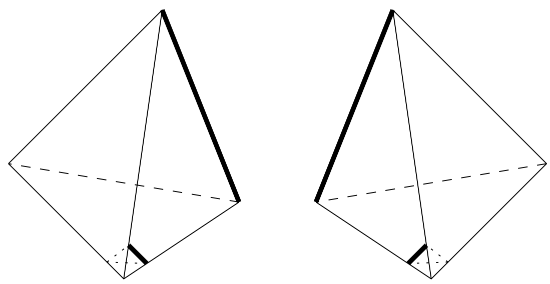


Figure 5.17: When the two edges of the two tetrahedra (long thick lines) are identified, we also know that the two frontier edges (short thick lines) will be identified.

are in each link. When frontier edges are identified together, the two edges either belong to the same link, or to two distinct links. If the two frontier edges belong to the same link (see Figure 5.18(a)), the number of frontier edges in the link is reduced by two. However if the frontier edges belong to two distinct links (see Figure 5.18(b)), with l_a and l_b frontier edges respectively, the resulting link has $l_a + l_b - 2$ frontier edges. Note that after this identification, two links have been joined together so we must not just track the number of frontier edges, but also which links have been identified.

Once a vertex link has no frontier edges, we consider it "closed off" as no other vertex links can be connected to it. If any other distinct vertex links exist, we know that the triangulation must have more than one vertex and we can prune the search tree.

Improvement 5.3.11. When initialising the algorithm, give each vertex a "frontier edges" variable initialised to three.

When an external arc is used in a walk, find the two faces f_1 and f_2 involved, as well as the pair of edges e_1 and e_2 being identified. Let v_1 (respectively v_2) be the vertex opposite edge e_1 on face f_1 (respectively e_2 on face f_2). If v_1 and v_2 are part of the same vertex of the triangulation, subtract two from the number of frontier edges of this vertex link. Otherwise, identify v_1 and v_2 as being part of the same vertex link, and set the number of frontier edges of this vertex link as the sum of the frontier edges of v_1 and v_2 minus two. If the resulting vertex link (from either case) has zero frontier edges remaining and there are unused external arcs, prune the current branch of the search tree.

The number of frontier edges of each vertex link, as well as which vertex links are identified together, are tracked via a union-find data structure. The data structure is slightly tweaked to allow back tracking (see [42] for details), storing the number of frontier edges at each node. For more details on the union-find algorithm in general, see [89].

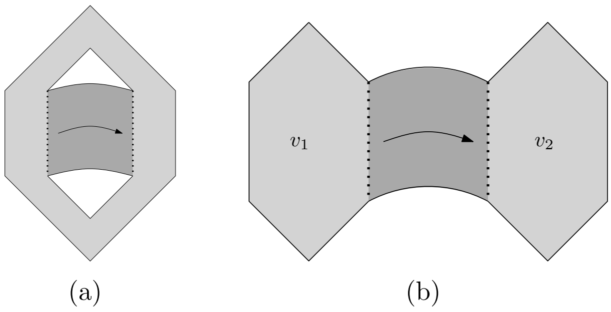


Figure 5.18: Two possibilities when identifying frontier edges of link vertices. The dark grey and the arrow indicate the two edges identified. In (a) the two frontier edges belong to the same vertex link, whereas in (b) the two edges belong to two different vertex links. Note that we are only interested in the number of frontier edges in the link, not its shape or the orientation of any identification.

5.3.5 Canonicity and Automorphisms

When running Algorithm 2, many equivalent manifold decompositions will be found. These decompositions may differ in the order of the walks found. Alternately, two walks may have different starting arcs or directions. For example, the two walks (a, b, c) and $(-b, -a, -c)$ are equivalent. The second starts on a different arc, and traverses the walk backwards, but neither of these change the manifold decomposition. Additionally, the underlying face pairing graph often has non-trivial automorphism group. Finding multiple equivalent manifold decompositions is unnecessary, so we instead only find *canonical* manifold decompositions. This requires the following definitions for comparing walks in an ordered decomposition. Recall that we store walks as a list of numbers (each referring to an external arc) along with the orientation of said arc in the walk.

Definition 5.3.12. *A walk $P = (x_1, x_2, \ldots, x_m)$ in an ordered decomposition is semi-canonical if*

- $x_1 > 0$ *; and*
- $|x_1| \leq |x_i|$ *for $i = 2, \ldots, m$.*

Definition 5.3.13. *A walk $P = (x_1, x_2, \ldots, x_m)$ in an ordered decomposition is canonical if*

- *P is semi-canonical; and*
- *for any semi-canonical $P' = (x'_1, x'_2, \ldots, x'_m)$ isomorphic (under cyclic permutation of the edges in the path or reversal of orientation) to P, either $|x_2| < |x'_2|$ or $|x_2| = |x'_2|$ and $x_2 > 0$.*

This definition of canonical simply says that we always start on the arc with lowest index, and take the arc in a forwards direction. If there are two or three such choices, we take the arc which results in the second arc in the walk having lowest index. If this still leaves us with two choices, we take the walk where we use said second arc in the "forwards" direction. Since there is exactly one internal arc between any two external arcs, we are guaranteed a unique choice at this stage.

Definition 5.3.14. *Given two walks $P_x = (x_1, \ldots, x_k)$ and $P_y = (y_1, \ldots, y_m)$ in canonical form, we say that $P_x < P_y$ if and only if*

- *$x_i = y_i$ for $i = 1, \ldots, n-1$ and $x_n < y_n$; or*
- *$x_i = y_i$ for $i = 1, \ldots, k$ and $k < m$.*

In plainer terms, pairs of arcs from each walk are compared in turn until one arc index is smaller in absolute value than the other, or until the end of one walk is reached in which case the shorter walk is considered "smaller".

Definition 5.3.15. *A manifold decomposition consisting of walks $P_1, P_2, \ldots, P_m$ is considered canonical if:*

- *P_i is canonical for $i = 1, \ldots, m$; and*
- *$P_i < P_{i+1}$ for $i = 1, \ldots, m-1$.*

Recall that we may have automorphisms of the underlying face pairing graph to consider. Each automorphism will relabel the arcs of the labelled fattened face pairing graph. Each relabelling changes any manifold decomposition by renumbering the arcs in the walks. We apply each automorphism to a manifold decomposition $\mathcal{D}$ to obtain a new decomposition $\mathcal{D}'$. Then $\mathcal{D}'$ is made canonical itself (by setting the first external arc in each walk and reordering the walks), and we compare $\mathcal{D}$ and $\mathcal{D}'$. If $\mathcal{D}' < \mathcal{D}$ then we can discard $\mathcal{D}$ and prune the search tree.

Algorithm 4 Is most canonical($\mathcal{D}$)

Require: $\mathcal{D}$ is a partial ordered decomposition
if $\mathcal{D}$ is not in canonical form **then**
  return False
end if
for each g in automorphism group of face pairing graph **do**
  Apply g to $\mathcal{D}$ to obtain $\mathcal{D}'$
  if $\mathcal{D}' < \mathcal{D}$ **then**
    return False
  end if
end for
return True

There are two times in the algorithm we can test for canonical decompositions.

Improvement 5.3.16. Every time an external arc is added to a walk, run Algorithm 4. If the result is false, disregard this choice of arc and prune the search tree here.

Improvement 5.3.17. Every time a walk is completed, run Algorithm 4. If the result is false, disregard this choice of arc and prune the search tree here.

However, Algorithm 4 is not computationally cheap. Experimentation showed that using Improvement 5.3.17 was significantly faster than using Improvement 5.3.16 as Algorithm 4 was not run as often.

5.4 Results and Timing

In this section we detail the results from testing the algorithm. We test the manifold decomposition algorithm from Section 5.3, with various improvements from that section, against *Regina*.

Regina is a suite of topological software and includes state of the art algorithms for census enumerations in various settings (including hyperbolic and normal surface settings). *Regina* is freely available, including source, allowing us to implement our algorithm without building any of the supporting framework. *Regina* also filters out invalid triangulations as a final stage, which allows us to test the efficiency of our improvements by disabling some of them. However like many algorithms in literature *Regina* builds triangulations by identifying faces two at a time. We find that while *Regina* outperforms MD overall, there are a non-trivial number of subcases for which MD is orders of magnitude faster. These are subcases which *Regina* finds extremely difficult, showing that MD is complements *Regina* very well and has great potential in census enumeration algorithms. However this comparison relies on running both algorithms and comparing the resulting running times. Despite looking at many graph parameters (including tree width), we have been unable to determine the heuristics necessary to decide on the best algorithm.

A full census enumeration involves generating all 4-regular multigraphs, and then for each such graph G, determining whether there exists some manifold M with G as its face pairing graph. In earlier sections, we only dealt with individual graphs but for the testing we ran each algorithm on all 4-regular multigraphs of a given order[3].

We are looking for triangulations of closed, minimal, irreducible and $\mathbb{P}^2$-irreducible 3-manifolds. We begin by pointing out that the results confirm the correctness of the MD algorithm. Both MD and *Regina* find a strict superset of the required triangulations, as perfect minimality tests can be computationally expensive. However, via a separate process we verified that any extra triangulations were not minimal.

One algorithmic improvement from Section 5.3 needs the resulting triangulation to be orientable. This gives us two settings to visit already, hence Algorithms MD and MD-o (where MD-o only finds orientable decompositions). Experimental testing with the various improvements from Section 5.3 also pointed towards Improvement 5.3.11 being computationally expensive, leading to Algorithms MD* and MD*-o. Note that these last two algorithms will now find ordered decompositions which are not necessarily manifold decompositions. These are instead filtered out as a final step of the enumeration process.

Algorithm 5 MD

Use Algorithms 1 and 2 with the following improvements:

- Ensure enough external arcs are unused to still finish each walk (5.3.7)
- Ensure the resulting triangulation has only one vertex (5.3.11)
- Ensure the partial decomposition is canonical after completing each walk (5.3.17)

The algorithms were tested on a cluster of Intel Xeon L5520s running at 2.27GHz. Times given are total CPU time; that is, a measure of how long the test would take if run as one single thread on one single core. The algorithms themselves, when run on all 4-regular multigraphs on n nodes, are trivially parallelisable which allows each census to complete much faster by taking advantage of available hardware.

[3] Recall that a triangulation on n tetrahedra has a face pairing graph on n nodes.

Algorithm 6 MD-o

Use Algorithms 1 and 2 with the following improvements:

- Ensure enough external arcs are unused to still finish each walk (5.3.7)
- Ensure the decomposition does not use two parallel external arcs in opposite directions (5.3.10).
- Ensure the resulting triangulation has only one vertex (5.3.11)
- Ensure the partial decomposition is canonical after completing each walk (5.3.17)

Algorithm 7 MD*

Use Algorithms 1 and 2 with the following improvements:

- Ensure enough external arcs are unused to still finish each walk (5.3.7)
- Ensure the partial decomposition is canonical after completing each walk (5.3.17)

Algorithm 8 MD*-o

Use Algorithms 1 and 2 with the following improvements:

- Ensure enough external arcs are unused to still finish each walk (5.3.7)
- Ensure the decomposition does not use two parallel external arcs in opposite directions (5.3.10).
- Ensure the partial decomposition is canonical after completing each walk (5.3.17)

5.4.1 Aggregate tests

In the general setting (where we allow orientable and non-orientable triangulations alike) Table 5.1 highlights that *Regina* outperforms MD. The difference seems to grow slightly as n increases, pointing to the possibility that more optimisations in this setting are possible. We suspect that tracking the orientability of vertex links is giving *Regina* an advantage here (see [42], Section 5). Tracking orientability at all is much harder with ordered decompositions, as the walks are built up one at a time. Each external arc represents an identification of edges, but does not specify the orientation of this identification. Thus no orientability information is given until at least two of any three parallel external arcs are used in walks.

Table 5.1: Running time in seconds of *Regina* and the manifold decomposition (MD) algorithms when searching for all manifold triangulations on on n tetrahedra.

n	*Regina*	MD
7	29	80
8	491	2453
9	11288	79685
10	323530	3406211

We also compare MD-o to *Regina*, where we ask both algorithms to only search for orientable triangulations. Both algorithms run significantly faster (demonstrating that Improvement 5.3.10 is a significant improvement). Table 5.2 shows that *Regina* outperforms MD-o roughly by a factor of four. This appears to be constant and we expect MD to be comparable to *Regina* after micro-optimisations (such as those *Regina* has received, see [41, 42]).

Table 5.2: Running time in seconds of *Regina* and the manifold decomposition (MD-o) algorithm when searching for all orientable manifold triangulations on n tetrahedra.

n	*Regina*	MD-o
7	< 1	25
8	147	535
9	3499	13161
10	90969	430162

To test Improvement 5.3.11, we compare MD* and MD*-o to MD and MD-o respectively. The timing data in Tables 5.3 and 5.4 shows that MD* and MD*-o out-performed MD and MD-o, demonstrating that Improvement 5.3.11 actually slows down the algorithm. We verified that Improvement 5.3.11 is indeed discarding unwanted triangulations; however, tracking the vertex links must be too computationally expensive. We instead use *Regina*'s framework to verify that each triangulation found is a 3-manifold triangulation with 1 vertex. The time for this is included in the timing results, which confirms that such a verification process is faster than the losses incurred by Improvement 5.3.11.

5.4.2 Graph by graph tests

The census enumeration problem requires running the appropriate algorithm (such as Algorithm 7) on all connected 4-regular multigraphs of a given order. Table 5.5 shows the running time of both *Regina* and MD* on a cherry-picked sample of such graphs on 10 tetrahedra. From these we can see that on some particular graphs, MD* outperforms *Regina* by an order of magnitude.

Table 5.3: Running time in seconds of MD-o and MD*-o when searching for orientable manifold triangulations on n tetrahedra.

n	MD-o	MD*-o
7	25	16
8	535	446
9	13161	10753
10	430162	291544

Table 5.4: Running time in seconds of MD and MD* when searching for all manifold triangulations on n tetrahedra.

n	MD	MD*
7	80	71
8	2453	1875
9	79685	58743
10	3406211	1624025

While these graphs were cherry-picked, they do display the shortfalls of *Regina*. There are 48432 4-regular multigraphs on 10 nodes, and it takes *Regina* 89.9 CPU-hours to complete this census.

Of the 48432 graphs, just 190 take over 300 seconds each for *Regina* to process. In total, it takes *Regina* 43.6 CPU-hours to process these 190 graphs. This accounts for 48.5% of the running time of *Regina*'s census on 10 tetrahedra triangulations. Running these graphs through MD takes 12.1 CPU-hours, for a saving of 31.5 CPU-hours. This would reduce the running time of the complete census from 89 hours to 58 hours, a 35% improvement.

Note that this improvement applies if we only consider this specific set of 190 graphs. MD* is slower in general, so for most graphs *Regina* is faster.

If we find the ideal heuristic which tells us exactly which of *Regina* or MD* will be faster on a given graph, we could always use the algorithm which is faster. This would save 40 hours of computing time for the 10 tetrahedra census, which would turn the running time from 90 CPU-hours down to 50 CPU-hours, a 44% improvement. Further work in this area involves identifying exactly which heuristics can be used to determine whether *Regina* or MD will analyse a given graph faster.

Table 5.5: Running time in seconds of MD* and *Regina* on particular graphs on 10 nodes. Here "Task" identifies the specific graph as being the i-th graph produced by *Regina*.

Task	*Regina*	MD*
48308	2476	142
48083	2487	192
48288	2164	118
47332	2141	229
47333	2003	134
47520	2083	221
46914	2108	302

Chapter 6

Conclusion

In Chapter 1 we completely solve a long standing and well-known problem on cycle decompositions of complete graphs; Alspach's conjecture. The obvious major remaining problem concerning cycle decompositions of complete graphs is the Oberwolfach Problem, which concerns decompositions into unions of disjoint cycles (or 2-factors). In Chapter 2 we obtain results on a variant of the problem; namely where we consider 2-factorisations of complete multipartite graphs. The Oberwolfach Problem itself remains open, although there has been considerable progress in recent times.

In Chapter 4 we present a metatheorem showing the existence of fixed parameter tractable algorithms for determining the p-admissibility of 4-regular multigraphs for a broad range of properties p. We also implement the generic version of this algorithm and show that the theoretical algorithm provides significant real-world improvements in running times. The framework for this result opens up the possibility of generating a 3-manifold census on n tetrahedra by building on a census of partial 3-manifold triangulations on $n-1$ tetrahedra. This has the potential to push beyond current limitations of census enumeration algorithms as this has never been achieved before. In Chapter 5 we look at alternate ways of enumerating 3-manifold triangulations. We give an algorithm which topologically identifies edges pairwise rather than faces; a paradigm shift from algorithms in literature. This algorithm is implemented, and results show that this new algorithm performs an order of magnitude faster precisely on subcases where existing state-of-the-art algorithms struggle. This demonstrates how the new algorithm complements existing state of the art algorithms, leading the way for further breakthroughs.

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Appendix A

Tables of Decompositions

A.1 Data for Section 1.6.1

$J_3 \to 3, 6$	$J_5 \to 4, 5, 6$	$J_6 \to 4^3, 6$	$J_7 \to 5^3, 6$
$J_5 \to 4^2, 7$	$J_5 \to 3, 5, 7$	$J_7 \to 4, 5^2, 7$	$J_9 \to 5^4, 7$
$J_7 \to 4^2, 5, 8$	$J_7 \to 3, 5^2, 8$	$J_7 \to 3^3, 4, 8$	$J_8 \to 4^4, 8$
$J_9 \to 4, 5^3, 8$	$J_{11} \to 5^5, 8$		
$J_8 \to 5^3, 9$	$J_8 \to 3, 4^3, 9$	$J_8 \to 3^2, 4, 5, 9$	$J_8 \to 3^5, 9$
$J_9 \to 4^2, 5^2, 9$	$J_{10} \to 4^4, 5, 9$	$J_{11} \to 4^6, 9$	
$J_9 \to 4^3, 5, 10$	$J_9 \to 3, 4, 5^2, 10$	$J_9 \to 3^3, 4^2, 10$	$J_9 \to 3^4, 5, 10$
$J_{10} \to 5^4, 10$	$J_{10} \to 4^5, 10$	$J_{11} \to 4^2, 5^3, 10$	

Table A.1: These decompositions are required for Lemma 1.6.2. The decompositions themselves are given in Table A.3.

$J_8 \to 4^2, 5, 11^*$	$J_8 \to 3, 5^2, 11^*$	$J_8 \to 3^3, 4, 11^*$	$J_9 \to 4^4, 11^*$
$J_{10} \to 4, 5^3, 11^*$	$J_{12} \to 5^5, 11^*$		
$J_9 \to 5^3, 12^*$	$J_9 \to 3, 4^3, 12^*$	$J_9 \to 3^2, 4, 5, 12^*$	$J_9 \to 3^5, 12^*$
$J_{10} \to 4^2, 5^2, 12^*$	$J_{11} \to 4^4, 5, 12^*$	$J_{12} \to 4^6, 12^*$	
$J_{10} \to 4^3, 5, 13^*$	$J_{10} \to 3, 4, 5^2, 13^*$	$J_{10} \to 3^3, 4^2, 13^*$	$J_{10} \to 3^4, 5, 13^*$
$J_{11} \to 5^4, 13^*$	$J_{11} \to 4^5, 13^*$	$J_{12} \to 4^2, 5^3, 13^*$	
$J_{12} \to 4^3, 5^2, 14^*$	$J_{12} \to 3, 4, 5^3, 14^*$	$J_{12} \to 3^2, 4^4, 14^*$	$J_{12} \to 3^3, 4^2, 5, 14^*$
$J_{12} \to 3^4, 5^2, 14^*$	$J_{12} \to 3^6, 4, 14^*$	$J_{13} \to 5^5, 14^*$	$J_{13} \to 4^5, 5, 14^*$
$J_{14} \to 4^2, 5^4, 14^*$	$J_{14} \to 4^7, 14^*$		
$J_{12} \to 4^4, 5, 15^*$	$J_{12} \to 3, 4^2, 5^2, 15^*$	$J_{12} \to 3^2, 5^3, 15^*$	$J_{12} \to 3^3, 4^3, 15^*$
$J_{12} \to 3^4, 4, 5, 15^*$	$J_{12} \to 3^7, 15^*$	$J_{13} \to 4, 5^4, 15^*$	$J_{13} \to 4^6, 15^*$
$J_{14} \to 4^3, 5^3, 15^*$	$J_{15} \to 5^6, 15^*$		
$J_{13} \to 4^2, 5^3, 16^*$	$J_{13} \to 3, 5^4, 16^*$	$J_{13} \to 3, 4^5, 16^*$	$J_{13} \to 3^2, 4^3, 5, 16^*$
$J_{13} \to 3^3, 4, 5^2, 16^*$	$J_{13} \to 3^5, 4^2, 16^*$	$J_{13} \to 3^6, 5, 16^*$	$J_{14} \to 4^4, 5^2, 16^*$
$J_{15} \to 4, 5^5, 16^*$	$J_{15} \to 4^6, 5, 16^*$	$J_{16} \to 4^8, 16^*$	$J_{17} \to 5^7, 16^*$

Table A.2: Decompositions required for Lemma 1.6.3. The decompositions themselves are given in full in Table A.3.

$J_1 \to 3$	$(0, 1, 3)$

Table A.3: Table of decompositions of $J_n^{\{1,2,3\}}$

$J_3 \to 4, 5$	$(0, 1, 4, 2, 3)$, $(1, 2, 5, 3)$
$J_4 \to 4^3$	$(0, 1, 2, 3)$, $(1, 3, 6, 4)$, $(2, 4, 3, 5)$
$J_5 \to 5^3$	$(0, 1, 4, 2, 3)$, $(1, 2, 5, 4, 3)$, $(3, 6, 4, 7, 5)$
$J_3 \to 3, 6$	$(0, 1, 4, 2, 5, 3)$, $(1, 2, 3)$
$J_5 \to 4, 5, 6$	$(0, 1, 4, 5, 2, 3)$, $(1, 2, 4, 3)$, $(3, 6, 4, 7, 5)$
$J_6 \to 4^3, 6$	$(0, 1, 4, 5, 2, 3)$, $(1, 2, 4, 3)$, $(3, 6, 8, 5)$, $(4, 7, 5, 6)$
$J_7 \to 5^3, 6$	$(0, 1, 4, 5, 2, 3)$, $(1, 2, 4, 6, 3)$, $(3, 4, 7, 6, 5)$, $(5, 8, 6, 9, 7)$
$J_5 \to 4^2, 7$	$(0, 1, 2, 5, 4, 6, 3)$, $(1, 4, 2, 3)$, $(3, 4, 7, 5)$
$J_5 \to 3, 5, 7$	$(0, 1, 2, 5, 4, 6, 3)$, $(1, 4, 3)$, $(2, 3, 5, 7, 4)$
$J_7 \to 4, 5^2, 7$	$(0, 1, 4, 6, 5, 2, 3)$, $(1, 2, 4, 3)$, $(4, 5, 8, 6, 7)$, $(3, 6, 9, 7, 5)$
$J_9 \to 5^4, 7$	$(0, 1, 4, 6, 5, 2, 3)$, $(1, 2, 4, 5, 3)$, $(3, 4, 7, 8, 6)$, $(5, 8, 9, 6, 7)$, $(7, 10, 8, 11, 9)$
$J_7 \to 4^2, 5, 8$	$(0, 1, 4, 7, 6, 5, 2, 3)$, $(1, 2, 4, 3)$, $(5, 8, 6, 9, 7)$, $(3, 6, 4, 5)$
$J_7 \to 3, 5^2, 8$	$(0, 1, 4, 7, 6, 5, 2, 3)$, $(1, 2, 4, 6, 3)$, $(3, 4, 5)$, $(5, 8, 6, 9, 7)$
$J_7 \to 3^3, 4, 8$	$(0, 1, 4, 2, 5, 7, 6, 3)$, $(1, 2, 3)$, $(3, 4, 5)$, $(4, 7, 9, 6)$, $(5, 6, 8)$
$J_8 \to 4^4, 8$	$(0, 1, 4, 7, 6, 5, 2, 3)$, $(1, 2, 4, 3)$, $(3, 6, 4, 5)$, $(6, 9, 7, 8)$, $(5, 8, 10, 7)$
$J_9 \to 4, 5^3, 8$	$(0, 1, 4, 6, 7, 5, 2, 3)$, $(1, 2, 4, 3)$, $(4, 5, 6, 8, 7)$, $(3, 6, 9, 8, 5)$, $(7, 10, 8, 11, 9)$
$J_{11} \to 5^5, 8$	$(0, 1, 4, 6, 7, 5, 2, 3)$, $(1, 2, 4, 5, 3)$, $(3, 4, 7, 8, 6)$, $(5, 6, 9, 10, 8)$, $(7, 10, 11, 8, 9)$, $(9, 12, 10, 13, 11)$
$J_8 \to 5^3, 9$	$(0, 1, 4, 6, 7, 8, 5, 2, 3)$, $(1, 2, 4, 5, 3)$, $(3, 4, 7, 5, 6)$, $(6, 9, 7, 10, 8)$
$J_8 \to 3, 4^3, 9$	$(0, 1, 4, 7, 8, 6, 5, 2, 3)$, $(1, 2, 4, 3)$, $(3, 6, 4, 5)$, $(6, 7, 9)$, $(5, 8, 10, 7)$
$J_8 \to 3^2, 4, 5, 9$	$(0, 1, 4, 2, 5, 8, 7, 6, 3)$, $(1, 2, 3)$, $(3, 4, 5)$, $(4, 7, 5, 6)$, $(6, 9, 7, 10, 8)$
$J_8 \to 3^5, 9$	$(0, 1, 2, 4, 7, 5, 8, 6, 3)$, $(2, 3, 5)$, $(1, 4, 3)$, $(4, 5, 6)$, $(6, 7, 9)$, $(7, 8, 10)$
$J_9 \to 4^2, 5^2, 9$	$(0, 1, 4, 6, 8, 7, 5, 2, 3)$, $(1, 2, 4, 3)$, $(4, 5, 6, 7)$, $(3, 6, 9, 8, 5)$, $(7, 10, 8, 11, 9)$
$J_{10} \to 4^4, 5, 9$	$(0, 1, 4, 6, 8, 7, 5, 2, 3)$, $(1, 2, 4, 5, 3)$, $(3, 4, 7, 6)$, $(5, 6, 9, 8)$, $(7, 10, 12, 9)$, $(8, 11, 9, 10)$
$J_{11} \to 4^6, 9$	$(0, 1, 4, 7, 6, 8, 5, 2, 3)$, $(1, 2, 4, 3)$, $(3, 6, 4, 5)$, $(5, 6, 9, 7)$, $(7, 8, 9, 10)$, $(8, 11, 13, 10)$, $(9, 12, 10, 11)$
$J_9 \to 4^3, 5, 10$	$(0, 1, 4, 6, 9, 8, 7, 5, 2, 3)$, $(1, 2, 4, 3)$, $(4, 5, 6, 7)$, $(3, 6, 8, 5)$, $(7, 10, 8, 11, 9)$
$J_9 \to 3, 4, 5^2, 10$	$(0, 1, 4, 2, 5, 7, 8, 9, 6, 3)$, $(1, 2, 3)$, $(3, 4, 7, 6, 5)$, $(4, 5, 8, 6)$, $(7, 10, 8, 11, 9)$
$J_9 \to 3^3, 4^2, 10$	$(0, 1, 4, 2, 5, 8, 9, 7, 6, 3)$, $(1, 2, 3)$, $(3, 4, 7, 5)$, $(4, 5, 6)$, $(7, 8, 10)$, $(6, 9, 11, 8)$
$J_9 \to 3^4, 5, 10$	$(0, 1, 4, 2, 5, 7, 8, 9, 6, 3)$, $(1, 2, 3)$, $(3, 4, 5)$, $(4, 7, 6)$, $(5, 6, 8)$, $(7, 10, 8, 11, 9)$
$J_{10} \to 5^4, 10$	$(0, 1, 4, 6, 9, 7, 8, 5, 2, 3)$, $(1, 2, 4, 5, 3)$, $(3, 4, 7, 5, 6)$, $(6, 7, 10, 9, 8)$, $(8, 11, 9, 12, 10)$
$J_{10} \to 4^5, 10$	$(0, 1, 4, 6, 9, 8, 7, 5, 2, 3)$, $(1, 2, 4, 3)$, $(3, 6, 8, 5)$, $(4, 5, 6, 7)$, $(7, 10, 12, 9)$, $(8, 11, 9, 10)$
$J_{11} \to 4^2, 5^3, 10$	$(0, 1, 4, 6, 9, 7, 8, 5, 2, 3)$, $(1, 2, 4, 5, 3)$, $(3, 4, 7, 5, 6)$, $(6, 7, 10, 9, 8)$, $(8, 11, 13, 10)$, $(9, 12, 10, 11)$
$J_8 \to 4^2, 5, 11^*$	$(0, 1, 2, 5, 8, 10, 7, 9, 6, 4, 3)$, $(1, 4, 7, 6, 3)$, $(2, 3, 5, 4)$, $(5, 6, 8, 7)$
$J_8 \to 3, 5^2, 11^*$	$(0, 1, 4, 2, 5, 8, 10, 7, 9, 6, 3)$, $(1, 2, 3)$, $(3, 4, 6, 7, 5)$, $(4, 5, 6, 8, 7)$
$J_8 \to 3^3, 4, 11^*$	$(0, 1, 4, 2, 5, 8, 10, 7, 9, 6, 3)$, $(1, 2, 3)$, $(3, 4, 5)$, $(4, 7, 5, 6)$, $(6, 7, 8)$
$J_9 \to 4^4, 11^*$	$(1, 2, 4, 6, 9, 11, 8, 10, 7, 5, 3)$, $(0, 1, 4, 3)$, $(2, 3, 6, 5)$, $(4, 5, 8, 7)$, $(6, 7, 9, 8)$
$J_{10} \to 4, 5^3, 11^*$	$(2, 3, 6, 4, 7, 10, 12, 9, 11, 8, 5)$, $(0, 1, 2, 4, 3)$, $(1, 4, 5, 3)$, $(5, 6, 9, 8, 7)$, $(6, 7, 9, 10, 8)$
$J_{12} \to 5^5, 11^*$	$(4, 5, 8, 6, 9, 12, 14, 11, 13, 10, 7)$, $(0, 1, 2, 4, 3)$, $(1, 4, 6, 5, 3)$, $(2, 3, 6, 7, 5)$, $(7, 8, 10, 11, 9)$, $(8, 9, 10, 12, 11)$

Table A.3: Table of decompositions of $J_n^{\{1,2,3\}}$

$J_9 \to 5^3, 12^*$	$(0,1,2,4,6,9,11,8,10,7,5,3)$, $(1,4,5,2,3)$, $(3,4,7,8,6)$, $(5,6,7,9,8)$
$J_9 \to 3, 4^3, 12^*$	$(0,1,4,7,10,8,11,9,6,5,2,3)$, $(1,2,4,3)$, $(3,6,4,5)$, $(5,8,6,7)$, $(7,8,9)$
$J_9 \to 3^2, 4, 5, 12^*$	$(0,1,4,7,10,8,11,9,6,5,2,3)$, $(1,2,4,5,3)$, $(3,4,6)$, $(5,8,6,7)$, $(7,8,9)$
$J_9 \to 3^5, 12^*$	$(0,1,2,4,5,7,10,8,11,9,6,3)$, $(2,3,5)$, $(1,4,3)$, $(5,6,8)$, $(4,7,6)$, $(7,8,9)$
$J_{10} \to 4^2, 5^2, 12^*$	$(1,2,5,8,11,9,12,10,7,4,6,3)$, $(0,1,4,3)$, $(2,3,5,4)$, $(5,6,8,9,7)$, $(6,7,8,10,9)$
$J_{11} \to 4^4, 5, 12^*$	$(2,3,6,8,11,13,10,12,9,7,5,4)$, $(0,1,4,3)$, $(1,2,5,3)$, $(4,7,8,5,6)$, $(6,7,10,9)$, $(8,9,11,10)$
$J_{12} \to 4^6, 12^*$	$(3,4,7,5,8,10,13,11,14,12,9,6)$, $(0,1,2,3)$, $(1,4,5,3)$, $(2,5,6,4)$, $(6,7,9,8)$, $(7,8,11,10)$, $(9,10,12,11)$
$J_{10} \to 4^3, 5, 13^*$	$(0,1,4,6,8,11,9,12,10,7,5,2,3)$, $(1,2,4,5,3)$, $(3,4,7,6)$, $(5,6,9,8)$, $(7,8,10,9)$
$J_{10} \to 3, 4, 5^2, 13^*$	$(0,1,2,5,8,11,9,12,10,7,4,6,3)$, $(1,4,2,3)$, $(3,4,5)$, $(5,6,8,9,7)$, $(6,7,8,10,9)$
$J_{10} \to 3^3, 4^2, 13^*$	$(0,1,2,5,8,11,9,12,10,7,4,6,3)$, $(1,4,2,3)$, $(3,4,5)$, $(5,6,7)$, $(6,9,7,8)$, $(8,9,10)$
$J_{10} \to 3^4, 5, 13^*$	$(0,1,2,4,7,10,12,9,11,8,5,6,3)$, $(1,4,3)$, $(2,3,5)$, $(4,5,7,9,6)$, $(6,7,8)$, $(8,9,10)$
$J_{11} \to 5^4, 13^*$	$(1,2,4,7,5,8,11,13,10,12,9,6,3)$, $(0,1,4,5,3)$, $(2,3,4,6,5)$, $(6,7,9,10,8)$, $(7,8,9,11,10)$
$J_{11} \to 4^5, 13^*$	$(1,4,2,5,7,9,12,10,13,11,8,6,3)$, $(0,1,2,3)$, $(3,4,6,5)$, $(4,5,8,7)$, $(6,7,10,9)$, $(8,9,11,10)$
$J_{12} \to 4^2, 5^3, 13^*$	$(2,3,5,8,6,9,12,14,11,13,10,7,4)$, $(1,2,5,6,3)$, $(0,1,4,3)$, $(4,5,7,6)$, $(7,8,10,11,9)$, $(8,9,10,12,11)$
$J_{12} \to 4^3, 5^2, 14^*$	$(1,2,4,7,10,13,11,14,12,9,6,8,5,3)$, $(0,1,4,3)$, $(2,3,6,5)$, $(4,5,7,6)$, $(7,8,10,11,9)$, $(8,9,10,12,11)$
$J_{12} \to 3, 4, 5^3, 14^*$	$(1,4,7,10,13,11,14,12,9,6,8,5,2,3)$, $(0,1,2,4,3)$, $(3,6,4,5)$, $(5,6,7)$, $(7,8,10,11,9)$, $(8,9,10,12,11)$
$J_{12} \to 3^2, 4^4, 14^*$	$(1,2,3,6,5,8,9,12,14,11,13,10,7,4)$, $(0,1,3)$, $(2,5,3,4)$, $(4,5,7,6)$, $(7,8,10,9)$, $(6,9,11,8)$, $(10,11,12)$
$J_{12} \to 3^3, 4^2, 5, 14^*$	$(1,4,7,10,13,11,14,12,9,6,8,5,2,3)$, $(0,1,2,4,3)$, $(3,6,4,5)$, $(5,6,7)$, $(7,8,9)$, $(8,11,9,10)$, $(10,11,12)$
$J_{12} \to 3^4, 5^2, 14^*$	$(1,2,3,5,8,6,9,12,14,11,13,10,7,4)$, $(0,1,3)$, $(3,4,6)$, $(2,5,4)$, $(5,6,7)$, $(7,8,11,10,9)$, $(8,9,11,12,10)$
$J_{12} \to 3^6, 4, 14^*$	$(1,2,3,5,8,6,9,12,14,11,13,10,7,4)$, $(0,1,3)$, $(2,5,4)$, $(3,4,6)$, $(5,6,7)$, $(7,8,10,9)$, $(8,9,11)$, $(10,11,12)$
$J_{13} \to 5^5, 14^*$	$(2,3,6,8,10,13,15,12,14,11,9,7,4,5)$, $(0,1,2,4,3)$, $(1,4,6,5,3)$, $(5,8,9,6,7)$, $(7,8,11,12,10)$, $(9,10,11,13,12)$
$J_{13} \to 4^5, 5, 14^*$	$(2,5,8,10,13,15,12,14,11,9,7,6,3,4)$, $(1,4,5,3)$, $(0,1,2,3)$, $(6,9,10,7,8)$, $(4,7,5,6)$, $(8,9,12,11)$, $(10,11,13,12)$
$J_{14} \to 4^2, 5^4, 14^*$	$(3,4,5,7,10,8,11,14,16,13,15,12,9,6)$, $(0,1,4,2,3)$, $(1,2,5,3)$, $(4,7,8,5,6)$, $(6,7,9,8)$, $(9,10,12,13,11)$, $(10,11,12,14,13)$
$J_{14} \to 4^7, 14^*$	$(3,4,7,5,8,10,12,15,13,16,14,11,9,6)$, $(0,1,2,3)$, $(1,4,5,3)$, $(2,5,6,4)$, $(6,7,9,8)$, $(7,8,11,10)$, $(9,10,13,12)$, $(11,12,14,13)$
$J_{12} \to 4^4, 5, 15^*$	$(0,1,4,7,9,12,14,11,13,10,8,6,5,2,3)$, $(1,2,4,3)$, $(5,8,9,6,7)$, $(3,6,4,5)$, $(7,8,11,10)$, $(9,10,12,11)$

Table A.3: Table of decompositions of $J_n^{\{1,2,3\}}$

$J_{12} \to 3, 4^2, 5^2, 15^*$	$(0,1,4,7,10,13,11,14,12,9,6,8,5,2,3)$, $(1,2,4,3)$, $(3,6,7,5)$, $(4,5,6)$, $(7,8,10,11,9)$, $(8,9,10,12,11)$
$J_{12} \to 3^2, 5^3, 15^*$	$(0,1,4,7,10,13,11,14,12,9,6,8,5,2,3)$, $(1,2,4,5,3)$, $(3,4,6)$, $(5,6,7)$, $(7,8,10,11,9)$, $(8,9,10,12,11)$
$J_{12} \to 3^3, 4^3, 15^*$	$(0,1,4,7,10,13,11,14,12,9,6,8,5,2,3)$, $(1,2,4,3)$, $(3,6,4,5)$, $(5,6,7)$, $(7,8,9)$, $(8,11,9,10)$, $(10,11,12)$
$J_{12} \to 3^4, 4, 5, 15^*$	$(0,1,4,7,10,13,11,14,12,9,6,8,5,2,3)$, $(1,2,4,5,3)$, $(3,4,6)$, $(5,6,7)$, $(7,8,9)$, $(8,11,9,10)$, $(10,11,12)$
$J_{12} \to 3^7, 15^*$	$(0,1,2,4,5,7,8,10,13,11,14,12,9,6,3)$, $(1,4,3)$, $(2,3,5)$, $(4,7,6)$, $(5,6,8)$, $(8,9,11)$, $(7,10,9)$, $(10,11,12)$
$J_{13} \to 4, 5^4, 15^*$	$(1,2,5,8,11,14,12,15,13,10,7,9,6,4,3)$, $(0,1,4,2,3)$, $(3,6,8,7,5)$, $(4,5,6,7)$, $(8,9,11,12,10)$, $(9,10,11,13,12)$
$J_{13} \to 4^6, 15^*$	$(1,2,5,8,10,13,15,12,14,11,9,7,4,6,3)$, $(0,1,4,3)$, $(2,3,5,4)$, $(5,6,8,7)$, $(6,7,10,9)$, $(8,9,12,11)$, $(10,11,13,12)$
$J_{14} \to 4^3, 5^3, 15^*$	$(2,3,6,9,12,15,13,16,14,11,8,10,7,5,4)$, $(0,1,4,3)$, $(1,2,5,3)$, $(4,7,8,5,6)$, $(6,7,9,8)$, $(9,10,12,13,11)$, $(10,11,12,14,13)$
$J_{15} \to 5^6, 15^*$	$(3,6,4,7,10,13,16,14,17,15,12,9,11,8,5)$, $(1,4,5,2,3)$, $(0,1,2,4,3)$, $(5,6,8,9,7)$, $(6,7,8,10,9)$, $(10,11,13,14,12)$, $(11,12,13,15,14)$
$J_{13} \to 4^2, 5^3, 16^*$	$(0,1,2,5,8,11,14,12,15,13,10,7,9,6,4,3)$, $(1,4,2,3)$, $(3,6,8,7,5)$, $(4,5,6,7)$, $(8,9,11,12,10)$, $(9,10,11,13,12)$
$J_{13} \to 3, 5^4, 16^*$	$(0,1,2,5,8,11,14,12,15,13,10,7,9,6,4,3)$, $(2,3,5,7,4)$, $(1,4,5,6,3)$, $(6,7,8)$, $(8,9,11,12,10)$, $(9,10,11,13,12)$
$J_{13} \to 3, 4^5, 16^*$	$(0,1,2,5,8,11,14,12,15,13,10,9,6,7,4,3)$, $(2,3,5,4)$, $(1,4,6,3)$, $(5,6,8,7)$, $(8,9,11,10)$, $(7,10,12,9)$, $(11,12,13)$
$J_{13} \to 3^2, 4^3, 5, 16^*$	$(0,1,2,5,8,11,14,12,15,13,10,9,7,4,6,3)$, $(1,4,2,3)$, $(3,4,5)$, $(5,6,8,10,7)$, $(6,7,8,9)$, $(9,12,13,11)$, $(10,11,12)$
$J_{13} \to 3^3, 4, 5^2, 16^*$	$(0,1,2,5,8,11,14,12,15,13,10,9,7,4,6,3)$, $(1,4,2,3)$, $(3,4,5)$, $(5,6,7)$, $(6,9,11,10,8)$, $(7,8,9,12,10)$, $(11,12,13)$
$J_{13} \to 3^5, 4^2, 16^*$	$(0,1,2,5,8,11,14,12,15,13,10,9,7,4,6,3)$, $(1,4,2,3)$, $(3,4,5)$, $(5,6,7)$, $(6,9,8)$, $(7,8,10)$, $(9,12,10,11)$, $(11,12,13)$
$J_{13} \to 3^6, 5, 16^*$	$(0,1,4,2,5,7,10,13,15,12,14,11,8,9,6,3)$, $(1,2,3)$, $(3,4,5)$, $(5,6,8)$, $(4,7,6)$, $(7,8,10,12,9)$, $(9,10,11)$, $(11,12,13)$
$J_{14} \to 4^4, 5^2, 16^*$	$(1,4,2,5,7,10,8,11,14,16,13,15,12,9,6,3)$, $(0,1,2,3)$, $(3,4,6,5)$, $(4,5,8,7)$, $(6,7,9,8)$, $(9,10,12,13,11)$, $(10,11,12,14,13)$
$J_{15} \to 4, 5^5, 16^*$	$(2,3,5,6,8,11,9,12,15,17,14,16,13,10,7,4)$, $(1,2,5,4,3)$, $(0,1,4,6,3)$, $(6,7,8,10,9)$, $(5,8,9,7)$, $(10,11,13,14,12)$, $(11,12,13,15,14)$
$J_{15} \to 4^6, 5, 16^*$	$(2,5,8,10,12,15,17,14,16,13,11,9,7,6,3,4)$, $(1,4,5,3)$, $(0,1,2,3)$, $(6,9,12,11,8)$, $(4,7,5,6)$, $(7,8,9,10)$, $(10,11,14,13)$, $(12,13,15,14)$
$J_{16} \to 4^8, 16^*$	$(3,4,7,5,8,10,12,14,17,15,18,16,13,11,9,6)$, $(0,1,2,3)$, $(1,4,5,3)$, $(2,5,6,4)$, $(6,7,9,8)$, $(7,8,11,10)$, $(9,10,13,12)$, $(11,12,15,14)$, $(13,14,16,15)$
$J_{17} \to 5^7, 16^*$	$(4,5,8,6,9,12,15,18,16,19,17,14,11,13,10,7)$, $(0,1,2,4,3)$, $(1,4,6,5,3)$, $(2,3,6,7,5)$, $(7,8,10,11,9)$, $(8,9,10,12,11)$, $(12,13,15,16,14)$, $(13,14,15,17,16)$
$J_3^+ \to 4^2, 1^+$	$[0,3,2]$, $(0,1,4,2)$, $(1,2,5,3)$
$J_4^+ \to 4, 5, 3^+$	$[0,3,6,4,2]$, $(0,1,3,5,2)$, $(1,2,3,4)$
$J_4^+ \to 5^2, 2^+$	$[0,3,4,2]$, $(0,1,3,5,2)$, $(1,2,3,6,4)$

Table A.3: Table of decompositions of $J_n^{\{1,2,3\}}$

$J_5^+ \to 5^2, 5^{+*}$	$[0,3,6,4,7,5,2]$, $(0,1,4,3,2)$, $(1,2,4,5,3)$
$J_6^+ \to 3^4, 6^{+*}$	$[0,3,6,8,5,7,4,2]$, $(0,2,3)$, $(1,3,4)$, $(2,3,5)$, $(5,6,7)$
$J_6^+ \to 4^3, 6^{+*}$	$[0,3,6,8,5,7,4,2]$, $(0,1,3,2)$, $(1,2,5,4)$, $(3,4,6,5)$
$J_6^+ \to 5^3, 3^+$	$[0,3,6,5,2]$, $(0,1,4,3,2)$, $(1,2,4,5,3)$, $(4,6,8,5,7)$
$J_7^+ \to 5^3, 6^+$	$[0,3,5,7,9,6,4,2]$, $(0,1,4,3,2)$, $(1,2,4,5,3)$, $(4,5,8,6,7)$
$J_8^+ \to 5^4, 4^+$	$[0,3,6,7,5,2]$, $(0,1,3,4,2)$, $(1,2,3,5,4)$, $(4,6,5,8,7)$, $(6,8,10,7,9)$

Table A.3: Table of decompositions of $J_n^{\{1,2,3\}}$

$J_3 \to 3^2, 6$	$J_4 \to 5^2, 6$	$J_4 \to 4, 5, 7$	$J_4 \to 3^3, 7$
$J_5 \to 4^3, 8$	$J_5 \to 3, 4, 5, 8$	$J_5 \to 3^4, 8$	$J_6 \to 3^2, 5^2, 8$
$J_7 \to 5^4, 8$			

Table A.4: These decompositions are required for Lemma 1.6.8. The decompositions themselves are given in Table A.6.

$J_6 \to 5^3, 9^*$	$J_6 \to 3, 4^3, 9^*$	$J_6 \to 3^2, 4, 5, 9^*$	$J_6 \to 3^5, 9^*$
$J_7 \to 3^3, 5^2, 9^*$			
$J_7 \to 4^2, 5^2, 10^*$	$J_7 \to 3, 5^3, 10^*$	$J_7 \to 3^2, 4^3, 10^*$	$J_7 \to 3^3, 4, 5, 10^*$
$J_7 \to 3^6, 10^*$	$J_8 \to 3^4, 5^2, 10^*$	$J_{10} \to 5^6, 10^*$	
$J_8 \to 4^4, 5, 11^*$	$J_8 \to 3, 4^2, 5^2, 11^*$	$J_8 \to 3^2, 5^3, 11^*$	$J_8 \to 3^3, 4^3, 11^*$
$J_8 \to 3^4, 4, 5, 11^*$	$J_8 \to 3^7, 11^*$	$J_9 \to 5^5, 11^*$	$J_9 \to 3^5, 5^2, 11^*$
$J_9 \to 4, 5^4, 12^*$	$J_9 \to 4^6, 12^*$	$J_9 \to 3, 4^4, 5, 12^*$	$J_9 \to 3^2, 4^2, 5^2, 12^*$
$J_9 \to 3^3, 5^3, 12^*$	$J_9 \to 3^4, 4^3, 12^*$	$J_9 \to 3^5, 4, 5, 12^*$	$J_9 \to 3^8, 12^*$
$J_{10} \to 3, 5^5, 12^*$	$J_{10} \to 3^6, 5^2, 12^*$	$J_{13} \to 5^8, 12^*$	
$J_{10} \to 4^3, 5^3, 13^*$	$J_{10} \to 3, 4, 5^4, 13^*$	$J_{10} \to 3, 4^6, 13^*$	$J_{10} \to 3^2, 4^4, 5, 13^*$
$J_{10} \to 3^3, 4^2, 5^2, 13^*$	$J_{10} \to 3^4, 5^3, 13^*$	$J_{10} \to 3^5, 4^3, 13^*$	$J_{10} \to 3^6, 4, 5, 13^*$
$J_{10} \to 3^9, 13^*$	$J_{11} \to 3^2, 5^5, 13^*$	$J_{11} \to 3^7, 5^2, 13^*$	$J_{12} \to 5^7, 13^*$

Table A.5: These decompositions are required for Lemma 1.6.9. The decompositions themselves are given in Table A.6.

$J_1 \to 4$	$(0,4,2,3)$
$J_2 \to 3, 5$	$(0,4,1,5,3)$, $(2,3,4)$
$J_3 \to 3^4$	$(0,4,3)$, $(1,5,4)$, $(2,6,4)$, $(2,5,3)$
$J_4 \to 3^2, 5^2$	$(0,4,2,5,3)$, $(1,5,7,3,4)$, $(2,6,3)$, $(4,5,6)$
$J_5 \to 5^4$	$(0,4,2,5,3)$, $(1,5,6,3,4)$, $(2,6,4,7,3)$, $(4,8,6,7,5)$
$J_3 \to 3^2, 6$	$(0,4,1,5,2,3)$, $(2,6,4)$, $(3,4,5)$
$J_4 \to 5^2, 6$	$(0,4,1,5,2,3)$, $(2,6,5,3,4)$, $(4,5,7,3,6)$
$J_4 \to 4, 5, 7$	$(0,4,1,5,2,6,3)$, $(2,3,5,6,4)$, $(3,7,5,4)$
$J_4 \to 3^3, 7$	$(0,4,1,5,2,6,3)$, $(2,3,4)$, $(3,7,5)$, $(4,5,6)$
$J_5 \to 4^3, 8$	$(0,4,1,5,2,6,7,3)$, $(2,3,6,4)$, $(3,4,7,5)$, $(4,8,6,5)$
$J_5 \to 3, 4, 5, 8$	$(0,4,1,5,2,6,7,3)$, $(2,3,4)$, $(3,6,8,4,5)$, $(4,7,5,6)$
$J_5 \to 3^4, 8$	$(0,4,1,5,2,6,7,3)$, $(2,3,4)$, $(3,6,5)$, $(4,8,6)$, $(4,7,5)$
$J_6 \to 3^2, 5^2, 8$	$(0,4,1,5,2,6,7,3)$, $(2,3,4)$, $(3,6,8,4,5)$, $(4,7,9,5,6)$, $(5,8,7)$
$J_7 \to 5^4, 8$	$(0,4,1,5,2,6,7,3)$, $(2,3,5,6,4)$, $(3,6,8,5,4)$, $(5,9,8,4,7)$, $(7,8,10,6,9)$

Table A.6: Table of decompositions of $J_n^{\{1,2,3,4\}}$

$J_6 \to 5^3, 9^*$	$(2,6,8,4,1,5,9,7,3)$, $(0,4,2,5,3)$, $(3,6,7,5,4)$, $(4,7,8,5,6)$
$J_6 \to 3, 4^3, 9^*$	$(2,6,8,4,1,5,9,7,3)$, $(0,4,3)$, $(2,5,6,4)$, $(3,6,7,5)$, $(4,7,8,5)$
$J_6 \to 3^2, 4, 5, 9^*$	$(2,6,8,4,1,5,9,7,3)$, $(0,4,2,5,3)$, $(3,6,4)$, $(4,7,5)$, $(5,8,7,6)$
$J_6 \to 3^5, 9^*$	$(2,6,8,4,1,5,9,7,3)$, $(0,4,3)$, $(2,5,4)$, $(3,6,5)$, $(4,7,6)$, $(5,8,7)$
$J_7 \to 3^3, 5^2, 9^*$	$(2,3,7,9,6,10,8,5,4)$, $(0,4,1,5,3)$, $(2,6,5)$, $(3,6,4)$, $(5,9,8,4,7)$, $(6,7,8)$
$J_7 \to 4^2, 5^2, 10^*$	$(2,3,7,9,6,10,8,5,1,4)$, $(0,4,6,5,3)$, $(3,6,2,5,4)$, $(4,8,6,7)$, $(5,9,8,7)$
$J_7 \to 3, 5^3, 10^*$	$(2,3,7,9,6,10,8,5,1,4)$, $(0,4,3)$, $(2,6,4,7,5)$, $(3,6,8,4,5)$, $(5,9,8,7,6)$
$J_7 \to 3^2, 4^3, 10^*$	$(2,3,7,9,6,10,8,5,1,4)$, $(0,4,3)$, $(2,6,4,5)$, $(3,6,5)$, $(4,8,6,7)$, $(5,9,8,7)$
$J_7 \to 3^3, 4, 5, 10^*$	$(2,3,7,9,6,10,8,5,1,4)$, $(0,4,3)$, $(2,6,4,7,5)$, $(3,6,5)$, $(4,8,9,5)$, $(6,7,8)$
$J_7 \to 3^6, 10^*$	$(2,5,1,4,8,10,6,9,7,3)$, $(0,4,3)$, $(3,6,5)$, $(4,7,5)$, $(2,6,4)$, $(5,9,8)$, $(6,7,8)$
$J_8 \to 3^4, 5^2, 10^*$	$(2,3,5,6,9,11,7,10,8,4)$, $(0,4,3)$, $(1,5,2,6,4)$, $(3,7,6)$, $(4,7,5)$, $(6,10,9,5,8)$, $(7,8,9)$
$J_{10} \to 5^6, 10^*$	$(4,5,7,8,11,13,9,12,10,6)$, $(0,4,2,5,3)$, $(1,5,6,3,4)$, $(2,6,9,7,3)$, $(4,8,10,11,7)$, $(5,9,11,12,8)$, $(6,7,10,9,8)$
$J_8 \to 4^4, 5, 11^*$	$(2,6,9,11,7,10,8,4,1,5,3)$, $(0,4,6,7,3)$, $(2,5,7,4)$, $(3,6,5,4)$, $(5,9,7,8)$, $(6,10,9,8)$
$J_8 \to 3, 4^2, 5^2, 11^*$	$(2,6,9,11,7,10,8,4,1,5,3)$, $(0,4,3)$, $(2,5,7,6,4)$, $(4,7,3,6,5)$, $(5,9,7,8)$, $(6,10,9,8)$
$J_8 \to 3^2, 5^3, 11^*$	$(2,6,9,11,7,10,8,4,1,5,3)$, $(0,4,3)$, $(2,5,4)$, $(3,7,5,8,6)$, $(4,7,9,5,6)$, $(6,10,9,8,7)$
$J_8 \to 3^3, 4^3, 11^*$	$(2,6,9,11,7,10,8,4,1,5,3)$, $(0,4,3)$, $(2,5,4)$, $(3,7,5,6)$, $(4,7,6)$, $(5,9,7,8)$, $(6,10,9,8)$
$J_8 \to 3^4, 4, 5, 11^*$	$(2,6,9,11,7,10,8,4,1,5,3)$, $(0,4,3)$, $(2,5,4)$, $(3,7,5,8,6)$, $(4,7,6)$, $(5,9,10,6)$, $(7,8,9)$
$J_8 \to 3^7, 11^*$	$(2,6,10,8,4,1,5,9,11,7,3)$, $(0,4,3)$, $(3,6,5)$, $(2,5,4)$, $(6,9,8)$, $(4,7,6)$, $(5,8,7)$, $(7,10,9)$
$J_9 \to 5^5, 11^*$	$(2,3,5,9,11,8,12,10,7,6,4)$, $(0,4,5,6,3)$, $(1,5,7,3,4)$, $(2,6,9,8,5)$, $(4,8,10,9,7)$, $(6,10,11,7,8)$
$J_9 \to 3^5, 5^2, 11^*$	$(2,3,5,9,11,8,12,10,7,6,4)$, $(0,4,3)$, $(1,5,4)$, $(2,6,3,7,5)$, $(4,8,7)$, $(5,8,6)$, $(7,11,10,6,9)$, $(8,9,10)$
$J_9 \to 4, 5^4, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,6,5,3)$, $(2,6,9,7,5)$, $(3,7,8,5,4)$, $(4,8,10,11,7)$, $(6,10,9,8)$
$J_9 \to 4^6, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,5,3)$, $(2,6,8,5)$, $(3,7,8,4)$, $(4,7,5,6)$, $(6,10,8,9)$, $(7,11,10,9)$
$J_9 \to 3, 4^4, 5, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,3)$, $(2,6,4,7,5)$, $(3,7,8,5)$, $(4,8,6,5)$, $(6,10,8,9)$, $(7,11,10,9)$
$J_9 \to 3^2, 4^2, 5^2, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,3)$, $(2,6,4,7,5)$, $(3,7,8,4,5)$, $(5,8,6)$, $(6,10,8,9)$, $(7,11,10,9)$
$J_9 \to 3^3, 5^3, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,3)$, $(2,6,4,7,5)$, $(3,7,8,4,5)$, $(5,8,6)$, $(6,10,9)$, $(7,11,10,8,9)$
$J_9 \to 3^4, 4^3, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,3)$, $(2,6,4,5)$, $(3,7,5)$, $(4,8,7)$, $(5,8,6)$, $(6,10,8,9)$, $(7,11,10,9)$

Table A.6: Table of decompositions of $J_n^{\{1,2,3,4\}}$

$J_9 \to 3^5, 4, 5, 12^*$	$(2,3,6,7,10,12,8,11,9,5,1,4)$, $(0,4,3)$, $(2,6,4,5)$, $(5,8,6)$, $(3,7,5)$, $(6,10,9)$, $(4,8,10,11,7)$, $(7,8,9)$
$J_9 \to 3^8, 12^*$	$(2,3,6,10,12,8,7,11,9,5,1,4)$, $(0,4,3)$, $(4,8,5)$, $(3,7,5)$, $(2,6,5)$, $(4,7,6)$, $(7,10,9)$, $(6,9,8)$, $(8,11,10)$
$J_{10} \to 3, 5^5, 12^*$	$(2,6,10,12,9,13,11,8,7,5,3,4)$, $(0,4,6,7,3)$, $(1,5,6,8,4)$, $(2,5,9,6,3)$, $(4,7,9,8,5)$, $(8,12,11,9,10)$, $(7,11,10)$
$J_{10} \to 3^6, 5^2, 12^*$	$(2,6,10,12,8,9,13,11,7,5,3,4)$, $(0,4,6,7,3)$, $(1,5,4)$, $(2,5,8,6,3)$, $(5,9,6)$, $(4,8,7)$, $(8,11,10)$, $(7,10,9)$, $(9,12,11)$
$J_{13} \to 5^8, 12^*$	$(5,9,13,15,12,16,14,11,10,8,6,7)$, $(0,4,2,5,3)$, $(1,5,6,3,4)$, $(2,6,9,7,3)$, $(4,8,7,10,6)$, $(4,7,11,8,5)$, $(8,12,14,10,9)$, $(9,12,10,13,11)$, $(11,15,14,13,12)$
$J_{10} \to 4^3, 5^3, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,5,2,3)$, $(3,6,5,7,4)$, $(4,8,5,9,6)$, $(6,7,9,8)$, $(7,11,9,10)$, $(8,12,11,10)$
$J_{10} \to 3, 4, 5^4, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,5,2,3)$, $(3,6,4)$, $(5,6,8,4,7)$, $(5,9,7,10,8)$, $(6,9,11,7)$, $(8,12,11,10,9)$
$J_{10} \to 3, 4^6, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,3)$, $(2,5,6,3)$, $(4,8,6,7)$, $(4,5,9,6)$, $(5,8,9,7)$, $(7,11,9,10)$, $(8,12,11,10)$
$J_{10} \to 3^2, 4^4, 5, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,5,2,3)$, $(3,6,4)$, $(4,8,6,7)$, $(5,9,7)$, $(5,8,9,6)$, $(7,11,9,10)$, $(8,12,11,10)$
$J_{10} \to 3^3, 4^2, 5^2, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,5,2,3)$, $(3,6,4)$, $(5,6,8,4,7)$, $(5,9,8)$, $(6,9,7)$, $(7,11,9,10)$, $(8,12,11,10)$
$J_{10} \to 3^4, 5^3, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,5,2,3)$, $(3,6,4)$, $(5,6,8,4,7)$, $(5,9,8)$, $(6,9,7)$, $(8,12,11,7,10)$, $(9,10,11)$
$J_{10} \to 3^5, 4^3, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,3)$, $(2,5,6,3)$, $(4,8,6)$, $(4,7,5)$, $(6,9,11,7)$, $(5,9,8)$, $(7,10,9)$, $(8,12,11,10)$
$J_{10} \to 3^6, 4, 5, 13^*$	$(2,6,10,12,9,13,11,8,7,3,5,1,4)$, $(0,4,3)$, $(2,5,6,3)$, $(5,9,7)$, $(4,8,5)$, $(4,7,6)$, $(8,12,11,7,10)$, $(6,9,8)$, $(9,10,11)$
$J_{10} \to 3^9, 13^*$	$(2,6,10,12,8,9,13,11,7,4,1,5,3)$, $(2,5,4)$, $(0,4,3)$, $(5,9,6)$, $(3,7,6)$, $(4,8,6)$, $(5,8,7)$, $(8,11,10)$, $(7,10,9)$, $(9,12,11)$
$J_{11} \to 3^2, 5^5, 13^*$	$(2,3,7,11,13,10,14,12,9,8,5,6,4)$, $(0,4,1,5,3)$, $(3,6,2,5,4)$, $(4,8,6,9,7)$, $(5,9,10,6,7)$, $(7,10,8)$, $(8,12,11)$, $(9,13,12,10,11)$
$J_{11} \to 3^7, 5^2, 13^*$	$(2,3,7,11,13,9,10,14,12,8,5,6,4)$, $(1,5,4)$, $(2,6,9,7,5)$, $(0,4,3)$, $(3,6,8,9,5)$, $(4,8,7)$, $(9,12,11)$, $(6,10,7)$, $(8,11,10)$, $(10,13,12)$
$J_{12} \to 5^7, 13^*$	$(3,4,8,12,14,11,15,13,10,9,6,7,5)$, $(0,4,2,6,3)$, $(1,5,8,6,4)$, $(2,5,4,7,3)$, $(5,9,7,10,6)$, $(7,11,13,9,8)$, $(8,11,9,12,10)$, $(10,14,13,12,11)$
$J_1^+ \to 3, 1^{+*}$	$[1,3]$, $[0,4,2]$, $(0,2,3)$
$J_3^+ \to 4, 5, 3^{+*}$	$[1,5,3]$, $[0,4,6,2]$, $(0,2,4,1,3)$, $(2,5,4,3)$
$J_4^+ \to 4^3, 4^{+*}$	$[1,5,7,3]$, $[0,4,6,2]$, $(1,3,5,4)$, $(2,5,6,3)$, $(0,2,4,3)$
$J_5^+ \to 5^3, 5^{+*}$	$[1,4,8,6,5,7,3]$, $[0,2]$, $(1,3,6,4,5)$, $(2,6,7,4,3)$, $(0,4,2,5,3)$
$J_3^+ \to 5^2, 2^+$	$[1,3]$, $[0,4,6,2]$, $(0,2,4,5,3)$, $(2,5,1,4,3)$
$J_6^+ \to 5^4, 4^+$	$[1,3]$, $[0,4,7,9,5,2]$, $(0,2,4,5,3)$, $(1,5,6,3,4)$, $(2,6,8,7,3)$, $(4,8,5,7,6)$
$J_4^+ \to 5^3, 1^+$	$[1,3]$, $[0,4,2]$, $(0,2,6,5,3)$, $(2,5,1,4,3)$, $(4,5,7,3,6)$
$J_7^+ \to 5^5, 3^+$	$[1,3]$, $[0,4,7,5,2]$, $(0,2,4,5,3)$, $(1,5,6,3,4)$, $(2,6,8,7,3)$, $(4,8,5,9,6)$, $(6,10,8,9,7)$

Table A.6: Table of decompositions of $J_n^{\{1,2,3,4\}}$

$J_2 \to 4, 6$	$J_3 \to 3^3, 6$	$J_4 \to 3^2, 4^2, 6$	$J_5 \to 3^5, 4, 6$
$J_6 \to 3^8, 6$			
$J_2 \to 3, 7$	$J_3 \to 4^2, 7$	$J_4 \to 3^3, 4, 7$	$J_5 \to 3^6, 7$
$J_7 \to 3^8, 4, 7$			
$J_3 \to 3, 4, 8$	$J_4 \to 4^3, 8$	$J_4 \to 3^4, 8$	$J_5 \to 3^3, 4^2, 8$
$J_6 \to 3^6, 4, 8$	$J_7 \to 3^9, 8$		
$J_3 \to 3^2, 9$	$J_4 \to 3, 4^2, 9$	$J_5 \to 4^4, 9$	$J_5 \to 3^4, 4, 9$
$J_6 \to 3^7, 9$	$J_8 \to 3^9, 4, 9$		
$J_4 \to 5^2, 10$	$J_4 \to 3^2, 4, 10$	$J_5 \to 3, 4^3, 10$	$J_5 \to 3^5, 10$
$J_6 \to 4^5, 10$	$J_6 \to 3^4, 4^2, 10$	$J_7 \to 3^7, 4, 10$	$J_8 \to 3^{10}, 10$

Table A.7: These decompositions are required for Lemma 1.6.14. The decompositions themselves are given in Table A.9.

$J_6 \to 4, 5^3, 11^*$	$J_6 \to 3, 4^4, 11^*$	$J_6 \to 3^2, 4^2, 5, 11^*$	$J_6 \to 3^3, 5^2, 11^*$
$J_6 \to 3^5, 4, 11^*$	$J_7 \to 4^6, 11^*$	$J_7 \to 3^4, 4^3, 11^*$	$J_7 \to 3^8, 11^*$
$J_8 \to 3^3, 4^5, 11^*$	$J_8 \to 3^7, 4^2, 11^*$	$J_9 \to 3^{10}, 4, 11^*$	$J_{10} \to 3^{13}, 11^*$
$J_6 \to 4^2, 5^2, 12^*$	$J_6 \to 3, 5^3, 12^*$	$J_6 \to 3^2, 4^3, 12^*$	$J_6 \to 3^3, 4, 5, 12^*$
$J_6 \to 3^6, 12^*$	$J_7 \to 3, 4^5, 12^*$	$J_7 \to 3^5, 4^2, 12^*$	$J_8 \to 4^7, 12^*$
$J_8 \to 3^4, 4^4, 12^*$	$J_8 \to 3^8, 4, 12^*$	$J_9 \to 3^{11}, 12^*$	$J_{11} \to 3^{13}, 4, 12^*$
$J_8 \to 4^3, 5^3, 13^*$	$J_8 \to 3, 4, 5^4, 13^*$	$J_8 \to 3, 4^6, 13^*$	$J_8 \to 3^2, 4^4, 5, 13^*$
$J_8 \to 3^3, 4^2, 5^2, 13^*$	$J_8 \to 3^4, 5^3, 13^*$	$J_8 \to 3^5, 4^3, 13^*$	$J_8 \to 3^6, 4, 5, 13^*$
$J_8 \to 3^9, 13^*$	$J_9 \to 4^8, 13^*$	$J_9 \to 3^4, 4^5, 13^*$	$J_9 \to 3^8, 4^2, 13^*$
$J_{10} \to 3^3, 4^7, 13^*$	$J_{10} \to 3^7, 4^4, 13^*$	$J_{10} \to 3^{11}, 4, 13^*$	$J_{11} \to 3^{14}, 13^*$
$J_{13} \to 3^{16}, 4, 13^*$			
$J_8 \to 4^4, 5^2, 14^*$	$J_8 \to 3, 4^2, 5^3, 14^*$	$J_8 \to 3^2, 5^4, 14^*$	$J_8 \to 3^2, 4^5, 14^*$
$J_8 \to 3^3, 4^3, 5, 14^*$	$J_8 \to 3^4, 4, 5^2, 14^*$	$J_8 \to 3^6, 4^2, 14^*$	$J_8 \to 3^7, 5, 14^*$
$J_9 \to 3, 4^7, 14^*$	$J_9 \to 3^5, 4^4, 14^*$	$J_9 \to 3^9, 4, 14^*$	$J_{10} \to 4^9, 14^*$
$J_{10} \to 3^4, 4^6, 14^*$	$J_{10} \to 3^8, 4^3, 14^*$	$J_{10} \to 3^{12}, 14^*$	$J_{11} \to 3^{11}, 4^2, 14^*$
$J_{12} \to 3^{14}, 4, 14^*$	$J_{13} \to 3^{17}, 14^*$		
$J_{10} \to 5^7, 15^*$	$J_{10} \to 3^{10}, 5, 15^*$	$J_{12} \to 3^{15}, 15^*$	$J_{15} \to 3^{20}, 15^*$
$J_{11} \to 3^{13}, 16^*$	$J_{14} \to 3^{18}, 16^*$		

Table A.8: These decompositions are required for Lemma 1.6.15. The decompositions themselves are given in Table A.9.

$J_1 \to 5$	$(0,6,3,2,4)$
$J_3 \to 3, 4^3$	$(0,6,2,4)$, $(1,7,4,5)$, $(2,8,5,3)$, $(3,4,6)$
$J_3 \to 3^2, 4, 5$	$(0,6,4)$, $(1,7,4,3,5)$, $(3,2,6)$, $(2,8,5,4)$
$J_4 \to 4^5$	$(0,6,2,4)$, $(1,7,4,5)$, $(2,8,5,3)$, $(3,7,5,6)$, $(3,9,6,4)$
$J_4 \to 3^4, 4^2$	$(0,6,2,4)$, $(1,7,5)$, $(2,8,5,3)$, $(4,3,7)$, $(3,9,6)$, $(4,6,5)$
$J_5 \to 3^3, 4^4$	$(0,6,2,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(3,9,6)$, $(4,6,5)$, $(4,10,7)$, $(5,7,6,8)$
$J_5 \to 3^5, 5^2$	$(0,6,7,3,4)$, $(2,8,6,9,3)$, $(2,6,4)$, $(5,4,8)$, $(1,7,5)$, $(3,5,6)$, $(4,10,7)$
$J_5 \to 3^7, 4$	$(0,6,4)$, $(3,9,6,5)$, $(4,10,7)$, $(1,7,5)$, $(3,7,6)$, $(2,4,3)$, $(5,4,8)$, $(2,8,6)$
$J_6 \to 3^2, 4^6$	$(0,6,2,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(3,9,6)$, $(4,10,7,6)$, $(5,6,8)$, $(4,5,9,7)$, $(5,11,8,7)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_6 \to 3^6, 4^3$	$(0,6,2,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(3,9,6)$, $(4,6,5)$, $(4,10,7)$, $(5,9,7)$, $(5,11,8)$, $(6,8,7)$
$J_6 \to 3^{10}$	$(0,6,4)$, $(1,7,5)$, $(2,8,4)$, $(3,2,6)$, $(3,9,7)$, $(3,5,4)$, $(4,10,7)$, $(5,11,8)$, $(6,5,9)$, $(6,8,7)$
$J_7 \to 3^5, 4^5$	$(0,6,2,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(3,9,6)$, $(4,6,5)$, $(5,9,7)$, $(5,11,8)$, $(6,12,9,8)$, $(7,6,10)$, $(4,10,8,7)$
$J_7 \to 3^9, 4^2$	$(0,6,2,4)$, $(1,7,5)$, $(2,8,4,3)$, $(3,7,6)$, $(4,6,5)$, $(5,11,8)$, $(3,9,5)$, $(6,12,9)$, $(6,10,8)$, $(4,10,7)$, $(7,9,8)$
$J_8 \to 3^{12}, 4$	$(0,6,7,4)$, $(2,8,4)$, $(3,2,6)$, $(3,9,7)$, $(3,5,4)$, $(4,10,6)$, $(5,6,8)$, $(5,11,9)$, $(1,7,5)$, $(6,12,9)$, $(7,13,10)$, $(8,7,11)$, $(8,10,9)$
$J_2 \to 4, 6$	$(0,6,3,4)$, $(2,4,7,1,5,3)$
$J_3 \to 3^3, 6$	$(0,6,4)$, $(2,8,5,1,7,4)$, $(3,2,6)$, $(3,5,4)$
$J_4 \to 3^2, 4^2, 6$	$(0,6,9,3,2,4)$, $(1,7,3,5)$, $(2,8,5,6)$, $(3,4,6)$, $(4,5,7)$
$J_5 \to 3^5, 4, 6$	$(0,6,9,3,2,4)$, $(1,7,3,5)$, $(2,8,6)$, $(3,4,6)$, $(4,10,7)$, $(5,4,8)$, $(5,7,6)$
$J_6 \to 3^8, 6$	$(0,6,7,8,2,4)$, $(3,9,7)$, $(1,7,5)$, $(3,5,4)$, $(3,2,6)$, $(4,10,7)$, $(5,11,8)$, $(4,8,6)$, $(6,5,9)$
$J_2 \to 3, 7$	$(0,6,3,5,1,7,4)$, $(2,4,3)$
$J_3 \to 4^2, 7$	$(0,6,3,5,8,2,4)$, $(1,7,4,5)$, $(2,6,4,3)$
$J_4 \to 3^3, 4, 7$	$(0,6,3,5,8,2,4)$, $(1,7,5)$, $(2,6,9,3)$, $(4,3,7)$, $(4,6,5)$
$J_5 \to 3^6, 7$	$(0,6,4)$, $(2,8,4)$, $(2,6,7,1,5,4,3)$, $(3,9,6)$, $(3,7,5)$, $(4,10,7)$, $(5,6,8)$
$J_7 \to 3^8, 4, 7$	$(0,6,9,5,3,2,4)$, $(4,6,5)$, $(1,7,5)$, $(2,8,6)$, $(3,9,12,6)$, $(4,3,7)$, $(7,6,10)$, $(4,10,8)$, $(7,9,8)$, $(5,11,8)$
$J_3 \to 3, 4, 8$	$(0,6,2,4)$, $(3,2,8,5,1,7,4,6)$, $(3,5,4)$
$J_4 \to 4^3, 8$	$(0,6,3,7,5,8,2,4)$, $(1,7,4,5)$, $(2,6,4,3)$, $(3,9,6,5)$
$J_4 \to 3^4, 8$	$(0,6,9,3,5,8,2,4)$, $(1,7,5)$, $(3,2,6)$, $(4,3,7)$, $(4,6,5)$
$J_5 \to 3^3, 4^2, 8$	$(0,6,3,7,5,8,2,4)$, $(1,7,6,5)$, $(3,5,4)$, $(2,6,9,3)$, $(4,8,6)$, $(4,10,7)$
$J_6 \to 3^6, 4, 8$	$(0,6,3,5,11,8,2,4)$, $(2,6,4,3)$, $(3,9,7)$, $(4,10,7)$, $(5,4,8)$, $(1,7,5)$, $(6,5,9)$, $(6,8,7)$
$J_7 \to 3^9, 8$	$(0,6,5,1,7,3,2,4)$, $(3,9,5)$, $(5,11,8)$, $(3,4,6)$, $(2,8,6)$, $(6,12,9)$, $(4,5,7)$, $(4,10,8)$, $(7,6,10)$, $(7,9,8)$
$J_3 \to 3^2, 9$	$(0,6,3,2,8,5,1,7,4)$, $(2,6,4)$, $(3,5,4)$
$J_4 \to 3, 4^2, 9$	$(0,6,3,7,1,5,8,2,4)$, $(2,6,4,3)$, $(3,9,6,5)$, $(4,5,7)$
$J_5 \to 4^4, 9$	$(0,6,3,7,1,5,8,2,4)$, $(2,6,4,3)$, $(3,9,6,5)$, $(4,8,6,7)$, $(4,10,7,5)$
$J_5 \to 3^4, 4, 9$	$(0,6,3,7,1,5,8,2,4)$, $(2,6,9,3)$, $(3,5,4)$, $(4,10,7)$, $(4,8,6)$, $(5,7,6)$
$J_6 \to 3^7, 9$	$(0,6,9,5,1,7,8,2,4)$, $(4,10,7)$, $(5,7,6)$, $(3,2,6)$, $(3,5,4)$, $(4,8,6)$, $(5,11,8)$, $(3,9,7)$
$J_8 \to 3^9, 4, 9$	$(0,6,5,3,7,11,8,2,4)$, $(4,3,9,7)$, $(5,4,8)$, $(3,2,6)$, $(6,12,9)$, $(1,7,5)$, $(6,8,7)$, $(4,10,6)$, $(7,13,10)$, $(5,11,9)$, $(8,10,9)$
$J_4 \to 5^2, 10$	$(0,6,9,3,7,1,5,8,2,4)$, $(3,5,7,4,6)$, $(2,6,5,4,3)$
$J_4 \to 3^2, 4, 10$	$(0,6,9,3,7,1,5,8,2,4)$, $(3,2,6)$, $(3,5,4)$, $(4,6,5,7)$
$J_5 \to 3, 4^3, 10$	$(0,6,9,3,7,1,5,8,2,4)$, $(3,2,6)$, $(3,5,6,4)$, $(4,8,6,7)$, $(4,10,7,5)$
$J_5 \to 3^5, 10$	$(0,6,9,3,7,1,5,8,2,4)$, $(3,2,6)$, $(3,5,4)$, $(4,8,6)$, $(4,10,7)$, $(5,7,6)$
$J_6 \to 4^5, 10$	$(0,6,3,7,1,5,11,8,2,4)$, $(2,6,4,3)$, $(3,9,6,5)$, $(4,8,6,7)$, $(4,10,7,5)$, $(5,9,7,8)$
$J_6 \to 3^4, 4^2, 10$	$(0,6,3,7,1,5,11,8,2,4)$, $(2,6,4,3)$, $(3,9,6,5)$, $(4,10,7)$, $(5,4,8)$, $(5,9,7)$, $(6,8,7)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_7 \to 3^7, 4, 10$	$(0,6,3,7,1,5,11,8,2,4)$, $(2,6,4,3)$, $(3,9,5)$, $(6,12,9)$, $(5,7,6)$, $(4,10,7)$, $(6,10,8)$, $(5,4,8)$, $(7,9,8)$
$J_8 \to 3^{10}, 10$	$(2,4,7,8,11,5,6,10,9,3)$, $(3,5,4)$, $(0,6,4)$, $(2,8,6)$, $(6,12,9)$, $(3,7,6)$, $(1,7,5)$, $(7,11,9)$, $(7,13,10)$, $(4,10,8)$, $(5,9,8)$
$J_6 \to 4, 5^3, 11^*$	$(2,8,11,5,1,7,10,4,6,9,3)$, $(0,6,2,4)$, $(3,4,7,5,6)$, $(3,7,6,8,5)$, $(4,8,7,9,5)$
$J_6 \to 3, 4^4, 11^*$	$(2,8,11,5,1,7,10,4,6,9,3)$, $(0,6,2,4)$, $(3,4,5,6)$, $(3,7,8,5)$, $(4,8,6,7)$, $(5,9,7)$
$J_6 \to 3^2, 4^2, 5, 11^*$	$(2,8,11,5,1,7,10,4,6,9,3)$, $(0,6,2,4)$, $(3,4,7,5,6)$, $(3,7,9,5)$, $(5,4,8)$, $(6,8,7)$
$J_6 \to 3^3, 5^2, 11^*$	$(2,8,11,5,1,7,10,4,6,9,3)$, $(0,6,3,7,4)$, $(2,6,5,3,4)$, $(5,4,8)$, $(5,9,7)$, $(6,8,7)$
$J_6 \to 3^5, 4, 11^*$	$(2,8,11,5,1,7,10,4,6,9,3)$, $(0,6,2,4)$, $(3,5,6)$, $(4,3,7)$, $(5,4,8)$, $(5,9,7)$, $(6,8,7)$
$J_7 \to 4^6, 11^*$	$(3,2,4,7,10,8,11,5,9,12,6)$, $(0,6,5,4)$, $(1,7,3,5)$, $(2,8,7,6)$, $(3,9,6,4)$, $(4,10,6,8)$, $(5,7,9,8)$
$J_7 \to 3^4, 4^3, 11^*$	$(3,2,4,7,10,8,11,5,9,12,6)$, $(0,6,4)$, $(1,7,3,5)$, $(2,8,6)$, $(4,10,6,5)$, $(3,9,8,4)$, $(5,7,8)$, $(6,7,9)$
$J_7 \to 3^8, 11^*$	$(2,8,11,5,3,9,12,6,7,10,4)$, $(3,2,6)$, $(0,6,4)$, $(6,10,8)$, $(6,5,9)$, $(4,3,7)$, $(1,7,5)$, $(5,4,8)$, $(7,9,8)$
$J_8 \to 3^3, 4^5, 11^*$	$(4,3,5,8,11,9,12,6,10,13,7)$, $(0,6,2,4)$, $(1,7,5)$, $(3,2,8,6)$, $(3,9,6,7)$, $(4,6,5)$, $(4,10,7,8)$, $(5,11,7,9)$, $(8,10,9)$
$J_8 \to 3^7, 4^2, 11^*$	$(4,3,5,8,11,9,12,6,10,13,7)$, $(0,6,4)$, $(1,7,11,5)$, $(4,10,7,5)$, $(2,8,4)$, $(3,2,6)$, $(6,8,7)$, $(3,9,7)$, $(6,5,9)$, $(8,10,9)$
$J_9 \to 3^{10}, 4, 11^*$	$(4,8,14,11,7,13,10,12,9,6,5)$, $(2,4,3)$, $(0,6,4)$, $(3,7,6)$, $(3,9,5)$, $(1,7,5)$, $(4,10,7)$, $(2,8,6)$, $(6,12,8,10)$, $(7,9,8)$, $(9,11,10)$, $(5,11,8)$
$J_{10} \to 3^{13}, 11^*$	$(5,9,15,12,8,14,11,7,13,10,6)$, $(3,2,6)$, $(3,5,4)$, $(1,7,5)$, $(0,6,4)$, $(6,12,9)$, $(2,8,4)$, $(6,8,7)$, $(3,9,7)$, $(4,10,7)$, $(9,13,11)$, $(8,10,9)$, $(10,12,11)$, $(5,11,8)$
$J_6 \to 4^2, 5^2, 12^*$	$(0,6,9,3,2,8,11,5,1,7,10,4)$, $(3,7,4,2,6)$, $(3,5,8,6,4)$, $(4,8,7,5)$, $(5,9,7,6)$
$J_6 \to 3, 5^3, 12^*$	$(0,6,9,3,2,8,11,5,1,7,10,4)$, $(3,7,4,2,6)$, $(3,5,4)$, $(4,8,7,5,6)$, $(5,9,7,6,8)$
$J_6 \to 3^2, 4^3, 12^*$	$(0,6,9,3,2,8,11,5,1,7,10,4)$, $(2,6,7,4)$, $(3,5,4,6)$, $(3,7,8,4)$, $(5,6,8)$, $(5,9,7)$
$J_6 \to 3^3, 4, 5, 12^*$	$(0,6,9,3,2,8,11,5,1,7,10,4)$, $(3,7,4,2,6)$, $(3,5,4)$, $(5,6,4,8)$, $(5,9,7)$, $(6,8,7)$
$J_6 \to 3^6, 12^*$	$(0,6,9,3,2,8,11,5,1,7,10,4)$, $(2,6,4)$, $(3,5,6)$, $(4,3,7)$, $(5,4,8)$, $(5,9,7)$, $(6,8,7)$
$J_7 \to 3, 4^5, 12^*$	$(1,7,10,4,3,9,12,6,2,8,11,5)$, $(0,6,7,4)$, $(2,4,5,3)$, $(3,7,9,6)$, $(5,6,4,8)$, $(5,9,8,7)$, $(6,10,8)$
$J_7 \to 3^5, 4^2, 12^*$	$(1,7,10,4,3,9,12,6,2,8,11,5)$, $(0,6,7,4)$, $(3,2,4,6)$, $(3,7,5)$, $(5,4,8)$, $(6,5,9)$, $(6,10,8)$, $(7,9,8)$
$J_8 \to 4^7, 12^*$	$(2,4,7,13,10,6,12,9,11,8,5,3)$, $(0,6,3,4)$, $(1,7,6,5)$, $(2,8,4,6)$, $(3,9,8,7)$, $(4,10,7,5)$, $(5,11,7,9)$, $(6,8,10,9)$
$J_8 \to 3^4, 4^4, 12^*$	$(2,4,7,13,10,6,12,9,11,8,5,3)$, $(0,6,3,4)$, $(1,7,5)$, $(2,8,4,6)$, $(3,9,7)$, $(5,11,7,6)$, $(4,10,9,5)$, $(6,8,9)$, $(7,8,10)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_8 \to 3^8, 4, 12^*$	$(2,8,11,5,3,9,12,6,10,13,7,4)$, $(0,6,4)$, $(1,7,5)$, $(5,4,8)$, $(3,7,10,4)$, $(3,2,6)$, $(6,8,7)$, $(6,5,9)$, $(7,11,9)$, $(8,10,9)$
$J_9 \to 3^{11}, 12^*$	$(3,9,12,6,10,13,7,11,14,8,4,5)$, $(1,7,5)$, $(5,11,8)$, $(0,6,4)$, $(2,8,6)$, $(2,4,3)$, $(6,5,9)$, $(3,7,6)$, $(7,9,8)$, $(4,10,7)$, $(8,12,10)$, $(9,11,10)$
$J_{11} \to 3^{13}, 4, 12^*$	$(5,6,7,10,16,13,9,15,12,14,11,8)$, $(3,9,5)$, $(4,10,11,5)$, $(0,6,4)$, $(1,7,5)$, $(4,3,7)$, $(3,2,6)$, $(2,8,4)$, $(8,14,10)$, $(10,12,13)$, $(7,13,11)$, $(7,9,8)$, $(6,10,9)$, $(9,11,12)$, $(6,12,8)$
$J_8 \to 4^3, 5^3, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,7,4)$, $(2,8,5,4,6)$, $(3,9,7,10,4)$, $(5,9,8,7)$, $(5,11,7,6)$, $(6,8,10,9)$
$J_8 \to 3, 4, 5^4, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,7,4)$, $(2,8,5,4,6)$, $(3,9,7,10,4)$, $(6,7,5,9)$, $(5,11,7,8,6)$, $(8,10,9)$
$J_8 \to 3, 4^6, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,4)$, $(2,8,9,6)$, $(3,9,5,7)$, $(4,6,5)$, $(4,10,9,7)$, $(5,11,7,8)$, $(7,6,8,10)$
$J_8 \to 3^2, 4^4, 5, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,7,4)$, $(2,8,9,6)$, $(3,9,5,4)$, $(4,10,7,6)$, $(5,6,8)$, $(5,11,7)$, $(7,9,10,8)$
$J_8 \to 3^3, 4^2, 5^2, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,7,4)$, $(2,8,5,4,6)$, $(3,9,10,4)$, $(5,11,7,6)$, $(5,9,7)$, $(6,8,9)$, $(7,8,10)$
$J_8 \to 3^4, 5^3, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,7,4)$, $(2,8,5,4,6)$, $(3,9,7,10,4)$, $(5,11,7)$, $(6,5,9)$, $(6,8,7)$, $(8,10,9)$
$J_8 \to 3^5, 4^3, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,4)$, $(2,8,9,6)$, $(3,9,7)$, $(5,11,7)$, $(5,6,8)$, $(4,6,7)$, $(4,10,9,5)$, $(7,8,10)$
$J_8 \to 3^6, 4, 5, 13^*$	$(2,4,8,11,9,12,6,10,13,7,1,5,3)$, $(0,6,3,7,4)$, $(2,8,6)$, $(3,9,10,4)$, $(4,6,5)$, $(5,11,7)$, $(7,8,10)$, $(6,7,9)$, $(5,9,8)$
$J_8 \to 3^9, 13^*$	$(2,8,11,9,12,6,10,13,7,1,5,3,4)$, $(0,6,4)$, $(5,11,7)$, $(6,5,9)$, $(5,4,8)$, $(3,2,6)$, $(6,8,7)$, $(4,10,7)$, $(8,10,9)$, $(3,9,7)$
$J_9 \to 4^8, 13^*$	$(3,2,8,14,11,7,13,10,12,9,5,4,6)$, $(0,6,2,4)$, $(1,7,3,5)$, $(4,3,9,7)$, $(4,10,7,8)$, $(5,7,6,8)$, $(6,5,11,9)$, $(6,12,8,10)$, $(8,9,10,11)$
$J_9 \to 3^4, 4^5, 13^*$	$(3,2,8,14,11,7,13,10,12,9,5,4,6)$, $(0,6,2,4)$, $(1,7,3,5)$, $(4,3,9,7)$, $(4,10,7,8)$, $(5,7,6)$, $(5,11,8)$, $(6,12,8,9)$, $(6,10,8)$, $(9,11,10)$
$J_9 \to 3^8, 4^2, 13^*$	$(3,2,8,14,11,7,13,10,12,9,5,4,6)$, $(0,6,2,4)$, $(1,7,5)$, $(3,9,6,5)$, $(4,3,7)$, $(6,12,8)$, $(7,6,10)$, $(5,11,8)$, $(7,9,8)$, $(4,10,8)$, $(9,11,10)$
$J_{10} \to 3^3, 4^7, 13^*$	$(3,9,15,12,8,14,11,13,10,7,5,4,6)$, $(0,6,2,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(4,10,6,7)$, $(5,9,6,8)$, $(5,11,12,6)$, $(7,13,9,8)$, $(7,11,9)$, $(8,10,11)$, $(9,10,12)$
$J_{10} \to 3^7, 4^4, 13^*$	$(3,9,15,12,8,14,11,13,10,7,5,4,6)$, $(0,6,2,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(4,10,6,7)$, $(5,6,8)$, $(5,11,9)$, $(6,12,9)$, $(7,13,9)$, $(8,7,11)$, $(8,10,9)$, $(10,12,11)$
$J_{10} \to 3^{11}, 4, 13^*$	$(3,9,15,12,8,14,11,7,13,10,4,5,6)$, $(3,7,9,5)$, $(2,4,3)$, $(1,7,5)$, $(0,6,4)$, $(2,8,6)$, $(4,8,7)$, $(7,6,10)$, $(9,13,11)$, $(6,12,9)$, $(5,11,8)$, $(8,10,9)$, $(10,12,11)$
$J_{11} \to 3^{14}, 13^*$	$(4,10,16,13,9,15,12,8,14,11,5,6,7)$, $(0,6,4)$, $(3,5,4)$, $(3,2,6)$, $(1,7,5)$, $(8,7,11)$, $(5,9,8)$, $(2,8,4)$, $(6,10,8)$, $(3,9,7)$, $(10,14,12)$, $(6,12,9)$, $(9,11,10)$, $(11,13,12)$, $(7,13,10)$
$J_{13} \to 3^{16}, 4, 13^*$	$(6,12,18,15,11,17,14,10,16,13,7,8,9)$, $(3,2,6)$, $(0,6,10,4)$, $(3,5,4)$, $(5,6,8)$, $(4,6,7)$, $(2,8,4)$, $(1,7,5)$, $(3,9,7)$, $(5,11,9)$, $(7,11,10)$, $(8,12,10)$, $(10,9,13)$, $(12,16,14)$, $(8,14,11)$, $(11,13,12)$, $(13,15,14)$, $(9,15,12)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_8 \to 4^4, 5^2, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(3,7,4,2,6)$, $(3,9,6,5,4)$, $(4,8,7,6)$, $(5,7,10,8)$, $(5,11,7,9)$, $(6,10,9,8)$
$J_8 \to 3, 4^2, 5^3, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(3,7,4,2,6)$, $(3,9,6,5,4)$, $(5,7,6,4,8)$, $(5,11,7,9)$, $(7,8,6,10)$, $(8,10,9)$
$J_8 \to 3^2, 5^4, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(3,7,4,2,6)$, $(3,9,6,5,4)$, $(5,7,6,4,8)$, $(5,11,7,8,9)$, $(6,10,8)$, $(7,9,10)$
$J_8 \to 3^2, 4^5, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(2,6,7,4)$, $(3,4,6)$, $(3,9,5,7)$, $(5,11,7,8)$, $(4,8,6,5)$, $(6,10,7,9)$, $(8,10,9)$
$J_8 \to 3^3, 4^3, 5, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(3,7,4,2,6)$, $(3,9,5,4)$, $(4,8,9,6)$, $(5,6,8)$, $(5,11,7)$, $(7,6,10)$, $(7,9,10,8)$
$J_8 \to 3^4, 4, 5^2, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(3,7,4,2,6)$, $(3,9,6,5,4)$, $(4,8,7,6)$, $(5,9,8)$, $(5,11,7)$, $(6,10,8)$, $(7,9,10)$
$J_8 \to 3^6, 4^2, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(2,6,7,4)$, $(3,4,6)$, $(3,9,10,7)$, $(5,11,7)$, $(5,4,8)$, $(6,5,9)$, $(6,10,8)$, $(7,9,8)$
$J_8 \to 3^7, 5, 14^*$	$(0,6,12,9,11,8,2,3,5,1,7,13,10,4)$, $(2,6,8,7,4)$, $(5,11,7)$, $(5,4,8)$, $(3,4,6)$, $(7,6,10)$, $(3,9,7)$, $(8,10,9)$, $(6,5,9)$
$J_9 \to 3, 4^7, 14^*$	$(2,8,14,11,5,1,7,13,10,12,9,6,4,3)$, $(0,6,2,4)$, $(3,9,5,6)$, $(3,7,8,5)$, $(4,5,7)$, $(4,10,11,8)$, $(6,12,8,10)$, $(6,8,9,7)$, $(7,11,9,10)$
$J_9 \to 3^5, 4^4, 14^*$	$(2,8,14,11,5,1,7,13,10,12,9,6,4,3)$, $(0,6,2,4)$, $(3,9,5,6)$, $(3,7,8,5)$, $(4,5,7)$, $(4,10,11,8)$, $(6,12,8)$, $(7,6,10)$, $(7,11,9)$, $(8,10,9)$
$J_9 \to 3^9, 4, 14^*$	$(2,8,14,11,5,1,7,13,10,12,9,6,4,3)$, $(0,6,2,4)$, $(3,9,7)$, $(4,5,7)$, $(3,5,6)$, $(6,12,8)$, $(7,6,10)$, $(8,7,11)$, $(5,9,8)$, $(4,10,8)$, $(9,11,10)$
$J_{10} \to 4^9, 14^*$	$(3,9,15,12,8,14,11,13,10,7,5,4,2,6)$, $(0,6,7,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(4,10,8,6)$, $(5,11,10,6)$, $(6,12,10,9)$, $(7,13,9,11)$, $(5,9,7,8)$, $(8,9,12,11)$
$J_{10} \to 3^4, 4^6, 14^*$	$(3,9,15,12,8,14,11,13,10,7,5,4,2,6)$, $(0,6,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(4,10,6,7)$, $(5,9,6,8)$, $(5,11,12,6)$, $(7,13,9,8)$, $(7,11,9)$, $(8,10,11)$, $(9,10,12)$
$J_{10} \to 3^8, 4^3, 14^*$	$(3,9,15,12,8,14,11,13,10,7,5,4,2,6)$, $(0,6,4)$, $(1,7,3,5)$, $(2,8,4,3)$, $(4,10,6,7)$, $(5,6,8)$, $(5,11,9)$, $(6,12,9)$, $(7,13,9)$, $(8,7,11)$, $(8,10,9)$, $(10,12,11)$
$J_{10} \to 3^{12}, 14^*$	$(2,4,7,10,13,11,14,8,5,6,12,15,9,3)$, $(3,7,6)$, $(3,5,4)$, $(0,6,4)$, $(2,8,6)$, $(1,7,5)$, $(7,13,9)$, $(5,11,9)$, $(8,7,11)$, $(4,10,8)$, $(6,10,9)$, $(9,8,12)$, $(10,12,11)$
$J_{11} \to 3^{11}, 4^2, 14^*$	$(3,7,4,10,16,13,9,15,12,14,11,8,5,6)$, $(0,6,4)$, $(2,6,9,3)$, $(2,8,4)$, $(3,5,4)$, $(1,7,11,5)$, $(8,14,10)$, $(9,8,12)$, $(6,8,7)$, $(5,9,7)$, $(9,11,10)$, $(6,12,10)$, $(11,13,12)$, $(7,13,10)$
$J_{12} \to 3^{14}, 4, 14^*$	$(4,8,5,11,17,14,10,16,13,15,12,6,9,7)$, $(2,8,6)$, $(3,9,5)$, $(2,4,3)$, $(0,6,5,4)$, $(1,7,5)$, $(3,7,6)$, $(4,10,6)$, $(9,15,11)$, $(9,8,12)$, $(12,14,13)$, $(8,14,11)$, $(7,8,10)$, $(10,9,13)$, $(7,13,11)$, $(10,12,11)$
$J_{13} \to 3^{17}, 14^*$	$(6,12,18,15,9,5,11,17,14,16,13,10,8,7)$, $(3,5,4)$, $(0,6,4)$, $(1,7,5)$, $(5,6,8)$, $(2,8,4)$, $(3,2,6)$, $(3,9,7)$, $(4,10,7)$, $(6,10,9)$, $(7,13,11)$, $(10,16,12)$, $(11,10,14)$, $(12,11,15)$, $(8,9,11)$, $(9,13,12)$, $(13,15,14)$, $(8,14,12)$
$J_{10} \to 5^7, 15^*$	$(3,9,15,12,8,14,11,13,10,7,1,5,4,2,6)$, $(0,6,5,7,4)$, $(2,8,6,4,3)$, $(3,7,6,9,5)$, $(4,10,9,7,8)$, $(6,12,9,11,10)$, $(8,9,13,7,11)$, $(5,11,12,10,8)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_{10} \to 3^{10}, 5, 15^*$	$(3,9,15,12,8,14,11,5,1,7,13,10,4,2,6)$, $(2,8,9,5,3)$, $(0,6,4)$, $(4,3,7)$, $(5,4,8)$, $(5,7,6)$, $(6,12,9)$, $(6,10,8)$, $(9,13,11)$, $(7,9,10)$, $(8,7,11)$, $(10,12,11)$
$J_{12} \to 3^{15}, 15^*$	$(3,7,13,16,10,6,12,15,9,8,14,17,11,5,4)$, $(3,2,6)$, $(0,6,4)$, $(3,9,5)$, $(1,7,5)$, $(6,7,9)$, $(5,6,8)$, $(2,8,4)$, $(4,10,7)$, $(8,7,11)$, $(11,15,13)$, $(9,11,10)$, $(8,12,10)$, $(9,13,12)$, $(10,14,13)$, $(11,12,14)$
$J_{15} \to 3^{20}, 15^*$	$(6,12,10,11,9,15,18,16,19,13,17,20,14,8,7)$, $(3,2,6)$, $(0,6,4)$, $(1,7,5)$, $(3,9,7)$, $(3,5,4)$, $(4,10,7)$, $(2,8,4)$, $(6,5,9)$, $(6,10,8)$, $(5,11,8)$, $(9,8,12)$, $(10,9,13)$, $(7,13,11)$, $(12,18,14)$, $(13,12,16)$, $(11,17,14)$, $(12,11,15)$, $(13,15,14)$, $(15,17,16)$, $(10,16,14)$
$J_{11} \to 3^{13}, 16^*$	$(3,2,4,10,16,13,9,15,12,14,11,8,7,1,5,6)$, $(0,6,4)$, $(2,8,6)$, $(3,9,5)$, $(4,3,7)$, $(5,4,8)$, $(5,11,7)$, $(6,7,9)$, $(6,12,10)$, $(7,13,10)$, $(8,14,10)$, $(9,8,12)$, $(9,11,10)$, $(11,13,12)$
$J_{14} \to 3^{18}, 16^*$	$(4,10,8,14,17,11,5,6,9,15,18,12,16,19,13,7)$, $(0,6,4)$, $(3,2,6)$, $(1,7,5)$, $(2,8,4)$, $(3,5,4)$, $(3,9,7)$, $(5,9,8)$, $(7,6,10)$, $(8,7,11)$, $(9,11,10)$, $(6,12,8)$, $(9,13,12)$, $(13,17,15)$, $(14,16,15)$, $(12,11,15)$, $(10,16,13)$, $(11,13,14)$, $(10,14,12)$
$J_{18} \to 3^{23}, 4, 17$	$(0,6,3,5,1,7,10,13,16,22,19,21,17,14,8,2,4)$, $(2,6,9,3)$, $(4,3,7)$, $(4,10,8)$, $(4,6,5)$, $(5,11,7)$, $(5,9,8)$, $(7,13,9)$, $(6,8,7)$, $(6,12,10)$, $(9,11,10)$, $(8,12,11)$, $(9,15,12)$, $(11,15,14)$, $(12,14,13)$, $(10,16,14)$, $(11,17,13)$, $(17,23,20)$, $(15,17,16)$, $(13,19,15)$, $(15,21,18)$, $(12,18,16)$, $(14,20,18)$, $(16,20,19)$, $(17,19,18)$
$J_4^+ \to 4^4, 4^{+*}$	$[2,8,5]$, $[1,7,4]$, $[0,6,9,3]$, $(2,5,1,4)$, $(2,6,5,3)$, $(0,3,6,4)$, $(3,7,5,4)$
$J_4^+ \to 3^2, 5^2, 4^{+*}$	$[2,8,5]$, $[1,7,4]$, $[0,6,9,3]$, $(1,4,3,7,5)$, $(0,3,5,2,4)$, $(3,2,6)$, $(4,6,5)$
$J_4^+ \to 3, 4^2, 5, 4^{+*}$	$[2,8,5]$, $[1,7,4]$, $[0,6,9,3]$, $(1,4,3,7,5)$, $(3,5,2,6)$, $(0,3,2,4)$, $(4,6,5)$
$J_4^+ \to 3^4, 4, 4^{+*}$	$[2,8,5]$, $[1,7,4]$, $[0,6,9,3]$, $(2,5,1,4)$, $(3,2,6)$, $(0,3,4)$, $(4,6,5)$, $(3,7,5)$
$J_5^+ \to 5^4, 5^{+*}$	$[2,8,5]$, $[1,7,10,4]$, $[0,6,9,3]$, $(1,4,3,7,5)$, $(2,5,4,8,6)$, $(0,3,5,6,4)$, $(3,2,4,7,6)$
$J_5^+ \to 3^5, 5, 5^{+*}$	$[2,8,5]$, $[1,7,10,4]$, $[0,6,9,3]$, $(1,4,7,3,5)$, $(0,3,4)$, $(2,5,4)$, $(3,2,6)$, $(4,8,6)$, $(5,7,6)$
$J_6^+ \to 3^8, 6^{+*}$	$[2,8,11,5]$, $[1,7,10,4]$, $[0,6,9,3]$, $(1,4,5)$, $(2,5,3)$, $(2,6,4)$, $(0,3,4)$, $(5,9,7)$, $(5,6,8)$, $(3,7,6)$, $(4,8,7)$
$J_4^+ \to 3^3, 4^2, 3^+$	$[2,8,5]$, $[1,7,4]$, $[0,6,3]$, $(0,3,2,4)$, $(1,4,5)$, $(3,9,6,4)$, $(2,5,6)$, $(3,7,5)$
$J_5^+ \to 3^2, 4^4, 3^+$	$[2,8,5]$, $[1,7,4]$, $[0,6,3]$, $(0,3,2,4)$, $(1,4,3,5)$, $(3,9,6,7)$, $(4,8,6)$, $(2,5,6)$, $(4,10,7,5)$
$J_7^+ \to 3^{10}, 5^+$	$[2,8,11,5]$, $[1,7,10,4]$, $[0,6,3]$, $(2,5,3)$, $(0,3,4)$, $(4,8,7)$, $(1,4,5)$, $(2,6,4)$, $(6,10,8)$, $(5,7,6)$, $(3,9,7)$, $(5,9,8)$, $(6,12,9)$
$J_8^+ \to 3^{11}, 7^+$	$[2,8,5]$, $[1,7,13,10,4]$, $[0,6,12,9,3]$, $(1,4,5)$, $(2,5,3)$, $(0,3,4)$, $(2,6,4)$, $(5,11,7)$, $(6,5,9)$, $(6,10,8)$, $(4,8,7)$, $(3,7,6)$, $(7,9,10)$, $(8,9,11)$
$J_9^+ \to 3^{13}, 6^+$	$[2,8,5]$, $[1,7,4]$, $[0,6,10,11,9,3]$, $(0,3,4)$, $(4,10,8)$, $(1,4,5)$, $(2,5,3)$, $(2,6,4)$, $(3,7,6)$, $(7,13,10)$, $(7,9,8)$, $(5,11,7)$, $(6,12,8)$, $(8,14,11)$, $(6,5,9)$, $(9,10,12)$
$J_9^+ \to 3^{12}, 4, 5^+$	$[2,8,5]$, $[1,7,4]$, $[0,6,12,9,3]$, $(1,4,5)$, $(2,5,3)$, $(0,3,4)$, $(2,6,8,4)$, $(3,7,6)$, $(7,13,10)$, $(8,12,10)$, $(4,10,6)$, $(6,5,9)$, $(5,11,7)$, $(7,9,8)$, $(8,14,11)$, $(9,11,10)$
$J_{10}^+ \to 3^{15}, 5^+$	$[2,8,5]$, $[1,7,4]$, $[0,6,10,9,3]$, $(0,3,4)$, $(1,4,5)$, $(2,5,3)$, $(2,6,4)$, $(3,7,6)$, $(7,9,8)$, $(6,12,8)$, $(4,10,8)$, $(6,5,9)$, $(9,15,12)$, $(7,13,10)$, $(5,11,7)$, $(8,14,11)$, $(9,13,11)$, $(10,12,11)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_{11}^{+} \to 3^{16}, 7^{+}$	$[2,8,5]$, $[1,7,4]$, $[0,6,10,16,13,9,3]$, $(0,3,4)$, $(2,6,4)$, $(2,5,3)$, $(1,4,5)$, $(3,7,6)$, $(7,13,10)$, $(6,5,9)$, $(7,9,8)$, $(5,11,7)$, $(6,12,8)$, $(4,10,8)$, $(10,14,12)$, $(8,14,11)$, $(9,11,10)$, $(9,15,12)$, $(11,13,12)$

Table A.9: Table of decompositions of $J_n^{\{1,2,3,4,6\}}$

$J_4 \to 3^6, 6$	$J_4 \to 3, 5^3, 6$	$J_4 \to 3^3, 4, 5, 6$	$J_4 \to 4^2, 5^2, 6$
$J_4 \to 3^2, 4^3, 6$	$J_5 \to 4, 5^4, 6$	$J_5 \to 3^2, 4^2, 5^2, 6$	$J_5 \to 3, 4^4, 5, 6$
$J_5 \to 4^6, 6$	$J_6 \to 5^6, 6$	$J_6 \to 4^5, 5^2, 6$	$J_6 \to 3^2, 4^6, 6$
$J_7 \to 4^9, 6$	$J_9 \to 4^{12}, 6$		

Table A.10: These decompositions are required for Lemma 1.6.20. The decompositions themselves are given in Table A.12.

$J_5 \to 3^6, 5, 7^*$	$J_5 \to 3, 5^4, 7^*$	$J_5 \to 3^3, 4, 5^2, 7^*$	$J_5 \to 3^5, 4^2, 7^*$
$J_5 \to 4^2, 5^3, 7^*$	$J_5 \to 3^2, 4^3, 5, 7^*$	$J_5 \to 3, 4^5, 7^*$	$J_6 \to 4, 5^5, 7^*$
$J_6 \to 3^2, 4^2, 5^3, 7^*$	$J_6 \to 3, 4^4, 5^2, 7^*$	$J_6 \to 4^6, 5, 7^*$	$J_7 \to 5^7, 7^*$
$J_7 \to 4^5, 5^3, 7^*$	$J_7 \to 3, 4^8, 7^*$	$J_8 \to 4^9, 5, 7^*$	$J_9 \to 3, 4^{11}, 7^*$
$J_{10} \to 4^{12}, 5, 7^*$			
$J_5 \to 3^4, 5^2, 8^*$	$J_5 \to 3^6, 4, 8^*$	$J_5 \to 3, 4, 5^3, 8^*$	$J_5 \to 3^3, 4^2, 5, 8^*$
$J_5 \to 4^3, 5^2, 8^*$	$J_5 \to 3^2, 4^4, 8^*$	$J_6 \to 3, 5^5, 8^*$	$J_6 \to 4^2, 5^4, 8^*$
$J_6 \to 3^2, 4^3, 5^2, 8^*$	$J_6 \to 3, 4^5, 5, 8^*$	$J_6 \to 4^7, 8^*$	$J_7 \to 4, 5^6, 8^*$
$J_7 \to 4^6, 5^2, 8^*$	$J_7 \to 3^2, 4^7, 8^*$	$J_8 \to 5^8, 8^*$	$J_8 \to 4^{10}, 8^*$
$J_5 \to 3^7, 9^*$	$J_5 \to 3^2, 5^3, 9^*$	$J_5 \to 3^4, 4, 5, 9^*$	$J_5 \to 3, 4^2, 5^2, 9^*$
$J_5 \to 3^3, 4^3, 9^*$	$J_5 \to 4^4, 5, 9^*$	$J_6 \to 3, 4, 5^4, 9^*$	$J_6 \to 4^3, 5^3, 9^*$
$J_6 \to 3^2, 4^4, 5, 9^*$	$J_6 \to 3, 4^6, 9^*$	$J_7 \to 3, 5^6, 9^*$	$J_7 \to 4^2, 5^5, 9^*$
$J_7 \to 4^7, 5, 9^*$	$J_8 \to 4, 5^7, 9^*$	$J_8 \to 3, 4^9, 9^*$	$J_9 \to 5^9, 9^*$
$J_6 \to 3^7, 5, 10^*$	$J_6 \to 3^2, 5^4, 10^*$	$J_6 \to 3^4, 4, 5^2, 10^*$	$J_6 \to 3^6, 4^2, 10^*$
$J_6 \to 3, 4^2, 5^3, 10^*$	$J_6 \to 3^3, 4^3, 5, 10^*$	$J_6 \to 4^4, 5^2, 10^*$	$J_6 \to 3^2, 4^5, 10^*$
$J_7 \to 3, 4, 5^5, 10^*$	$J_7 \to 4^3, 5^4, 10^*$	$J_7 \to 3^2, 4^4, 5^2, 10^*$	$J_7 \to 3, 4^6, 5, 10^*$
$J_7 \to 4^8, 10^*$	$J_8 \to 3, 5^7, 10^*$	$J_8 \to 4^2, 5^6, 10^*$	$J_8 \to 4^7, 5^2, 10^*$
$J_8 \to 3^2, 4^8, 10^*$	$J_9 \to 4, 5^8, 10^*$	$J_9 \to 4^{11}, 10^*$	$J_{10} \to 5^{10}, 10^*$
$J_6 \to 3^5, 5^2, 11^*$	$J_6 \to 5^5, 11^*$	$J_6 \to 3^7, 4, 11^*$	$J_6 \to 3^2, 4, 5^3, 11^*$
$J_6 \to 3^4, 4^2, 5, 11^*$	$J_6 \to 3, 4^3, 5^2, 11^*$	$J_6 \to 3^3, 4^4, 11^*$	$J_6 \to 4^5, 5, 11^*$
$J_7 \to 3^2, 5^5, 11^*$	$J_7 \to 3, 4^2, 5^4, 11^*$	$J_7 \to 4^4, 5^3, 11^*$	$J_7 \to 3^2, 4^5, 5, 11^*$
$J_7 \to 3, 4^7, 11^*$	$J_8 \to 4^3, 5^5, 11^*$	$J_8 \to 4^8, 5, 11^*$	$J_9 \to 3, 4^{10}, 11^*$
$J_{10} \to 4, 5^9, 11^*$			
$J_7 \to 3^{10}, 12^*$	$J_7 \to 3^5, 5^3, 12^*$	$J_7 \to 5^6, 12^*$	$J_7 \to 3^7, 4, 5, 12^*$
$J_7 \to 3^2, 4, 5^4, 12^*$	$J_7 \to 3^4, 4^2, 5^2, 12^*$	$J_7 \to 3^6, 4^3, 12^*$	$J_7 \to 3, 4^3, 5^3, 12^*$
$J_7 \to 3^3, 4^4, 5, 12^*$	$J_7 \to 4^5, 5^2, 12^*$	$J_7 \to 3^2, 4^6, 12^*$	$J_8 \to 3^2, 5^6, 12^*$
$J_8 \to 3, 4^2, 5^5, 12^*$	$J_8 \to 4^4, 5^4, 12^*$	$J_8 \to 3^2, 4^5, 5^2, 12^*$	$J_8 \to 3, 4^7, 5, 12^*$
$J_8 \to 4^9, 12^*$	$J_9 \to 4^3, 5^6, 12^*$	$J_9 \to 4^8, 5^2, 12^*$	$J_9 \to 3^2, 4^9, 12^*$
$J_{10} \to 4^{12}, 12^*$	$J_{11} \to 4, 5^{10}, 12^*$		

Table A.11: These decompositions are required for Lemma 1.6.21. The decompositions themselves are given in Table A.12.

$J_1 \to 3^2$	$(0,7,5)$, $(6,3,7)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_2 \to 3,4,5$	$(0,7,5)$, $(3,7,8,1,6)$, $(6,8,4,7)$
$J_3 \to 4^2,5^2$	$(0,7,6,8,5)$, $(3,7,8,1,6)$, $(5,9,2,7)$, $(4,8,9,7)$
$J_3 \to 3,5^3$	$(0,7,6,8,5)$, $(3,7,8,1,6)$, $(4,8,9,2,7)$, $(5,9,7)$
$J_3 \to 3^2,4^3$	$(0,7,8,5)$, $(6,3,7)$, $(1,8,6)$, $(4,8,9,7)$, $(5,9,2,7)$
$J_4 \to 4,5^4$	$(0,7,6,8,5)$, $(3,7,8,1,6)$, $(5,9,2,7)$, $(8,3,10,6,9)$, $(4,8,10,9,7)$
$J_4 \to 4^6$	$(0,7,8,5)$, $(1,8,9,6)$, $(5,9,2,7)$, $(3,10,9,7)$, $(6,8,4,7)$, $(3,8,10,6)$
$J_4 \to 3,4^4,5$	$(0,7,6,8,5)$, $(1,8,9,6)$, $(5,9,2,7)$, $(3,8,4,7)$, $(3,10,6)$, $(7,9,10,8)$
$J_5 \to 5^6$	$(0,7,6,8,5)$, $(3,7,8,1,6)$, $(2,9,8,10,7)$, $(4,9,10,3,8)$, $(5,9,11,4,7)$, $(7,11,10,6,9)$
$J_5 \to 4^5,5^2$	$(0,7,6,8,5)$, $(3,7,8,1,6)$, $(5,9,2,7)$, $(8,3,10,9)$, $(4,8,10,7)$, $(6,10,11,9)$, $(7,11,4,9)$
$J_6 \to 4^9$	$(0,7,8,5)$, $(1,8,9,6)$, $(5,9,2,7)$, $(3,10,11,8)$, $(3,7,10,6)$, $(6,8,4,7)$, $(7,11,4,9)$, $(9,11,12,10)$, $(8,12,5,10)$
$J_4 \to 3^6,6$	$(6,9,5,8,4,7)$, $(0,7,5)$, $(1,8,6)$, $(2,9,7)$, $(3,8,7)$, $(3,10,6)$, $(8,10,9)$
$J_4 \to 3,5^3,6$	$(5,9,6,8,4,7)$, $(0,7,3,8,5)$, $(6,1,8,9,7)$, $(7,2,9,10,8)$, $(3,10,6)$
$J_4 \to 3^3,4,5,6$	$(5,9,6,8,4,7)$, $(0,7,3,8,5)$, $(6,1,8,7)$, $(2,9,7)$, $(3,10,6)$, $(8,10,9)$
$J_4 \to 4^2,5^2,6$	$(5,9,6,8,4,7)$, $(0,7,3,8,5)$, $(3,10,8,1,6)$, $(7,2,9,8)$, $(6,10,9,7)$
$J_4 \to 3^2,4^3,6$	$(5,9,6,8,4,7)$, $(0,7,8,5)$, $(3,8,1,6)$, $(2,9,7)$, $(6,10,3,7)$, $(8,10,9)$
$J_5 \to 4,5^4,6$	$(5,9,6,8,4,7)$, $(0,7,3,8,5)$, $(1,8,7,10,6)$, $(2,9,4,11,7)$, $(6,3,10,9,7)$, $(8,10,11,9)$
$J_5 \to 3^2,4^2,5^2,6$	$(5,9,6,8,4,7)$, $(0,7,3,8,5)$, $(6,1,8,7)$, $(3,10,6)$, $(8,10,11,4,9)$, $(2,9,11,7)$, $(7,10,9)$
$J_5 \to 3,4^4,5,6$	$(5,9,6,8,4,7)$, $(0,7,3,8,5)$, $(6,1,8,7)$, $(2,9,10,7)$, $(3,10,6)$, $(7,11,4,9)$, $(8,10,11,9)$
$J_5 \to 4^6,6$	$(5,9,6,8,4,7)$, $(0,7,8,5)$, $(3,8,1,6)$, $(2,9,10,7)$, $(6,10,3,7)$, $(7,11,4,9)$, $(8,10,11,9)$
$J_6 \to 5^6,6$	$(5,9,6,8,4,7)$, $(0,7,6,10,5)$, $(3,7,8,1,6)$, $(2,9,8,11,7)$, $(7,10,11,4,9)$, $(9,11,12,8,10)$, $(5,12,10,3,8)$
$J_6 \to 4^5,5^2,6$	$(5,9,6,8,4,7)$, $(0,7,6,10,5)$, $(3,7,8,1,6)$, $(2,9,10,7)$, $(3,10,11,8)$, $(7,11,4,9)$, $(5,12,10,8)$, $(8,12,11,9)$
$J_6 \to 3^2,4^6,6$	$(5,9,6,8,4,7)$, $(0,7,8,5)$, $(3,8,1,6)$, $(2,9,7)$, $(6,10,3,7)$, $(9,4,11,10)$, $(8,12,5,10)$, $(7,11,12,10)$, $(8,11,9)$
$J_7 \to 4^9,6$	$(5,9,6,8,4,7)$, $(0,7,8,5)$, $(3,8,1,6)$, $(2,9,10,7)$, $(6,10,3,7)$, $(7,11,4,9)$, $(10,5,12,11)$, $(9,13,6,11)$, $(8,12,13,11)$, $(8,10,12,9)$
$J_9 \to 4^{12},6$	$(5,9,6,8,4,7)$, $(0,7,8,5)$, $(3,8,1,6)$, $(2,9,10,7)$, $(6,10,3,7)$, $(7,11,4,9)$, $(10,5,12,11)$, $(9,13,6,11)$, $(12,7,14,13)$, $(8,12,14,11)$, $(8,10,12,9)$, $(11,15,8,13)$, $(10,14,15,13)$
$J_5 \to 3^6,5,7^*$	$(6,10,11,9,5,8,7)$, $(0,7,5)$, $(3,7,2,9,6)$, $(1,8,6)$, $(4,11,7)$, $(7,10,9)$, $(4,9,8)$, $(3,10,8)$
$J_5 \to 3,5^4,7^*$	$(6,10,11,9,8,5,7)$, $(0,7,4,9,5)$, $(3,7,8,1,6)$, $(2,9,7)$, $(6,9,10,3,8)$, $(4,11,7,10,8)$
$J_5 \to 3^3,4,5^2,7^*$	$(6,10,11,9,8,5,7)$, $(0,7,4,9,5)$, $(1,8,6)$, $(3,7,2,9,6)$, $(3,10,8)$, $(7,11,4,8)$, $(7,10,9)$
$J_5 \to 3^5,4^2,7^*$	$(6,10,11,9,5,8,7)$, $(0,7,5)$, $(3,8,10,7)$, $(2,9,7)$, $(4,9,8)$, $(1,8,6)$, $(3,10,9,6)$, $(4,11,7)$
$J_5 \to 4^2,5^3,7^*$	$(6,10,11,9,8,5,7)$, $(0,7,4,9,5)$, $(1,8,7,9,6)$, $(2,9,10,7)$, $(3,8,4,11,7)$, $(3,10,8,6)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_5 \to 3^2, 4^3, 5, 7^*$	$(6,10,11,9,8,5,7)$, $(0,7,4,9,5)$, $(1,8,6)$, $(2,9,7)$, $(3,8,10,7)$, $(3,10,9,6)$, $(7,11,4,8)$
$J_5 \to 3, 4^5, 7^*$	$(6,10,11,9,8,5,7)$, $(0,7,9,5)$, $(1,8,6)$, $(4,9,2,7)$, $(3,10,9,6)$, $(7,11,4,8)$, $(3,8,10,7)$
$J_6 \to 4, 5^5, 7^*$	$(6,8,9,10,12,11,7)$, $(0,7,3,8,5)$, $(3,10,8,1,6)$, $(7,9,4,11,8)$, $(2,9,5,10,7)$, $(5,12,8,4,7)$, $(6,10,11,9)$
$J_6 \to 3^2, 4^2, 5^3, 7^*$	$(6,8,9,10,12,11,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(5,9,2,7)$, $(3,10,6)$, $(4,9,11,10,7)$, $(4,11,8)$, $(8,12,5,10)$
$J_6 \to 3, 4^4, 5^2, 7^*$	$(6,8,9,10,12,11,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(5,9,2,7)$, $(3,10,6)$, $(4,11,10,7)$, $(4,9,11,8)$, $(8,12,5,10)$
$J_6 \to 4^6, 5, 7^*$	$(6,8,9,10,12,11,7)$, $(0,7,3,10,5)$, $(3,8,1,6)$, $(4,11,10,8)$, $(5,12,8,7)$, $(7,10,6,9)$, $(4,9,2,7)$, $(5,9,11,8)$
$J_7 \to 5^7, 7^*$	$(7,9,10,11,13,12,8)$, $(0,7,6,8,5)$, $(1,8,4,9,6)$, $(3,8,9,2,7)$, $(3,10,8,11,6)$, $(5,9,11,4,7)$, $(7,11,12,5,10)$, $(6,13,9,12,10)$
$J_7 \to 4^5, 5^3, 7^*$	$(7,9,10,11,13,12,8)$, $(0,7,6,8,5)$, $(1,8,4,11,6)$, $(4,9,2,7)$, $(3,10,5,7)$, $(3,8,9,13,6)$, $(8,11,7,10)$, $(6,10,12,9)$, $(5,12,11,9)$
$J_7 \to 3, 4^8, 7^*$	$(7,9,10,11,13,12,8)$, $(0,7,5)$, $(3,8,1,6)$, $(6,10,3,7)$, $(5,10,12,9)$, $(6,11,4,9)$, $(6,13,9,8)$, $(2,9,11,7)$, $(4,8,10,7)$, $(5,12,11,8)$
$J_8 \to 4^9, 5, 7^*$	$(8,10,11,12,14,13,9)$, $(0,7,6,8,5)$, $(3,8,1,6)$, $(5,9,2,7)$, $(3,10,9,7)$, $(7,4,11,8)$, $(6,11,7,10)$, $(6,13,11,9)$, $(4,9,12,8)$, $(10,14,7,12)$, $(5,12,13,10)$
$J_9 \to 3, 4^{11}, 7^*$	$(9,11,12,13,15,14,10)$, $(0,7,5)$, $(6,1,8,7)$, $(4,9,2,7)$, $(6,10,3,8)$, $(3,7,9,6)$, $(4,11,10,8)$, $(8,5,12,9)$, $(7,12,8,11)$, $(7,14,12,10)$, $(5,10,13,9)$, $(11,15,8,13)$, $(6,13,14,11)$
$J_{10} \to 4^{12}, 5, 7^*$	$(10,12,13,14,16,15,11)$, $(0,7,6,8,5)$, $(1,8,9,6)$, $(5,9,2,7)$, $(3,10,8,7)$, $(3,8,13,6)$, $(4,8,12,7)$, $(7,11,4,9)$, $(9,16,12,11)$, $(6,11,13,10)$, $(9,13,15,12)$, $(9,14,7,10)$, $(8,15,14,11)$, $(5,12,14,10)$
$J_5 \to 3^4, 5^2, 8^*$	$(6,10,8,5,9,11,4,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(2,9,7)$, $(4,9,10,3,8)$, $(6,9,8)$, $(10,7,11)$
$J_5 \to 3^6, 4, 8^*$	$(6,10,8,5,9,11,4,7)$, $(0,7,5)$, $(1,8,6)$, $(2,9,7)$, $(3,10,9,6)$, $(4,9,8)$, $(3,8,7)$, $(10,7,11)$
$J_5 \to 3, 4, 5^3, 8^*$	$(6,10,8,5,9,11,4,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(2,9,10,11,7)$, $(8,3,10,7,9)$, $(4,9,6,8)$
$J_5 \to 3^3, 4^2, 5, 8^*$	$(6,10,8,5,9,11,4,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(2,9,7)$, $(8,3,10,9)$, $(4,9,6,8)$, $(10,7,11)$
$J_5 \to 4^3, 5^2, 8^*$	$(5,8,6,10,9,11,4,7)$, $(0,7,8,9,5)$, $(3,8,1,6)$, $(6,9,2,7)$, $(3,10,11,7)$, $(4,9,7,10,8)$
$J_5 \to 3^2, 4^4, 8^*$	$(6,10,8,5,9,11,4,7)$, $(0,7,5)$, $(1,8,6)$, $(2,9,10,7)$, $(3,8,9,6)$, $(7,9,4,8)$, $(3,10,11,7)$
$J_6 \to 3, 5^5, 8^*$	$(6,8,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(2,9,5,10,7)$, $(3,10,6)$, $(9,4,11,8,10)$, $(4,8,12,11,7)$
$J_6 \to 4^2, 5^4, 8^*$	$(6,8,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(1,8,7,10,6)$, $(3,10,5,9,6)$, $(4,11,12,8)$, $(4,9,2,7)$, $(7,11,8,10,9)$
$J_6 \to 3^2, 4^3, 5^2, 8^*$	$(6,8,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(4,9,2,7)$, $(3,10,6)$, $(5,10,9)$, $(8,11,7,10)$, $(4,11,12,8)$
$J_6 \to 3, 4^5, 5, 8^*$	$(6,8,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(1,8,10,6)$, $(4,9,2,7)$, $(3,10,9,6)$, $(4,11,12,8)$, $(5,10,7,9)$, $(7,11,8)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_6 \to 4^7, 8^*$	$(6,8,9,11,10,12,5,7)$, $(0,7,8,5)$, $(3,8,1,6)$, $(4,9,2,7)$, $(3,10,9,7)$, $(4,11,12,8)$, $(5,10,6,9)$, $(8,11,7,10)$
$J_7 \to 4, 5^6, 8^*$	$(6,13,11,12,10,9,8,7)$, $(0,7,3,8,5)$, $(1,8,4,9,6)$, $(4,11,9,2,7)$, $(3,10,11,8,6)$, $(5,10,8,12,9)$, $(6,11,7,10)$, $(5,12,13,9,7)$
$J_7 \to 4^6, 5^2, 8^*$	$(6,13,11,12,10,9,8,7)$, $(0,7,3,8,5)$, $(1,8,4,11,6)$, $(3,10,8,6)$, $(5,10,11,7)$, $(7,10,6,9)$, $(4,9,2,7)$, $(9,12,8,11)$, $(5,12,13,9)$
$J_7 \to 3^2, 4^7, 8^*$	$(6,13,11,12,10,9,8,7)$, $(0,7,5)$, $(1,8,6)$, $(4,9,2,7)$, $(5,10,3,8)$, $(4,11,10,8)$, $(5,12,13,9)$, $(3,7,9,6)$, $(6,11,7,10)$, $(9,12,8,11)$
$J_8 \to 5^8, 8^*$	$(7,14,12,13,11,10,9,8)$, $(0,7,6,8,5)$, $(1,8,4,9,6)$, $(2,9,5,10,7)$, $(3,8,12,5,7)$, $(3,10,14,13,6)$, $(6,11,9,13,10)$, $(8,11,7,12,10)$, $(4,11,12,9,7)$
$J_8 \to 4^{10}, 8^*$	$(7,14,12,13,11,10,9,8)$, $(0,7,9,5)$, $(3,8,1,6)$, $(6,9,2,7)$, $(3,10,5,7)$, $(4,11,6,8)$, $(4,9,11,7)$, $(5,12,11,8)$, $(8,12,7,10)$, $(10,13,9,12)$, $(6,13,14,10)$
$J_5 \to 3^7, 9^*$	$(6,3,10,8,5,9,11,4,7)$, $(0,7,5)$, $(1,8,6)$, $(2,9,7)$, $(3,8,7)$, $(4,9,8)$, $(9,6,10)$, $(10,7,11)$
$J_5 \to 3^2, 5^3, 9^*$	$(6,3,10,8,5,9,11,4,7)$, $(0,7,5)$, $(1,8,7,9,6)$, $(3,8,9,2,7)$, $(4,9,10,6,8)$, $(10,7,11)$
$J_5 \to 3^4, 4, 5, 9^*$	$(6,3,10,8,5,9,11,4,7)$, $(0,7,5)$, $(1,8,4,9,6)$, $(2,9,7)$, $(6,10,9,8)$, $(3,8,7)$, $(10,7,11)$
$J_5 \to 3, 4^2, 5^2, 9^*$	$(6,3,10,8,5,9,11,4,7)$, $(0,7,5)$, $(1,8,7,10,6)$, $(3,8,9,2,7)$, $(4,9,6,8)$, $(7,11,10,9)$
$J_5 \to 3^3, 4^3, 9^*$	$(6,10,3,8,5,9,11,4,7)$, $(0,7,5)$, $(2,9,10,7)$, $(4,9,8)$, $(3,7,9,6)$, $(1,8,6)$, $(7,11,10,8)$
$J_5 \to 4^4, 5, 9^*$	$(3,8,5,7,4,11,9,10,6)$, $(0,7,2,9,5)$, $(1,8,9,6)$, $(7,9,4,8)$, $(6,8,10,7)$, $(3,10,11,7)$
$J_6 \to 3, 4, 5^4, 9^*$	$(6,8,4,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(1,8,9,10,6)$, $(2,9,7)$, $(3,10,5,9,6)$, $(7,4,11,12,8)$, $(8,11,7,10)$
$J_6 \to 4^3, 5^3, 9^*$	$(6,8,4,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(3,10,8,1,6)$, $(5,10,6,9)$, $(8,12,11,7,9)$, $(2,9,10,7)$, $(7,4,11,8)$
$J_6 \to 3^2, 4^4, 5, 9^*$	$(6,8,4,9,11,10,12,5,7)$, $(0,7,3,8,5)$, $(1,8,10,6)$, $(3,10,9,6)$, $(5,10,7,9)$, $(7,2,9,8)$, $(4,11,7)$, $(11,8,12)$
$J_6 \to 3, 4^6, 9^*$	$(6,8,4,11,9,10,12,5,7)$, $(4,9,2,7)$, $(1,8,10,6)$, $(3,8,9,6)$, $(7,9,5,8)$, $(0,7,10,5)$, $(3,10,11,7)$, $(11,8,12)$
$J_7 \to 3, 5^6, 9^*$	$(6,13,11,12,10,9,8,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,4,11,7)$, $(3,8,10,11,6)$, $(4,8,6,10,7)$, $(5,12,8,11,9)$, $(12,9,13)$
$J_7 \to 4^2, 5^5, 9^*$	$(6,13,11,12,10,9,8,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,4,11,7)$, $(3,8,11,10,6)$, $(4,8,10,7)$, $(5,12,13,9)$, $(6,11,9,12,8)$
$J_7 \to 4^7, 5, 9^*$	$(6,13,11,12,10,9,8,5,7)$, $(0,7,3,10,5)$, $(3,8,1,6)$, $(4,9,2,7)$, $(4,11,10,8)$, $(5,12,13,9)$, $(7,10,6,8)$, $(7,11,6,9)$, $(9,12,8,11)$
$J_8 \to 4, 5^7, 9^*$	$(6,8,9,10,11,13,12,14,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(2,9,5,10,7)$, $(3,10,8,11,6)$, $(4,11,7,12,8)$, $(5,12,9,4,7)$, $(6,13,14,10)$, $(9,13,10,12,11)$
$J_8 \to 3, 4^9, 9^*$	$(6,8,9,10,11,13,12,14,7)$, $(0,7,5)$, $(3,8,1,6)$, $(4,9,2,7)$, $(3,10,8,7)$, $(4,11,12,8)$, $(5,9,11,8)$, $(6,13,14,10)$, $(7,11,6,9)$, $(10,13,9,12)$, $(7,12,5,10)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_9 \to 5^9, 9^*$	$(7,9,10,11,12,14,13,15,8)$, $(0,7,6,8,5)$, $(1,8,4,9,6)$, $(3,8,9,2,7)$, $(3,10,8,11,6)$, $(5,9,11,4,7)$, $(5,12,8,13,10)$, $(6,13,9,12,10)$, $(12,7,14,11,13)$, $(7,11,15,14,10)$
$J_6 \to 3^7, 5, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,3,10,5)$, $(1,8,6)$, $(2,9,7)$, $(4,11,7)$, $(7,10,8)$, $(8,5,9)$, $(9,6,10)$, $(11,8,12)$
$J_6 \to 3^2, 5^4, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,8,10,7)$, $(4,11,7)$, $(6,10,9,5,8)$, $(11,8,12)$
$J_6 \to 3^4, 4, 5^2, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,10,7)$, $(4,11,7)$, $(6,10,8)$, $(8,5,9)$, $(11,8,12)$
$J_6 \to 3^6, 4^2, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,8,5)$, $(1,8,6)$, $(2,9,7)$, $(3,10,7)$, $(4,11,7)$, $(8,10,5,9)$, $(9,6,10)$, $(11,8,12)$
$J_6 \to 3, 4^2, 5^3, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,3,10,5)$, $(1,8,6)$, $(2,9,8,11,7)$, $(7,4,11,12,8)$, $(5,9,10,8)$, $(7,10,6,9)$
$J_6 \to 3^3, 4^3, 5, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,3,10,5)$, $(1,8,6)$, $(7,2,9,8)$, $(4,11,7)$, $(5,9,10,8)$, $(7,10,6,9)$, $(11,8,12)$
$J_6 \to 4^4, 5^2, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,3,10,5)$, $(1,8,10,6)$, $(4,11,8,9,7)$, $(6,9,5,8)$, $(2,9,10,7)$, $(7,11,12,8)$
$J_6 \to 3^2, 4^5, 10^*$	$(6,3,8,4,9,11,10,12,5,7)$, $(0,7,9,5)$, $(3,10,8,7)$, $(4,11,7)$, $(2,9,10,7)$, $(6,10,5,8)$, $(1,8,9,6)$, $(11,8,12)$
$J_7 \to 3, 4, 5^5, 10^*$	$(6,13,11,12,10,9,4,8,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,8,10,7)$, $(3,8,11,10,6)$, $(4,11,7)$, $(5,12,13,9)$, $(6,11,9,12,8)$
$J_7 \to 4^3, 5^4, 10^*$	$(6,13,11,12,10,9,4,8,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,8,11,7)$, $(4,11,10,7)$, $(5,12,13,9)$, $(3,8,10,6)$, $(6,11,9,12,8)$
$J_7 \to 3^2, 4^4, 5^2, 10^*$	$(6,13,11,12,10,9,4,8,5,7)$, $(0,7,3,10,5)$, $(1,8,7,9,6)$, $(2,9,11,7)$, $(3,8,6)$, $(4,11,10,7)$, $(8,12,5,9)$, $(6,11,8,10)$, $(12,9,13)$
$J_7 \to 3, 4^6, 5, 10^*$	$(6,13,11,12,10,9,4,8,5,7)$, $(0,7,3,10,5)$, $(1,8,6)$, $(7,2,9,8)$, $(3,8,10,6)$, $(4,11,10,7)$, $(7,11,6,9)$, $(9,12,8,11)$, $(5,12,13,9)$
$J_7 \to 4^8, 10^*$	$(6,13,11,12,10,9,4,8,5,7)$, $(0,7,10,5)$, $(7,4,11,8)$, $(3,8,9,7)$, $(2,9,11,7)$, $(1,8,10,6)$, $(3,10,11,6)$, $(6,9,12,8)$, $(5,12,13,9)$
$J_8 \to 3, 5^7, 10^*$	$(6,8,9,5,10,11,13,12,14,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(2,9,10,12,7)$, $(3,10,14,13,6)$, $(5,12,9,4,7)$, $(4,11,8)$, $(8,12,11,7,10)$, $(6,11,9,13,10)$
$J_8 \to 4^2, 5^6, 10^*$	$(6,8,9,5,10,11,13,12,14,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(2,9,10,12,7)$, $(3,10,8,11,6)$, $(4,11,9,12,8)$, $(4,9,13,10,7)$, $(5,12,11,7)$, $(6,13,14,10)$
$J_8 \to 4^7, 5^2, 10^*$	$(6,8,9,5,10,11,13,12,14,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(4,9,2,7)$, $(5,12,10,7)$, $(6,11,8,10)$, $(3,10,13,6)$, $(4,11,12,8)$, $(7,12,9,11)$, $(9,13,14,10)$
$J_8 \to 3^2, 4^8, 10^*$	$(6,8,9,5,10,11,13,12,14,7)$, $(0,7,5)$, $(3,8,1,6)$, $(2,9,7)$, $(3,10,8,7)$, $(4,8,12,7)$, $(5,12,11,8)$, $(6,11,4,9)$, $(9,11,7,10)$, $(10,13,9,12)$, $(6,13,14,10)$
$J_9 \to 4, 5^8, 10^*$	$(6,9,10,11,12,14,13,15,8,7)$, $(0,7,3,8,5)$, $(1,8,4,11,6)$, $(2,9,8,10,7)$, $(3,10,12,8,6)$, $(5,12,9,4,7)$, $(5,10,6,13,9)$, $(7,12,13,8,11)$, $(7,14,15,11,9)$, $(11,14,10,13)$
$J_9 \to 4^{11}, 10^*$	$(6,9,10,11,12,14,13,15,8,7)$, $(0,7,9,5)$, $(3,8,1,6)$, $(4,9,2,7)$, $(3,10,5,7)$, $(4,11,6,8)$, $(8,5,12,9)$, $(6,13,12,10)$, $(8,12,7,10)$, $(9,13,8,11)$, $(11,14,10,13)$, $(7,14,15,11)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_{10} \to 5^{10}, 10^*$	$(7,10,11,12,13,15,14,16,9,8)$, $(0,7,6,8,5)$, $(1,8,4,9,6)$, $(2,9,5,12,7)$, $(3,7,5,10,6)$, $(3,10,9,11,8)$, $(4,11,13,9,7)$, $(7,14,13,6,11)$, $(8,12,9,14,10)$, $(10,13,8,15,12)$, $(12,16,15,11,14)$
$J_6 \to 3^5, 5^2, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,6,10,5)$, $(1,8,6)$, $(5,9,11,12,8)$, $(4,9,7)$, $(7,11,8)$, $(3,10,7)$, $(8,10,9)$
$J_6 \to 5^5, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,6,8,5)$, $(1,8,7,10,6)$, $(3,10,9,4,7)$, $(8,12,11,7,9)$, $(5,10,8,11,9)$
$J_6 \to 3^7, 4, 11^*$	$(3,10,12,5,8,11,4,7,2,9,6)$, $(0,7,5)$, $(6,10,7)$, $(1,8,6)$, $(3,8,7)$, $(4,9,8)$, $(5,10,9)$, $(7,11,9)$, $(8,12,11,10)$
$J_6 \to 3^2, 4, 5^3, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,6,8,5)$, $(1,8,7,10,6)$, $(3,10,9,4,7)$, $(7,11,9)$, $(8,10,5,9)$, $(11,8,12)$
$J_6 \to 3^4, 4^2, 5, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,6,8,5)$, $(1,8,10,6)$, $(3,10,7)$, $(5,10,9)$, $(7,4,9,8)$, $(7,11,9)$, $(11,8,12)$
$J_6 \to 3, 4^3, 5^2, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,6,8,5)$, $(1,8,7,10,6)$, $(3,10,9,7)$, $(4,9,11,7)$, $(8,10,5,9)$, $(11,8,12)$
$J_6 \to 3^3, 4^4, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,8,5)$, $(1,8,6)$, $(3,10,9,7)$, $(6,10,7)$, $(4,9,11,7)$, $(8,10,5,9)$, $(11,8,12)$
$J_6 \to 4^5, 5, 11^*$	$(3,8,4,11,10,12,5,7,2,9,6)$, $(0,7,3,10,5)$, $(1,8,10,6)$, $(4,9,10,7)$, $(7,9,5,8)$, $(6,8,11,7)$, $(8,12,11,9)$
$J_7 \to 3^2, 5^5, 11^*$	$(6,13,11,12,10,5,9,4,8,3,7)$, $(0,7,5)$, $(1,8,7,9,6)$, $(2,9,10,11,7)$, $(3,10,6)$, $(4,11,8,10,7)$, $(6,11,9,12,8)$, $(8,5,12,13,9)$
$J_7 \to 3, 4^2, 5^4, 11^*$	$(6,13,11,12,10,5,9,4,8,3,7)$, $(0,7,5)$, $(1,8,7,9,6)$, $(2,9,10,11,7)$, $(3,10,8,6)$, $(4,11,6,10,7)$, $(8,5,12,13,9)$, $(9,12,8,11)$
$J_7 \to 4^4, 5^3, 11^*$	$(6,13,11,12,10,5,9,4,8,3,7)$, $(0,7,8,5)$, $(1,8,9,10,6)$, $(3,10,8,11,6)$, $(5,12,13,9,7)$, $(6,9,12,8)$, $(2,9,11,7)$, $(4,11,10,7)$
$J_7 \to 3^2, 4^5, 5, 11^*$	$(6,13,11,12,10,5,9,4,8,3,7)$, $(0,7,5)$, $(1,8,6)$, $(4,11,10,7)$, $(3,10,9,6)$, $(6,11,8,10)$, $(7,9,12,5,8)$, $(2,9,11,7)$, $(8,12,13,9)$
$J_7 \to 3, 4^7, 11^*$	$(6,13,11,12,10,5,9,4,8,3,7)$, $(0,7,8,5)$, $(3,10,8,6)$, $(1,8,11,6)$, $(4,11,7)$, $(2,9,10,7)$, $(5,12,9,7)$, $(6,10,11,9)$, $(8,12,13,9)$
$J_8 \to 4^3, 5^5, 11^*$	$(6,8,4,9,5,10,11,13,12,14,7)$, $(0,7,3,8,5)$, $(1,8,7,9,6)$, $(2,9,8,10,7)$, $(3,10,9,11,6)$, $(5,12,11,4,7)$, $(6,13,14,10)$, $(7,12,8,11)$, $(10,13,9,12)$
$J_8 \to 4^8, 5, 11^*$	$(6,8,4,9,5,10,11,13,12,14,7)$, $(0,7,3,8,5)$, $(1,8,9,6)$, $(2,9,11,7)$, $(3,10,13,6)$, $(6,11,12,10)$, $(7,4,11,8)$, $(5,12,9,7)$, $(8,12,7,10)$, $(9,13,14,10)$
$J_9 \to 3, 4^{10}, 11^*$	$(6,9,5,10,11,12,14,13,15,8,7)$, $(0,7,5)$, $(3,8,1,6)$, $(2,9,10,7)$, $(3,10,12,7)$, $(4,8,9,7)$, $(7,14,15,11)$, $(5,12,13,8)$, $(9,12,8,11)$, $(4,11,13,9)$, $(6,13,10,8)$, $(6,11,14,10)$
$J_{10} \to 4, 5^9, 11^*$	$(6,10,11,12,13,15,14,16,9,8,7)$, $(0,7,3,8,5)$, $(1,8,4,9,6)$, $(2,9,5,10,7)$, $(3,10,12,8,6)$, $(4,11,9,14,7)$, $(5,12,9,7)$, $(8,11,6,13,10)$, $(13,8,15,12,14)$, $(9,13,11,14,10)$, $(7,12,16,15,11)$
$J_7 \to 3^{10}, 12^*$	$(3,7,2,9,10,5,12,8,4,11,13,6)$, $(3,10,8)$, $(1,8,6)$, $(0,7,5)$, $(4,9,7)$, $(7,11,8)$, $(6,10,7)$, $(8,5,9)$, $(6,11,9)$, $(10,12,11)$, $(12,9,13)$
$J_7 \to 3^5, 5^3, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(4,9,7)$, $(6,10,11,9,8)$, $(5,10,8)$, $(9,6,11,7,10)$, $(11,8,12)$, $(12,9,13)$
$J_7 \to 5^6, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,8,10,5)$, $(1,8,9,10,6)$, $(3,7,5,8,6)$, $(4,9,11,10,7)$, $(7,11,8,12,9)$, $(6,11,12,13,9)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_7 \to 3^7, 4, 5, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(4,9,7)$, $(6,10,9,8)$, $(5,10,8)$, $(6,11,9)$, $(10,7,11)$, $(11,8,12)$, $(12,9,13)$
$J_7 \to 3^2, 4, 5^4, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(4,9,7)$, $(6,10,11,9,8)$, $(5,10,7,11,8)$, $(9,13,12,8,10)$, $(6,11,12,9)$
$J_7 \to 3^4, 4^2, 5^2, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(4,9,7)$, $(6,10,9,8)$, $(6,11,12,13,9)$, $(5,10,8)$, $(10,7,11)$, $(9,12,8,11)$
$J_7 \to 3^6, 4^3, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(1,8,6)$, $(3,7,9,6)$, $(7,4,9,8)$, $(5,10,8)$, $(9,11,6,10)$, $(10,7,11)$, $(11,8,12)$, $(12,9,13)$
$J_7 \to 3, 4^3, 5^3, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(4,9,8,10,7)$, $(6,10,5,8)$, $(7,11,12,13,9)$, $(9,12,8,11)$, $(9,6,11,10)$
$J_7 \to 3^3, 4^4, 5, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(3,7,8,1,6)$, $(4,9,7)$, $(6,9,8)$, $(5,10,11,8)$, $(6,11,7,10)$, $(9,12,8,10)$, $(9,13,12,11)$
$J_7 \to 4^5, 5^2, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,8,10,5)$, $(1,8,9,6)$, $(4,9,10,7)$, $(3,7,11,6)$, $(5,8,12,9,7)$, $(6,10,11,8)$, $(9,13,12,11)$
$J_7 \to 3^2, 4^6, 12^*$	$(6,13,11,4,8,3,10,12,5,9,2,7)$, $(0,7,5)$, $(1,8,6)$, $(3,7,9,6)$, $(7,4,9,8)$, $(5,10,11,8)$, $(6,11,7,10)$, $(9,12,8,10)$, $(9,13,12,11)$
$J_8 \to 3^2, 5^6, 12^*$	$(6,3,8,4,9,5,10,11,13,12,14,7)$, $(0,7,5)$, $(1,8,7,9,6)$, $(2,9,8,10,7)$, $(3,10,9,11,7)$, $(4,11,8,12,7)$, $(6,11,12,5,8)$, $(6,13,9,12,10)$, $(13,10,14)$
$J_8 \to 3, 4^2, 5^5, 12^*$	$(6,3,8,4,9,5,10,11,13,12,14,7)$, $(0,7,5)$, $(1,8,7,9,6)$, $(2,9,8,10,7)$, $(3,10,9,11,7)$, $(4,11,8,12,7)$, $(6,11,12,5,8)$, $(6,13,14,10)$, $(10,13,9,12)$
$J_8 \to 4^4, 5^4, 12^*$	$(6,3,8,4,9,5,10,11,13,12,14,7)$, $(0,7,8,12,5)$, $(1,8,9,10,6)$, $(4,11,9,2,7)$, $(5,8,11,7)$, $(6,11,12,9)$, $(6,13,14,10,8)$, $(3,10,12,7)$, $(7,10,13,9)$
$J_8 \to 3^2, 4^5, 5^2, 12^*$	$(6,3,8,4,9,5,10,11,13,12,14,7)$, $(0,7,5)$, $(1,8,7,9,6)$, $(3,10,9,2,7)$, $(4,11,7)$, $(5,12,11,8)$, $(6,11,9,8)$, $(10,13,9,12)$, $(8,12,7,10)$, $(6,13,14,10)$
$J_8 \to 3, 4^7, 5, 12^*$	$(6,3,8,4,9,5,10,11,13,12,14,7)$, $(0,7,5)$, $(1,8,7,11,6)$, $(4,11,9,7)$, $(3,10,12,7)$, $(2,9,10,7)$, $(6,9,12,8)$, $(5,12,11,8)$, $(8,10,13,9)$, $(6,13,14,10)$
$J_8 \to 4^9, 12^*$	$(6,3,8,4,9,5,10,11,13,12,14,7)$, $(0,7,8,5)$, $(1,8,11,6)$, $(5,12,11,7)$, $(2,9,10,7)$, $(4,11,9,7)$, $(3,10,12,7)$, $(6,9,12,8)$, $(8,10,13,9)$, $(6,13,14,10)$
$J_9 \to 4^3, 5^6, 12^*$	$(6,9,4,11,10,5,12,14,13,15,8,7)$, $(0,7,3,8,5)$, $(1,8,9,10,6)$, $(5,9,2,7)$, $(3,10,8,11,6)$, $(4,8,13,9,7)$, $(6,13,11,12,8)$, $(7,14,15,11)$, $(7,12,13,10)$, $(9,12,10,14,11)$
$J_9 \to 4^8, 5^2, 12^*$	$(6,9,4,11,10,5,12,14,13,15,8,7)$, $(0,7,3,8,5)$, $(1,8,9,10,6)$, $(5,9,2,7)$, $(3,10,8,6)$, $(4,8,12,7)$, $(7,10,12,9)$, $(11,6,13,12)$, $(11,14,10,13)$, $(9,13,8,11)$, $(7,14,15,11)$
$J_9 \to 3^2, 4^9, 12^*$	$(6,9,4,11,10,5,12,14,13,15,8,7)$, $(0,7,5)$, $(1,8,6)$, $(2,9,10,7)$, $(3,8,4,7)$, $(3,10,13,6)$, $(6,11,8,10)$, $(7,14,15,11)$, $(5,9,13,8)$, $(8,12,7,9)$, $(11,14,10,12)$, $(9,12,13,11)$
$J_{10} \to 4^{12}, 12^*$	$(6,10,5,12,11,13,15,14,16,9,8,7)$, $(0,7,9,5)$, $(3,8,1,6)$, $(2,9,10,7)$, $(3,10,11,7)$, $(5,8,4,7)$, $(6,13,10,8)$, $(6,11,4,9)$, $(8,13,14,11)$, $(12,14,9,13)$, $(10,14,7,12)$, $(9,12,15,11)$, $(8,15,16,12)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_{11} \to 4, 5^{10}, 12^*$	$(6,11,12,13,14,16,15,17,10,9,8,7)$, $(0,7,3,8,5)$, $(1,8,4,9,6)$, $(2,9,5,10,7)$, $(3,10,8,13,6)$, $(4,11,10,12,7)$, $(5,12,8,11,7)$, $(6,10,13,15,8)$, $(7,14,15,11,9)$, $(9,14,10,15,12)$, $(11,14,12,16,13)$, $(9,16,17,13)$
$J_4^+ \to 5^4, 4^{+*}$	$[4,7,8,10,9,6]$, $(0,7,2,9,5)$, $(3,7,5,8,6)$, $(4,6,10,3,8)$, $(6,1,8,9,7)$
$J_4^+ \to 4^5, 4^{+*}$	$[4,7,8,9,10,6]$, $(6,9,2,7)$, $(0,7,9,5)$, $(1,8,4,6)$, $(3,8,5,7)$, $(3,10,8,6)$
$J_6^+ \to 3^{10}, 6^{+*}$	$[4,7,8,9,11,12,10,6]$, $(3,10,8)$, $(1,8,6)$, $(0,7,5)$, $(5,12,8)$, $(4,11,8)$, $(2,9,7)$, $(6,4,9)$, $(6,3,7)$, $(10,7,11)$, $(5,10,9)$
$J_2^+ \to 3, 4^2, 1^+$	$[6,7,4]$, $(0,7,5)$, $(1,8,4,6)$, $(3,7,8,6)$
$J_3^+ \to 3, 4^2, 5, 2^+$	$[4,7,8,6]$, $(0,7,2,9,5)$, $(6,3,7)$, $(1,8,4,6)$, $(5,8,9,7)$
$J_3^+ \to 4^3, 5, 1^+$	$[6,7,4]$, $(0,7,8,5)$, $(5,9,2,7)$, $(3,7,9,8,6)$, $(1,8,4,6)$
$J_4^+ \to 3, 4^5, 1^+$	$[6,7,4]$, $(0,7,5)$, $(1,8,4,6)$, $(7,2,9,8)$, $(3,7,9,6)$, $(5,9,10,8)$, $(6,10,3,8)$
$J_4^+ \to 3, 5^4, 1^+$	$[6,7,4]$, $(0,7,3,8,5)$, $(1,8,9,10,6)$, $(3,10,8,4,6)$, $(5,9,6,8,7)$, $(2,9,7)$
$J_5^+ \to 4^7, 2^+$	$[4,7,8,6]$, $(0,7,9,5)$, $(1,8,9,6)$, $(2,9,11,7)$, $(3,8,4,6)$, $(9,4,11,10)$, $(5,8,10,7)$, $(6,10,3,7)$
$J_5^+ \to 3, 4^6, 3^+$	$[4,7,8,9,6]$, $(0,7,5)$, $(3,8,1,6)$, $(2,9,11,7)$, $(6,4,11,10)$, $(6,8,10,7)$, $(3,10,9,7)$, $(4,9,5,8)$
$J_6^+ \to 4^8, 4^+$	$[4,7,8,9,10,6]$, $(0,7,10,5)$, $(3,8,5,7)$, $(7,11,4,9)$, $(1,8,4,6)$, $(6,9,2,7)$, $(3,10,8,6)$, $(5,12,11,9)$, $(10,12,8,11)$
$J_7^+ \to 4^9, 6^+$	$[4,7,8,9,10,11,13,6]$, $(0,7,9,5)$, $(3,8,1,6)$, $(6,9,2,7)$, $(3,10,5,7)$, $(6,11,7,10)$, $(6,4,11,8)$, $(4,9,12,8)$, $(5,12,10,8)$, $(9,13,12,11)$

Table A.12: Table of decompositions of $J_n^{\{1,2,3,4,5,7\}}$

$J_7 \to 3^{11}, 5^2, 6$	$J_7 \to 3^6, 5^5, 6$	$J_7 \to 3, 5^8, 6$	$J_7 \to 3^{13}, 4, 6$
$J_7 \to 3^8, 4, 5^3, 6$	$J_7 \to 3^3, 4, 5^6, 6$	$J_7 \to 3^{10}, 4^2, 5, 6$	$J_7 \to 3^5, 4^2, 5^4, 6$
$J_7 \to 4^2, 5^7, 6$	$J_7 \to 3^7, 4^3, 5^2, 6$	$J_7 \to 3^2, 4^3, 5^5, 6$	$J_7 \to 3^9, 4^4, 6$
$J_7 \to 3^4, 4^4, 5^3, 6$	$J_7 \to 3^6, 4^5, 5, 6$	$J_7 \to 3, 4^5, 5^4, 6$	$J_7 \to 3^3, 4^6, 5^2, 6$
$J_7 \to 3^5, 4^7, 6$	$J_7 \to 4^7, 5^3, 6$	$J_7 \to 3^2, 4^8, 5, 6$	$J_7 \to 3, 4^{10}, 6$
$J_8 \to 3^{15}, 5, 6$	$J_8 \to 3^{10}, 5^4, 6$	$J_8 \to 3^5, 5^7, 6$	$J_8 \to 5^{10}, 6$
$J_8 \to 4^5, 5^6, 6$	$J_8 \to 4^{10}, 5^2, 6$	$J_9 \to 3^{19}, 6$	$J_9 \to 4^{13}, 5, 6$
$J_{10} \to 4^{16}, 6$	$J_{12} \to 3^{26}, 6$		
$J_8 \to 3^{13}, 5^2, 7$	$J_8 \to 3^8, 5^5, 7$	$J_8 \to 3^3, 5^8, 7$	$J_8 \to 3^{15}, 4, 7$
$J_8 \to 3^{10}, 4, 5^3, 7$	$J_8 \to 3^5, 4, 5^6, 7$	$J_8 \to 4, 5^9, 7$	$J_8 \to 3^{12}, 4^2, 5, 7$
$J_8 \to 3^7, 4^2, 5^4, 7$	$J_8 \to 3^2, 4^2, 5^7, 7$	$J_8 \to 3^9, 4^3, 5^2, 7$	$J_8 \to 3^4, 4^3, 5^5, 7$
$J_8 \to 3^{11}, 4^4, 7$	$J_8 \to 3^6, 4^4, 5^3, 7$	$J_8 \to 3, 4^4, 5^6, 7$	$J_8 \to 3^8, 4^5, 5, 7$
$J_8 \to 3^3, 4^5, 5^4, 7$	$J_8 \to 3^5, 4^6, 5^2, 7$	$J_8 \to 4^6, 5^5, 7$	$J_8 \to 3^7, 4^7, 7$
$J_8 \to 3^2, 4^7, 5^3, 7$	$J_8 \to 3^4, 4^8, 5, 7$	$J_8 \to 3, 4^9, 5^2, 7$	$J_8 \to 3^3, 4^{10}, 7$
$J_8 \to 4^{11}, 5, 7$	$J_9 \to 3^{17}, 5, 7$	$J_9 \to 3^{12}, 5^4, 7$	$J_9 \to 3^7, 5^7, 7$
$J_9 \to 3^2, 5^{10}, 7$	$J_9 \to 4^4, 5^8, 7$	$J_9 \to 4^9, 5^4, 7$	$J_9 \to 4^{14}, 7$
$J_{10} \to 3^{21}, 7$	$J_{10} \to 3, 5^{12}, 7$	$J_{11} \to 5^{14}, 7$	$J_{13} \to 3^{28}, 7$

Table A.13: These decompositions required for Lemma 1.6.26. For all these decompositions the k-cycle is incident upon vertices $\{4, 5, \ldots, k+3\}$. The decompositions themselves are given in full in table A.15.

$J_6 \to 3^8, 5^2, 8^*$	$J_6 \to 3^3, 5^5, 8^*$	$J_6 \to 3^{10}, 4, 8^*$	$J_6 \to 3^5, 4, 5^3, 8^*$

Table A.14: These decompositions are required for Lemma 1.6.27. These are sorted by the value of k in the decomposition, and are given in table A.15.

$J_6 \to 4, 5^6, 8^*$	$J_6 \to 3^7, 4^2, 5, 8^*$	$J_6 \to 3^2, 4^2, 5^4, 8^*$	$J_6 \to 3^4, 4^3, 5^2, 8^*$
$J_6 \to 3^6, 4^4, 8^*$	$J_6 \to 3, 4^4, 5^3, 8^*$	$J_6 \to 3^3, 4^5, 5, 8^*$	$J_6 \to 4^6, 5^2, 8^*$
$J_6 \to 3^2, 4^7, 8^*$	$J_7 \to 3^{12}, 5, 8^*$	$J_7 \to 3^7, 5^4, 8^*$	$J_7 \to 3^2, 5^7, 8^*$
$J_7 \to 4^4, 5^5, 8^*$	$J_7 \to 4^9, 5, 8^*$	$J_8 \to 3^{16}, 8^*$	$J_8 \to 3, 5^9, 8^*$
$J_8 \to 4^{12}, 8^*$	$J_9 \to 5^{11}, 8^*$	$J_{11} \to 3^{23}, 8^*$	
$J_7 \to 3^{10}, 5^2, 9^*$	$J_7 \to 3^5, 5^5, 9^*$	$J_7 \to 5^8, 9^*$	$J_7 \to 3^{12}, 4, 9^*$
$J_7 \to 3^7, 4, 5^3, 9^*$	$J_7 \to 3^2, 4, 5^6, 9^*$	$J_7 \to 3^9, 4^2, 5, 9^*$	$J_7 \to 3^4, 4^2, 5^4, 9^*$
$J_7 \to 3^6, 4^3, 5^2, 9^*$	$J_7 \to 3, 4^3, 5^5, 9^*$	$J_7 \to 3^8, 4^4, 9^*$	$J_7 \to 3^3, 4^4, 5^3, 9^*$
$J_7 \to 3^5, 4^5, 5, 9^*$	$J_7 \to 4^5, 5^4, 9^*$	$J_7 \to 3^2, 4^6, 5^2, 9^*$	$J_7 \to 3^4, 4^7, 9^*$
$J_7 \to 3, 4^8, 5, 9^*$	$J_7 \to 4^{10}, 9^*$	$J_8 \to 3^{14}, 5, 9^*$	$J_8 \to 3^9, 5^4, 9^*$
$J_8 \to 3^4, 5^7, 9^*$	$J_8 \to 4^3, 5^7, 9^*$	$J_8 \to 4^8, 5^3, 9^*$	$J_9 \to 3^{18}, 9^*$
$J_{11} \to 3, 5^{13}, 9^*$	$J_{12} \to 3^{25}, 9^*$		
$J_3 \to 3^2, 5, 10^*$	$J_3 \to 3, 4^2, 10^*$	$J_4 \to 3^6, 10^*$	$J_4 \to 3, 5^3, 10^*$
$J_4 \to 4^2, 5^2, 10^*$	$J_5 \to 5^5, 10^*$	$J_5 \to 4^5, 5, 10^*$	$J_6 \to 4^8, 10^*$
$J_7 \to 3^{13}, 10^*$			
$J_4 \to 3^4, 5, 11^*$	$J_4 \to 3, 4, 5^2, 11^*$	$J_4 \to 3^3, 4^2, 11^*$	$J_4 \to 4^3, 5, 11^*$
$J_5 \to 3^8, 11^*$	$J_5 \to 3^3, 5^3, 11^*$	$J_5 \to 4, 5^4, 11^*$	$J_5 \to 4^6, 11^*$
$J_6 \to 3^2, 5^5, 11^*$	$J_7 \to 3, 5^7, 11^*$	$J_8 \to 3^{15}, 11^*$	$J_8 \to 5^9, 11^*$
$J_5 \to 3^6, 5, 12^*$	$J_5 \to 3, 5^4, 12^*$	$J_5 \to 3^3, 4, 5^2, 12^*$	$J_5 \to 3^5, 4^2, 12^*$
$J_5 \to 4^2, 5^3, 12^*$	$J_5 \to 3^2, 4^3, 5, 12^*$	$J_5 \to 3, 4^5, 12^*$	$J_6 \to 3^{10}, 12^*$
$J_6 \to 3^5, 5^3, 12^*$	$J_6 \to 5^6, 12^*$	$J_6 \to 4^5, 5^2, 12^*$	$J_7 \to 4^8, 5, 12^*$
$J_8 \to 4^{11}, 12^*$	$J_9 \to 3^{17}, 12^*$		
$J_6 \to 3^8, 5, 13^*$	$J_6 \to 3^3, 5^4, 13^*$	$J_6 \to 3^5, 4, 5^2, 13^*$	$J_6 \to 4, 5^5, 13^*$
$J_6 \to 3^7, 4^2, 13^*$	$J_6 \to 3^2, 4^2, 5^3, 13^*$	$J_6 \to 3^4, 4^3, 5, 13^*$	$J_6 \to 3, 4^4, 5^2, 13^*$
$J_6 \to 3^3, 4^5, 13^*$	$J_6 \to 4^6, 5, 13^*$	$J_7 \to 3^{12}, 13^*$	$J_7 \to 3^7, 5^3, 13^*$
$J_7 \to 3^2, 5^6, 13^*$	$J_7 \to 4^4, 5^4, 13^*$	$J_7 \to 4^9, 13^*$	$J_8 \to 3, 5^8, 13^*$
$J_9 \to 5^{10}, 13^*$	$J_{10} \to 3^{19}, 13^*$		
$J_7 \to 3^{10}, 5, 14^*$	$J_7 \to 3^5, 5^4, 14^*$	$J_7 \to 5^7, 14^*$	$J_7 \to 3^7, 4, 5^2, 14^*$
$J_7 \to 3^2, 4, 5^5, 14^*$	$J_7 \to 3^9, 4^2, 14^*$	$J_7 \to 3^4, 4^2, 5^3, 14^*$	$J_7 \to 3^6, 4^3, 5, 14^*$
$J_7 \to 3, 4^3, 5^4, 14^*$	$J_7 \to 3^3, 4^4, 5^2, 14^*$	$J_7 \to 3^5, 4^5, 14^*$	$J_7 \to 4^5, 5^3, 14^*$
$J_7 \to 3^2, 4^6, 5, 14^*$	$J_7 \to 3, 4^8, 14^*$	$J_8 \to 3^{14}, 14^*$	$J_8 \to 3^9, 5^3, 14^*$
$J_8 \to 3^4, 5^6, 14^*$	$J_8 \to 4^3, 5^6, 14^*$	$J_8 \to 4^8, 5^2, 14^*$	$J_9 \to 4^{11}, 5, 14^*$
$J_{10} \to 4^{14}, 14^*$	$J_{11} \to 3^{21}, 14^*$	$J_{11} \to 3, 5^{12}, 14^*$	
$J_8 \to 3^{12}, 5, 15^*$	$J_8 \to 3^7, 5^4, 15^*$	$J_8 \to 3^2, 5^7, 15^*$	$J_8 \to 3^9, 4, 5^2, 15^*$
$J_8 \to 3^4, 4, 5^5, 15^*$	$J_8 \to 3^{11}, 4^2, 15^*$	$J_8 \to 3^6, 4^2, 5^3, 15^*$	$J_8 \to 3, 4^2, 5^6, 15^*$
$J_8 \to 3^8, 4^3, 5, 15^*$	$J_8 \to 3^3, 4^3, 5^4, 15^*$	$J_8 \to 3^5, 4^4, 5^2, 15^*$	$J_8 \to 4^4, 5^5, 15^*$
$J_8 \to 3^7, 4^5, 15^*$	$J_8 \to 3^2, 4^5, 5^3, 15^*$	$J_8 \to 3^4, 4^6, 5, 15^*$	$J_8 \to 3, 4^7, 5^2, 15^*$
$J_8 \to 3^3, 4^8, 15^*$	$J_8 \to 4^9, 5, 15^*$	$J_9 \to 3^{16}, 15^*$	$J_9 \to 3^{11}, 5^3, 15^*$
$J_9 \to 3^6, 5^6, 15^*$	$J_9 \to 3, 5^9, 15^*$	$J_9 \to 4^2, 5^8, 15^*$	$J_9 \to 4^7, 5^4, 15^*$
$J_9 \to 4^{12}, 15^*$	$J_{10} \to 5^{11}, 15^*$	$J_{12} \to 3^{23}, 15^*$	
$J_9 \to 3^{14}, 5, 16^*$	$J_9 \to 3^9, 5^4, 16^*$	$J_9 \to 3^4, 5^7, 16^*$	$J_9 \to 3^{11}, 4, 5^2, 16^*$
$J_9 \to 3^6, 4, 5^5, 16^*$	$J_9 \to 3, 4, 5^8, 16^*$	$J_9 \to 3^{13}, 4^2, 16^*$	$J_9 \to 3^8, 4^2, 5^3, 16^*$
$J_9 \to 3^3, 4^2, 5^6, 16^*$	$J_9 \to 3^{10}, 4^3, 5, 16^*$	$J_9 \to 3^5, 4^3, 5^4, 16^*$	$J_9 \to 4^3, 5^7, 16^*$
$J_9 \to 3^7, 4^4, 5^2, 16^*$	$J_9 \to 3^2, 4^4, 5^5, 16^*$	$J_9 \to 3^9, 4^5, 16^*$	$J_9 \to 3^4, 4^5, 5^3, 16^*$
$J_9 \to 3^6, 4^6, 5, 16^*$	$J_9 \to 3, 4^6, 5^4, 16^*$	$J_9 \to 3^3, 4^7, 5^2, 16^*$	$J_9 \to 3^5, 4^8, 16^*$
$J_9 \to 4^8, 5^3, 16^*$	$J_9 \to 3^2, 4^9, 5, 16^*$	$J_9 \to 3, 4^{11}, 16^*$	$J_{10} \to 3^{18}, 16^*$
$J_{10} \to 3^{13}, 5^3, 16^*$	$J_{10} \to 3^8, 5^6, 16^*$	$J_{10} \to 3^3, 5^9, 16^*$	$J_{10} \to 4, 5^{10}, 16^*$
$J_{10} \to 4^6, 5^6, 16^*$	$J_{10} \to 4^{11}, 5^2, 16^*$	$J_{11} \to 3^2, 5^{11}, 16^*$	$J_{11} \to 4^{14}, 5, 16^*$
$J_{12} \to 3, 5^{13}, 16^*$	$J_{12} \to 4^{17}, 16^*$	$J_{13} \to 3^{25}, 16^*$	$J_{13} \to 5^{15}, 16^*$

Table A.14: These decompositions are required for Lemma 1.6.27. These are sorted by the value of k in the decomposition, and are given in table A.15.

$J_{10} \to 3^{16}, 5, 17^*$	$J_{10} \to 3^{11}, 5^4, 17^*$	$J_{10} \to 3^6, 5^7, 17^*$	$J_{10} \to 3, 5^{10}, 17^*$
$J_{10} \to 3^{13}, 4, 5^2, 17^*$	$J_{10} \to 3^8, 4, 5^5, 17^*$	$J_{10} \to 3^3, 4, 5^8, 17^*$	$J_{10} \to 3^{15}, 4^2, 17^*$
$J_{10} \to 3^{10}, 4^2, 5^3, 17^*$	$J_{10} \to 3^5, 4^2, 5^6, 17^*$	$J_{10} \to 4^2, 5^9, 17^*$	$J_{10} \to 3^{12}, 4^3, 5, 17^*$
$J_{10} \to 3^7, 4^3, 5^4, 17^*$	$J_{10} \to 3^2, 4^3, 5^7, 17^*$	$J_{10} \to 3^9, 4^4, 5^2, 17^*$	$J_{10} \to 3^4, 4^4, 5^5, 17^*$
$J_{10} \to 3^{11}, 4^5, 17^*$	$J_{10} \to 3^6, 4^5, 5^3, 17^*$	$J_{10} \to 3, 4^5, 5^6, 17^*$	$J_{10} \to 3^8, 4^6, 5, 17^*$
$J_{10} \to 3^3, 4^6, 5^4, 17^*$	$J_{10} \to 3^5, 4^7, 5^2, 17^*$	$J_{10} \to 4^7, 5^5, 17^*$	$J_{10} \to 3^7, 4^8, 17^*$
$J_{10} \to 3^2, 4^8, 5^3, 17^*$	$J_{10} \to 3^4, 4^9, 5, 17^*$	$J_{10} \to 3, 4^{10}, 5^2, 17^*$	$J_{10} \to 3^3, 4^{11}, 17^*$
$J_{10} \to 4^{12}, 5, 17^*$	$J_{11} \to 3^{20}, 17^*$	$J_{11} \to 3^{15}, 5^3, 17^*$	$J_{11} \to 3^{10}, 5^6, 17^*$
$J_{11} \to 3^5, 5^9, 17^*$	$J_{11} \to 5^{12}, 17^*$	$J_{11} \to 4^5, 5^8, 17^*$	$J_{11} \to 4^{10}, 5^4, 17^*$
$J_{11} \to 4^{15}, 17^*$	$J_{14} \to 3^{27}, 17^*$		

Table A.14: These decompositions are required for Lemma 1.6.27. These are sorted by the value of k in the decomposition, and are given in table A.15.

$J_1 \to 3, 4$	$(0,6,3,4)$, $(0,5,7)$
$J_2 \to 4, 5^2$	$(0,5,7,4)$, $(0,6,8,1,7)$, $(3,4,5,1,6)$
$J_2 \to 3^3, 5$	$(0,7,4)$, $(1,6,8)$, $(0,6,3,4,5)$, $(5,1,7)$
$J_3 \to 4^4, 5$	$(0,6,3,4)$, $(1,6,8,5)$, $(1,7,9,2,8)$, $(0,5,4,7)$, $(5,6,2,7)$
$J_3 \to 3^2, 5^3$	$(0,5,6,3,4)$, $(0,6,1,5,7)$, $(1,7,4,5,8)$, $(6,2,8)$, $(2,7,9)$
$J_4 \to 4^7$	$(0,7,3,4)$, $(0,6,1,5)$, $(3,10,8,6)$, $(2,8,3,9)$, $(5,8,1,7)$, $(4,5,6,7)$, $(2,7,9,6)$
$J_4 \to 3, 5^5$	$(0,7,9,3,4)$, $(5,0,6,3,7)$, $(1,7,2,6,8)$, $(1,6,7,4,5)$, $(2,8,5,6,9)$, $(3,8,10)$
$J_5 \to 5^7$	$(0,7,9,3,4)$, $(5,0,6,3,7)$, $(1,7,4,8,5)$, $(1,6,2,7,8)$, $(6,7,10,3,8)$, $(2,8,10,4,9)$, $(4,11,9,6,5)$
$J_6 \to 3^{14}$	$(0,7,4)$, $(0,6,5)$, $(1,8,5)$, $(2,9,6)$, $(1,7,6)$, $(3,8,6)$, $(3,9,4)$, $(5,9,7)$, $(2,8,7)$, $(3,10,7)$, $(5,10,12)$, $(8,4,10)$, $(9,8,11)$, $(4,11,5)$
$J_9 \to 3^{21}$	$(0,7,4)$, $(0,6,5)$, $(1,6,8)$, $(2,8,7)$, $(2,9,6)$, $(3,8,4)$, $(3,7,6)$, $(3,9,10)$, $(4,9,11)$, $(5,1,7)$, $(4,10,5)$, $(5,11,12)$, $(7,12,9)$, $(7,14,11)$, $(6,11,13)$, $(7,13,10)$, $(5,9,8)$, $(8,11,10)$, $(8,13,15)$, $(10,6,12)$, $(12,8,14)$
$J_7 \to 3^{11}, 5^2, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,2,7)$, $(7,3,9)$, $(1,7,8)$, $(9,5,11)$, $(4,11,8)$, $(6,10,7)$, $(6,11,13)$, $(2,8,9)$, $(3,8,10)$, $(5,10,12)$, $(4,10,9)$, $(6,12,9)$
$J_7 \to 3^6, 5^5, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,7)$, $(4,10,5,11,8)$, $(10,6,12)$, $(3,9,10)$, $(6,11,13)$, $(4,9,11)$, $(5,12,9)$
$J_7 \to 3, 5^8, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,9,10)$, $(3,9,11,8,7)$, $(4,10,12,5,11)$, $(5,10,6,12,9)$, $(6,11,13)$
$J_7 \to 3^{13}, 4, 6$	$(5,8,6,3,4,7)$, $(0,6,2,7)$, $(0,5,4)$, $(4,11,8)$, $(1,6,5)$, $(3,8,10)$, $(1,7,8)$, $(2,8,9)$, $(6,12,9)$, $(9,5,11)$, $(5,10,12)$, $(6,11,13)$, $(6,10,7)$, $(7,3,9)$, $(4,10,9)$
$J_7 \to 3^8, 4, 5^3, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,3,10,8)$, $(5,10,12)$, $(2,9,12,6)$, $(6,11,13)$, $(2,8,7)$, $(4,11,8)$, $(3,9,8)$, $(6,10,7)$, $(4,10,9)$, $(9,5,11)$
$J_7 \to 3^3, 4, 5^6, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,9,10)$, $(3,9,11,8,7)$, $(4,10,5,11)$, $(5,12,9)$, $(6,11,13)$, $(10,6,12)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_7 \to 3^{10}, 4^2, 5, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,7)$, $(2,7,10,6)$, $(4,10,9)$, $(1,7,8)$, $(2,8,9)$, $(7,3,9)$, $(9,5,11)$, $(5,10,12)$, $(3,8,10)$, $(6,12,9)$, $(6,11,13)$, $(4,11,8)$
$J_7 \to 3^5, 4^2, 5^4, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,7)$, $(4,10,5,11)$, $(5,12,9)$, $(3,9,10)$, $(4,9,11,8)$, $(10,6,12)$, $(6,11,13)$
$J_7 \to 4^2, 5^7, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,9,10)$, $(3,9,11,8,7)$, $(4,10,6,13,11)$, $(5,10,12,9)$, $(5,11,6,12)$
$J_7 \to 3^7, 4^3, 5^2, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,8)$, $(2,9,12,6)$, $(3,8,7)$, $(3,9,8,10)$, $(5,10,12)$, $(6,10,7)$, $(4,10,9)$, $(6,11,13)$, $(4,11,8)$, $(9,5,11)$
$J_7 \to 3^2, 4^3, 5^5, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,10)$, $(3,9,11,8,7)$, $(5,10,12)$, $(6,12,9,10)$, $(4,9,5,11)$, $(6,11,13)$
$J_7 \to 3^9, 4^4, 6$	$(5,8,6,3,4,7)$, $(0,6,1,7)$, $(0,5,4)$, $(1,8,10,5)$, $(2,8,3,9)$, $(2,7,6)$, $(5,11,13,6)$, $(3,10,7)$, $(7,8,9)$, $(6,11,9)$, $(4,11,8)$, $(5,12,9)$, $(4,10,9)$, $(10,6,12)$
$J_7 \to 3^4, 4^4, 5^3, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,7,6)$, $(3,8,10)$, $(3,9,10,7)$, $(4,10,5,11)$, $(4,9,11,8)$, $(5,12,9)$, $(6,11,13)$, $(10,6,12)$
$J_7 \to 3^6, 4^5, 5, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,9,7)$, $(1,7,2,8)$, $(2,9,6)$, $(3,10,6,7)$, $(3,9,11,8)$, $(4,10,5,11)$, $(4,9,8)$, $(5,12,6)$, $(6,11,13)$, $(8,7,10)$, $(10,9,12)$
$J_7 \to 3, 4^5, 5^4, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,3,7,6)$, $(4,10,7,8)$, $(6,11,13)$, $(4,9,5,11)$, $(8,11,9,10)$, $(3,9,12,10)$, $(5,10,6,12)$
$J_7 \to 3^3, 4^6, 5^2, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,8)$, $(2,9,12,6)$, $(4,9,5,11)$, $(5,10,12)$, $(6,11,13)$, $(7,8,9,10)$, $(3,9,11,8)$, $(8,4,10)$, $(3,10,6,7)$
$J_7 \to 3^5, 4^7, 6$	$(5,8,6,3,4,7)$, $(0,6,1,7)$, $(0,5,4)$, $(1,8,10,5)$, $(2,8,3,9)$, $(2,7,9,6)$, $(3,10,7)$, $(4,10,9,8)$, $(4,9,11)$, $(6,11,8,7)$, $(5,11,13,6)$, $(10,6,12)$, $(5,12,9)$
$J_7 \to 4^7, 5^3, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,8)$, $(2,9,11,13,6)$, $(4,9,5,11)$, $(3,8,4,10)$, $(6,11,8,7)$, $(3,9,10,7)$, $(8,9,12,10)$, $(5,10,6,12)$
$J_7 \to 3^2, 4^8, 5, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,9,7)$, $(1,7,2,8)$, $(2,9,6)$, $(3,10,6,7)$, $(3,9,4,8)$, $(5,11,13,6)$, $(9,12,6,11)$, $(8,11,4,10)$, $(7,8,9,10)$, $(5,10,12)$
$J_7 \to 3, 4^{10}, 6$	$(5,8,6,3,4,7)$, $(0,6,1,7)$, $(0,5,4)$, $(1,8,10,5)$, $(2,8,3,9)$, $(2,7,9,6)$, $(3,10,6,7)$, $(4,10,7,8)$, $(4,9,8,11)$, $(5,12,10,9)$, $(5,11,13,6)$, $(9,12,6,11)$
$J_8 \to 3^{15}, 5, 6$	$(5,8,6,3,4,7)$, $(0,6,9,11,7)$, $(0,5,4)$, $(7,12,14)$, $(4,11,8)$, $(2,8,9)$, $(1,7,8)$, $(3,8,10)$, $(1,6,5)$, $(5,11,10)$, $(2,7,6)$, $(10,6,12)$, $(6,11,13)$, $(7,3,9)$, $(5,12,9)$, $(4,10,9)$, $(7,13,10)$
$J_8 \to 3^{10}, 5^4, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,3,7,6)$, $(8,4,10)$, $(3,9,10)$, $(6,11,13)$, $(10,6,12)$, $(7,13,10)$, $(7,11,8)$, $(7,12,14)$, $(5,12,9)$, $(4,9,11)$, $(5,11,10)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_8 \to 3^5, 5^7, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,11,7)$, $(3,9,12,5,10)$, $(7,13,6,11,8)$, $(9,5,11)$, $(4,10,9)$, $(10,6,12)$, $(7,12,14)$, $(11,10,13)$
$J_8 \to 5^{10}, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,11,7)$, $(3,9,12,5,10)$, $(6,12,14,7,13)$, $(7,12,10,11,8)$, $(5,11,13,10,9)$, $(9,4,10,6,11)$
$J_8 \to 4^5, 5^6, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,9,8)$, $(2,8,10,7,6)$, $(3,8,4,9,10)$, $(5,11,7,14,12)$, $(3,9,12,7)$, $(4,10,6,11)$, $(10,13,6,12)$, $(7,13,11,8)$, $(9,5,10,11)$
$J_8 \to 4^{10}, 5^2, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,6,9,7)$, $(1,7,2,8)$, $(2,9,11,6)$, $(3,9,4,10)$, $(3,8,10,7)$, $(4,11,7,8)$, $(5,11,8,9)$, $(5,10,9,12)$, $(6,13,11,10)$, $(6,12,14,7)$, $(10,13,7,12)$
$J_9 \to 3^{19}, 6$	$(3,7,4,8,5,6)$, $(0,5,4)$, $(0,6,7)$, $(1,6,8)$, $(5,1,7)$, $(2,8,7)$, $(2,9,6)$, $(7,12,9)$, $(5,11,12)$, $(10,6,12)$, $(8,13,15)$, $(3,8,10)$, $(12,8,14)$, $(3,9,4)$, $(9,8,11)$, $(4,10,11)$, $(5,10,9)$, $(7,13,10)$, $(6,11,13)$, $(7,14,11)$
$J_9 \to 4^{13}, 5, 6$	$(5,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,9,7)$, $(1,7,2,8)$, $(2,9,11,6)$, $(3,9,6,7)$, $(3,8,4,10)$, $(4,9,8,11)$, $(5,12,10,6)$, $(5,11,13,10)$, $(6,12,8,13)$, $(7,13,15,8)$, $(7,12,9,10)$, $(8,14,11,10)$, $(7,14,12,11)$
$J_{10} \to 4^{16}, 6$	$(5,8,6,3,4,7)$, $(0,6,1,7)$, $(0,5,9,4)$, $(1,8,10,5)$, $(2,8,3,9)$, $(2,7,9,6)$, $(3,10,12,7)$, $(4,11,6,5)$, $(4,10,7,8)$, $(9,12,5,11)$, $(6,13,9,10)$, $(6,12,11,7)$, $(11,14,7,13)$, $(12,15,9,14)$, $(8,15,13,12)$, $(8,14,16,9)$, $(8,13,10,11)$
$J_{12} \to 3^{26}, 6$	$(3,7,4,8,5,6)$, $(0,5,4)$, $(5,1,7)$, $(1,6,8)$, $(0,6,7)$, $(2,8,7)$, $(2,9,6)$, $(3,9,8)$, $(4,9,11)$, $(5,10,9)$, $(9,14,16)$, $(13,9,15)$, $(3,10,4)$, $(10,16,13)$, $(10,15,17)$, $(5,11,12)$, $(8,15,11)$, $(11,17,14)$, $(8,14,10)$, $(7,11,10)$, $(11,16,18)$, $(10,6,12)$, $(7,12,9)$, $(12,15,14)$, $(8,13,12)$, $(6,11,13)$, $(7,13,14)$
$J_8 \to 3^{13}, 5^2, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,7)$, $(1,7,10,13,6)$, $(6,2,8)$, $(7,12,14)$, $(2,7,9)$, $(4,9,11)$, $(3,8,7)$, $(5,12,9)$, $(3,9,10)$, $(4,10,5)$, $(5,11,6)$, $(10,6,12)$, $(8,11,10)$, $(11,7,13)$
$J_8 \to 3^8, 5^5, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,2,9,7)$, $(6,1,7,2,8)$, $(3,10,12,6,7)$, $(8,3,9,11,10)$, $(4,11,5)$, $(4,10,9)$, $(5,12,9)$, $(5,10,6)$, $(6,11,13)$, $(7,13,10)$, $(7,11,8)$, $(7,12,14)$
$J_8 \to 3^3, 5^8, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,2,9,7)$, $(6,1,7,2,8)$, $(3,10,12,6,7)$, $(8,3,9,11,10)$, $(4,9,12,5,11)$, $(4,10,13,6,5)$, $(5,10,9)$, $(7,8,11,6,10)$, $(7,12,14)$, $(11,7,13)$
$J_8 \to 3^{15}, 4, 7$	$(5,8,9,6,3,4,7)$, $(0,5,4)$, $(1,7,3,8)$, $(1,6,5)$, $(6,2,8)$, $(0,6,7)$, $(7,11,8)$, $(8,4,10)$, $(6,11,13)$, $(7,13,10)$, $(7,12,14)$, $(2,7,9)$, $(4,9,11)$, $(5,11,10)$, $(3,9,10)$, $(10,6,12)$, $(5,12,9)$
$J_8 \to 3^{10}, 4, 5^3, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(2,9,7,10,6)$, $(3,8,10)$, $(4,10,5)$, $(3,9,5,11,7)$, $(6,13,7)$, $(5,12,6)$, $(2,8,7)$, $(7,12,14)$, $(6,11,8)$, $(10,9,12)$, $(4,9,11)$, $(11,10,13)$
$J_8 \to 3^5, 4, 5^6, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,7,6,5)$, $(5,10,12)$, $(6,12,9,10,11)$, $(9,5,11)$, $(6,13,10)$, $(11,7,13)$, $(7,12,14)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_8 \to 4, 5^9, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,7,6,5)$, $(9,5,12,10,11)$, $(5,11,13,6,10)$, $(6,12,14,7,11)$, $(9,10,13,7,12)$
$J_8 \to 3^{12}, 4^2, 5, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(2,7,9)$, $(6,2,8)$, $(3,8,11,7)$, $(3,9,10)$, $(5,11,6)$, $(4,10,5)$, $(8,7,10)$, $(7,12,14)$, $(5,12,9)$, $(10,6,12)$, $(4,9,11)$, $(11,10,13)$, $(6,13,7)$
$J_8 \to 3^7, 4^2, 5^4, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,5)$, $(7,14,12,10)$, $(5,12,9)$, $(6,12,7)$, $(9,10,11)$, $(11,7,13)$, $(5,11,6)$, $(6,13,10)$
$J_8 \to 3^2, 4^2, 5^7, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,7,6,5)$, $(9,5,12,10,11)$, $(5,11,6,10)$, $(11,7,13)$, $(6,12,9,10,13)$, $(7,12,14)$
$J_8 \to 3^9, 4^3, 5^2, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,3,7)$, $(3,9,10)$, $(4,10,5)$, $(6,12,14,7)$, $(4,9,11)$, $(5,12,9)$, $(7,12,10)$, $(5,11,6)$, $(11,7,13)$, $(6,13,10)$, $(8,11,10)$
$J_8 \to 3^4, 4^3, 5^5, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,7,6,5)$, $(5,12,10,9)$, $(5,11,10)$, $(6,13,10)$, $(11,7,13)$, $(7,12,14)$, $(9,12,6,11)$
$J_8 \to 3^{11}, 4^4, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,7)$, $(0,6,5,4)$, $(1,6,8)$, $(2,9,7,6)$, $(2,8,3,7)$, $(3,9,10)$, $(4,9,11)$, $(5,11,10)$, $(5,12,9)$, $(6,11,13)$, $(7,13,10)$, $(7,12,14)$, $(8,4,10)$, $(7,11,8)$, $(10,6,12)$
$J_8 \to 3^6, 4^4, 5^3, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,11,8)$, $(4,9,5,11)$, $(4,10,5)$, $(6,11,10,7)$, $(5,12,6)$, $(7,12,14)$, $(11,7,13)$, $(10,9,12)$, $(6,13,10)$
$J_8 \to 3, 4^4, 5^6, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,7,6,5)$, $(9,5,12,10,11)$, $(5,11,13,10)$, $(6,11,7,13)$, $(7,12,14)$, $(6,12,9,10)$
$J_8 \to 3^8, 4^5, 5, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(2,9,7,6)$, $(2,8,3,7)$, $(3,9,4,10)$, $(4,11,9,5)$, $(5,11,10)$, $(6,11,13)$, $(5,12,6)$, $(6,10,8)$, $(7,11,8)$, $(7,13,10)$, $(7,12,14)$, $(10,9,12)$
$J_8 \to 3^3, 4^5, 5^4, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,6,5)$, $(5,12,10,9)$, $(9,12,6,11)$, $(5,11,13,10)$, $(6,13,7)$, $(7,11,10)$, $(7,12,14)$
$J_8 \to 3^5, 4^6, 5^2, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,3,7)$, $(3,9,4,10)$, $(4,11,6,5)$, $(5,10,12)$, $(6,13,10)$, $(6,12,14,7)$, $(7,12,9,10)$, $(9,5,11)$, $(8,11,10)$, $(11,7,13)$
$J_8 \to 4^6, 5^5, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,4,11,8)$, $(4,10,13,6,5)$, $(9,10,6,11)$, $(5,10,12,9)$, $(6,12,14,7)$, $(5,11,7,12)$, $(7,13,11,10)$
$J_8 \to 3^7, 4^7, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,7)$, $(0,6,5,4)$, $(1,6,8)$, $(2,9,7,6)$, $(2,8,3,7)$, $(3,9,4,10)$, $(4,11,8)$, $(5,12,10,9)$, $(9,12,6,11)$, $(5,11,10)$, $(6,13,10)$, $(7,12,14)$, $(8,7,10)$, $(11,7,13)$
$J_8 \to 3^2, 4^7, 5^3, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(6,2,9,7,8)$, $(2,8,10,3,7)$, $(3,9,11,8)$, $(4,9,5,11)$, $(4,10,6,5)$, $(5,10,12)$, $(6,12,14,7)$, $(6,11,13)$, $(7,12,9,10)$, $(7,13,10,11)$
$J_8 \to 3^4, 4^8, 5, 7$	$(5,8,9,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(2,9,7,6)$, $(2,8,3,7)$, $(3,9,4,10)$, $(4,11,6,5)$, $(5,10,12)$, $(9,5,11)$, $(6,13,7,8)$, $(6,12,9,10)$, $(7,12,14)$, $(7,11,13,10)$, $(8,11,10)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_8 \to 3, 4^9, 5^2, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 1, 7), (6, 2, 9, 7, 8), (2, 8, 3, 7), (3, 9, 4, 10), (4, 11, 6, 5), (5, 12, 10, 9), (6, 12, 14, 7), (6, 13, 11, 10), (7, 13, 10), (8, 11, 5, 10), (9, 12, 7, 11)
$J_8 \to 3^3, 4^{10}, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 7), (0, 6, 5, 4), (1, 6, 8), (2, 9, 7, 6), (2, 8, 3, 7), (3, 9, 4, 10), (4, 11, 8), (5, 12, 10, 9), (9, 12, 6, 11), (5, 11, 13, 10), (7, 13, 6, 10), (8, 7, 11, 10), (7, 12, 14)
$J_8 \to 4^{11}, 5, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 1, 7), (2, 9, 7, 6), (2, 8, 3, 7), (3, 9, 4, 10), (4, 11, 6, 5), (5, 10, 6, 12), (7, 14, 12, 10), (6, 13, 10, 8), (9, 12, 7, 11), (5, 11, 10, 9), (7, 13, 11, 8)
$J_9 \to 3^{17}, 5, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 4), (6, 2, 8), (0, 6, 7), (2, 7, 13, 11, 9), (6, 13, 10), (1, 7, 8), (1, 6, 5), (8, 13, 15), (4, 10, 9), (5, 12, 9), (5, 11, 10), (6, 12, 11), (7, 3, 9), (3, 8, 10), (4, 11, 8), (12, 8, 14), (7, 12, 10), (7, 14, 11)
$J_9 \to 3^{12}, 5^4, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 2, 9, 7), (6, 1, 7, 2, 8), (3, 8, 12, 6, 7), (8, 13, 15), (3, 9, 10), (4, 9, 11), (4, 10, 5), (5, 12, 9), (5, 11, 6), (11, 7, 13), (6, 13, 10), (8, 14, 11), (8, 7, 10), (7, 12, 14), (10, 11, 12)
$J_9 \to 3^7, 5^7, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 2, 9, 7), (6, 1, 7, 2, 8), (3, 10, 12, 6, 7), (8, 3, 9, 11, 10), (4, 11, 5), (7, 14, 11, 13, 10), (5, 10, 6), (4, 10, 9), (7, 11, 8), (6, 11, 12, 7, 13), (12, 8, 14), (8, 13, 15), (5, 12, 9)
$J_9 \to 3^2, 5^{10}, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 2, 9, 7), (6, 1, 7, 2, 8), (3, 10, 12, 6, 7), (8, 3, 9, 11, 10), (4, 9, 12, 5, 11), (4, 10, 13, 6, 5), (5, 10, 9), (7, 8, 11, 6, 10), (11, 12, 14, 7, 13), (7, 12, 8, 14, 11), (8, 13, 15)
$J_9 \to 4^4, 5^8, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 1, 7), (6, 2, 9, 7, 8), (2, 8, 10, 3, 7), (3, 9, 11, 12, 8), (4, 9, 12, 5, 11), (4, 10, 9, 5), (5, 10, 7, 12, 6), (10, 11, 7, 14, 12), (6, 10, 13, 7), (11, 14, 8, 15, 13), (8, 13, 6, 11)
$J_9 \to 4^9, 5^4, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 1, 7), (6, 2, 9, 7, 8), (2, 8, 10, 3, 7), (3, 9, 11, 12, 8), (4, 9, 5, 11), (4, 10, 6, 5), (5, 10, 9, 12), (6, 13, 10, 7), (7, 13, 8, 14), (8, 15, 13, 11), (10, 11, 6, 12), (7, 12, 14, 11)
$J_9 \to 4^{14}, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 7), (0, 6, 5, 4), (1, 6, 2, 8), (2, 7, 3, 9), (3, 8, 4, 10), (4, 9, 5, 11), (5, 10, 6, 12), (6, 7, 10, 8), (6, 11, 7, 13), (7, 14, 11, 9), (8, 15, 13, 11), (10, 13, 8, 12), (9, 10, 11, 12), (7, 12, 14, 8)
$J_{10} \to 3^{21}, 7$	(3, 4, 8, 5, 9, 7, 6), (0, 7, 4), (0, 6, 5), (2, 9, 6), (2, 8, 7), (5, 1, 7), (1, 6, 8), (3, 9, 8), (3, 10, 7), (9, 14, 16), (12, 8, 14), (7, 14, 11), (4, 9, 11), (5, 11, 12), (6, 11, 13), (7, 13, 12), (10, 6, 12), (8, 13, 15), (8, 11, 10), (9, 13, 10), (4, 10, 5), (9, 15, 12)
$J_{10} \to 3, 5^{12}, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 2, 9, 7), (6, 1, 7, 2, 8), (3, 10, 12, 6, 7), (8, 3, 9, 11, 10), (4, 9, 12, 5, 11), (4, 10, 13, 6, 5), (5, 10, 9), (7, 8, 11, 6, 10), (11, 7, 14, 12, 13), (7, 13, 15, 8, 12), (8, 14, 16, 9, 13), (11, 12, 15, 9, 14)
$J_{11} \to 5^{14}, 7$	(5, 8, 9, 6, 3, 4, 7), (0, 5, 1, 8, 4), (0, 6, 2, 9, 7), (6, 1, 7, 2, 8), (3, 10, 12, 6, 7), (8, 3, 9, 11, 10), (4, 9, 12, 5, 11), (4, 10, 13, 6, 5), (7, 13, 9, 5, 10), (6, 11, 14, 9, 10), (7, 14, 12, 11, 8), (11, 7, 12, 8, 13), (8, 14, 16, 13, 15), (9, 15, 17, 10, 16), (12, 13, 14, 10, 15)
$J_{13} \to 3^{28}, 7$	(3, 4, 8, 5, 9, 7, 6), (0, 6, 5), (0, 7, 4), (5, 1, 7), (1, 6, 8), (2, 8, 7), (2, 9, 6), (3, 9, 8), (3, 10, 7), (4, 10, 9), (9, 14, 13), (4, 11, 5), (5, 10, 12), (6, 12, 11), (6, 13, 10), (7, 13, 12), (12, 8, 14), (8, 13, 11), (8, 15, 10), (12, 17, 19), (10, 17, 11), (16, 12, 18), (9, 16, 11), (13, 16, 15), (14, 10, 16), (7, 14, 11), (9, 15, 12), (11, 18, 15), (15, 14, 17)

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_6 \to 3^8, 5^2, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,6,2,8,5)$, $(1,7,8)$, $(1,6,5)$, $(2,7,9)$, $(0,7,4)$, $(3,10,4)$, $(3,8,6)$, $(4,11,5)$, $(4,9,8)$, $(3,9,5,10,7)$
$J_6 \to 3^3, 5^5, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,8)$, $(2,7,10,5,9)$, $(4,8,2,6,5)$, $(3,10,4)$, $(5,11,4,9,8)$, $(3,8,7)$
$J_6 \to 3^{10}, 4, 8^*$	$(6,7,10,12,5,9,11,8)$, $(0,6,5)$, $(3,8,1,6)$, $(2,9,6)$, $(3,10,4)$, $(4,11,5)$, $(0,7,4)$, $(5,1,7)$, $(5,10,8)$, $(2,8,7)$, $(4,9,8)$, $(7,3,9)$
$J_6 \to 3^5, 4, 5^3, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,8)$, $(2,7,10,5,9)$, $(2,8,5,6)$, $(3,10,4)$, $(4,11,5)$, $(4,9,8)$, $(3,8,7)$
$J_6 \to 4, 5^6, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,8)$, $(3,7,2,9,4)$, $(4,10,3,8,5)$, $(6,5,11,4,8)$, $(7,8,9,5,10)$
$J_6 \to 3^7, 4^2, 5, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,8,4)$, $(4,11,5)$, $(0,6,1,7)$, $(3,10,5,6)$, $(4,10,7)$, $(3,8,7)$, $(2,7,9)$, $(5,9,8)$, $(3,9,4)$, $(6,2,8)$
$J_6 \to 3^2, 4^2, 5^4, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,8)$, $(3,7,2,9,4)$, $(4,8,2,6,5)$, $(3,8,7,10)$, $(4,10,5,11)$, $(5,9,8)$
$J_6 \to 3^4, 4^3, 5^2, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,8)$, $(2,8,3,7)$, $(2,9,5,6)$, $(3,10,4)$, $(4,11,5)$, $(4,9,8)$, $(5,10,7,8)$
$J_6 \to 3^6, 4^4, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7)$, $(0,6,3,4)$, $(1,6,8)$, $(2,9,5,6)$, $(2,8,7)$, $(4,10,5,11)$, $(3,10,7)$, $(4,8,5)$, $(4,9,7)$, $(3,9,8)$
$J_6 \to 3, 4^4, 5^3, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,8)$, $(3,7,2,9,4)$, $(2,8,5,6)$, $(3,8,7,10)$, $(4,10,5,11)$, $(4,8,9,5)$
$J_6 \to 3^3, 4^5, 5, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,7)$, $(1,6,8)$, $(2,9,5,6)$, $(2,8,7)$, $(7,10,3,9)$, $(5,11,4,8)$, $(4,10,5)$, $(3,8,9,4)$
$J_6 \to 4^6, 5^2, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,8)$, $(2,7,8,9)$, $(5,10,3,8)$, $(3,7,10,4)$, $(4,9,5,11)$, $(6,5,4,8)$
$J_6 \to 3^2, 4^7, 8^*$	$(5,12,10,8,11,9,6,7)$, $(0,5,1,7)$, $(0,6,3,4)$, $(1,6,8)$, $(2,9,5,6)$, $(2,8,3,7)$, $(3,9,4,10)$, $(5,11,4,8)$, $(4,5,10,7)$, $(7,8,9)$
$J_7 \to 3^{12}, 5, 8^*$	$(6,13,11,8,7,10,12,9)$, $(2,8,9)$, $(4,11,6,2,7)$, $(7,3,9)$, $(5,1,7)$, $(0,6,7)$, $(0,5,4)$, $(5,12,6)$, $(9,5,11)$, $(1,6,8)$, $(3,8,4)$, $(5,10,8)$, $(3,10,6)$, $(4,10,9)$
$J_7 \to 3^7, 5^4, 8^*$	$(6,13,11,9,12,10,7,8)$, $(0,6,3,7,4)$, $(0,5,7)$, $(1,6,2,8,5)$, $(1,7,2,9,8)$, $(5,12,6,10,9)$, $(7,6,9)$, $(3,9,4)$, $(5,11,6)$, $(4,11,8)$, $(4,10,5)$, $(3,8,10)$
$J_7 \to 3^2, 5^7, 8^*$	$(6,13,11,9,12,10,7,8)$, $(0,6,3,7,4)$, $(0,5,7)$, $(1,6,2,8,5)$, $(1,7,9,10,8)$, $(2,7,6,5,9)$, $(3,8,11,5,10)$, $(3,9,4)$, $(6,11,4,8,9)$, $(4,10,6,12,5)$
$J_7 \to 4^4, 5^5, 8^*$	$(6,13,11,9,12,10,7,8)$, $(0,6,3,7,4)$, $(0,5,8,1,7)$, $(5,1,6,2,7)$, $(2,8,10,3,9)$, $(3,8,11,5,4)$, $(4,10,9,8)$, $(6,11,4,9)$, $(5,10,6,12)$, $(7,6,5,9)$
$J_7 \to 4^9, 5, 8^*$	$(6,13,11,9,12,10,7,8)$, $(0,6,3,7,4)$, $(0,5,1,7)$, $(1,6,2,8)$, $(2,7,6,9)$, $(3,10,5,4)$, $(8,11,4,10)$, $(3,9,4,8)$, $(5,11,6,12)$, $(5,8,9,7)$, $(5,9,10,6)$
$J_8 \to 3^{16}, 8^*$	$(8,9,12,14,7,13,10,11)$, $(3,8,6)$, $(2,8,7)$, $(3,9,4)$, $(0,6,5)$, $(0,7,4)$, $(4,11,5)$, $(1,7,6)$, $(6,11,13)$, $(7,11,9)$, $(1,8,5)$, $(2,9,6)$, $(8,4,10)$, $(5,10,9)$, $(10,6,12)$, $(5,12,7)$, $(3,10,7)$
$J_8 \to 3, 5^9, 8^*$	$(7,14,12,10,13,11,8,9)$, $(0,7,6,3,4)$, $(5,0,6,8,7)$, $(1,6,2,8,5)$, $(1,7,4,10,8)$, $(2,7,10,3,9)$, $(3,8,4,11,7)$, $(4,9,11,6,5)$, $(5,12,6,10,9)$, $(5,11,10)$, $(6,13,7,12,9)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_8 \to 4^{12}, 8^*$	$(7,14,12,10,13,11,8,9)$, $(0,7,3,4)$, $(0,6,8,5)$, $(1,6,2,8)$, $(2,7,6,9)$, $(3,10,5,6)$, $(3,9,4,8)$, $(4,11,9,5)$, $(5,11,10,7)$, $(1,7,12,5)$, $(4,10,8,7)$, $(6,11,7,13)$, $(6,12,9,10)$
$J_9 \to 5^{11}, 8^*$	$(8,15,13,11,14,12,9,10)$, $(0,7,9,3,4)$, $(5,0,6,3,7)$, $(1,7,4,8,5)$, $(1,6,9,2,8)$, $(2,7,10,5,6)$, $(3,8,11,4,10)$, $(4,9,11,12,5)$, $(6,11,5,9,8)$, $(10,6,13,7,12)$, $(6,12,8,14,7)$, $(7,11,10,13,8)$
$J_{11} \to 3^{23}, 8^*$	$(10,17,15,13,16,14,11,12)$, $(0,7,4)$, $(0,6,5)$, $(1,8,5)$, $(3,9,4)$, $(4,10,5)$, $(5,12,7)$, $(1,7,6)$, $(2,8,7)$, $(3,8,6)$, $(6,11,10)$, $(2,9,6)$, $(11,7,13)$, $(9,5,11)$, $(3,10,7)$, $(4,11,8)$, $(9,16,10)$, $(10,14,13)$, $(7,14,9)$, $(12,8,14)$, $(6,12,13)$, $(8,13,9)$, $(9,15,12)$, $(8,15,10)$
$J_7 \to 3^{10}, 5^2, 9^*$	$(6,13,11,9,7,10,12,5,8)$, $(1,6,10,4,5)$, $(0,6,2,7,4)$, $(1,7,8)$, $(0,5,7)$, $(3,7,6)$, $(5,10,9)$, $(6,12,9)$, $(5,11,6)$, $(2,8,9)$, $(3,9,4)$, $(3,8,10)$, $(4,11,8)$
$J_7 \to 3^5, 5^5, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(2,8,10,6,9)$, $(3,10,4)$, $(3,8,11,6,7)$, $(4,11,5)$, $(4,9,8)$, $(5,12,6)$, $(5,10,9)$
$J_7 \to 5^8, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(2,8,10,6,9)$, $(5,10,3,7,6)$, $(4,11,6,12,5)$, $(3,8,9,10,4)$, $(4,9,5,11,8)$
$J_7 \to 3^{12}, 4, 9^*$	$(6,13,11,9,7,10,12,5,8)$, $(5,1,7)$, $(2,7,6)$, $(3,8,7)$, $(0,7,4)$, $(0,6,5)$, $(1,6,11,8)$, $(2,8,9)$, $(3,10,6)$, $(4,11,5)$, $(8,4,10)$, $(3,9,4)$, $(6,12,9)$, $(5,10,9)$
$J_7 \to 3^7, 4, 5^3, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,5,10,8)$, $(5,11,6,12)$, $(4,9,5)$, $(6,10,9)$, $(2,8,9)$, $(3,10,4)$, $(2,7,6)$, $(3,8,7)$, $(4,11,8)$
$J_7 \to 3^2, 4, 5^6, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(2,8,10,6,9)$, $(3,8,11,4,10)$, $(3,7,6,5,4)$, $(4,9,8)$, $(5,11,6,12)$, $(5,10,9)$
$J_7 \to 3^9, 4^2, 5, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,2,7)$, $(4,11,5)$, $(1,6,11,8)$, $(2,8,9)$, $(5,12,6)$, $(3,8,7)$, $(8,4,10)$, $(3,9,4)$, $(7,6,9)$, $(5,10,9)$, $(3,10,6)$
$J_7 \to 3^4, 4^2, 5^4, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(2,8,11,6,9)$, $(3,10,6,7)$, $(3,8,4)$, $(4,10,5,11)$, $(5,12,6)$, $(4,9,5)$, $(8,9,10)$
$J_7 \to 3^6, 4^3, 5^2, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,8)$, $(2,7,6,9)$, $(3,10,4)$, $(3,8,7)$, $(4,11,8)$, $(5,10,6,12)$, $(5,11,6)$, $(8,9,10)$, $(4,9,5)$
$J_7 \to 3, 4^3, 5^5, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(2,8,10,6,9)$, $(3,8,4,10)$, $(3,7,6,5,4)$, $(5,11,6,12)$, $(5,10,9)$, $(4,9,8,11)$
$J_7 \to 3^8, 4^4, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7)$, $(0,6,3,4)$, $(1,6,10,8)$, $(2,7,6)$, $(4,10,5)$, $(3,9,10)$, $(4,9,7)$, $(5,11,6,12)$, $(4,11,8)$, $(5,9,6)$, $(2,8,9)$, $(3,8,7)$
$J_7 \to 3^3, 4^4, 5^3, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(2,8,4,9)$, $(3,8,11,4)$, $(3,10,6,7)$, $(6,11,5,9)$, $(5,12,6)$, $(8,9,10)$, $(4,10,5)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_7 \to 3^5, 4^5, 5, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,7)$, $(1,6,2,8)$, $(2,7,9)$, $(3,10,5,4)$, $(3,9,8)$, $(6,12,5,9)$, $(5,11,6)$, $(8,7,6,10)$, $(4,11,8)$, $(4,10,9)$
$J_7 \to 4^5, 5^4, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,7,8)$, $(3,8,2,9,4)$, $(3,10,6,7)$, $(8,11,4,10)$, $(5,11,6,12)$, $(4,8,9,5)$, $(5,10,9,6)$
$J_7 \to 3^2, 4^6, 5^2, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,9,7)$, $(1,6,2,8)$, $(2,7,6,9)$, $(3,10,5,4)$, $(3,8,7)$, $(4,9,5,11)$, $(5,12,6)$, $(8,11,6,10)$, $(4,10,9,8)$
$J_7 \to 3^4, 4^7, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7)$, $(0,6,3,4)$, $(1,6,2,8)$, $(2,7,9)$, $(3,9,6,7)$, $(3,8,10)$, $(4,10,5)$, $(4,11,8,7)$, $(5,11,6,12)$, $(5,9,10,6)$, $(4,9,8)$
$J_7 \to 3, 4^8, 5, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7,4)$, $(0,6,3,7)$, $(1,6,2,8)$, $(2,7,9)$, $(3,10,5,4)$, $(3,9,4,8)$, $(8,11,4,10)$, $(5,11,6,12)$, $(5,9,10,6)$, $(6,7,8,9)$
$J_7 \to 4^{10}, 9^*$	$(5,8,6,13,11,9,12,10,7)$, $(0,5,1,7)$, $(0,6,3,4)$, $(1,6,2,8)$, $(2,7,3,9)$, $(3,8,4,10)$, $(4,9,5,11)$, $(4,5,6,7)$, $(5,10,6,12)$, $(7,8,10,9)$, $(6,11,8,9)$
$J_8 \to 3^{14}, 5, 9^*$	$(6,13,10,12,14,7,9,11,8)$, $(0,7,4)$, $(3,9,6)$, $(3,8,4)$, $(5,0,6,12,7)$, $(3,10,7)$, $(1,7,8)$, $(1,6,5)$, $(2,8,9)$, $(2,7,6)$, $(5,12,9)$, $(4,10,9)$, $(5,10,8)$, $(4,11,5)$, $(6,11,10)$, $(11,7,13)$
$J_8 \to 3^9, 5^4, 9^*$	$(6,9,7,14,12,10,13,11,8)$, $(0,6,3,7,4)$, $(0,5,7)$, $(1,6,2,8,5)$, $(1,7,2,9,8)$, $(3,8,4)$, $(4,9,11)$, $(5,11,7,12,9)$, $(3,9,10)$, $(5,12,6)$, $(4,10,5)$, $(6,13,7)$, $(6,11,10)$, $(8,7,10)$
$J_8 \to 3^4, 5^7, 9^*$	$(6,9,7,14,12,10,13,11,8)$, $(0,6,3,7,4)$, $(0,5,7)$, $(1,6,2,8,5)$, $(1,7,2,9,8)$, $(3,10,6,5,4)$, $(3,9,11,4,8)$, $(8,7,11,5,10)$, $(4,10,9)$, $(5,12,9)$, $(7,12,6,11,10)$, $(6,13,7)$
$J_8 \to 4^3, 5^7, 9^*$	$(6,9,7,14,12,10,13,11,8)$, $(0,6,3,7,4)$, $(0,5,8,1,7)$, $(5,1,6,2,7)$, $(2,8,10,3,9)$, $(3,8,7,11,4)$, $(4,9,12,6,5)$, $(4,10,9,8)$, $(6,11,5,12,7)$, $(7,13,6,10)$, $(9,5,10,11)$
$J_8 \to 4^8, 5^3, 9^*$	$(6,9,7,14,12,10,13,11,8)$, $(0,6,3,7,4)$, $(0,5,8,1,7)$, $(5,1,6,2,7)$, $(2,8,3,9)$, $(3,10,5,4)$, $(4,9,5,11)$, $(5,12,7,6)$, $(7,13,6,10)$, $(9,12,6,11)$, $(8,7,11,10)$, $(4,10,9,8)$
$J_9 \to 3^{18}, 9^*$	$(8,15,13,7,14,11,9,12,10)$, $(3,9,6)$, $(4,9,8)$, $(3,8,7)$, $(0,7,4)$, $(0,6,5)$, $(6,2,8)$, $(1,8,5)$, $(1,7,6)$, $(12,8,14)$, $(2,7,9)$, $(3,10,4)$, $(5,10,9)$, $(6,13,10)$, $(8,13,11)$, $(4,11,5)$, $(6,12,11)$, $(5,12,7)$, $(7,11,10)$
$J_{11} \to 3, 5^{13}, 9^*$	$(9,12,10,17,15,13,16,14,11)$, $(0,7,9,3,4)$, $(5,0,6,3,7)$, $(1,7,4,8,5)$, $(1,6,8)$, $(4,9,2,6,5)$, $(2,8,10,6,7)$, $(3,8,11,4,10)$, $(5,12,6,13,9)$, $(7,13,11,5,10)$, $(6,11,7,8,9)$, $(7,12,13,8,14)$, $(10,16,9,14,13)$, $(9,15,12,11,10)$, $(12,8,15,10,14)$
$J_{12} \to 3^{25}, 9^*$	$(11,18,16,14,12,15,17,10,13)$, $(5,1,7)$, $(0,6,7)$, $(0,5,4)$, $(1,6,8)$, $(2,9,6)$, $(7,3,9)$, $(3,10,6)$, $(3,8,4)$, $(2,8,7)$, $(4,10,9)$, $(4,11,7)$, $(5,10,8)$, $(5,11,6)$, $(5,12,9)$, $(11,17,14)$, $(9,16,13)$, $(10,16,11)$, $(6,12,13)$, $(7,12,10)$, $(8,12,11)$, $(8,13,15)$, $(7,13,14)$, $(10,15,14)$, $(8,14,9)$, $(9,15,11)$
$J_3 \to 3^2, 5, 10^*$	$(0,5,8,1,7,9,2,6,3,4)$, $(0,6,8,2,7)$, $(1,6,5)$, $(5,4,7)$
$J_3 \to 3, 4^2, 10^*$	$(0,5,1,7,9,2,8,6,3,4)$, $(1,6,5,8)$, $(0,6,2,7)$, $(5,4,7)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_4 \to 3^6, 10^*$	$(5,4,3,10,8,2,9,6,1,7)$, $(0,7,4)$, $(2,7,6)$, $(7,3,9)$, $(0,6,5)$, $(1,8,5)$, $(3,8,6)$
$J_4 \to 3, 5^3, 10^*$	$(3,10,8,2,9,7,4,5,1,6)$, $(0,5,7,3,4)$, $(0,6,8,1,7)$, $(2,7,6)$, $(5,6,9,3,8)$
$J_4 \to 4^2, 5^2, 10^*$	$(3,10,8,2,9,7,4,5,1,6)$, $(0,5,7,3,4)$, $(0,6,2,7)$, $(5,6,9,3,8)$, $(1,7,6,8)$
$J_5 \to 5^5, 10^*$	$(5,6,3,4,11,9,2,8,10,7)$, $(0,5,1,7,4)$, $(0,6,1,8,7)$, $(4,10,3,8,5)$, $(2,7,3,9,6)$, $(6,7,9,4,8)$
$J_5 \to 4^5, 5, 10^*$	$(5,6,3,4,11,9,2,8,10,7)$, $(0,5,1,8,4)$, $(1,7,2,6)$, $(3,9,4,10)$, $(4,5,8,7)$, $(6,7,3,8)$, $(0,6,9,7)$
$J_6 \to 4^8, 10^*$	$(5,12,10,8,11,9,6,3,4,7)$, $(0,5,1,7)$, $(0,6,5,4)$, $(1,6,2,8)$, $(2,7,3,9)$, $(3,8,4,10)$, $(4,9,5,11)$, $(6,7,9,8)$, $(5,10,7,8)$
$J_7 \to 3^{13}, 10^*$	$(4,10,7,8,11,13,6,12,5,9)$, $(2,7,6)$, $(5,1,7)$, $(0,7,4)$, $(7,3,9)$, $(0,6,5)$, $(1,6,8)$, $(2,8,9)$, $(3,8,4)$, $(4,11,5)$, $(6,11,9)$, $(10,9,12)$, $(5,10,8)$, $(3,10,6)$
$J_4 \to 3^4, 5, 11^*$	$(0,6,3,10,8,2,9,7,1,5,4)$, $(0,5,7)$, $(1,6,8)$, $(2,7,6)$, $(5,6,9,3,8)$, $(3,7,4)$
$J_4 \to 3, 4, 5^2, 11^*$	$(0,6,3,10,8,2,9,7,1,5,4)$, $(0,5,7)$, $(1,6,5,8)$, $(3,9,6,7,4)$, $(6,2,7,3,8)$
$J_4 \to 3^3, 4^2, 11^*$	$(0,6,3,10,8,2,9,7,1,5,4)$, $(0,5,7)$, $(1,6,5,8)$, $(6,9,3,8)$, $(2,7,6)$, $(3,7,4)$
$J_4 \to 4^3, 5, 11^*$	$(0,5,1,6,3,10,8,2,9,7,4)$, $(1,7,2,6,8)$, $(3,8,5,4)$, $(3,9,6,7)$, $(0,6,5,7)$
$J_5 \to 3^8, 11^*$	$(3,10,7,2,9,11,4,5,8,1,6)$, $(5,1,7)$, $(0,6,5)$, $(0,7,4)$, $(3,9,4)$, $(8,4,10)$, $(7,6,9)$, $(6,2,8)$, $(3,8,7)$
$J_5 \to 3^3, 5^3, 11^*$	$(5,1,6,3,4,11,9,2,8,10,7)$, $(0,7,1,8,4)$, $(0,6,9,4,5)$, $(4,10,3,8,7)$, $(6,5,8)$, $(2,7,6)$, $(7,3,9)$
$J_5 \to 4, 5^4, 11^*$	$(5,1,6,3,4,11,9,2,8,10,7)$, $(0,7,1,8,4)$, $(0,6,9,4,5)$, $(4,10,3,9,7)$, $(6,2,7,3,8)$, $(5,6,7,8)$
$J_5 \to 4^6, 11^*$	$(5,1,6,3,4,11,9,2,8,10,7)$, $(0,5,4,7)$, $(0,6,8,4)$, $(1,7,3,8)$, $(2,7,9,6)$, $(3,9,4,10)$, $(5,6,7,8)$
$J_6 \to 3^2, 5^5, 11^*$	$(5,12,10,8,11,9,2,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,8,9,7)$, $(1,7,6)$, $(5,11,4,9,6)$, $(4,10,5)$, $(5,9,3,7,8)$, $(2,8,3,10,7)$
$J_7 \to 3, 5^7, 11^*$	$(5,8,10,12,9,11,13,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,8,9,7)$, $(1,7,2,9,6)$, $(2,8,3,7,6)$, $(4,9,3,10,5)$, $(4,10,9,5,11)$, $(5,12,6)$, $(7,8,11,6,10)$
$J_8 \to 3^{15}, 11^*$	$(5,4,11,8,9,10,13,6,12,14,7)$, $(1,8,5)$, $(9,5,11)$, $(3,9,4)$, $(2,9,6)$, $(2,8,7)$, $(8,4,10)$, $(0,7,4)$, $(3,8,6)$, $(0,6,5)$, $(7,12,9)$, $(5,10,12)$, $(1,7,6)$, $(11,7,13)$, $(3,10,7)$, $(6,11,10)$
$J_8 \to 5^9, 11^*$	$(5,4,8,6,9,11,13,10,12,14,7)$, $(0,5,8,1,7)$, $(0,6,3,7,4)$, $(1,6,2,9,5)$, $(7,2,8,10,9)$, $(3,9,8,7,10)$, $(3,8,11,10,4)$, $(4,9,12,6,11)$, $(5,11,7,13,6)$, $(6,10,5,12,7)$
$J_5 \to 3^6, 5, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,9,4,5)$, $(4,10,7)$, $(3,8,4)$, $(6,5,8)$, $(1,7,6)$, $(2,8,7)$, $(7,3,9)$
$J_5 \to 3, 5^4, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(2,8,5,6,7)$, $(3,8,7,10,4)$, $(7,3,9)$
$J_5 \to 3^3, 4, 5^2, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,8,4,5)$, $(1,7,9,6)$, $(3,9,4)$, $(2,8,5,6,7)$, $(3,8,7)$, $(4,10,7)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_5 \to 3^5, 4^2, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,8,5)$, $(1,7,6)$, $(3,8,4)$, $(2,8,7)$, $(4,9,6,5)$, $(4,10,7)$, $(7,3,9)$
$J_5 \to 4^2, 5^3, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(2,8,5,6,7)$, $(7,8,3,9)$, $(3,7,10,4)$
$J_5 \to 3^2, 4^3, 5, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,8,4,5)$, $(1,7,9,6)$, $(2,8,3,7)$, $(3,9,4)$, $(4,10,7)$, $(5,6,7,8)$
$J_5 \to 3, 4^5, 12^*$	$(0,7,5,1,8,10,3,6,2,9,11,4)$, $(0,6,8,5)$, $(1,7,9,6)$, $(2,8,3,7)$, $(3,9,4)$, $(4,5,6,7)$, $(4,10,7,8)$
$J_6 \to 3^{10}, 12^*$	$(5,12,10,3,6,2,9,4,11,8,1,7)$, $(0,5,4)$, $(1,6,5)$, $(9,5,11)$, $(0,6,7)$, $(2,8,7)$, $(5,10,8)$, $(4,10,7)$, $(3,8,4)$, $(6,9,8)$, $(7,3,9)$
$J_6 \to 3^5, 5^3, 12^*$	$(5,12,10,3,4,11,9,2,6,8,1,7)$, $(0,5,1,6,7)$, $(0,6,3,7,4)$, $(7,2,8,3,9)$, $(5,11,8)$, $(5,9,6)$, $(4,9,8)$, $(4,10,5)$, $(8,7,10)$
$J_6 \to 5^6, 12^*$	$(5,12,10,3,4,11,9,2,6,8,1,7)$, $(0,5,1,6,7)$, $(0,6,3,7,4)$, $(7,2,8,5,9)$, $(8,3,9,4,10)$, $(4,8,9,6,5)$, $(7,8,11,5,10)$
$J_6 \to 4^5, 5^2, 12^*$	$(5,12,10,3,4,11,9,2,6,8,1,7)$, $(0,5,1,6,7)$, $(0,6,3,7,4)$, $(2,8,10,7)$, $(5,10,4,8)$, $(7,8,3,9)$, $(4,9,6,5)$, $(5,11,8,9)$
$J_7 \to 4^8, 5, 12^*$	$(5,12,10,8,2,9,11,13,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,1,7)$, $(3,9,4,10)$, $(4,11,8,5)$, $(6,11,5,9)$, $(6,2,7,8)$, $(5,10,7,6)$, $(6,12,9,10)$, $(7,3,8,9)$
$J_8 \to 4^{11}, 12^*$	$(5,4,3,6,8,10,13,11,9,12,14,7)$, $(0,6,1,7)$, $(0,5,8,4)$, $(1,8,11,5)$, $(2,8,3,9)$, $(2,7,9,6)$, $(3,10,6,7)$, $(5,12,10,9)$, $(6,12,7,13)$, $(5,10,11,6)$, $(4,10,7,11)$, $(4,9,8,7)$
$J_9 \to 3^{17}, 12^*$	$(6,12,10,9,4,5,11,14,7,13,15,8)$, $(0,7,4)$, $(0,6,5)$, $(8,13,10)$, $(6,11,13)$, $(3,10,6)$, $(1,7,6)$, $(2,9,6)$, $(7,3,9)$, $(3,8,4)$, $(1,8,5)$, $(2,8,7)$, $(12,8,14)$, $(5,12,9)$, $(7,12,11)$, $(9,8,11)$, $(4,10,11)$, $(5,10,7)$
$J_6 \to 3^8, 5, 13^*$	$(0,7,5,12,10,3,6,1,8,2,9,11,4)$, $(1,7,4,9,5)$, $(0,6,5)$, $(6,9,8)$, $(2,7,6)$, $(7,3,9)$, $(3,8,4)$, $(4,10,5)$, $(5,11,8)$, $(8,7,10)$
$J_6 \to 3^3, 5^4, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,4,7,8,6)$, $(2,8,5,10,7)$, $(7,3,9)$, $(4,10,8,11,5)$, $(4,9,8)$, $(5,9,6)$
$J_6 \to 3^5, 4, 5^2, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,4,7,8,6)$, $(2,8,10,7)$, $(4,10,5)$, $(5,11,8)$, $(4,9,8)$, $(5,9,6)$, $(7,3,9)$
$J_6 \to 4, 5^5, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,4,7,8,6)$, $(7,2,8,5,9)$, $(3,9,4,10,7)$, $(4,8,9,6,5)$, $(8,11,5,10)$
$J_6 \to 3^7, 4^2, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,5)$, $(3,4,7,6)$, $(2,8,7)$, $(7,3,9)$, $(1,7,10,5)$, $(8,4,10)$, $(5,11,8)$, $(4,9,5)$, $(6,9,8)$
$J_6 \to 3^2, 4^2, 5^3, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,4,7,8,6)$, $(2,8,5,10,7)$, $(7,3,9)$, $(4,9,6,5)$, $(5,11,8,9)$, $(8,4,10)$
$J_6 \to 3^4, 4^3, 5, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,9,7,4)$, $(2,8,10,7)$, $(3,7,8,6)$, $(4,10,5)$, $(5,11,8)$, $(4,9,8)$, $(5,9,6)$
$J_6 \to 3, 4^4, 5^2, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,4,7,8,6)$, $(2,8,10,7)$, $(5,10,4,8)$, $(4,9,6,5)$, $(5,11,8,9)$, $(7,3,9)$
$J_6 \to 3^3, 4^5, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,8,5)$, $(1,7,4,5)$, $(2,8,7)$, $(3,7,10,4)$, $(8,11,5,10)$, $(4,9,8)$, $(5,9,6)$, $(3,9,7,6)$
$J_6 \to 4^6, 5, 13^*$	$(0,7,5,12,10,3,8,1,6,2,9,11,4)$, $(0,6,7,1,5)$, $(3,9,7,4)$, $(2,8,10,7)$, $(3,7,8,6)$, $(5,10,4,8)$, $(4,9,6,5)$, $(5,11,8,9)$
$J_7 \to 3^{12}, 13^*$	$(3,4,10,9,12,5,1,7,2,8,11,13,6)$, $(0,5,4)$, $(10,6,12)$, $(3,10,7)$, $(5,10,8)$, $(0,6,7)$, $(1,6,8)$, $(2,9,6)$, $(4,9,11)$, $(5,11,6)$, $(5,9,7)$, $(3,9,8)$, $(4,8,7)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_7 \to 3^7, 5^3, 13^*$	$(3,4,7,1,5,12,10,8,2,9,11,13,6)$, $(0,7,9,5,4)$, $(5,0,6,8,7)$, $(1,6,10,5,8)$, $(3,10,7)$, $(2,7,6)$, $(4,10,9)$, $(5,11,6)$, $(4,11,8)$, $(6,12,9)$, $(3,9,8)$
$J_7 \to 3^2, 5^6, 13^*$	$(3,4,7,1,5,12,10,8,2,9,11,13,6)$, $(0,7,9,5,4)$, $(5,0,6,8,7)$, $(1,6,5,11,8)$, $(2,7,3,10,6)$, $(6,11,4,10,7)$, $(5,10,9,3,8)$, $(6,12,9)$, $(4,9,8)$
$J_7 \to 4^4, 5^4, 13^*$	$(3,4,7,1,5,12,10,8,2,9,11,13,6)$, $(0,7,9,5,4)$, $(5,0,6,8,7)$, $(1,6,5,11,8)$, $(2,7,10,6)$, $(4,9,12,6,11)$, $(3,9,6,7)$, $(4,10,9,8)$, $(5,10,3,8)$
$J_7 \to 4^9, 13^*$	$(3,4,7,1,5,12,10,8,2,9,11,13,6)$, $(0,7,5,4)$, $(0,6,8,5)$, $(1,6,9,8)$, $(2,7,10,6)$, $(7,3,10,9)$, $(4,10,5,11)$, $(5,9,12,6)$, $(3,9,4,8)$, $(6,11,8,7)$
$J_8 \to 3, 5^8, 13^*$	$(5,4,3,6,8,2,9,11,13,10,12,14,7)$, $(0,6,1,7,4)$, $(0,5,1,8,7)$, $(2,7,9,5,6)$, $(3,9,6,10,7)$, $(3,8,9,4,10)$, $(6,13,7)$, $(5,12,6,11,8)$, $(9,10,11,7,12)$, $(8,4,11,5,10)$
$J_9 \to 5^{10}, 13^*$	$(5,9,11,14,12,10,13,15,8,6,3,4,7)$, $(0,6,1,5,4)$, $(0,5,8,1,7)$, $(2,9,12,5,6)$, $(7,2,8,10,9)$, $(3,10,4,8,7)$, $(3,9,6,11,8)$, $(4,9,8,13,11)$, $(7,12,11,5,10)$, $(6,13,7,11,10)$, $(6,12,8,14,7)$
$J_{10} \to 3^{19}, 13^*$	$(5,4,11,6,10,12,15,13,8,14,16,9,7)$, $(0,7,4)$, $(0,6,5)$, $(1,8,5)$, $(3,8,6)$, $(3,10,7)$, $(3,9,4)$, $(1,7,6)$, $(8,15,9)$, $(8,4,10)$, $(2,9,6)$, $(2,8,7)$, $(9,13,10)$, $(5,11,10)$, $(6,12,13)$, $(5,12,9)$, $(11,7,13)$, $(9,14,11)$, $(7,12,14)$, $(8,12,11)$
$J_7 \to 3^{10}, 5, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,8,7,2,6)$, $(4,9,7)$, $(5,11,6)$, $(4,11,8)$, $(4,10,5)$, $(6,12,9)$, $(3,9,10)$, $(5,9,8)$, $(6,10,7)$
$J_7 \to 3^5, 5^4, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(4,10,5,9,7)$, $(4,9,3,8,5)$, $(4,11,8)$, $(5,11,6)$, $(6,12,9,8,7)$, $(6,10,9)$
$J_7 \to 5^7, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,8,4,7)$, $(1,6,9,3,8)$, $(3,10,4,5,6)$, $(5,11,4,9,7)$, $(2,7,8,11,6)$, $(6,10,5,9,8)$, $(6,12,9,10,7)$
$J_7 \to 3^7, 4, 5^2, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(4,10,5,8,7)$, $(3,9,8)$, $(4,9,5)$, $(7,6,12,9)$, $(6,10,9)$, $(4,11,8)$, $(5,11,6)$
$J_7 \to 3^2, 4, 5^5, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(4,10,5,9,7)$, $(5,6,9,3,8)$, $(6,11,4,8,7)$, $(4,9,8,11,5)$, $(6,12,9,10)$
$J_7 \to 3^9, 4^2, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,5,6)$, $(2,7,6)$, $(5,11,8)$, $(4,10,6,11)$, $(6,12,9)$, $(4,8,7)$, $(3,9,8)$, $(7,10,9)$, $(4,9,5)$
$J_7 \to 3^4, 4^2, 5^3, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(4,10,5,9,7)$, $(5,6,9,3,8)$, $(4,11,5)$, $(4,9,8)$, $(6,11,8,7)$, $(6,12,9,10)$
$J_7 \to 3^6, 4^3, 5, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(5,10,4,8)$, $(3,9,8)$, $(4,9,7)$, $(6,12,9,10)$, $(5,9,6)$, $(4,11,5)$, $(6,11,8,7)$
$J_7 \to 3, 4^3, 5^4, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,9,5,8)$, $(3,10,4,8,6)$, $(4,5,6,2,7)$, $(3,9,10,7,8)$, $(4,9,8,11)$, $(5,11,6,10)$, $(7,6,12,9)$
$J_7 \to 3^3, 4^4, 5^2, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(4,10,5,9,7)$, $(3,9,4,8)$, $(4,11,5)$, $(5,6,7,8)$, $(6,11,8,9)$, $(6,12,9,10)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_7 \to 3^5, 4^5, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,5,6)$, $(2,7,9,6)$, $(3,9,4,8)$, $(4,11,5)$, $(4,10,7)$, $(6,12,9,10)$, $(5,9,8)$, $(6,11,8,7)$
$J_7 \to 4^5, 5^3, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,8,4,7)$, $(1,6,9,3,8)$, $(3,10,4,5,6)$, $(4,9,5,11)$, $(5,10,9,7)$, $(2,7,10,6)$, $(6,12,9,8)$, $(6,11,8,7)$
$J_7 \to 3^2, 4^6, 5, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,8)$, $(3,10,7,2,6)$, $(5,10,4,8)$, $(7,8,3,9)$, $(4,9,5,11)$, $(4,5,6,7)$, $(6,11,8,9)$, $(6,12,9,10)$
$J_7 \to 3, 4^8, 14^*$	$(0,6,13,11,9,2,8,10,12,5,1,7,3,4)$, $(0,5,7)$, $(1,6,5,8)$, $(3,8,4,10)$, $(3,9,8,6)$, $(2,7,9,6)$, $(4,9,5,11)$, $(4,5,10,7)$, $(6,11,8,7)$, $(6,12,9,10)$
$J_8 \to 3^{14}, 14^*$	$(1,7,14,12,10,13,6,2,8,3,9,11,4,5)$, $(0,7,4)$, $(0,6,5)$, $(3,7,6)$, $(3,10,4)$, $(6,12,9)$, $(1,6,8)$, $(5,12,7)$, $(2,7,9)$, $(11,7,13)$, $(5,11,8)$, $(4,9,8)$, $(8,7,10)$, $(5,10,9)$, $(6,11,10)$
$J_8 \to 3^9, 5^3, 14^*$	$(5,1,8,6,2,9,3,4,11,13,10,12,14,7)$, $(0,5,6,1,7)$, $(0,6,3,7,4)$, $(2,8,5,12,7)$, $(6,13,7)$, $(6,12,9)$, $(7,11,8)$, $(4,9,8)$, $(3,8,10)$, $(9,5,11)$, $(4,10,5)$, $(6,11,10)$, $(7,10,9)$
$J_8 \to 3^4, 5^6, 14^*$	$(5,1,8,6,2,9,3,4,11,13,10,12,14,7)$, $(0,5,6,1,7)$, $(0,6,3,7,4)$, $(7,2,8,5,9)$, $(3,8,10)$, $(4,10,7,12,5)$, $(4,9,11,7,8)$, $(6,13,7)$, $(8,9,10,6,11)$, $(5,11,10)$, $(6,12,9)$
$J_8 \to 4^3, 5^6, 14^*$	$(5,1,8,6,2,9,3,4,11,13,10,12,14,7)$, $(0,5,6,1,7)$, $(0,6,3,7,4)$, $(7,2,8,5,9)$, $(3,8,9,4,10)$, $(4,8,7,12,5)$, $(6,13,7,11,9)$, $(8,11,5,10)$, $(6,12,9,10)$, $(6,11,10,7)$
$J_8 \to 4^8, 5^2, 14^*$	$(5,1,8,6,2,9,3,4,11,13,10,12,14,7)$, $(0,5,6,1,7)$, $(0,6,3,7,4)$, $(2,8,10,7)$, $(3,8,4,10)$, $(4,9,8,5)$, $(5,10,6,12)$, $(6,11,7,13)$, $(6,7,12,9)$, $(7,8,11,9)$, $(5,11,10,9)$
$J_9 \to 4^{11}, 5, 14^*$	$(3,4,7,2,9,11,14,12,5,10,13,15,8,6)$, $(0,6,1,5,4)$, $(0,5,9,7)$, $(5,8,1,7)$, $(2,8,9,6)$, $(3,9,4,10)$, $(3,8,10,7)$, $(4,11,7,8)$, $(5,11,13,6)$, $(7,13,8,14)$, $(6,11,12,7)$, $(10,11,8,12)$, $(6,12,9,10)$
$J_{10} \to 4^{14}, 14^*$	$(5,4,3,6,8,10,12,15,13,11,14,16,9,7)$, $(0,6,1,7)$, $(0,5,8,4)$, $(1,8,11,5)$, $(2,7,3,9)$, $(2,8,7,6)$, $(3,8,9,10)$, $(4,10,7,11)$, $(4,9,14,7)$, $(9,5,12,11)$, $(5,10,11,6)$, $(6,10,13,9)$, $(6,12,7,13)$, $(9,15,8,12)$, $(12,13,8,14)$
$J_{11} \to 3^{21}, 14^*$	$(5,4,11,14,16,9,15,17,10,8,6,13,12,7)$, $(0,7,4)$, $(0,6,5)$, $(2,9,6)$, $(1,7,6)$, $(1,8,5)$, $(3,10,6)$, $(3,8,4)$, $(7,3,9)$, $(2,8,7)$, $(4,10,9)$, $(5,11,10)$, $(5,12,9)$, $(7,14,10)$, $(6,12,11)$, $(11,7,13)$, $(12,8,14)$, $(9,8,11)$, $(9,14,13)$, $(10,16,13)$, $(10,15,12)$, $(8,13,15)$
$J_{11} \to 3, 5^{12}, 14^*$	$(5,6,8,10,17,15,13,16,14,12,9,11,4,7)$, $(0,5,1,8,4)$, $(0,6,3,9,7)$, $(1,7,2,9,6)$, $(2,8,5,10,6)$, $(3,8,9,4,10)$, $(3,7,11,5,4)$, $(5,12,10,13,9)$, $(7,12,6,13,8)$, $(6,11,14,10,7)$, $(8,14,7,13,11)$, $(10,15,8,12,11)$, $(9,16,10)$, $(9,15,12,13,14)$
$J_8 \to 3^{12}, 5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,2,7)$, $(8,4,10)$, $(6,13,7)$, $(4,11,7)$, $(4,9,5)$, $(1,7,8)$, $(3,10,7)$, $(6,10,9)$, $(3,9,8)$, $(5,11,10)$, $(5,12,6)$, $(7,12,9)$, $(6,11,8)$
$J_8 \to 3^7, 5^4, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,11,5,6)$, $(3,9,8)$, $(4,9,7,12,5)$, $(6,13,7)$, $(3,10,7)$, $(4,11,8)$, $(6,12,9)$, $(5,10,9)$, $(6,11,10)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_8 \to 3^2, 5^7, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,8,4,11,7)$, $(4,9,3,10,5)$, $(5,11,8,9,12)$, $(7,12,6,11,10)$, $(6,13,7)$, $(6,10,9)$
$J_8 \to 3^9, 4, 5^2, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,6)$, $(3,9,8)$, $(6,11,7,13)$, $(4,11,8)$, $(4,9,5)$, $(5,11,10)$, $(3,10,7)$, $(7,12,9)$, $(5,12,6)$, $(6,10,9)$
$J_8 \to 3^4, 4, 5^5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,8,4,11,7)$, $(3,9,10)$, $(4,9,8,11,5)$, $(6,13,7)$, $(7,12,5,10)$, $(6,11,10)$, $(6,12,9)$
$J_8 \to 3^{11}, 4^2, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(0,6,5)$, $(1,7,6,8)$, $(2,7,13,6)$, $(6,12,9)$, $(5,10,9)$, $(3,8,10)$, $(4,11,5)$, $(6,11,10)$, $(4,10,7)$, $(5,12,7)$, $(7,3,9)$, $(4,9,8)$, $(7,11,8)$
$J_8 \to 3^6, 4^2, 5^3, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,8,11,7)$, $(4,11,5)$, $(6,13,7)$, $(6,11,10)$, $(7,12,5,10)$, $(3,9,10)$, $(4,9,8)$, $(6,12,9)$
$J_8 \to 3, 4^2, 5^6, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,8,4,11,7)$, $(3,9,6,7,10)$, $(4,9,8,11,5)$, $(5,10,9,12)$, $(6,12,7,13)$, $(6,11,10)$
$J_8 \to 3^8, 4^3, 5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,8)$, $(2,7,9,6)$, $(3,9,8)$, $(3,10,7)$, $(6,11,7,13)$, $(5,10,6)$, $(6,12,7)$, $(8,11,10)$, $(4,11,5)$, $(4,10,9)$, $(5,12,9)$
$J_8 \to 3^3, 4^3, 5^4, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,8,4,11,7)$, $(3,9,10)$, $(4,9,12,5)$, $(6,12,7,13)$, $(6,10,7)$, $(5,11,10)$, $(6,11,8,9)$
$J_8 \to 3^5, 4^4, 5^2, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,6)$, $(3,9,4,8)$, $(3,10,7)$, $(4,11,5)$, $(5,12,9)$, $(6,11,7,13)$, $(5,10,6)$, $(6,12,7)$, $(8,9,10,11)$
$J_8 \to 4^4, 5^5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,8,4,11,7)$, $(3,9,6,7,10)$, $(4,9,12,5)$, $(5,11,6,10)$, $(6,12,7,13)$, $(8,9,10,11)$
$J_8 \to 3^7, 4^5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(0,6,5)$, $(1,7,6,8)$, $(2,7,9,6)$, $(4,9,3,7)$, $(3,8,10)$, $(4,11,5)$, $(6,12,7,13)$, $(5,12,9)$, $(5,10,7)$, $(4,10,9,8)$, $(6,11,10)$, $(7,11,8)$
$J_8 \to 3^2, 4^5, 5^3, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,5,6)$, $(3,9,4,8)$, $(3,10,6,7)$, $(4,11,5)$, $(5,10,9,12)$, $(6,12,7,13)$, $(6,11,8,9)$, $(7,11,10)$
$J_8 \to 3^4, 4^6, 5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,8)$, $(2,7,9,6)$, $(3,9,4,10)$, $(3,8,10,7)$, $(4,11,5)$, $(5,12,9)$, $(6,12,7,11)$, $(6,13,7)$, $(5,10,6)$, $(8,9,10,11)$
$J_8 \to 3, 4^7, 5^2, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,10,8)$, $(2,7,9,6)$, $(3,9,4,8)$, $(3,10,6,7)$, $(4,11,5)$, $(7,12,5,10)$, $(5,9,12,6)$, $(6,11,7,13)$, $(8,9,10,11)$
$J_8 \to 3^3, 4^8, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(0,6,5)$, $(1,7,6,8)$, $(2,7,9,6)$, $(4,9,3,7)$, $(3,8,10)$, $(4,10,5,11)$, $(5,4,8,7)$, $(5,12,9)$, $(7,12,6,10)$, $(6,11,7,13)$, $(8,9,10,11)$
$J_8 \to 4^9, 5, 15^*$	$(0,7,14,12,10,13,11,9,2,8,5,1,6,3,4)$, $(5,0,6,8,7)$, $(1,7,4,8)$, $(2,7,9,6)$, $(3,9,4,10)$, $(3,8,10,7)$, $(4,11,6,5)$, $(5,10,9,12)$, $(6,12,7,13)$, $(6,10,11,7)$, $(5,11,8,9)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_9 \to 3^{16}, 15^*$	$(5,1,8,15,13,6,3,4,10,12,14,11,9,2,7)$, $(0,7,4)$, $(0,6,5)$, $(1,7,6)$, $(7,14,8)$, $(11,7,13)$, $(6,2,8)$, $(3,9,8)$, $(3,10,7)$, $(8,13,10)$, $(5,12,8)$, $(4,11,8)$, $(5,11,10)$, $(6,12,11)$, $(4,9,5)$, $(6,10,9)$, $(7,12,9)$
$J_9 \to 3^{11}, 5^3, 15^*$	$(5,1,8,15,13,10,12,14,11,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,9,5,4)$, $(7,14,8,3,9)$, $(5,10,8)$, $(3,10,7)$, $(6,11,10)$, $(4,10,9)$, $(6,12,7)$, $(2,8,7)$, $(11,7,13)$, $(4,11,8)$, $(5,11,12)$, $(6,13,8)$, $(8,12,9)$
$J_9 \to 3^6, 5^6, 15^*$	$(5,1,8,15,13,10,12,14,11,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,10,9)$, $(3,10,4,11,7)$, $(4,9,5,12,8)$, $(3,9,8)$, $(7,12,11,5,10)$, $(6,12,9)$, $(6,13,7)$, $(7,14,8)$, $(8,13,11)$, $(6,11,10)$
$J_9 \to 3, 5^9, 15^*$	$(5,1,8,15,13,10,12,14,11,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,10,9)$, $(3,10,4,8,7)$, $(3,9,6,11,8)$, $(4,9,5,10,11)$, $(5,11,13,6,12)$, $(6,10,7)$, $(7,13,8,12,11)$, $(8,14,7,12,9)$
$J_9 \to 4^2, 5^8, 15^*$	$(5,1,8,15,13,10,12,14,11,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,10,9)$, $(3,10,4,8,7)$, $(3,9,6,11,8)$, $(4,9,5,10,11)$, $(6,12,5,11,7)$, $(7,13,6,10)$, $(8,14,7,12,9)$, $(11,12,8,13)$
$J_9 \to 4^7, 5^4, 15^*$	$(5,1,8,15,13,10,12,14,11,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,10,9)$, $(3,10,4,8,7)$, $(3,9,12,8)$, $(4,9,5,11)$, $(5,10,6,12)$, $(6,13,8,9)$, $(6,11,10,7)$, $(8,14,7,11)$, $(11,12,7,13)$
$J_9 \to 4^{12}, 15^*$	$(5,1,8,15,13,10,12,14,11,9,2,6,3,4,7)$, $(0,6,1,7)$, $(0,5,8,4)$, $(3,10,8,7)$, $(4,9,5,11)$, $(4,10,6,5)$, $(5,10,9,12)$, $(6,9,3,8)$, $(6,12,7,13)$, $(7,2,8,9)$, $(8,14,7,11)$, $(6,11,10,7)$, $(11,12,8,13)$
$J_{10} \to 5^{11}, 15^*$	$(5,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,5,1,8,4)$, $(0,6,8,9,7)$, $(1,7,3,9,6)$, $(2,8,3,10,7)$, $(4,10,5,9,11)$, $(4,9,12,6,5)$, $(5,11,8,14,12)$, $(6,13,7,11,10)$, $(6,11,12,8,7)$, $(7,12,13,9,14)$, $(9,15,8,13,10)$
$J_{12} \to 3^{23}, 15^*$	$(5,8,10,6,13,12,15,17,14,9,16,18,11,4,7)$, $(0,5,4)$, $(6,2,8)$, $(0,6,7)$, $(2,7,9)$, $(1,7,8)$, $(1,6,5)$, $(3,8,4)$, $(3,9,6)$, $(3,10,7)$, $(8,14,13)$, $(4,10,9)$, $(9,5,11)$, $(5,10,12)$, $(10,16,13)$, $(11,7,13)$, $(13,9,15)$, $(8,12,9)$, $(10,15,14)$, $(8,15,11)$, $(10,17,11)$, $(6,12,11)$, $(11,16,14)$, $(7,12,14)$
$J_9 \to 3^{14}, 5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,5)$, $(6,1,7,14,8)$, $(3,9,8)$, $(8,4,10)$, $(2,8,7)$, $(3,10,7)$, $(6,13,7)$, $(4,11,7)$, $(8,13,11)$, $(4,9,5)$, $(7,12,9)$, $(5,12,8)$, $(6,12,11)$, $(6,10,9)$, $(5,11,10)$
$J_9 \to 3^9, 5^4, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(2,8,5,10,7)$, $(7,14,8)$, $(3,8,13,6,10)$, $(5,11,6)$, $(6,12,7)$, $(7,3,9)$, $(11,7,13)$, $(4,10,11)$, $(8,9,10)$, $(5,12,9)$, $(8,12,11)$
$J_9 \to 3^4, 5^7, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(3,8,10,6,7)$, $(5,11,7,12,6)$, $(4,10,5,12,11)$, $(7,14,8,11,10)$, $(3,9,10)$, $(7,13,8)$, $(8,12,9)$, $(6,11,13)$
$J_9 \to 3^{11}, 4, 5^2, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(2,8,7)$, $(6,11,13)$, $(7,13,8,14)$, $(7,3,9)$, $(5,12,6)$, $(4,10,11)$, $(6,10,7)$, $(5,11,8)$, $(3,8,10)$, $(5,10,9)$, $(8,12,9)$, $(7,12,11)$
$J_9 \to 3^6, 4, 5^5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(3,8,12,6,7)$, $(5,11,6)$, $(4,10,11)$, $(3,9,12,5,10)$, $(7,14,8)$, $(7,12,11)$, $(7,13,6,10)$, $(8,9,10)$, $(8,13,11)$
$J_9 \to 3, 4, 5^8, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(5,10,3,7,6)$, $(3,9,8)$, $(4,10,7,8,11)$, $(5,11,13,6,12)$, $(8,12,11,6,10)$, $(7,13,8,14)$, $(9,10,11,7,12)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_9 \to 3^{13}, 4^2, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,5)$, $(1,7,6)$, $(2,8,14,7)$, $(7,12,9)$, $(5,12,6)$, $(6,11,13)$, $(6,10,9)$, $(4,9,5)$, $(3,9,8)$, $(3,10,7)$, $(4,11,7)$, $(7,13,8)$, $(8,4,10)$, $(5,11,10)$, $(8,12,11)$
$J_9 \to 3^8, 4^2, 5^3, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(3,8,10,7)$, $(4,10,5,11)$, $(5,12,6)$, $(7,14,8)$, $(7,12,11)$, $(8,13,11)$, $(6,13,7)$, $(6,11,10)$, $(8,12,9)$, $(3,9,10)$
$J_9 \to 3^3, 4^2, 5^6, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(5,10,3,7,6)$, $(3,9,8)$, $(4,10,7,8,11)$, $(5,11,13,6,12)$, $(6,11,10)$, $(7,13,8,14)$, $(7,12,11)$, $(8,12,9,10)$
$J_9 \to 3^{10}, 4^3, 5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,9,6)$, $(2,8,3,7)$, $(3,9,10)$, $(7,14,8)$, $(4,9,5,11)$, $(4,10,7)$, $(5,10,8)$, $(5,12,6)$, $(6,13,7)$, $(8,12,9)$, $(6,11,10)$, $(7,12,11)$, $(8,13,11)$
$J_9 \to 3^5, 4^3, 5^4, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(5,10,3,7,6)$, $(3,9,8)$, $(8,11,4,10)$, $(5,11,12)$, $(7,13,8,14)$, $(7,11,10)$, $(7,12,8)$, $(6,12,9,10)$, $(6,11,13)$
$J_9 \to 4^3, 5^7, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(5,10,3,7,6)$, $(3,9,12,7,8)$, $(4,10,9,8,11)$, $(5,11,13,6,12)$, $(7,11,6,10)$, $(7,13,8,14)$, $(8,12,11,10)$
$J_9 \to 3^7, 4^4, 5^2, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(2,8,10,7)$, $(5,10,3,8)$, $(4,10,6,11)$, $(5,12,6)$, $(5,11,10,9)$, $(7,3,9)$, $(7,14,8)$, $(6,13,7)$, $(8,12,9)$, $(7,12,11)$, $(8,13,11)$
$J_9 \to 3^2, 4^4, 5^5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(5,10,3,7,6)$, $(3,9,8)$, $(4,10,7,8,11)$, $(5,11,6,12)$, $(6,13,11,10)$, $(7,13,8,14)$, $(7,12,11)$, $(8,12,9,10)$
$J_9 \to 3^9, 4^5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,5)$, $(1,7,9,6)$, $(2,8,10,7)$, $(4,10,3,7)$, $(3,9,4,8)$, $(4,11,5)$, $(5,12,6)$, $(5,10,9)$, $(6,13,7)$, $(6,11,10)$, $(7,14,8)$, $(8,12,9)$, $(7,12,11)$, $(8,13,11)$
$J_9 \to 3^4, 4^5, 5^3, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(3,10,6,7)$, $(3,9,8)$, $(4,10,5,11)$, $(5,12,6)$, $(6,11,13)$, $(8,13,7,10)$, $(8,14,7,11)$, $(7,12,8)$, $(9,10,11,12)$
$J_9 \to 3^6, 4^6, 5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,9,6)$, $(2,8,10,7)$, $(4,10,3,7)$, $(5,9,3,8)$, $(4,9,10,11)$, $(5,11,6,10)$, $(5,12,6)$, $(7,14,8)$, $(6,13,7)$, $(8,12,9)$, $(7,12,11)$, $(8,13,11)$
$J_9 \to 3, 4^6, 5^4, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(5,10,3,7,6)$, $(3,9,8)$, $(8,11,4,10)$, $(5,11,6,12)$, $(6,13,11,10)$, $(7,13,8,14)$, $(7,12,9,10)$, $(7,11,12,8)$
$J_9 \to 3^3, 4^7, 5^2, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(2,8,10,7)$, $(5,10,3,8)$, $(4,10,6,11)$, $(5,12,6)$, $(5,11,8,9)$, $(6,13,11,7)$, $(7,3,9)$, $(7,13,8,14)$, $(7,12,8)$, $(9,10,11,12)$
$J_9 \to 3^5, 4^8, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,5)$, $(1,7,9,6)$, $(2,8,10,7)$, $(4,10,3,7)$, $(3,9,4,8)$, $(4,11,5)$, $(5,12,6)$, $(5,10,9)$, $(6,13,8,7)$, $(8,14,7,11)$, $(8,12,9)$, $(11,12,7,13)$, $(6,11,10)$
$J_9 \to 4^8, 5^3, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,4,9,6)$, $(7,2,8,5,9)$, $(3,10,6,7)$, $(3,9,12,8)$, $(4,10,5,11)$, $(5,12,11,6)$, $(6,12,7,13)$, $(8,14,7,10)$, $(7,11,13,8)$, $(8,9,10,11)$
$J_9 \to 3^2, 4^9, 5, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,4,5)$, $(1,7,9,6)$, $(2,8,10,7)$, $(4,10,3,7)$, $(5,9,3,8)$, $(4,9,8,11)$, $(5,10,6,12)$, $(5,11,7,6)$, $(6,11,13)$, $(7,13,8,14)$, $(7,12,8)$, $(9,10,11,12)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_9 \to 3, 4^{11}, 16^*$	$(0,7,5,1,8,15,13,10,12,14,11,9,2,6,3,4)$, $(0,6,8,5)$, $(1,7,9,6)$, $(2,8,10,7)$, $(4,10,3,7)$, $(3,9,4,8)$, $(4,11,5)$, $(5,12,8,9)$, $(6,13,11,7)$, $(5,10,11,6)$, $(7,13,8,14)$, $(7,12,11,8)$, $(6,12,9,10)$
$J_{10} \to 3^{18}, 16^*$	$(3,4,7,1,5,8,15,13,11,10,12,14,16,9,2,6)$, $(5,10,6)$, $(0,5,4)$, $(1,6,8)$, $(0,6,7)$, $(2,8,7)$, $(7,13,10)$, $(7,3,9)$, $(6,13,9)$, $(8,14,9)$, $(7,14,11)$, $(5,12,7)$, $(3,8,10)$, $(8,13,12)$, $(4,10,9)$, $(9,5,11)$, $(9,15,12)$, $(4,11,8)$, $(6,12,11)$
$J_{10} \to 3^{13}, 5^3, 16^*$	$(5,1,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,7,2,8,4)$, $(4,10,5,12,11)$, $(0,6,5)$, $(4,9,13,8,5)$, $(3,8,7)$, $(1,7,6)$, $(8,15,9)$, $(7,14,9)$, $(3,9,10)$, $(12,8,14)$, $(6,11,8)$, $(6,12,9)$, $(7,13,12)$, $(9,5,11)$, $(6,13,10)$, $(7,11,10)$
$J_{10} \to 3^8, 5^6, 16^*$	$(5,1,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,3,9)$, $(3,10,4,8,7)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,10)$, $(6,13,9)$, $(6,11,8,14,7)$, $(7,13,10)$, $(8,15,9)$, $(9,14,12)$, $(8,13,12)$, $(7,12,11)$
$J_{10} \to 3^3, 5^9, 16^*$	$(5,1,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,3,9)$, $(3,10,4,8,7)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(6,13,7,12,9)$, $(6,11,7)$, $(7,14,12,11,10)$, $(8,14,9,13,12)$, $(8,15,9)$
$J_{10} \to 4, 5^{10}, 16^*$	$(5,1,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,3,9)$, $(3,10,4,8,7)$, $(4,9,12,5,11)$, $(6,13,7,11,9)$, $(5,10,13,8,9)$, $(6,11,10,7)$, $(6,12,14,9,10)$, $(8,14,7,12,11)$, $(8,15,9,13,12)$
$J_{10} \to 4^6, 5^6, 16^*$	$(5,1,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(7,2,8,3,9)$, $(3,10,4,8,7)$, $(4,9,12,5,11)$, $(6,13,7,10,9)$, $(6,12,13,10)$, $(9,5,10,11)$, $(8,13,9,15)$, $(8,12,14,9)$, $(8,14,7,11)$, $(6,11,12,7)$
$J_{10} \to 4^{11}, 5^2, 16^*$	$(5,1,8,10,12,15,13,11,14,16,9,2,6,3,4,7)$, $(0,5,6,1,7)$, $(0,6,8,5,4)$, $(2,8,3,7)$, $(3,9,4,10)$, $(4,11,7,8)$, $(6,12,5,9)$, $(7,13,6,10)$, $(9,10,5,11)$, $(7,14,8,9)$, $(6,11,12,7)$, $(9,15,8,12)$, $(8,13,10,11)$, $(12,13,9,14)$
$J_{11} \to 3^2, 5^{11}, 16^*$	$(5,4,11,13,16,14,12,15,17,10,3,6,8,2,9,7)$, $(0,6,1,7,4)$, $(0,5,1,8,7)$, $(2,7,3,9,6)$, $(3,8,4)$, $(5,6,7,10,8)$, $(9,4,10,12,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(6,11,7,12,13)$, $(7,13,15,8,14)$, $(8,12,9)$, $(9,15,10,14,13)$, $(10,16,9,14,11)$
$J_{11} \to 4^{14}, 5, 16^*$	$(5,4,11,13,16,14,12,15,17,10,3,6,8,2,9,7)$, $(0,6,1,7,4)$, $(0,5,8,7)$, $(1,8,9,5)$, $(2,7,10,6)$, $(3,9,6,7)$, $(3,8,10,4)$, $(4,9,11,8)$, $(5,11,6,12)$, $(5,10,13,6)$, $(7,12,8,14)$, $(7,13,14,11)$, $(8,13,9,15)$, $(9,14,10,16)$, $(9,10,11,12)$, $(10,15,13,12)$
$J_{12} \to 3, 5^{13}, 16^*$	$(5,8,6,3,4,11,18,16,14,17,15,13,10,12,9,7)$, $(0,6,1,7,4)$, $(0,5,1,8,7)$, $(2,7,3,9,6)$, $(2,8,10,11,9)$, $(3,8,9,4,10)$, $(4,8,11,6,5)$, $(5,12,6,10,9)$, $(7,13,11,5,10)$, $(6,13,8,14,7)$, $(7,12,13,14,11)$, $(12,11,16,9,14)$, $(8,15,12)$, $(10,17,11,15,14)$, $(9,15,10,16,13)$
$J_{12} \to 4^{17}, 16^*$	$(5,8,6,3,4,11,18,16,14,17,15,13,10,12,9,7)$, $(0,6,1,7)$, $(0,5,9,4)$, $(1,8,10,5)$, $(2,8,3,9)$, $(2,7,10,6)$, $(4,10,3,7)$, $(4,8,12,5)$, $(5,11,9,6)$, $(6,13,8,7)$, $(6,12,14,11)$, $(8,14,7,11)$, $(11,12,7,13)$, $(8,15,10,9)$, $(9,14,13,16)$, $(10,16,11,17)$, $(10,14,15,11)$, $(9,15,12,13)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_{13} \to 3^{25}, 16^*$	$(5,12,19,17,14,8,13,9,16,18,15,10,6,11,4,7)$, $(0,5,4)$, $(1,6,5)$, $(0,6,7)$, $(3,10,7)$, $(3,9,6)$, $(3,8,4)$, $(6,2,8)$, $(2,7,9)$, $(1,7,8)$, $(9,5,11)$, $(5,10,8)$, $(4,10,9)$, $(11,18,12)$, $(6,12,13)$, $(7,12,14)$, $(12,17,15)$, $(10,17,11)$, $(11,7,13)$, $(10,14,13)$, $(8,15,11)$, $(8,12,9)$, $(9,15,14)$, $(13,16,15)$, $(11,16,14)$, $(10,16,12)$
$J_{13} \to 5^{15}, 16^*$	$(5,12,19,17,15,18,16,14,10,13,11,9,6,8,4,7)$, $(0,5,1,6,7)$, $(0,6,3,9,4)$, $(1,7,9,2,8)$, $(2,7,3,10,6)$, $(3,8,10,5,4)$, $(4,10,9,5,11)$, $(5,6,13,7,8)$, $(7,14,12,11,10)$, $(8,9,12,6,11)$, $(7,12,8,14,11)$, $(8,13,16,9,15)$, $(9,14,15,12,13)$, $(10,15,16,12,17)$, $(10,16,11,18,12)$, $(13,14,17,11,15)$
$J_{10} \to 3^{16}, 5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,8,2,7)$, $(1,7,6)$, $(7,14,9)$, $(8,15,9)$, $(12,8,14)$, $(3,9,10)$, $(3,8,6)$, $(4,9,5)$, $(9,13,12)$, $(6,11,9)$, $(6,13,10)$, $(5,11,10)$, $(4,10,7)$, $(7,13,8)$, $(4,11,8)$, $(7,12,11)$, $(5,12,6)$
$J_{10} \to 3^{11}, 5^4, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,4,11,8)$, $(5,12,9)$, $(6,11,9)$, $(8,15,9)$, $(9,13,10)$, $(6,10,7)$, $(7,13,8)$, $(12,8,14)$, $(7,14,9)$, $(5,11,10)$, $(6,12,13)$, $(7,12,11)$
$J_{10} \to 3^6, 5^7, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(7,13,6,11,10)$, $(7,14,9)$, $(7,12,11)$, $(8,15,9)$, $(9,13,12)$, $(12,8,14)$
$J_{10} \to 3, 5^{10}, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(7,13,6,11,10)$, $(7,11,12,8,9)$, $(7,12,13,9,14)$, $(9,15,8,14,12)$
$J_{10} \to 3^{13}, 4, 5^2, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,9,6)$, $(2,8,7)$, $(3,9,15,8)$, $(4,8,5)$, $(6,13,8)$, $(6,10,7)$, $(4,11,7)$, $(9,13,10)$, $(5,11,10)$, $(7,13,12)$, $(6,12,11)$, $(9,8,11)$, $(12,8,14)$, $(5,12,9)$, $(7,14,9)$
$J_{10} \to 3^8, 4, 5^5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,10)$, $(8,13,6,11)$, $(8,15,9)$, $(7,13,10)$, $(9,13,12)$, $(7,14,9)$, $(7,12,11)$, $(12,8,14)$
$J_{10} \to 3^3, 4, 5^8, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(7,13,6,11,10)$, $(7,11,12,8,9)$, $(7,12,14)$, $(8,14,9,15)$, $(9,13,12)$
$J_{10} \to 3^{15}, 4^2, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,4,7)$, $(1,7,6)$, $(5,11,10)$, $(2,8,7)$, $(5,9,15,8)$, $(3,9,8)$, $(3,10,6)$, $(4,10,9)$, $(7,13,10)$, $(4,11,8)$, $(5,12,6)$, $(7,12,11)$, $(6,11,9)$, $(9,13,12)$, $(6,13,8)$, $(12,8,14)$, $(7,14,9)$
$J_{10} \to 3^{10}, 4^2, 5^3, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(7,8,3,9)$, $(4,9,11)$, $(5,12,9)$, $(5,11,10)$, $(6,11,7)$, $(8,14,9,15)$, $(7,12,14)$, $(8,12,11)$, $(8,13,9)$, $(6,10,9)$, $(7,13,10)$, $(6,12,13)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_{10} \to 3^5, 4^2, 5^6, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(6,11,7,13)$, $(7,14,9)$, $(7,12,11,10)$, $(8,15,9)$, $(12,8,14)$, $(9,13,12)$
$J_{10} \to 4^2, 5^9, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,12,5,11)$, $(7,13,6,10,9)$, $(9,14,12,6,11)$, $(5,10,13,8,9)$, $(8,14,7,11)$, $(7,12,11,10)$, $(8,15,9,13,12)$
$J_{10} \to 3^{12}, 4^3, 5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,8,4,10)$, $(3,9,6)$, $(7,2,8,9)$, $(9,15,8,11)$, $(4,11,7)$, $(6,10,7)$, $(4,9,5)$, $(9,13,10)$, $(5,11,10)$, $(6,12,11)$, $(7,13,12)$, $(6,13,8)$, $(5,12,8)$, $(9,14,12)$, $(7,14,8)$
$J_{10} \to 3^7, 4^3, 5^4, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,10,6,12)$, $(5,11,10,9)$, $(7,13,10)$, $(8,13,6,11)$, $(8,15,9)$, $(7,14,9)$, $(12,8,14)$, $(9,13,12)$, $(7,12,11)$
$J_{10} \to 3^2, 4^3, 5^7, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(7,13,6,11,10)$, $(7,14,8,9)$, $(7,12,11)$, $(9,15,8,12)$, $(12,13,9,14)$
$J_{10} \to 3^9, 4^4, 5^2, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(2,8,7)$, $(4,11,8,5)$, $(3,9,8)$, $(4,9,7)$, $(6,12,5,9)$, $(7,13,6,10)$, $(5,11,10)$, $(6,11,7)$, $(8,14,9,15)$, $(7,12,14)$, $(9,12,11)$, $(9,13,10)$, $(8,13,12)$
$J_{10} \to 3^4, 4^4, 5^5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(7,11,5,10)$, $(6,11,10,13)$, $(7,13,8,9)$, $(7,12,14)$, $(8,14,9,15)$, $(8,12,11)$, $(9,13,12)$
$J_{10} \to 3^{11}, 4^5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,4,7)$, $(1,7,9,6)$, $(6,7,2,8)$, $(3,8,4,10)$, $(3,9,13,6)$, $(5,10,9)$, $(5,12,6)$, $(5,11,8)$, $(6,11,10)$, $(4,9,11)$, $(7,12,11)$, $(8,15,9)$, $(9,14,12)$, $(7,14,8)$, $(8,13,12)$, $(7,13,10)$
$J_{10} \to 3^6, 4^5, 5^3, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(7,8,3,9)$, $(4,9,11)$, $(6,12,5,9)$, $(5,11,6,10)$, $(6,13,7)$, $(7,14,9,10)$, $(8,13,10,11)$, $(8,15,9)$, $(9,13,12)$, $(7,12,11)$, $(12,8,14)$
$J_{10} \to 3, 4^5, 5^6, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,12,6,10,9)$, $(5,11,8,13,10)$, $(6,11,7,13)$, $(7,14,8,9)$, $(7,12,11,10)$, $(9,15,8,12)$, $(12,13,9,14)$
$J_{10} \to 3^8, 4^6, 5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,8,4,10)$, $(3,9,8,6)$, $(2,8,7)$, $(4,11,8,5)$, $(4,9,7)$, $(6,12,5,9)$, $(7,13,6,10)$, $(5,11,10)$, $(6,11,7)$, $(8,14,9,15)$, $(7,12,14)$, $(9,12,11)$, $(9,13,10)$, $(8,13,12)$
$J_{10} \to 3^3, 4^6, 5^4, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,11)$, $(5,10,6,12)$, $(5,11,8,9)$, $(6,11,7,13)$, $(7,14,9,10)$, $(7,12,9)$, $(8,13,9,15)$, $(10,11,12,13)$, $(12,8,14)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_{10} \to 3^5, 4^7, 5^2, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(2,8,7)$, $(4,11,8,5)$, $(3,9,8)$, $(4,9,7)$, $(6,12,5,9)$, $(7,13,6,10)$, $(9,10,5,11)$, $(6,11,7)$, $(7,12,14)$, $(9,14,8,12)$, $(8,13,9,15)$, $(10,11,12,13)$
$J_{10} \to 4^7, 5^5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(3,9,6,7,8)$, $(4,9,12,5,11)$, $(7,13,6,10)$, $(9,5,10,11)$, $(8,14,9,15)$, $(8,13,10,9)$, $(7,12,13,9)$, $(8,12,6,11)$, $(7,14,12,11)$
$J_{10} \to 3^7, 4^8, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,4,7)$, $(1,7,9,6)$, $(6,7,2,8)$, $(3,8,4,10)$, $(3,9,5,6)$, $(4,9,11)$, $(5,10,6,12)$, $(5,11,8)$, $(9,15,8,12)$, $(6,11,12,13)$, $(7,11,10)$, $(9,13,10)$, $(7,13,8)$, $(8,14,9)$, $(7,12,14)$
$J_{10} \to 3^2, 4^8, 5^3, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(4,5,8,2,7)$, $(7,8,3,9)$, $(4,9,11)$, $(6,12,5,9)$, $(5,11,6,10)$, $(6,13,7)$, $(7,14,9,10)$, $(8,12,7,11)$, $(8,13,9,15)$, $(8,14,12,9)$, $(10,11,12,13)$
$J_{10} \to 3^4, 4^9, 5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,8,4,10)$, $(3,9,8,6)$, $(2,8,7)$, $(4,11,8,5)$, $(4,9,7)$, $(6,12,5,9)$, $(7,13,6,10)$, $(9,10,5,11)$, $(6,11,7)$, $(7,12,14)$, $(9,14,8,12)$, $(8,13,9,15)$, $(10,11,12,13)$
$J_{10} \to 3, 4^{10}, 5^2, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,10,4,8,6)$, $(2,8,7)$, $(4,11,8,5)$, $(3,9,12,8)$, $(4,9,6,7)$, $(5,10,6,12)$, $(6,11,7,13)$, $(7,12,13,10)$, $(5,11,10,9)$, $(8,13,9,15)$, $(7,14,8,9)$, $(9,14,12,11)$
$J_{10} \to 3^3, 4^{11}, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,4,7)$, $(1,7,9,6)$, $(6,7,2,8)$, $(3,8,4,10)$, $(3,9,5,6)$, $(4,9,11)$, $(5,10,6,12)$, $(5,11,7,8)$, $(6,11,10,13)$, $(7,13,8,14)$, $(7,12,9,10)$, $(8,15,9)$, $(8,12,11)$, $(12,13,9,14)$
$J_{10} \to 4^{12}, 5, 17^*$	$(0,6,2,9,16,14,11,13,15,12,10,8,1,5,7,3,4)$, $(0,5,6,1,7)$, $(3,8,4,10)$, $(3,9,8,6)$, $(2,8,11,7)$, $(4,5,8,7)$, $(4,9,5,11)$, $(5,10,6,12)$, $(7,13,6,9)$, $(6,11,10,7)$, $(7,12,8,14)$, $(8,13,9,15)$, $(9,14,12,11)$, $(9,10,13,12)$
$J_{11} \to 3^{20}, 17^*$	$(3,10,17,15,13,16,14,12,5,11,4,9,2,7,8,1,6)$, $(0,7,4)$, $(5,1,7)$, $(0,6,5)$, $(6,2,8)$, $(7,3,9)$, $(7,12,11)$, $(3,8,4)$, $(10,6,12)$, $(7,14,10)$, $(8,15,10)$, $(5,9,8)$, $(4,10,5)$, $(9,14,13)$, $(8,14,11)$, $(9,15,12)$, $(9,16,10)$, $(6,11,9)$, $(6,13,7)$, $(8,13,12)$, $(11,10,13)$
$J_{11} \to 3^{15}, 5^3, 17^*$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7,4)$, $(0,5,4,3,7)$, $(2,8,15,9,6)$, $(3,9,8)$, $(9,16,10)$, $(5,12,9)$, $(8,4,10)$, $(5,10,6)$, $(5,11,8)$, $(10,11,12)$, $(10,15,13)$, $(9,14,11)$, $(6,11,7)$, $(7,13,9)$, $(7,14,10)$, $(7,12,8)$, $(6,12,13)$, $(8,14,13)$
$J_{11} \to 3^{10}, 5^6, 17^*$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7,4)$, $(0,5,4,3,7)$, $(2,8,10,5,6)$, $(4,10,7,11,8)$, $(5,12,9,3,8)$, $(10,6,13,7,12)$, $(8,13,12)$, $(7,6,9)$, $(9,16,10)$, $(9,5,11)$, $(10,15,13)$, $(8,15,9)$, $(9,14,13)$, $(6,12,11)$, $(10,14,11)$, $(7,14,8)$
$J_{11} \to 3^5, 5^9, 17^*$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7,4)$, $(0,5,4,3,7)$, $(2,8,10,5,6)$, $(4,10,7,11,8)$, $(5,12,9,3,8)$, $(6,13,7,8,9)$, $(7,14,8,15,9)$, $(6,11,10,12,7)$, $(8,13,12)$, $(9,16,10)$, $(6,12,11,14,10)$, $(9,5,11)$, $(9,14,13)$, $(10,15,13)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_{11} \to 5^{12}, 17^{*}$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7,4)$, $(0,5,4,3,7)$, $(2,8,10,5,6)$, $(4,10,7,11,8)$, $(5,12,9,3,8)$, $(6,13,7,8,9)$, $(7,14,8,13,9)$, $(6,10,13,12,7)$, $(9,15,8,12,11)$, $(5,11,10,16,9)$, $(10,14,11,6,12)$, $(9,14,13,15,10)$
$J_{11} \to 4^{5}, 5^{8}, 17^{*}$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7,4)$, $(0,5,4,3,7)$, $(2,8,10,5,6)$, $(4,10,7,11,8)$, $(5,12,9,3,8)$, $(6,13,7,8,9)$, $(7,14,8,13,9)$, $(6,11,12,7)$, $(5,11,14,10,9)$, $(6,12,13,10)$, $(10,15,8,12)$, $(9,16,10,11)$, $(13,14,9,15)$
$J_{11} \to 4^{10}, 5^{4}, 17^{*}$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7,4)$, $(0,5,4,3,7)$, $(2,8,10,5,6)$, $(4,10,7,11,8)$, $(5,9,3,8)$, $(5,11,10,12)$, $(7,13,6,9)$, $(6,12,8,7)$, $(9,10,6,11)$, $(7,12,11,14)$, $(8,14,9,15)$, $(8,13,12,9)$, $(10,16,9,13)$, $(13,14,10,15)$
$J_{11} \to 4^{15}, 17^{*}$	$(5,1,8,6,3,10,17,15,12,14,16,13,11,4,9,2,7)$, $(0,6,1,7)$, $(0,5,8,4)$, $(2,8,9,6)$, $(3,8,7,4)$, $(4,10,6,5)$, $(5,12,10,9)$, $(6,11,7,13)$, $(7,6,12,9)$, $(3,9,14,7)$, $(8,12,7,10)$, $(5,11,14,10)$, $(8,14,13,15)$, $(9,13,8,11)$, $(9,15,10,16)$, $(10,11,12,13)$
$J_{14} \to 3^{27}, 17^{*}$	$(6,13,20,18,16,19,17,12,9,15,14,7,11,4,5,10,8)$, $(0,7,4)$, $(3,8,4)$, $(0,6,5)$, $(2,9,6)$, $(1,7,6)$, $(1,8,5)$, $(7,3,9)$, $(3,10,6)$, $(2,8,7)$, $(4,10,9)$, $(9,5,11)$, $(5,12,7)$, $(11,18,13)$, $(6,12,11)$, $(12,18,15)$, $(13,16,15)$, $(9,14,16)$, $(10,15,17)$, $(8,15,11)$, $(8,13,9)$, $(10,16,12)$, $(12,8,14)$, $(7,13,10)$, $(12,19,13)$, $(10,14,11)$, $(13,17,14)$, $(11,17,16)$
$J_7^{+} \to 4^{3}, 5^{6}, 7^{+*}$	$[4,7,10,12,9,8,11,13,6]$, $(0,6,3,8,4)$, $(0,5,8,1,7)$, $(5,1,6,2,7)$, $(2,8,10,3,9)$, $(3,7,6,5,4)$, $(4,6,10,9,11)$, $(4,10,5,9)$, $(5,11,6,12)$, $(6,9,7,8)$
$J_7^{+} \to 3^{4}, 5^{6}, 7^{+*}$	$[4,7,10,12,9,8,11,13,6]$, $(0,6,3,8,4)$, $(0,5,7)$, $(1,7,2,8,5)$, $(1,6,5,10,8)$, $(3,9,2,6,4)$, $(6,10,3,7,8)$, $(4,10,9)$, $(4,11,6,12,5)$, $(9,5,11)$, $(7,6,9)$
$J_7^{+} \to 3^{6}, 4^{6}, 7^{+*}$	$[4,7,10,12,9,8,11,13,6]$, $(0,6,3,4)$, $(0,5,7)$, $(1,7,2,8)$, $(1,6,5)$, $(2,9,6)$, $(3,9,5,10)$, $(6,7,3,8)$, $(4,8,5)$, $(5,11,6,12)$, $(4,6,10)$, $(7,8,10,9)$, $(4,9,11)$
$J_7^{+} \to 3^{14}, 7^{+*}$	$[4,7,10,12,9,8,11,13,6]$, $(0,6,4)$, $(0,5,7)$, $(1,7,6)$, $(1,8,5)$, $(6,2,8)$, $(2,7,9)$, $(3,8,7)$, $(4,11,5)$, $(3,9,4)$, $(8,4,10)$, $(5,12,6)$, $(6,11,9)$, $(3,10,6)$, $(5,10,9)$
$J_8^{+} \to 4^{12}, 8^{+*}$	$[6,8,11,13,10,9,12,14,7,4]$, $(0,6,3,4)$, $(0,5,1,7)$, $(1,6,2,8)$, $(2,7,3,9)$, $(5,10,3,8)$, $(8,4,6,10)$, $(4,9,5,11)$, $(5,12,7,6)$, $(5,4,10,7)$, $(7,13,6,9)$, $(9,8,7,11)$, $(10,11,6,12)$
$J_{10}^{+} \to 5^{12}, 10^{+*}$	$[4,7,10,9,16,14,12,11,8,15,13,6]$, $(0,6,3,8,4)$, $(0,5,8,1,7)$, $(5,1,6,2,7)$, $(2,8,10,3,9)$, $(3,7,6,5,4)$, $(4,6,10,5,11)$, $(4,10,12,5,9)$, $(6,12,9,7,8)$, $(6,11,7,14,9)$, $(10,11,14,8,13)$, $(9,8,12,13,11)$, $(7,13,9,15,12)$
$J_5^{+} \to 3^{2}, 5^{5}, 4^{+}$	$[6,8,7,9,11,4]$, $(0,6,3,10,4)$, $(0,5,7)$, $(1,7,2,8,5)$, $(1,6,5,4,8)$, $(2,9,3,7,6)$, $(3,8,10,7,4)$, $(6,4,9)$
$J_5^{+} \to 3^{7}, 5^{2}, 4^{+}$	$[6,8,7,9,11,4]$, $(0,6,3,7,4)$, $(2,8,3,10,7)$, $(1,7,6)$, $(0,5,7)$, $(1,8,5)$, $(8,4,10)$, $(2,9,6)$, $(3,9,4)$, $(4,6,5)$
$J_6^{+} \to 4^{8}, 5, 5^{+}$	$[6,7,10,8,9,11,4]$, $(0,6,3,7,4)$, $(5,6,1,7)$, $(6,2,7,8)$, $(0,5,9,7)$, $(3,9,6,4)$, $(2,8,4,9)$, $(1,8,11,5)$, $(5,10,3,8)$, $(4,10,12,5)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_8^+ \to 4, 5^9, 7^+$	$[4, 7, 10, 12, 9, 8, 11, 13, 6]$, $(0, 6, 3, 8, 4)$, $(0, 5, 8, 1, 7)$, $(5, 1, 6, 2, 7)$, $(2, 8, 10, 3, 9)$, $(3, 7, 6, 5, 4)$, $(4, 6, 10, 5, 11)$, $(7, 14, 12, 5, 9)$, $(6, 11, 10, 4, 9)$, $(6, 12, 7, 8)$, $(9, 10, 13, 7, 11)$
$J_9^+ \to 5^{11}, 8^+$	$[4, 7, 10, 9, 12, 14, 8, 15, 13, 6]$, $(0, 6, 3, 8, 4)$, $(0, 5, 8, 1, 7)$, $(5, 1, 6, 2, 7)$, $(2, 8, 10, 3, 9)$, $(3, 7, 6, 5, 4)$, $(4, 6, 10, 5, 11)$, $(4, 10, 12, 5, 9)$, $(6, 12, 11, 13, 8)$, $(7, 12, 8, 11, 9)$, $(6, 11, 7, 8, 9)$, $(7, 13, 10, 11, 14)$
$J_5^+ \to 4^8, 3^+$	$[6, 7, 9, 11, 4]$, $(0, 6, 3, 4)$, $(0, 5, 1, 7)$, $(1, 6, 2, 8)$, $(2, 7, 3, 9)$, $(3, 8, 4, 10)$, $(4, 9, 6, 5)$, $(5, 8, 10, 7)$, $(6, 4, 7, 8)$
$J_8^+ \to 5^{10}, 6^+$	$[4, 7, 14, 12, 10, 11, 13, 6]$, $(0, 6, 3, 8, 4)$, $(0, 5, 8, 1, 7)$, $(5, 1, 6, 2, 7)$, $(2, 8, 10, 3, 9)$, $(3, 7, 6, 5, 4)$, $(4, 6, 10, 5, 11)$, $(7, 12, 5, 9, 10)$, $(7, 13, 10, 4, 9)$, $(6, 9, 11, 7, 8)$, $(8, 9, 12, 6, 11)$
$J_5^+ \to 3^{11}, 2^+$	$[6, 9, 11, 4]$, $(0, 6, 4)$, $(2, 7, 9)$, $(6, 2, 8)$, $(3, 7, 6)$, $(1, 6, 5)$, $(3, 9, 4)$, $(4, 10, 7)$, $(3, 8, 10)$, $(0, 5, 7)$, $(1, 7, 8)$, $(4, 8, 5)$
$J_7^+ \to 5^9, 4^+$	$[4, 7, 10, 12, 9, 6]$, $(0, 6, 3, 8, 4)$, $(0, 5, 8, 1, 7)$, $(5, 1, 6, 2, 7)$, $(2, 8, 10, 3, 9)$, $(3, 7, 9, 5, 4)$, $(9, 4, 6, 13, 11)$, $(6, 12, 5, 11, 8)$, $(5, 10, 4, 11, 6)$, $(6, 10, 9, 8, 7)$
$J_6^+ \to 5^8, 2^+$	$[6, 9, 11, 4]$, $(0, 6, 3, 10, 4)$, $(0, 5, 8, 1, 7)$, $(5, 1, 6, 2, 7)$, $(2, 8, 3, 7, 9)$, $(4, 9, 5, 6, 7)$, $(3, 9, 8, 6, 4)$, $(4, 8, 10, 12, 5)$, $(7, 8, 11, 5, 10)$

Table A.15: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7\}}$

$J_3 \to 3^6, 6$	$J_3 \to 3, 5^3, 6$	$J_3 \to 3^3, 4, 5, 6$	$J_3 \to 4^2, 5^2, 6$
$J_3 \to 3^2, 4^3, 6$	$J_5 \to 4, 5^6, 6$	$J_6 \to 3^{14}, 6$	
$J_3 \to 3^4, 5, 7$	$J_3 \to 3, 4, 5^2, 7$	$J_3 \to 3^3, 4^2, 7$	$J_3 \to 4^3, 5, 7$
$J_4 \to 5^5, 7$	$J_4 \to 3^7, 4, 7$	$J_5 \to 3^{11}, 7$	$J_3 \to 3^6, 6$
$J_3 \to 3, 5^3, 6$	$J_3 \to 3^3, 4, 5, 6$	$J_3 \to 4^2, 5^2, 6$	$J_3 \to 3^2, 4^3, 6$
$J_5 \to 4, 5^6, 6$	$J_6 \to 3^{14}, 6$	$J_3 \to 3^4, 5, 7$	$J_3 \to 3, 4, 5^2, 7$
$J_3 \to 3^3, 4^2, 7$	$J_3 \to 4^3, 5, 7$	$J_4 \to 5^5, 7$	$J_4 \to 3^7, 4, 7$
$J_5 \to 3^{11}, 7$			
$J_3 \to 3^2, 5^2, 8$	$J_3 \to 3^4, 4, 8$	$J_3 \to 3, 4^2, 5, 8$	$J_3 \to 4^4, 8$
$J_4 \to 3^8, 8$	$J_4 \to 4, 5^4, 8$	$J_6 \to 5^8, 8$	$J_7 \to 3^{16}, 8$

Table A.16: These decompositions are required for Lemma 1.6.32. These have been generated such that the k-cycle is incident on some subset of the vertices $\{n, n+1, \ldots, n+9\}$, and are given in table A.18.

$J_4 \to 4^2, 5^3, 9^*$	$J_4 \to 3, 5^4, 9^*$	$J_4 \to 3, 4^5, 9^*$	$J_4 \to 3^2, 4^3, 5, 9^*$
$J_4 \to 3^3, 4, 5^2, 9^*$	$J_4 \to 3^5, 4^2, 9^*$	$J_4 \to 3^6, 5, 9^*$	$J_5 \to 3^9, 4, 9^*$
$J_6 \to 4, 5^7, 9^*$	$J_6 \to 3^{13}, 9^*$	$J_8 \to 5^{11}, 9^*$	$J_9 \to 3^{21}, 9^*$
$J_4 \to 4^3, 5^2, 10^*$	$J_4 \to 3, 4, 5^3, 10^*$	$J_4 \to 3^2, 4^4, 10^*$	$J_4 \to 3^3, 4^2, 5, 10^*$
$J_4 \to 3^4, 5^2, 10^*$	$J_4 \to 3^6, 4, 10^*$	$J_5 \to 5^6, 10^*$	$J_5 \to 3^{10}, 10^*$
$J_8 \to 4, 5^{10}, 10^*$	$J_8 \to 3^{18}, 10^*$		
$J_5 \to 4, 5^5, 11^*$	$J_5 \to 4^6, 5, 11^*$	$J_5 \to 3, 4^4, 5^2, 11^*$	$J_5 \to 3^2, 4^2, 5^3, 11^*$
$J_5 \to 3^3, 5^4, 11^*$	$J_5 \to 3^3, 4^5, 11^*$	$J_5 \to 3^4, 4^3, 5, 11^*$	$J_5 \to 3^5, 4, 5^2, 11^*$
$J_5 \to 3^7, 4^2, 11^*$	$J_5 \to 3^8, 5, 11^*$	$J_6 \to 3^{11}, 4, 11^*$	$J_7 \to 5^9, 11^*$
$J_7 \to 3^{15}, 11^*$	$J_{10} \to 3^{23}, 11^*$		
$J_7 \to 4, 5^8, 12^*$	$J_7 \to 4^6, 5^4, 12^*$	$J_7 \to 4^{11}, 12^*$	$J_7 \to 3, 4^4, 5^5, 12^*$
$J_7 \to 3, 4^9, 5, 12^*$	$J_7 \to 3^2, 4^2, 5^6, 12^*$	$J_7 \to 3^2, 4^7, 5^2, 12^*$	$J_7 \to 3^3, 5^7, 12^*$

Table A.17: These decompositions are required for Lemma 1.6.33. These are sorted by the value of k in the decomposition, and are given in table A.18.

$J_7 \to 3^3, 4^5, 5^3, 12^*$	$J_7 \to 3^4, 4^3, 5^4, 12^*$	$J_7 \to 3^4, 4^8, 12^*$	$J_7 \to 3^5, 4, 5^5, 12^*$
$J_7 \to 3^5, 4^6, 5, 12^*$	$J_7 \to 3^6, 4^4, 5^2, 12^*$	$J_7 \to 3^7, 4^2, 5^3, 12^*$	$J_7 \to 3^8, 5^4, 12^*$
$J_7 \to 3^8, 4^5, 12^*$	$J_7 \to 3^9, 4^3, 5, 12^*$	$J_7 \to 3^{10}, 4, 5^2, 12^*$	$J_7 \to 3^{12}, 4^2, 12^*$
$J_7 \to 3^{13}, 5, 12^*$	$J_8 \to 3^{16}, 4, 12^*$	$J_9 \to 5^{12}, 12^*$	$J_9 \to 3^{20}, 12^*$
$J_{12} \to 3^{28}, 12^*$			
$J_7 \to 4^2, 5^7, 13^*$	$J_7 \to 4^7, 5^3, 13^*$	$J_7 \to 3, 5^8, 13^*$	$J_7 \to 3, 4^5, 5^4, 13^*$
$J_7 \to 3, 4^{10}, 13^*$	$J_7 \to 3^2, 4^3, 5^5, 13^*$	$J_7 \to 3^2, 4^8, 5, 13^*$	$J_7 \to 3^3, 4, 5^6, 13^*$
$J_7 \to 3^3, 4^6, 5^2, 13^*$	$J_7 \to 3^4, 4^4, 5^3, 13^*$	$J_7 \to 3^5, 4^2, 5^4, 13^*$	$J_7 \to 3^5, 4^7, 13^*$
$J_7 \to 3^6, 5^5, 13^*$	$J_7 \to 3^6, 4^5, 5, 13^*$	$J_7 \to 3^7, 4^3, 5^2, 13^*$	$J_7 \to 3^8, 4, 5^3, 13^*$
$J_7 \to 3^9, 4^4, 13^*$	$J_7 \to 3^{10}, 4^2, 5, 13^*$	$J_7 \to 3^{11}, 5^2, 13^*$	$J_7 \to 3^{13}, 4, 13^*$
$J_8 \to 3^{17}, 13^*$	$J_9 \to 4, 5^{11}, 13^*$	$J_{11} \to 5^{15}, 13^*$	$J_{11} \to 3^{25}, 13^*$
$J_8 \to 5^{10}, 14^*$	$J_8 \to 4^5, 5^6, 14^*$	$J_8 \to 4^{10}, 5^2, 14^*$	$J_8 \to 3, 4^3, 5^7, 14^*$
$J_8 \to 3, 4^8, 5^3, 14^*$	$J_8 \to 3^2, 4, 5^8, 14^*$	$J_8 \to 3^2, 4^6, 5^4, 14^*$	$J_8 \to 3^2, 4^{11}, 14^*$
$J_8 \to 3^3, 4^4, 5^5, 14^*$	$J_8 \to 3^3, 4^9, 5, 14^*$	$J_8 \to 3^4, 4^2, 5^6, 14^*$	$J_8 \to 3^4, 4^7, 5^2, 14^*$
$J_8 \to 3^5, 5^7, 14^*$	$J_8 \to 3^5, 4^5, 5^3, 14^*$	$J_8 \to 3^6, 4^3, 5^4, 14^*$	$J_8 \to 3^6, 4^8, 14^*$
$J_8 \to 3^7, 4, 5^5, 14^*$	$J_8 \to 3^7, 4^6, 5, 14^*$	$J_8 \to 3^8, 4^4, 5^2, 14^*$	$J_8 \to 3^9, 4^2, 5^3, 14^*$
$J_8 \to 3^{10}, 5^4, 14^*$	$J_8 \to 3^{10}, 4^5, 14^*$	$J_8 \to 3^{11}, 4^3, 5, 14^*$	$J_8 \to 3^{12}, 4, 5^2, 14^*$
$J_8 \to 3^{14}, 4^2, 14^*$	$J_8 \to 3^{15}, 5, 14^*$	$J_9 \to 3^{18}, 4, 14^*$	$J_{10} \to 3^{22}, 14^*$
$J_{11} \to 4, 5^{14}, 14^*$	$J_{13} \to 3^{30}, 14^*$		

Table A.17: These decompositions are required for Lemma 1.6.33. These are sorted by the value of k in the decomposition, and are given in table A.18.

$J_1 \to 4^2$	$(0,5,8,7)$, $(5,9,1,7)$
$J_1 \to 3, 5$	$(0,5,7)$, $(7,1,9,5,8)$
$J_2 \to 3^4, 4$	$(0,5,7)$, $(1,6,9)$, $(7,1,8)$, $(6,10,2,8)$, $(5,9,8)$
$J_3 \to 4, 5^4$	$(0,5,9,8,7)$, $(1,8,6,9)$, $(1,6,10,2,7)$, $(5,8,2,9,7)$, $(9,3,11,7,10)$
$J_4 \to 3^9, 5$	$(0,5,7)$, $(1,6,9)$, $(6,10,4,12,8)$, $(7,11,10)$, $(7,1,8)$, $(2,8,10)$, $(2,7,9)$, $(3,8,11)$, $(9,3,10)$, $(5,9,8)$
$J_5 \to 5^8$	$(0,5,9,8,7)$, $(1,8,10,6,9)$, $(5,8,6,1,7)$, $(2,8,3,9,10)$, $(2,7,11,4,9)$, $(7,10,3,11,9)$, $(10,4,12,8,11)$, $(11,5,13,9,12)$
$J_5 \to 3^{12}, 4$	$(0,5,7)$, $(1,6,8)$, $(5,13,9)$, $(6,10,4,9)$, $(7,11,10)$, $(7,2,8)$, $(1,7,9)$, $(8,12,9)$, $(4,11,12)$, $(5,11,8)$, $(3,9,11)$, $(3,8,10)$, $(2,9,10)$
$J_6 \to 3^{16}$	$(0,5,7)$, $(1,6,8)$, $(5,13,9)$, $(6,12,9)$, $(3,9,11)$, $(8,2,9)$, $(1,7,9)$, $(9,4,10)$, $(6,14,10)$, $(5,12,8)$, $(7,11,8)$, $(2,7,10)$, $(3,8,10)$, $(10,5,11)$, $(4,11,12)$, $(10,13,12)$
$J_9 \to 3^{24}$	$(0,5,7)$, $(1,6,9)$, $(7,1,8)$, $(8,2,9)$, $(2,7,10)$, $(9,17,13)$, $(4,10,12)$, $(4,9,11)$, $(3,8,11)$, $(9,3,10)$, $(10,5,11)$, $(5,13,8)$, $(5,12,9)$, $(12,16,13)$, $(8,15,16)$, $(13,15,14)$, $(7,13,11)$, $(6,13,10)$, $(7,15,9)$, $(7,12,14)$, $(6,12,8)$, $(8,14,10)$, $(6,11,14)$, $(11,15,12)$
$J_3 \to 3^6, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,9)$, $(1,6,8)$, $(2,7,10)$, $(6,10,9)$, $(8,2,9)$
$J_3 \to 3, 5^3, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(6,10,7,9,8)$
$J_3 \to 3^3, 4, 5, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,2,8,9)$, $(1,6,8)$, $(6,10,2,9)$, $(7,10,9)$
$J_3 \to 4^2, 5^2, 6$	$(5,8,9,3,11,7)$, $(0,5,9,1,7)$, $(1,6,9,2,8)$, $(2,7,9,10)$, $(7,10,6,8)$
$J_3 \to 3^2, 4^3, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,2,9)$, $(1,6,8)$, $(6,10,7,9)$, $(8,2,10,9)$
$J_5 \to 4, 5^6, 6$	$(7,10,6,9,5,8)$, $(1,8,9,11,7)$, $(1,6,8,10,9)$, $(2,9,3,10)$, $(0,5,13,9,7)$, $(3,8,12,4,11)$, $(5,11,8,2,7)$, $(10,4,9,12,11)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_6 \to 3^{14}, 6$	$(7,11,12,13,10,8)$, $(1,7,9)$, $(2,7,10)$, $(0,5,7)$, $(3,8,11)$, $(10,5,11)$, $(4,9,11)$, $(5,13,9)$, $(6,12,9)$, $(1,6,8)$, $(9,3,10)$, $(8,2,9)$, $(6,14,10)$, $(5,12,8)$, $(4,10,12)$
$J_3 \to 3^4, 5, 7$	$(6,10,7,11,3,9,8)$, $(0,5,7)$, $(5,9,6,1,8)$, $(1,7,9)$, $(7,2,8)$, $(2,9,10)$
$J_3 \to 3, 4, 5^2, 7$	$(6,10,7,11,3,9,8)$, $(0,5,7)$, $(1,6,9,2,7)$, $(7,9,10,2,8)$, $(5,9,1,8)$
$J_3 \to 3^3, 4^2, 7$	$(6,10,7,11,3,9,8)$, $(0,5,7)$, $(1,6,9,7)$, $(2,9,10)$, $(5,9,1,8)$, $(7,2,8)$
$J_3 \to 4^3, 5, 7$	$(6,10,7,11,3,9,8)$, $(0,5,8,7)$, $(1,8,2,10,9)$, $(1,6,9,7)$, $(5,9,2,7)$
$J_4 \to 5^5, 7$	$(7,9,6,10,4,12,8)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(5,9,10,2,7)$, $(7,11,8,3,10)$, $(8,10,11,3,9)$
$J_4 \to 3^7, 4, 7$	$(5,9,6,10,4,12,8)$, $(0,5,7)$, $(1,6,8,9)$, $(2,7,9)$, $(2,8,10)$, $(7,1,8)$, $(3,8,11)$, $(7,11,10)$, $(9,3,10)$
$J_5 \to 3^{11}, 7$	$(7,11,10,6,9,12,8)$, $(1,6,8)$, $(2,7,10)$, $(1,7,9)$, $(0,5,7)$, $(3,8,10)$, $(9,4,10)$, $(4,11,12)$, $(8,2,9)$, $(5,11,8)$, $(3,9,11)$, $(5,13,9)$
$J_3 \to 3^6, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,9)$, $(1,6,8)$, $(2,7,10)$, $(6,10,9)$, $(8,2,9)$
$J_3 \to 3, 5^3, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(6,10,7,9,8)$
$J_3 \to 3^3, 4, 5, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,2,8,9)$, $(1,6,8)$, $(6,10,2,9)$, $(7,10,9)$
$J_3 \to 4^2, 5^2, 6$	$(5,8,9,3,11,7)$, $(0,5,9,1,7)$, $(1,6,9,2,8)$, $(2,7,9,10)$, $(7,10,6,8)$
$J_3 \to 3^2, 4^3, 6$	$(7,11,3,9,5,8)$, $(0,5,7)$, $(1,7,2,9)$, $(1,6,8)$, $(6,10,7,9)$, $(8,2,10,9)$
$J_5 \to 4, 5^6, 6$	$(7,10,6,9,5,8)$, $(1,8,9,11,7)$, $(1,6,8,10,9)$, $(2,9,3,10)$, $(0,5,13,9,7)$, $(3,8,12,4,11)$, $(5,11,8,2,7)$, $(10,4,9,12,11)$
$J_6 \to 3^{14}, 6$	$(7,11,12,13,10,8)$, $(1,7,9)$, $(2,7,10)$, $(0,5,7)$, $(3,8,11)$, $(10,5,11)$, $(4,9,11)$, $(5,13,9)$, $(6,12,9)$, $(1,6,8)$, $(9,3,10)$, $(8,2,9)$, $(6,14,10)$, $(5,12,8)$, $(4,10,12)$
$J_3 \to 3^4, 5, 7$	$(6,10,7,11,3,9,8)$, $(0,5,7)$, $(5,9,6,1,8)$, $(1,7,9)$, $(7,2,8)$, $(2,9,10)$
$J_3 \to 3, 4, 5^2, 7$	$(6,10,7,11,3,9,8)$, $(0,5,7)$, $(1,6,9,2,7)$, $(7,9,10,2,8)$, $(5,9,1,8)$
$J_3 \to 3^3, 4^2, 7$	$(6,10,7,11,3,9,8)$, $(0,5,7)$, $(1,6,9,7)$, $(2,9,10)$, $(5,9,1,8)$, $(7,2,8)$
$J_3 \to 4^3, 5, 7$	$(6,10,7,11,3,9,8)$, $(0,5,8,7)$, $(1,8,2,10,9)$, $(1,6,9,7)$, $(5,9,2,7)$
$J_4 \to 5^5, 7$	$(7,9,6,10,4,12,8)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(5,9,10,2,7)$, $(7,11,8,3,10)$, $(8,10,11,3,9)$
$J_4 \to 3^7, 4, 7$	$(5,9,6,10,4,12,8)$, $(0,5,7)$, $(1,6,8,9)$, $(2,7,9)$, $(2,8,10)$, $(7,1,8)$, $(3,8,11)$, $(7,11,10)$, $(9,3,10)$
$J_5 \to 3^{11}, 7$	$(7,11,10,6,9,12,8)$, $(1,6,8)$, $(2,7,10)$, $(1,7,9)$, $(0,5,7)$, $(3,8,10)$, $(9,4,10)$, $(4,11,12)$, $(8,2,9)$, $(5,11,8)$, $(3,9,11)$, $(5,13,9)$
$J_3 \to 3^2, 5^2, 8$	$(5,9,3,11,7,10,6,8)$, $(0,5,7)$, $(1,7,2,8,9)$, $(7,9,6,1,8)$, $(2,9,10)$
$J_3 \to 3^4, 4, 8$	$(5,9,3,11,7,10,6,8)$, $(0,5,7)$, $(1,8,9)$, $(1,6,9,7)$, $(7,2,8)$, $(2,9,10)$
$J_3 \to 3, 4^2, 5, 8$	$(5,9,3,11,7,10,6,8)$, $(0,5,7)$, $(7,1,9,8)$, $(1,6,9,2,8)$, $(2,7,9,10)$
$J_3 \to 4^4, 8$	$(5,8,6,10,9,3,11,7)$, $(0,5,9,7)$, $(1,7,2,9)$, $(1,6,9,8)$, $(7,10,2,8)$
$J_4 \to 3^8, 8$	$(5,9,7,11,10,4,12,8)$, $(0,5,7)$, $(1,6,9)$, $(2,7,10)$, $(7,1,8)$, $(8,2,9)$, $(3,8,11)$, $(9,3,10)$, $(6,10,8)$
$J_4 \to 4, 5^4, 8$	$(5,9,6,10,4,12,8,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(2,7,9,10)$, $(7,11,8,3,10)$, $(8,10,11,3,9)$
$J_6 \to 5^8, 8$	$(7,11,12,13,10,9,6,8)$, $(0,5,8,1,7)$, $(5,10,11,3,9)$, $(1,6,10,2,9)$, $(4,9,13,5,12)$, $(5,11,8,9,7)$, $(2,7,10,3,8)$, $(4,11,9,12,10)$, $(8,12,6,14,10)$
$J_7 \to 3^{16}, 8$	$(7,15,11,12,13,14,10,8)$, $(0,5,7)$, $(6,13,9)$, $(8,2,9)$, $(9,3,10)$, $(4,9,12)$, $(1,7,9)$, $(5,11,9)$, $(1,6,8)$, $(6,11,14)$, $(2,7,10)$, $(3,8,11)$, $(6,12,10)$, $(5,12,8)$, $(5,10,13)$, $(10,4,11)$, $(7,13,11)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_4 \to 4^2, 5^3, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(7,2,10,9,8)$, $(3,8,10,11)$, $(7,10,3,9)$
$J_4 \to 3, 5^4, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(7,2,10,9,8)$, $(7,10,8,3,9)$, $(3,10,11)$
$J_4 \to 3, 4^5, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(8,2,10,9)$, $(3,8,10,11)$, $(7,10,3,9)$
$J_4 \to 3^2, 4^3, 5, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,1,7)$, $(1,6,8,9)$, $(2,9,7,10)$, $(9,3,11,10)$, $(3,8,10)$, $(7,2,8)$
$J_4 \to 3^3, 4, 5^2, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(2,7,9,10)$, $(3,10,11)$, $(7,10,8)$, $(8,3,9)$
$J_4 \to 3^5, 4^2, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(2,8,10)$, $(3,10,11)$, $(8,3,9)$, $(7,10,9)$
$J_4 \to 3^6, 5, 9^*$	$(5,9,6,10,4,12,8,11,7)$, $(0,5,8,2,7)$, $(1,6,8)$, $(1,7,9)$, $(2,9,10)$, $(3,10,11)$, $(7,10,8)$, $(8,3,9)$
$J_5 \to 3^9, 4, 9^*$	$(5,13,9,6,10,7,11,12,8)$, $(0,5,7)$, $(1,6,8,9)$, $(2,8,10)$, $(10,4,11)$, $(9,3,10)$, $(4,9,12)$, $(5,11,9)$, $(7,1,8)$, $(2,7,9)$, $(3,8,11)$
$J_6 \to 4, 5^7, 9^*$	$(6,14,10,7,11,8,12,13,9)$, $(0,5,9,8,7)$, $(1,8,10,2,9)$, $(5,8,6,1,7)$, $(2,7,9,3,8)$, $(9,4,11,3,10)$, $(4,10,5,11,12)$, $(6,12,5,13,10)$, $(10,12,9,11)$
$J_6 \to 3^{13}, 9^*$	$(6,14,10,7,11,8,12,13,9)$, $(0,5,7)$, $(1,6,8)$, $(3,9,11)$, $(11,5,12)$, $(10,4,11)$, $(5,10,13)$, $(7,2,8)$, $(1,7,9)$, $(3,8,10)$, $(5,9,8)$, $(6,12,10)$, $(2,9,10)$, $(4,9,12)$
$J_8 \to 5^{11}, 9^*$	$(8,16,12,11,15,14,10,13,9)$, $(0,5,11,8,7)$, $(5,9,1,6,8)$, $(1,8,2,9,7)$, $(7,11,3,8,10)$, $(5,12,10,2,7)$, $(9,12,13,5,10)$, $(3,10,11,4,9)$, $(6,14,12,4,10)$, $(6,13,14,11,9)$, $(8,14,7,15,12)$, $(6,11,13,7,12)$
$J_9 \to 3^{21}, 9^*$	$(9,17,13,12,16,15,11,14,10)$, $(0,5,7)$, $(1,6,9)$, $(7,1,8)$, $(8,13,16)$, $(7,15,9)$, $(8,2,9)$, $(3,8,10)$, $(5,13,9)$, $(4,9,12)$, $(3,9,11)$, $(2,7,10)$, $(6,13,10)$, $(10,4,11)$, $(5,10,12)$, $(5,11,8)$, $(13,15,14)$, $(8,15,12)$, $(6,14,8)$, $(7,13,11)$, $(11,6,12)$, $(7,12,14)$
$J_4 \to 4^3, 5^2, 10^*$	$(5,9,6,10,4,12,8,3,11,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(2,7,9,10)$, $(7,10,11,8)$, $(8,10,3,9)$
$J_4 \to 3, 4, 5^3, 10^*$	$(5,9,6,10,4,12,8,3,11,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(7,2,10,9,8)$, $(7,10,3,9)$, $(8,11,10)$
$J_4 \to 3^2, 4^4, 10^*$	$(5,9,6,10,4,12,8,3,11,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(8,2,10,9)$, $(7,10,3,9)$, $(8,11,10)$
$J_4 \to 3^3, 4^2, 5, 10^*$	$(5,9,6,10,4,12,8,3,11,7)$, $(0,5,8,1,7)$, $(1,6,8,9)$, $(2,9,7,10)$, $(7,2,8)$, $(9,3,10)$, $(8,11,10)$
$J_4 \to 3^4, 5^2, 10^*$	$(5,9,6,10,4,12,8,3,11,7)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(2,7,10)$, $(7,9,8)$, $(9,3,10)$, $(8,11,10)$
$J_4 \to 3^6, 4, 10^*$	$(5,9,6,10,4,12,8,3,11,7)$, $(0,5,8,7)$, $(1,7,9)$, $(1,6,8)$, $(2,7,10)$, $(9,3,10)$, $(8,2,9)$, $(8,11,10)$
$J_5 \to 5^6, 10^*$	$(5,13,9,6,10,7,11,4,12,8)$, $(0,5,9,8,7)$, $(1,8,10,2,9)$, $(1,6,8,2,7)$, $(3,9,4,10,11)$, $(9,11,8,3,10)$, $(5,11,12,9,7)$
$J_5 \to 3^{10}, 10^*$	$(6,10,7,11,5,13,9,4,12,8)$, $(0,5,7)$, $(1,6,9)$, $(7,1,8)$, $(5,9,8)$, $(2,8,10)$, $(10,4,11)$, $(2,7,9)$, $(9,3,10)$, $(9,12,11)$, $(3,8,11)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_8 \to 4, 5^{10}, 10^*$	$(7,15,11,12,16,8,14,10,13,9)$, $(0,5,9,8,7)$, $(1,7,10,6,9)$, $(1,6,14,11,8)$, $(2,8,3,9,10)$, $(2,7,11,4,9)$, $(5,11,3,10,8)$, $(5,13,12,14,7)$, $(9,12,6,13,11)$, $(6,11,10,12,8)$, $(4,10,5,12)$, $(13,7,12,15,14)$
$J_8 \to 3^{18}, 10^*$	$(7,15,11,12,16,8,14,10,13,9)$, $(0,5,7)$, $(1,6,9)$, $(8,2,9)$, $(9,3,10)$, $(4,9,11)$, $(5,12,9)$, $(5,11,13)$, $(6,11,10)$, $(3,8,11)$, $(5,10,8)$, $(7,1,8)$, $(4,10,12)$, $(2,7,10)$, $(7,14,11)$, $(6,12,8)$, $(12,7,13)$, $(6,13,14)$, $(12,15,14)$
$J_5 \to 4, 5^5, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,9,8,7)$, $(1,6,8,2,9)$, $(5,11,9,10,8)$, $(2,7,9,4,10)$, $(3,9,12,11)$, $(1,8,11,10,7)$
$J_5 \to 4^6, 5, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,11,8,7)$, $(1,6,8,9)$, $(2,9,11,10)$, $(3,9,12,11)$, $(7,10,4,9)$, $(1,8,2,7)$, $(5,9,10,8)$
$J_5 \to 3, 4^4, 5^2, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,9,8,7)$, $(1,6,8,2,9)$, $(2,7,9,10)$, $(1,8,10,7)$, $(3,9,12,11)$, $(10,4,9,11)$, $(5,11,8)$
$J_5 \to 3^2, 4^2, 5^3, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,9,12,11)$, $(10,4,9,11)$, $(5,11,8)$
$J_5 \to 3^3, 5^4, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,9,4,10,11)$, $(5,11,8)$, $(9,12,11)$
$J_5 \to 3^3, 4^5, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(8,2,10,9)$, $(3,9,11)$, $(5,11,12,9)$, $(7,10,4,9)$, $(8,11,10)$
$J_5 \to 3^4, 4^3, 5, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(2,8,10)$, $(9,3,11,10)$, $(5,11,8)$, $(7,10,4,9)$, $(9,12,11)$
$J_5 \to 3^5, 4, 5^2, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(3,9,4,10,11)$, $(5,11,8)$, $(2,8,10)$, $(7,10,9)$, $(9,12,11)$
$J_5 \to 3^7, 4^2, 11^*$	$(5,13,9,6,10,3,8,12,4,11,7)$, $(0,5,8,7)$, $(1,7,9)$, $(1,6,8)$, $(5,11,12,9)$, $(9,4,10)$, $(3,9,11)$, $(8,2,9)$, $(2,7,10)$, $(8,11,10)$
$J_5 \to 3^8, 5, 11^*$	$(5,13,9,3,10,6,8,12,4,11,7)$, $(7,1,8)$, $(1,6,9)$, $(0,5,11,10,7)$, $(9,4,10)$, $(2,8,10)$, $(5,9,8)$, $(3,8,11)$, $(2,7,9)$, $(9,12,11)$
$J_6 \to 3^{11}, 4, 11^*$	$(6,14,10,7,11,5,13,9,4,12,8)$, $(7,1,8)$, $(0,5,7)$, $(2,7,9)$, $(5,9,8)$, $(2,8,10)$, $(1,6,9)$, $(9,3,10)$, $(3,8,11)$, $(10,4,11)$, $(9,12,11)$, $(5,10,13,12)$, $(6,12,10)$
$J_7 \to 5^9, 11^*$	$(5,13,9,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(6,11,12,4,10)$, $(5,11,7,10,8)$, $(3,8,11,4,9)$, $(5,10,13,6,12)$, $(7,13,14,11,9)$, $(3,10,12,13,11)$
$J_7 \to 3^{15}, 11^*$	$(5,11,15,7,13,9,6,14,10,12,8)$, $(5,10,9)$, $(0,5,7)$, $(3,9,11)$, $(11,6,12)$, $(5,12,13)$, $(4,9,12)$, $(1,7,9)$, $(8,2,9)$, $(1,6,8)$, $(3,8,10)$, $(2,7,10)$, $(10,4,11)$, $(6,13,10)$, $(7,11,8)$, $(11,14,13)$
$J_{10} \to 3^{23}, 11^*$	$(8,12,16,9,17,13,15,11,14,18,10)$, $(0,5,7)$, $(1,7,9)$, $(1,6,8)$, $(11,6,12)$, $(4,9,11)$, $(10,5,11)$, $(7,13,11)$, $(3,8,11)$, $(2,7,10)$, $(8,2,9)$, $(9,3,10)$, $(6,13,9)$, $(5,13,8)$, $(14,9,15)$, $(4,10,12)$, $(6,14,10)$, $(5,12,9)$, $(7,12,15)$, $(8,15,16)$, $(10,16,13)$, $(12,14,13)$, $(7,14,8)$, $(14,17,16)$
$J_7 \to 4, 5^8, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,9,3,8)$, $(6,12,9,7,10)$, $(8,11,12,5,10)$, $(4,11,6,13,10)$, $(7,13,14,11)$, $(3,10,12,13,11)$
$J_7 \to 4^6, 5^4, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,9,3,8)$, $(6,11,3,10)$, $(7,10,12,9)$, $(8,11,4,10)$, $(5,10,13,12)$, $(11,13,6,12)$, $(7,13,14,11)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_7 \to 4^{11}, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,9,8)$, $(2,8,3,10)$, $(5,11,7,9)$, $(3,9,12,11)$, $(6,12,5,10)$, $(7,13,12,10)$, $(9,11,8,10)$, $(4,11,13,10)$, $(6,11,14,13)$
$J_7 \to 3, 4^4, 5^5, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,9,3,8)$, $(6,12,11,7,10)$, $(3,10,8,11)$, $(4,11,13,10)$, $(5,10,12)$, $(6,11,14,13)$, $(7,13,12,9)$
$J_7 \to 3, 4^9, 5, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,10,8)$, $(2,8,3,10)$, $(5,12,11,8)$, $(6,13,10,9)$, $(4,11,5,10)$, $(9,12,6,11)$, $(7,11,3,9)$, $(7,13,12,10)$, $(11,14,13)$
$J_7 \to 3^2, 4^2, 5^6, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,9,3,8)$, $(6,12,11,7,10)$, $(3,10,8,11)$, $(4,11,6,13,10)$, $(5,10,12)$, $(7,13,12,9)$, $(11,14,13)$
$J_7 \to 3^2, 4^7, 5^2, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,10,8)$, $(5,11,3,8)$, $(2,9,11,8)$, $(6,13,7,9)$, $(3,10,12,9)$, $(7,11,4,10)$, $(5,10,13,12)$, $(11,6,12)$, $(11,14,13)$
$J_7 \to 3^3, 5^7, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,3,10,8)$, $(5,10,12)$, $(6,11,9,7,10)$, $(3,8,11,12,9)$, $(7,13,10,4,11)$, $(12,6,13)$, $(11,14,13)$
$J_7 \to 3^3, 4^5, 5^3, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,3,8)$, $(6,13,7,10)$, $(3,10,12,9)$, $(7,11,9)$, $(8,11,4,10)$, $(5,10,13,12)$, $(11,6,12)$, $(11,14,13)$
$J_7 \to 3^4, 4^3, 5^4, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,9,3,8)$, $(6,11,3,10)$, $(7,11,12,9)$, $(12,6,13)$, $(5,10,12)$, $(8,11,4,10)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^4, 4^8, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,9,8)$, $(2,8,3,10)$, $(5,11,7,9)$, $(3,9,11)$, $(6,12,9,10)$, $(7,13,10)$, $(5,10,12)$, $(8,11,4,10)$, $(6,11,14,13)$, $(11,13,12)$
$J_7 \to 3^5, 4, 5^5, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,11,6,10,8)$, $(3,8,11,4,10)$, $(5,10,12)$, $(3,9,12,11)$, $(12,6,13)$, $(7,11,9)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^5, 4^6, 5, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,10,8)$, $(2,8,3,10)$, $(5,12,11,8)$, $(6,11,7,9)$, $(9,12,10)$, $(3,9,11)$, $(4,11,5,10)$, $(7,13,10)$, $(12,6,13)$, $(11,14,13)$
$J_7 \to 3^6, 4^4, 5^2, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,10,8)$, $(5,11,8)$, $(2,9,3,8)$, $(6,11,12,9)$, $(7,11,9)$, $(12,6,13)$, $(7,13,10)$, $(3,10,4,11)$, $(5,10,12)$, $(11,14,13)$
$J_7 \to 3^7, 4^2, 5^3, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,10,8)$, $(6,11,4,10)$, $(12,6,13)$, $(3,8,11)$, $(3,10,12,9)$, $(7,11,9)$, $(11,5,12)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^8, 5^4, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,8,2,9,7)$, $(3,8,11,4,10)$, $(5,10,8)$, $(1,6,9)$, $(6,11,7,2,10)$, $(3,9,11)$, $(7,13,10)$, $(11,5,12)$, $(11,14,13)$, $(12,6,13)$, $(9,12,10)$
$J_7 \to 3^8, 4^5, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,9,8)$, $(2,8,10)$, $(6,11,4,10)$, $(7,11,9)$, $(3,8,11)$, $(3,10,12,9)$, $(5,10,9)$, $(7,13,10)$, $(11,5,12)$, $(12,6,13)$, $(11,14,13)$
$J_7 \to 3^9, 4^3, 5, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(1,8,2,7)$, $(7,11,4,10)$, $(2,9,10)$, $(5,11,8)$, $(1,6,9)$, $(7,13,12,9)$, $(3,9,11)$, $(3,8,10)$, $(5,10,12)$, $(6,13,10)$, $(11,6,12)$, $(11,14,13)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_7 \to 3^{10}, 4, 5^2, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,9,8,7)$, $(5,11,7,1,8)$, $(3,8,11)$, $(2,8,10)$, $(6,11,4,10)$, $(1,6,9)$, $(5,10,12)$, $(9,3,10)$, $(2,7,9)$, $(7,13,10)$, $(11,14,13)$, $(9,12,11)$, $(12,6,13)$
$J_7 \to 3^{12}, 4^2, 12^*$	$(5,13,9,4,12,8,6,14,10,11,15,7)$, $(0,5,11,7)$, $(7,1,8)$, $(6,11,4,10)$, $(9,3,10)$, $(1,6,9)$, $(2,8,10)$, $(2,7,9)$, $(3,8,11)$, $(5,9,8)$, $(5,10,12)$, $(9,12,11)$, $(12,6,13)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^{13}, 5, 12^*$	$(5,8,12,6,9,13,14,10,4,11,15,7)$, $(0,5,12,13,7)$, $(6,13,10)$, $(5,11,13)$, $(1,7,9)$, $(1,6,8)$, $(5,10,9)$, $(8,2,9)$, $(2,7,10)$, $(6,11,14)$, $(7,11,8)$, $(3,8,10)$, $(3,9,11)$, $(10,12,11)$, $(4,9,12)$
$J_8 \to 3^{16}, 4, 12^*$	$(5,9,13,6,10,14,8,16,12,11,15,7)$, $(5,10,8)$, $(0,5,12,7)$, $(3,8,11)$, $(7,1,8)$, $(1,6,9)$, $(8,2,9)$, $(4,9,12)$, $(2,7,10)$, $(9,3,10)$, $(10,4,11)$, $(7,11,9)$, $(5,11,13)$, $(6,12,8)$, $(10,13,12)$, $(13,7,14)$, $(6,11,14)$, $(12,15,14)$
$J_9 \to 5^{12}, 12^*$	$(7,10,14,6,11,15,9,17,13,12,16,8)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(5,9,10,2,7)$, $(8,3,10,6,9)$, $(7,13,11,3,9)$, $(8,12,4,9,11)$, $(7,15,12,10,11)$, $(4,11,12,5,10)$, $(5,11,14,8,13)$, $(8,15,16,13,10)$, $(9,13,14,7,12)$, $(6,13,15,14,12)$
$J_9 \to 3^{20}, 12^*$	$(7,15,11,14,10,6,9,17,13,12,16,8)$, $(9,3,10)$, $(0,5,7)$, $(6,11,13)$, $(1,6,8)$, $(7,12,11)$, $(4,9,11)$, $(3,8,11)$, $(10,5,11)$, $(1,7,9)$, $(8,2,9)$, $(2,7,10)$, $(5,12,8)$, $(4,10,12)$, $(8,13,10)$, $(5,13,9)$, $(6,12,14)$, $(13,7,14)$, $(14,8,15)$, $(9,15,12)$, $(13,16,15)$
$J_{12} \to 3^{28}, 12^*$	$(10,12,20,16,13,17,9,14,18,19,15,11)$, $(0,5,7)$, $(1,6,9)$, $(7,1,8)$, $(2,7,10)$, $(8,2,9)$, $(3,8,10)$, $(3,9,11)$, $(9,4,10)$, $(4,11,12)$, $(6,11,13)$, $(5,12,9)$, $(5,11,8)$, $(5,10,13)$, $(6,12,8)$, $(6,14,10)$, $(12,18,16)$, $(7,14,11)$, $(7,13,9)$, $(8,13,15)$, $(7,12,15)$, $(12,14,13)$, $(10,15,18)$, $(17,11,18)$, $(14,17,15)$, $(16,10,17)$, $(8,14,16)$, $(15,9,16)$, $(11,16,19)$
$J_7 \to 4^2, 5^7, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(10,12,13,5,11)$, $(6,10,3,8)$, $(7,13,11,4,9)$, $(7,11,12,5,10)$, $(9,12,6,11)$, $(8,11,14,13,10)$
$J_7 \to 4^7, 5^3, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,9,2,8)$, $(5,10,12,13)$, $(6,10,3,8)$, $(4,9,12,11)$, $(5,12,6,11)$, $(7,13,11,9)$, $(7,11,8,10)$, $(10,13,14,11)$
$J_7 \to 3, 5^8, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(7,11,8,3,10)$, $(10,12,13,5,11)$, $(4,9,12,6,11)$, $(11,13,10,5,12)$, $(7,13,14,11,9)$
$J_7 \to 3, 4^5, 5^4, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(7,11,8,3,10)$, $(5,10,12,13)$, $(4,9,12,11)$, $(5,12,6,11)$, $(7,13,11,9)$, $(10,13,14,11)$
$J_7 \to 3, 4^{10}, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,7)$, $(7,1,9,8)$, $(1,6,8)$, $(2,8,3,10)$, $(2,7,10,9)$, $(11,5,13,12)$, $(6,11,4,9)$, $(6,12,5,10)$, $(7,13,14,11)$, $(8,11,13,10)$, $(10,12,9,11)$
$J_7 \to 3^2, 4^3, 5^5, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(7,11,8,3,10)$, $(5,11,6,12,13)$, $(7,13,11,9)$, $(4,9,12,11)$, $(5,10,12)$, $(10,13,14,11)$
$J_7 \to 3^2, 4^8, 5, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(2,8,3,10)$, $(11,5,13,12)$, $(6,10,7,9)$, $(7,13,14,11)$, $(5,10,12)$, $(9,12,6,11)$, $(9,4,11,10)$, $(8,11,13,10)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_7 \to 3^3, 4, 5^6, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(7,11,8,3,10)$, $(9,12,13,5,11)$, $(7,13,11,4,9)$, $(5,10,12)$, $(11,6,12)$, $(10,13,14,11)$
$J_7 \to 3^3, 4^6, 5^2, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(2,9,11,8)$, $(3,8,10)$, $(11,5,13,12)$, $(6,11,4,9)$, $(7,10,12,9)$, $(6,12,5,10)$, $(10,13,11)$, $(7,13,14,11)$
$J_7 \to 3^4, 4^4, 5^3, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(3,8,11,10)$, $(11,5,13,12)$, $(7,11,4,9)$, $(7,13,10)$, $(5,10,12)$, $(9,12,6,11)$, $(11,14,13)$
$J_7 \to 3^5, 4^2, 5^4, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(7,11,8,3,10)$, $(10,13,5,11)$, $(4,9,11)$, $(5,10,12)$, $(11,6,12)$, $(7,13,12,9)$, $(11,14,13)$
$J_7 \to 3^5, 4^7, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,7)$, $(7,1,9,8)$, $(1,6,8)$, $(2,8,3,10)$, $(2,7,10,9)$, $(11,5,13,12)$, $(6,12,9)$, $(4,9,11)$, $(5,10,12)$, $(10,13,14,11)$, $(7,13,11)$, $(6,11,8,10)$
$J_7 \to 3^6, 5^5, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(6,10,8,2,9)$, $(7,11,8,3,10)$, $(5,12,13)$, $(7,13,10,12,9)$, $(4,9,11)$, $(10,5,11)$, $(11,6,12)$, $(11,14,13)$
$J_7 \to 3^6, 4^5, 5, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(2,8,3,10)$, $(11,5,13,12)$, $(6,11,7,9)$, $(4,9,11)$, $(6,12,10)$, $(9,12,5,10)$, $(8,11,10)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^7, 4^3, 5^2, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(2,9,11,8)$, $(3,8,10)$, $(11,5,13,12)$, $(6,12,9)$, $(5,10,12)$, $(7,11,4,9)$, $(7,13,10)$, $(6,11,10)$, $(11,14,13)$
$J_7 \to 3^8, 4, 5^3, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,10,9)$, $(1,6,8)$, $(2,9,4,11,8)$, $(3,8,10)$, $(11,6,12)$, $(6,10,12,9)$, $(7,11,9)$, $(5,12,13)$, $(10,5,11)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^9, 4^4, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,7)$, $(7,1,9,8)$, $(1,6,8)$, $(2,7,11,8)$, $(3,8,10)$, $(6,11,12,9)$, $(4,9,11)$, $(2,9,10)$, $(6,12,10)$, $(5,12,13)$, $(10,5,11)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^{10}, 4^2, 5, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,9,8,7)$, $(1,8,2,7)$, $(7,11,4,9)$, $(3,8,10)$, $(1,6,9)$, $(6,11,8)$, $(2,9,10)$, $(6,12,10)$, $(9,12,11)$, $(5,12,13)$, $(10,5,11)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^{11}, 5^2, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,11,8,7)$, $(8,2,9)$, $(1,6,8)$, $(1,7,9)$, $(2,7,11,12,10)$, $(3,8,10)$, $(5,10,9)$, $(4,9,11)$, $(6,12,9)$, $(6,11,10)$, $(5,12,13)$, $(7,13,10)$, $(11,14,13)$
$J_7 \to 3^{13}, 4, 13^*$	$(5,8,12,4,10,14,6,13,9,3,11,15,7)$, $(0,5,13,7)$, $(1,6,8)$, $(1,7,9)$, $(2,7,10)$, $(7,11,8)$, $(8,2,9)$, $(3,8,10)$, $(4,9,11)$, $(6,12,9)$, $(5,10,9)$, $(6,11,10)$, $(11,5,12)$, $(10,13,12)$, $(11,14,13)$
$J_8 \to 3^{17}, 13^*$	$(5,11,15,7,13,9,6,14,10,4,12,16,8)$, $(0,5,7)$, $(5,12,9)$, $(1,7,9)$, $(8,2,9)$, $(4,9,11)$, $(9,3,10)$, $(1,6,8)$, $(2,7,10)$, $(5,10,13)$, $(6,11,10)$, $(8,12,10)$, $(3,8,11)$, $(12,6,13)$, $(11,14,13)$, $(7,12,11)$, $(7,14,8)$, $(12,15,14)$
$J_9 \to 4, 5^{11}, 13^*$	$(5,13,17,9,12,16,15,11,14,10,6,8,7)$, $(0,5,8,1,7)$, $(1,6,14,8,9)$, $(2,8,3,9,10)$, $(5,10,7,2,9)$, $(3,10,8,12,11)$, $(10,4,12,5,11)$, $(6,12,13,7,9)$, $(8,13,9,4,11)$, $(7,12,10,13,11)$, $(13,16,8,15,14)$, $(9,15,13,6,11)$, $(7,14,12,15)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_{11} \to 5^{15}, 13^*$	$(7,15,19,11,14,18,17,13,16,12,10,9,8)$, $(0,5,8,1,7)$, $(1,6,8,2,9)$, $(7,11,3,8,10)$, $(5,13,10,2,7)$, $(5,11,10,6,9)$, $(3,10,4,11,9)$, $(8,14,10,5,12)$, $(4,9,13,11,12)$, $(6,12,7,13,14)$, $(7,14,12,15,9)$, $(8,15,13,6,11)$, $(9,16,8,13,12)$, $(14,9,17,16,15)$, $(11,17,10,18,15)$, $(14,17,15,10,16)$
$J_{11} \to 3^{25}, 13^*$	$(7,15,19,11,8,12,16,10,13,17,18,14,9)$, $(0,5,7)$, $(7,1,8)$, $(2,7,10)$, $(8,2,9)$, $(1,6,9)$, $(9,4,10)$, $(3,9,11)$, $(3,8,10)$, $(11,17,15)$, $(7,14,11)$, $(4,11,12)$, $(6,11,13)$, $(10,5,11)$, $(5,13,8)$, $(5,12,9)$, $(6,12,10)$, $(6,14,8)$, $(12,7,13)$, $(8,15,16)$, $(13,16,14)$, $(9,16,17)$, $(10,17,14)$, $(9,15,13)$, $(10,15,18)$, $(12,15,14)$
$J_8 \to 5^{10}, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(5,10,6,1,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(6,12,9,11,8)$, $(7,14,8,12,11)$, $(13,7,12,15,14)$, $(5,12,10,13,11)$, $(12,14,11,6,13)$
$J_8 \to 4^5, 5^6, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(5,10,6,1,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(6,12,9,11,8)$, $(7,14,12,11)$, $(6,11,14,13)$, $(8,14,15,12)$, $(10,13,7,12)$, $(5,12,13,11)$
$J_8 \to 4^{10}, 5^2, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,10,8)$, $(2,7,11,8)$, $(5,11,6,8)$, $(7,12,10,9)$, $(4,9,12,11)$, $(5,10,13,12)$, $(6,13,14,12)$, $(8,14,15,12)$, $(3,10,11,9)$, $(11,14,7,13)$
$J_8 \to 3, 4^3, 5^7, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,12,9,11,8)$, $(6,12,11,5,10)$, $(7,14,13,6,11)$, $(8,14,15,12)$, $(10,13,7,12)$, $(12,14,11,13)$
$J_8 \to 3, 4^8, 5^3, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,9)$, $(4,9,12,11)$, $(5,10,12,8)$, $(5,12,14,11)$, $(6,11,13,10)$, $(6,13,7,12)$, $(7,14,8,11)$, $(12,15,14,13)$
$J_8 \to 3^2, 4, 5^8, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,12,9,11,8)$, $(6,12,11,5,10)$, $(7,14,13,6,11)$, $(8,14,11,13,12)$, $(10,13,7,12)$, $(12,15,14)$
$J_8 \to 3^2, 4^6, 5^4, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,12,11,8)$, $(6,11,5,10)$, $(7,12,9,11)$, $(8,14,13,12)$, $(10,13,6,12)$, $(11,14,7,13)$, $(12,15,14)$
$J_8 \to 3^2, 4^{11}, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(5,11,8,9)$, $(2,8,10)$, $(6,12,9,10)$, $(7,11,4,9)$, $(8,14,15,12)$, $(3,10,11,9)$, $(11,14,7,13)$, $(11,6,13,12)$, $(7,12,5,10)$, $(10,13,14,12)$
$J_8 \to 3^3, 4^4, 5^5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,12,9,11,8)$, $(6,11,5,10)$, $(7,13,12,11)$, $(8,14,7,12)$, $(10,13,6,12)$, $(11,14,13)$, $(12,15,14)$
$J_8 \to 3^3, 4^9, 5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(5,10,2,8)$, $(6,12,9,10)$, $(7,10,3,9)$, $(4,9,11)$, $(7,13,10,11)$, $(8,12,11)$, $(6,11,14,13)$, $(5,12,13,11)$, $(7,12,15,14)$, $(8,14,12,10)$
$J_8 \to 3^4, 4^2, 5^6, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,12,9,11,8)$, $(6,12,11,5,10)$, $(7,13,6,11)$, $(8,14,7,12)$, $(10,13,12)$, $(11,14,13)$, $(12,15,14)$
$J_8 \to 3^4, 4^7, 5^2, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,11,8)$, $(5,11,10,8)$, $(4,9,11)$, $(6,12,5,10)$, $(3,10,12,9)$, $(7,13,10,9)$, $(8,14,11,12)$, $(6,11,13)$, $(12,7,14,13)$, $(12,15,14)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_8 \to 3^5, 5^7, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,12,9,11,8)$, $(6,12,11,5,10)$, $(7,12,15,14,11)$, $(6,11,13)$, $(8,14,12)$, $(13,7,14)$, $(10,13,12)$
$J_8 \to 3^5, 4^5, 5^3, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,9)$, $(4,9,12,11)$, $(5,11,8)$, $(6,12,5,10)$, $(7,13,6,11)$, $(8,14,7,12)$, $(10,13,12)$, $(11,14,13)$, $(12,15,14)$
$J_8 \to 3^6, 4^3, 5^4, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,11,8)$, $(5,10,12)$, $(6,11,13,10)$, $(7,12,9,11)$, $(12,6,13)$, $(13,7,14)$, $(8,14,15,12)$, $(11,14,12)$
$J_8 \to 3^6, 4^8, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(5,11,8,9)$, $(2,8,10)$, $(6,12,9,10)$, $(7,10,3,9)$, $(4,9,11)$, $(7,13,10,11)$, $(5,10,12)$, $(8,14,11,12)$, $(6,11,13)$, $(12,7,14,13)$, $(12,15,14)$
$J_8 \to 3^7, 4, 5^5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,11,4,9)$, $(5,11,8)$, $(6,12,7,13,10)$, $(5,10,12)$, $(9,12,11)$, $(6,11,13)$, $(7,14,11)$, $(12,15,14,13)$, $(8,14,12)$
$J_8 \to 3^7, 4^6, 5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(5,10,2,8)$, $(6,12,9,10)$, $(7,10,3,9)$, $(4,9,11)$, $(7,13,10,11)$, $(8,12,10)$, $(7,12,15,14)$, $(11,5,12)$, $(12,14,13)$, $(6,11,13)$, $(8,14,11)$
$J_8 \to 3^8, 4^4, 5^2, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,11,8)$, $(5,11,10,8)$, $(4,9,11)$, $(6,12,5,10)$, $(9,3,10)$, $(7,12,9)$, $(6,11,13)$, $(13,7,14)$, $(10,13,12)$, $(11,14,15,12)$, $(8,14,12)$
$J_8 \to 3^9, 4^2, 5^3, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(3,10,12,9)$, $(4,9,11)$, $(6,11,5,10)$, $(5,12,8)$, $(12,6,13)$, $(7,12,11)$, $(13,7,14)$, $(8,14,11)$, $(10,13,11)$, $(12,15,14)$
$J_8 \to 3^{10}, 5^4, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,10,2,9)$, $(1,6,8)$, $(2,7,9,10,8)$, $(6,12,9,3,10)$, $(4,9,11)$, $(5,12,8)$, $(10,5,11)$, $(6,11,13)$, $(10,13,12)$, $(13,7,14)$, $(7,12,11)$, $(8,14,11)$, $(12,15,14)$
$J_8 \to 3^{10}, 4^5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(5,11,8,9)$, $(2,8,10)$, $(6,13,7,10)$, $(4,9,11)$, $(7,12,9)$, $(9,3,10)$, $(5,10,12)$, $(11,6,12)$, $(10,13,11)$, $(7,14,11)$, $(12,15,14,13)$, $(8,14,12)$
$J_8 \to 3^{11}, 4^3, 5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,7,2,9)$, $(1,6,8)$, $(5,10,2,8)$, $(6,11,7,10)$, $(9,3,10)$, $(4,9,11)$, $(8,14,11)$, $(10,13,11)$, $(13,7,14)$, $(11,5,12)$, $(12,6,13)$, $(7,12,9)$, $(8,12,10)$, $(12,15,14)$
$J_8 \to 3^{12}, 4, 5^2, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,8,7)$, $(1,6,10,2,9)$, $(1,8,2,7)$, $(5,10,8)$, $(9,3,10)$, $(4,9,11)$, $(6,11,13)$, $(6,12,8)$, $(7,12,9)$, $(11,5,12)$, $(10,13,12)$, $(7,11,10)$, $(13,7,14)$, $(8,14,11)$, $(12,15,14)$
$J_8 \to 3^{14}, 4^2, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,8,7)$, $(1,7,9)$, $(8,2,9)$, $(1,6,12,8)$, $(2,7,10)$, $(6,10,8)$, $(9,3,10)$, $(4,9,11)$, $(10,5,11)$, $(5,12,9)$, $(10,13,12)$, $(7,12,11)$, $(6,11,13)$, $(8,14,11)$, $(13,7,14)$, $(12,15,14)$
$J_8 \to 3^{15}, 5, 14^*$	$(5,13,9,6,14,10,4,12,16,8,3,11,15,7)$, $(0,5,9,1,7)$, $(9,3,10)$, $(1,6,8)$, $(2,7,9)$, $(4,9,11)$, $(2,8,10)$, $(6,13,10)$, $(11,6,12)$, $(8,12,9)$, $(5,10,12)$, $(5,11,8)$, $(7,11,10)$, $(11,14,13)$, $(12,7,13)$, $(7,14,8)$, $(12,15,14)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_9 \to 3^{18}, 4, 14^*$	$(5, 13, 17, 9, 4, 10, 14, 6, 12, 16, 15, 11, 8, 7)$, $(0, 5, 11, 7)$, $(1, 7, 9)$, $(1, 6, 8)$, $(5, 12, 8)$, $(8, 13, 16)$, $(3, 9, 11)$, $(2, 7, 10)$, $(8, 2, 9)$, $(3, 8, 10)$, $(5, 10, 9)$, $(6, 11, 10)$, $(6, 13, 9)$, $(4, 11, 12)$, $(10, 13, 12)$, $(9, 15, 12)$, $(11, 14, 13)$, $(7, 13, 15)$, $(7, 12, 14)$, $(14, 8, 15)$
$J_{10} \to 3^{22}, 14^*$	$(5, 12, 16, 17, 13, 9, 15, 11, 7, 10, 18, 14, 6, 8)$, $(0, 5, 7)$, $(7, 1, 8)$, $(1, 6, 9)$, $(2, 7, 9)$, $(2, 8, 10)$, $(10, 16, 14)$, $(5, 10, 13)$, $(6, 12, 10)$, $(10, 4, 11)$, $(9, 3, 10)$, $(3, 8, 11)$, $(5, 11, 9)$, $(4, 9, 12)$, $(6, 11, 13)$, $(8, 13, 12)$, $(9, 14, 17)$, $(8, 16, 9)$, $(11, 14, 12)$, $(13, 7, 14)$, $(7, 12, 15)$, $(14, 8, 15)$, $(13, 16, 15)$
$J_{11} \to 4, 5^{14}, 14^*$	$(7, 15, 19, 11, 14, 18, 17, 13, 16, 12, 10, 9, 6, 8)$, $(0, 5, 8, 1, 7)$, $(5, 10, 6, 1, 9)$, $(8, 2, 10, 7, 9)$, $(2, 7, 11, 3, 9)$, $(3, 8, 12, 11, 10)$, $(4, 10, 13, 9, 12)$, $(4, 9, 17, 15, 11)$, $(5, 13, 12, 14, 7)$, $(5, 12, 6, 13, 11)$, $(13, 7, 12, 15, 14)$, $(8, 11, 9, 14, 10)$, $(6, 11, 17, 16, 14)$, $(8, 15, 10, 17, 14)$, $(8, 13, 15, 9, 16)$, $(15, 18, 10, 16)$
$J_{13} \to 3^{30}, 14^*$	$(8, 12, 16, 20, 19, 15, 10, 18, 14, 11, 13, 21, 17, 9)$, $(0, 5, 7)$, $(7, 1, 8)$, $(1, 6, 9)$, $(2, 8, 10)$, $(3, 8, 11)$, $(2, 7, 9)$, $(9, 3, 10)$, $(10, 4, 11)$, $(4, 9, 12)$, $(5, 11, 9)$, $(5, 10, 12)$, $(5, 13, 8)$, $(6, 13, 10)$, $(6, 14, 8)$, $(11, 6, 12)$, $(7, 15, 11)$, $(12, 7, 13)$, $(7, 14, 10)$, $(8, 15, 16)$, $(9, 14, 16)$, $(9, 15, 13)$, $(12, 17, 20)$, $(13, 17, 14)$, $(12, 15, 14)$, $(11, 17, 19)$, $(16, 10, 17)$, $(18, 12, 19)$, $(13, 19, 16)$, $(15, 18, 17)$, $(11, 16, 18)$
$J_4^+ \to 4^7, 4^{+*}$	$[8, 12, 4]$, $[7, 11, 3]$, $[6, 10, 2]$, $[5, 9, 1]$, $(0, 5, 1, 7)$, $(5, 8, 9, 7)$, $(2, 7, 3, 9)$, $(1, 6, 2, 8)$, $(6, 9, 10, 8)$, $(10, 4, 8, 11)$, $(7, 10, 3, 8)$
$J_5^+ \to 5^7, 5^{+*}$	$[8, 12, 4]$, $[7, 11, 3]$, $[6, 10, 2]$, $[1, 9, 13, 5]$, $(0, 5, 8, 10, 7)$, $(3, 10, 4, 11, 9)$, $(5, 1, 8, 2, 9)$, $(1, 6, 8, 3, 7)$, $(7, 2, 6, 9, 8)$, $(5, 11, 12, 9, 7)$, $(9, 4, 8, 11, 10)$
$J_6^+ \to 3^{14}, 6^{+*}$	$[8, 12, 4]$, $[7, 11, 3]$, $[2, 9, 13, 10, 14, 6]$, $[1, 5]$, $(1, 7, 9)$, $(7, 2, 8)$, $(0, 5, 7)$, $(7, 3, 10)$, $(6, 2, 10)$, $(1, 6, 8)$, $(8, 3, 9)$, $(5, 11, 8)$, $(8, 4, 10)$, $(4, 9, 11)$, $(5, 12, 13)$, $(6, 12, 9)$, $(5, 10, 9)$, $(10, 12, 11)$
$J_2^+ \to 3^5, 1^+$	$[4, 8]$, $[3, 7]$, $[6, 10, 2]$, $[1, 5]$, $(0, 5, 7)$, $(7, 1, 8)$, $(5, 9, 8)$, $(6, 2, 8)$, $(1, 6, 9)$
$J_2^+ \to 5^3, 1^+$	$[4, 8]$, $[3, 7]$, $[2, 6]$, $[5, 9, 1]$, $(0, 5, 1, 8, 7)$, $(5, 8, 6, 1, 7)$, $(8, 2, 10, 6, 9)$
$J_4^+ \to 3^{10}, 2^+$	$[8, 12, 4]$, $[3, 7]$, $[6, 9, 2]$, $[1, 5]$, $(0, 5, 7)$, $(6, 2, 10)$, $(1, 6, 8)$, $(1, 7, 9)$, $(7, 11, 10)$, $(7, 2, 8)$, $(9, 3, 10)$, $(8, 4, 10)$, $(3, 8, 11)$, $(5, 9, 8)$
$J_4^+ \to 5^6, 2^+$	$[4, 8]$, $[3, 7]$, $[6, 10, 2]$, $[5, 9, 1]$, $(0, 5, 1, 8, 7)$, $(5, 8, 6, 1, 7)$, $(8, 10, 7, 2, 9)$, $(7, 11, 3, 10, 9)$, $(6, 2, 8, 3, 9)$, $(10, 4, 12, 8, 11)$
$J_6^+ \to 3^{15}, 3^+$	$[4, 8]$, $[3, 7]$, $[2, 9, 13, 12, 6]$, $[1, 5]$, $(0, 5, 7)$, $(11, 5, 12)$, $(5, 10, 13)$, $(9, 3, 10)$, $(6, 2, 8)$, $(7, 11, 9)$, $(7, 1, 8)$, $(3, 8, 11)$, $(10, 4, 11)$, $(2, 7, 10)$, $(8, 12, 10)$, $(4, 9, 12)$, $(5, 9, 8)$, $(1, 6, 9)$, $(6, 14, 10)$
$J_6^+ \to 5^9, 3^+$	$[4, 8]$, $[3, 7]$, $[2, 6]$, $[1, 9, 10, 11, 5]$, $(0, 5, 1, 8, 7)$, $(5, 8, 6, 1, 7)$, $(6, 10, 2, 7, 9)$, $(7, 11, 3, 8, 10)$, $(3, 10, 4, 11, 9)$, $(8, 11, 12, 4, 9)$, $(5, 12, 8, 2, 9)$, $(12, 6, 14, 10, 13)$, $(9, 13, 5, 10, 12)$
$J_5^+ \to 3^{13}, 1^+$	$[4, 8]$, $[3, 7]$, $[6, 9, 2]$, $[1, 5]$, $(1, 7, 9)$, $(0, 5, 7)$, $(1, 6, 8)$, $(7, 11, 10)$, $(7, 2, 8)$, $(6, 2, 10)$, $(9, 4, 10)$, $(3, 8, 10)$, $(8, 12, 9)$, $(4, 11, 12)$, $(5, 11, 8)$, $(3, 9, 11)$, $(5, 13, 9)$
$J_8^+ \to 5^{12}, 4^+$	$[4, 8]$, $[3, 7]$, $[2, 6]$, $[1, 9, 10, 14, 13, 5]$, $(0, 5, 1, 8, 7)$, $(5, 8, 6, 1, 7)$, $(6, 10, 2, 7, 9)$, $(7, 13, 6, 14, 11)$, $(2, 9, 3, 11, 8)$, $(4, 11, 6, 12, 10)$, $(7, 12, 8, 3, 10)$, $(8, 16, 12, 13, 10)$, $(5, 10, 11, 13, 9)$, $(4, 9, 11, 5, 12)$, $(11, 15, 7, 14, 12)$, $(8, 14, 15, 12, 9)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

$J_7^+ \to 3^{18}, 2^+$	$[4,8]$, $[3,7]$, $[2,10,14,6]$, $[1,5]$, $(0,5,7)$, $(7,1,8)$, $(1,6,9)$, $(2,7,9)$, $(7,15,11)$, $(8,12,9)$, $(4,10,12)$, $(6,11,10)$, $(6,2,8)$, $(7,13,10)$, $(5,10,8)$, $(9,3,10)$, $(3,8,11)$, $(4,9,11)$, $(12,6,13)$, $(11,5,12)$, $(5,13,9)$, $(11,14,13)$

Table A.18: Table of decompositions of $J_n^{\{1,2,3,4,5,6,7,8\}}$

A.2 Data for Section 1.6.2

$L_2 \to 3, 4^+, 1^H$	$[2,4,1,0,3]$, $[3,4]$, $(1,2,3)$
$L_2 \to 4, 3^+, 1^H$	$[2,4,1,3]$, $[3,4]$, $(0,1,2,3)$
$L_2 \to 5, 2^+, 1^H$	$[2,1,3]$, $[3,4]$, $(0,1,5,2,3)$
$L_2 \to 3^2, 1^+, 1^H$	$[2,3]$, $[3,4]$, $(0,1,3)$, $(1,2,5)$
$L_3 \to 4, 5^+, 2^H$	$[3,0,1,2,5,4]$, $[5,3,4]$, $(1,3,2,4)$
$L_3 \to 3^2, 3^+, 2^H$	$[3,2,5,4]$, $[5,3,4]$, $(0,1,3)$, $(1,2,4)$
$L_3 \to 3, 4, 2^+, 2^H$	$[3,2,4]$, $[5,3,4]$, $(0,1,3)$, $(1,2,5,4)$
$L_4 \to 5, 6^+, 3^H$	$[4,2,1,0,3,6,5]$, $[6,4,3,5]$, $(1,3,2,5,4)$
$L_4 \to 4, 5, 2^+, 3^H$	$[4,2,5]$, $[6,4,3,5]$, $(0,1,2,3)$, $(1,3,6,5,4)$
$L_4 \to 3^3, 2^+, 3^H$	$[4,6,5]$, $[6,3,4,5]$, $(0,1,3)$, $(1,2,4)$, $(2,3,5)$
$L_5 \to 3^2, 4, 3^+, 4^H$	$[5,2,3,6]$, $[7,5,3,4,6]$, $(0,1,3)$, $(1,2,4)$, $(4,5,6,7)$
$P_3 \to 4, 2^+, 3^H$	$[0,3]$, $[1,4]$, $[1,3,5]$, $[2,4]$, $(2,3,4,5)$
$P_4 \to 4, 4^+, 4^H$	$[0,3,6,5]$, $[1,4]$, $[1,3,5]$, $[2,4,6]$, $(2,3,4,5)$
$P_4 \to 5, 3^+, 4^H$	$[0,3,5]$, $[1,4]$, $[1,3,4,6]$, $[2,5]$, $(2,3,6,5,4)$
$P_5 \to 5, 5^+, 5^H$	$[0,3,5]$, $[1,4,7,6]$, $[1,3,4,6]$, $[2,5,7]$, $(2,3,6,5,4)$
$P_5 \to 3^2, 4^+, 5^H$	$[0,3,5]$, $[1,4,6]$, $[1,3,6]$, $[2,5,4,7]$, $(2,3,4)$, $(5,6,7)$
$P_6 \to 3^2, 6^+, 6^H$	$[0,3,5,8,7]$, $[1,4,6]$, $[1,3,6,8]$, $[2,5,4,7]$, $(2,3,4)$, $(5,6,7)$
$Q_4 \to 3, 2^+, 5^H$	$[0,3,1]$, $[1,4,6,3,5,2]$, $(2,3,4)$
$Q_5 \to 4, 3^+, 6^H$	$[0,3,4,1]$, $[1,3,6,4,7,5,2]$, $(2,3,5,4)$
$Q_6 \to 5, 4^+, 7^H$	$[0,3,5,4,1]$, $[1,3,6,8,5,7,4,2]$, $(2,3,4,6,5)$
$R_5 \to 5^2, 5^H$	$[0,3,6,4,7,5,2]$, $(0,1,4,3,2)$, $(1,2,4,5,3)$
$R_6 \to 3, 4, 5, 6^H$	$[0,3,6,8,5,7,4,2]$, $(0,1,3,2)$, $(1,2,5,3,4)$, $(4,5,6)$
$R_6 \to 4^3, 6^H$	$[0,3,6,8,5,7,4,2]$, $(0,1,3,2)$, $(1,2,5,4)$, $(3,4,6,5)$
$R_6 \to 3^4, 6^H$	$[0,3,6,8,5,7,4,2]$, $(0,1,2)$, $(1,3,4)$, $(2,3,5)$, $(4,5,6)$

Table A.19: Decompositions of graphs for section 1.6.2

The next five tables contain all decompositions required for Lemma 1.6.39. We make extensive use of concatenation to obtain the results presented in these tables. We therefore use the notation $(G \to M) \oplus (H \to M')$ for the concatenation of the decomposition $G \to M$ with the decomposition $H \to M'$. Various different forms of concatenation are defined and used in Section 1.6.2. In the table, the particular form of concatenation being used is well-defined by the decompositions it involves. We also define the notation $a \cdot (G \to M)$ to mean $(G \to M) \oplus (G \to M) \oplus \cdots \oplus (G \to M)$, where a is the number of copies of the decomposition $G \to M$ involved.

$J_7 \to 4^2, 6, 7$	$(L_2 \to 4, 3^+, 1^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_7 \to 3, 5, 6, 7$	$(L_3 \to 5, 4^+, 2^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_8 \to 4, 5, 7, 8$	$(L_3 \to 5, 4^+, 2^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_8 \to 3^3, 7, 8$	$(5,2,0,3,1,4,6,7)$, $(5,3,6)$, $(2,3,4)$, $(9,6,8,5,4,7,10)$, $(7,8,9)$
$J_9 \to 5^2, 8, 9$	$(L_3 \to 5, 4^+, 2^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_9 \to 3^2, 4, 8, 9$	$(L_2 \to 3, 4^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{10} \to 3, 4^2, 9, 10$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{10} \to 3^2, 5, 9, 10$	$(L_2 \to 3, 4^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{11} \to 4^3, 10, 11$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{11} \to 3, 4, 5, 10, 11$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.20: Table of decompositions for Lemma 1.6.39 with $j = 1$

$J_{11} \to 3^4, 10, 11$	$(L_2 \to 3, 4^+, 1^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 4^2, 5, 11, 12$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{12} \to 3, 5^2, 11, 12$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 3^3, 4, 11, 12$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 4, 5^2, 12, 13$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3^2, 4^2, 12, 13$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3^3, 5, 12, 13$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 5^3, 13, 14$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{14} \to 3, 4^3, 13, 14$	$(L_2 \to 4, 3^+, 1^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 3^2, 4, 5, 13, 14$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3^5, 13, 14$	$(11, 8, 6, 4, 1, 3, 0, 2, 5, 7, 10, 9, 12, 13), (2, 3, 4), (5, 3, 6),$ $(15, 12, 14, 11, 9, 6, 7, 4, 5, 8, 10, 13, 16), (7, 8, 9), (10, 11, 12), (13, 14, 15)$
$J_{15} \to 4^4, 14, 15$	$(L_2 \to 4, 3^+, 1^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{15} \to 3, 4^2, 5, 14, 15$	$(L_3 \to 5, 4^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^2, 5^2, 14, 15$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{15} \to 3^4, 4, 14, 15$	$(L_2 \to 3, 4^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 4^3, 5, 15, 16$	$(L_3 \to 5, 4^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3, 4, 5^2, 15, 16$	$(L_3 \to 5, 4^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^3, 4^2, 15, 16$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^4, 5, 15, 16$	$(L_2 \to 3, 4^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.20: Table of decompositions for Lemma 1.6.39 with $j = 1$

$J_8 \to 5^2, 6, 8$	$(L_2 \to 5, 2^+, 1^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_8 \to 3^2, 4, 6, 8$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_9 \to 3, 4^2, 7, 9$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_9 \to 3^2, 5, 7, 9$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{10} \to 4^3, 8, 10$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{10} \to 3, 4, 5, 8, 10$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{10} \to 3^4, 8, 10$	$(8, 5, 2, 0, 3, 1, 4, 7, 6, 9), (2, 3, 4), (5, 3, 6), (10, 9, 7, 5, 4, 6, 8, 11),$ $(8, 7, 10), (11, 9, 12)$
$J_{11} \to 4^2, 5, 9, 11$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{11} \to 3, 5^2, 9, 11$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{11} \to 3^3, 4, 9, 11$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 4, 5^2, 10, 12$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{12} \to 3^2, 4^2, 10, 12$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{12} \to 3^3, 5, 10, 12$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 5^3, 11, 13$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{13} \to 3, 4^3, 11, 13$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 3^2, 4, 5, 11, 13$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3^5, 11, 13$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 4^4, 12, 14$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.21: Table of decompositions for Lemma 1.6.39 with $j = 2$

$J_{14} \to 3, 4^2, 5, 12, 14$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 3^2, 5^2, 12, 14$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{14} \to 3^4, 4, 12, 14$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{15} \to 4^3, 5, 13, 15$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{15} \to 3, 4, 5^2, 13, 15$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^3, 4^2, 13, 15$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^4, 5, 13, 15$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{16} \to 4^2, 5^2, 14, 16$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3, 5^3, 14, 16$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^2, 4^3, 14, 16$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3^3, 4, 5, 14, 16$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^6, 14, 16$	$(L_2 \to 3, 4^+, 1^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 4, 5^3, 15, 17$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3, 4^4, 15, 17$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^2, 4^2, 5, 15, 17$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3^3, 5^2, 15, 17$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^5, 4, 15, 17$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 5^4, 16, 18$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{18} \to 4^5, 16, 18$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3, 4^3, 5, 16, 18$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^2, 4, 5^2, 16, 18$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3^4, 4^2, 16, 18$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3^5, 5, 16, 18$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 4^4, 5, 17, 19$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3, 4^2, 5^2, 17, 19$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^2, 5^3, 17, 19$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{19} \to 3^3, 4^3, 17, 19$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^4, 4, 5, 17, 19$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3^7, 17, 19$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 4^3, 5^2, 18, 20$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3, 4, 5^3, 18, 20$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3^2, 4^4, 18, 20$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3^3, 4^2, 5, 18, 20$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.21: Table of decompositions for Lemma 1.6.39 with $j = 2$

$J_{20} \to 3^4, 5^2, 18, 20$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{20} \to 3^6, 4, 18, 20$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 4^2, 5^3, 19, 21$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3, 5^4, 19, 21$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3, 4^5, 19, 21$	$(L_2 \to 4, 3^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^2, 4^3, 5, 19, 21$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3^3, 4, 5^2, 19, 21$	$(L_3 \to 3^2, 3^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^5, 4^2, 19, 21$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^6, 5, 19, 21$	$(L_3 \to 3^2, 3^+, 2^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$

Table A.21: Table of decompositions for Lemma 1.6.39 with $j = 2$

$J_9 \to 4^3, 6, 9$	$(7, 6, 4, 1, 3, 0, 2, 5, 8), (3, 2, 4, 5), (8, 6, 3, 4, 7, 10), (7, 5, 6, 9), (10, 9, 8, 11)$
$J_9 \to 3, 4, 5, 6, 9$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_9 \to 3^4, 6, 9$	$(6, 7, 4, 1, 3, 0, 2, 5, 8), (2, 3, 4), (5, 3, 6), (8, 7, 5, 4, 6, 9), (9, 7, 10), (10, 8, 11)$
$J_{10} \to 4^2, 5, 7, 10$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{10} \to 3, 5^2, 7, 10$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{10} \to 3^3, 4, 7, 10$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{11} \to 4, 5^2, 8, 11$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{11} \to 3^2, 4^2, 8, 11$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{11} \to 3^3, 5, 8, 11$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 5^3, 9, 12$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{12} \to 3, 4^3, 9, 12$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{12} \to 3^2, 4, 5, 9, 12$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 3^5, 9, 12$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 4^4, 10, 13$	$(L_2 \to 4, 3^+, 1^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3, 4^2, 5, 10, 13$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3^2, 5^2, 10, 13$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{13} \to 3^4, 4, 10, 13$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 4^3, 5, 11, 14$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3, 4, 5^2, 11, 14$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3^3, 4^2, 11, 14$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3^4, 5, 11, 14$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{15} \to 4^2, 5^2, 12, 15$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{15} \to 3, 5^3, 12, 15$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^2, 4^3, 12, 15$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^3, 4, 5, 12, 15$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.22: Table of decompositions for Lemma 1.6.39 with $j = 3$

$J_{15} \to 3^6, 12, 15$	$(13,10,8,5,2,0,3,1,4,7,6,9,12,11,14), (2,3,4), (5,3,6), (16,13,11,8,6,4,5,7,10,12,14,17), (7,8,9), (9,10,11), (13,12,15), (14,15,16)$
$J_{16} \to 4, 5^3, 13, 16$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3, 4^4, 13, 16$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3^2, 4^2, 5, 13, 16$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^3, 5^2, 13, 16$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^5, 4, 13, 16$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 5^4, 14, 17$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{17} \to 4^5, 14, 17$	$(L_2 \to 4, 3^+, 1^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3, 4^3, 5, 14, 17$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3^2, 4, 5^2, 14, 17$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^4, 4^2, 14, 17$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^5, 5, 14, 17$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 4^4, 5, 15, 18$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3, 4^2, 5^2, 15, 18$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3^2, 5^3, 15, 18$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{18} \to 3^3, 4^3, 15, 18$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3^4, 4, 5, 15, 18$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^7, 15, 18$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 4^3, 5^2, 16, 19$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3, 4, 5^3, 16, 19$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3^2, 4^4, 16, 19$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^3, 4^2, 5, 16, 19$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3^4, 5^2, 16, 19$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{19} \to 3^6, 4, 16, 19$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 4^2, 5^3, 17, 20$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3, 5^4, 17, 20$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3, 4^5, 17, 20$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3^2, 4^3, 5, 17, 20$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.22: Table of decompositions for Lemma 1.6.39 with $j = 3$

$J_{20} \to 3^3, 4, 5^2, 17, 20$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3^5, 4^2, 17, 20$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3^6, 5, 17, 20$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{21} \to 4, 5^4, 18, 21$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 4^6, 18, 21$	$(L_2 \to 4, 3^+, 1^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3, 4^4, 5, 18, 21$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3^2, 4^2, 5^2, 18, 21$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^3, 5^3, 18, 21$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^4, 4^3, 18, 21$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^5, 4, 5, 18, 21$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3^8, 18, 21$	$(L_2 \to 3, 4^+, 1^H) \oplus 3 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 5^5, 19, 22$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{22} \to 4^5, 5, 19, 22$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3, 4^3, 5^2, 19, 22$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3^2, 4, 5^3, 19, 22$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^3, 4^4, 19, 22$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3^4, 4^2, 5, 19, 22$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^5, 5^2, 19, 22$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^7, 4, 19, 22$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 4^4, 5^2, 20, 23$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3, 4^2, 5^3, 20, 23$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3^2, 5^4, 20, 23$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{23} \to 3^2, 4^5, 20, 23$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^3, 4^3, 5, 20, 23$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3^4, 4, 5^2, 20, 23$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^6, 4^2, 20, 23$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^7, 5, 20, 23$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 4^3, 5^3, 21, 24$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3, 4, 5^4, 21, 24$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.22: Table of decompositions for Lemma 1.6.39 with $j = 3$

$J_{24} \to 3, 4^6, 21, 24$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3^2, 4^4, 5, 21, 24$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^3, 4^2, 5^2, 21, 24$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3^4, 5^3, 21, 24$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{24} \to 3^5, 4^3, 21, 24$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3^6, 4, 5, 21, 24$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^9, 21, 24$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 4^2, 5^4, 22, 25$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 4^7, 22, 25$	$(L_2 \to 4, 3^+, 1^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3, 5^5, 22, 25$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3, 4^5, 5, 22, 25$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^2, 4^3, 5^2, 22, 25$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^3, 4, 5^3, 22, 25$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^4, 4^4, 22, 25$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^5, 4^2, 5, 22, 25$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^6, 5^2, 22, 25$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{25} \to 3^8, 4, 22, 25$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 4, 5^5, 23, 26$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 4^6, 5, 23, 26$	$(L_2 \to 5, 2^+, 1^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3, 4^4, 5^2, 23, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^2, 4^2, 5^3, 23, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^3, 5^4, 23, 26$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^3, 4^5, 23, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^4, 4^3, 5, 23, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^5, 4, 5^2, 23, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^7, 4^2, 23, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^8, 5, 23, 26$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$

Table A.22: Table of decompositions for Lemma 1.6.39 with $j = 3$

$J_{10} \to 4, 5^2, 6, 10$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{10} \to 3^2, 4^2, 6, 10$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{10} \to 3^3, 5, 6, 10$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{11} \to 5^3, 7, 11$	$(8, 5, 2, 0, 3, 1, 4, 7, 6, 9, 10), (5, 4, 2, 3, 6), (7, 5, 3, 4, 6, 8, 9),$ $(10, 7, 8, 11, 12), (12, 9, 11, 10, 13)$
$J_{11} \to 3, 4^3, 7, 11$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{11} \to 3^2, 4, 5, 7, 11$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{11} \to 3^5, 7, 11$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 4^4, 8, 12$	$(10, 9, 7, 6, 4, 1, 3, 0, 2, 5, 8, 11), (3, 2, 4, 5), (11, 9, 6, 3, 4, 7, 10, 13),$ $(6, 5, 7, 8), (10, 8, 9, 12), (13, 12, 11, 14)$
$J_{12} \to 3, 4^2, 5, 8, 12$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 3^2, 5^2, 8, 12$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{12} \to 3^4, 4, 8, 12$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 4^3, 5, 9, 13$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3, 4, 5^2, 9, 13$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 3^3, 4^2, 9, 13$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 3^4, 5, 9, 13$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 4^2, 5^2, 10, 14$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3, 5^3, 10, 14$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 3^2, 4^3, 10, 14$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3^3, 4, 5, 10, 14$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 3^6, 10, 14$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 4, 5^3, 11, 15$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{15} \to 3, 4^4, 11, 15$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to$ $4, 3^+, 6^H)$
$J_{15} \to 3^2, 4^2, 5, 11, 15$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{15} \to 3^3, 5^2, 11, 15$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{15} \to 3^5, 4, 11, 15$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 5^4, 12, 16$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{16} \to 4^5, 12, 16$	$(L_2 \to 4, 3^+, 1^H) \oplus 3 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3, 4^3, 5, 12, 16$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^2, 4, 5^2, 12, 16$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{16} \to 3^4, 4^2, 12, 16$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^5, 5, 12, 16$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{17} \to 4^4, 5, 13, 17$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3, 4^2, 5^2, 13, 17$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to$ $3, 2^+, 5^H)$
$J_{17} \to 3^2, 5^3, 13, 17$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to$ $5, 4^+, 7^H)$
$J_{17} \to 3^3, 4^3, 13, 17$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^4, 4, 5, 13, 17$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to$ $3, 2^+, 5^H)$
$J_{17} \to 3^7, 13, 17$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to$ $3, 2^+, 5^H)$
$J_{18} \to 4^3, 5^2, 14, 18$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3, 4, 5^3, 14, 18$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^2, 4^4, 14, 18$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.23: Table of decompositions for Lemma 1.6.39 with $j = 4$

$J_{18} \to 3^3, 4^2, 5, 14, 18$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^4, 5^2, 14, 18$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^6, 4, 14, 18$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 4^2, 5^3, 15, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3, 5^4, 15, 19$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3, 4^5, 15, 19$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3^2, 4^3, 5, 15, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3^3, 4, 5^2, 15, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^5, 4^2, 15, 19$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^6, 5, 15, 19$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 4, 5^4, 16, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{20} \to 4^6, 16, 20$	$(L_2 \to 4, 3^+, 1^H) \oplus 3 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3, 4^4, 5, 16, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3^2, 4^2, 5^2, 16, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3^3, 5^3, 16, 20$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{20} \to 3^4, 4^3, 16, 20$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3^5, 4, 5, 16, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3^8, 16, 20$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 5^5, 17, 21$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{21} \to 4^5, 5, 17, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3, 4^3, 5^2, 17, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^2, 4, 5^3, 17, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{21} \to 3^3, 4^4, 17, 21$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^4, 4^2, 5, 17, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3^5, 5^2, 17, 21$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{21} \to 3^7, 4, 17, 21$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 4^4, 5^2, 18, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3, 4^2, 5^3, 18, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^2, 5^4, 18, 22$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{22} \to 3^2, 4^5, 18, 22$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3^3, 4^3, 5, 18, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.23: Table of decompositions for Lemma 1.6.39 with $j = 4$

$J_{22} \to 3^4, 4, 5^2, 18, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{22} \to 3^6, 4^2, 18, 22$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^7, 5, 18, 22$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{23} \to 4^3, 5^3, 19, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3, 4, 5^4, 19, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3, 4^6, 19, 23$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3^2, 4^4, 5, 19, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3^3, 4^2, 5^2, 19, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^4, 5^3, 19, 23$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^5, 4^3, 19, 23$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^6, 4, 5, 19, 23$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^9, 19, 23$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 4^2, 5^4, 20, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 4^7, 20, 24$	$(L_2 \to 4, 3^+, 1^H) \oplus 3 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3, 5^5, 20, 24$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3, 4^5, 5, 20, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^2, 4^3, 5^2, 20, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3^3, 4, 5^3, 20, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^4, 4^4, 20, 24$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^5, 4^2, 5, 20, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^6, 5^2, 20, 24$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^8, 4, 20, 24$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 4, 5^5, 21, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{25} \to 4^6, 5, 21, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3, 4^4, 5^2, 21, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^2, 4^2, 5^3, 21, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^3, 5^4, 21, 25$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{25} \to 3^3, 4^5, 21, 25$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^4, 4^3, 5, 21, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^5, 4, 5^2, 21, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.23: Table of decompositions for Lemma 1.6.39 with $j = 4$

$J_{25} \to 3^7, 4^2, 21, 25$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^8, 5, 21, 25$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 5^6, 22, 26$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{26} \to 4^5, 5^2, 22, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3, 4^3, 5^3, 22, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^2, 4, 5^4, 22, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{26} \to 3^2, 4^6, 22, 26$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^3, 4^4, 5, 22, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^4, 4^2, 5^2, 22, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^5, 5^3, 22, 26$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{26} \to 3^6, 4^3, 22, 26$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^7, 4, 5, 22, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^{10}, 22, 26$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 4^4, 5^3, 23, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 3, 4^2, 5^4, 23, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3, 4^7, 23, 27$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 3^2, 5^5, 23, 27$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{27} \to 3^2, 4^5, 5, 23, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 3^3, 4^3, 5^2, 23, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^4, 4, 5^3, 23, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{27} \to 3^5, 4^4, 23, 27$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^6, 4^2, 5, 23, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 3^7, 5^2, 23, 27$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{27} \to 3^9, 4, 23, 27$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 4^3, 5^4, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{28} \to 4^8, 24, 28$	$(L_2 \to 4, 3^+, 1^H) \oplus 3 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{28} \to 3, 4, 5^5, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3, 4^6, 5, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.23: Table of decompositions for Lemma 1.6.39 with $j = 4$

$J_{28} \to 3^2, 4^4, 5^2, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{28} \to 3^3, 4^2, 5^3, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^4, 5^4, 24, 28$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^4, 4^5, 24, 28$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^5, 4^3, 5, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^6, 4, 5^2, 24, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{28} \to 3^8, 4^2, 24, 28$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^9, 5, 24, 28$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{29} \to 4^2, 5^5, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 4^7, 5, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3, 5^6, 25, 29$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3, 4^5, 5^2, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^2, 4^3, 5^3, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3^3, 4, 5^4, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^3, 4^6, 25, 29$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^4, 4^4, 5, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3^5, 4^2, 5^2, 25, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^6, 5^3, 25, 29$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^7, 4^3, 25, 29$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^8, 4, 5, 25, 29$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^{11}, 25, 29$	$(L_2 \to 3^2, 1^+, 1^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 4, 5^6, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{30} \to 4^6, 5^2, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3, 4^4, 5^3, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^2, 4^2, 5^4, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3^2, 4^7, 26, 30$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3^3, 5^5, 26, 30$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{30} \to 3^3, 4^5, 5, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^4, 4^3, 5^2, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.23: Table of decompositions for Lemma 1.6.39 with $j = 4$

$J_{30} \to 3^5, 4, 5^3, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^6, 4^4, 26, 30$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^7, 4^2, 5, 26, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^8, 5^2, 26, 30$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^{10}, 4, 26, 30$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 5^7, 27, 31$	$(L_2 \to 5, 2^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{31} \to 4^5, 5^3, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3, 4^3, 5^4, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3, 4^8, 27, 31$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3^2, 4, 5^5, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{31} \to 3^2, 4^6, 5, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3^3, 4^4, 5^2, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^4, 4^2, 5^3, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3^5, 5^4, 27, 31$	$(L_4 \to 3^3, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{31} \to 3^5, 4^5, 27, 31$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^6, 4^3, 5, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3^7, 4, 5^2, 27, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^9, 4^2, 27, 31$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^{10}, 5, 27, 31$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.23: Table of decompositions for Lemma 1.6.39 with $j = 4$

$J_{11} \to 4^4, 6, 11$	$(9, 8, 6, 4, 1, 3, 0, 2, 5, 7, 10), (3, 2, 4, 5), (6, 3, 4, 7), (9, 7, 8, 11),$ $(10, 8, 5, 6, 9, 12), (12, 11, 10, 13)$
$J_{11} \to 3, 4^2, 5, 6, 11$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{11} \to 3^2, 5^2, 6, 11$	$(8, 5, 2, 0, 3, 1, 4, 7, 6, 9, 10), (2, 3, 4), (5, 3, 6), (6, 4, 5, 7, 8),$ $(12, 11, 9, 7, 10, 13), (10, 11, 8, 9, 12)$
$J_{11} \to 3^4, 4, 6, 11$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 4^3, 5, 7, 12$	$(10, 9, 6, 7, 4, 1, 3, 0, 2, 5, 8, 11), (6, 3, 2, 4, 5, 7, 8), (4, 3, 5, 6),$ $(10, 7, 9, 12), (11, 9, 8, 10, 13), (13, 12, 11, 14)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{12} \to 3, 4, 5^2, 7, 12$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 3^3, 4^2, 7, 12$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{12} \to 3^4, 5, 7, 12$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 4^2, 5^2, 8, 13$	$(11, 10, 7, 4, 1, 3, 0, 2, 5, 6, 8, 9, 12), (8, 5, 3, 2, 4, 6, 9, 10), (5, 4, 3, 6, 7), (9, 7, 8, 11), (12, 10, 13, 11, 14), (14, 13, 12, 15)$
$J_{13} \to 3, 5^3, 8, 13$	$(11, 10, 7, 4, 1, 3, 0, 2, 5, 6, 8, 9, 12), (7, 8, 5, 3, 2, 4, 6, 9), (5, 4, 3, 6, 7), (13, 10, 8, 11, 14), (11, 9, 10, 12, 13), (14, 12, 15)$
$J_{13} \to 3^2, 4^3, 8, 13$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{13} \to 3^3, 4, 5, 8, 13$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{13} \to 3^6, 8, 13$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{14} \to 4, 5^3, 9, 14$	$(12, 11, 8, 5, 2, 0, 3, 1, 4, 7, 6, 9, 10, 13), (5, 4, 2, 3, 6), (9, 8, 6, 4, 3, 5, 7, 10, 11), (10, 8, 7, 9, 12), (13, 11, 14, 12, 15), (15, 14, 13, 16)$
$J_{14} \to 3, 4^4, 9, 14$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3^2, 4^2, 5, 9, 14$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{14} \to 3^3, 5^2, 9, 14$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{14} \to 3^5, 4, 9, 14$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 5^4, 10, 15$	$(12, 10, 9, 6, 7, 4, 1, 3, 0, 2, 5, 8, 11, 13, 14), (4, 2, 3, 5, 6), (12, 9, 7, 5, 4, 3, 6, 8, 10, 13), (9, 8, 7, 10, 11), (14, 11, 12, 15, 16), (16, 13, 15, 14, 17)$
$J_{15} \to 4^5, 10, 15$	$(13, 10, 12, 9, 6, 7, 4, 1, 3, 0, 2, 5, 8, 11, 14), (9, 8, 6, 3, 2, 4, 5, 7, 10, 11), (4, 3, 5, 6), (8, 7, 9, 10), (13, 11, 12, 15), (14, 12, 13, 16), (16, 15, 14, 17)$
$J_{15} \to 3, 4^3, 5, 10, 15$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^2, 4, 5^2, 10, 15$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{15} \to 3^4, 4^2, 10, 15$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{15} \to 3^5, 5, 10, 15$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{16} \to 4^4, 5, 11, 16$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{16} \to 3, 4^2, 5^2, 11, 16$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^2, 5^3, 11, 16$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{16} \to 3^3, 4^3, 11, 16$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^4, 4, 5, 11, 16$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{16} \to 3^7, 11, 16$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 4^3, 5^2, 12, 17$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3, 4, 5^3, 12, 17$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^2, 4^4, 12, 17$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{17} \to 3^3, 4^2, 5, 12, 17$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^4, 5^2, 12, 17$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{17} \to 3^6, 4, 12, 17$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{18} \to 4^2, 5^3, 13, 18$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3, 5^4, 13, 18$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3, 4^5, 13, 18$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3^2, 4^3, 5, 13, 18$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{18} \to 3^3, 4, 5^2, 13, 18$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^5, 4^2, 13, 18$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{18} \to 3^6, 5, 13, 18$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 4, 5^4, 14, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{19} \to 4^6, 14, 19$	$(L_2 \to 4, 3^+, 1^H) \oplus 4 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3, 4^4, 5, 14, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^2, 4^2, 5^2, 14, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{19} \to 3^3, 5^3, 14, 19$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{19} \to 3^4, 4^3, 14, 19$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^5, 4, 5, 14, 19$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{19} \to 3^8, 14, 19$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 5^5, 15, 20$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{20} \to 4^5, 5, 15, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3, 4^3, 5^2, 15, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3^2, 4, 5^3, 15, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{20} \to 3^3, 4^4, 15, 20$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{20} \to 3^4, 4^2, 5, 15, 20$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{20} \to 3^5, 5^2, 15, 20$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{20} \to 3^7, 4, 15, 20$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 4^4, 5^2, 16, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{21} \to 3, 4^2, 5^3, 16, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^2, 5^4, 16, 21$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{21} \to 3^2, 4^5, 16, 21$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{21} \to 3^3, 4^3, 5, 16, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^4, 4, 5^2, 16, 21$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{21} \to 3^6, 4^2, 16, 21$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{21} \to 3^7, 5, 16, 21$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{22} \to 4^3, 5^3, 17, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3, 4, 5^4, 17, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3, 4^6, 17, 22$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3^2, 4^4, 5, 17, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{22} \to 3^3, 4^2, 5^2, 17, 22$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^4, 5^3, 17, 22$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^5, 4^3, 17, 22$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^6, 4, 5, 17, 22$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{22} \to 3^9, 17, 22$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 4^2, 5^4, 18, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 4^7, 18, 23$	$(L_2 \to 4, 3^+, 1^H) \oplus 4 \cdot (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3, 5^5, 18, 23$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3, 4^5, 5, 18, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^2, 4^3, 5^2, 18, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{23} \to 3^3, 4, 5^3, 18, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^4, 4^4, 18, 23$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^5, 4^2, 5, 18, 23$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^6, 5^2, 18, 23$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{23} \to 3^8, 4, 18, 23$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 4, 5^5, 19, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{24} \to 4^6, 5, 19, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3, 4^4, 5^2, 19, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^2, 4^2, 5^3, 19, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3^3, 5^4, 19, 24$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{24} \to 3^3, 4^5, 19, 24$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^4, 4^3, 5, 19, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{24} \to 3^5, 4, 5^2, 19, 24$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^7, 4^2, 19, 24$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{24} \to 3^8, 5, 19, 24$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 5^6, 20, 25$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{25} \to 4^5, 5^2, 20, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3, 4^3, 5^3, 20, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^2, 4, 5^4, 20, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{25} \to 3^2, 4^6, 20, 25$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^3, 4^4, 5, 20, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^4, 4^2, 5^2, 20, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{25} \to 3^5, 5^3, 20, 25$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{25} \to 3^6, 4^3, 20, 25$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^7, 4, 5, 20, 25$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{25} \to 3^{10}, 20, 25$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 4^4, 5^3, 21, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3, 4^2, 5^4, 21, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3, 4^7, 21, 26$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^2, 5^5, 21, 26$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{26} \to 3^2, 4^5, 5, 21, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^3, 4^3, 5^2, 21, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^4, 4, 5^3, 21, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{26} \to 3^5, 4^4, 21, 26$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{26} \to 3^6, 4^2, 5, 21, 26$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{26} \to 3^7, 5^2, 21, 26$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{26} \to 3^9, 4, 21, 26$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 4^3, 5^4, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 4^8, 22, 27$	$(L_2 \to 4, 3^+, 1^H) \oplus 4 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 3, 4, 5^5, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3, 4^6, 5, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^2, 4^4, 5^2, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{27} \to 3^3, 4^2, 5^3, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^4, 5^4, 22, 27$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^4, 4^5, 22, 27$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^5, 4^3, 5, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^6, 4, 5^2, 22, 27$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{27} \to 3^8, 4^2, 22, 27$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{27} \to 3^9, 5, 22, 27$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{28} \to 4^2, 5^5, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{28} \to 4^7, 5, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{28} \to 3, 5^6, 23, 28$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3, 4^5, 5^2, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^2, 4^3, 5^3, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{28} \to 3^3, 4, 5^4, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^3, 4^6, 23, 28$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^4, 4^4, 5, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{28} \to 3^5, 4^2, 5^2, 23, 28$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^6, 5^3, 23, 28$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^7, 4^3, 23, 28$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^8, 4, 5, 23, 28$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{28} \to 3^{11}, 23, 28$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 4, 5^6, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{29} \to 4^6, 5^2, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3, 4^4, 5^3, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^2, 4^2, 5^4, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3^2, 4^7, 24, 29$	$(L_5 \to 3^2, 4, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3^3, 5^5, 24, 29$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{29} \to 3^3, 4^5, 5, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^4, 4^3, 5^2, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{29} \to 3^5, 4, 5^3, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^6, 4^4, 24, 29$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^7, 4^2, 5, 24, 29$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^8, 5^2, 24, 29$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{29} \to 3^{10}, 4, 24, 29$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 5^7, 25, 30$	$(L_2 \to 5, 2^+, 1^H) \oplus 3 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{30} \to 4^5, 5^3, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3, 4^3, 5^4, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{30} \to 3, 4^8, 25, 30$	$(L_3 \to 3, 4, 2^+, 2^H) \oplus 2 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3^2, 4, 5^5, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{30} \to 3^2, 4^6, 5, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3^3, 4^4, 5^2, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^4, 4^2, 5^3, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3^5, 5^4, 25, 30$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{30} \to 3^5, 4^5, 25, 30$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^6, 4^3, 5, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{30} \to 3^7, 4, 5^2, 25, 30$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^9, 4^2, 25, 30$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{30} \to 3^{10}, 5, 25, 30$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 4^4, 5^4, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 4^9, 26, 31$	$(L_2 \to 4, 3^+, 1^H) \oplus 4 \cdot (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3, 4^2, 5^5, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3, 4^7, 5, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^2, 5^6, 26, 31$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{31} \to 3^2, 4^5, 5^2, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3^3, 4^3, 5^3, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^4, 4, 5^4, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{31} \to 3^4, 4^6, 26, 31$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^5, 4^4, 5, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^6, 4^2, 5^2, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{31} \to 3^7, 5^3, 26, 31$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{31} \to 3^8, 4^3, 26, 31$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

$J_{31} \to 3^9, 4, 5, 26, 31$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{31} \to 3^{12}, 26, 31$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 4^3, 5^5, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{32} \to 4^8, 5, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{32} \to 3, 4, 5^6, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3, 4^6, 5^2, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 4 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3^2, 4^4, 5^3, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{32} \to 3^3, 4^2, 5^4, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3^3, 4^7, 27, 32$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 5 \cdot (P_4 \to 4, 4^+, 4^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3^4, 5^5, 27, 32$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 4 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3^4, 4^5, 5, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{32} \to 3^5, 4^3, 5^2, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus (P_4 \to 4, 4^+, 4^H) \oplus (P_5 \to 5, 5^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3^6, 4, 5^3, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{32} \to 3^7, 4^4, 27, 32$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_3 \to 4, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 4, 4^+, 4^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{32} \to 3^8, 4^2, 5, 27, 32$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus 2 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{32} \to 3^9, 5^2, 27, 32$	$(L_4 \to 3^3, 2^+, 3^H) \oplus (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_6 \to 5, 4^+, 7^H)$
$J_{32} \to 3^{11}, 4, 27, 32$	$(L_5 \to 4, 3^2, 3^+, 4^H) \oplus (P_5 \to 3^2, 4^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$
$J_{33} \to 4^2, 5^6, 28, 33$	$(L_4 \to 4, 5, 2^+, 3^H) \oplus 2 \cdot (P_4 \to 5, 3^+, 4^H) \oplus 3 \cdot (P_5 \to 5, 5^+, 5^H) \oplus (Q_5 \to 4, 3^+, 6^H)$
$J_{34} \to 3^{13}, 29, 34$	$(L_2 \to 3^2, 1^+, 1^H) \oplus 2 \cdot (P_5 \to 3^2, 4^+, 5^H) \oplus 3 \cdot (P_6 \to 3^2, 6^+, 6^H) \oplus (Q_4 \to 3, 2^+, 5^H)$

Table A.24: Table of decompositions for Lemma 1.6.39 with $j = 5$

t	Difference triples
11	$\{5, 24, 29\}$, $\{7, 20, 27\}$, $\{8, 23, 31\}$, $\{9, 21, 30\}$, $\{10, 26, 36\}$, $\{11, 28, 39\}$, $\{12, 25, 37\}$, $\{13, 22, 35\}$, $\{14, 18, 32\}$, $\{15, 19, 34\}$, $\{16, 17, 33\}$
12	$\{5, 21, 26\}$, $\{7, 27, 34\}$, $\{8, 23, 31\}$, $\{9, 24, 33\}$, $\{10, 28, 38\}$, $\{11, 29, 40\}$, $\{12, 30, 42\}$, $\{13, 19, 32\}$, $\{14, 25, 39\}$, $\{15, 22, 37\}$, $\{16, 20, 36\}$, $\{17, 18, 35\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
13	$\{5,20,25\}$, $\{7,28,35\}$, $\{8,26,34\}$, $\{9,29,38\}$, $\{10,23,33\}$, $\{11,31,42\}$, $\{12,32,44\}$, $\{13,30,43\}$, $\{14,27,41\}$, $\{15,21,36\}$, $\{16,24,40\}$, $\{17,22,39\}$, $\{18,19,37\}$
14	$\{5,18,23\}$, $\{7,30,37\}$, $\{8,31,39\}$, $\{9,26,35\}$, $\{10,28,38\}$, $\{11,25,36\}$, $\{12,33,45\}$, $\{13,34,47\}$, $\{14,32,46\}$, $\{15,29,44\}$, $\{16,27,43\}$, $\{17,24,41\}$, $\{19,21,40\}$, $\{20,22,42\}$
15	$\{5,33,38\}$, $\{7,13,20\}$, $\{8,34,42\}$, $\{9,28,37\}$, $\{10,31,41\}$, $\{11,29,40\}$, $\{12,27,39\}$, $\{14,35,49\}$, $\{15,36,51\}$, $\{16,32,48\}$, $\{17,30,47\}$, $\{18,26,44\}$, $\{19,24,43\}$, $\{21,25,46\}$, $\{22,23,45\}$
16	$\{5,12,17\}$, $\{7,34,41\}$, $\{8,31,39\}$, $\{9,35,44\}$, $\{10,30,40\}$, $\{11,32,43\}$, $\{13,29,42\}$, $\{14,36,50\}$, $\{15,37,52\}$, $\{16,38,54\}$, $\{18,33,51\}$, $\{19,27,46\}$, $\{20,25,45\}$, $\{21,28,49\}$, $\{22,26,48\}$, $\{23,24,47\}$
17	$\{5,9,14\}$, $\{7,34,41\}$, $\{8,37,45\}$, $\{10,36,46\}$, $\{11,31,42\}$, $\{12,32,44\}$, $\{13,30,43\}$, $\{15,39,54\}$, $\{16,40,56\}$, $\{17,38,55\}$, $\{18,35,53\}$, $\{19,33,52\}$, $\{20,28,48\}$, $\{21,26,47\}$, $\{22,29,51\}$, $\{23,27,50\}$, $\{24,25,49\}$
18	$\{5,30,35\}$, $\{7,29,36\}$, $\{8,31,39\}$, $\{9,41,50\}$, $\{10,28,38\}$, $\{11,44,55\}$, $\{12,46,58\}$, $\{13,43,56\}$, $\{14,45,59\}$, $\{15,42,57\}$, $\{16,24,40\}$, $\{17,37,54\}$, $\{18,33,51\}$, $\{19,34,53\}$, $\{20,32,52\}$, $\{21,26,47\}$, $\{22,27,49\}$, $\{23,25,48\}$
19	$\{5,32,37\}$, $\{7,31,38\}$, $\{8,33,41\}$, $\{9,30,39\}$, $\{10,46,56\}$, $\{11,47,58\}$, $\{12,49,61\}$, $\{13,29,42\}$, $\{14,45,59\}$, $\{15,48,63\}$, $\{16,44,60\}$, $\{17,40,57\}$, $\{18,35,53\}$, $\{19,36,55\}$, $\{20,34,54\}$, $\{21,22,43\}$, $\{23,27,50\}$, $\{24,28,52\}$, $\{25,26,51\}$
20	$\{5,33,38\}$, $\{7,32,39\}$, $\{8,35,43\}$, $\{9,31,40\}$, $\{10,48,58\}$, $\{11,30,41\}$, $\{12,47,59\}$, $\{13,51,64\}$, $\{14,49,63\}$, $\{15,46,61\}$, $\{16,50,66\}$, $\{17,45,62\}$, $\{18,42,60\}$, $\{19,36,55\}$, $\{20,37,57\}$, $\{21,23,44\}$, $\{22,34,56\}$, $\{24,28,52\}$, $\{25,29,54\}$, $\{26,27,53\}$
21	$\{5,35,40\}$, $\{7,34,41\}$, $\{8,37,45\}$, $\{9,33,42\}$, $\{10,51,61\}$, $\{11,32,43\}$, $\{12,54,66\}$, $\{13,50,63\}$, $\{14,53,67\}$, $\{15,49,64\}$, $\{16,52,68\}$, $\{17,48,65\}$, $\{18,44,62\}$, $\{19,27,46\}$, $\{20,38,58\}$, $\{21,39,60\}$, $\{22,25,47\}$, $\{23,36,59\}$, $\{24,31,55\}$, $\{26,30,56\}$, $\{28,29,57\}$
22	$\{5,29,34\}$, $\{7,30,37\}$, $\{8,28,36\}$, $\{9,45,54\}$, $\{10,46,56\}$, $\{11,27,38\}$, $\{12,43,55\}$, $\{13,44,57\}$, $\{14,51,65\}$, $\{15,52,67\}$, $\{16,42,58\}$, $\{17,49,66\}$, $\{18,53,71\}$, $\{19,50,69\}$, $\{20,41,61\}$, $\{21,47,68\}$, $\{22,48,70\}$, $\{23,39,62\}$, $\{24,40,64\}$, $\{25,35,60\}$, $\{26,33,59\}$, $\{31,32,63\}$
23	$\{5,29,34\}$, $\{7,28,35\}$, $\{8,30,38\}$, $\{9,47,56\}$, $\{10,27,37\}$, $\{11,46,57\}$, $\{12,48,60\}$, $\{13,45,58\}$, $\{14,54,68\}$, $\{15,44,59\}$, $\{16,53,69\}$, $\{17,55,72\}$, $\{18,43,61\}$, $\{19,51,70\}$, $\{20,42,62\}$, $\{21,50,71\}$, $\{22,41,63\}$, $\{23,52,75\}$, $\{24,49,73\}$, $\{25,39,64\}$, $\{26,40,66\}$, $\{31,36,67\}$, $\{32,33,65\}$
24	$\{5,29,34\}$, $\{7,28,35\}$, $\{8,30,38\}$, $\{9,27,36\}$, $\{10,48,58\}$, $\{11,49,60\}$, $\{12,47,59\}$, $\{13,50,63\}$, $\{14,51,65\}$, $\{15,46,61\}$, $\{16,54,70\}$, $\{17,45,62\}$, $\{18,55,73\}$, $\{19,57,76\}$, $\{20,44,64\}$, $\{21,53,74\}$, $\{22,56,78\}$, $\{23,52,75\}$, $\{24,42,66\}$, $\{25,43,68\}$, $\{26,41,67\}$, $\{31,40,71\}$, $\{32,37,69\}$, $\{33,39,72\}$
25	$\{5,29,34\}$, $\{7,28,35\}$, $\{8,52,60\}$, $\{9,27,36\}$, $\{10,51,61\}$, $\{11,26,37\}$, $\{12,50,62\}$, $\{13,53,66\}$, $\{14,49,63\}$, $\{15,55,70\}$, $\{16,48,64\}$, $\{17,56,73\}$, $\{18,47,65\}$, $\{19,57,76\}$, $\{20,59,79\}$, $\{21,46,67\}$, $\{22,58,80\}$, $\{23,45,68\}$, $\{24,54,78\}$, $\{25,44,69\}$, $\{30,41,71\}$, $\{31,43,74\}$, $\{32,40,72\}$, $\{33,42,75\}$, $\{38,39,77\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
26	$\{5,28,33\}$, $\{7,27,34\}$, $\{8,29,37\}$, $\{9,53,62\}$, $\{10,26,36\}$, $\{11,52,63\}$, $\{12,54,66\}$, $\{13,51,64\}$, $\{14,55,69\}$, $\{15,50,65\}$, $\{16,56,72\}$, $\{17,57,74\}$, $\{18,49,67\}$, $\{19,59,78\}$, $\{20,48,68\}$, $\{21,60,81\}$, $\{22,61,83\}$, $\{23,47,70\}$, $\{24,58,82\}$, $\{25,46,71\}$, $\{30,43,73\}$, $\{31,44,75\}$, $\{32,45,77\}$, $\{35,41,76\}$, $\{38,42,80\}$, $\{39,40,79\}$
27	$\{5,27,32\}$, $\{7,28,35\}$, $\{8,26,34\}$, $\{9,55,64\}$, $\{10,56,66\}$, $\{11,25,36\}$, $\{12,53,65\}$, $\{13,54,67\}$, $\{14,57,71\}$, $\{15,58,73\}$, $\{16,52,68\}$, $\{17,59,76\}$, $\{18,51,69\}$, $\{19,60,79\}$, $\{20,50,70\}$, $\{21,61,82\}$, $\{22,62,84\}$, $\{23,49,72\}$, $\{24,63,87\}$, $\{29,45,74\}$, $\{30,47,77\}$, $\{31,44,75\}$, $\{33,48,81\}$, $\{37,41,78\}$, $\{38,42,80\}$, $\{39,46,85\}$, $\{40,43,83\}$
28	$\{5,27,32\}$, $\{7,26,33\}$, $\{8,58,66\}$, $\{9,25,34\}$, $\{10,57,67\}$, $\{11,24,35\}$, $\{12,56,68\}$, $\{13,59,72\}$, $\{14,55,69\}$, $\{15,60,75\}$, $\{16,54,70\}$, $\{17,62,79\}$, $\{18,53,71\}$, $\{19,63,82\}$, $\{20,64,84\}$, $\{21,52,73\}$, $\{22,65,87\}$, $\{23,51,74\}$, $\{28,48,76\}$, $\{29,61,90\}$, $\{30,47,77\}$, $\{31,49,80\}$, $\{36,42,78\}$, $\{37,44,81\}$, $\{38,50,88\}$, $\{39,46,85\}$, $\{40,43,83\}$, $\{41,45,86\}$
29	$\{5,44,49\}$, $\{7,72,79\}$, $\{8,42,50\}$, $\{9,71,80\}$, $\{10,41,51\}$, $\{11,70,81\}$, $\{12,40,52\}$, $\{13,69,82\}$, $\{14,39,53\}$, $\{15,60,75\}$, $\{16,68,84\}$, $\{17,38,55\}$, $\{18,67,85\}$, $\{19,37,56\}$, $\{20,66,86\}$, $\{21,36,57\}$, $\{22,65,87\}$, $\{23,35,58\}$, $\{24,64,88\}$, $\{25,48,73\}$, $\{26,63,89\}$, $\{27,47,74\}$, $\{28,62,90\}$, $\{29,54,83\}$, $\{30,61,91\}$, $\{31,45,76\}$, $\{32,46,78\}$, $\{33,59,92\}$, $\{34,43,77\}$
30	$\{5,46,51\}$, $\{7,78,85\}$, $\{8,45,53\}$, $\{9,77,86\}$, $\{10,44,54\}$, $\{11,76,87\}$, $\{12,43,55\}$, $\{13,75,88\}$, $\{14,42,56\}$, $\{15,59,74\}$, $\{16,57,73\}$, $\{17,65,82\}$, $\{18,72,90\}$, $\{19,41,60\}$, $\{20,71,91\}$, $\{21,40,61\}$, $\{22,70,92\}$, $\{23,39,62\}$, $\{24,69,93\}$, $\{25,38,63\}$, $\{26,68,94\}$, $\{27,52,79\}$, $\{28,36,64\}$, $\{29,66,95\}$, $\{30,37,67\}$, $\{31,58,89\}$, $\{32,49,81\}$, $\{33,47,80\}$, $\{34,50,84\}$, $\{35,48,83\}$
31	$\{5,84,89\}$, $\{7,51,58\}$, $\{8,82,90\}$, $\{9,50,59\}$, $\{10,81,91\}$, $\{11,49,60\}$, $\{12,80,92\}$, $\{13,48,61\}$, $\{14,79,93\}$, $\{15,47,62\}$, $\{16,54,70\}$, $\{17,46,63\}$, $\{18,76,94\}$, $\{19,77,96\}$, $\{20,75,95\}$, $\{21,43,64\}$, $\{22,44,66\}$, $\{23,42,65\}$, $\{24,73,97\}$, $\{25,74,99\}$, $\{26,41,67\}$, $\{27,56,83\}$, $\{28,40,68\}$, $\{29,57,86\}$, $\{30,55,85\}$, $\{31,38,69\}$, $\{32,39,71\}$, $\{33,45,78\}$, $\{34,53,87\}$, $\{35,37,72\}$, $\{36,52,88\}$
32	$\{5,85,90\}$, $\{7,49,56\}$, $\{8,83,91\}$, $\{9,48,57\}$, $\{10,82,92\}$, $\{11,47,58\}$, $\{12,81,93\}$, $\{13,46,59\}$, $\{14,80,94\}$, $\{15,63,78\}$, $\{16,44,60\}$, $\{17,62,79\}$, $\{18,54,72\}$, $\{19,42,61\}$, $\{20,76,96\}$, $\{21,74,95\}$, $\{22,75,97\}$, $\{23,43,66\}$, $\{24,41,65\}$, $\{25,39,64\}$, $\{26,73,99\}$, $\{27,40,67\}$, $\{28,70,98\}$, $\{29,71,100\}$, $\{30,38,68\}$, $\{31,53,84\}$, $\{32,45,77\}$, $\{33,69,102\}$, $\{34,55,89\}$, $\{35,51,86\}$, $\{36,52,88\}$, $\{37,50,87\}$
33	$\{5,50,55\}$, $\{7,82,89\}$, $\{8,48,56\}$, $\{9,81,90\}$, $\{10,47,57\}$, $\{11,80,91\}$, $\{12,46,58\}$, $\{13,79,92\}$, $\{14,45,59\}$, $\{15,78,93\}$, $\{16,44,60\}$, $\{17,68,85\}$, $\{18,77,95\}$, $\{19,43,62\}$, $\{20,76,96\}$, $\{21,42,63\}$, $\{22,75,97\}$, $\{23,41,64\}$, $\{24,74,98\}$, $\{25,40,65\}$, $\{26,73,99\}$, $\{27,39,66\}$, $\{28,72,100\}$, $\{29,54,83\}$, $\{30,71,101\}$, $\{31,53,84\}$, $\{32,70,102\}$, $\{33,61,94\}$, $\{34,69,103\}$, $\{35,51,86\}$, $\{36,52,88\}$, $\{37,67,104\}$, $\{38,49,87\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
34	$\{5,52,57\}$, $\{7,88,95\}$, $\{8,51,59\}$, $\{9,87,96\}$, $\{10,50,60\}$, $\{11,86,97\}$, $\{12,49,61\}$, $\{13,85,98\}$, $\{14,48,62\}$, $\{15,84,99\}$, $\{16,47,63\}$, $\{17,66,83\}$, $\{18,64,82\}$, $\{19,73,92\}$, $\{20,81,101\}$, $\{21,46,67\}$, $\{22,80,102\}$, $\{23,45,68\}$, $\{24,79,103\}$, $\{25,44,69\}$, $\{26,78,104\}$, $\{27,43,70\}$, $\{28,77,105\}$, $\{29,42,71\}$, $\{30,76,106\}$, $\{31,58,89\}$, $\{32,40,72\}$, $\{33,74,107\}$, $\{34,41,75\}$, $\{35,65,100\}$, $\{36,55,91\}$, $\{37,53,90\}$, $\{38,56,94\}$, $\{39,54,93\}$
35	$\{5,94,99\}$, $\{7,57,64\}$, $\{8,92,100\}$, $\{9,56,65\}$, $\{10,91,101\}$, $\{11,55,66\}$, $\{12,90,102\}$, $\{13,54,67\}$, $\{14,89,103\}$, $\{15,53,68\}$, $\{16,88,104\}$, $\{17,52,69\}$, $\{18,60,78\}$, $\{19,86,105\}$, $\{20,51,71\}$, $\{21,49,70\}$, $\{22,84,106\}$, $\{23,85,108\}$, $\{24,83,107\}$, $\{25,47,72\}$, $\{26,48,74\}$, $\{27,46,73\}$, $\{28,81,109\}$, $\{29,82,111\}$, $\{30,45,75\}$, $\{31,62,93\}$, $\{32,44,76\}$, $\{33,63,96\}$, $\{34,61,95\}$, $\{35,42,77\}$, $\{36,43,79\}$, $\{37,50,87\}$, $\{38,59,97\}$, $\{39,41,80\}$, $\{40,58,98\}$
35	$\{5,94,99\}$, $\{7,58,65\}$, $\{8,92,100\}$, $\{9,55,64\}$, $\{10,56,66\}$, $\{11,91,102\}$, $\{12,89,101\}$, $\{13,90,103\}$, $\{14,54,68\}$, $\{15,52,67\}$, $\{16,88,104\}$, $\{17,61,78\}$, $\{18,51,69\}$, $\{19,53,72\}$, $\{20,85,105\}$, $\{21,87,108\}$, $\{22,84,106\}$, $\{23,48,71\}$, $\{24,46,70\}$, $\{25,49,74\}$, $\{26,83,109\}$, $\{27,80,107\}$, $\{28,47,75\}$, $\{29,82,111\}$, $\{30,43,73\}$, $\{31,45,76\}$, $\{32,63,95\}$, $\{33,60,93\}$, $\{34,62,96\}$, $\{35,42,77\}$, $\{36,50,86\}$, $\{37,44,81\}$, $\{38,41,79\}$, $\{39,59,98\}$, $\{40,57,97\}$
36	$\{5,95,100\}$, $\{7,56,63\}$, $\{8,93,101\}$, $\{9,53,62\}$, $\{10,54,64\}$, $\{11,92,103\}$, $\{12,90,102\}$, $\{13,91,104\}$, $\{14,52,66\}$, $\{15,50,65\}$, $\{16,89,105\}$, $\{17,70,87\}$, $\{18,49,67\}$, $\{19,69,88\}$, $\{20,48,68\}$, $\{21,58,79\}$, $\{22,85,107\}$, $\{23,83,106\}$, $\{24,84,108\}$, $\{25,47,72\}$, $\{26,45,71\}$, $\{27,46,73\}$, $\{28,82,110\}$, $\{29,80,109\}$, $\{30,81,111\}$, $\{31,44,75\}$, $\{32,42,74\}$, $\{33,61,94\}$, $\{34,78,112\}$, $\{35,51,86\}$, $\{36,41,77\}$, $\{37,59,96\}$, $\{38,76,114\}$, $\{39,60,99\}$, $\{40,57,97\}$, $\{43,55,98\}$
37	$\{5,56,61\}$, $\{7,92,99\}$, $\{8,54,62\}$, $\{9,91,100\}$, $\{10,53,63\}$, $\{11,90,101\}$, $\{12,52,64\}$, $\{13,89,102\}$, $\{14,51,65\}$, $\{15,88,103\}$, $\{16,50,66\}$, $\{17,87,104\}$, $\{18,49,67\}$, $\{19,76,95\}$, $\{20,86,106\}$, $\{21,48,69\}$, $\{22,85,107\}$, $\{23,47,70\}$, $\{24,84,108\}$, $\{25,46,71\}$, $\{26,83,109\}$, $\{27,45,72\}$, $\{28,82,110\}$, $\{29,44,73\}$, $\{30,81,111\}$, $\{31,43,74\}$, $\{32,80,112\}$, $\{33,60,93\}$, $\{34,79,113\}$, $\{35,59,94\}$, $\{36,78,114\}$, $\{37,68,105\}$, $\{38,77,115\}$, $\{39,57,96\}$, $\{40,58,98\}$, $\{41,75,116\}$, $\{42,55,97\}$
38	$\{5,58,63\}$, $\{7,98,105\}$, $\{8,57,65\}$, $\{9,97,106\}$, $\{10,56,66\}$, $\{11,96,107\}$, $\{12,55,67\}$, $\{13,95,108\}$, $\{14,54,68\}$, $\{15,94,109\}$, $\{16,53,69\}$, $\{17,93,110\}$, $\{18,52,70\}$, $\{19,73,92\}$, $\{20,71,91\}$, $\{21,81,102\}$, $\{22,90,112\}$, $\{23,51,74\}$, $\{24,89,113\}$, $\{25,50,75\}$, $\{26,88,114\}$, $\{27,49,76\}$, $\{28,87,115\}$, $\{29,48,77\}$, $\{30,86,116\}$, $\{31,47,78\}$, $\{32,85,117\}$, $\{33,46,79\}$, $\{34,84,118\}$, $\{35,64,99\}$, $\{36,44,80\}$, $\{37,82,119\}$, $\{38,45,83\}$, $\{39,72,111\}$, $\{40,61,101\}$, $\{41,59,100\}$, $\{42,62,104\}$, $\{43,60,103\}$
39	$\{5,33,38\}$, $\{7,32,39\}$, $\{8,35,43\}$, $\{9,31,40\}$, $\{10,34,44\}$, $\{11,30,41\}$, $\{12,78,90\}$, $\{13,79,92\}$, $\{14,77,91\}$, $\{15,80,95\}$, $\{16,81,97\}$, $\{17,76,93\}$, $\{18,82,100\}$, $\{19,75,94\}$, $\{20,83,103\}$, $\{21,84,105\}$, $\{22,74,96\}$, $\{23,85,108\}$, $\{24,86,110\}$, $\{25,73,98\}$, $\{26,88,114\}$, $\{27,72,99\}$, $\{28,87,115\}$, $\{29,89,118\}$, $\{36,65,101\}$, $\{37,67,104\}$, $\{42,60,102\}$, $\{45,61,106\}$, $\{46,63,109\}$, $\{47,64,111\}$, $\{48,59,107\}$, $\{49,70,119\}$, $\{50,66,116\}$, $\{51,69,120\}$, $\{52,71,123\}$, $\{53,68,121\}$, $\{54,58,112\}$, $\{55,62,117\}$, $\{56,57,113\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
40	$\{5,39,44\}$, $\{7,38,45\}$, $\{8,41,49\}$, $\{9,37,46\}$, $\{10,40,50\}$, $\{11,36,47\}$, $\{12,81,93\}$, $\{13,82,95\}$, $\{14,80,94\}$, $\{15,83,98\}$, $\{16,35,51\}$, $\{17,79,96\}$, $\{18,84,102\}$, $\{19,78,97\}$, $\{20,87,107\}$, $\{21,88,109\}$, $\{22,77,99\}$, $\{23,85,108\}$, $\{24,76,100\}$, $\{25,86,111\}$, $\{26,75,101\}$, $\{27,89,116\}$, $\{28,90,118\}$, $\{29,74,103\}$, $\{30,91,121\}$, $\{31,92,123\}$, $\{32,72,104\}$, $\{33,73,106\}$, $\{34,71,105\}$, $\{42,68,110\}$, $\{43,69,112\}$, $\{48,65,113\}$, $\{52,62,114\}$, $\{53,64,117\}$, $\{54,61,115\}$, $\{55,67,122\}$, $\{56,70,126\}$, $\{57,63,120\}$, $\{58,66,124\}$, $\{59,60,119\}$
41	$\{5,62,67\}$, $\{7,102,109\}$, $\{8,60,68\}$, $\{9,101,110\}$, $\{10,59,69\}$, $\{11,100,111\}$, $\{12,58,70\}$, $\{13,99,112\}$, $\{14,57,71\}$, $\{15,98,113\}$, $\{16,56,72\}$, $\{17,97,114\}$, $\{18,55,73\}$, $\{19,96,115\}$, $\{20,54,74\}$, $\{21,84,105\}$, $\{22,95,117\}$, $\{23,53,76\}$, $\{24,94,118\}$, $\{25,52,77\}$, $\{26,93,119\}$, $\{27,51,78\}$, $\{28,92,120\}$, $\{29,50,79\}$, $\{30,91,121\}$, $\{31,49,80\}$, $\{32,90,122\}$, $\{33,48,81\}$, $\{34,89,123\}$, $\{35,47,82\}$, $\{36,88,124\}$, $\{37,66,103\}$, $\{38,87,125\}$, $\{39,65,104\}$, $\{40,86,126\}$, $\{41,75,116\}$, $\{42,85,127\}$, $\{43,63,106\}$, $\{44,64,108\}$, $\{45,83,128\}$, $\{46,61,107\}$
42	$\{5,64,69\}$, $\{7,108,115\}$, $\{8,63,71\}$, $\{9,107,116\}$, $\{10,62,72\}$, $\{11,106,117\}$, $\{12,61,73\}$, $\{13,105,118\}$, $\{14,60,74\}$, $\{15,104,119\}$, $\{16,59,75\}$, $\{17,103,120\}$, $\{18,58,76\}$, $\{19,102,121\}$, $\{20,57,77\}$, $\{21,80,101\}$, $\{22,78,100\}$, $\{23,89,112\}$, $\{24,99,123\}$, $\{25,56,81\}$, $\{26,98,124\}$, $\{27,55,82\}$, $\{28,97,125\}$, $\{29,54,83\}$, $\{30,96,126\}$, $\{31,53,84\}$, $\{32,95,127\}$, $\{33,52,85\}$, $\{34,94,128\}$, $\{35,51,86\}$, $\{36,93,129\}$, $\{37,50,87\}$, $\{38,92,130\}$, $\{39,70,109\}$, $\{40,48,88\}$, $\{41,90,131\}$, $\{42,49,91\}$, $\{43,79,122\}$, $\{44,67,111\}$, $\{45,65,110\}$, $\{46,68,114\}$, $\{47,66,113\}$
43	$\{5,114,119\}$, $\{7,113,120\}$, $\{8,68,76\}$, $\{9,69,78\}$, $\{10,67,77\}$, $\{11,110,121\}$, $\{12,111,123\}$, $\{13,109,122\}$, $\{14,65,79\}$, $\{15,66,81\}$, $\{16,64,80\}$, $\{17,107,124\}$, $\{18,108,126\}$, $\{19,106,125\}$, $\{20,62,82\}$, $\{21,63,84\}$, $\{22,61,83\}$, $\{23,71,94\}$, $\{24,103,127\}$, $\{25,104,129\}$, $\{26,102,128\}$, $\{27,60,87\}$, $\{28,57,85\}$, $\{29,101,130\}$, $\{30,56,86\}$, $\{31,100,131\}$, $\{32,58,90\}$, $\{33,99,132\}$, $\{34,55,89\}$, $\{35,98,133\}$, $\{36,52,88\}$, $\{37,75,112\}$, $\{38,54,92\}$, $\{39,96,135\}$, $\{40,51,91\}$, $\{41,74,115\}$, $\{42,53,95\}$, $\{43,73,116\}$, $\{44,49,93\}$, $\{45,72,117\}$, $\{46,59,105\}$, $\{47,50,97\}$, $\{48,70,118\}$
44	$\{5,115,120\}$, $\{7,67,74\}$, $\{8,113,121\}$, $\{9,66,75\}$, $\{10,112,122\}$, $\{11,65,76\}$, $\{12,111,123\}$, $\{13,64,77\}$, $\{14,110,124\}$, $\{15,63,78\}$, $\{16,109,125\}$, $\{17,62,79\}$, $\{18,108,126\}$, $\{19,61,80\}$, $\{20,107,127\}$, $\{21,84,105\}$, $\{22,59,81\}$, $\{23,83,106\}$, $\{24,72,96\}$, $\{25,57,82\}$, $\{26,103,129\}$, $\{27,101,128\}$, $\{28,102,130\}$, $\{29,58,87\}$, $\{30,56,86\}$, $\{31,54,85\}$, $\{32,100,132\}$, $\{33,98,131\}$, $\{34,99,133\}$, $\{35,55,90\}$, $\{36,53,89\}$, $\{37,51,88\}$, $\{38,97,135\}$, $\{39,52,91\}$, $\{40,94,134\}$, $\{41,95,136\}$, $\{42,50,92\}$, $\{43,71,114\}$, $\{44,60,104\}$, $\{45,93,138\}$, $\{46,73,119\}$, $\{47,69,116\}$, $\{48,70,118\}$, $\{49,68,117\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
45	$\{5,68,73\}$, $\{7,112,119\}$, $\{8,66,74\}$, $\{9,111,120\}$, $\{10,65,75\}$, $\{11,110,121\}$, $\{12,64,76\}$, $\{13,109,122\}$, $\{14,63,77\}$, $\{15,108,123\}$, $\{16,62,78\}$, $\{17,107,124\}$, $\{18,61,79\}$, $\{19,106,125\}$, $\{20,60,80\}$, $\{21,105,126\}$, $\{22,59,81\}$, $\{23,92,115\}$, $\{24,104,128\}$, $\{25,58,83\}$, $\{26,103,129\}$, $\{27,57,84\}$, $\{28,102,130\}$, $\{29,56,85\}$, $\{30,101,131\}$, $\{31,55,86\}$, $\{32,100,132\}$, $\{33,54,87\}$, $\{34,99,133\}$, $\{35,53,88\}$, $\{36,98,134\}$, $\{37,52,89\}$, $\{38,97,135\}$, $\{39,51,90\}$, $\{40,96,136\}$, $\{41,72,113\}$, $\{42,95,137\}$, $\{43,71,114\}$, $\{44,94,138\}$, $\{45,82,127\}$, $\{46,93,139\}$, $\{47,69,116\}$, $\{48,70,118\}$, $\{49,91,140\}$, $\{50,67,117\}$
46	$\{5,70,75\}$, $\{7,118,125\}$, $\{8,69,77\}$, $\{9,117,126\}$, $\{10,68,78\}$, $\{11,116,127\}$, $\{12,67,79\}$, $\{13,115,128\}$, $\{14,66,80\}$, $\{15,114,129\}$, $\{16,65,81\}$, $\{17,113,130\}$, $\{18,64,82\}$, $\{19,112,131\}$, $\{20,63,83\}$, $\{21,111,132\}$, $\{22,62,84\}$, $\{23,87,110\}$, $\{24,85,109\}$, $\{25,97,122\}$, $\{26,108,134\}$, $\{27,61,88\}$, $\{28,107,135\}$, $\{29,60,89\}$, $\{30,106,136\}$, $\{31,59,90\}$, $\{32,105,137\}$, $\{33,58,91\}$, $\{34,104,138\}$, $\{35,57,92\}$, $\{36,103,139\}$, $\{37,56,93\}$, $\{38,102,140\}$, $\{39,55,94\}$, $\{40,101,141\}$, $\{41,54,95\}$, $\{42,100,142\}$, $\{43,76,119\}$, $\{44,52,96\}$, $\{45,98,143\}$, $\{46,53,99\}$, $\{47,86,133\}$, $\{48,73,121\}$, $\{49,71,120\}$, $\{50,74,124\}$, $\{51,72,123\}$
47	$\{5,124,129\}$, $\{7,75,82\}$, $\{8,122,130\}$, $\{9,74,83\}$, $\{10,121,131\}$, $\{11,73,84\}$, $\{12,120,132\}$, $\{13,72,85\}$, $\{14,119,133\}$, $\{15,71,86\}$, $\{16,118,134\}$, $\{17,70,87\}$, $\{18,117,135\}$, $\{19,69,88\}$, $\{20,116,136\}$, $\{21,68,89\}$, $\{22,115,137\}$, $\{23,67,90\}$, $\{24,78,102\}$, $\{25,113,138\}$, $\{26,66,92\}$, $\{27,64,91\}$, $\{28,111,139\}$, $\{29,112,141\}$, $\{30,110,140\}$, $\{31,62,93\}$, $\{32,63,95\}$, $\{33,61,94\}$, $\{34,108,142\}$, $\{35,109,144\}$, $\{36,107,143\}$, $\{37,59,96\}$, $\{38,60,98\}$, $\{39,58,97\}$, $\{40,105,145\}$, $\{41,106,147\}$, $\{42,57,99\}$, $\{43,80,123\}$, $\{44,56,100\}$, $\{45,81,126\}$, $\{46,79,125\}$, $\{47,54,101\}$, $\{48,55,103\}$, $\{49,65,114\}$, $\{50,77,127\}$, $\{51,53,104\}$, $\{52,76,128\}$
48	$\{5,125,130\}$, $\{7,74,81\}$, $\{8,123,131\}$, $\{9,71,80\}$, $\{10,72,82\}$, $\{11,122,133\}$, $\{12,120,132\}$, $\{13,121,134\}$, $\{14,70,84\}$, $\{15,68,83\}$, $\{16,69,85\}$, $\{17,119,136\}$, $\{18,117,135\}$, $\{19,118,137\}$, $\{20,67,87\}$, $\{21,65,86\}$, $\{22,116,138\}$, $\{23,91,114\}$, $\{24,64,88\}$, $\{25,90,115\}$, $\{26,63,89\}$, $\{27,76,103\}$, $\{28,112,140\}$, $\{29,110,139\}$, $\{30,111,141\}$, $\{31,62,93\}$, $\{32,60,92\}$, $\{33,61,94\}$, $\{34,109,143\}$, $\{35,107,142\}$, $\{36,108,144\}$, $\{37,59,96\}$, $\{38,57,95\}$, $\{39,58,97\}$, $\{40,106,146\}$, $\{41,104,145\}$, $\{42,105,147\}$, $\{43,56,99\}$, $\{44,54,98\}$, $\{45,79,124\}$, $\{46,102,148\}$, $\{47,66,113\}$, $\{48,53,101\}$, $\{49,77,126\}$, $\{50,100,150\}$, $\{51,78,129\}$, $\{52,75,127\}$, $\{55,73,128\}$
49	$\{5,74,79\}$, $\{7,122,129\}$, $\{8,72,80\}$, $\{9,121,130\}$, $\{10,71,81\}$, $\{11,120,131\}$, $\{12,70,82\}$, $\{13,119,132\}$, $\{14,69,83\}$, $\{15,118,133\}$, $\{16,68,84\}$, $\{17,117,134\}$, $\{18,67,85\}$, $\{19,116,135\}$, $\{20,66,86\}$, $\{21,115,136\}$, $\{22,65,87\}$, $\{23,114,137\}$, $\{24,64,88\}$, $\{25,100,125\}$, $\{26,113,139\}$, $\{27,63,90\}$, $\{28,112,140\}$, $\{29,62,91\}$, $\{30,111,141\}$, $\{31,61,92\}$, $\{32,110,142\}$, $\{33,60,93\}$, $\{34,109,143\}$, $\{35,59,94\}$, $\{36,108,144\}$, $\{37,58,95\}$, $\{38,107,145\}$, $\{39,57,96\}$, $\{40,106,146\}$, $\{41,56,97\}$, $\{42,105,147\}$, $\{43,55,98\}$, $\{44,104,148\}$, $\{45,78,123\}$, $\{46,103,149\}$, $\{47,77,124\}$, $\{48,102,150\}$, $\{49,89,138\}$, $\{50,101,151\}$, $\{51,75,126\}$, $\{52,76,128\}$, $\{53,99,152\}$, $\{54,73,127\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
50	$\{5,76,81\}$, $\{7,128,135\}$, $\{8,75,83\}$, $\{9,127,136\}$, $\{10,74,84\}$, $\{11,126,137\}$, $\{12,73,85\}$, $\{13,125,138\}$, $\{14,72,86\}$, $\{15,124,139\}$, $\{16,71,87\}$, $\{17,123,140\}$, $\{18,70,88\}$, $\{19,122,141\}$, $\{20,69,89\}$, $\{21,121,142\}$, $\{22,68,90\}$, $\{23,120,143\}$, $\{24,67,91\}$, $\{25,94,119\}$, $\{26,92,118\}$, $\{27,105,132\}$, $\{28,117,145\}$, $\{29,66,95\}$, $\{30,116,146\}$, $\{31,65,96\}$, $\{32,115,147\}$, $\{33,64,97\}$, $\{34,114,148\}$, $\{35,63,98\}$, $\{36,113,149\}$, $\{37,62,99\}$, $\{38,112,150\}$, $\{39,61,100\}$, $\{40,111,151\}$, $\{41,60,101\}$, $\{42,110,152\}$, $\{43,59,102\}$, $\{44,109,153\}$, $\{45,58,103\}$, $\{46,108,154\}$, $\{47,82,129\}$, $\{48,56,104\}$, $\{49,106,155\}$, $\{50,57,107\}$, $\{51,93,144\}$, $\{52,79,131\}$, $\{53,77,130\}$, $\{54,80,134\}$, $\{55,78,133\}$
51	$\{5,39,44\}$, $\{7,38,45\}$, $\{8,40,48\}$, $\{9,37,46\}$, $\{10,42,52\}$, $\{11,36,47\}$, $\{12,104,116\}$, $\{13,105,118\}$, $\{14,35,49\}$, $\{15,102,117\}$, $\{16,34,50\}$, $\{17,103,120\}$, $\{18,101,119\}$, $\{19,106,125\}$, $\{20,107,127\}$, $\{21,100,121\}$, $\{22,108,130\}$, $\{23,99,122\}$, $\{24,109,133\}$, $\{25,98,123\}$, $\{26,110,136\}$, $\{27,97,124\}$, $\{28,111,139\}$, $\{29,112,141\}$, $\{30,96,126\}$, $\{31,114,145\}$, $\{32,115,147\}$, $\{33,95,128\}$, $\{41,88,129\}$, $\{43,113,156\}$, $\{51,80,131\}$, $\{53,79,132\}$, $\{54,81,135\}$, $\{55,82,137\}$, $\{56,78,134\}$, $\{57,83,140\}$, $\{58,90,148\}$, $\{59,92,151\}$, $\{60,89,149\}$, $\{61,77,138\}$, $\{62,91,153\}$, $\{63,87,150\}$, $\{64,93,157\}$, $\{65,94,159\}$, $\{66,76,142\}$, $\{67,85,152\}$, $\{68,75,143\}$, $\{69,86,155\}$, $\{70,84,154\}$, $\{71,73,144\}$, $\{72,74,146\}$
52	$\{5,38,43\}$, $\{7,37,44\}$, $\{8,39,47\}$, $\{9,36,45\}$, $\{10,40,50\}$, $\{11,35,46\}$, $\{12,106,118\}$, $\{13,107,120\}$, $\{14,34,48\}$, $\{15,104,119\}$, $\{16,105,121\}$, $\{17,108,125\}$, $\{18,33,51\}$, $\{19,103,122\}$, $\{20,109,129\}$, $\{21,102,123\}$, $\{22,110,132\}$, $\{23,101,124\}$, $\{24,111,135\}$, $\{25,112,137\}$, $\{26,100,126\}$, $\{27,113,140\}$, $\{28,99,127\}$, $\{29,114,143\}$, $\{30,98,128\}$, $\{31,116,147\}$, $\{32,117,149\}$, $\{41,89,130\}$, $\{42,115,157\}$, $\{49,82,131\}$, $\{52,81,133\}$, $\{53,83,136\}$, $\{54,80,134\}$, $\{55,84,139\}$, $\{56,85,141\}$, $\{57,93,150\}$, $\{58,94,152\}$, $\{59,79,138\}$, $\{60,91,151\}$, $\{61,97,158\}$, $\{62,92,154\}$, $\{63,90,153\}$, $\{64,78,142\}$, $\{65,95,160\}$, $\{66,96,162\}$, $\{67,77,144\}$, $\{68,87,155\}$, $\{69,76,145\}$, $\{70,86,156\}$, $\{71,88,159\}$, $\{72,74,146\}$, $\{73,75,148\}$

Table A.25: Difference triples for Lemma 1.3.6

t	Difference triples
13	$\{6,26,32\}$, $\{8,27,35\}$, $\{9,25,34\}$, $\{10,31,41\}$, $\{11,33,44\}$, $\{12,24,36\}$, $\{13,29,42\}$, $\{14,23,37\}$, $\{15,28,43\}$, $\{16,30,46\}$, $\{17,21,38\}$, $\{18,22,40\}$, $\{19,20,39\}$
14	$\{6,25,31\}$, $\{8,28,36\}$, $\{9,29,38\}$, $\{10,27,37\}$, $\{11,33,44\}$, $\{12,35,47\}$, $\{13,32,45\}$, $\{14,34,48\}$, $\{15,24,39\}$, $\{16,30,46\}$, $\{17,26,43\}$, $\{18,22,40\}$, $\{19,23,42\}$, $\{20,21,41\}$
15	$\{6,23,29\}$, $\{8,30,38\}$, $\{9,31,40\}$, $\{10,36,46\}$, $\{11,28,39\}$, $\{12,35,47\}$, $\{13,37,50\}$, $\{14,27,41\}$, $\{15,34,49\}$, $\{16,32,48\}$, $\{17,25,42\}$, $\{18,33,51\}$, $\{19,26,45\}$, $\{20,24,44\}$, $\{21,22,43\}$
16	$\{6,20,26\}$, $\{8,32,40\}$, $\{9,33,42\}$, $\{10,31,41\}$, $\{11,37,48\}$, $\{12,38,50\}$, $\{13,30,43\}$, $\{14,39,53\}$, $\{15,29,44\}$, $\{16,35,51\}$, $\{17,28,45\}$, $\{18,34,52\}$, $\{19,36,55\}$, $\{21,25,46\}$, $\{22,27,49\}$, $\{23,24,47\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
17	$\{6,17,23\}$, $\{8,34,42\}$, $\{9,35,44\}$, $\{10,33,43\}$, $\{11,36,47\}$, $\{12,41,53\}$, $\{13,32,45\}$, $\{14,40,54\}$, $\{15,31,46\}$, $\{16,39,55\}$, $\{18,30,48\}$, $\{19,37,56\}$, $\{20,38,58\}$, $\{21,28,49\}$, $\{22,29,51\}$, $\{24,26,50\}$, $\{25,27,52\}$
18	$\{6,14,20\}$, $\{8,36,44\}$, $\{9,37,46\}$, $\{10,35,45\}$, $\{11,38,49\}$, $\{12,43,55\}$, $\{13,34,47\}$, $\{15,33,48\}$, $\{16,41,57\}$, $\{17,42,59\}$, $\{18,40,58\}$, $\{19,31,50\}$, $\{21,39,60\}$, $\{22,29,51\}$, $\{23,30,53\}$, $\{24,32,56\}$, $\{25,27,52\}$, $\{26,28,54\}$
19	$\{6,10,16\}$, $\{8,38,46\}$, $\{9,39,48\}$, $\{11,36,47\}$, $\{12,37,49\}$, $\{13,44,57\}$, $\{14,45,59\}$, $\{15,35,50\}$, $\{17,34,51\}$, $\{18,42,60\}$, $\{19,43,62\}$, $\{20,32,52\}$, $\{21,40,61\}$, $\{22,41,63\}$, $\{23,30,53\}$, $\{24,31,55\}$, $\{25,33,58\}$, $\{26,28,54\}$, $\{27,29,56\}$
20	$\{6,9,15\}$, $\{8,39,47\}$, $\{10,38,48\}$, $\{11,41,52\}$, $\{12,37,49\}$, $\{13,40,53\}$, $\{14,36,50\}$, $\{16,51,67\}$, $\{17,44,61\}$, $\{18,46,64\}$, $\{19,43,62\}$, $\{20,45,65\}$, $\{21,42,63\}$, $\{22,32,54\}$, $\{23,35,58\}$, $\{24,31,55\}$, $\{25,34,59\}$, $\{26,30,56\}$, $\{27,33,60\}$, $\{28,29,57\}$
21	$\{6,8,14\}$, $\{9,40,49\}$, $\{10,41,51\}$, $\{11,39,50\}$, $\{12,42,54\}$, $\{13,57,70\}$, $\{15,37,52\}$, $\{16,46,62\}$, $\{17,36,53\}$, $\{18,48,66\}$, $\{19,45,64\}$, $\{20,35,55\}$, $\{21,47,68\}$, $\{22,43,65\}$, $\{23,44,67\}$, $\{24,32,56\}$, $\{25,38,63\}$, $\{26,34,60\}$, $\{27,31,58\}$, $\{28,33,61\}$, $\{29,30,59\}$
22	$\{6,65,71\}$, $\{8,10,18\}$, $\{9,41,50\}$, $\{11,40,51\}$, $\{12,42,54\}$, $\{13,39,52\}$, $\{14,43,57\}$, $\{15,49,64\}$, $\{16,53,69\}$, $\{17,38,55\}$, $\{19,37,56\}$, $\{20,46,66\}$, $\{21,47,68\}$, $\{22,36,58\}$, $\{23,44,67\}$, $\{24,48,72\}$, $\{25,45,70\}$, $\{26,33,59\}$, $\{27,35,62\}$, $\{28,32,60\}$, $\{29,34,63\}$, $\{30,31,61\}$
23	$\{6,21,27\}$, $\{8,20,28\}$, $\{9,40,49\}$, $\{10,46,56\}$, $\{11,47,58\}$, $\{12,45,57\}$, $\{13,48,61\}$, $\{14,50,64\}$, $\{15,44,59\}$, $\{16,54,70\}$, $\{17,43,60\}$, $\{18,55,73\}$, $\{19,53,72\}$, $\{22,52,74\}$, $\{23,39,62\}$, $\{24,51,75\}$, $\{25,38,63\}$, $\{26,42,68\}$, $\{29,36,65\}$, $\{30,41,71\}$, $\{31,35,66\}$, $\{32,37,69\}$, $\{33,34,67\}$
24	$\{6,22,28\}$, $\{8,21,29\}$, $\{9,35,44\}$, $\{10,48,58\}$, $\{11,49,60\}$, $\{12,47,59\}$, $\{13,50,63\}$, $\{14,51,65\}$, $\{15,46,61\}$, $\{16,55,71\}$, $\{17,45,62\}$, $\{18,57,75\}$, $\{19,54,73\}$, $\{20,56,76\}$, $\{23,41,64\}$, $\{24,42,66\}$, $\{25,52,77\}$, $\{26,53,79\}$, $\{27,40,67\}$, $\{30,38,68\}$, $\{31,43,74\}$, $\{32,37,69\}$, $\{33,39,72\}$, $\{34,36,70\}$
25	$\{6,23,29\}$, $\{8,22,30\}$, $\{9,51,60\}$, $\{10,52,62\}$, $\{11,28,39\}$, $\{12,49,61\}$, $\{13,50,63\}$, $\{14,53,67\}$, $\{15,54,69\}$, $\{16,48,64\}$, $\{17,57,74\}$, $\{18,47,65\}$, $\{19,59,78\}$, $\{20,46,66\}$, $\{21,58,79\}$, $\{24,44,68\}$, $\{25,55,80\}$, $\{26,56,82\}$, $\{27,43,70\}$, $\{31,40,71\}$, $\{32,45,77\}$, $\{33,42,75\}$, $\{34,38,72\}$, $\{35,41,76\}$, $\{36,37,73\}$
26	$\{6,24,30\}$, $\{8,23,31\}$, $\{9,25,34\}$, $\{10,52,62\}$, $\{11,53,64\}$, $\{12,51,63\}$, $\{13,54,67\}$, $\{14,55,69\}$, $\{15,50,65\}$, $\{16,56,72\}$, $\{17,49,66\}$, $\{18,60,78\}$, $\{19,61,80\}$, $\{20,48,68\}$, $\{21,58,79\}$, $\{22,59,81\}$, $\{26,44,70\}$, $\{27,57,84\}$, $\{28,43,71\}$, $\{29,45,74\}$, $\{32,41,73\}$, $\{33,42,75\}$, $\{35,47,82\}$, $\{36,40,76\}$, $\{37,46,83\}$, $\{38,39,77\}$
27	$\{6,22,28\}$, $\{8,23,31\}$, $\{9,55,64\}$, $\{10,56,66\}$, $\{11,21,32\}$, $\{12,53,65\}$, $\{13,54,67\}$, $\{14,57,71\}$, $\{15,58,73\}$, $\{16,52,68\}$, $\{17,63,80\}$, $\{18,51,69\}$, $\{19,62,81\}$, $\{20,50,70\}$, $\{24,48,72\}$, $\{25,60,85\}$, $\{26,61,87\}$, $\{27,59,86\}$, $\{29,45,74\}$, $\{30,46,76\}$, $\{33,42,75\}$, $\{34,43,77\}$, $\{35,49,84\}$, $\{36,47,83\}$, $\{37,41,78\}$, $\{38,44,82\}$, $\{39,40,79\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
28	$\{6,15,21\}$, $\{8,24,32\}$, $\{9,57,66\}$, $\{10,23,33\}$, $\{11,56,67\}$, $\{12,58,70\}$, $\{13,55,68\}$, $\{14,59,73\}$, $\{16,53,69\}$, $\{17,54,71\}$, $\{18,62,80\}$, $\{19,65,84\}$, $\{20,52,72\}$, $\{22,64,86\}$, $\{25,49,74\}$, $\{26,63,89\}$, $\{27,48,75\}$, $\{28,60,88\}$, $\{29,47,76\}$, $\{30,61,91\}$, $\{31,46,77\}$, $\{34,44,78\}$, $\{35,50,85\}$, $\{36,51,87\}$, $\{37,42,79\}$, $\{38,45,83\}$, $\{39,43,82\}$, $\{40,41,81\}$
29	$\{6,25,31\}$, $\{8,26,34\}$, $\{9,24,33\}$, $\{10,55,65\}$, $\{11,57,68\}$, $\{12,82,94\}$, $\{13,56,69\}$, $\{14,58,72\}$, $\{15,59,74\}$, $\{16,54,70\}$, $\{17,60,77\}$, $\{18,53,71\}$, $\{19,61,80\}$, $\{20,64,84\}$, $\{21,52,73\}$, $\{22,66,88\}$, $\{23,67,90\}$, $\{27,48,75\}$, $\{28,63,91\}$, $\{29,47,76\}$, $\{30,62,92\}$, $\{32,46,78\}$, $\{35,44,79\}$, $\{36,51,87\}$, $\{37,49,86\}$, $\{38,43,81\}$, $\{39,50,89\}$, $\{40,45,85\}$, $\{41,42,83\}$
30	$\{6,18,24\}$, $\{8,26,34\}$, $\{9,58,67\}$, $\{10,25,35\}$, $\{11,59,70\}$, $\{12,84,96\}$, $\{13,60,73\}$, $\{14,57,71\}$, $\{15,61,76\}$, $\{16,56,72\}$, $\{17,62,79\}$, $\{19,55,74\}$, $\{20,69,89\}$, $\{21,54,75\}$, $\{22,65,87\}$, $\{23,68,91\}$, $\{27,50,77\}$, $\{28,66,94\}$, $\{29,49,78\}$, $\{30,63,93\}$, $\{31,64,95\}$, $\{32,48,80\}$, $\{33,53,86\}$, $\{36,45,81\}$, $\{37,46,83\}$, $\{38,44,82\}$, $\{39,51,90\}$, $\{40,52,92\}$, $\{41,47,88\}$, $\{42,43,85\}$
31	$\{6,30,36\}$, $\{8,29,37\}$, $\{9,61,70\}$, $\{10,28,38\}$, $\{11,63,74\}$, $\{12,27,39\}$, $\{13,59,72\}$, $\{14,62,76\}$, $\{15,60,75\}$, $\{16,64,80\}$, $\{17,65,82\}$, $\{18,66,84\}$, $\{19,68,87\}$, $\{20,58,78\}$, $\{21,77,98\}$, $\{22,57,79\}$, $\{23,71,94\}$, $\{24,73,97\}$, $\{25,56,81\}$, $\{26,69,95\}$, $\{31,52,83\}$, $\{32,67,99\}$, $\{33,53,86\}$, $\{34,51,85\}$, $\{35,54,89\}$, $\{40,48,88\}$, $\{41,55,96\}$, $\{42,50,92\}$, $\{43,47,90\}$, $\{44,49,93\}$, $\{45,46,91\}$
32	$\{6,31,37\}$, $\{8,30,38\}$, $\{9,54,63\}$, $\{10,29,39\}$, $\{11,65,76\}$, $\{12,28,40\}$, $\{13,61,74\}$, $\{14,64,78\}$, $\{15,62,77\}$, $\{16,66,82\}$, $\{17,67,84\}$, $\{18,68,86\}$, $\{19,69,88\}$, $\{20,60,80\}$, $\{21,79,100\}$, $\{22,59,81\}$, $\{23,72,95\}$, $\{24,73,97\}$, $\{25,58,83\}$, $\{26,75,101\}$, $\{27,71,98\}$, $\{32,53,85\}$, $\{33,70,103\}$, $\{34,55,89\}$, $\{35,52,87\}$, $\{36,56,92\}$, $\{41,49,90\}$, $\{42,57,99\}$, $\{43,48,91\}$, $\{44,50,94\}$, $\{45,51,96\}$, $\{46,47,93\}$
33	$\{6,32,38\}$, $\{8,31,39\}$, $\{9,47,56\}$, $\{10,30,40\}$, $\{11,65,76\}$, $\{12,29,41\}$, $\{13,66,79\}$, $\{14,64,78\}$, $\{15,67,82\}$, $\{16,68,84\}$, $\{17,63,80\}$, $\{18,69,87\}$, $\{19,70,89\}$, $\{20,71,91\}$, $\{21,62,83\}$, $\{22,81,103\}$, $\{23,77,100\}$, $\{24,61,85\}$, $\{25,74,99\}$, $\{26,60,86\}$, $\{27,75,102\}$, $\{28,73,101\}$, $\{33,55,88\}$, $\{34,72,106\}$, $\{35,57,92\}$, $\{36,54,90\}$, $\{37,58,95\}$, $\{42,51,93\}$, $\{43,53,96\}$, $\{44,50,94\}$, $\{45,59,104\}$, $\{46,52,98\}$, $\{48,49,97\}$
34	$\{6,33,39\}$, $\{8,32,40\}$, $\{9,69,78\}$, $\{10,31,41\}$, $\{11,38,49\}$, $\{12,30,42\}$, $\{13,67,80\}$, $\{14,68,82\}$, $\{15,66,81\}$, $\{16,70,86\}$, $\{17,71,88\}$, $\{18,72,90\}$, $\{19,65,84\}$, $\{20,73,93\}$, $\{21,64,85\}$, $\{22,74,96\}$, $\{23,75,98\}$, $\{24,63,87\}$, $\{25,83,108\}$, $\{26,77,103\}$, $\{27,62,89\}$, $\{28,79,107\}$, $\{29,76,105\}$, $\{34,57,91\}$, $\{35,59,94\}$, $\{36,56,92\}$, $\{37,58,95\}$, $\{43,54,97\}$, $\{44,60,104\}$, $\{45,61,106\}$, $\{46,53,99\}$, $\{47,55,102\}$, $\{48,52,100\}$, $\{50,51,101\}$
35	$\{6,35,41\}$, $\{8,34,42\}$, $\{9,36,45\}$, $\{10,33,43\}$, $\{11,69,80\}$, $\{12,32,44\}$, $\{13,68,81\}$, $\{14,70,84\}$, $\{15,67,82\}$, $\{16,71,87\}$, $\{17,66,83\}$, $\{18,72,90\}$, $\{19,73,92\}$, $\{20,74,94\}$, $\{21,65,86\}$, $\{22,89,111\}$, $\{23,75,98\}$, $\{24,64,88\}$, $\{25,85,110\}$, $\{26,76,102\}$, $\{27,78,105\}$, $\{28,63,91\}$, $\{29,79,108\}$, $\{30,77,107\}$, $\{31,62,93\}$, $\{37,58,95\}$, $\{38,59,97\}$, $\{39,57,96\}$, $\{40,60,100\}$, $\{46,53,99\}$, $\{47,54,101\}$, $\{48,61,109\}$, $\{49,55,104\}$, $\{50,56,106\}$, $\{51,52,103\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
36	$\{6,35,41\}$, $\{8,34,42\}$, $\{9,40,49\}$, $\{10,33,43\}$, $\{11,69,80\}$, $\{12,32,44\}$, $\{13,71,84\}$, $\{14,68,82\}$, $\{15,70,85\}$, $\{16,72,88\}$, $\{17,73,90\}$, $\{18,97,115\}$, $\{19,67,86\}$, $\{20,74,94\}$, $\{21,75,96\}$, $\{22,77,99\}$, $\{23,66,89\}$, $\{24,87,111\}$, $\{25,83,108\}$, $\{26,65,91\}$, $\{27,79,106\}$, $\{28,64,92\}$, $\{29,78,107\}$, $\{30,63,93\}$, $\{31,81,112\}$, $\{36,59,95\}$, $\{37,76,113\}$, $\{38,60,98\}$, $\{39,61,100\}$, $\{45,56,101\}$, $\{46,57,103\}$, $\{47,55,102\}$, $\{48,62,110\}$, $\{50,54,104\}$, $\{51,58,109\}$, $\{52,53,105\}$
37	$\{6,34,40\}$, $\{8,35,43\}$, $\{9,33,42\}$, $\{10,72,82\}$, $\{11,73,84\}$, $\{12,32,44\}$, $\{13,74,87\}$, $\{14,31,45\}$, $\{15,71,86\}$, $\{16,75,91\}$, $\{17,76,93\}$, $\{18,70,88\}$, $\{19,99,118\}$, $\{20,77,97\}$, $\{21,69,90\}$, $\{22,79,101\}$, $\{23,80,103\}$, $\{24,68,92\}$, $\{25,85,110\}$, $\{26,89,115\}$, $\{27,67,94\}$, $\{28,81,109\}$, $\{29,66,95\}$, $\{30,83,113\}$, $\{36,60,96\}$, $\{37,61,98\}$, $\{38,78,116\}$, $\{39,63,102\}$, $\{41,59,100\}$, $\{46,58,104\}$, $\{47,64,111\}$, $\{48,57,105\}$, $\{49,65,114\}$, $\{50,62,112\}$, $\{51,55,106\}$, $\{52,56,108\}$, $\{53,54,107\}$
38	$\{6,37,43\}$, $\{8,36,44\}$, $\{9,39,48\}$, $\{10,35,45\}$, $\{11,38,49\}$, $\{12,34,46\}$, $\{13,76,89\}$, $\{14,77,91\}$, $\{15,75,90\}$, $\{16,78,94\}$, $\{17,79,96\}$, $\{18,74,92\}$, $\{19,80,99\}$, $\{20,73,93\}$, $\{21,81,102\}$, $\{22,82,104\}$, $\{23,72,95\}$, $\{24,83,107\}$, $\{25,84,109\}$, $\{26,71,97\}$, $\{27,85,112\}$, $\{28,70,98\}$, $\{29,86,115\}$, $\{30,87,117\}$, $\{31,69,100\}$, $\{32,88,120\}$, $\{33,68,101\}$, $\{40,63,103\}$, $\{41,64,105\}$, $\{42,66,108\}$, $\{47,59,106\}$, $\{50,60,110\}$, $\{51,67,118\}$, $\{52,62,114\}$, $\{53,58,111\}$, $\{54,65,119\}$, $\{55,61,116\}$, $\{56,57,113\}$
39	$\{6,36,42\}$, $\{8,35,43\}$, $\{9,37,46\}$, $\{10,38,48\}$, $\{11,34,45\}$, $\{12,79,91\}$, $\{13,80,93\}$, $\{14,33,47\}$, $\{15,77,92\}$, $\{16,78,94\}$, $\{17,81,98\}$, $\{18,82,100\}$, $\{19,76,95\}$, $\{20,83,103\}$, $\{21,75,96\}$, $\{22,84,106\}$, $\{23,74,97\}$, $\{24,85,109\}$, $\{25,86,111\}$, $\{26,73,99\}$, $\{27,87,114\}$, $\{28,90,118\}$, $\{29,72,101\}$, $\{30,89,119\}$, $\{31,71,102\}$, $\{32,88,120\}$, $\{39,65,104\}$, $\{40,67,107\}$, $\{41,64,105\}$, $\{44,66,110\}$, $\{49,59,108\}$, $\{50,62,112\}$, $\{51,70,121\}$, $\{52,61,113\}$, $\{53,69,122\}$, $\{54,63,117\}$, $\{55,68,123\}$, $\{56,60,116\}$, $\{57,58,115\}$
40	$\{6,47,53\}$, $\{8,46,54\}$, $\{9,48,57\}$, $\{10,45,55\}$, $\{11,50,61\}$, $\{12,44,56\}$, $\{13,82,95\}$, $\{14,83,97\}$, $\{15,43,58\}$, $\{16,80,96\}$, $\{17,42,59\}$, $\{18,81,99\}$, $\{19,79,98\}$, $\{20,84,104\}$, $\{21,91,112\}$, $\{22,78,100\}$, $\{23,90,113\}$, $\{24,77,101\}$, $\{25,89,114\}$, $\{26,76,102\}$, $\{27,92,119\}$, $\{28,75,103\}$, $\{29,87,116\}$, $\{30,94,124\}$, $\{31,74,105\}$, $\{32,93,125\}$, $\{33,73,106\}$, $\{34,86,120\}$, $\{35,88,123\}$, $\{36,85,121\}$, $\{37,70,107\}$, $\{38,72,110\}$, $\{39,69,108\}$, $\{40,71,111\}$, $\{41,68,109\}$, $\{49,66,115\}$, $\{51,67,118\}$, $\{52,65,117\}$, $\{60,62,122\}$, $\{63,64,127\}$
41	$\{6,35,41\}$, $\{8,34,42\}$, $\{9,36,45\}$, $\{10,33,43\}$, $\{11,84,95\}$, $\{12,32,44\}$, $\{13,83,96\}$, $\{14,85,99\}$, $\{15,31,46\}$, $\{16,81,97\}$, $\{17,86,103\}$, $\{18,80,98\}$, $\{19,82,101\}$, $\{20,87,107\}$, $\{21,79,100\}$, $\{22,88,110\}$, $\{23,89,112\}$, $\{24,78,102\}$, $\{25,91,116\}$, $\{26,94,120\}$, $\{27,77,104\}$, $\{28,93,121\}$, $\{29,76,105\}$, $\{30,92,122\}$, $\{37,69,106\}$, $\{38,70,108\}$, $\{39,72,111\}$, $\{40,90,130\}$, $\{47,62,109\}$, $\{48,65,113\}$, $\{49,66,115\}$, $\{50,64,114\}$, $\{51,75,126\}$, $\{52,71,123\}$, $\{53,74,127\}$, $\{54,63,117\}$, $\{55,73,128\}$, $\{56,68,124\}$, $\{57,61,118\}$, $\{58,67,125\}$, $\{59,60,119\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
42	$\{6,51,57\}$, $\{8,50,58\}$, $\{9,52,61\}$, $\{10,49,59\}$, $\{11,53,64\}$, $\{12,48,60\}$, $\{13,87,100\}$, $\{14,88,102\}$, $\{15,47,62\}$, $\{16,85,101\}$, $\{17,46,63\}$, $\{18,86,104\}$, $\{19,84,103\}$, $\{20,89,109\}$, $\{21,45,66\}$, $\{22,83,105\}$, $\{23,96,119\}$, $\{24,82,106\}$, $\{25,95,120\}$, $\{26,81,107\}$, $\{27,94,121\}$, $\{28,80,108\}$, $\{29,93,122\}$, $\{30,99,129\}$, $\{31,79,110\}$, $\{32,98,130\}$, $\{33,78,111\}$, $\{34,97,131\}$, $\{35,91,126\}$, $\{36,92,128\}$, $\{37,90,127\}$, $\{38,74,112\}$, $\{39,77,116\}$, $\{40,73,113\}$, $\{41,76,117\}$, $\{42,72,114\}$, $\{43,75,118\}$, $\{44,71,115\}$, $\{54,69,123\}$, $\{55,70,125\}$, $\{56,68,124\}$, $\{65,67,132\}$
43	$\{6,40,46\}$, $\{8,41,49\}$, $\{9,39,48\}$, $\{10,42,52\}$, $\{11,89,100\}$, $\{12,38,50\}$, $\{13,88,101\}$, $\{14,37,51\}$, $\{15,87,102\}$, $\{16,90,106\}$, $\{17,36,53\}$, $\{18,85,103\}$, $\{19,86,105\}$, $\{20,84,104\}$, $\{21,91,112\}$, $\{22,92,114\}$, $\{23,93,116\}$, $\{24,83,107\}$, $\{25,94,119\}$, $\{26,82,108\}$, $\{27,95,122\}$, $\{28,81,109\}$, $\{29,96,125\}$, $\{30,80,110\}$, $\{31,97,128\}$, $\{32,79,111\}$, $\{33,98,131\}$, $\{34,99,133\}$, $\{35,78,113\}$, $\{43,72,115\}$, $\{44,73,117\}$, $\{45,75,120\}$, $\{47,71,118\}$, $\{54,67,121\}$, $\{55,68,123\}$, $\{56,74,130\}$, $\{57,77,134\}$, $\{58,66,124\}$, $\{59,76,135\}$, $\{60,69,129\}$, $\{61,65,126\}$, $\{62,70,132\}$, $\{63,64,127\}$
44	$\{6,40,46\}$, $\{8,39,47\}$, $\{9,41,50\}$, $\{10,38,48\}$, $\{11,42,53\}$, $\{12,37,49\}$, $\{13,89,102\}$, $\{14,90,104\}$, $\{15,36,51\}$, $\{16,87,103\}$, $\{17,88,105\}$, $\{18,91,109\}$, $\{19,92,111\}$, $\{20,86,106\}$, $\{21,93,114\}$, $\{22,85,107\}$, $\{23,94,117\}$, $\{24,84,108\}$, $\{25,95,120\}$, $\{26,96,122\}$, $\{27,83,110\}$, $\{28,97,125\}$, $\{29,98,127\}$, $\{30,82,112\}$, $\{31,100,131\}$, $\{32,81,113\}$, $\{33,99,132\}$, $\{34,101,135\}$, $\{35,80,115\}$, $\{43,73,116\}$, $\{44,74,118\}$, $\{45,76,121\}$, $\{52,67,119\}$, $\{54,69,123\}$, $\{55,71,126\}$, $\{56,68,124\}$, $\{57,77,134\}$, $\{58,72,130\}$, $\{59,78,137\}$, $\{60,79,139\}$, $\{61,75,136\}$, $\{62,66,128\}$, $\{63,70,133\}$, $\{64,65,129\}$
46	$\{6,58,64\}$, $\{8,57,65\}$, $\{9,59,68\}$, $\{10,56,66\}$, $\{11,60,71\}$, $\{12,55,67\}$, $\{13,61,74\}$, $\{14,96,110\}$, $\{15,54,69\}$, $\{16,95,111\}$, $\{17,97,114\}$, $\{18,94,112\}$, $\{19,53,72\}$, $\{20,93,113\}$, $\{21,52,73\}$, $\{22,98,120\}$, $\{23,92,115\}$, $\{24,51,75\}$, $\{25,91,116\}$, $\{26,106,132\}$, $\{27,90,117\}$, $\{28,107,135\}$, $\{29,89,118\}$, $\{30,104,134\}$, $\{31,88,119\}$, $\{32,109,141\}$, $\{33,103,136\}$, $\{34,87,121\}$, $\{35,108,143\}$, $\{36,86,122\}$, $\{37,100,137\}$, $\{38,101,139\}$, $\{39,105,144\}$, $\{40,102,142\}$, $\{41,99,140\}$, $\{42,81,123\}$, $\{43,82,125\}$, $\{44,80,124\}$, $\{45,84,129\}$, $\{46,85,131\}$, $\{47,83,130\}$, $\{48,78,126\}$, $\{49,79,128\}$, $\{50,77,127\}$, $\{62,76,138\}$, $\{63,70,133\}$
47	$\{6,71,77\}$, $\{8,117,125\}$, $\{9,69,78\}$, $\{10,116,126\}$, $\{11,68,79\}$, $\{12,115,127\}$, $\{13,67,80\}$, $\{14,114,128\}$, $\{15,66,81\}$, $\{16,113,129\}$, $\{17,65,82\}$, $\{18,112,130\}$, $\{19,64,83\}$, $\{20,111,131\}$, $\{21,63,84\}$, $\{22,110,132\}$, $\{23,62,85\}$, $\{24,98,122\}$, $\{25,109,134\}$, $\{26,61,87\}$, $\{27,108,135\}$, $\{28,60,88\}$, $\{29,107,136\}$, $\{30,59,89\}$, $\{31,106,137\}$, $\{32,58,90\}$, $\{33,105,138\}$, $\{34,57,91\}$, $\{35,104,139\}$, $\{36,56,92\}$, $\{37,103,140\}$, $\{38,55,93\}$, $\{39,102,141\}$, $\{40,54,94\}$, $\{41,101,142\}$, $\{42,76,118\}$, $\{43,100,143\}$, $\{44,75,119\}$, $\{45,99,144\}$, $\{46,74,120\}$, $\{47,86,133\}$, $\{48,97,145\}$, $\{49,72,121\}$, $\{50,96,146\}$, $\{51,73,124\}$, $\{52,95,147\}$, $\{53,70,123\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
48	$\{6,125,131\}$, $\{8,124,132\}$, $\{9,74,83\}$, $\{10,72,82\}$, $\{11,70,81\}$, $\{12,121,133\}$, $\{13,122,135\}$, $\{14,120,134\}$, $\{15,71,86\}$, $\{16,69,85\}$, $\{17,67,84\}$, $\{18,118,136\}$, $\{19,119,138\}$, $\{20,117,137\}$, $\{21,68,89\}$, $\{22,65,87\}$, $\{23,80,103\}$, $\{24,115,139\}$, $\{25,88,113\}$, $\{26,78,104\}$, $\{27,102,129\}$, $\{28,114,142\}$, $\{29,63,92\}$, $\{30,64,94\}$, $\{31,62,93\}$, $\{32,111,143\}$, $\{33,112,145\}$, $\{34,110,144\}$, $\{35,60,95\}$, $\{36,61,97\}$, $\{37,59,96\}$, $\{38,108,146\}$, $\{39,109,148\}$, $\{40,107,147\}$, $\{41,57,98\}$, $\{42,58,100\}$, $\{43,56,99\}$, $\{44,105,149\}$, $\{45,106,151\}$, $\{46,77,123\}$, $\{47,54,101\}$, $\{48,79,127\}$, $\{49,91,140\}$, $\{50,66,116\}$, $\{51,90,141\}$, $\{52,76,128\}$, $\{53,73,126\}$, $\{55,75,130\}$
49	$\{6,43,49\}$, $\{8,42,50\}$, $\{9,44,53\}$, $\{10,45,55\}$, $\{11,41,52\}$, $\{12,101,113\}$, $\{13,102,115\}$, $\{14,40,54\}$, $\{15,99,114\}$, $\{16,100,116\}$, $\{17,39,56\}$, $\{18,103,121\}$, $\{19,38,57\}$, $\{20,97,117\}$, $\{21,98,119\}$, $\{22,96,118\}$, $\{23,104,127\}$, $\{24,105,129\}$, $\{25,95,120\}$, $\{26,106,132\}$, $\{27,107,134\}$, $\{28,94,122\}$, $\{29,108,137\}$, $\{30,93,123\}$, $\{31,109,140\}$, $\{32,92,124\}$, $\{33,112,145\}$, $\{34,91,125\}$, $\{35,111,146\}$, $\{36,90,126\}$, $\{37,110,147\}$, $\{46,82,128\}$, $\{47,83,130\}$, $\{48,85,133\}$, $\{51,80,131\}$, $\{58,77,135\}$, $\{59,79,138\}$, $\{60,76,136\}$, $\{61,78,139\}$, $\{62,86,148\}$, $\{63,87,150\}$, $\{64,88,152\}$, $\{65,89,154\}$, $\{66,75,141\}$, $\{67,84,151\}$, $\{68,81,149\}$, $\{69,73,142\}$, $\{70,74,144\}$, $\{71,72,143\}$
50	$\{6,77,83\}$, $\{8,129,137\}$, $\{9,76,85\}$, $\{10,128,138\}$, $\{11,75,86\}$, $\{12,127,139\}$, $\{13,74,87\}$, $\{14,126,140\}$, $\{15,73,88\}$, $\{16,125,141\}$, $\{17,72,89\}$, $\{18,124,142\}$, $\{19,71,90\}$, $\{20,123,143\}$, $\{21,70,91\}$, $\{22,122,144\}$, $\{23,69,92\}$, $\{24,121,145\}$, $\{25,93,118\}$, $\{26,105,131\}$, $\{27,107,134\}$, $\{28,80,108\}$, $\{29,66,95\}$, $\{30,117,147\}$, $\{31,65,96\}$, $\{32,116,148\}$, $\{33,64,97\}$, $\{34,115,149\}$, $\{35,63,98\}$, $\{36,114,150\}$, $\{37,62,99\}$, $\{38,113,151\}$, $\{39,61,100\}$, $\{40,112,152\}$, $\{41,60,101\}$, $\{42,111,153\}$, $\{43,59,102\}$, $\{44,110,154\}$, $\{45,58,103\}$, $\{46,109,155\}$, $\{47,57,104\}$, $\{48,82,130\}$, $\{49,84,133\}$, $\{50,106,156\}$, $\{51,68,119\}$, $\{52,94,146\}$, $\{53,67,120\}$, $\{54,78,132\}$, $\{55,81,136\}$, $\{56,79,135\}$
51	$\{6,77,83\}$, $\{8,127,135\}$, $\{9,75,84\}$, $\{10,126,136\}$, $\{11,74,85\}$, $\{12,125,137\}$, $\{13,73,86\}$, $\{14,124,138\}$, $\{15,72,87\}$, $\{16,123,139\}$, $\{17,71,88\}$, $\{18,122,140\}$, $\{19,70,89\}$, $\{20,121,141\}$, $\{21,69,90\}$, $\{22,120,142\}$, $\{23,68,91\}$, $\{24,119,143\}$, $\{25,67,92\}$, $\{26,106,132\}$, $\{27,118,145\}$, $\{28,66,94\}$, $\{29,117,146\}$, $\{30,65,95\}$, $\{31,116,147\}$, $\{32,64,96\}$, $\{33,115,148\}$, $\{34,63,97\}$, $\{35,114,149\}$, $\{36,62,98\}$, $\{37,113,150\}$, $\{38,61,99\}$, $\{39,112,151\}$, $\{40,60,100\}$, $\{41,111,152\}$, $\{42,59,101\}$, $\{43,110,153\}$, $\{44,58,102\}$, $\{45,109,154\}$, $\{46,82,128\}$, $\{47,108,155\}$, $\{48,81,129\}$, $\{49,107,156\}$, $\{50,80,130\}$, $\{51,93,144\}$, $\{52,105,157\}$, $\{53,78,131\}$, $\{54,104,158\}$, $\{55,79,134\}$, $\{56,103,159\}$, $\{57,76,133\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
52	$\{6,135,141\}$, $\{8,79,87\}$, $\{9,133,142\}$, $\{10,78,88\}$, $\{11,132,143\}$, $\{12,77,89\}$, $\{13,131,144\}$, $\{14,76,90\}$, $\{15,130,145\}$, $\{16,75,91\}$, $\{17,129,146\}$, $\{18,74,92\}$, $\{19,128,147\}$, $\{20,73,93\}$, $\{21,127,148\}$, $\{22,72,94\}$, $\{23,126,149\}$, $\{24,71,95\}$, $\{25,86,111\}$, $\{26,124,150\}$, $\{27,96,123\}$, $\{28,84,112\}$, $\{29,110,139\}$, $\{30,68,98\}$, $\{31,69,100\}$, $\{32,121,153\}$, $\{33,122,155\}$, $\{34,120,154\}$, $\{35,66,101\}$, $\{36,67,103\}$, $\{37,65,102\}$, $\{38,118,156\}$, $\{39,119,158\}$, $\{40,117,157\}$, $\{41,63,104\}$, $\{42,64,106\}$, $\{43,62,105\}$, $\{44,115,159\}$, $\{45,116,161\}$, $\{46,114,160\}$, $\{47,60,107\}$, $\{48,61,109\}$, $\{49,59,108\}$, $\{50,113,163\}$, $\{51,83,134\}$, $\{52,85,137\}$, $\{53,99,152\}$, $\{54,97,151\}$, $\{55,70,125\}$, $\{56,80,136\}$, $\{57,81,138\}$, $\{58,82,140\}$
53	$\{6,40,46\}$, $\{8,39,47\}$, $\{9,41,50\}$, $\{10,38,48\}$, $\{11,42,53\}$, $\{12,109,121\}$, $\{13,110,123\}$, $\{14,37,51\}$, $\{15,107,122\}$, $\{16,36,52\}$, $\{17,108,125\}$, $\{18,106,124\}$, $\{19,35,54\}$, $\{20,111,131\}$, $\{21,105,126\}$, $\{22,112,134\}$, $\{23,104,127\}$, $\{24,113,137\}$, $\{25,103,128\}$, $\{26,114,140\}$, $\{27,102,129\}$, $\{28,115,143\}$, $\{29,101,130\}$, $\{30,116,146\}$, $\{31,119,150\}$, $\{32,100,132\}$, $\{33,120,153\}$, $\{34,99,133\}$, $\{43,92,135\}$, $\{44,118,162\}$, $\{45,91,136\}$, $\{49,117,166\}$, $\{55,83,138\}$, $\{56,85,141\}$, $\{57,82,139\}$, $\{58,84,142\}$, $\{59,86,145\}$, $\{60,87,147\}$, $\{61,94,155\}$, $\{62,95,157\}$, $\{63,81,144\}$, $\{64,96,160\}$, $\{65,98,163\}$, $\{66,90,156\}$, $\{67,97,164\}$, $\{68,93,161\}$, $\{69,89,158\}$, $\{70,78,148\}$, $\{71,88,159\}$, $\{72,77,149\}$, $\{73,79,152\}$, $\{74,80,154\}$, $\{75,76,151\}$
54	$\{6,83,89\}$, $\{8,139,147\}$, $\{9,82,91\}$, $\{10,138,148\}$, $\{11,81,92\}$, $\{12,137,149\}$, $\{13,80,93\}$, $\{14,136,150\}$, $\{15,79,94\}$, $\{16,135,151\}$, $\{17,78,95\}$, $\{18,134,152\}$, $\{19,77,96\}$, $\{20,133,153\}$, $\{21,76,97\}$, $\{22,132,154\}$, $\{23,75,98\}$, $\{24,131,155\}$, $\{25,74,99\}$, $\{26,130,156\}$, $\{27,100,127\}$, $\{28,113,141\}$, $\{29,115,144\}$, $\{30,86,116\}$, $\{31,71,102\}$, $\{32,126,158\}$, $\{33,70,103\}$, $\{34,125,159\}$, $\{35,69,104\}$, $\{36,124,160\}$, $\{37,68,105\}$, $\{38,123,161\}$, $\{39,67,106\}$, $\{40,122,162\}$, $\{41,66,107\}$, $\{42,121,163\}$, $\{43,65,108\}$, $\{44,120,164\}$, $\{45,64,109\}$, $\{46,119,165\}$, $\{47,63,110\}$, $\{48,118,166\}$, $\{49,62,111\}$, $\{50,117,167\}$, $\{51,61,112\}$, $\{52,88,140\}$, $\{53,90,143\}$, $\{54,114,168\}$, $\{55,73,128\}$, $\{56,101,157\}$, $\{57,72,129\}$, $\{58,84,142\}$, $\{59,87,146\}$, $\{60,85,145\}$
55	$\{6,46,52\}$, $\{8,45,53\}$, $\{9,47,56\}$, $\{10,48,58\}$, $\{11,44,55\}$, $\{12,49,61\}$, $\{13,113,126\}$, $\{14,43,57\}$, $\{15,112,127\}$, $\{16,114,130\}$, $\{17,42,59\}$, $\{18,110,128\}$, $\{19,41,60\}$, $\{20,109,129\}$, $\{21,111,132\}$, $\{22,115,137\}$, $\{23,108,131\}$, $\{24,116,140\}$, $\{25,117,142\}$, $\{26,107,133\}$, $\{27,118,145\}$, $\{28,106,134\}$, $\{29,119,148\}$, $\{30,105,135\}$, $\{31,120,151\}$, $\{32,104,136\}$, $\{33,121,154\}$, $\{34,122,156\}$, $\{35,103,138\}$, $\{36,124,160\}$, $\{37,102,139\}$, $\{38,123,161\}$, $\{39,125,164\}$, $\{40,101,141\}$, $\{50,93,143\}$, $\{51,95,146\}$, $\{54,90,144\}$, $\{62,85,147\}$, $\{63,86,149\}$, $\{64,88,152\}$, $\{65,97,162\}$, $\{66,84,150\}$, $\{67,96,163\}$, $\{68,87,155\}$, $\{69,100,169\}$, $\{70,83,153\}$, $\{71,99,170\}$, $\{72,94,166\}$, $\{73,98,171\}$, $\{74,91,165\}$, $\{75,82,157\}$, $\{76,92,168\}$, $\{77,81,158\}$, $\{78,89,167\}$, $\{79,80,159\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
56	$\{6,145,151\}$, $\{8,85,93\}$, $\{9,143,152\}$, $\{10,84,94\}$, $\{11,142,153\}$, $\{12,83,95\}$, $\{13,141,154\}$, $\{14,82,96\}$, $\{15,140,155\}$, $\{16,81,97\}$, $\{17,139,156\}$, $\{18,80,98\}$, $\{19,138,157\}$, $\{20,79,99\}$, $\{21,137,158\}$, $\{22,78,100\}$, $\{23,136,159\}$, $\{24,77,101\}$, $\{25,135,160\}$, $\{26,76,102\}$, $\{27,92,119\}$, $\{28,133,161\}$, $\{29,103,132\}$, $\{30,90,120\}$, $\{31,118,149\}$, $\{32,73,105\}$, $\{33,74,107\}$, $\{34,130,164\}$, $\{35,131,166\}$, $\{36,129,165\}$, $\{37,71,108\}$, $\{38,72,110\}$, $\{39,70,109\}$, $\{40,127,167\}$, $\{41,128,169\}$, $\{42,126,168\}$, $\{43,68,111\}$, $\{44,69,113\}$, $\{45,67,112\}$, $\{46,124,170\}$, $\{47,125,172\}$, $\{48,123,171\}$, $\{49,65,114\}$, $\{50,66,116\}$, $\{51,64,115\}$, $\{52,121,173\}$, $\{53,122,175\}$, $\{54,63,117\}$, $\{55,89,144\}$, $\{56,91,147\}$, $\{57,106,163\}$, $\{58,104,162\}$, $\{59,75,134\}$, $\{60,86,146\}$, $\{61,87,148\}$, $\{62,88,150\}$
57	$\{6,150,156\}$, $\{8,91,99\}$, $\{9,149,158\}$, $\{10,147,157\}$, $\{11,89,100\}$, $\{12,90,102\}$, $\{13,88,101\}$, $\{14,145,159\}$, $\{15,146,161\}$, $\{16,144,160\}$, $\{17,86,103\}$, $\{18,87,105\}$, $\{19,85,104\}$, $\{20,142,162\}$, $\{21,143,164\}$, $\{22,141,163\}$, $\{23,83,106\}$, $\{24,84,108\}$, $\{25,82,107\}$, $\{26,139,165\}$, $\{27,140,167\}$, $\{28,138,166\}$, $\{29,80,109\}$, $\{30,81,111\}$, $\{31,93,124\}$, $\{32,78,110\}$, $\{33,135,168\}$, $\{34,136,170\}$, $\{35,134,169\}$, $\{36,76,112\}$, $\{37,77,114\}$, $\{38,75,113\}$, $\{39,132,171\}$, $\{40,133,173\}$, $\{41,131,172\}$, $\{42,73,115\}$, $\{43,74,117\}$, $\{44,72,116\}$, $\{45,129,174\}$, $\{46,130,176\}$, $\{47,128,175\}$, $\{48,70,118\}$, $\{49,71,120\}$, $\{50,98,148\}$, $\{51,68,119\}$, $\{52,69,121\}$, $\{53,125,178\}$, $\{54,97,151\}$, $\{55,67,122\}$, $\{56,96,152\}$, $\{57,66,123\}$, $\{58,79,137\}$, $\{59,94,153\}$, $\{60,95,155\}$, $\{61,65,126\}$, $\{62,92,154\}$, $\{63,64,127\}$
58	$\{6,89,95\}$, $\{8,149,157\}$, $\{9,88,97\}$, $\{10,148,158\}$, $\{11,87,98\}$, $\{12,147,159\}$, $\{13,86,99\}$, $\{14,146,160\}$, $\{15,85,100\}$, $\{16,145,161\}$, $\{17,84,101\}$, $\{18,144,162\}$, $\{19,83,102\}$, $\{20,143,163\}$, $\{21,82,103\}$, $\{22,142,164\}$, $\{23,81,104\}$, $\{24,141,165\}$, $\{25,80,105\}$, $\{26,140,166\}$, $\{27,79,106\}$, $\{28,139,167\}$, $\{29,107,136\}$, $\{30,121,151\}$, $\{31,123,154\}$, $\{32,92,124\}$, $\{33,76,109\}$, $\{34,135,169\}$, $\{35,75,110\}$, $\{36,134,170\}$, $\{37,74,111\}$, $\{38,133,171\}$, $\{39,73,112\}$, $\{40,132,172\}$, $\{41,72,113\}$, $\{42,131,173\}$, $\{43,71,114\}$, $\{44,130,174\}$, $\{45,70,115\}$, $\{46,129,175\}$, $\{47,69,116\}$, $\{48,128,176\}$, $\{49,68,117\}$, $\{50,127,177\}$, $\{51,67,118\}$, $\{52,126,178\}$, $\{53,66,119\}$, $\{54,125,179\}$, $\{55,65,120\}$, $\{56,94,150\}$, $\{57,96,153\}$, $\{58,122,180\}$, $\{59,78,137\}$, $\{60,108,168\}$, $\{61,77,138\}$, $\{62,90,152\}$, $\{63,93,156\}$, $\{64,91,155\}$
59	$\{6,89,95\}$, $\{8,147,155\}$, $\{9,87,96\}$, $\{10,146,156\}$, $\{11,86,97\}$, $\{12,145,157\}$, $\{13,85,98\}$, $\{14,144,158\}$, $\{15,84,99\}$, $\{16,143,159\}$, $\{17,83,100\}$, $\{18,142,160\}$, $\{19,82,101\}$, $\{20,141,161\}$, $\{21,81,102\}$, $\{22,140,162\}$, $\{23,80,103\}$, $\{24,139,163\}$, $\{25,79,104\}$, $\{26,138,164\}$, $\{27,78,105\}$, $\{28,137,165\}$, $\{29,77,106\}$, $\{30,122,152\}$, $\{31,136,167\}$, $\{32,76,108\}$, $\{33,135,168\}$, $\{34,75,109\}$, $\{35,134,169\}$, $\{36,74,110\}$, $\{37,133,170\}$, $\{38,73,111\}$, $\{39,132,171\}$, $\{40,72,112\}$, $\{41,131,172\}$, $\{42,71,113\}$, $\{43,130,173\}$, $\{44,70,114\}$, $\{45,129,174\}$, $\{46,69,115\}$, $\{47,128,175\}$, $\{48,68,116\}$, $\{49,127,176\}$, $\{50,67,117\}$, $\{51,126,177\}$, $\{52,66,118\}$, $\{53,125,178\}$, $\{54,94,148\}$, $\{55,124,179\}$, $\{56,93,149\}$, $\{57,123,180\}$, $\{58,92,150\}$, $\{59,107,166\}$, $\{60,121,181\}$, $\{61,90,151\}$, $\{62,120,182\}$, $\{63,91,154\}$, $\{64,119,183\}$, $\{65,88,153\}$

Table A.26: Difference triples for Lemma 1.3.6

t	Difference triples
60	$\{6,155,161\}$, $\{8,154,162\}$, $\{9,92,101\}$, $\{10,90,100\}$, $\{11,88,99\}$, $\{12,151,163\}$, $\{13,152,165\}$, $\{14,150,164\}$, $\{15,89,104\}$, $\{16,87,103\}$, $\{17,85,102\}$, $\{18,148,166\}$, $\{19,149,168\}$, $\{20,147,167\}$, $\{21,86,107\}$, $\{22,84,106\}$, $\{23,82,105\}$, $\{24,145,169\}$, $\{25,146,171\}$, $\{26,144,170\}$, $\{27,83,110\}$, $\{28,80,108\}$, $\{29,98,127\}$, $\{30,142,172\}$, $\{31,109,140\}$, $\{32,96,128\}$, $\{33,126,159\}$, $\{34,141,175\}$, $\{35,78,113\}$, $\{36,79,115\}$, $\{37,77,114\}$, $\{38,138,176\}$, $\{39,139,178\}$, $\{40,137,177\}$, $\{41,75,116\}$, $\{42,76,118\}$, $\{43,74,117\}$, $\{44,135,179\}$, $\{45,136,181\}$, $\{46,134,180\}$, $\{47,72,119\}$, $\{48,73,121\}$, $\{49,71,120\}$, $\{50,132,182\}$, $\{51,133,184\}$, $\{52,131,183\}$, $\{53,69,122\}$, $\{54,70,124\}$, $\{55,68,123\}$, $\{56,129,185\}$, $\{57,130,187\}$, $\{58,95,153\}$, $\{59,66,125\}$, $\{60,97,157\}$, $\{61,112,173\}$, $\{62,81,143\}$, $\{63,111,174\}$, $\{64,94,158\}$, $\{65,91,156\}$, $\{67,93,160\}$

Table A.26: Difference triples for Lemma 1.3.6

r	π_r
1	$\{\{5,6,7,8,10\}\}$
2	$\{\{5,7,8,9,11\}, \{6,10,12,13,15\}\}$
3	$\{\{5,8,11,16,18\}, \{6,9,12,13,14\}, \{7,10,15,17,19\}\}$
4	$\{\{5,11,13,21,24\}, \{6,8,17,19,22\}, \{7,14,18,20,23\}, \{9,10,12,15,16\}\}$
5	$\{\{5,13,22,23,27\}, \{6,14,15,16,21\}, \{7,8,19,20,24\}, \{9,11,17,25,28\}, \{10,12,18,26,30\}\}$
6	$\{\{5,6,21,22,32\}, \{7,8,20,23,28\}, \{9,12,19,31,33\}, \{10,14,26,27,29\}, \{11,15,16,18,24\}, \{13,17,25,30,35\}\}$

Table A.27: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4\}$

r	π_r
1	$\{\{5,7,8,9,11\}\}$
2	$\{\{5,8,11,13,15\}, \{7,9,10,12,14\}\}$
3	$\{\{5,7,14,18,20\}, \{8,10,15,16,17\}, \{9,11,12,13,19\}\}$
4	$\{\{5,7,15,23,26\}, \{8,9,20,21,24\}, \{10,13,16,17,22\}, \{11,12,14,18,19\}\}$
5	$\{\{5,11,19,26,29\}, \{7,9,23,24,31\}, \{8,16,25,27,28\}, \{10,13,15,18,20\}, \{12,14,17,21,22\}\}$
6	$\{\{5,13,21,30,33\}, \{7,12,25,29,35\}, \{8,9,22,26,31\}, \{10,11,27,28,34\}, \{14,15,18,23,24\}, \{16,17,19,20,32\}\}$
7	$\{\{5,10,25,30,40\}, \{7,12,23,34,38\}, \{8,14,27,32,37\}, \{9,19,31,36,39\}, \{11,20,29,33,35\}, \{13,15,18,22,24\}, \{16,17,21,26,28\}\}$
8	$\{\{5,17,31,35,44\}, \{7,12,29,30,40\}, \{8,24,36,37,41\}, \{9,11,25,38,43\}, \{10,15,33,34,42\}, \{13,19,22,26,28\}, \{14,16,20,23,27\}, \{18,21,32,39,46\}\}$
9	$\{\{5,13,26,37,45\}, \{7,14,27,41,47\}, \{8,10,30,34,46\}, \{9,16,36,38,49\}, \{11,17,29,43,44\}, \{12,23,25,28,32\}, \{15,18,35,40,42\}, \{19,21,24,31,33\}, \{20,22,39,48,51\}\}$
10	$\{\{5,16,28,47,54\}, \{7,15,29,46,53\}, \{8,17,30,45,50\}, \{9,14,31,44,52\}, \{10,13,32,42,51\}, \{11,18,36,41,48\}, \{12,19,34,40,43\}, \{20,25,27,33,39\}, \{21,23,38,49,55\}, \{22,24,26,35,37\}\}$

Table A.28: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,6\}$

r	π_r
11	$\{\{5,17,30,51,59\}, \{7,16,31,50,58\}, \{8,18,32,49,55\}, \{9,15,33,48,57\}, \{10,19,34,47,52\}, \{11,14,35,46,56\}, \{12,20,40,45,53\}, \{13,22,36,43,44\}, \{21,27,42,54,60\}, \{23,26,29,37,41\}, \{24,25,28,38,39\}\}$
12	$\{\{5,18,32,55,64\}, \{7,17,33,54,63\}, \{8,19,34,53,60\}, \{9,16,35,52,62\}, \{10,20,36,51,57\}, \{11,15,37,50,61\}, \{12,21,40,49,56\}, \{13,14,38,48,59\}, \{22,23,43,45,47\}, \{24,30,31,39,46\}, \{25,27,44,58,66\}, \{26,28,29,41,42\}\}$
13	$\{\{5,19,34,59,69\}, \{7,18,35,58,68\}, \{8,20,36,57,65\}, \{9,17,37,56,67\}, \{10,21,38,55,62\}, \{11,16,39,54,66\}, \{12,22,41,53,60\}, \{13,15,40,52,64\}, \{14,23,47,51,61\}, \{24,25,44,45,50\}, \{26,31,49,63,71\}, \{27,30,33,42,48\}, \{28,29,32,43,46\}\}$
14	$\{\{5,20,36,63,74\}, \{7,19,37,62,73\}, \{8,21,38,61,70\}, \{9,18,39,60,72\}, \{10,22,40,59,67\}, \{11,17,41,58,71\}, \{12,23,42,57,64\}, \{13,24,46,56,65\}, \{14,15,43,55,69\}, \{16,25,53,54,66\}, \{26,27,44,45,52\}, \{28,30,51,68,75\}, \{29,33,35,47,50\}, \{31,32,34,48,49\}\}$
15	$\{\{5,21,38,66,78\}, \{7,20,39,65,77\}, \{8,22,40,64,74\}, \{9,19,41,63,76\}, \{10,23,42,62,71\}, \{11,18,43,61,75\}, \{12,24,44,60,68\}, \{13,25,46,59,67\}, \{14,16,45,58,73\}, \{15,17,47,57,72\}, \{26,27,52,55,56\}, \{28,35,54,70,79\}, \{29,36,37,49,53\}, \{30,32,51,69,80\}, \{31,33,34,48,50\}\}$
16	$\{\{5,22,40,70,83\}, \{7,21,41,69,82\}, \{8,23,42,68,79\}, \{9,20,43,67,81\}, \{10,24,44,66,76\}, \{11,19,45,65,80\}, \{12,25,46,64,73\}, \{13,26,47,63,71\}, \{14,18,48,62,78\}, \{15,27,53,61,72\}, \{16,17,50,60,77\}, \{28,30,54,55,59\}, \{29,38,58,75,84\}, \{31,36,39,49,57\}, \{32,34,37,51,52\}, \{33,35,56,74,86\}\}$
17	$\{\{5,23,42,74,88\}, \{7,22,43,73,87\}, \{8,24,44,72,84\}, \{9,21,45,71,86\}, \{10,25,46,70,81\}, \{11,20,47,69,85\}, \{12,26,48,68,78\}, \{13,27,49,67,76\}, \{14,19,50,66,83\}, \{15,28,53,65,75\}, \{16,17,51,64,82\}, \{18,29,61,63,77\}, \{30,33,54,55,62\}, \{31,38,60,80,89\}, \{32,39,40,52,59\}, \{34,36,58,79,91\}, \{35,37,41,56,57\}\}$
18	$\{\{5,22,43,75,91\}, \{7,21,44,74,90\}, \{8,23,45,73,87\}, \{9,20,46,72,89\}, \{10,24,47,71,84\}, \{11,19,48,70,88\}, \{12,25,50,69,82\}, \{13,26,51,68,80\}, \{14,27,52,67,78\}, \{15,18,53,66,86\}, \{16,28,55,65,76\}, \{17,29,59,64,77\}, \{30,38,49,54,63\}, \{31,39,62,85,93\}, \{32,40,61,81,92\}, \{33,41,42,56,60\}, \{34,35,58,83,94\}, \{36,37,57,79,95\}\}$
19	$\{\{5,23,45,79,96\}, \{7,22,46,78,95\}, \{8,24,48,77,93\}, \{9,25,49,76,91\}, \{10,21,50,75,94\}, \{11,26,51,74,88\}, \{12,27,52,73,86\}, \{13,20,53,72,92\}, \{14,28,54,71,83\}, \{15,29,55,70,81\}, \{16,19,56,69,90\}, \{17,30,59,68,80\}, \{18,31,64,67,82\}, \{32,42,66,89,97\}, \{33,43,65,87,98\}, \{34,39,47,57,63\}, \{35,41,44,58,62\}, \{36,40,61,84,99\}, \{37,38,60,85,100\}\}$
20	$\{\{5,24,47,82,100\}, \{7,23,48,81,99\}, \{8,25,49,80,96\}, \{9,22,50,79,98\}, \{10,26,51,78,93\}, \{11,21,52,77,97\}, \{12,27,53,76,90\}, \{13,28,54,75,88\}, \{14,20,55,74,95\}, \{15,29,56,73,85\}, \{16,19,57,72,94\}, \{17,30,59,71,83\}, \{18,31,65,70,86\}, \{32,34,60,63,69\}, \{33,44,68,92,101\}, \{35,43,67,91,102\}, \{36,42,46,58,66\}, \{37,41,64,89,103\}, \{38,40,45,61,62\}, \{39,84,87,104,106\}\}$

Table A.28: Table of partitions for Lemma 1.3.7 when $S = \{1, 2, 3, 4, 6\}$

r	π_r
21	$\{\{5,25,49,86,105\}$, $\{7,24,50,85,104\}$, $\{8,26,51,84,101\}$, $\{9,23,52,83,103\}$, $\{10,27,53,82,98\}$, $\{11,22,54,81,102\}$, $\{12,28,55,80,95\}$, $\{13,29,56,79,93\}$, $\{14,21,57,78,100\}$, $\{15,30,58,77,90\}$, $\{16,20,59,76,99\}$, $\{17,31,60,75,87\}$, $\{18,32,64,74,88\}$, $\{19,33,68,73,89\}$, $\{34,39,61,62,72\}$, $\{35,45,71,97,106\}$, $\{36,46,70,96,108\}$, $\{37,47,48,63,69\}$, $\{38,42,67,94,107\}$, $\{40,43,66,92,109\}$, $\{41,44,65,91,111\}\}$
22	$\{\{5,26,51,90,110\}$, $\{7,25,52,89,109\}$, $\{8,27,53,88,106\}$, $\{9,24,54,87,108\}$, $\{10,28,55,86,103\}$, $\{11,23,56,85,107\}$, $\{12,29,57,84,100\}$, $\{13,30,59,83,99\}$, $\{14,31,60,82,97\}$, $\{15,22,61,81,105\}$, $\{16,32,62,80,94\}$, $\{17,21,63,79,104\}$, $\{18,33,64,78,91\}$, $\{19,34,68,77,92\}$, $\{20,35,72,76,93\}$, $\{36,46,58,65,75\}$, $\{37,47,74,101,111\}$, $\{38,45,73,102,112\}$, $\{39,48,50,66,71\}$, $\{40,49,70,95,114\}$, $\{41,43,69,98,113\}$, $\{42,44,67,96,115\}\}$
23	$\{\{5,27,53,93,114\}$, $\{7,26,54,92,113\}$, $\{8,28,55,91,110\}$, $\{9,25,56,90,112\}$, $\{10,29,57,89,107\}$, $\{11,24,58,88,111\}$, $\{12,30,59,87,104\}$, $\{13,31,60,86,102\}$, $\{14,23,61,85,109\}$, $\{15,32,62,84,99\}$, $\{16,22,63,83,108\}$, $\{17,33,64,82,96\}$, $\{18,34,65,81,94\}$, $\{19,21,66,80,106\}$, $\{20,35,71,79,95\}$, $\{36,38,72,76,78\}$, $\{37,50,77,105,115\}$, $\{39,49,75,103,116\}$, $\{40,51,52,69,74\}$, $\{41,48,73,101,117\}$, $\{42,46,70,100,118\}$, $\{43,45,47,67,68\}$, $\{44,97,98,119,120\}\}$
24	$\{\{5,28,55,97,119\}$, $\{7,27,56,96,118\}$, $\{8,29,57,95,115\}$, $\{9,26,58,94,117\}$, $\{10,30,59,93,112\}$, $\{11,25,60,92,116\}$, $\{12,31,61,91,109\}$, $\{13,32,62,90,107\}$, $\{14,24,63,89,114\}$, $\{15,33,64,88,104\}$, $\{16,23,65,87,113\}$, $\{17,34,66,86,101\}$, $\{18,35,67,85,99\}$, $\{19,22,68,84,111\}$, $\{20,36,71,83,98\}$, $\{21,37,76,82,100\}$, $\{38,39,70,74,81\}$, $\{40,50,80,110,120\}$, $\{41,51,79,108,121\}$, $\{42,52,53,69,78\}$, $\{43,54,77,102,122\}$, $\{44,48,75,106,123\}$, $\{45,47,73,105,124\}$, $\{46,49,72,103,126\}\}$
25	$\{\{5,29,57,101,124\}$, $\{7,28,58,100,123\}$, $\{8,30,59,99,120\}$, $\{9,27,60,98,122\}$, $\{10,31,61,97,117\}$, $\{11,26,62,96,121\}$, $\{12,32,63,95,114\}$, $\{13,33,64,94,112\}$, $\{14,25,65,93,119\}$, $\{15,34,66,92,109\}$, $\{16,24,67,91,118\}$, $\{17,35,68,90,106\}$, $\{18,36,69,89,104\}$, $\{19,23,70,88,116\}$, $\{20,37,72,87,102\}$, $\{21,38,76,86,103\}$, $\{22,39,81,85,105\}$, $\{40,46,71,73,84\}$, $\{41,52,83,115,125\}$, $\{42,53,82,113,126\}$, $\{43,54,80,110,127\}$, $\{44,55,79,108,128\}$, $\{45,51,56,74,78\}$, $\{47,48,77,111,129\}$, $\{49,50,75,107,131\}\}$
26	$\{\{5,30,59,105,129\}$, $\{7,29,60,104,128\}$, $\{8,31,61,103,125\}$, $\{9,28,62,102,127\}$, $\{10,32,63,101,122\}$, $\{11,27,64,100,126\}$, $\{12,33,65,99,119\}$, $\{13,34,66,98,117\}$, $\{14,35,68,97,116\}$, $\{15,26,69,96,124\}$, $\{16,36,70,95,113\}$, $\{17,25,71,94,123\}$, $\{18,37,72,93,110\}$, $\{19,38,73,92,108\}$, $\{20,24,74,91,121\}$, $\{21,39,76,90,106\}$, $\{22,40,80,89,107\}$, $\{23,41,85,88,109\}$, $\{42,53,67,75,87\}$, $\{43,54,86,120,131\}$, $\{44,52,84,118,130\}$, $\{45,55,83,115,132\}$, $\{46,56,57,77,82\}$, $\{47,58,81,111,135\}$, $\{48,50,79,114,133\}$, $\{49,51,78,112,134\}\}$

Table A.28: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,6\}$

r	π_r
27	$\{\{5,31,61,108,133\}$, $\{7,30,62,107,132\}$, $\{8,32,63,106,129\}$, $\{9,29,64,105,131\}$, $\{10,33,65,104,126\}$, $\{11,28,66,103,130\}$, $\{12,34,67,102,123\}$, $\{13,35,68,101,121\}$, $\{14,27,69,100,128\}$, $\{15,36,70,99,118\}$, $\{16,26,71,98,127\}$, $\{17,37,72,97,115\}$, $\{18,38,73,96,113\}$, $\{19,25,74,95,125\}$, $\{20,39,75,94,110\}$, $\{21,24,76,93,124\}$, $\{22,40,79,92,109\}$, $\{23,41,84,91,111\}$, $\{42,43,83,88,90\}$, $\{44,57,89,122,134\}$, $\{45,58,87,119,135\}$, $\{46,56,86,120,136\}$, $\{47,55,60,77,85\}$, $\{48,53,59,78,82\}$, $\{49,52,81,117,137\}$, $\{50,54,80,114,138\}$, $\{51,112,116,139,140\}\}$
28	$\{\{5,34,64,115,140\}$, $\{7,33,65,114,139\}$, $\{8,35,66,113,136\}$, $\{9,32,67,112,138\}$, $\{10,36,68,111,133\}$, $\{11,31,69,110,137\}$, $\{12,37,70,109,130\}$, $\{13,38,71,108,128\}$, $\{14,30,72,107,135\}$, $\{15,39,73,106,125\}$, $\{16,29,74,105,134\}$, $\{17,40,75,104,122\}$, $\{18,41,76,103,120\}$, $\{19,28,77,102,132\}$, $\{20,42,78,101,117\}$, $\{21,27,79,100,131\}$, $\{22,43,82,99,116\}$, $\{23,26,80,98,129\}$, $\{24,44,89,97,118\}$, $\{25,45,93,96,119\}$, $\{46,47,83,85,95\}$, $\{48,60,94,127,141\}$, $\{49,59,92,126,142\}$, $\{50,61,91,123,143\}$, $\{51,57,63,81,90\}$, $\{52,56,88,124,144\}$, $\{53,58,62,86,87\}$, $\{54,55,84,121,146\}\}$
29	$\{\{5,35,66,119,145\}$, $\{7,34,67,118,144\}$, $\{8,36,68,117,141\}$, $\{9,33,69,116,143\}$, $\{10,37,70,115,138\}$, $\{11,32,71,114,142\}$, $\{12,38,72,113,135\}$, $\{13,39,73,112,133\}$, $\{14,31,74,111,140\}$, $\{15,40,75,110,130\}$, $\{16,30,76,109,139\}$, $\{17,41,77,108,127\}$, $\{18,42,78,107,125\}$, $\{19,29,79,106,137\}$, $\{20,43,80,105,122\}$, $\{21,44,82,104,121\}$, $\{22,28,83,103,136\}$, $\{23,45,86,102,120\}$, $\{24,27,84,101,134\}$, $\{25,46,94,100,123\}$, $\{26,47,98,99,124\}$, $\{48,53,81,85,97\}$, $\{49,61,96,132,146\}$, $\{50,62,95,131,148\}$, $\{51,60,93,129,147\}$, $\{52,63,64,87,92\}$, $\{54,58,91,128,149\}$, $\{55,59,65,89,90\}$, $\{56,57,88,126,151\}\}$
30	$\{\{5,53,97,103,142\}$, $\{7,59,83,118,135\}$, $\{8,22,72,105,147\}$, $\{9,21,71,100,141\}$, $\{10,14,69,101,146\}$, $\{11,36,86,106,145\}$, $\{12,34,73,121,148\}$, $\{13,33,92,98,144\}$, $\{15,66,104,109,132\}$, $\{16,17,68,99,134\}$, $\{18,52,94,116,140\}$, $\{19,51,87,110,127\}$, $\{20,65,111,112,138\}$, $\{23,27,82,93,125\}$, $\{24,30,75,108,129\}$, $\{25,61,74,139,151\}$, $\{26,43,89,117,137\}$, $\{28,54,85,123,126\}$, $\{29,49,88,120,130\}$, $\{31,35,102,114,150\}$, $\{32,45,107,113,143\}$, $\{37,63,77,131,154\}$, $\{38,46,95,122,133\}$, $\{39,57,90,149,155\}$, $\{40,50,70,76,96\}$, $\{41,67,84,128,152\}$, $\{42,56,81,136,153\}$, $\{44,55,58,78,79\}$, $\{47,60,64,80,91\}$, $\{48,62,115,119,124\}\}$

Table A.28: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,6\}$

r	π_r
2	$\{\{6,9,12,13,16\}$, $\{8,10,11,14,15\}\}$
3	$\{\{6,8,13,18,19\}$, $\{9,12,16,17,22\}$, $\{10,11,14,15,20\}\}$
4	$\{\{6,12,15,16,19\}$, $\{8,14,17,22,27\}$, $\{9,11,18,23,25\}$, $\{10,13,20,21,24\}\}$
5	$\{\{6,8,19,22,27\}$, $\{9,17,21,25,30\}$, $\{10,11,18,26,29\}$, $\{12,14,23,28,31\}$, $\{13,15,16,20,24\}\}$

Table A.29: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,5,7\}$

r	π_r
6	$\{\{6,9,21,24,30\}, \{8,19,25,33,35\}, \{10,11,22,27,28\}, \{12,17,26,31,34\}, \{13,14,16,20,23\}, \{15,18,29,32,36\}\}$
7	$\{\{6,13,26,27,34\}, \{8,19,23,36,40\}, \{9,18,32,33,38\}, \{10,12,24,29,31\}, \{11,16,20,22,25\}, \{14,21,30,37,42\}, \{15,17,28,35,39\}\}$
8	$\{\{6,11,24,31,38\}, \{8,9,27,30,40\}, \{10,14,28,29,33\}, \{12,23,32,42,45\}, \{13,17,36,37,43\}, \{15,19,26,39,47\}, \{16,21,34,41,44\}, \{18,20,22,25,35\}\}$
9	$\{\{6,11,26,36,45\}, \{8,19,33,37,43\}, \{9,12,27,40,46\}, \{10,13,29,38,44\}, \{14,18,28,47,51\}, \{15,21,25,30,31\}, \{16,22,39,41,42\}, \{17,34,48,49,50\}, \{20,23,24,32,35\}\}$
10	$\{\{6,12,28,42,52\}, \{8,20,33,44,49\}, \{9,13,34,38,50\}, \{10,11,29,40,48\}, \{14,26,39,53,54\}, \{15,24,30,47,56\}, \{16,19,31,51,55\}, \{17,21,41,43,46\}, \{18,23,27,32,36\}, \{22,25,35,37,45\}\}$
11	$\{\{6,23,39,42,52\}, \{8,14,33,47,58\}, \{9,13,30,43,51\}, \{10,17,40,41,54\}, \{11,12,34,44,55\}, \{15,21,35,36,37\}, \{16,25,38,56,59\}, \{18,26,46,48,50\}, \{19,20,32,53,60\}, \{22,28,45,57,62\}, \{24,27,29,31,49\}\}$
12	$\{\{6,22,40,50,62\}, \{8,18,39,41,54\}, \{9,16,42,43,60\}, \{10,17,38,47,58\}, \{11,19,35,51,56\}, \{12,30,32,36,46\}, \{13,28,37,63,67\}, \{14,25,45,53,59\}, \{15,31,49,52,55\}, \{20,21,26,33,34\}, \{23,24,44,61,64\}, \{27,29,48,57,65\}\}$
13	$\{\{6,30,48,56,68\}, \{8,21,39,57,67\}, \{9,17,42,44,60\}, \{10,15,40,43,58\}, \{11,31,53,54,65\}, \{12,29,49,55,63\}, \{13,35,41,62,69\}, \{14,26,33,36,37\}, \{16,19,50,51,66\}, \{18,22,47,52,59\}, \{20,24,38,64,70\}, \{23,32,45,61,71\}, \{25,27,28,34,46\}\}$
14	$\{\{6,23,42,58,71\}, \{8,20,48,50,70\}, \{9,36,56,62,73\}, \{10,28,51,55,68\}, \{11,34,52,57,64\}, \{12,31,33,37,39\}, \{13,16,40,49,60\}, \{14,25,47,59,67\}, \{15,21,45,54,63\}, \{17,19,41,61,66\}, \{18,29,44,72,75\}, \{22,30,43,65,74\}, \{24,32,35,38,53\}, \{26,27,46,69,76\}\}$
15	$\{\{6,21,50,54,77\}, \{8,23,51,56,76\}, \{9,24,44,60,71\}, \{10,36,48,65,67\}, \{11,30,46,63,68\}, \{12,20,45,62,75\}, \{13,37,42,72,80\}, \{14,34,55,66,73\}, \{15,17,47,59,74\}, \{16,18,43,52,61\}, \{19,27,57,58,69\}, \{22,26,33,40,41\}, \{25,32,53,78,82\}, \{28,29,31,39,49\}, \{35,38,64,70,79\}\}$
16	$\{\{6,20,48,56,78\}, \{8,32,42,68,70\}, \{9,34,54,66,77\}, \{10,16,47,53,74\}, \{11,12,43,59,79\}, \{13,30,55,57,69\}, \{14,19,46,60,73\}, \{15,23,51,63,76\}, \{17,24,40,82,83\}, \{18,38,50,81,87\}, \{21,29,58,64,72\}, \{22,27,44,80,85\}, \{25,45,61,75,84\}, \{26,31,36,41,52\}, \{28,33,65,67,71\}, \{35,37,39,49,62\}\}$
17	$\{\{6,18,46,62,84\}, \{8,29,54,68,85\}, \{9,20,47,70,88\}, \{10,12,44,59,81\}, \{11,21,53,56,77\}, \{13,14,52,55,80\}, \{15,60,64,79,90\}, \{16,37,58,71,76\}, \{17,24,51,72,82\}, \{19,28,61,73,87\}, \{22,32,41,45,50\}, \{23,30,65,66,78\}, \{25,40,57,83,91\}, \{26,31,35,43,49\}, \{27,34,67,69,75\}, \{33,36,42,48,63\}, \{38,39,74,86,89\}\}$
18	$\{\{6,34,48,75,83\}, \{8,17,50,61,86\}, \{9,41,56,71,77\}, \{10,30,52,68,80\}, \{11,39,58,70,78\}, \{12,33,66,69,90\}, \{13,38,65,67,81\}, \{14,35,44,87,92\}, \{15,26,60,63,82\}, \{16,28,55,74,85\}, \{18,42,49,84,95\}, \{19,20,54,73,88\}, \{21,29,62,64,76\}, \{22,32,51,91,94\}, \{23,31,37,45,46\}, \{24,27,47,89,93\}, \{25,43,53,57,72\}, \{36,40,59,79,96\}\}$

Table A.29: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,5,7\}$

r	π_r
19	$\{\{6,25,52,59,80\}, \{8,36,50,79,85\}, \{9,17,57,58,89\}, \{10,37,68,75,96\}, \{11,20,61,64,94\}, \{12,22,48,70,84\}, \{13,27,54,78,92\}, \{14,31,49,77,81\}, \{15,16,51,66,86\}, \{18,30,63,67,82\}, \{19,35,69,76,91\}, \{21,39,73,74,87\}, \{23,45,60,90,98\}, \{24,38,46,56,72\}, \{26,83,93,100,102\}, \{28,40,44,47,65\}, \{29,33,53,88,97\}, \{32,42,43,55,62\}, \{34,41,71,95,99\}\}$
20	$\{\{6,25,61,70,100\}, \{8,24,60,68,96\}, \{9,13,57,58,93\}, \{10,31,62,73,94\}, \{11,21,59,71,98\}, \{12,28,55,74,89\}, \{14,44,64,80,86\}, \{15,29,66,79,101\}, \{16,36,48,99,103\}, \{17,18,51,75,91\}, \{19,54,63,92,102\}, \{20,35,65,78,88\}, \{22,27,67,69,87\}, \{23,37,52,97,105\}, \{26,47,77,81,85\}, \{30,34,50,90,104\}, \{32,38,82,83,95\}, \{33,46,56,84,107\}, \{39,40,42,49,72\}, \{41,43,45,53,76\}\}$
21	$\{\{6,27,51,81,99\}, \{8,87,93,97,111\}, \{9,33,55,85,98\}, \{10,39,69,82,102\}, \{11,15,59,62,95\}, \{12,29,66,76,101\}, \{13,35,73,75,100\}, \{14,42,74,86,104\}, \{16,19,52,89,106\}, \{17,25,63,70,91\}, \{18,38,54,107,109\}, \{20,44,58,71,77\}, \{21,37,72,80,94\}, \{22,34,67,79,90\}, \{23,31,68,78,92\}, \{24,43,84,88,105\}, \{26,47,61,96,108\}, \{28,36,57,103,110\}, \{30,40,50,56,64\}, \{32,41,45,53,65\}, \{46,48,49,60,83\}\}$
22	$\{\{6,50,72,84,100\}, \{8,28,68,74,106\}, \{9,42,80,81,110\}, \{10,107,111,113,115\}, \{11,21,70,71,109\}, \{12,46,59,93,94\}, \{13,33,66,85,105\}, \{14,29,67,79,103\}, \{15,30,57,86,98\}, \{16,25,54,91,104\}, \{17,18,65,78,108\}, \{19,41,64,92,96\}, \{20,37,75,77,95\}, \{22,26,62,83,97\}, \{23,51,56,69,87\}, \{24,44,47,55,60\}, \{27,48,49,61,63\}, \{31,32,73,89,99\}, \{34,38,39,53,58\}, \{35,36,82,90,101\}, \{40,52,88,112,116\}, \{43,45,76,102,114\}\}$
23	$\{\{6,13,61,63,105\}, \{8,46,79,89,114\}, \{9,42,68,82,99\}, \{10,40,74,80,104\}, \{11,24,70,78,113\}, \{12,36,71,93,116\}, \{14,49,72,91,100\}, \{15,29,73,81,110\}, \{16,25,65,88,112\}, \{17,31,57,94,103\}, \{18,45,56,115,122\}, \{19,51,59,107,118\}, \{20,43,77,84,98\}, \{21,34,66,95,106\}, \{22,50,83,90,101\}, \{23,26,67,69,87\}, \{27,38,48,55,58\}, \{28,52,86,96,102\}, \{30,54,75,111,120\}, \{32,53,76,108,117\}, \{33,44,47,60,64\}, \{35,37,62,109,119\}, \{39,41,85,92,97\}\}$
24	$\{\{6,101,105,117,127\}, \{8,17,69,77,121\}, \{9,29,76,82,120\}, \{10,40,71,92,113\}, \{11,14,65,72,112\}, \{12,42,75,86,107\}, \{13,35,67,99,118\}, \{15,41,78,97,119\}, \{16,51,85,96,114\}, \{18,36,68,94,108\}, \{19,27,66,95,115\}, \{20,28,73,79,104\}, \{21,23,62,98,116\}, \{22,39,81,83,103\}, \{24,49,56,59,70\}, \{25,46,89,91,109\}, \{26,45,47,58,60\}, \{30,31,80,87,106\}, \{32,34,63,122,125\}, \{33,44,57,64,84\}, \{37,43,55,61,74\}, \{38,50,90,100,102\}, \{48,54,88,110,124\}, \{52,53,93,111,123\}\}$
25	$\{\{6,23,58,91,120\}, \{8,19,57,76,106\}, \{9,35,66,83,105\}, \{10,11,59,61,99\}, \{12,29,65,92,116\}, \{13,33,73,74,101\}, \{14,21,63,90,118\}, \{15,36,71,95,115\}, \{16,39,75,77,97\}, \{17,25,64,80,102\}, \{18,26,70,84,110\}, \{20,38,67,89,98\}, \{22,37,72,87,100\}, \{24,46,88,96,114\}, \{27,28,78,81,104\}, \{30,108,117,125,130\}, \{31,109,111,124,127\}, \{32,45,56,107,128\}, \{34,54,85,119,122\}, \{40,44,55,60,79\}, \{41,47,62,103,129\}, \{42,43,68,69,86\}, \{48,50,93,121,126\}, \{49,52,82,112,131\}, \{51,53,94,113,123\}\}$

Table A.29: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,5,7\}$

r	π_r
26	$\{\{6,42,69,107,128\}, \{8,9,67,68,118\}, \{10,21,76,77,122\}, \{11,47,86,89,117\}, \{12,35,73,100,126\}, \{13,16,66,83,120\}, \{14,37,64,106,119\}, \{15,19,70,80,116\}, \{17,50,71,108,112\}, \{18,48,61,130,135\}, \{20,40,74,101,115\}, \{22,31,84,96,127\}, \{23,54,87,103,113\}, \{24,53,62,121,136\}, \{25,33,88,95,125\}, \{26,60,93,104,111\}, \{27,51,85,102,109\}, \{28,30,91,98,131\}, \{29,41,65,129,134\}, \{32,44,92,94,110\}, \{34,52,78,124,132\}, \{36,79,105,123,133\}, \{38,57,59,72,82\}, \{39,43,97,99,114\}, \{45,55,56,75,81\}, \{46,49,58,63,90\}\}$
27	$\{\{6,46,81,90,119\}, \{8,42,73,106,129\}, \{9,62,63,134,142\}, \{10,39,84,95,130\}, \{11,22,64,83,114\}, \{12,37,71,91,113\}, \{13,56,99,102,132\}, \{14,16,68,78,116\}, \{15,20,65,87,117\}, \{17,26,72,93,122\}, \{18,48,92,101,127\}, \{19,44,88,103,128\}, \{21,60,85,111,115\}, \{23,29,66,110,124\}, \{24,25,80,105,136\}, \{27,47,89,108,123\}, \{28,45,97,109,133\}, \{30,33,82,112,131\}, \{31,51,69,125,138\}, \{32,58,77,126,139\}, \{34,50,61,70,75\}, \{35,38,96,98,121\}, \{36,53,100,107,118\}, \{40,59,79,120,140\}, \{41,55,57,67,86\}, \{43,49,74,76,94\}, \{52,54,104,135,137\}\}$
28	$\{\{6,46,92,95,135\}, \{8,39,80,91,124\}, \{9,16,71,75,121\}, \{10,29,82,89,132\}, \{11,53,69,115,120\}, \{12,13,65,96,136\}, \{14,51,99,103,137\}, \{15,19,73,86,125\}, \{17,28,84,88,127\}, \{18,43,85,93,117\}, \{20,24,83,94,133\}, \{21,130,140,144,147\}, \{22,64,102,112,128\}, \{23,40,67,114,118\}, \{25,47,104,107,139\}, \{26,60,97,108,119\}, \{27,55,98,113,129\}, \{30,54,63,68,79\}, \{31,33,66,70,72\}, \{32,61,77,126,142\}, \{34,42,101,116,141\}, \{35,58,59,74,78\}, \{36,49,109,110,134\}, \{37,62,87,131,143\}, \{38,50,100,111,123\}, \{41,48,105,106,122\}, \{44,56,57,76,81\}, \{45,52,90,138,145\}\}$
29	$\{\{6,49,85,105,135\}, \{8,11,66,67,114\}, \{9,61,78,111,119\}, \{10,63,93,110,130\}, \{12,26,71,82,115\}, \{13,32,70,99,124\}, \{14,30,74,101,131\}, \{15,56,96,104,129\}, \{16,43,97,100,138\}, \{17,58,95,108,128\}, \{18,22,80,81,121\}, \{19,20,89,90,140\}, \{21,44,87,91,113\}, \{23,38,98,102,139\}, \{24,62,75,134,145\}, \{25,35,72,106,118\}, \{27,28,86,94,125\}, \{29,41,83,103,116\}, \{31,33,69,107,112\}, \{34,45,68,133,144\}, \{36,126,136,148,150\}, \{37,48,65,123,143\}, \{39,60,84,132,147\}, \{40,42,77,137,142\}, \{46,51,55,64,88\}, \{47,50,52,73,76\}, \{53,117,127,146,151\}, \{54,57,92,122,141\}, \{59,79,109,120,149\}\}$
30	$\{\{6,24,83,90,143\}, \{8,56,88,103,127\}, \{9,35,76,120,152\}, \{10,119,129,136,156\}, \{11,58,97,112,140\}, \{12,54,100,113,147\}, \{13,28,71,116,146\}, \{14,22,72,95,131\}, \{15,30,85,93,133\}, \{16,33,73,126,150\}, \{17,49,86,114,134\}, \{18,66,80,149,153\}, \{19,25,91,104,151\}, \{20,27,84,101,138\}, \{21,34,77,110,132\}, \{23,53,96,117,137\}, \{26,61,106,109,128\}, \{29,47,105,115,144\}, \{31,50,64,70,75\}, \{32,40,89,122,139\}, \{36,55,111,121,141\}, \{37,46,107,124,148\}, \{38,39,99,108,130\}, \{41,65,118,123,135\}, \{42,60,67,82,87\}, \{43,92,125,145,155\}, \{44,62,94,142,154\}, \{45,51,57,74,79\}, \{48,52,59,78,81\}, \{63,68,69,98,102\}\}$

Table A.29: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,5,7\}$

r	π_r
31	$\{\{6,45,73,120,142\}, \{8,55,85,118,140\}, \{9,31,83,109,152\}, \{10,43,79,127,153\}, \{11,64,90,119,134\}, \{12,27,84,101,146\}, \{13,21,91,92,149\}, \{14,25,96,100,157\}, \{15,30,81,102,138\}, \{16,18,82,89,137\}, \{17,53,104,105,139\}, \{19,20,76,117,154\}, \{22,58,99,126,145\}, \{23,52,98,113,136\}, \{24,40,80,115,131\}, \{26,44,103,123,156\}, \{28,39,108,114,155\}, \{29,54,86,129,132\}, \{32,57,78,151,162\}, \{33,67,69,75,94\}, \{34,51,66,74,77\}, \{35,36,97,122,148\}, \{37,56,112,116,135\}, \{38,60,107,124,133\}, \{41,62,87,143,159\}, \{42,61,95,150,158\}, \{46,65,125,130,144\}, \{47,59,93,147,160\}, \{48,49,110,128,141\}, \{50,63,70,72,111\}, \{68,71,88,106,121\}\}$
32	$\{\{6,23,77,102,150\}, \{8,36,94,109,159\}, \{9,60,92,131,154\}, \{10,16,84,91,149\}, \{11,72,103,117,137\}, \{12,34,76,130,160\}, \{13,71,108,123,147\}, \{14,64,81,133,136\}, \{15,45,99,104,143\}, \{17,44,98,111,148\}, \{18,31,86,120,157\}, \{19,37,85,127,156\}, \{20,42,97,100,135\}, \{21,29,79,126,155\}, \{22,49,93,119,141\}, \{24,38,101,112,151\}, \{25,66,114,115,138\}, \{26,46,96,129,153\}, \{27,68,106,134,145\}, \{28,63,65,74,82\}, \{30,51,75,83,89\}, \{32,50,116,128,162\}, \{33,41,90,124,140\}, \{35,47,78,161,165\}, \{39,62,95,158,164\}, \{40,53,118,121,146\}, \{43,58,80,142,163\}, \{48,55,122,125,144\}, \{52,54,113,132,139\}, \{56,69,70,88,107\}, \{57,67,73,87,110\}, \{59,61,105,152,167\}\}$

Table A.29: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,5,7\}$

r	π_r
2	$\{\{8,11,13,14,18\}, \{9,10,12,15,16\}\}$
3	$\{\{8,11,14,18,23\}, \{9,10,16,17,20\}, \{12,13,15,19,21\}\}$
4	$\{\{8,13,20,23,24\}, \{9,11,16,21,25\}, \{10,12,15,18,19\}, \{14,17,22,26,27\}\}$
5	$\{\{8,16,21,28,31\}, \{9,11,23,24,27\}, \{10,14,20,26,30\}, \{12,13,15,18,22\}, \{17,19,25,29,32\}\}$
6	$\{\{8,11,22,27,30\}, \{9,12,25,28,32\}, \{10,18,23,29,34\}, \{13,15,26,33,35\}, \{14,17,24,31,38\}, \{16,19,20,21,36\}\}$
7	$\{\{8,10,27,29,38\}, \{9,12,26,28,33\}, \{11,19,24,37,43\}, \{13,16,23,35,41\}, \{14,22,32,36,40\}, \{15,20,21,25,31\}, \{17,18,30,34,39\}\}$
8	$\{\{8,20,32,34,38\}, \{9,22,25,41,47\}, \{10,11,26,30,35\}, \{12,18,27,43,46\}, \{13,23,33,42,45\}, \{14,19,36,37,40\}, \{15,21,24,29,31\}, \{16,17,28,39,44\}\}$
9	$\{\{8,18,32,39,45\}, \{9,12,30,35,44\}, \{10,19,33,38,42\}, \{11,26,34,46,49\}, \{13,20,40,41,48\}, \{14,22,24,29,31\}, \{15,16,28,47,50\}, \{17,21,25,27,36\}, \{23,37,43,51,52\}\}$
10	$\{\{8,45,46,47,56\}, \{9,13,30,44,52\}, \{10,14,35,38,49\}, \{11,26,33,39,43\}, \{12,16,36,40,48\}, \{15,20,28,29,34\}, \{17,22,31,50,58\}, \{18,24,27,32,37\}, \{19,25,42,53,55\}, \{21,23,41,51,54\}\}$
11	$\{\{8,15,38,42,57\}, \{9,22,37,45,51\}, \{10,23,43,44,54\}, \{11,28,32,56,63\}, \{12,19,33,48,50\}, \{13,24,41,49,53\}, \{14,20,31,58,61\}, \{16,30,39,40,47\}, \{17,26,35,52,60\}, \{18,25,27,34,36\}, \{21,29,46,55,59\}\}$

Table A.30: Table of partitions for Lemma 12 when $S = \{1,2,3,4,5,6,7\}$

r	π_r
12	$\{\{8,11,37,38,56\}, \{9,13,36,41,55\}, \{10,18,39,49,60\}, \{12,31,35,40,48\}, \{14,16,42,51,63\}, \{15,25,27,33,34\}, \{17,57,58,65,67\}, \{19,20,46,47,54\}, \{21,23,50,53,59\}, \{22,24,44,62,64\}, \{26,30,32,43,45\}, \{28,29,52,61,66\}\}$

Table A.30: Table of partitions for Lemma 12 when $S = \{1,2,3,4,5,6,7\}$

r	π_r
3	$\{\{9,13,15,16,21\}, \{10,14,17,18,23\}, \{11,12,19,20,22\}\}$
4	$\{\{9,12,17,18,20\}, \{10,15,16,19,22\}, \{11,21,23,27,28\}, \{13,14,24,25,26\}\}$
5	$\{\{9,16,20,25,30\}, \{10,14,21,29,32\}, \{11,18,19,22,26\}, \{12,17,23,28,34\}, \{13,15,24,27,31\}\}$
6	$\{\{9,16,29,30,34\}, \{10,11,23,26,28\}, \{12,17,20,22,27\}, \{13,19,25,32,39\}, \{14,15,24,31,36\}, \{18,21,33,35,37\}\}$
7	$\{\{9,10,24,31,36\}, \{11,18,23,25,27\}, \{12,21,29,37,41\}, \{13,20,28,38,43\}, \{14,15,26,30,33\}, \{16,17,32,39,40\}, \{19,22,34,35,42\}\}$
8	$\{\{9,13,27,38,43\}, \{10,23,35,37,39\}, \{11,21,31,32,33\}, \{12,15,26,44,45\}, \{14,20,29,41,46\}, \{16,17,25,28,30\}, \{18,24,36,42,48\}, \{19,22,34,40,47\}\}$
9	$\{\{9,46,49,50,54\}, \{10,23,26,29,30\}, \{11,13,33,38,47\}, \{12,25,31,45,51\}, \{14,15,35,42,48\}, \{16,17,28,34,39\}, \{18,19,40,41,44\}, \{20,21,27,32,36\}, \{22,24,37,43,52\}\}$
10	$\{\{9,27,30,35,41\}, \{10,25,43,46,54\}, \{11,13,36,37,49\}, \{12,14,32,44,50\}, \{15,20,38,45,48\}, \{16,28,29,33,40\}, \{17,19,34,53,55\}, \{18,21,31,51,59\}, \{22,26,39,47,56\}, \{23,24,42,52,57\}\}$
11	$\{\{9,19,37,43,52\}, \{10,31,34,53,60\}, \{11,24,44,45,54\}, \{12,13,33,48,56\}, \{14,30,46,49,51\}, \{15,22,32,58,63\}, \{16,25,36,57,62\}, \{17,21,47,50,59\}, \{18,27,35,38,42\}, \{20,26,40,55,61\}, \{23,28,29,39,41\}\}$
12	$\{\{9,12,42,43,64\}, \{10,25,40,52,57\}, \{11,24,44,54,63\}, \{13,21,46,47,59\}, \{14,22,45,49,58\}, \{15,28,48,51,56\}, \{16,29,39,61,67\}, \{17,35,41,55,66\}, \{18,27,30,37,38\}, \{19,20,31,34,36\}, \{23,33,53,62,65\}, \{26,32,50,60,68\}\}$
13	$\{\{9,15,41,46,63\}, \{10,22,42,48,58\}, \{11,30,38,67,70\}, \{12,25,47,51,61\}, \{13,17,45,49,64\}, \{14,20,44,52,62\}, \{16,27,33,37,39\}, \{18,28,40,65,71\}, \{19,29,57,59,68\}, \{21,23,54,56,66\}, \{24,32,53,69,72\}, \{26,31,36,43,50\}, \{34,35,55,60,74\}\}$
14	$\{\{9,22,42,61,72\}, \{10,12,44,47,69\}, \{11,27,50,62,74\}, \{13,15,43,49,64\}, \{14,25,46,60,67\}, \{16,19,48,57,70\}, \{17,34,41,65,75\}, \{18,21,54,56,71\}, \{20,35,37,40,52\}, \{23,36,38,39,58\}, \{24,29,45,68,76\}, \{26,33,55,59,63\}, \{28,31,53,73,79\}, \{30,32,51,66,77\}\}$

Table A.31: Table of partitions for Lemma 1.3.7 when $S = \{1,2,3,4,5,6,7,8\}$