Cycle decompositions V: Complete graphs into cycles of arbitrary lengths

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Abstract

We show that the complete graph on n vertices can be decomposed into t cycles of specified lengths $m_1, \ldots, m_t$ if and only if n is odd, $3 \le m_i \le n$ for $i = 1, \ldots, t$, and $m_1 + \cdots + m_t = \binom{n}{2}$. We also show that the complete graph on n vertices can be decomposed into a perfect matching and t cycles of specified lengths $m_1, \ldots, m_t$ if and only if n is even, $3 \le m_i \le n$ for $i = 1, \ldots, t$, and $m_1 + \ldots + m_t = \binom{n}{2} - \frac{n}{2}$.

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1 Introduction

A decomposition of a graph K is a set of subgraphs of K whose edge sets partition the edge set of K. In 1981, Alspach [3] asked whether it is possible to decompose the complete graph on n vertices, denoted K_n, into t cycles of specified lengths $m_1, \ldots, m_t$ whenever the obvious necessary conditions are satisfied; namely that n is odd, $3 \le m_i \le n$, and $m_1 + \cdots + m_t = \binom{n}{2}$. He also asked whether it is possible to decompose K_n into a perfect matching and t cycles of specified lengths $m_1, \ldots, m_t$ whenever n is even, $3 \le m_i \le n$, and $m_1 + \cdots + m_t = \binom{n}{2} - \frac{n}{2}$. Again, these conditions are obviously necessary.

Here we solve Alspach's problem by proving the following theorem.

Theorem 1 *There is a decomposition $\{G_1, \ldots, G_t\}$ of K_n in which G_i is an m_i-cycle for $i = 1, \ldots, t$ if and only if n is odd, $3 \le m_i \le n$ for $i = 1, \ldots, t$, and $m_1 + \cdots + m_t = \frac{n(n-1)}{2}$. There is a decomposition $\{G_1, \ldots, G_t, I\}$ of K_n in which G_i is an m_i-cycle for $i = 1, \ldots, t$ and I is a perfect matching if and only if n is even, $3 \le m_i \le n$ for $i = 1, \ldots, t$, and $m_1 + \cdots + m_t = \frac{n(n-2)}{2}$.*

Let K be a graph and let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \ge 3$ for $i = 1, \ldots, t$. If each vertex of K has even degree, then an (M)-decomposition of K is a decomposition $\{G_1, \ldots, G_t\}$ such that G_i is an m_i-cycle for $i = 1, \ldots, t$. If each vertex of K has odd degree, then an (M)-decomposition of K is a decomposition $\{G_1, \ldots, G_t, I\}$ such that G_i is an m_i-cycle for $i = 1, \ldots, t$ and I is a perfect matching in K.

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We say that a list $(m_1, \ldots, m_t)$ of integers is *n-admissible* if $3 \le m_1, \ldots, m_t \le n$ and $m_1 + \cdots + m_t = n\lfloor \frac{n-1}{2} \rfloor$. Note that $n\lfloor \frac{n-1}{2} \rfloor = \binom{n}{2}$ if n is odd, and $n\lfloor \frac{n-1}{2} \rfloor = \binom{n}{2} - \frac{n}{2}$ if n is even. Thus, we can rephrase Alspach's question as follows. *Prove that for each n-admissible list M, there exists an (M)-decomposition of K_n.*

A decomposition of K_n into 3-cycles is equivalent to a *Steiner triple system of order* n, and a decomposition of K_n into n-cycles is a *Hamilton decomposition*. Thus, the work of Kirkman [37] and Walecki (see [5, 40]) from the 1800s addresses Alspach's problem in the cases where M is of the form $(3, 3, \ldots, 3)$ or $(n, n, \ldots, n)$. The next results on Alspach's problem appeared in the 1960s [38, 43, 44], and a multitude of results have appeared since then. Many of these focussed on the case of decompositions into cycles of uniform length [7, 8, 12, 14, 32, 34, 35, 45], and a complete solution in this case was eventually obtained [6, 46].

There have also been many papers on the case where the lengths of the cycles in the decomposition may vary. In recent work [18, 20, 21], the first two authors have made progress by developing methods introduced in [19] and [22]. In [20], Alspach's problem is settled in the case where all the cycle lengths are greater than about $\frac{n}{2}$, and in [21] the problem is completely settled for sufficiently large odd n. Earlier results for the case of cycles of varying lengths can be found in [1, 2, 9, 24, 25, 32, 33, 36]. See [16] for a survey on Alspach's problem, and see [28] for a survey on cycle decompositions generally.

The analogous problems on decompositions of complete graphs into matchings, stars or paths have all been completely solved, see [11], [39] and [17] respectively. It is also worth mentioning that the easier problems in which each G_i is required only to be a closed trail of length m_i or each G_i is required only to be a 2-regular graph of order m_i have been solved in [10], [22] and [26]. Decompositions of complete multigraphs into cycles are considered in [23].

Balister [9] has verified by computer that Theorem 1 holds for $n \le 14$, and we include this result as a lemma for later reference.

Lemma 2 ([9]) *Theorem 1 holds for $n \le 14$.*

Our proof of Theorem 1 relies heavily on the reduction of Alspach's problem obtained in [21], see Theorem 3 below. Throughout the paper, we use the notation $\nu_i(M)$ to denote the number of occurrences of i in a given list M.

Definition A list M is an *n-ancestor* list if it is n-admissible and satisfies

(1) $\nu_6(M) + \nu_7(M) + \cdots + \nu_{n-1}(M) \in \{0, 1\}$;

(2) if $\nu_5(M) \ge 3$, then $2\nu_4(M) \le n - 6$;

(3) if $\nu_5(M) \ge 2$, then $3\nu_3(M) \le n - 10$;

(4) if $\nu_4(M) \ge 1$ and $\nu_5(M) \ge 1$, then $3\nu_3(M) \le n - 9$;

(5) if $\nu_4(M) \ge 1$, then $\nu_i(M) = 0$ for $i \in \{n - 2, n - 1\}$; and

(6) if $\nu_5(M) \ge 1$, then $\nu_i(M) = 0$ for $i \in \{n - 4, n - 3, n - 2, n - 1\}$.

Thus, an n-ancestor list is of the form

$$(3, 3, \ldots, 3, 4, 4, \ldots, 4, 5, 5, \ldots, 5, k, n, n, \ldots, n)$$

where k is either absent or in the range $6 \leq k \leq n-1$, and there are additional constraints involving the number of occurrences of cycle lengths in the list. The following theorem was proved in [21].

Theorem 3 ([21], Theorem 4.1) *For each positive integer n, if there exists an (M)-decomposition of K_n for each n-ancestor list M, then there exists an (M)-decomposition of K_n for each n-admissible list M.*

Our goal is to construct an (M)-decomposition of K_n for each n-ancestor list M. We split this problem into two cases: the case where $\nu_n(M) \geq 2$ and the case where $\nu_n(M) \leq 1$. In particular, we prove the following two results.

Lemma 4 *If M is an n-ancestor list with $\nu_n(M) \geq 2$, then there is an (M)-decomposition of K_n.*

Proof See Section 3. $\square$

Lemma 5 *If Theorem 1 holds for K_{n-3}, K_{n-2} and K_{n-1}, then there is an (M)-decomposition of K_n for each n-ancestor list M satisfying $\nu_n(M) \leq 1$.*

Proof The case $\nu_n(M) = 0$ is proved in Section 4 (see Lemma 23) and the case $\nu_n(M) = 1$ is proved in Section 5 (see Lemma 45). $\square$

Lemmas 4 and 5 allow us to prove our main result using induction on n.

Proof of Theorem 1 The proof is by induction on n. By Lemma 2, Theorem 1 holds for $n \leq 14$. So let $n \geq 15$ and assume Theorem 1 holds for complete graphs having fewer than n vertices. By Theorem 3, it suffices to prove the existence of an (M)-decomposition of K_n for each n-ancestor list M. Lemma 4 covers each n-ancestor list M with $\nu_n(M) \geq 2$, and using the inductive hypothesis, Lemma 5 covers those with $\nu_n(M) \leq 1$. $\square$

2 Notation

We shall sometimes use superscripts to specify the number of occurrences of a particular integer in a list. That is, we define $(m_1^{\alpha_1}, \ldots, m_t^{\alpha_t})$ to be the list comprised of α_i occurrences of m_i for $i = 1, \ldots, t$. Let $M = (m_1^{\alpha_1}, \ldots, m_t^{\alpha_t})$ and let $M' = (m_1^{\beta_1}, \ldots, m_t^{\beta_t})$, where $m_1, \ldots, m_t$ are distinct. Then (M, M') is the list $(m_1^{\alpha_1+\beta_1}, \ldots, m_t^{\alpha_t+\beta_t})$ and, if $0 \leq \beta_i \leq \alpha_i$ for $i = 1, \ldots, t$, $M - M'$ is the list $(m_1^{\alpha_1-\beta_1}, \ldots, m_t^{\alpha_t-\beta_t})$.

Let Γ be a finite group and let S be a subset of Γ such that the identity of Γ is not in S and such that the inverse of any element of S is also in S. The *Cayley graph* on Γ with *connection set* S, denoted $\mathrm{Cay}(\Gamma, S)$, has the elements of Γ as its vertices and there is an edge between vertices g and h if and only if $g = hs$ for some $s \in S$.

A Cayley graph on a cyclic group is called a *circulant graph*. For any graph with vertex set $\mathbb{Z}_n$, we define the *length* of an edge xy to be $x - y$ or $y - x$, whichever is in $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. It is convenient to be able to describe the connection set of a circulant graph on $\mathbb{Z}_n$ by listing only one of s and $n - s$. Thus, we use the following notation. For any subset S of $\mathbb{Z}_n \setminus \{0\}$ such that $s \in S$ and $n - s \in S$ implies $n = 2s$, we define $\langle S \rangle_n$ to be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S \cup -S)$.

Let $m \in \{3, 4, 5\}$ and let $D = \{a_1, \ldots, a_m\}$ where $a_1, \ldots, a_m$ are positive integers. If there is a partition $\{D_1, D_2\}$ of D such that $\sum D_1 - \sum D_2 = 0$, then D is called a *difference m-tuple*. If there is a partition $\{D_1, D_2\}$ of D such that $\sum D_1 - \sum D_2 = 0 \pmod{n}$, then D is called a *modulo n difference m-tuple*. Clearly, any difference m-tuple is also a modulo n difference m-tuple for all n. We may use the terms difference triple, quadruple and quintuple respectively rather than 3-tuple, 4-tuple and 5-tuple. For $m \in \{3, 4, 5\}$, it is clear that if D is a difference m-tuple, then there is an (m^n)-decomposition of $\langle D\rangle_n$ for all $n \geq 2\max(D) + 1$, and that if D is a modulo n difference m-tuple, then there is an (m^n)-decomposition of $\langle D\rangle_n$.

We denote the complete graph with vertex set V by K_V and the complete bipartite graph with parts U and V by $K_{U,V}$. If G and H are graphs then $G-H$ is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G) \setminus E(H)$. If G and H are graphs whose vertex sets are disjoint then $G \vee H$ is the graph with vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)\cup\{xy : x \in V(G), y \in V(H)\}$. A cycle with m edges is called an *m-cycle* and is denoted $(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m$ are the vertices of the cycle and $x_1x_2, \ldots, x_{m-1}x_m, x_mx_1$ are the edges. A path with m edges is called an *m-path* and is denoted $[x_0, \ldots, x_m]$, where $x_0, \ldots, x_m$ are the vertices of the path and $x_0x_1, \ldots, x_{m-1}x_m$ are the edges. A graph is said to be *even* if every vertex of the graph has even degree and is said to be *odd* if every vertex of the graph has odd degree.

A *packing* of a graph K is a decomposition of some subgraph G of K, and the graph $K - G$ is called the *leave* of the packing. An (M)*-packing* of K_n is an (M)-decomposition of some subgraph G of K_n such that G is an even graph if n is odd and G is an odd graph if n is even (recall that an (M)-decomposition of an odd graph contains a perfect matching). Thus, the leave of an (M)-packing of K_n is an even graph and, like an (M)-decomposition of K_n, an (M)-packing of K_n contains a perfect matching if and only if n is even. A decomposition of a graph into Hamilton cycles is called a *Hamilton decomposition*.

3 The case of at least two Hamilton cycles

The purpose of this section is to prove Lemma 4 which states that there is an (M)-decomposition of K_n for each n-ancestor list M with $\nu_n(M) \geq 2$. We first give a general outline of this proof. Theorem 1 has been proved in the case where $M = (3^a, n^b)$ for some $a, b \geq 0$ [25], so we will restrict our attention to ancestor lists which are not of this form. The basic construction involves decomposing K_n into $\langle S\rangle_n$ and $K_n - \langle S\rangle_n$ where, for some $x \leq 8$, the connection set S is either $\{1, \ldots, x\}$ or $\{1, \ldots, x-1\} \cup \{x+1\}$ so that $\sum S$ is even. We partition any given n-ancestor list M into two lists M_s and $\overline{M_s} = M - M_s$, and construct an (M_s)-decomposition of $\langle S\rangle_n$ and an $(\overline{M_s})$-decomposition of $K_n - \langle S\rangle_n$. This yields the desired (M)-decomposition of K_n. Taking $S = \{1, \ldots, x-1\} \cup \{x+1\}$, rather than $S = \{1, \ldots, x\}$, is necessary when $1 + \cdots + x$ is odd as many desired cycle decompositions of $\langle\{1, \ldots, x\}\rangle_n$ do not exist when $1 + \cdots + x$ is odd, see [27].

If $M = (3^{\alpha_3 n+\beta_3}, 4^{\alpha_4 n+\beta_4}, 5^{\alpha_5 n+\beta_5}, k^\gamma, n^\delta)$ where $\alpha_i \geq 0$ and $0 \leq \beta_i \leq n-1$ for $i \in \{3, 4, 5\}$, $6 \leq k \leq n-1$, $\gamma \in \{0, 1\}$, and $\delta \geq 2$, then we usually choose $M_s = (3^{\beta_3}, 4^{\beta_4}, 5^{\beta_5}, k^\gamma)$. However, if this would result in $\sum M_s$ being less than $4n$, then we sometimes adjust this definition slightly. We always choose M_s such that $\sum M_s$ is at most $8n$, which explains why we have $|S| \leq 8$.

Our (M_s)-decompositions of $\langle S\rangle_n$ will be constructed using adaptations of techniques used in [27] and [29]. We construct our $(\overline{M_s})$-decompositions of $K_n - \langle S\rangle_n$ using a combination of difference methods and results on Hamilton decompositions of circulant graphs. In general, we split the problem into the case $\nu_5(M) \leq 2$ and the case $\nu_5(M) \geq 3$. In the former case it will follow from our choice of M_s that $\overline{M_s} = (3^{tn}, 4^{qn}, n^h)$ for some $t, q, h \geq 0$ and in the latter case it will follow from our choice of M_s that $\overline{M_s} = (5^{rn}, n^h)$ for some $r, h \geq 0$.

The precise definition of M_s is given in Lemma 6, which details the properties that we require of our partition of M into M_s and $\overline{M_s}$, and establishes its existence. The definition includes several minor technicalities in order to deal with complications and exceptions that arise in the above-described approach. Throughout the remainder of this section, for a given n-ancestor list M such that $\nu_n(M) \geq 2$ and $M \neq (3^a, n^b)$ for any $a, b \geq 0$, we shall use the notation M_s and $\overline{M_s}$ to denote the lists constructed in the proof of Lemma 6. If $\nu_n(M) \leq 1$ or $M = (3^a, n^b)$ for some $a, b \geq 0$, then M_s and $\overline{M_s}$ are not defined.

Lemma 6 *If M is any n-ancestor list such that $\nu_n(M) \geq 2$ and $M \neq (3^a, n^b)$ for any $a, b \geq 0$, then there exists a partition of M into sublists M_s and $\overline{M_s}$ such that*

(1) $\sum M_s \in \{2n, 3n, \ldots, 8n\}$ and $\sum M_s \neq 8n$ when $\nu_5(M) \leq 2$;

(2) if $\sum M_s = 2n$, then $\nu_n(M_s) = 1$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$;

(3) if $\sum M_s = 3n$, then $\nu_n(M_s) \in \{0, 1\}$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$;

(4) if $\sum M_s \in \{4n, 5n, \ldots, 8n\}$ and $\nu_5(M) \geq 3$, then $\nu_n(M_s) = 0$ and $\overline{M_s} = (5^{rn}, n^h)$ for some $r \geq 0$, $h \geq 2$;

(5) if $\sum M_s \in \{4n, 5n, \ldots, 7n\}$ and $\nu_5(M) \leq 2$, then $\nu_n(M_s) = 0$ and $\overline{M_s} = (3^{tn}, 4^{qn}, n^h)$ for some $t, q \geq 0$, $h \geq 2$; and

(6) $M_s \neq (3^{\frac{5n}{3}})$.

Proof Let M be an n-ancestor list. The conditions of the lemma imply $n \geq 7$. We will first define a list M_e which in many cases will serve as M_s, but will sometimes need to be adjusted slightly.

If
$$M = (3^{\alpha_3 n+\beta_3}, 4^{\alpha_4 n+\beta_4}, 5^{\alpha_5 n+\beta_5}, k^\gamma, n^\delta)$$
where $\alpha_i \geq 0$ and $0 \leq \beta_i \leq n-1$ for $i \in \{3,4,5\}$, $6 \leq k \leq n-1$, $\gamma \in \{0,1\}$, and $\delta \geq 2$, then
$$M_e = (3^{\beta_3}, 4^{\beta_4}, 5^{\beta_5}, k^\gamma).$$

It is clear from the definition of n-ancestor list that if we take $M_s = M_e$, then (4) and (5) are satisfied.

We now show that $\sum M_e \in \{0, n, 2n, \ldots, 8n\}$, and that $\sum M_e \neq 8n$ when $\nu_5(M) \leq 2$. Noting that $\sum M_e \leq 3\beta_3 + 4\beta_4 + 5\beta_5 + (n-1)$ and separately considering the cases $\nu_5(M) \geq 3$, $\nu_5(M) \in \{1,2\}$ and $\nu_5(M) = 0$, it is routine to use the definition of (M)-ancestor lists to show that $\sum M_e < 9n$, and that $\sum M_e < 8n$ when $\nu_5(M) \leq 2$. Thus, because it follows from $\sum M = n\lfloor \frac{n-1}{2} \rfloor$ and the definition of M_e that n divides $\sum M_e$, we have that $\sum M_e \in \{0, n, 2n, \ldots, 8n\}$, and that $\sum M_e \neq 8n$ when $\nu_5(M) \leq 2$.

If $\sum M_e \in \{4n, 5n, 6n, 7n, 8n\}$, then we let $M_s = M_e$. If $\sum M_e \in \{0, n, 2n, 3n\}$, then we define M_s by
$$M_s = \begin{cases} (M_e, 4^n) & \text{if } \alpha_4 > 0; \\ (M_e, 5^n) & \text{if } \alpha_4 = 0 \text{ and } \alpha_5 > 0; \\ (M_e, 3^n) & \text{if } \alpha_4 = \alpha_5 = 0 \text{ and } \alpha_3 > 0; \\ (M_e, n) & \text{if } \alpha_3 = \alpha_4 = \alpha_5 = 0 \text{ and } \sum M_e \in \{n, 2n\}; \\ M_e & \text{otherwise.} \end{cases}$$

Using the definition of M_s and the fact that M is an n-ancestor list with $M \neq (3^a, n^b)$ for any $a, b \geq 0$, it is routine to check that M_s satisfies (1)–(6). $\square$

Before proving Lemma 4, we need a number of preliminary lemmas. The first three give us the necessary decompositions of $\langle S \rangle_n$ where $S = \{1, \ldots, x\}$ or $S = \{1, \ldots, x-1\} \cup \{x+1\}$ for some $x \leq 8$. Lemma 8 was proven independently in [15] and [42], and is a special case of Theorem 5 in [27]. Lemmas 7 and 9 will be proved in Section 6.

Lemma 7 *If*

$$S \in \{\{1,2,3\}, \{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}, \{1,2,3,4,5,6,7,8\}\},$$

$n \geq 2\max(S)+1$, *and* $M = (m_1, \ldots, m_t, k)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i = 1, \ldots, t$, $3 \leq k \leq n$, *and* $\sum M = |S|n$, *then there is an* (M)*-decomposition of* $\langle S \rangle_n$, *except possibly when*

- $S = \{1,2,3,4,6\}$, $n \equiv 3 \pmod 6$ *and* $M = (3^{\frac{5n}{3}})$*; or*
- $S = \{1,2,3,4,6\}$, $n \equiv 4 \pmod 6$ *and* $M = (3^{\frac{5n-5}{3}}, 5)$.

Proof See Section 6. $\square$

Lemma 8 ([15, 42]) *If* $n \geq 5$ *and* $M = (m_1, \ldots, m_t, n)$ *is any list satisfying* $m_i \in \{3, \ldots, n\}$ *for* $i = 1, \ldots, t$, *and* $\sum M = 2n$, *then there is an* (M)*-decomposition of* $\langle \{1,2\} \rangle_n$.

Lemma 9 *If* $n \geq 7$ *and* $M = (m_1, \ldots, m_t, k, n)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i = 1, \ldots, t$, $3 \leq k \leq n$, *and* $\sum M = 3n$, *then there is an* (M)*-decomposition of* $\langle \{1,2,3\} \rangle_n$.

Proof See Section 6. $\square$

We now present the lemmas which give us the necessary decompositions of $K_n - \langle S \rangle_n$. Lemma 10 was proved in [25] where it was used to prove Theorem 1 in the case where $M = (3^a, n^b)$ for some $a, b \geq 0$. Lemmas 11 and 12 give our main results on decompositions of $K_n - \langle S \rangle_n$. Lemma 11 is for the case $\nu_5(M) \leq 2$ and Lemma 12 is for the case $\nu_5(M) \geq 3$.

Lemma 10 ([25], Lemma 3.1) *If* $1 \leq h \leq \lfloor \frac{n-1}{2} \rfloor$, *then there is an* (n^h)*-decomposition of* $K_n - \langle \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor - h\} \rangle_n$.

Lemma 11 *If* $S \in \{\{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}\}$ *and* $n \geq 2\max(S)+1$, $t \geq 0$, $q \geq 0$ *and* $h \geq 2$ *are integers satisfying* $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - |S|$, *then there is a* $(3^{tn}, 4^{qn}, n^h)$*-decomposition of* $K_n - \langle S \rangle_n$, *except possibly when* $h = 2$, $S = \{1,2,3,4,5,6,7\}$ *and*

- $n \in \{25, 26\}$ *and* $t = 1$*; or*
- $n = 31$ *and* $t = 2$.

Proof See Section 7. $\square$

Lemma 12 *If* $S \in \{\{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}, \{1,2,3,4,5,6,7,8\}\}$ *and* $n \geq 2\max(S)+1$, $r \geq 0$ *and* $h \geq 2$ *are integers satisfying* $5r + h = \lfloor \frac{n-1}{2} \rfloor - |S|$, *then there is a* $(5^{rn}, n^h)$*-decomposition of* $K_n - \langle S \rangle_n$.

Proof See Section 7. □

We also need Lemmas 14 and 15 below to deal with cases arising from the possible exceptions in Lemmas 7 and 11 respectively. To prove Lemma 14 we use the following special case of Lemma 2.8 in [21].

Lemma 13 *If there exists an $(M, 4^2)$-decomposition of K_n in which there are two 4-cycles intersecting in exactly one vertex, then there exists an $(M, 3, 5)$-decomposition of K_n.*

Lemma 14 *If M is an n-ancestor list such that $\nu_n(M) \geq 2$, $M_s = (3^{\frac{5n-5}{3}}, 5)$ and $n \equiv 4 \pmod 6$ then there is an (M)-decomposition of K_n.*

Proof We will construct an $(\overline{M_s}, 3^{\frac{5n-8}{3}}, 4^2)$-decomposition of K_n in which two 4-cycles intersect in exactly one vertex. The required (M)-decomposition of K_n can then be obtained by applying Lemma 13.

By Lemma 11 there is an $(\overline{M_s})$-decomposition of $K_n - \langle\{1,2,3,4,6\}\rangle_n$, so it suffices to construct a $(3^{\frac{5n-8}{3}}, 4^2)$-decomposition of $\langle\{1,2,3,4,6\}\rangle_n$ in which the two 4-cycles intersect in exactly one vertex for all $n \equiv 4 \pmod 6$ with $n \geq 16$ (note that the conditions of the lemma imply $n \geq 16$). The union of the following two sets of cycles gives such a decomposition.

$$\{\ (0,4,2,6), (2,3,5,8), (1,5,7), (3,4,7), (3,6,9), (4,5,6)\ \}$$

$$\begin{aligned}\{\ (x+6i, y+6i, z+6i)\ :\ i \in \{0,\ldots,\tfrac{n-10}{6}\},\ (x,y,z) \in \{&(4,8,10),(5,9,11),(6,8,12),(6,7,10),\\ &(7,11,13),(7,8,9),(9,12,15),(9,10,13),(10,11,12),(8,11,14)\}\ \}\end{aligned}$$

□

Lemma 15 *If M is an n-ancestor list such that $\nu_5(M) \leq 2$, $\nu_n(M) = 2$, $\sum M_s = 7n$, and*

- *$n = 25$ and $\nu_3(\overline{M_s}) = 25$;*
- *$n = 26$ and $\nu_3(\overline{M_s}) = 26$; or*
- *$n = 31$ and $\nu_3(\overline{M_s}) = 62$;*

then there is an (M)-decomposition of K_n.

Proof We begin by showing that it is possible to partition M_s into two lists M_s^1 and M_s^2 such that $\sum M_s^1 = 3n$ and $\sum M_s^2 = 4n$. If $\nu_3(M_s) \geq n$ or $\nu_4(M_s) \geq n$, then clearly such a partition exists. Otherwise, $\nu_n(M_s) = 0$ by Property (5) of Lemma 6, and so by the definition of n-ancestor list and the hypotheses of this lemma, we have that

$$7n = \sum M_s \leq 3\nu_3(M_s) + 4\nu_4(M_s) + 10 + (n-1).$$

It is routine to check, using $3\nu_3(M_s) \leq 3n-3$ and $4\nu_4(M_s) \leq 4n-4$, that $\nu_4(M_s) \geq \frac{3n-6}{4}$ and $\nu_3(M_s) \geq \frac{2n-5}{3}$. Thus for $n = 25$, $n = 26$ and $n = 31$, we can choose $M_s^1 = (3, 4^{18})$, $M_s^1 = (3^2, 4^{18})$, and $M_s^1 = (3^3, 4^{21})$ respectively. This yields the desired partition of M_s.

For $n = 25$ we note that $\langle\{1,2,3\}\rangle_n \cong \langle\{2,4,6\}\rangle_n$ (with $x \mapsto 2x$ being an isomorphism) and $\langle\{1,2,3,4\}\rangle_n \cong \langle\{1,7,8,9\}\rangle_n$ (with $x \mapsto 8x$ being an isomorphism). Since $\{3,10,12\}$ is a modulo 25 difference triple and $\langle\{5,11\}\rangle_{25}$ has a Hamilton decomposition (by a result of Bermond et al [13], see Lemma 53), this gives us a decomposition of K_{25} into a copy of $\langle\{1,2,3\}\rangle_{25}$, a copy of

$\langle\{1,2,3,4\}\rangle_{25}$, twenty-five 3-cycles and two Hamilton cycles. By Lemma 7, there is an (M_s^1)-decomposition of $\langle\{1,2,3\}\rangle_{25}$ and an (M_s^2)-decomposition of $\langle\{1,2,3,4\}\rangle_{25}$, and this gives us the required (M)-decomposition of K_{25}.

For $n = 26$ we note that $\langle\{1,2,3,4\}\rangle_n \cong \langle\{5,6,10,11\}\rangle_n$ (with $x \mapsto 5x$ being an isomorphism). Since $\{4,8,12\}$ is a difference triple and $\langle\{7,9\}\rangle_{26}$ has a Hamilton decomposition (by a result of Bermond et al [13], see Lemma 53), this gives us a decomposition of K_{26} into a copy of $\langle\{1,2,3\}\rangle_{26}$, a copy of $\langle\{1,2,3,4\}\rangle_{26}$, twenty-six 3-cycles and two Hamilton cycles. By Lemma 7, there is an (M_s^1)-decomposition of $\langle\{1,2,3\}\rangle_{26}$ and an (M_s^2)-decomposition of $\langle\{1,2,3,4\}\rangle_{26}$, and this gives us the required (M)-decomposition of K_{26}.

For $n = 31$ we note that $\langle\{1,2,3,4\}\rangle_n \cong \langle\{4,8,12,15\}\rangle_n$ (with $x \mapsto 4x$ being an isomorphism). Since $\{5,6,11\}$ is a difference triple, $\{7,10,14\}$ is a modulo 31 difference triple, and $\langle\{9,13\}\rangle_{31}$ has a Hamilton decomposition (by a result of Bermond et al [13], see Lemma 53), this gives us a decomposition of K_{31} into a copy of $\langle\{1,2,3\}\rangle_{31}$, a copy of $\langle\{1,2,3,4\}\rangle_{31}$, sixty-two 3-cycles and two Hamilton cycles. By Lemma 7, there is an (M_s^1)-decomposition of $\langle\{1,2,3\}\rangle_{31}$ and an (M_s^2)-decomposition of $\langle\{1,2,3,4\}\rangle_{31}$, which yields required (M)-decomposition of K_{31}. $\square$

We can now prove Lemma 4 which states that if M is an n-ancestor list with $\nu_n(M) \geq 2$, then there is an (M)-decomposition of K_n.

Proof of Lemma 4 If $M = (3^a, n^b)$ for some $a, b \geq 0$, then we can use the main result from [25] to obtain an (M)-decomposition of K_n, so we can assume that $M \neq (3^a, n^b)$ for any $a, b \geq 0$. By Lemma 2 we can assume that $n \geq 15$. Partition M into M_s and $\overline{M_s}$. The proof splits into cases according to the value of $\sum M_s$, which by Lemma 6 is in $\{2n, 3n, \ldots, 8n\}$.

Case 1 Suppose that $\sum M_s = 2n$. In this case, from Property (2) of Lemma 6 we have $\nu_n(M_s) = 1$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$. The required decomposition of K_n can be obtained by combining an (M_s)-decomposition of $\langle\{1,2\}\rangle_n$ (which exists by Lemma 8) with a Hamilton decomposition of $K_n - \langle\{1,2\}\rangle_n$ (which exists by Lemma 10).

Case 2 Suppose that $\sum M_s = 3n$. In this case, from Property (3) of Lemma 6 we have $\nu_n(M_s) \in \{0,1\}$ and $\overline{M_s} = (n^h)$ for some $h \geq 1$. The required decomposition of K_n can be obtained by combining an (M_s)-decomposition of $\langle\{1,2,3\}\rangle_n$ (which exists by Lemma 7 or 9) with a Hamilton decomposition of $K_n - \langle\{1,2,3\}\rangle_n$ (which exists by Lemma 10).

Case 3 Suppose that $\sum M_s \in \{4n, 5n, 6n, 7n, 8n\}$ and $\nu_5(M) \geq 3$. In this case, from Property (4) of Lemma 6 we have $\nu_n(M_s) = 0$ and $\overline{M_s} = (5^{rn}, n^h)$ for some $r \geq 0$, $h \geq 2$, and we also have $3\nu_3(M) \leq n - 10$ from the definition of n-ancestor list. We let

$$S \in \{\{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}, \{1,2,3,4,5,6,7,8\}\}$$

such that $|S| = \frac{1}{n}\sum M_s$ and obtain the required decomposition of K_n by combining an (M_s)-decomposition of $\langle S\rangle_n$ (which exists by Lemma 7), with an $(\overline{M_s})$-decomposition of $K_n - \langle S\rangle_n$ (which exists by Lemma 12). Note that the condition $3\nu_3(M) \leq n - 10$ implies that the required (M_s)-decomposition of $\langle S\rangle_n$ is not among the listed possible exceptions in Lemma 7. Note also that the condition $n \geq 2\max(S) + 1$ required in Lemmas 7 and 12 is easily seen to be satisfied because $n \geq 15$ and $\sum M_s \leq n\lfloor\frac{n-1}{2}\rfloor$.

Case 4 Suppose that $\sum M_s \in \{4n, 5n, 6n, 7n, 8n\}$ and $\nu_5(M) \leq 2$. In this case we have $\nu_n(M_s) = 0$ and $\overline{M_s} = (3^{tn}, 4^{qn}, n^h)$ for some $t, q \geq 0$, $h \geq 2$ (see Property (5) in Lemma 6), and $\sum M_s \neq 8n$ (see Property (1) in Lemma 6). We let

$$S \in \{\{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}\}$$

such that $|S| = \frac{1}{n}\sum M_s$. If Lemma 7 gives us an (M_s)-decomposition of $\langle S\rangle_n$ and Lemma 11 gives us an $(\overline{M_s})$-decomposition of $K_n - \langle S\rangle_n$, then we have the required decomposition of K_n. The condition $n \geq 2\max(S)+1$ required in Lemmas 7 and 11 is satisfied because $n \geq 15$. This leaves only the cases arising from the possible exceptions in Lemma 7 and Lemma 11, and these are covered by Lemmas 14 and 15 respectively. □

4 The case of no Hamilton cycles

In this section we prove that Lemma 5 holds in the case $\nu_n(M) = 0$. In this case, for $n \geq 15$, one of $\nu_3(M)$, $\nu_4(M)$ and $\nu_5(M)$ must be sizable, and the proof splits into three cases accordingly. Each of these three cases splits into subcases according to whether n is even or odd. In each case we construct the required decomposition of K_n from a suitable decomposition of K_{n-1} or K_{n-2}.

4.1 Many 3-cycles and no Hamilton cycles

In Lemma 16 we construct the required decompositions of complete graphs of odd order and in Lemma 17 we construct the required decompositions of complete graphs of even order.

Lemma 16 *If n is odd, Theorem 1 holds for K_{n-1}, and $(M, 3^{\frac{n-1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-1}{2}})$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that M is $(n-1)$-admissible. Thus, by assumption there is an (M)-decomposition $\mathcal{D}$ of K_U. Let I be the perfect matching in $\mathcal{D}$. Then

$$\mathcal{D} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-1}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(3^{\frac{n-1}{2}})$-decomposition of $K_{\{\infty\}} \vee I$. □

Lemma 17 *If n is even, Theorem 1 holds for K_{n-1}, and $(M, 3^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-2}{2}})$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-2)$ is $(n-1)$-admissible and so by assumption there is an $(M, n-2)$-decomposition $\mathcal{D}$ of K_U. Let C be an $(n-2)$-cycle in $\mathcal{D}$, let $\{I, I_1\}$ be a decomposition of C into two matchings, and let x be the vertex in $U \setminus V(C)$. Then

$$(\mathcal{D} \setminus \{C\}) \cup \{I + \infty x\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-2}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(3^{\frac{n-2}{2}})$-decomposition of $K_{\{\infty\}} \vee I_1$. □

4.2 Many 4-cycles and no Hamilton cycles

Lemma 18 *If n is odd, Theorem 1 holds for K_{n-2}, and $(M, 4^{\frac{n+1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n+1}{2}})$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n - 2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n+1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n - 3$. Thus, it is easily seen that $(M, 5)$ is $(n - 2)$-admissible and by assumption there is an $(M, 5)$-decomposition $\mathcal{D}$ of K_U.

Let C be a 5-cycle in $\mathcal{D}$ and let x, y and z be vertices of C such that x and y are adjacent in C and z is not adjacent to either x or y in C. Then

$$(\mathcal{D} \setminus \{C\}) \cup \mathcal{D}_1 \cup \mathcal{D}_2$$

is an $(M, 4^{\frac{n+1}{2}})$-decomposition of K_V, where

- $\mathcal{D}_1$ is a $(4^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U\setminus\{x,y,z\}}$; and
- $\mathcal{D}_2$ is a (4^3)-decomposition of $K_{\{\infty_1,\infty_2\},\{x,y,z\}} \cup [\infty_1, \infty_2] \cup C$.

These decompositions are straightforward to construct. □

Lemma 19 *If n is even, Theorem 1 holds for K_{n-2}, and $(M, 4^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n-2}{2}})$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n - 2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n-2}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n - 3$. Thus, it is easily seen that M is $(n - 2)$-admissible and by assumption there is an (M)-decomposition $\mathcal{D}$ of K_U. Let I be the perfect matching in $\mathcal{D}$. Then

$$(\mathcal{D} \setminus \{I\}) \cup \{I + \infty_1\infty_2\} \cup \mathcal{D}_1$$

is an $(M, 4^{\frac{n-2}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(4^{\frac{n-2}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U}$. □

4.3 Many 5-cycles and no Hamilton cycles

We will make use of the following lemma in this subsection and in Subsection 5.5.

Lemma 20 *If G is a 3-regular graph which contains a perfect matching and ∞ is a vertex not in $V(G)$, then there is a decomposition of $K_{\{\infty\}} \vee G$ into $\frac{1}{2}|V(G)|$ 5-cycles.*

Proof Let I be a perfect matching in G. Then $G - I$ is a 2-regular graph on the vertex set $V(G)$ and hence it can be given a coherent orientation O. Let

$$\mathcal{D} = \{(\infty, a, b, c, d) : bc \in E(I) \text{ and } (b, a), (c, d) \in E(O)\}$$

be a set of (undirected) 5-cycles. Because O contains exactly one arc directed from each vertex of $V(G)$, $|\mathcal{D}| = |E(I)| = \frac{1}{2}|V(G)|$ and each edge of G appears in exactly one cycle in $\mathcal{D}$. Further, because O contains exactly one arc directed to each vertex of $V(G)$, each edge of $K_{\{\infty\},V}$ appears in exactly one cycle in $\mathcal{D}$. Thus $\mathcal{D}$ is a decomposition of $K_{\{\infty\}} \vee G$ into $\frac{1}{2}|V(G)|$ 5-cycles. □

Lemma 21 *If n is odd, Theorem 1 holds for K_{n-1}, and $(M, 5^{\frac{n-1}{2}})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{\frac{n-1}{2}})$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n - 1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since the list $(M, n-1)$ is easily seen to be $(n-1)$-admissible, by assumption there is an $(M, n-1)$-decomposition $\mathcal{D}$ of K_U. Let C be an $(n-1)$-cycle in $\mathcal{D}$ and let I be the perfect matching in $\mathcal{D}$. Then

$$(\mathcal{D} \setminus \{C, I\}) \cup \mathcal{D}_1$$

is an $(M, 5^{\frac{n-1}{2}})$-decomposition of K_V, where $\mathcal{D}_1$ is a $(5^{\frac{n-1}{2}})$-decomposition of $K_{\{\infty\}} \vee (C \cup I)$ (this exists by Lemma 20). □

Lemma 22 *If n is even, Theorem 1 holds for K_{n-2}, and $(M, 5^{n-2})$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{n-2})$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n - 2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 5^{n-2})$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (6) in the definition of ancestor lists that any cycle length in M is at most $n - 5$. Thus, it is easily seen that the list $(M, (n-2)^3)$ is $(n-2)$-admissible and by assumption there is an $(M, (n-2)^3)$-decomposition $\mathcal{D}$ of K_U. Let C_1, C_2 and C_3 be distinct $(n-2)$-cycles in $\mathcal{D}$ and let I be the perfect matching in $\mathcal{D}$. Let $\{I_1, I_2\}$ be a decomposition of C_3 into two perfect matchings. Then

$$(\mathcal{D} \setminus \{C_1, C_2, C_3, I\}) \cup \{I + \infty_1\infty_2\} \cup \mathcal{D}_1 \cup \mathcal{D}_2$$

is an $(M, 5^{n-2})$-decomposition of K_V, where for $i = 1, 2$, $\mathcal{D}_i$ is a $(5^{\frac{n-2}{2}})$-decomposition of $K_{\{\infty_i\}} \vee (C_i \cup I_i)$ (these exist by Lemma 20). □

4.4 Proof of Lemma 5 in the case of no Hamilton cycles

Lemma 23 *If Theorem 1 holds for K_{n-1} and K_{n-2}, then there is an (M)-decomposition of K_n for each n-ancestor list M satisfying $\nu_n(M) = 0$.*

Proof By Lemma 2 we can assume that $n \geq 15$. If there is a cycle length in M which is at least 6 and at most $n-1$, then let k be this cycle length. Otherwise let $k = 0$. We deal separately with the case n is odd and the case n is even.

Case 1 Suppose that n is odd. Since $n \geq 15$ and $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k = \frac{n(n-1)}{2}$, it can be seen that either $\nu_3(M) \geq \frac{n-1}{2}$, $\nu_4(M) \geq \frac{n+1}{2}$ or $\nu_5(M) \geq \frac{n-1}{2}$. If $\nu_3(M) \geq \frac{n-1}{2}$, then the result follows by Lemma 16. If $\nu_4(M) \geq \frac{n+1}{2}$, then the result follows by Lemma 18. If $\nu_5(M) \geq \frac{n-1}{2}$, then the result follows by Lemma 21.

Case 2 Suppose that n is even. Since $n \geq 16$, $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k = \frac{n(n-2)}{2}$ and $k \leq n-1$, it can be seen that either $\nu_3(M) \geq \frac{n-2}{2}$, $\nu_4(M) \geq \frac{n-2}{2}$ or $\nu_5(M) \geq n-2$. (To see this consider the cases $\nu_5(M) \geq 3$ and $\nu_5(M) \leq 2$ separately and use the definition of n-ancestor list.) If $\nu_3(M) \geq \frac{n-2}{2}$, then the result follows by Lemma 17. If $\nu_4(M) \geq \frac{n-2}{2}$, then the result follows by Lemma 19. If $\nu_5(M) \geq n-2$, then the result follows by Lemma 22. □

5 The case of exactly one Hamilton cycle

In this section we prove that Lemma 5 holds in the case $\nu_n(M) = 1$. Again in this case, for $n \geq 15$, one of $\nu_3(M)$, $\nu_4(M)$ and $\nu_5(M)$ must be sizable, and the proof splits into cases accordingly. The case in which $\nu_3(M)$ is sizable further splits according to whether $\nu_4(M) \geq 1$, $\nu_5(M) \geq 1$, or $\nu_4(M) = \nu_5(M) = 0$. We first require some preliminary definitions and results.

5.1 Preliminaries

Let $\mathcal{P}$ be an (M)-packing of K_n, let $\mathcal{P}'$ be an (M')-packing of K_n and let S be a subset of $V(K_n)$. We say that $\mathcal{P}$ and $\mathcal{P}'$ are *equivalent on* S if we can write $\{G \in \mathcal{P} : V(G) \cap S \neq \emptyset\} = \{G_1, \ldots, G_t\}$ and $\{G \in \mathcal{P}' : V(G) \cap S \neq \emptyset\} = \{G'_1, \ldots, G'_t\}$ such that

- for $i \in \{1, \ldots, t\}$, G_i is isomorphic to G'_i;
- for each $x \in S$ and for $i \in \{1, \ldots, t\}$, $x \in V(G_i)$ if and only if $x \in V(G'_i)$; and
- for all distinct $x, y \in S$ and for $i \in \{1, \ldots, t\}$, $xy \in E(G_i)$ if and only if $xy \in E(G'_i)$.

The following lemma is from [20]. It encapsulates a key *edge swapping* technique which was used in many of the proofs in [21], and which we shall make use of in this section.

Lemma 24 ([20], Lemma 2.1) *Let n be a positive integer, let M be a list of integers, let $\mathcal{P}$ be an (M)-packing of K_n with a leave, L say, let α and β be vertices of L, let π be the transposition $(\alpha\beta)$, and let $Z = Z(\mathcal{P}, \alpha, \beta) = (\mathrm{Nbd}_L(\alpha) \cup \mathrm{Nbd}_L(\beta)) \setminus ((\mathrm{Nbd}_L(\alpha) \cap \mathrm{Nbd}_L(\beta)) \cup \{\alpha, \beta\})$. Then there exists a partition of the set Z into pairs such that for each pair $\{u, v\}$ of the partition, there exists an (M)-packing of K_n, $\mathcal{P}'$ say, with a leave, L' say, which differs from L only in that αu, αv, βu and βv are edges in L' if and only if they are not edges in L. Furthermore, if $\mathcal{P} = \{C_1, \ldots, C_t\}$ (n odd) or $\mathcal{P} = \{I, C_1, \ldots, C_t\}$ (n even) where $C_1, \ldots, C_t$ are cycles and I is a perfect matching, then $\mathcal{P}' = \{C'_1, \ldots, C'_t\}$ (n odd) or $\mathcal{P}' = \{I', C'_1, \ldots, C'_t\}$ (n even) where for $i = 1, \ldots, t$, C'_i is a cycle of the same length as C_i and I' is a perfect matching such that*

- *either $I' = I$ or $I' = \pi(I)$;*
- *for $i = 1, \ldots, t$ if neither α nor β is in $V(C_i)$ then $C'_i = C_i$;*
- *for $i = 1, \ldots, t$ if exactly one of α and β is in $V(C_i)$ then either $C'_i = C_i$ or $C'_i = \pi(C_i)$; and*
- *for $i = 1, \ldots, t$ if both α and β are in $V(C_i)$ then $C'_i \in \{C_i, \pi(C_i), \pi(P_i) \cup P_i^\dagger, P_i \cup \pi(P_i^\dagger)\}$ where P_i and $P_i^\dagger$ are the two paths in C_i which have endpoints α and β.*

We say that $\mathcal{P}'$ is the (M)-packing obtained from $\mathcal{P}$ by performing the (α, β)-switch with origin u and terminus v (we could equally call v the origin and u the terminus). For our purposes here, it is important to note that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V(L) \setminus \{\alpha, \beta\}$.

We will also make use of three lemmas from [21]. The original version of Lemma 25 (Lemma 2.15 in [21]) does not include the claim that $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V(L) \setminus \{a, b\}$, this follows directly from the fact that the proof uses only (a, b)-switches.

Lemma 25 *Let n be a positive integer and let M be a list of integers. Suppose that there exists an (M)-packing $\mathcal{P}$ of K_n with a leave L which contains two vertices a and b such that $\deg_L(a) + 2 \le \deg_L(b)$. Then there exists an (M)-packing $\mathcal{P}'$ of K_n, which is equivalent to $\mathcal{P}$ on $V(L) \setminus \{a, b\}$, and which has a leave L' such that $\deg_{L'}(a) = \deg_L(a) + 2$, $\deg_{L'}(b) = \deg_L(b) - 2$ and $\deg_{L'}(x) = \deg_L(x)$ for all $x \in V(L) \setminus \{a, b\}$. Furthermore,*

(i) if a and b are adjacent in L, then L' has the same number of non-trivial components as L;

(ii) if $\deg_L(a) = 0$ and b is not a cut-vertex of L, then L' has the same number of non-trivial components as L; and

(iii) if $\deg_L(a) = 0$, then either L' has the same number of non-trivial components as L, or L' has one more non-trivial component than L.

Similarly, the original versions of Lemmas 26 and 27 (Lemmas 2.14 and 2.11 respectively in [21]) did not include the claims that the final decompositions are equivalent to the initial packings on $V \setminus U$. However, these claims can be seen to hold as the proofs of the lemmas given in [21] require switching only on vertices of positive degree in the leave, with one exception which we discuss shortly. The lemmas below each contain the additional hypothesis that $\deg_L(x) = 0$ for all $x \in V \setminus U$, and this ensures all the switches are on vertices of U and hence that the final decomposition is equivalent to the initial packing on $V \setminus U$.

The exception mentioned above occurs in the proof of the original version of Lemma 27 where a switch on a vertex of degree 0 in the leave is required when $3 \in \{m_1, m_2\}$. We can ensure this switch is on a vertex in U because we have the additional hypothesis that $\deg_L(x) = 0$ for some $x \in U$ when $3 \in \{m_1, m_2\}$. This additional hypothesis also allows us to omit the hypothesis, included in the original version of Lemma 27, that the size of the leave be at most $n + 1$, because in the proof this was used only to ensure the existence of a vertex of degree 0 in the leave when $3 \in \{m_1, m_2\}$. Thus the modified versions stated below hold by the proofs presented in [21].

Lemma 26 *Let V be a vertex set and let U be a subset of V. Let M be a list of integers and let k, m_1 and m_2 be positive integers such that $m_1, m_2 \ge \max(\{3, k + 1\})$. Suppose that there exists an (M)-packing $\mathcal{P}$ of K_V with a leave L of size $m_1 + m_2$ such that $\Delta(L) = 4$, exactly one vertex of L has degree 4, L has exactly k non-trivial components, L does not have a decomposition into two odd cycles if m_1 and m_2 are both even, and $\deg_L(x) = 0$ for all $x \in V \setminus U$. Then there exists an (M, m_1, m_2)-decomposition of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.*

Lemma 27 *Let V be a vertex set and let U be a subset of V. Let M be a list of integers and let m_1 and m_2 be integers such that $m_1, m_2 \ge 3$. Suppose that there exists (M)-packing $\mathcal{P}$ of K_V with a leave L of size $m_1 + m_2$ such that $\Delta(L) = 4$, exactly two vertices of L have degree 4, L has exactly one non-trivial component, $\deg_L(x) = 0$ for all $x \in V \setminus U$, and $\deg_L(x) = 0$ for some $x \in U$ if $3 \in \{m_1, m_2\}$. Then there exists an (M, m_1, m_2)-decomposition of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.*

We also require Lemma 28, which deals with some small order cases.

Lemma 28 *Let n be an integer such that $n \in \{15, 16, 17, 18, 19, 20, 22, 24, 26\}$ and let M be an n-ancestor list such that $\nu_n(M) = 1$, $\nu_5(M) \ge 3$, and $\nu_4(M) \ge 2$ if $n = 24$. Then there is an (M)-decomposition of K_n.*

Proof If there is a cycle length in M which is at least 6 and at most $n-1$, then let k be this cycle length. Otherwise let $k = 0$. Note that $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k + n = n\lfloor \frac{n-1}{2} \rfloor$ and that, because M is an n-ancestor list with $\nu_5(M) \geq 3$, it follows that $3\nu_3(M) \leq n-10$, $2\nu_4(M) \leq n-6$, and $k \leq n-5$.

Using this, it is routine to check that if $n = 15$ then M must be one of 12 possible lists and if $n = 16$ then M must be one of 26 possible lists. In each of these cases we have constructed an (M)-decomposition of K_n by computer search.

If $n \in \{17, 18, 19, 20, 22, 24, 26\}$, then we partition $\{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ into sets S_1, S_2 and S_3 according to the following table.

n	S_1	S_2	S_3
17	$\{1,2,3,4,5,6,7\}$	$\emptyset$	$\{8\}$
18	$\{1,2,3,4,5,6,7\}$	$\emptyset$	$\{8,9\}$
19	$\{1,2,3,4,5,6,7,8\}$	$\emptyset$	$\{9\}$
20	$\{1,2,3,4,5,6,7,8\}$	$\emptyset$	$\{9,10\}$
22	$\{1,2,3,4\}$	$\{6,7,8,9,10\}$	$\{5,11\}$
24	$\{1,2,3\}$	$\{4,6,7,8,11\}$	$\{5,9,10,12\}$
26	$\{1,2,3,4,5,7\}$	$\{6,8,9,10,11\}$	$\{12,13\}$

Using $3\nu_3(M) \leq n-10$, $2\nu_4(M) \leq n-6$ and $k \leq n-5$, it is routine to check that $\nu_5(M) \geq n$ when $n \in \{22, 26\}$, and that $\nu_5(M) \geq n+8$ when $n = 24$. By Lemma 7, there is an (M')-decomposition of $\langle S_1 \rangle_n$, where $M = (M', n)$ when $n \in \{17, 18, 19, 20\}$, $M = (M', 5^n, n)$ when $n \in \{22, 26\}$, and $M = (M', 4^2, 5^{n+8}, n)$ when $n = 24$. For $n \in \{22, 24, 26\}$, it is easy to see that S_2 is a modulo n difference 5-tuple, and so there is a (5^n)-decomposition of $\langle S_2 \rangle_n$. For $n \in \{17, 18, 19, 20, 22, 26\}$, there is an (n)-decomposition of the graph $\langle S_3 \rangle_n$, as it is either an n-cycle or a connected 3-regular Cayley graph on a cyclic group, and the latter are well known to contain a Hamilton cycle, see [30]. For $n = 24$, $\langle \{5, 9, 10\} \rangle_n \cong \langle \{1, 2, 3\} \rangle_n$ (with $x \mapsto 5x$ being an isomorphism). Thus, by Lemma 9 there is a $(4^2, 5^8, 24)$-decomposition of $\langle \{5, 9, 10, 12\} \rangle_{24}$ (as $\langle \{12\} \rangle_{24}$ is a perfect matching). Combining these decompositions of $\langle S_1 \rangle_n$, $\langle S_2 \rangle_n$ and $\langle S_3 \rangle_n$ gives us the required (M)-decomposition of K_n. $\square$

5.2 Many 3-cycles, one Hamilton cycle, and at least one 4- or 5-cycle

In Lemmas 29 and 30 we construct the required decompositions of complete graphs of odd order in the cases where the decomposition contains at least one 4-cycle or at least one 5-cycle, respectively. In Lemmas 31 and 32 we construct the required decompositions of complete graphs of even order in the cases where the decomposition contains at least one 4-cycle or at least one 5-cycle, respectively. These results are proved by constructing the required decomposition of K_n from a suitable decomposition of K_{n-1}, K_{n-2} or K_{n-3}.

Lemma 29 *If n is odd, Theorem 1 holds for K_{n-1}, and $(M, 3^{\frac{n-5}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-5}{2}}, 4, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-5}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-2)$ is $(n-1)$-admissible and by assumption there is an $(M, n-2)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-2)$-cycle in $\mathcal{D}$, let I be the perfect matching in $\mathcal{D}$, and let $[w,x,y,z]$ be a path in $I \cup H$ such that $w \notin V(H)$, $wx, yz \in E(I)$ and $xy \in E(H)$. Then

$$(\mathcal{D} \setminus \{I, H\}) \cup \{H', (\infty, x, y, z)\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-5}{2}}, 4, n)$-decomposition of K_V, where

- $H' = (H - [x,y]) \cup [x, w, \infty, y]$; and
- $\mathcal{D}_1$ is a $(3^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty\}, U \setminus \{w,x,y,z\}} \cup (I - \{wx, yz\})$. □

Lemma 30 *If n is odd, Theorem 1 holds for K_{n-1}, and $(M, 3^{\frac{n-5}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-5}{2}}, 5, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-5}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-1)$ is $(n-1)$-admissible and so by assumption there is an $(M, n-1)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-1)$-cycle in $\mathcal{D}$, let I be the perfect matching in $\mathcal{D}$, and let $[w,x,y,z]$ be a path in $I \cup H$ such that $wx, yz \in E(I)$ and $xy \in E(H)$. Then

$$(\mathcal{D} \setminus \{I, H\}) \cup \{H', (\infty, w, x, y, z)\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-5}{2}}, 5, n)$-decomposition of K_V, where

- $H' = (H - [x,y]) \cup [x, \infty, y]$; and
- $\mathcal{D}_1$ is a $(3^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty\}, U \setminus \{w,x,y,z\}} \cup (I - \{wx, yz\})$. □

Lemma 31 *If n is even, Theorem 1 holds for K_{n-3}, and $(M, 3^{\frac{3n-14}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{3n-14}{2}}, 4, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n-3$, let ∞_1, ∞_2 and ∞_3 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2, \infty_3\}$. Since $(M, 3^{\frac{3n-14}{2}}, 4, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n-3$. Thus, it is easily seen that $(M, (n-4)^2, n-3)$ is $(n-3)$-admissible and so by assumption there is an $(M, (n-4)^2, n-3)$-decomposition $\mathcal{D}$ of K_U.

Let C_1 and C_2 be distinct $(n-4)$-cycles in $\mathcal{D}$, let H be an $(n-3)$-cycle in $\mathcal{D}$, let $\{I, I_1\}$ be a decomposition of C_1 into two matchings, let $\{I_2, I_3\}$ be a decomposition of C_2 into two matchings, let w be the vertex in $U \setminus V(C_1)$, and let $[x,y,z]$ be a path in $H \cup I_3$ such that $x \notin V(C_2)$, $xy \in E(H)$ and $yz \in E(I_3)$ (possibly $w \in \{x,y,z\}$). Then

$$(\mathcal{D} \setminus \{H, C_1, C_2\}) \cup \{I + \{\infty_1 w, \infty_2\infty_3\}, H', (\infty_3, x, y, z)\} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

is an $(M, 3^{\frac{3n-14}{2}}, 4, n)$-decomposition of K_V, where

- $H' = (H - [x,y]) \cup [x, \infty_2, \infty_1, \infty_3, y]$;
- for $i = 1, 2$, $\mathcal{D}_i$ is a $(3^{\frac{n-4}{2}})$-decomposition of $K_{\{\infty_i\}} \vee I_i$; and

- $\mathcal{D}_3$ is a $(3^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty_3\},U\setminus\{x,y,z\}} \cup (I_3 - yz)$. □

Lemma 32 *If n is even, Theorem 1 holds for K_{n-1}, and $(M, 3^{\frac{n-6}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 3^{\frac{n-6}{2}}, 5, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n-1$, let ∞ be a vertex not in U, and let $V = U \cup \{\infty\}$. Since $(M, 3^{\frac{n-6}{2}}, 5, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows that $(M, n-2, n-1)$ is $(n-1)$-admissible and so by assumption there is an $(M, n-2, n-1)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-1)$-cycle in $\mathcal{D}$, let C be an $(n-2)$-cycle in $\mathcal{D}$, let $\{I, I_1\}$ be a decomposition of C into two matchings, let $[w, x, y, z]$ be a path in $I_1 \cup H$ such that $wx, yz \in E(I_1)$ and $xy \in E(H)$, and let v be the vertex in $U \setminus V(C)$. Then

$$(\mathcal{D} \setminus \{C, H\}) \cup \{I + v\infty, H', (\infty, w, x, y, z)\} \cup \mathcal{D}_1$$

is an $(M, 3^{\frac{n-6}{2}}, 5, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty, y]$; and
- $\mathcal{D}_1$ is a $(3^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty\},U\setminus\{v,w,x,y,z\}} \cup (I - \{wx, yz\})$. □

5.3 Many 3-cycles, one Hamilton cycle and no 4- or 5-cycles

We show that the required decompositions exist in Lemma 36. We first require three preliminary lemmas. These results are proved using the edge swapping techniques mentioned previously.

Lemma 33 *Let n and k be positive integers, and let M be a list of integers. If there exists an (M)-packing of K_n whose leave has a decomposition into two 3-cycles, T_1 and T_2, and a k-cycle C such that $|V(T_1) \cap V(T_2)| = 1$, $|V(T_2) \cap V(C)| = 1$ and $V(T_1) \cap V(C) = \emptyset$, then there exists an $(M, 3, k+3)$-decomposition of K_n.*

Proof Let $\mathcal{P}$ be an (M)-packing of K_n which satisfies the conditions of the lemma and let L be its leave. Let $[w, x, y, z]$ be a path in L such that $w \in V(T_1) \setminus V(T_2)$, $V(T_1) \cap V(T_2) = \{x\}$, $V(T_2) \cap V(C) = \{y\}$, and $z \in V(C) \setminus V(T_2)$. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (w, z)-switch S with origin x. If the terminus of S is y, then the leave of $\mathcal{P}'$ has a decomposition into the 3-cycle T_2 and a $(k+3)$-cycle. Otherwise the terminus of S is not y and the leave of $\mathcal{P}'$ has a decomposition into the 3-cycle (x, y, z) and a $(k+3)$-cycle. In either case we complete the proof by adding these cycles to $\mathcal{P}'$. □

Lemma 34 *Let n and k be positive integers such that $k \leq n-4$, and let M be a list of integers. Suppose that there exists an (M)-packing of K_n whose leave has a decomposition into two 3-cycles, T_1 and T_2, and a k-cycle C such that $|V(T_1) \cap V(C)| \leq 1$, $|V(T_2) \cap V(C)| \leq 1$, and $|V(T_1) \cap V(T_2)| = 1$ if $k = n-4$. Then there exists an $(M, 3, k+3)$-decomposition of K_n.*

Proof Let $\mathcal{P}$ be an (M)-packing of K_n which satisfies the conditions of the lemma and let L be its leave.

Case 1 Suppose that $\Delta(L) = 2$. Then T_1, T_2 and C are pairwise vertex disjoint. Let $x \in V(T_1)$ and $y \in V(C)$ and let z be a neighbour in T_1 of x. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (x, y)-switch S with origin z, and let L' be the leave of $\mathcal{P}'$. Then either the non-trivial components of L' are a 3-cycle and a $(k+3)$-cycle or $\Delta(L') = 4$, exactly one vertex of L' has degree 4, and L' has exactly two nontrivial components. In the former case we can add these cycles to $\mathcal{P}'$ to complete the proof. In the latter case we can apply Lemma 26 to complete the proof.

Case 2 Suppose that $\Delta(L) = 4$. If exactly one vertex of L has degree 4, then L has exactly two nontrivial components and we can complete the proof by applying Lemma 26. Thus we can assume that L has at least two vertices of degree 4. We can further assume that $\mathcal{P}$ does not satisfy the conditions of Lemma 33, for otherwise we can complete the proof by applying Lemma 33. Noting that $|V(T_1) \cap V(T_2)| \in \{0, 1\}$, it follows that $|V(T_1) \cap V(C)| = 1$ and $|V(T_2) \cap V(C)| = 1$. Because $k \le n - 4$ and $|V(T_1) \cap V(T_2)| = 1$ if $k = n - 4$, there is an isolated vertex z in L. Let w be the vertex in $V(T_1) \cap V(C)$, let x and y be the neighbours in T_1 of w. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (w, z)-switch S with origin x, and let L' be the leave of $\mathcal{P}'$. If the terminus of S is not y, then L' has a decomposition into a $(k+3)$-cycle and a 3-cycle, and we complete the proof by adding these cycles to $\mathcal{P}'$. Otherwise the terminus of S is y and either $\Delta(L') = 4$, exactly one vertex of L' has degree 4, and L' has exactly two nontrivial components (this occurs when $|V(T_1) \cap V(T_2)| = 0$) or $\mathcal{P}'$ satisfies the conditions of Lemma 33 (this occurs when $|V(T_1) \cap V(T_2)| = 1$). Thus we can complete the proof by applying Lemma 26 or Lemma 33.

Case 3 Suppose that $\Delta(L) \ge 6$. In this case, exactly one vertex of L has degree 6 and every other vertex of L has degree at most 2. Let w be the vertex of degree 6 in L, let x be a neighbour in T_2 of w, and let y and z be the neighbours in T_1 of w. Let $\mathcal{P}'$ be the (M)-packing of K_n obtained from $\mathcal{P}$ by performing the (w, x)-switch S with origin y, and let L' be the leave of $\mathcal{P}'$. If the terminus of S is z, then $\mathcal{P}'$ satisfies the conditions of Lemma 33 and we complete the proof by applying Lemma 33. Otherwise the terminus of S is not z and L' has a decomposition into the 3-cycle T_2 and a $(k+3)$-cycle, and we complete the proof by adding these cycles to $\mathcal{P}'$. $\Box$

Lemma 35 *Let n, k and t be positive integers such that $3 \le k \le n-4$. If there exists a $(3^t, k, n)$-decomposition of K_n, then there exists a $(3^{t-1}, k+3, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \ge 15$. Let V be a vertex set with $|V| = n$, let $\mathcal{D}$ be a $(3^t, k, n)$-decomposition of K_V, and let C be a k-cycle in $\mathcal{D}$. Let $U = V \setminus V(C)$. The n-cycle in $\mathcal{D}$ contains at most $|U| - 1$ edges of K_U (as the subgraph of the n-cycle induced by U is a forest). Also, if n is even, then the perfect matching in $\mathcal{D}$ contains at most $\lfloor \frac{1}{2}|U| \rfloor$ edges of K_U. The proof now splits into two cases depending on whether $k = n - 4$.

Case 1 Suppose that $k \le n-5$. Then $|U| \ge 5$ and, by the comments in the preceding paragraph, the 3-cycles in $\mathcal{D}$ contain at least four edges of K_U. Thus there are distinct 3-cycles $T_1, T_2 \in \mathcal{D}$ such that each contains at least one edge of K_U. We can remove C, T_1 and T_2 from $\mathcal{D}$ and apply Lemma 34 to the resulting packing to complete the proof.

Case 2 Suppose that $k = n - 4$. Then $|U| = 4$, the n-cycle in $\mathcal{D}$ contains at most three edges of K_U and the perfect matching in $\mathcal{D}$ contains at most two edges of K_U. This leaves at least one edge of K_U which occurs in a 3-cycle $T_1 \in \mathcal{D}$. Let $U = \{u_1, u_2, u_3, u_4\}$ and let H be the n-cycle in $\mathcal{D}$. This case now splits into two subcases depending on whether $V(T_1) \cap V(C) = \emptyset$.

Case 2a Suppose that $V(T_1) \cap V(C) = \emptyset$. Then we can assume without loss of generality that $T_1 = (u_1, u_2, u_3)$. If any of the three edges u_1u_4, u_2u_4, u_3u_4 is in a 3-cycle $T_2 \in \mathcal{D}$, then we can remove C, T_1 and T_2 from $\mathcal{D}$ and apply Lemma 34 to the resulting packing to complete the proof. Thus, we assume there is no such 3-cycle in $\mathcal{D}$. Without loss of generality, it follows that n is even, that u_1u_4 is an edge of the perfect matching in $\mathcal{D}$, and that $u_2u_4, u_3u_4 \in E(H)$. Let z be a vertex in C which is not adjacent in H to a vertex in U (such a vertex exists as $n \ge 15$ implies

$|V(C)| \geq 11$ and there are only at most four vertices of C which are adjacent in H to vertices in U).

Now let $\mathcal{P}'$ be the $(3^{t-1}, n)$-packing of K_V obtained from $\mathcal{D} \setminus \{C, T_1\}$ by performing the (u_1, z)-switch S with origin u_2. If the terminus of S is not u_3, then the only non-trivial component in the leave of $\mathcal{P}'$ is a $(k+3)$-cycle and we can complete the proof by adding this cycle to $\mathcal{P}'$. Otherwise, the terminus of S is u_3 and the only non-trivial component in the leave of $\mathcal{P}'$ is $(u_2, u_3, z) \cup C$. Furthermore, since z is not adjacent in H to a vertex in U, the final dot point in Lemma 24 guarantees that neither u_1u_2 nor u_1u_3 is an edge of the n-cycle in $\mathcal{P}'$. Since u_1u_2 and u_1u_3 cannot both be edges of the perfect matching in $\mathcal{P}'$, this means that one of them must be in a 3-cycle $T_2' \in \mathcal{P}'$. Thus, we can remove T_2' from $\mathcal{P}'$ and apply Lemma 34 to the resulting packing to complete the proof.

Case 2b Suppose that $|V(T_1) \cap V(C)| = 1$. Let $T_1 = (x, y, z)$ with $x \in V(C)$ and $y, z \in U$, and let $w \in U \setminus \{y, z\}$. Let $\mathcal{P}'$ be the $(3^{t-1}, n)$-packing of K_V obtained from $\mathcal{D} \setminus \{C, T_1\}$ by performing the (w, x)-switch S with origin y, and let L' be the leave of $\mathcal{P}'$. If the terminus of S is not z, then the only non-trivial component in the leave of $\mathcal{P}'$ is a $(k+3)$-cycle and we can complete the proof by adding this cycle to $\mathcal{P}'$. Otherwise the terminus of S is z, and the only non-trivial components in the leave of $\mathcal{P}'$ are C and (w, y, z). By adding these cycles to $\mathcal{P}'$ we obtain a $(3^t, k, n)$-decomposition of K_V which contains a 3-cycle and an $(n-4)$-cycle which are vertex disjoint, and we can proceed as we did in Case 2a. $\square$

We are now ready to prove the main result of this subsection.

Lemma 36 *If n, k and t are positive integers such that $3 \leq k \leq n-1$, Theorem 1 holds for K_{n-3}, K_{n-2} and K_{n-1}, and $(3^t, k, n)$ is an n-ancestor list, then there is a $(3^t, k, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let $r \in \{3, 4, 5\}$ such that $r \equiv k \pmod 3$. It suffices to find a $(3^{t+\frac{k-r}{3}}, r, n)$-decomposition of K_n, since we can then obtain a $(3^t, k, n)$-decomposition of K_n by repeatedly applying Lemma 35 ($\frac{k-r}{3}$ times). If $r = 3$ then the existence of a $(3^{t+\frac{k-r}{3}}, r, n)$-decomposition of K_n follows from the main result of [25], so we may assume $r \in \{4, 5\}$. Thus, the existence of the required $(3^{t+\frac{k-r}{3}}, r, n)$-decomposition of K_n is given by one of Lemmas 29, 30, 31 and 32, provided that $t + \frac{k-r}{3} \geq \frac{3n-14}{2}$ (the number of 3-cycles in the decompositions given by Lemma 31 is at least $\frac{3n-14}{2}$ and the number is smaller for the other three lemmas for $n \geq 15$). However, it follows from $3t + k + n = n\lfloor \frac{n-1}{2} \rfloor$, $k \leq n-1$ and $n \geq 15$ that $t \geq \frac{3n-14}{2}$. $\square$

5.4 Many 4-cycles and one Hamilton cycle

In Lemma 37 we construct the required decompositions of complete graphs of odd order and in Lemma 38 we construct the required decompositions of complete graphs of even order. In each case we construct the required decomposition of K_n from a suitable decomposition of K_{n-2}.

Lemma 37 *If n is odd, Theorem 1 holds for K_{n-2}, and $(M, 4^{\frac{n-3}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n-3}{2}}, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 15$. Let U be a vertex set with $|U| = n-2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n-3}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n-3$. Thus, it is easily seen that $(M, n-3)$ is $(n-2)$-admissible and so by assumption there is an $(M, n-3)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n-3)$-cycle in $\mathcal{D}$, let z be the vertex in $U \setminus V(H)$, and let x and y be adjacent vertices in H. Then

$$(\mathcal{D} \setminus \{H\}) \cup \{H', (\infty_1, y, x, \infty_2)\} \cup \mathcal{D}_1$$

is an $(M, 4^{\frac{n-3}{2}}, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty_1, z, \infty_2, y]$; and
- $\mathcal{D}_1$ is a $(4^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U\setminus\{x,y,z\}}$. □

Lemma 38 *If n is even, Theorem 1 holds for K_{n-2}, and $(M, 4^{\frac{n-2}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 4^{\frac{n-2}{2}}, n)$-decomposition of K_n.*

Proof By Lemma 2 we can assume that $n \geq 16$. Let U be a vertex set with $|U| = n - 2$, let ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1, \infty_2\}$. Since $(M, 4^{\frac{n-2}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (5) in the definition of ancestor lists that any cycle length in M is at most $n - 3$. Thus, it is easily seen that $(M, 3, n - 3)$ is $(n - 2)$-admissible and so by assumption there is an $(M, 3, n - 3)$-decomposition $\mathcal{D}$ of K_U.

Let H be an $(n - 3)$-cycle in $\mathcal{D}$, let C be a 3-cycle in $\mathcal{D}$, and let I be the perfect matching in $\mathcal{D}$. Let z be the vertex in $U \setminus V(H)$, let w and x be distinct vertices in $V(C) \cap V(H)$, let u be the vertex in $V(C) \setminus \{w, x\}$ (possibly $u = z$), and let y be a vertex adjacent to x in H. Then

$$(\mathcal{D} \setminus \{I, C, H\}) \cup \{I + \infty_1\infty_2, H', (\infty_1, y, x, w), (\infty_2, x, u, w)\} \cup \mathcal{D}_1$$

is an $(M, 4^{\frac{n-2}{2}}, n)$-decomposition of K_V, where

- $H' = (H - [x, y]) \cup [x, \infty_1, z, \infty_2, y]$; and
- $\mathcal{D}_1$ is a $(4^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty_1,\infty_2\},U\setminus\{w,x,y,z\}}$. □

5.5 Many 5-cycles and one Hamilton cycle

In Lemma 43 we construct the required decompositions of complete graphs of odd order and in Lemma 44 we construct the required decompositions of complete graphs of even order. We first require four preliminary lemmas.

Lemma 39 *Every even graph has a decomposition into cycles such that any two cycles in the decomposition share at most two vertices.*

Proof It is well known that every even graph has a decomposition into cycles. Let G be an even graph. Amongst all decompositions of G into cycles, let $\mathcal{D}$ be one with a maximum number of cycles. We claim that any pair of cycles in $\mathcal{D}$ shares at most two vertices. Suppose otherwise. That is, there are distinct cycles A and B in $\mathcal{D}$ and distinct vertices x, y and z of G such that $\{x, y, z\} \subseteq V(A) \cap V(B)$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, where A_1, A_2, B_1 and B_2 are paths from x to y such that $z \in V(A_2)$ and $z \in V(B_2)$. Then it is easy to see that $A_1 \cup B_1$ and $A_2 \cup B_2$ are both nonempty even graphs. For $i = 1, 2$, let $\mathcal{D}_i$ be a decomposition of $A_i \cup B_i$ into cycles, and note that $|\mathcal{D}_2| \geq 2$ because $\deg_{A_2 \cup B_2}(z) = 4$. Then $(\mathcal{D} \setminus \{A, B\}) \cup \mathcal{D}_1 \cup \mathcal{D}_2$ is a decomposition of G into cycles which contains more cycles than $\mathcal{D}$, contradicting our definition of $\mathcal{D}$. □

Lemma 40 *Let V be a vertex set and let U be a subset of V such that $|U| \geq 10$. Let M be a list of integers, let $m \in \{3,4,5\}$, and let $\mathcal{P}$ be an (M)-packing of K_V with a leave L such that $\deg_L(x) = 0$ for all $x \in V \setminus U$. If there exists a spanning even subgraph G of L such that at least one of the following holds,*

(i) $\Delta(G) = 4$, *exactly one vertex of G has degree 4, G has at most two nontrivial components, $|E(G)| \geq m+3$, and G does not have a decomposition into two odd cycles if $m = 4$;*

(ii) $\Delta(G) = 4$, *exactly two vertices of G have degree 4, G has exactly one nontrivial component, $|E(G)| \geq m+3$, and $\deg_G(x) = 0$ for some $x \in U$ if $m = 3$;*

(iii) $m = 4$, *G has exactly one nontrivial component, G has a decomposition into three cycles each pair of which intersect in exactly one vertex, and $\deg_G(x) = 0$ for some $x \in U$; or*

(iv) $m = 5$, $\Delta(G) \geq 4$, *and G has a decomposition into three cycles such that any two intersect in at most two vertices, and such that any two which intersect have lengths adding to 6 or 7;*

then there exists (M,m)-packing $\mathcal{P}'$ of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.

Proof Suppose that there is a spanning even subgraph G of L which satisfies one of (i), (ii), (iii) or (iv). Because L and G are both even graphs, it follows that $L - G$ is an even graph and hence has a decomposition $\mathcal{A} = \{A_1, \ldots, A_t\}$ into cycles. Let $a_i = |V(A_i)|$ for $i = 1, \ldots, t$ and let $M^\dagger = (a_1, \ldots, a_t)$. So $\mathcal{P} \cup \mathcal{A}$ is an $(M, M^\dagger)$-packing of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$. The leave of $\mathcal{P} \cup \mathcal{A}$ is G. Let $e = |E(G)|$.

If we can produce an $(M, M^\dagger, m, e-m)$-decomposition $\mathcal{D}$ of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$, then there will be cycles in $\mathcal{D}$ with lengths $a_1, \ldots, a_t, e-m$ whose vertex sets are subsets of U, and we can complete the proof by removing these cycles from $\mathcal{D}$. So it suffices to find such a decomposition. The proof now splits into cases.

Case 1 Suppose that G satisfies (i). Then we can apply Lemma 26 to obtain the required decomposition.

Case 2 Suppose that G satisfies (ii). Then we can apply Lemma 27 to obtain the required decomposition. The only non-trivial thing to check is that there is an $x \in U$ with $\deg_G(x) = 0$ when $m \in \{4,5\}$ and $e - m = 3$. In this case we have $e \in \{7,8\}$ and, because G is even and $|U| \geq 10$, there is indeed an $x \in U$ with $\deg_G(x) = 0$.

Case 3 Suppose that G satisfies (iii). Either exactly one vertex of G has degree 6 and every other vertex of G has degree at most 2, or exactly three vertices of G have degree 4 and every other vertex of G has degree at most 2. In the former case we apply Lemma 25, choosing b to be the vertex of degree 6 in G and a to be a neighbour in G of b. In the latter case we apply Lemma 25, choosing b to be a vertex of degree 4 in G and a to be a vertex in U which has degree 0 in G. In either case we obtain an $(M, M^\dagger)$-packing $\mathcal{P}'$ of K_V, which is equivalent to $\mathcal{P}$ on $V \setminus U$, with a leave G' such that $\Delta(G') = 4$, exactly two vertices of G' have degree 4, G' has exactly one nontrivial component, and $|E(G')| \geq 9$. Thus we can apply Lemma 27 to obtain the required decomposition.

Case 4 Suppose that G satisfies (iv). Since $\Delta(G) \geq 4$ there is at least one pair of intersecting cycles in any cycle decomposition of G. Thus, there exists a decomposition $\{B_1, B_2, B_3\}$ of G into three cycles such that $|V(B_1) \cap V(B_2)| \in \{1,2\}$ and $|E(B_1)| + |E(B_2)| \in \{6,7\}$.

Case 4a Suppose that B_3 is a component of G. If $|V(B_1) \cap V(B_2)| = 1$, then we can apply Lemma 26 to obtain the required decomposition, so we may assume that $|V(B_1) \cap V(B_2)| = 2$. Let $x \in V(B_1) \cap V(B_2)$, let $y \in V(B_3)$, and let z be a neighbour in G of x. Let $\mathcal{P}'$ be the

$(M, M^\dagger)$-packing of K_n obtained from $\mathcal{P}$ by performing the (x, y)-switch S with origin z and let G' be its leave. Then $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V \setminus U$, G' has exactly one nontrivial component, $\Delta(G') = 4$, exactly two vertices of G' have degree 4, and $|E(G')| \geq 9$. Thus we can apply Lemma 27 to obtain the required decomposition.

Case 4b Suppose that B_3 is not a component of G. Then B_3 intersects with B_1 or B_2 and so $|E(B_3)| \in \{3, 4\}$. Thus, $e \in \{9, 10, 11\}$ as we have $|E(B_1)| + |E(B_2)| \in \{6, 7\}$. Let $\mathcal{P}'$ be the $(M, M^\dagger)$-packing of K_n obtained from $\mathcal{P}$ by repeatedly applying Lemma 25, each time choosing b to be a vertex maximum degree in the leave and a to be a vertex in U of degree 0 in the leave, until the leave has maximum degree 4 and has exactly one vertex of degree 4 (a suitable choice for a will exist each time since $e \leq 11$ and $|U| \geq 10$). Let G' be the leave of $\mathcal{P}'$. Then $\mathcal{P}'$ is equivalent to $\mathcal{P}$ on $V \setminus U$, $\Delta(G') = 4$, exactly one vertex of G' has degree 4, $|E(G')| \in \{9, 10, 11\}$, and G' has at most two components (because $|E(G')| \leq 11$). Thus we can apply Lemma 26 to obtain the required decomposition. □

Lemma 41 *Let V be a vertex set and let U be a subset of V such that $|U| \geq 10$. Let $m \in \{3, 4, 5\}$, let M be a list of integers, and let $\mathcal{P}$ be an (M)-packing of K_V with a leave L such that $|E(L)| \geq |U| + m$ and $\deg_L(x) = 0$ for all $x \in V \setminus U$. Then there exists an (M, m)-packing of K_V which is equivalent to $\mathcal{P}$ on $V \setminus U$.*

Proof Since L is an even graph, Lemma 39 guarantees that there is a decomposition $\mathcal{D}$ of L such that any pair of cycles in $\mathcal{D}$ intersect in at most two vertices. Let $e = |E(L)|$. Since $e \geq |U| + m \geq 13$ it follows that $\mathcal{D}$ contains at least three cycles. Also, since $e > |U|$, there is at least one pair of intersecting cycles in $\mathcal{D}$. We now consider separately the cases $m = 3$, $m = 4$ and $m = 5$.

Case 1 Suppose that $m = 3$. We can assume that there are no 3-cycles in $\mathcal{D}$ (otherwise we can simply add one to $\mathcal{P}$ to complete the proof). Let C_1, C_2 and C_3 be distinct cycles in $\mathcal{D}$ such that C_1 and C_2 intersect. If $|V(C_1) \cap V(C_2)| = 1$, then we can apply Lemma 40 (i) (with $E(G) = E(C_1 \cup C_2)$) to complete the proof, so we may assume that $|V(C_1) \cap V(C_2)| = 2$. If $|E(C_1)| + |E(C_2)| \leq |U| + 1$, then there is at least one vertex of U that is not in $V(C_1) \cup V(C_2)$ and we can apply Lemma 40 (ii) (with $E(G) = E(C_1 \cup C_2)$) to complete the proof. Thus, we may assume $|E(C_1)| + |E(C_2)| = |U| + 2$, and it follows from this that $V(C_1) \cup V(C_2) = U$. This means that $V(C_3) \subseteq V(C_1) \cup V(C_2)$. Thus, since we have $V(C_3) \geq 4$, $V(C_1) \cap V(C_3) \leq 2$ and $V(C_2) \cap V(C_3) \leq 2$, it follows that $V(C_3) = 4$, $V(C_1) \cap V(C_3) = 2$ and $V(C_2) \cap V(C_3) = 2$. We can assume without loss of generality that $|E(C_1)| \leq |E(C_2)|$ and hence that $|E(C_2)| \geq 5$ (since $|U| \geq 10$). This means that there is at least one vertex of U that is in neither C_1 nor C_3, and so we can apply Lemma 40 (ii) (with $E(G) = E(C_1 \cup C_3)$) to complete the proof.

Case 2 Suppose that $m = 4$. If two cycles in $\mathcal{D}$ intersect in exactly two vertices, then we can apply Lemma 40 (ii) (with the edges of G being the edges of two such cycles) to complete the proof. So we may assume that any two cycles in $\mathcal{D}$ intersect in at most one vertex. Let $\{C_1, C_2\}$ be a pair of intersecting cycles in $\mathcal{D}$ such that $|E(C_1 \cup C_2)| \leq |E(C_i \cup C_j)|$ for any pair $\{C_i, C_j\}$ of intersecting cycles in $\mathcal{D}$. If there is a cycle in $\mathcal{D}$ which is vertex disjoint from $C_1 \cup C_2$, then we can apply Lemma 40 (i) (with the edges of G being the edges of C_1, C_2 and this cycle) to complete the proof. If there is a cycle in $\mathcal{D}$ which intersects with exactly one of C_1 and C_2, then we can apply Lemma 40 (ii) (with the edges of G being the edges of C_1, C_2 and this cycle) to complete the proof. So we may assume that every cycle in $\mathcal{D} \setminus \{C_1, C_2\}$ intersects (in exactly one vertex) with C_1 and with C_2. Let C_3 be a shortest cycle in $\mathcal{D} \setminus \{C_1, C_2\}$ and note that $|V(C_i)| \leq |V(C_3)|$ for $i = 1, 2$ by our definition of C_1 and C_2. If $V(C_1 \cup C_2 \cup C_3) \neq U$, then we can apply Lemma

40 (iii) (with $E(G) = E(C_1 \cup C_2 \cup C_3)$) to complete the proof. Otherwise $V(C_1 \cup C_2 \cup C_3) = U$ which means that $|V(C_1)| + |V(C_2)| + |V(C_3)| \in \{|U|+2, |U|+3\}$. However, we have $e \geq |U|+4$ and so there is a cycle $C_4 \in \mathcal{D} \setminus \{C_1, C_2, C_3\}$. Thus C_4 is a 3-cycle (as $V(C_1 \cup C_2 \cup C_3) = U$ and C_4 intersects each of C_1, C_2 and C_3 in exactly one vertex). It then follows from the minimality of C_3 and from $|V(C_i)| \leq |V(C_3)|$ for $i = 1, 2$ that C_1, C_2 and C_3 are also 3-cycles. Since $|U| \geq 10$, this is a contradiction and the result is proved.

Case 3 Suppose that $m = 5$. Let C_1, C_2 and C_3 be three cycles in $\mathcal{D}$ such that C_1 and C_2 intersect. If there are a pair of cycles in $\{C_1, C_2, C_3\}$ which intersect and whose lengths add to at least 8, then the union of this pair of cycles has at least $m+3$ edges and we can apply Lemma 40 (i) or Lemma 40 (ii) (with the edges of G being the edges of this pair of cycles) to complete the proof. Otherwise we can apply Lemma 40 (iv) to complete the proof. □

Lemma 42 *Let u and k be integers such that u is even, $u \geq 16$ and $6 \leq k \leq u-1$, let U be a vertex set such that $|U| = u$, and let x and y be distinct vertices in U. Then there exists a packing of K_U with a perfect matching, a u-cycle, a $(u-1)$-path from x to y, a k-cycle, three $(u-2)$-cycles each having vertex set $U \setminus \{x, y\}$, and a 2-path from x to y.*

Proof Let $U = \mathbb{Z}_{u-3} \cup \{\infty, x, y\}$. For $i = 0, \dots, 5$, let

$$H_i = (\infty, i, i+1, i+(u-4), i+2, i+(u-5), \dots, i+\tfrac{u-6}{2}, i+\tfrac{u}{2}, i+\tfrac{u-4}{2}, i+\tfrac{u-2}{2}),$$

and let

$$I = [\tfrac{u-6}{2}, \tfrac{u-2}{2}] \cup [\tfrac{u-8}{2}, \tfrac{u}{2}] \cup \cdots \cup [0, u-4] \cup [\infty, \tfrac{u-4}{2}],$$

so that $\{I, H_0, \dots, H_5\}$ is a packing of $K_{\mathbb{Z}_{u-3} \cup \{\infty\}}$ with one perfect matching and six $(u-2)$-cycles (recall that $u \geq 16$). Then

$$\{I + xy, (H_0 - [\infty, 0, 1]) \cup [\infty, x, 0, y, 1], (H_1 - [1, 2]) \cup [1, x] \cup [2, y], P \cup [a, x, b], H_3, H_4, H_5, [x, c, y]\}$$

is the required packing, where P is a $(k-2)$-path in H_2 with endpoints a and b such that $a, b \in \mathbb{Z}_{u-3} \setminus \{0, 1\}$ (P exists as there are $u-2$ distinct paths of length $k-2$ in H_2, and at most six having ∞, 0 or 1 as an endpoint), and c is any vertex in $\mathbb{Z}_{u-3} \setminus \{0, 1, 2, a, b\}$. □

Lemma 43 *If n is odd and $(M, 5^{\frac{3n-11}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{\frac{3n-11}{2}}, n)$-decomposition of K_n.*

Proof By Lemma 2 (for $n \leq 13$) and Lemma 28 (for $n \in \{15, 17\}$) we may assume that $n \geq 19$ (Lemma 28 can indeed be applied as $\nu_5(M, 5^{\frac{3n-11}{2}}, n) \geq 3$ when $n \in \{15, 17\}$). Let U be a vertex set with $|U| = n-3$, let x and y be distinct vertices in U, let $\infty^\dagger$, ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty^\dagger, \infty_1, \infty_2\}$.

Since $(M, 5^{\frac{3n-11}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (6) in the definition of ancestor lists that any cycle length in M is at most $n-5$. If there is a cycle length in M which is at least 6, then let k be this cycle length. Otherwise let $k = 0$ (so $k \in \{0\} \cup \{6, \dots, n-5\}$). By Lemma 42, there exists a packing $\mathcal{P}$ of K_U with a perfect matching I, an $(n-4)$-path P_1 from x to y, a k-cycle (if $k \neq 0$), three $(n-5)$-cycles C_1, C_2 and C_3 each having vertex set $U \setminus \{x, y\}$, and a 2-path P_2 from x to y. Let $\{I_1, I_2\}$ be a decomposition of C_3 into two matchings.

Let

$$\mathcal{P}' = (\mathcal{P} \setminus \{I, P_1, P_2, C_1, C_2, C_3\}) \cup \{P_1 \cup [x, \infty_1, \infty^\dagger, \infty_2, y], P_2 \cup [x, \infty_2, \infty_1, y]\} \cup \mathcal{D} \cup \mathcal{D}_1 \cup \mathcal{D}_2,$$

where

- $\mathcal{D}$ is a $(3^{\frac{n-3}{2}})$-decomposition of $K_{\{\infty^\dagger\}} \vee I$;

- for $i = 1, 2$, $\mathcal{D}_i$ is a $(5^{\frac{n-5}{2}})$-decomposition of $K_{\{\infty_i\}} \vee (C_i \cup I_i)$ (this exists by Lemma 20).

Then $\mathcal{P}'$ is a $(3^{\frac{n-3}{2}}, 5^{n-4}, k, n)$-packing of K_V (a $(3^{\frac{n-3}{2}}, 5^{n-4}, n)$-packing of K_V if $k = 0$) such that

(i) $\frac{n-3}{2}$ 3-cycles in $\mathcal{P}'$ contain the vertex $\infty^\dagger$;

(ii) $\infty^\dagger\infty_1$ and $\infty^\dagger\infty_2$ are edges of the n-cycle in $\mathcal{P}'$; and

(iii) $\infty^\dagger$, ∞_1 and ∞_2 all have degree 0 in the leave of $\mathcal{P}'$.

Since $(M, 5^{\frac{3n-11}{2}}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it can be seen that by beginning with $\mathcal{P}'$ and repeatedly applying Lemma 41 we can obtain an $(M, 3^{\frac{n-3}{2}}, 5^{n-4}, n)$-packing of K_V which is equivalent to $\mathcal{P}'$ on $V \setminus U$. Note that the leave of this packing has $n-3$ edges. Thus, by then repeatedly applying Lemma 25 we can obtain an $(M, 3^{\frac{n-3}{2}}, 5^{n-4}, n)$-packing $\mathcal{P}''$ of K_V which is equivalent to $\mathcal{P}'$ on $V \setminus U$ and whose leave L'' has the property that $\deg_{L''}(x) = 0$ for each $x \in \{\infty^\dagger, \infty_1, \infty_2\}$ and $\deg_{L''}(x) = 2$ for each $x \in U$. Because $\mathcal{P}''$ is equivalent to $\mathcal{P}'$ on $V \setminus U$, it follows from (i) and (ii) that there is a set $\mathcal{T}$ of $\frac{n-3}{2}$ 3-cycles in $\mathcal{P}''$ each of which contains the vertex $\infty^\dagger$ and two vertices in U. Let T be the union of the 3-cycles in $\mathcal{T}$. Then

$$(\mathcal{P}'' \setminus \mathcal{T}) \cup \mathcal{D}''$$

is an $(M, 5^{\frac{3n-11}{2}}, n)$-decomposition of K_V where $\mathcal{D}''$ is a $(5^{\frac{n-3}{2}})$-decomposition of $T \cup L''$ (this exists by Lemma 20, noting that $E(T \cup L'') = E(K_{\{\infty^\dagger\}} \vee G)$ for some 3-regular graph G on vertex set U which contains a perfect matching). $\square$

Lemma 44 *If n is even and $(M, 5^{2n-9}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, then there is an $(M, 5^{2n-9}, n)$-decomposition of K_n.*

Proof By Lemma 2 (for $n \leq 14$) and Lemma 28 (for $n \in \{16, 18\}$) we may assume that $n \geq 20$ (Lemma 28 can indeed be applied as $\nu_5(M, 5^{2n-9}, n) \geq 3$ when $n \in \{16, 18\}$). Let U be a vertex set with $|U| = n - 4$, let x and y be distinct vertices in U, let $\infty_1^\dagger$, $\infty_2^\dagger$, ∞_1 and ∞_2 be distinct vertices not in U, and let $V = U \cup \{\infty_1^\dagger, \infty_2^\dagger, \infty_1, \infty_2\}$.

Since $(M, 5^{2n-9}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, it follows from (6) in the definition of ancestor lists that any cycle length in M is at most $n-5$. If there is a cycle length in M which is at least 6 then let k be this cycle length. Otherwise let $k = 0$ (so $k \in \{0\} \cup \{6, \ldots, n-5\}$). By Lemma 42, there exists a packing $\mathcal{P}$ of K_U with a perfect matching I, an $(n-4)$-cycle B, an $(n-5)$-path P_1 from x to y, a k-cycle (if $k \neq 0$), and three $(n-6)$-cycles C_1, C_2 and C_3 each having vertex set $U \setminus \{x, y\}$, and a 2-path P_2 from x to y. Let $\{I_1^\dagger, I_2^\dagger\}$ be a decomposition of B into two matchings and $\{I_1, I_2\}$ be a decomposition of C_3 into two matchings.

Let

$$\begin{aligned}\mathcal{P}' = \ & (\mathcal{P} \setminus \{B, P_1, P_2, C_1, C_2, C_3\}) \cup \\ & \{I + \{\infty_1\infty_2^\dagger, \infty_2\infty_1^\dagger\}, P_1 \cup [x, \infty_1, \infty_1^\dagger, \infty_2^\dagger, \infty_2, y], P_2 \cup [x, \infty_2, \infty_1, y]\} \cup \mathcal{D}_1^\dagger \cup \mathcal{D}_2^\dagger \cup \mathcal{D}_1 \cup \mathcal{D}_2,\end{aligned}$$

where

- for $i = 1, 2$, $\mathcal{D}_i^\dagger$ is a $(3^{\frac{n-4}{2}})$-decomposition of $K_{\{\infty_i^\dagger\}} \vee I_i^\dagger$;

- for $i = 1,2$, $\mathcal{D}_i$ is a $(5^{\frac{n-6}{2}})$-decomposition of $K_{\{\infty_i\}} \vee (C_i \cup I_i)$ (this exists by Lemma 20).

Then $\mathcal{P}'$ is a $(3^{n-4}, 5^{n-5}, k, n)$-packing of K_V (a $(3^{n-4}, 5^{n-5}, n)$-packing of K_V if $k = 0$) such that

(i) for $i = 1,2$, $\frac{n-4}{2}$ 3-cycles in $\mathcal{P}'$ contain the vertex $\infty_i^\dagger$;

(ii) $\infty_1^\dagger\infty_1$, $\infty_1^\dagger\infty_2^\dagger$ and $\infty_2^\dagger\infty_2$ are edges of the n-cycle in $\mathcal{P}'$;

(iii) $\infty_1^\dagger\infty_2$ and $\infty_2^\dagger\infty_1$ are edges of the perfect matching in $\mathcal{P}'$; and

(iv) $\infty_1^\dagger$, $\infty_2^\dagger$, ∞_1 and ∞_2 all have degree 0 in the leave of $\mathcal{P}'$.

Since $(M, 5^{2n-9}, n)$ is an n-ancestor list with $\nu_n(M) = 0$, by beginning with $\mathcal{P}'$ and repeatedly applying Lemma 41 we can obtain an $(M, 3^{n-4}, 5^{n-5}, n)$-packing of K_V, which is equivalent to $\mathcal{P}'$ on $V \setminus U$. Note that the leave of this packing has $2n-8$ edges. Thus, by then repeatedly applying Lemma 25 we can obtain an $(M, 3^{n-4}, 5^{n-5}, n)$-packing $\mathcal{P}''$ of K_V which is equivalent to $\mathcal{P}'$ on $V \setminus U$ and whose leave L'' has the property that $\deg_{L''}(x) = 0$ for $x \in \{\infty_1^\dagger, \infty_2^\dagger, \infty_1, \infty_2\}$ and $\deg_{L''}(x) = 4$ for all $x \in U$. By Petersen's Theorem [41], L'' has a decomposition $\{H_1, H_2\}$ into two 2-regular graphs, each with vertex set U. Because $\mathcal{P}''$ is equivalent to $\mathcal{P}'$ on $V \setminus U$, it follows from (i), (ii) and (iii) that, for $i = 1,2$ there is a set $\mathcal{T}_i$ of $\frac{n-4}{2}$ 3-cycles in $\mathcal{P}''$ each of which contains the vertex $\infty_i^\dagger$ and two vertices in U. For $i = 1,2$, let T_i be the union of the 3-cycles in $\mathcal{T}_i$. Then

$$(\mathcal{P}'' \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)) \cup \mathcal{D}_1'' \cup \mathcal{D}_2''$$

is an $(M, 5^{2n-9}, n)$-decomposition of K_V where, for $i = 1,2$, $\mathcal{D}_i''$ is a $(5^{\frac{n-4}{2}})$-decomposition of $T_i \cup H_i$ (these decompositions exist by Lemma 20, noting that for $i = 1,2$, $E(T_i \cup H_i) = E(K_{\{\infty_i^\dagger\}} \vee G)$ for some 3-regular graph G with vertex set U that contains a perfect matching). □

5.6 Proof of Lemma 5 in the case of one Hamilton cycle

Lemma 45 *If Theorem 1 holds for K_{n-1}, K_{n-2} and K_{n-3}, then there is an (M)-decomposition of K_n for each n-ancestor list M satisfying $\nu_n(M) = 1$.*

Proof By Lemma 2 we can assume that $n \geq 15$. If there is a cycle length in M which is at least 6 and at most $n-1$ then let k be this cycle length. Otherwise let $k = 0$. We deal separately with the case n is odd and the case n is even.

Case 1 Suppose that n is odd. Since $n \geq 15$ and $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k + n = \frac{n(n-1)}{2}$, it can be seen that either

(i) $n \in \{15, 17, 19\}$ and $\nu_5(M) \geq 3$;

(ii) $\nu_3(M) \geq \frac{n-5}{2}$;

(iii) $\nu_4(M) \geq \frac{n-3}{2}$; or

(iv) $\nu_5(M) \geq \frac{3n-11}{2}$.

(To see this consider the cases $\nu_5(M) \geq 3$ and $\nu_5(M) \leq 2$ separately and use the definition of n-ancestor list.) If (i) holds, then the result follows by Lemma 28. If (ii) holds, then the result follows by one of Lemmas 29, 30 or 36. If (iii) holds, then the result follows by Lemma 37. If (iv) holds, then the result follows by Lemma 43.

Case 2 Suppose that n is even. Since $n \geq 16$ and $3\nu_3(M) + 4\nu_4(M) + 5\nu_5(M) + k + n = \frac{n(n-2)}{2}$, it can be seen that either

(i) $n \in \{16, 18, 20, 22, 24, 26\}$, $\nu_5(M) \geq 3$, and $\nu_4(M) \geq 2$ if $n = 24$;

(ii) $\nu_3(M) \geq \frac{3n-14}{2}$;

(iii) $\nu_4(M) \geq \frac{n-2}{2}$; or

(iv) $\nu_5(M) \geq 2n - 9$.

(To see this consider the cases $\nu_5(M) \geq 3$ and $\nu_5(M) \leq 2$ separately and use the definition of n-ancestor list.) If (i) holds, then the result follows by Lemma 28. If (ii) holds, then the result follows by one of Lemmas 31, 32 or 36 (note that $\frac{3n-14}{2} \geq \frac{n-6}{2}$ for $n \geq 16$). If (iii) holds, then the result follows by Lemma 38. If (iv) holds, then the result follows by Lemma 44. □

6 Decompositions of $\langle S \rangle_n$

Our general approach to constructing decompositions of $\langle S \rangle_n$ follows the approach used in [27] and [29]. For each connection set S in which we are interested, we define a graph J_n for each positive integer n such that there is a natural bijection between $E(J_n)$ and $E(\langle S \rangle_n)$, and such that $\langle S \rangle_n$ can be obtained from J_n by identifying a small number (approximately $|S|$) of pairs of vertices. Thus, decompositions of J_n yield decompositions of $\langle S \rangle_n$.

The key property of the graph J_n is that it can be decomposed into a copy of J_{n-y} and a copy of J_y for any positive integer y such that $1 \leq y < n$, and this facilitates the construction of desired decompositions of J_n for arbitrarily large n from decompositions of J_i for various small values of i. For example, in the case $S = \{1, 2, 3\}$ we define J_n by $V(J_n) = \{0, \ldots, n+2\}$ and $E(J_n) = \{\{i, i+1\}, \{i+1, i+3\}, \{i, i+3\}\} : i = 0, \ldots n-1\}$. It is straightforward to construct a (3)-decomposition of J_1, a (4, 5)-decomposition of J_3, a (4^3)-decomposition of J_4 and a (5^3)-decomposition of J_5. Moreover, since J_n decomposes into J_{n-y} and J_y, it is easy to see that these decompositions can be combined to produce an (M)-decomposition of J_n for any list $M = (m_1, \ldots, m_t)$ satisfying $\sum M = 3n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$. For all $n \geq 7$, an (M)-decomposition of $\langle \{1, 2, 3\} \rangle_n$ can be obtained from an (M)-decomposition of J_n by identifying vertex i with vertex $i + n$ for $i = 0, 1, 2$.

In what follows, this general approach is modified to allow for the construction of decompositions which, in addition to cycles of lengths 3, 4 and 5, contain one arbitrarily long cycle or, in the case $S = \{1, 2, 3\}$, one arbitrarily long cycle and one Hamilton cycle. The constructions used to prove Lemma 7 proceed in a similar fashion for each connection set S. We include the proof only for the most interesting and difficult case, namely the case $S = \{1, 2, 3, 4, 6\}$ where there are some exceptional decompositions which appear not to exist. Many decompositions of various J_n with n small are required as ingredients in our constructions. All of these ingredients have been constructed, in most cases with the aid of a computer, but due to the large number of them, they are not presented here.

6.1 Proof of Lemma 7

In this section we prove Lemma 7, which we restate here for convenience.

Lemma 7 *If*

$$S \in \{\{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 6\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\},$$

$n \geq 2\max(S)+1$, *and* $M = (m_1, \ldots, m_t, k)$ *is any list satisfying* $m_i \in \{3,4,5\}$ *for* $i = 1, \ldots, t$, $3 \leq k \leq n$, *and* $\sum M = |S|n$, *then there is an* (M)*-decomposition of* $\langle S \rangle_n$, *except possibly when*

- $S = \{1,2,3,4,6\}$, $n \equiv 3 \pmod 6$ *and* $M = (3^{\frac{5n}{3}})$*; or*
- $S = \{1,2,3,4,6\}$, $n \equiv 4 \pmod 6$ *and* $M = (3^{\frac{5n-5}{3}}, 5)$.

Let

$$\mathcal{S} = \{\{1,2,3\}, \{1,2,3,4\}, \{1,2,3,4,6\}, \{1,2,3,4,5,7\}, \{1,2,3,4,5,6,7\}, \{1,2,3,4,5,6,7,8\}\}.$$

For $n \geq 1$ and for each $S \in \mathcal{S}$, we define the graph J_n^S according to the following table. We shall give the proof of Lemma 7 for $S = \{1,2,3,4,6\}$ only. Thus, we shall be working with the graph $J_n^{\{1,2,3,4,6\}}$, which in this subsection we denote by just J_n. For each other $S \in \mathcal{S}$, we use the graph J_n^S given in the table, and then the proof proceeds along similar lines.

S	$V(J_n^S)$	$E(J_n^S)$
$\{1,2,3\}$	$\{0,\ldots,n+2\}$	$\{\{i,i+1\},\{i+1,i+3\},$ $\{i,i+3\} : i = 0,\ldots,n-1\}$
$\{1,2,3,4\}$	$\{0,\ldots,n+3\}$	$\{\{i+2,i+3\},\{i+2,i+4\},\{i,i+3\},$ $\{i,i+4\} : i = 0,\ldots,n-1\}$
$\{1,2,3,4,6\}$	$\{0,\ldots,n+5\}$	$\{\{i+2,i+3\},\{i+2,i+4\},\{i+3,i+6\},\{i,i+4\},$ $\{i,i+6\} : i = 0,\ldots,n-1\}$
$\{1,2,3,4,5,7\}$	$\{0,\ldots,n+6\}$	$\{\{i+6,i+7\},\{i+5,i+7\},\{i+3,i+6\},\{i+3,i+7\},$ $\{i,i+5\},\{i,i+7\} : i = 0,\ldots,n-1\}$
$\{1,2,3,4,5,6,7\}$	$\{0,\ldots,n+6\}$	$\{\{i+3,i+4\},\{i+5,i+7\},\{i+3,i+6\},\{i,i+4\},$ $\{i,i+5\},\{i,i+6\},\{i,i+7\} : i = 0,\ldots,n-1\}$
$\{1,2,3,4,5,6,7,8\}$	$\{0,\ldots,n+8\}$	$\{\{i+7,i+8\},\{i+5,i+7\},\{i+5,i+8\},\{i+5,i+9\},$ $\{i,i+5\},\{i+1,i+7\},\{i,i+7\},$ $\{i+1,i+9\} : i = 0,\ldots,n-1\}$

For a list of integers M, an (M)-decomposition of J_n will be denoted by $J_n \to M$. We note the following basic properties of J_n.

- For $n \geq 13$, if for each $i \in \{0,1,2,3,4,5\}$ we identify vertex i of J_n with vertex $i+n$ of J_n then the resulting graph is $\langle\{1,2,3,4,6\}\rangle_n$. This means that for $n \geq 13$, we can obtain an (M)-decomposition of $\langle\{1,2,3,4,6\}\rangle_n$ from a decomposition $J_n \to M$, provided that for each $i \in \{0,1,2,3,4,5\}$, no cycle in the decomposition of J_n contains both vertex i and vertex $i+n$.

- For any integers y and n such that $1 \leq y < n$, the graph J_n is the union of J_{n-y} and the graph obtained from J_y by applying the permutation $x \mapsto x + (n-y)$. Thus, if there is a decomposition $J_{n-y} \to M$ and a decomposition $J_y \to M'$, then there is a decomposition $J_n \to M, M'$. We will call this construction, and the similar constructions that follow, *concatenations*.

We prove Lemma 7 by constructing various decompositions of graphs J_n and related graphs, then concatenating these decompositions to produce a much wider range of decompositions of graphs J_n, and finally identifying pairs of vertices to produce the decompositions of $\langle S \rangle_n$ that we require.

Lemma 46 *If n is a positive integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 5n$, $m_i \in \{3,4,5\}$ for $i = 1, \ldots, t$, and $M \notin \mathcal{E}$ where*

$$\mathcal{E} = \{(3^2, 4)\} \cup \{(3^{5i}) : i \geq 1 \text{ is odd}\} \cup \{(3^{5i}, 5) : i \geq 1 \text{ is odd}\},$$

then there is a decomposition $J_n \to M$.

Proof We have verified by computer search and concatenation that the result holds for $n \leq 10$. So assume $n \geq 11$ and suppose by induction that the result holds for each integer n' in the range $1 \leq n' < n$. It is routine to check that if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $J_y \to Y$ is one of the six decompositions below (which exist because $y \leq 10$).

$$J_1 \to 5 \qquad J_4 \to 4^5 \qquad J_6 \to 3^{10} \qquad J_5 \to 3^7, 4 \qquad J_4 \to 3^4, 4^2 \qquad J_3 \to 3, 4^3$$

If $X \notin \mathcal{E}$, then we can obtain a decomposition $J_n \to M$ by concatenating a decomposition $J_{n-y} \to X$ (which exists by our inductive hypothesis) with a decomposition $J_y \to Y$. Thus, we can assume $X \in \mathcal{E}$. But since $n \geq 11$ and $\sum Y \leq 30$, we have $\sum X \geq 25$ which implies $X \in \{(3^{5i}) : i \geq 3 \text{ is odd}\} \cup \{(3^{5i}, 5) : i \geq 3 \text{ is odd}\}$. It follows that $M = (3^{10}, X')$ for some nonempty list $X' \notin \mathcal{E}$ (because $M \notin \mathcal{E}$) and we can obtain a decomposition $J_n \to M$ by concatenating a decomposition $J_{n-6} \to X'$ (which exists by our inductive hypothesis) with a decomposition $J_6 \to 3^{10}$. □

Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, C\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- C is a k-cycle such that $V(C) = \{n-k+6, \ldots, n+5\}$ and $\{\{n, n+3\}, \{n+1, n+4\}, \{n+2, n+5\}\} \subseteq E(C)$;

will be denoted $J_n \to M, k^*$.

In Lemma 47 we will form new decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs J_y^+ which we will now define. For $y \in \{4, \ldots, 11\}$, the graph obtained from J_y by adding the three edges $\{0,3\}$, $\{1,4\}$ and $\{2,5\}$ will be denoted J_y^+. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A_1, A_2, A_3\}$ of J_y^+ such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A_1, A_2 and A_3 are vertex-disjoint paths, one from 0 to 3, one from 1 to 4, and one from 2 to 5; and
- $|E(A_1)| + |E(A_2)| + |E(A_3)| = l + 3$;

will be denoted $J_y^+ \to M, l^+$. Moreover, if $l = y$ and $\{\{n, n+3\}, \{n+1, n+4\}, \{n+2, n+5\}\} \subseteq E(A_1) \cup E(A_2) \cup E(A_3)$, then the decomposition will be denoted $J_y^+ \to M, y^{+*}$.

For $y \in \{4, \ldots, 11\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by deleting the edges in $\{\{n-y, n-y+3\}, \{n-y+1, n-y+4\}, \{n-y+2, n-y+5\}\}$ and the graph obtained from J_y^+ applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', l^+$, then there is a decomposition $J_n \to M, M', k+l$. The three edges $\{n, n+3\}$, $\{n+1, n+4\}$ and $\{n+2, n+5\}$ of the k-cycle in the decomposition of J_{n-y} are replaced by the three paths in the decomposition of J_y^+ to form the $(k+l)$-cycle in the new decomposition. Similarly, if there is a decomposition $J_{n-y} \to M, k^*$ and a decomposition $J_y^+ \to M', y^{+*}$, then there is a decomposition $J_n \to M, M', (k+y)^*$.

Lemma 47 *If $n \geq 11$ is an integer and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 4n$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, n^*$.*

Proof We have verified using computer search and concatenation that the result holds for $11 \leq n \leq 16$. So let $n \geq 17$ and suppose by induction that the result holds for each integer n' in the range $11 \leq n' < n$. We have verified that the following decompositions exist.

$$J_4^+ \to 4^4, 4^{+*} \qquad J_5^+ \to 5^4, 5^{+*} \qquad J_6^+ \to 3^8, 6^{+*}$$

It is routine to check, using $\sum M = 4n \geq 68$, that M can be written as $M = (X, Y)$ where $J_y^+ \to Y, y^{+*}$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, n^*$ by concatenating a decomposition $J_{n-y} \to X, (n-y)^*$ (which exists by our inductive hypothesis, since $n - y \geq n - 6 \geq 11$) with a decomposition $J_y^+ \to Y, y^{+*}$. $\square$

Lemma 48 *If n and k are integers such that $6 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 5n - k$ and $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k$. Furthermore, for $n \geq 13$ any cycle in this decomposition having a vertex in $\{0, \ldots, 5\}$ has no vertex in $\{n, \ldots, n+5\}$.*

Proof We have verified using computer search and concatenation that the result holds for $6 \leq n \leq 21$. So let $n \geq 22$ and suppose by induction that the result holds for each positive integer n' in the range $6 \leq n' < n$. By Lemma 47 we can assume that $k \leq n-1$. The following decompositions exist by Lemma 46.

$$J_1 \to 5^1 \qquad J_3 \to 3^1 4^3 \qquad J_4 \to 4^5 \qquad J_6 \to 3^{10}$$

Case 1 Suppose that $k \leq n-6$. Then it is routine to check, using $\sum M = 5n-k \geq 4n+6 \geq 94$, that $M = (X, Y)$ where $J_y \to Y$ is one of the decompositions above and X is some nonempty list. We can obtain a decomposition $J_n \to M, k$ by concatenating a decomposition $J_{n-y} \to X, k$ (which exists by our inductive hypothesis, since $k \leq n - 6 \leq n - y$) with a decomposition $J_y \to Y$. Since $n \geq 22$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, \ldots, 5\}$ has no vertex in $\{n, \ldots, n+5\}$, and by our inductive hypothesis the same holds for the k-cycle.
Case 2 Suppose that $n-5 \leq k \leq n-1$. In a similar manner to Case 1, we can obtain the required decomposition $J_n \to M, k$ if $M = (X, 5)$ for some list X, if $k \in \{n-5, n-4, n-3\}$ and $M = (X, 3, 4^3)$ for some list X, and if $k \in \{n-5, n-4\}$ and $M = (X, 4^5)$ for some list Y. So we may assume that none of these hold.

Given this, using $\sum M = 5n - k \geq 4n + 1 \geq 89$, it is routine to check that the required decomposition $J_n \to M, k$ can be obtained using one of the concatenations given in the table below (note that, since $\nu_5(M) = 0$, in each case we can deduce the given value of $\nu_3(M) \pmod 4$ from $\sum M = 5n - k$). The decompositions in the third column exist by Lemma 47, and we have verified the existence of the decompositions listed in the last column.

k	$\nu_3(M)$ (mod 4)	first decomposition	second decomposition
$n-5$	3	$J_{n-10} \to (M-(3^{15})), (n-10)^*$	$J_{10}^+ \to 3^{15}, 5^+$
$n-4$	0	$J_{n-11} \to (M-(3^{16})), (n-11)^*$	$J_{11}^+ \to 3^{16}, 7^+$
$n-3$	1	$J_{n-9} \to (M-(3^{13})), (n-9)^*$	$J_9^+ \to 3^{13}, 6^+$
$n-2$	2	$J_{n-7} \to (M-(3^{10})), (n-7)^*$ $J_{n-5} \to (M-(3^2, 4^4)), (n-5)^*$	$J_7^+ \to 3^{10}, 5^+$ $J_5^+ \to 3^2, 4^4, 3^+$
$n-1$	3	$J_{n-8} \to (M-(3^{11})), (n-8)^*$ $J_{n-4} \to (M-(3^3, 4^2)), (n-4)^*$	$J_8^+ \to 3^{11}, 7^+$ $J_4^+ \to 3^3, 4^2, 3^+$

Since $n \geq 22$ it is clear that any 3-, 4- or 5-cycle in this decomposition having a vertex in $\{0, \ldots, 5\}$ has no vertex in $\{n, \ldots, n+5\}$, and by the definition of the decompositions given in the third column the k-cycle has no vertex in $\{0, \ldots, 5\}$. $\square$

Proof of Lemma 7 We give the proof for $S = \{1, 2, 3, 4, 6\}$ only. For each other connection set S the proof proceeds along similar lines. As noted above, for $n \geq 13$ we can obtain an (M)-decomposition of $\langle\{1, 2, 3, 4, 6\}\rangle_n$ from an (M)-decomposition of J_n, provided that for each $i \in \{0, \ldots, 5\}$, no cycle contains both vertex i and vertex $i + n$. Thus, for $S = \{1, 2, 3, 4, 6\}$, Lemma 7 follows by Lemma 46 for $k \in \{3, 4, 5\}$ (since $n \geq 13$ it can be seen that no cycle in this decomposition contains both vertex i and vertex $i + n$ for $i \in \{0, \ldots, 5\}$) and by Lemma 48 for $6 \leq k \leq n$. $\square$

6.2 Proof of Lemma 9

In this section we prove Lemma 9, which we restate here for convenience.

Lemma 9 *If $n \geq 7$ and $M = (m_1, \ldots, m_t, k, n)$ is any list satisfying $m_i \in \{3, 4, 5\}$ for $i = 1, \ldots, t$, $3 \leq k \leq n$, and $\sum M = 3n$, then there is an (M)-decomposition of $\langle\{1, 2, 3\}\rangle_n$.*

The proof of Lemma 9 proceeds along similar lines to the proof of Lemma 7. We make use of the graphs $J_n^{\{1,2,3\}}$ defined in the proof of Lemma 7, which in this subsection we denote by just J_n. Recall that $J_n^{\{1,2,3\}}$ is the graph with vertex set $\{0, \ldots, n+2\}$ and edge set

$$\{\{i, i+1\}, \{i+1, i+3\}, \{i, i+3\} : i = 0, \ldots, n-1\}.$$

We first construct decompositions of graphs which are related to the graphs J_n, then concatenate these decompositions to produce decompositions of the graphs J_n, and finally identify pairs of vertices to produce the required decompositions of $\langle\{1, 2, 3\}\rangle_n$.

For $n \geq 1$, the graph obtained from J_n by adding the edges $\{n, n+1\}$ and $\{n+1, n+2\}$ will be denoted L_n. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A, B\}$ of L_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A is a path of length k from n to $n+1$; and
- B is a path of length $n-1$ from $n+1$ to $n+2$ such that $0, 1, 2 \notin V(B)$;

will be denoted $L_n \to M, k^+, (n-1)^H$.

In Lemmas 49 and 50 we form new decompositions of graphs L_n by concatenating decompositions of L_{n-y} with decompositions of graphs P_y which we will now define. For $y \in \{3,4,5,6\}$, the graph obtained from J_y by deleting the edges in $\{\{0,1\},\{1,2\}\}$ and adding the edges in $\{\{y,y+1\},\{y+1,y+2\}\}$ will be denoted P_y. Let $M=(m_1,\ldots,m_t)$ be a list of integers with $m_i \geq 3$ for $i=1,\ldots,t$. A decomposition $\{G_1,\ldots,G_t,A_1,A_2,B_1,B_2\}$ of P_y such that

- G_i is an m_i-cycle for $i=1,2,\ldots,t$;
- A_1 and A_2 are vertex-disjoint paths with endpoints 0, 1, y and $y+1$, such that A_1 has endpoints 0 and y or 0 and $y+1$;
- $|E(A_1)|+|E(A_2)|=k'$ and $2 \notin V(A_1) \cup V(A_2)$;
- B_1 and B_2 are vertex-disjoint paths with endpoints 1, 2, $y+1$ and $y+2$, such that B_1 has endpoints 1 and $y+1$ or 1 and $y+2$; and
- $|E(B_1)|+|E(B_2)|=y$, and $0 \notin V(B_1) \cup V(B_2)$;

will be denoted $P_y \to M, k'^{+}, y^H$.

For $y \in \{3,4,5,6\}$ and $n>y$, the graph L_n is the union of the graph L_{n-y} and the graph obtained from P_y by applying the permutation $x \mapsto x+(n-y)$. It follows that if there is a decomposition $L_{n-y} \to M, k^+, (n-y-1)^H$ and a decomposition $P_y \to M', k'^{+}, y^H$, then there is a decomposition $L_n \to M, M', (k+k')^+, (n-1)^H$.

Lemma 49 *If $n \geq 2$ is an integer and $M=(m_1,\ldots,m_t)$ is a list such that $\sum M = n+1$, $M \notin \{(3^i) : i \text{ is even}\}$ and $m_i \in \{3,4,5\}$ for $i=1,\ldots,t$, then there is a decomposition $L_n \to M, (n+2)^+, (n-1)^H$.*

Proof We have verified that the following decompositions exist, thus verifying the lemma for $n \in \{2,3,4\}$.

$$L_2 \to 3, 4^+, 1^H \qquad L_3 \to 4, 5^+, 2^H \qquad L_4 \to 5, 6^+, 3^H$$
$$P_4 \to 4, 4^+, 4^H \qquad P_5 \to 5, 5^+, 5^H \qquad P_6 \to 3^2, 6^+, 6^H$$

So let $n \geq 5$ and assume by induction that the result holds for each integer n' in the range $2 \leq n' < n$. It is routine to check that for $n \geq 5$, if M satisfies the hypotheses of the lemma, then M can be written as $M=(X,Y)$ where $n-y \geq 2$, $X \notin \{(3^i) : i \text{ is even}\}$, and $P_y \to Y, y^+, y^H$ is one of the decompositions above. We can obtain the required decomposition $L_n \to M, (n+2)^+, (n-1)^H$ by concatenating a decomposition $L_{n-y} \to X, (n-y+2)^+, (n-y-1)^H$ (which exists by our inductive hypothesis) with a decomposition $P_y \to Y, y^+, y^H$. □

Lemma 50 *If n and k are positive integers with $\frac{4n+12}{5} \leq k \leq n+2$, and $M=(m_1,\ldots,m_t)$ is a list such that $\sum M = 2n-k+3$, $M \notin \{(3^i) : i \text{ is even}\}$ and $m_i \in \{3,4,5\}$ for $i=1,\ldots,t$, then there is a decomposition $L_n \to M, k^+, (n-1)^H$.*

Proof The proof will be by induction on $j=n-k+2$. For a given n we need to prove the result for each integer j in the range $0 \leq j \leq \frac{n-2}{5}$. The case $j=0$ is covered in Lemma 49, so assume $1 \leq j \leq \frac{n-2}{5}$ and that the result holds for each integer j' in the range $0 \leq j' < j$. Note that, since $\frac{4n+12}{5} \leq k$ and $j \geq 1$, we have $n \geq 7$. We have verified that the following decompositions exist.

$$P_3 \to 4, 2^+, 3^H \qquad P_4 \to 5, 3^+, 4^H \qquad P_5 \to 3^2, 4^+, 5^H$$

It is routine to check that for $j \leq \frac{n-2}{5}$, if M satisfies the hypotheses of the lemma, then M can be written as $M = (X, Y)$ where $X \notin \{(3^i) : i \text{ is even}\}$ and $P_y \to Y, (y-1)^+, y^H$ is one of the decompositions listed above. A decomposition $L_{n-y} \to X, (k-y+1)^+, (n-y-1)^H$ will exist by our inductive hypothesis provided that

$$\tfrac{4(n-y)+12}{5} \leq k-y+1 \leq n-y+2$$

and it is routine to check that this holds using $\frac{4n+12}{5} \leq k$, $j \geq 1$ and $y \in \{3,4,5\}$. Thus, the required decomposition $L_n \to M, k^+, (n-1)^H$ can be obtained by concatenating a decomposition $L_{n-y} \to X, (k-y+1)^+, (n-y-1)^H$ with a decomposition $P_y \to Y, (y-1)^+, y^H$. □

Let $(m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, H\}$ of J_n such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$; and
- H is an n-cycle such that $0, 1, 2 \notin V(H)$ and $\{n, n+2\} \in E(H)$;

will be denoted $J_n \to m_1, \ldots, m_t, n^H$.

In Lemma 51 we will form decompositions of graphs J_n by concatenating decompositions of graphs L_{n-y} obtained from Lemma 50 with decompositions of graphs Q_y which we will now define. For each $y \in \{4,5,6\}$, the graph obtained from J_y by deleting the edges $\{0,1\}$ and $\{1,2\}$ will be denoted by Q_y. Let $M = (m_1, \ldots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \ldots, t$. A decomposition $\{G_1, \ldots, G_t, A, B\}$ of Q_y such that

- G_i is an m_i-cycle for $i = 1, \ldots, t$;
- A is a path of length k' from 0 to 1 such that $\{2, y, y+1, y+2\} \notin V(A)$; and
- B is a path of length $y+1$ from 1 to 2 such that $0 \notin V(B)$ and $\{y, y+2\} \in E(B)$;

will be denoted $Q_y \to M, k'^+, (y+1)^H$.

For $y \in \{4,5,6\}$ and $n > y$, the graph J_n is the union of the graph L_{n-y} and the graph obtained from Q_y by applying the permutation $x \mapsto x + (n-y)$. It follows that if there is a decomposition $L_{n-y} \to M, k^+, (n-y-1)^H$ and a decomposition $Q_y \to M', k'^+, (y+1)^H$, then there is a decomposition $J_n \to M, M', k+k', n^H$. Note that, for $y \in \{4,5,6\}$ and $n - y \geq 3$, no cycle of this decomposition contains both vertex i and vertex $i+n$ for $i \in \{0,1,2\}$.

Lemma 51 *If n and k are integers with $n \geq 6$, $k \geq 3$ and $n-5 \leq k \leq n$ and $M = (m_1, \ldots, m_t)$ is a list such that $\sum M = 2n - k$, $m_i \in \{3,4,5\}$ for $i = 1, \ldots, t$, then there is a decomposition $J_n \to M, k, n^H$ such that for $i \in \{0,1,2\}$ no cycle of the decomposition contains both vertex i and vertex $i+n$.*

Proof For $6 \leq n \leq 32$ we have verified the result by computer search and concatenation. So assume $n \geq 33$. The special case where $M \in \{(3^i) : i \text{ is odd}\}$ will be dealt with separately in a moment.

Case 1 Suppose that $M \notin \{(3^i) : i \text{ is odd}\}$. We have verified that the following decompositions exist.

$$Q_4 \to 3, 2^+, 5^H \qquad Q_5 \to 4, 3^+, 6^H \qquad Q_6 \to 5, 4^+, 7^H$$

It is routine to check for $n \geq 33$, if M satisfies the hypotheses of the lemma (and $M \notin \{(3^i) : i \text{ is odd}\}$), then M can be written as $M = (X, y-1)$ where $X \notin \{(3^i) : i \text{ is even}\}$ and $Q_y \to (y-1), (y-2)^+, (y+1)^H$ is one of the decompositions listed above. Using $n \geq 33$ and $y \in \{4, 5, 6\}$, it can be verified that a decomposition $L_{n-y} \to X, (k-y+2)^+, (n-y-1)^H$ exists by Lemma 50. Concatenation of this decomposition with $Q_y \to (y-1), (y-2)^+, (y+1)^H$ yields the required decomposition $J_n \to M, k, n^H$.

Case 2 Suppose that $M \in \{(3^i) : i \text{ is odd}\}$. Let $p = \frac{i-3}{2} - (n-k)$. We deal separately with the case $n = k$ and the case $n \in \{k+1, k+2, k+3, k+4, k+5\}$.

Case 2a Suppose that $n = k$. Note that since $n \geq 33$ and $\sum M = 3i = 2n - k$, we have $p \geq 4$ when $n = k$. The set of 3-cycles in the decomposition is the union of the following two sets.

$$\{(0,1,3), (2,4,5), (n-3, n-2, n-1)\}$$
$$\{(6j+6, 6j+7, 6j+8), (6j+9, 6j+10, 6j+11) : j \in \{0, \dots, p-1\}\}$$

The edge set of one n-cycle is $E_1 \cup E_2 \cup E_3$ where

$$\begin{aligned}
E_1 = \; & \{\{5,3\}, \{3,4\}, \{4,6\}\}, \\
E_2 = \; & \{\{n-4, n-2\}, \{n-2, n+1\}, \{n+1, n-1\}, \{n-1, n+2\}, \{n+2, n\}, \{n, n-3\}\}, \\
E_3 = \; & \{\{6j+6, 6j+9\}, \{6j+9, 6j+8\}, \{6j+8, 6j+11\}, \\
& \{6j+5, 6j+7\}, \{6j+7, 6j+10\}, \{6j+10, 6j+12\} : j \in \{0, \dots, p-1\}\}.
\end{aligned}$$

Note that this n-cycle contains the edge $\{n, n+2\}$ and does not contain any of the vertices in $\{0, 1, 2\}$. The remaining edges form the edge set of the other n-cycle (here $n = k$).

Case 2b Suppose that $n \in \{k+1, k+2, k+3, k+4, k+5\}$. Since $n \geq 33$, it is routine to verify that for any integers n and k and list $M \in \{(3^i) : i \text{ is odd}\}$ which satisfy the conditions of the lemma we have $p \geq 1$, except in the case where $M = (3^{13})$ and $n = k+5$. In this special case we have $(n, k) = (34, 29)$ and we have constructed the decomposition required in this case explicitly. Thus we can assume $p \geq 1$. Let $l = 5(n-k)$. The set of 3-cycles in the decomposition is the union of the following three sets.

$$\{(0,1,3), (2,4,5), (n-3, n-2, n-1)\}$$
$$\{(5j+6, 5j+7, 5j+8), (5j+9, 5j+10, 5j+11) : j \in \{0, \dots, (n-k)-1\}\}$$
$$\{(6j+l+6, 6j+l+7, 6j+l+8), (6j+l+9, 6j+l+10, 6j+l+11) : j \in \{0, \dots, p-1\}\}$$

The edge set of the n-cycle is $E_1 \cup E_2 \cup E_3 \cup E_4$ where

$$\begin{aligned}
E_1 = \; & \{\{5,3\}, \{3,4\}, \{4,6\}\}, \\
E_2 = \; & \{\{n-4, n-2\}, \{n-2, n+1\}, \{n+1, n-1\}, \{n-1, n+2\}, \{n+2, n\}, \{n, n-3\}\}, \\
E_3 = \; & \{\{5j+6, 5j+9\}, \{5j+9, 5j+8\}, \{5j+8, 5j+11\}, \\
& \{5j+5, 5j+7\}, \{5j+7, 5j+10\} : j \in \{0, \dots, (n-k)-1\}\}, \\
E_4 = \; & \{\{6j+l+6, 6j+l+9\}, \{6j+l+9, 6j+l+8\}, \\
& \{6j+l+8, 6j+l+11\}, \{6j+l+5, 6j+l+7\}, \\
& \{6j+l+7, 6j+l+10\}, \{6j+l+10, 6j+l+12\} : j \in \{0, \dots, p-1\}\}.
\end{aligned}$$

Note that this n-cycle contains the edge $\{n, n+2\}$ and does not contain any of the vertices in $\{0, 1, 2\}$. The remaining edges form the edge set of the cycle of length k. □

In Lemma 52 we will form decompositions of graphs J_n by concatenating decompositions of J_{n-y} with decompositions of graphs R_y which we will now define. For $y \in \{5, 6\}$, the graph obtained from J_y by adding the edge $\{0, 2\}$ will be denoted by R_y. Let $M = (m_1, \dots, m_t)$ be a list of integers with $m_i \geq 3$ for $i = 1, \dots, t$. A decomposition $\{G_1, \dots, G_t, A\}$ of R_y such that

- G_i is an m_i-cycle for $i = 1, \dots, t$; and
- A is a path of length $y+1$ from 0 to 2 such that $1 \notin V(A)$ and $\{y, y+2\} \in E(A)$;

will be denoted $R_y \to M, y^H$.

For $y \in \{5,6\}$ and $n > y$, the graph J_n is the union of the graph obtained from J_{n-y} by removing the edge $\{n-y, n-y+2\}$ and the graph obtained from R_y by applying the permutation $x \mapsto x+(n-y)$. It follows that if there is a decomposition $J_{n-y} \to M, k, (n-y)^H$ and a decomposition $R_y \to M', y^H$, then there is a decomposition $J_n \to M, M', k, n^H$. In this construction the edge $\{n-y, n-y+2\}$ in the $(n-y)$-cycle of the decomposition of J_{n-y} is replaced with the path from the decomposition of R_y to form the n-cycle in the decomposition of J_n. Note that, for $y \in \{5,6\}$ and $n-y \geq 3$, no cycle of the decomposition contains both vertex i and vertex $i+n$ for $i \in \{0,1,2\}$.

Lemma 52 *Let n and k be integers with $6 \leq k \leq n$. If $M = (m_1, \dots, m_t)$ is a list such that $\sum M = 2n - k$ and $m_i \in \{3,4,5\}$ for $i = 1, \dots, t$, then there is a decomposition $J_n \to M, k, n^H$ such that for $i \in \{0,1,2\}$ no cycle of the decomposition contains both vertex i and vertex $i+n$.*

Proof If $k \geq n-5$, then the result follows by Lemma 51, which means the result holds for $n \leq 11$. We can therefore assume that $k \leq n-6$, $n \geq 12$ and, by induction, that the result holds for each integer n' in the range $6 \leq n' < n$.

We have verified that the following decompositions exist.

$$R_5 \to 5^2, 5^H \qquad R_6 \to 3,4,5,6^H \qquad R_6 \to 4^3, 6^H \qquad R_6 \to 3^4, 6^H$$

It is routine to check that if M satisfies the conditions of the lemma, then M can be written as $M = (X, Y)$ where $R_y \to Y, y^H$ is one of the decompositions above. The required decomposition can be obtained by concatenating a decomposition $J_{n-y} \to X, k, (n-y)^H$ (which exists by our inductive hypothesis since $k \leq n-6 \leq n-y$) with a decomposition $R_y \to Y, y^H$. $\square$

Proof of Lemma 9 If $k \in \{3,4,5\}$, then the result follows by Lemma 7. So we can assume $k \geq 6$. Since $n \geq 7$, we can obtain an (M)-decomposition of $\langle\{1,2,3\}\rangle_n$ from an (M)-decomposition of J_n by identifying vertex i with vertex $i+n$ for each $i \in \{0,1,2\}$, provided that for each $i \in \{0,1,2\}$, no cycle of our decomposition contains both vertex i and vertex $i+n$. Thus, Lemma 9 follows immediately from Lemma 52. $\square$

7 Decompositions of $K_n - \langle S \rangle_n$

The purpose of this section is to prove Lemmas 11 and 12, and these proofs are given in Subsections 7.2 and 7.3 respectively. In Subsection 7.1 we present results on Hamilton decompositions of circulant graphs that we will require.

To prove Lemma 11, we require a $(3^{tn}, 4^{qn}, n^h)$-decomposition of $\langle \overline{S} \rangle_n$, where $\overline{S} = \{1, \dots, \lfloor \frac{n}{2} \rfloor\} \setminus S$, for almost all n, t, q and h satisfying $h \geq 2$, $n \geq 2\max(S) + 1$ and $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - |S|$. To construct this, $\overline{S}$ will be partitioned into three subsets S_1, S_2 and S_3 such that there is a (3^{tn})-decomposition of $\langle S_1 \rangle_n$, a (4^{qn})-decomposition of $\langle S_2 \rangle_n$, and an (n^h)-decomposition of $\langle S_3 \rangle_n$. Our (3^{tn})-decompositions of $\langle S_1 \rangle_n$ are constructed by partitioning S_1 into modulo n difference triples, our (4^{qn})-decompositions of $\langle S_2 \rangle_n$ are constructed by partitioning S_2 into modulo n difference quadruples, and our (n^h)-decompositions of $\langle S_3 \rangle_n$ are constructed by partitioning S_3 into sets of size at most 3 to yield connected circulant graphs of degree at most 6 that are known to have Hamilton decompositions. Lemma 12 is proved in an analogous manner.

7.1 Decompositions of circulant graphs into Hamilton cycles

Theorems 53–55 address the open problem of whether every connected Cayley graph on a finite abelian group has a Hamilton decomposition [4]. Note that $\langle S\rangle_n$ is connected if and only if $\gcd(S \cup \{n\}) = 1$.

Theorem 53 ([13]) *Every connected* 4*-regular Cayley graph on a finite abelian group has a decomposition into two Hamilton cycles.*

The following theorem is an easy corollary of Theorem 53.

Theorem 54 *Every connected* 5*-regular Cayley graph on a finite abelian group has a decomposition into two Hamilton cycles and a perfect matching.*

Proof Let the graph be $X = \mathrm{Cay}(\Gamma, S)$. Since each vertex of X has odd degree, S contains an element s of order 2 in Γ. Let F be the perfect matching of X generated by s. If $\mathrm{Cay}(\Gamma, S \setminus \{s\})$ is connected then, as it is also 4-regular, the result follows immediately from Theorem 53. On the other hand, if $\mathrm{Cay}(\Gamma, S \setminus \{s\})$ is not connected, then it consists of two isomorphic connected components, with $x \mapsto sx$ being an isomorphism. These components are 4-regular and so by Theorem 53, each can be decomposed into two Hamilton cycles. Moreover, since $x \mapsto sx$ is an isomorphism, there exists a Hamilton decomposition $\{H_1, H_1'\}$ of the first and a Hamilton decomposition $\{H_2, H_2'\}$ of the second such that there is a pair of vertex-disjoint 4-cycles (x_1, y_1, y_2, x_2) and (x_1', y_1', y_2', x_2') in X with $x_1y_1 \in E(H_1)$, $x_1'y_1' \in E(H_1')$, $x_2y_2 \in E(H_2)$, $x_2'y_2' \in E(H_2')$, and $x_1x_2, y_1y_2, x_1'x_2', y_1'y_2' \in E(F)$. It follows that if we let G be the graph with edge set

$$(E(H_1) \cup E(H_2) \cup \{x_1x_2, y_1y_2\}) \setminus \{x_1y_1, x_2y_2\}$$

and let G' be the graph with edge set

$$(E(H_1') \cup E(H_2') \cup \{x_1'x_2', y_1'y_2'\}) \setminus \{x_1'y_1', x_2'y_2'\},$$

then G and G' are edge-disjoint Hamilton cycles in X. This proves the result. □

Theorem 55 ([31]) *Every* 6*-regular Cayley graph on a group which is generated by an element of the connection set has a decomposition into three Hamilton cycles.*

This theorem implies that, for distinct $a, b, c \in \{1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$, the graph $\langle \{a, b, c\}\rangle_n$ has a decomposition into three Hamilton cycles if $\gcd(x, n) = 1$ for some $x \in \{a, b, c\}$.

In the next two lemmas we give results similar to that of Lemma 10, but for the case where the connection set is of the form $\{x-1\} \cup \{x+1, \dots, \lfloor \frac{n}{2} \rfloor\}$ rather than $\{x, \dots, \lfloor \frac{n}{2} \rfloor\}$. Lemma 56 deals with the case n is odd, and Lemma 57 deals with the case n is even.

Lemma 56 *If n is odd and $1 \le h \le \frac{n-3}{2}$, then there is an (n^h)-decomposition of $\langle \{\frac{n-1}{2} - h\} \cup \{\frac{n-1}{2} - h + 2, \dots, \frac{n-1}{2}\}\rangle_n$; except when $h = 1$ and $n \equiv 3 \pmod 6$ in which case the graph is not connected.*

Proof If $h = 1$, then the graph is $\langle \frac{n-3}{2}\rangle_n$. If $n \equiv 1, 5 \pmod 6$, then $\gcd(\frac{n-3}{2}, n) = 1$ and $\langle \frac{n-3}{2}\rangle_n$ is an n-cycle. If $n \equiv 3 \pmod 6$, then $\gcd(\frac{n-3}{2}, n) = 3$ and $\langle \frac{n-3}{2}\rangle_n$ is not connected. Thus the result holds for $h = 1$. In the remainder of the proof we assume $h \ge 2$.

We first decompose $\langle\{\frac{n-1}{2}-h\}\cup\{\frac{n-1}{2}-h+2,\ldots,\frac{n-1}{2}\}\rangle_n$ into circulant graphs by partitioning the connection set, and then decompose the resulting circulant graphs into n-cycles using Theorems 53 and 55.

If h is even, then we partition the connection set into pairs by pairing $\frac{n-1}{2}-h$ with $\frac{n-1}{2}$ and partitioning $\{\frac{n-1}{2}-h+2,\ldots,\frac{n-3}{2}\}$ into consecutive pairs (if $h=2$, then our partition is just $\{\{\frac{n-5}{2},\frac{n-1}{2}\}\}$). Each of the resulting circulant graphs is 4-regular and connected and thus can be decomposed into two n-cycles by Theorem 53. If h is odd, then we partition the connection set into the triple $\{\frac{n-1}{2}-h,\frac{n-1}{2}-h+2,\frac{n-1}{2}\}$ and consecutive pairs from $\{\frac{n-1}{2}-h+3,\ldots,\frac{n-3}{2}\}$ (if $h=3$, then our partition is just $\{\{\frac{n-7}{2},\frac{n-3}{2},\frac{n-1}{2}\}\}$). Since $\gcd(\frac{n-1}{2},n)=1$, the graph $\langle\{\frac{n-1}{2}-h,\frac{n-1}{2}-h+2,\frac{n-1}{2}\}\rangle_n$ can be decomposed into three n-cycles by Theorem 55. Any other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 53. □

Lemma 57 *If n is even and $1\le h\le\frac{n-4}{2}$, then there is an (n^h)-decomposition of $\langle\{\frac{n}{2}-h-1\}\cup\{\frac{n}{2}-h+1,\ldots,\frac{n}{2}\}\rangle_n$; except when $h=1$ and $n\equiv 0 \pmod 4$ in which case the graph is not connected.*

Proof If $h=1$, then the graph is $\langle\{\frac{n-4}{2},\frac{n}{2}\}\rangle_n$. If $n\equiv 2\pmod 4$, then $\gcd(\frac{n-4}{2},n)=1$, $\langle\frac{n-4}{2}\rangle_n$ is an n-cycle, and $\langle\{\frac{n}{2}\}\rangle_n$ is a perfect matching. If $n\equiv 0\pmod 4$, then $\gcd(\frac{n-4}{2},\frac{n}{2},n)=2$ and $\langle\{\frac{n-4}{2},\frac{n}{2}\}\rangle_n$ is not connected. Thus the result holds for $h=1$. In the remainder of the proof we assume $h\ge 2$.

We first decompose $\langle\{\frac{n}{2}-h-1\}\cup\{\frac{n}{2}-h+1,\ldots,\frac{n}{2}\}\rangle_n$ into circulant graphs by partitioning the connection set, and then decompose the resulting circulant graphs into n-cycles using Theorems 53, 54 and 55.

If $n\equiv 0\pmod 4$ and h is even, then we partition the connection set into pairs and the singleton $\{\frac{n}{2}\}$ by pairing $\frac{n}{2}-h-1$ with $\frac{n-2}{2}$ and partitioning $\{\frac{n}{2}-h+1,\ldots,\frac{n-4}{2}\}$ into pairs of consecutive integers (if $h=2$, then our partition is just $\{\{\frac{n}{2}\},\{\frac{n-6}{2},\frac{n-2}{2}\}\}$). The graph $\langle\{\frac{n}{2}\}\rangle_n$ is a perfect matching. The other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 53 (note that $\gcd(\frac{n-2}{2},n)=1$).

If $n\equiv 0\pmod 4$ and h is odd, then we partition the connection set into pairs, the triple $\{\frac{n}{2}-h-1,\frac{n}{2}-h+1,\frac{n-2}{2}\}$ and the singleton $\{\frac{n}{2}\}$ by partitioning $\{\frac{n}{2}-h+2,\ldots,\frac{n-4}{2}\}$ into pairs of consecutive integers (if $h=3$, then our partition is just $\{\{\frac{n}{2}\},\{\frac{n-8}{2},\frac{n-4}{2},\frac{n-2}{2}\}\}$). The graph $\langle\{\frac{n}{2}\}\rangle_n$ is a perfect matching and, since $\gcd(\frac{n-2}{2},n)=1$, the graph $\langle\{\frac{n}{2}-h-1,\frac{n}{2}-h+1,\frac{n-2}{2}\}\rangle_n$ can be decomposed into three n-cycles using Theorem 55. Any other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 53.

If $n\equiv 2\pmod 4$, then we partition the connection set into pairs, the triple $\{\frac{n}{2}-h-1,\frac{n-2}{2},\frac{n}{2}\}$ and, when h is odd, the singleton $\{\frac{n-4}{2}\}$ by partitioning $\{\frac{n}{2}-h+1,\ldots,\frac{n-4}{2}\}$ into pairs of consecutive integers (when h is even) or into pairs of consecutive integers and the singleton $\{\frac{n-4}{2}\}$ (when h is odd). (Our partition is just $\{\{\frac{n-6}{2},\frac{n-2}{2},\frac{n}{2}\}\}$ if $h=2$, and just $\{\{\frac{n-8}{2},\frac{n-2}{2},\frac{n}{2}\},\{\frac{n-4}{2}\}\}$ if $h=3$.) Since $\gcd(\frac{n-2}{2},\frac{n}{2})=1$, the graph $\langle\{\frac{n}{2}-h-1,\frac{n-2}{2},\frac{n}{2}\}\rangle_n$ can be decomposed into two n-cycles and a perfect matching using Theorem 54. When h is odd, $\langle\{\frac{n-4}{2}\}\rangle_n$ is an n-cycle (note that $\gcd(\frac{n-4}{2},n)=1$). Any other resulting circulant graphs are 4-regular and connected and thus can each be decomposed into two n-cycles by Theorem 53. □

7.2 Proof of Lemma 11

Before we prove Lemma 11, we require three preliminary lemmas which establish the existence of various $(4^{qn},n^h)$-decompositions of circulant graphs.

Lemma 58 *If* $S \subseteq \{1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$ *such that*

- $S = \{x+1, \ldots, x+4q\}$ *for some* x*;*
- $S = \{x\} \cup \{x+2, \ldots, x+4q-1\} \cup \{x+4q+1\}$ *for some* x*; or*
- $S = \{\frac{n-1}{2} - 4q\} \cup \{\frac{n-1}{2} - 4q + 2, \ldots, \frac{n-1}{2}\}$ *where* n *is odd;*

then there is a (4^{qn})*-decomposition of* $\langle S \rangle_n$*.*

Proof It is sufficient to partition S into q modulo n difference quadruples. If $S = \{x+1, \ldots, x+4q\}$, then we partition S into q sets of the form $\{y, y+1, y+2, y+3\}$, each of which is a difference quadruple. If $S = \{x\} \cup \{x+2, \ldots, x+4q-1\} \cup \{x+4q+1\}$, then we partition S into q sets of the form $\{y, y+2, y+3, y+5\}$, each of which is a difference quadruple. If $S = \{\frac{n-1}{2} - 4q\} \cup \{\frac{n-1}{2} - 4q + 2, \ldots, \frac{n-1}{2}\}$ and n is odd, then we partition S into $q-1$ sets of the form $\{y, y+2, y+3, y+5\}$, each of which is a difference quadruple, and the set $\{\frac{n-9}{2}, \frac{n-5}{2}, \frac{n-3}{2}, \frac{n-1}{2}\}$, which is a modulo n difference quadruple (note that $\frac{n-5}{2} + \frac{n-3}{2} + \frac{n-1}{2} - \frac{n-9}{2} = n$). $\square$

Lemma 59 *If* h*,* q *and* n *are non-negative integers with* $1 \leq 4q + h \leq \lfloor \frac{n-1}{2} \rfloor$*, then there is a* $(4^{qn}, n^h)$*-decomposition of* $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 1, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$*.*

Proof If $h = 0$ then the result follows immediately by Lemma 58, and if $q = 0$ then the result follows immediately by Lemma 10. For $q, h \geq 1$ we partition the connection set into the set $\{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - h\}$ and the set $\{\lfloor \frac{n-1}{2} \rfloor - h + 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 1, \ldots, \lfloor \frac{n-1}{2} \rfloor - h\} \rangle_n$ has a (4^{qn})-decomposition by Lemma 58, and $\langle \{\lfloor \frac{n-1}{2} \rfloor - h + 1, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ has an (n^h)-decomposition by Lemma 10. $\square$

Lemma 60 *If* h*,* q *and* n *are non-negative integers with* $1 \leq 4q + h \leq \lfloor \frac{n-3}{2} \rfloor$ *such that* n *is odd when* $h = 0$ *and* $n \equiv 1, 2, 5, 6, 7, 10, 11 \pmod{12}$ *when* $h = 1$*, then there is a* $(4^{qn}, n^h)$*-decomposition of* $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$*.*

Proof If $h = 0$, then the result follows immediately by Lemma 58. If $q = 0$, then the result follows immediately by Lemma 56 (n odd) or Lemma 57 (n even). For $h, q \geq 1$ we partition the connection set into the set $\{\lfloor \frac{n-1}{2} \rfloor - h - 4q\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - h - 1\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 1\}$ and the set $\{\lfloor \frac{n-1}{2} \rfloor - h\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $\langle \{\lfloor \frac{n-1}{2} \rfloor - h - 4q\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h - 4q + 2, \ldots, \lfloor \frac{n-1}{2} \rfloor - h - 1\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 1\} \rangle_n$ has a (4^{qn})-decomposition by Lemma 58, and $\langle \{\lfloor \frac{n-1}{2} \rfloor - h\} \cup \{\lfloor \frac{n-1}{2} \rfloor - h + 2, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ has an (n^h)-decomposition by Lemma 56 (n odd) or Lemma 57 (n even). $\square$

Proof of Lemma 11 We give the proof of Lemma 11 for the case $S = \{1, 2, 3, 4, 6\}$ only. The cases

$$S \in \{\{1, 2, 3, 4\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6, 7\}\}$$

are similar.

The conditions $h \geq 2$ and $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 5$ imply $n \geq 6t + 15$. If $t = 0$, then the result follows immediately by Lemma 60. We deal separately with the three cases $t \in \{1, 2, 3, 4, 5, 6\}$, $t \in \{7, 8, 9, 10\}$, and $t \geq 11$.

Case 1 Suppose that $t \in \{1, 2, 3, 4, 5, 6\}$. The cases $6t + 15 \leq n \leq 6t + 18$ and the cases

$$(n, t) \in \{(38, 3), (39, 3), (40, 3), (44, 4), (45, 4)\}$$

are dealt with first. Since $h \geq 2$, it follows from $3t + 4q + h = \lfloor \frac{n-1}{2} \rfloor - 5$ that in each of these cases we have $q = 0$. Thus, the value of h is uniquely determined by the values of n and t. The required decompositions are obtained by partitioning $\{5\} \cup \{7, \ldots, \lfloor \frac{n}{2} \rfloor\}$ into t modulo n difference triples and a collection of connection sets for circulant graphs such that the circulant graphs can be decomposed into Hamilton cycles (or Hamilton cycles and a perfect matching) using the results in Section 7.1. Suitable partitions are given in the following tables.

t=1:

n	modulo n difference triples	connection sets
21	$\{5,7,9\}$	$\{8,10\}$
22	$\{5,7,10\}$	$\{8,9\},\{11\}$
23	$\{5,8,10\}$	$\{7,9\},\{11\}$
24	$\{5,9,10\}$	$\{7,8\},\{11\},\{12\}$

t=2:

n	modulo n difference triples	connection sets
27	$\{5,7,12\},\{8,9,10\}$	$\{11,13\}$
28	$\{5,7,12\},\{8,9,11\}$	$\{10,13\},\{14\}$
29	$\{5,7,12\},\{8,10,11\}$	$\{9\},\{13,14\}$
30	$\{5,9,14\},\{8,10,12\}$	$\{7\},\{11,13\},\{15\}$

t=3:

n	modulo n difference triples	connection sets
33	$\{5,8,13\},\{7,9,16\},\{10,11,12\}$	$\{14,15\}$
34	$\{5,10,15\},\{7,9,16\},\{8,12,14\}$	$\{11,13\},\{17\}$
35	$\{5,8,13\},\{7,9,16\},\{10,11,14\}$	$\{12,15\},\{17\}$
36	$\{5,8,13\},\{7,9,16\},\{10,12,14\}$	$\{11,15\},\{17\},\{18\}$
38	$\{5,12,17\},\{7,9,16\},\{8,10,18\}$	$\{11,13\},\{14,15\},\{19\}$
39	$\{5,12,17\},\{7,9,16\},\{8,10,18\}$	$\{11,13\},\{14,15\},\{19\}$
40	$\{5,12,17\},\{7,9,16\},\{8,10,18\}$	$\{11,13\},\{14,15\},\{19\},\{20\}$

t=4:

n	modulo n difference triples	connection sets
39	$\{5,11,16\},\{7,8,15\},\{9,10,19\},\{12,13,14\}$	$\{17,18\}$
40	$\{5,8,13\},\{7,9,16\},\{10,12,18\},\{11,14,15\}$	$\{17,19\},\{20\}$
41	$\{5,14,19\},\{7,8,15\},\{9,11,20\},\{12,13,16\}$	$\{10,17\},\{18\}$
42	$\{5,10,15\},\{7,11,18\},\{8,9,17\},\{12,14,16\}$	$\{13\},\{19,20\},\{21\}$
44	$\{5,11,16\},\{7,13,20\},\{8,10,18\},\{9,12,21\}$	$\{14,15\},\{17,19\},\{22\}$
45	$\{5,11,16\},\{7,13,20\},\{8,10,18\},\{9,12,21\}$	$\{14,15\},\{17,19\},\{22\}$

t=5:

n	modulo n difference triples	connection sets
45	$\{5,17,22\},\{7,13,20\},\{8,10,18\},\{9,12,21\},\{14,15,16\}$	$\{11,19\}$
46	$\{5,17,22\},\{7,13,20\},\{8,10,18\},\{9,12,21\},\{11,16,19\}$	$\{14,15\},\{23\}$
47	$\{5,18,23\},\{7,13,20\},\{8,11,19\},\{10,12,22\},\{14,16,17\}$	$\{9,15\},\{21\}$
48	$\{5,17,22\},\{7,13,20\},\{8,10,18\},\{9,12,21\},\{11,14,23\}$	$\{15,16\},\{19\},\{24\}$

t=6:

n	modulo n difference triples	connection sets
51	$\{5,13,18\},\{7,15,22\},\{8,16,24\},$ $\{9,10,19\},\{11,12,23\},\{14,17,20\}$	$\{21,25\}$
52	$\{5,13,18\},\{7,15,22\},\{8,16,24\},$ $\{9,10,19\},\{11,12,23\},\{14,17,21\}$	$\{20,25\},\{26\}$
53	$\{5,13,18\},\{7,12,19\},\{8,14,22\},$ $\{9,15,24\},\{10,11,21\},\{16,17,20\}$	$\{23\},\{25,26\}$
54	$\{5,17,22\},\{7,8,15\},\{9,10,19\},$ $\{11,13,24\},\{12,14,26\},\{16,18,20\}$	$\{21,23\},\{25\},\{27\}$

We now deal with $n \geq 6t+19$ which implies $4q+h \geq 4$. Define S_t by $S_t = \{5\} \cup \{7,\ldots,3t+9\}$ for $t \in \{1,2,5,6\}$, and $S_t = \{5\} \cup \{7,\ldots,3t+8\} \cup \{3t+10\}$ when $t \in \{3,4\}$. The following table gives a partition π_t of S_t into difference triples and a difference quadruple Q_t such that Q_t can be partitioned into two pairs of relatively prime integers.

t	π_t
1	$\{\{5,7,12\},\{8,9,10,11\}\}$
2	$\{\{5,9,14\},\{7,8,15\},\{10,11,12,13\}\}$
3	$\{\{5,9,14\},\{7,10,17\},\{8,11,19\},\{12,13,15,16\}\}$
4	$\{\{5,9,14\},\{7,13,20\},\{8,11,19\},\{10,12,22\},\{15,16,17,18\}\}$
5	$\{\{5,14,19\},\{7,13,20\},\{8,10,18\},\{9,15,24\},\{11,12,23\},\{16,17,21,22\}\}$
6	$\{\{5,15,20\},\{7,16,23\},\{8,14,22\},\{9,12,21\},\{10,17,27\},\{11,13,24\},$ $\{18,19,25,26\}\}$

Thus, $\langle Q_t \rangle_n$ can be decomposed into two connected 4-regular Cayley graphs, which in turn can be decomposed into Hamilton cycles using Theorem 53. It follows that there is both a $(3^{tn},4^n)$-decomposition and a $(3^{tn},n^4)$-decomposition of $\langle S_t \rangle_n$. If $q=0$, then we use the $(3^{tn},n^4)$-decomposition of $\langle S_t \rangle_n$ and if $q \geq 1$, then we use the $(3^{tn},4^n)$-decomposition of $\langle S_t \rangle_n$. This leaves us needing an (n^{h-4})-decomposition of $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$ when $q=0$, and a $(4^{(q-1)n},n^h)$-decomposition of $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$ when $q \geq 1$. Note that $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$ is isomorphic to

- $\langle \{3t+10,\ldots,\lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \in \{1,2,5,6\}$; and
- $\langle \{3t+9\} \cup \{3t+11,\ldots,\lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \in \{3,4\}$.

When $t \in \{1,2,5,6\}$ the required decomposition exists by Lemma 59. When $t \in \{3,4\}$ and the required number of Hamilton cycles (that is, $h-4$ when $q=0$ and h when $q \geq 1$) is at least 2, the required decomposition exists by Lemma 60. So we need to consider only the cases where $q=0$, $h \in \{4,5\}$ and $t \in \{3,4\}$.

Since $3t+4q+h = \lfloor \frac{n-1}{2} \rfloor - 5$, and since we have already dealt with the cases where $(n,t) \in \{(38,3),(39,3),(40,3),(44,4),(45,4)\}$, this leaves us with only the three cases where $(n,t,h) \in \{(37,3,4),(43,4,4),(46,4,5)\}$. In the cases $(n,t,h) \in \{(37,3,4),(43,4,4)\}$ we have that $h-4$ (the required number of Hamilton cycles) is 0 and n is odd, and in the case $(n,t,h)=(46,4,5)$ we have that $h-4$ (the required number of Hamilton cycles) is 1 and $n \equiv 10 \pmod{12}$. So in all these cases the required decompositions exist by Lemma 60.

Case 2 Suppose that $t \in \{7,8,9,10\}$. Redefine S_t by $S_t = \{5\} \cup \{7,\ldots,3t+7\}$. The following table gives a partition of S_t into difference triples and a set R_t consisting of a pair of relatively prime integers. Thus, $\langle R_t \rangle_n$ is a connected 4-regular Cayley graph, and so can be decomposed into two Hamilton cycles using Theorem 53. Thus, we have a $(3^{tn}, n^2)$-decomposition of $\langle S_t \rangle_n$.

t	difference triples	R_t
7	$\{5,17,22\},\{7,16,23\},\{8,20,28\},\{9,18,27\},\{10,14,24\},\{11,15,26\},\{12,13,25\}$	$\{19,21\}$
8	$\{5,17,22\},\{7,19,26\},\{8,16,24\},\{9,20,29\},\{10,21,31\},\{11,14,25\},\{12,18,30\},\{13,15,28\}$	$\{23,27\}$
9	$\{5,19,24\},\{7,20,27\},\{8,21,29\},\{9,22,31\},\{10,23,33\},\{11,15,26\},\{12,16,28\},\{13,17,30\},\{14,18,32\}$	$\{25,34\}$
10	$\{5,21,26\},\{7,22,29\},\{8,23,31\},\{9,24,33\},\{10,25,35\},\{11,17,28\},\{12,18,30\},\{13,14,27\},\{15,19,34\},\{16,20,36\}$	$\{32,37\}$

Thus, we only require a $(4^{qn}, n^{h-2})$-decomposition of $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$. But $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$ is isomorphic to $\langle \{3t+8,\ldots,\lfloor \frac{n}{2} \rfloor\} \rangle_n$ and so this decomposition exists by Lemma 59.

Case 3 Suppose that $t \geq 11$. Redefine S_t by $S_t = \{5\} \cup \{7,\ldots,3t+5\}$ when $t \equiv 1,2 \pmod 4$, and $S_t = \{5\} \cup \{7,\ldots,3t+4\} \cup \{3t+6\}$ when $t \equiv 0,3 \pmod 4$. We now obtain a (3^{tn})-decomposition of $\langle S_t \rangle_n$. For $11 \leq t \leq 101$, we have found such a decomposition by partitioning S_t into difference triples with the aid of a computer. For $t \geq 102$ we first decompose $\langle S_t \rangle_n$ into $\langle \{5\} \cup \{7,\ldots,44\} \rangle_n = \langle S_{13} \rangle_n$ and $\langle S_t \setminus S_{13} \rangle_n$. We have already noted the existence of a (3^{13n})-decomposition of $\langle S_{13} \rangle_n$. For $t \equiv 1,2 \pmod 4$ (respectively $t \equiv 0,3 \pmod 4$), we can obtain a $(3^{(t-13)n})$-decomposition of $\langle S_t \setminus S_{13} \rangle_n$ by using a Langford sequence (respectively hooked Langford sequence) of order $t-13$ and defect 45, which exists since $t \geq 102$, to partition $S_t \setminus S_{13}$ into difference triples (see [47, 48]). So we have a (3^{tn})-decomposition of $\langle S_t \rangle_n$, and require a $(4^{qn}, n^h)$-decomposition of $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$. Since $K_n - \langle \{1,2,3,4,6\} \cup S_t \rangle_n$ is isomorphic to

- $\langle \{3t+6,\ldots,\lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \equiv 1,2 \pmod 4$; and
- $\langle \{3t+5\} \cup \{3t+7,\ldots,\lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $t \equiv 0,3 \pmod 4$;

this decomposition exists by Lemma 59 or 60. □

7.3 Proof of Lemma 12

Proof of Lemma 12 We give the proof for the case $S = \{1, 2, 3, 4, 6\}$ only. The proofs for the cases

$$S \in \{\{1, 2, 3, 4\}, \{1, 2, 3, 4, 5, 7\}, \{1, 2, 3, 4, 5, 6, 7\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}$$

are similar.

The conditions $h \geq 2$ and $5r + h = \lfloor \frac{n-1}{2} \rfloor - 5$ imply $n \geq 10r + 15$. If $r = 0$, then the result follows immediately by Lemma 56 (n odd) or Lemma 57 (n even). Thus, we assume $r \geq 1$.

Define S_r by $S_r = \{5\} \cup \{7, \ldots, 5r + 5\}$ when $r \equiv 2, 3 \pmod 4$, and $S_r = \{5\} \cup \{7, \ldots, 5r + 4\} \cup \{5r + 6\}$ when $r \equiv 0, 1 \pmod 4$. We now obtain a (5^{rn})-decomposition $\langle S_r \rangle_n$ by partitioning S_r into difference quintuples. We have constructed such a partition of S for $1 \leq r \leq 30$ with the aid of a computer.

For $r \geq 31$ we first partition S_r into $\{5\} \cup \{7, \ldots, 15\} = S_2$ and $S_r \setminus S_2$. We have already noted that S_2 can be partitioned into difference quintuples, so we only need to partition $S_r \setminus S_2$ into difference quintuples. Note that $S_r \setminus S_2 = \{16, \ldots, 5r + 5\}$ when $r \equiv 2, 3 \pmod 4$, and $S_r \setminus S_2 = \{16, \ldots, 5r + 4\} \cup \{5r + 6\}$ when $r \equiv 0, 1 \pmod 4$.

For $r \equiv 2, 3 \pmod 4$, we take a Langford sequence of order $r - 2$ and defect 15, which exists since $r \geq 31$ (see [47, 48]), and use it to partition $\{15, \ldots, 3r + 8\}$ into $r - 2$ difference triples. We then add 1 to each element of each of these triples to obtain a partition of $\{16, \ldots, 3r + 9\}$ into $r - 2$ triples of the form $\{a, b, c\}$ where $a + b = c + 1$. It is easy to construct the required partition of $\{16, \ldots, 5r + 5\}$ into difference quintuples from this by partitioning $\{3r + 10, \ldots, 5r + 5\}$ into $r - 2$ pairs of consecutive integers.

For $r \equiv 0, 1 \pmod 4$, we take a hooked Langford sequence of order $r - 2$ and defect 15, which exists since $r \geq 31$ (see [47, 48]), and use it to partition $\{15, \ldots, 3r + 7\} \cup \{3r + 9\}$ into $r - 2$ difference triples. We then add 1 to each element of each of these triples, except the element $3r + 9$, to obtain a partition of $\{16, \ldots, 3r + 9\}$ into $r - 3$ triples of the form $\{a, b, c\}$ where $a + b = c + 1$, and one triple of the form $\{a, b, c\}$ where $a + b = c + 2$. It is easy to construct the required partition of $\{16, \ldots, 5r + 4\} \cup \{5r + 6\}$ into difference quintuples from this by partitioning $\{3r + 10, \ldots, 5r + 4\} \cup \{5r + 6\}$ into $r - 3$ pairs of consecutive integers, and the pair $\{5r + 4, 5r + 6\}$. The pair $\{5r + 4, 5r + 6\}$ combines with the triple of the form $\{a, b, c\}$ where $a + b = c + 2$ to form a difference quintuple.

So we have a (5^{rn})-decomposition of $\langle S_r \rangle_n$, and require an (n^h)-decomposition of $K_n - \langle \{1, 2, 3, 4, 6\} \cup S_r \rangle_n$. Note that $K_n - \langle \{1, 2, 3, 4, 6\} \cup S_r \rangle_n$ is isomorphic to

- $\langle \{5r + 6, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 2, 3 \pmod 4$; and
- $\langle \{5r + 5\} \cup \{5r + 7, \ldots, \lfloor \frac{n}{2} \rfloor\} \rangle_n$ when $r \equiv 0, 1 \pmod 4$.

If $r \equiv 2, 3 \pmod 4$, then the decomposition exists by Lemma 10. If $r \equiv 0, 1 \pmod 4$, then the decomposition exists by Lemma 56 (n odd) or Lemma 57 (n even). $\square$

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