

Deriving tight error-trade-off relations for approximate joint measurements of incompatible quantum observables

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The quantification of the “measurement uncertainty” aspect of Heisenberg’s uncertainty principle—that is, the study of trade-offs between accuracy and disturbance, or between accuracies in an approximate joint measurement on two incompatible observables—has regained a lot of interest recently. Several approaches have been proposed and debated. In this paper we consider Ozawa’s definitions for inaccuracies (as root-mean-square errors) in approximate joint measurements, and study how these are constrained in different cases, whether one specifies certain properties of the approximations—namely their standard deviations and/or their bias—or not. Extending our previous work [C. Branciard, *Proc. Natl. Acad. Sci. USA* **110**, 6742 (2013)], we derive error-trade-off relations, which we prove to be tight for pure states. We show explicitly how all previously known relations for Ozawa’s inaccuracies follow from ours. While our relations are in general not tight for mixed states, we show how these can be strengthened and how tight relations can still be obtained in that case.

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I. INTRODUCTION

Heisenberg’s uncertainty principle [1] is undoubtedly one of the most famous characteristics of quantum theory. It is not only celebrated as a deep, fundamental feature of the quantum world by physicists, but has also entered popular culture in many (though sometimes improper) ways [2].

It is worth emphasizing, however, that the general term “uncertainty principle” encompasses in fact different properties of quantum theory. It says, on one hand, that a quantum system cannot be prepared in such a way that a pair of incompatible—i.e., noncommuting—observables are arbitrarily well defined (“preparation uncertainty”). Another aspect is that the measurement of one observable to a certain accuracy in general disturbs the subsequent measurement of another incompatible observable accordingly, or that one cannot in general jointly measure two incompatible observables to a perfect accuracy (“measurement uncertainty”). Interestingly, while the latter measurement uncertainty aspect corresponds to Heisenberg’s initial intuition [1], the best known—and historically the first formally derived [3–5]—uncertainty relations quantify instead the former preparation uncertainty aspect. This has led to some confusion in their interpretation, which it is essential to clarify.

Surprisingly little work has been done in the previous century on the problem of quantifying the trade-off between (in)accuracy and disturbance, or between (in)accuracies in approximate joint measurements. The past two decades have however witnessed a renewed interest in these questions, motivated in particular by the development of quantum information science and by its implications for our understanding of quantum foundations. An important milestone was the derivation in 2003–2004 by Ozawa [6,7] and by Hall [8] of universally valid “uncertainty relations” for measurement-disturbance scenarios and for approximate joint measurements—or, as we will call them in this paper, of *error-disturbance* and *error-trade-off relations*. These came back in the spotlight in the past couple of years as they were experimentally tested [9–15], and strengthened [11,16].

This measurement uncertainty aspect of Heisenberg’s principle has also recently sparked an active debate in the scientific community [17–20], on the question whether Heisenberg’s heuristic argument [1]—that the product of the inaccuracy ϵ_q of the measurement of a particle’s position and the disturbance η_p induced on its momentum is at least of the order of Planck’s constant—is correct or not. Such a relation—precisely, $\epsilon_q \eta_p \geq \hbar/2$ —has only been formally proven a few months ago [17,20], while some authors still argue that this relation may in fact not always be satisfied [18,20]. What is at stake in this controversy is the rigorous definition to be given to the inaccuracy and disturbance (in particular, whether these are state dependent or not)—which Heisenberg was not clear about—and more generally the proper way to quantify the measurement uncertainty aspect of Heisenberg’s principle. The large number of recent publications on this subject [9–31] illustrates how topical these questions are, and suggests that these are still far from being settled.

Agreeing on a framework and definitions is thus a necessary starting point to derive and study error-trade-off relations. Here we consider Ozawa’s (state-dependent) definitions [6,7] for the inaccuracies in approximate joint measurements—for which (in the case of error disturbance) the above relation $\epsilon_q \eta_p \geq \hbar/2$ can indeed be violated—and extend our previous analysis of Ref. [16] to characterize the allowed values of inaccuracies in a number of different cases. The paper is structured as follows. In Sec. II we introduce the general framework for approximate joint measurements that we consider. Section III presents the derivation of our error-trade-off relations, for cases where certain properties of the approximations—namely, their standard deviations and/or their bias with respect to the ideal measurements—are assumed to take some given values; when none of these properties are specified, we show that we obtain the relations previously derived in [16]. In Sec. IV we give an alternative, equivalent form for our relations. Section V, completed by the Appendix, contains the proof that our error-trade-off relations are tight for pure states (at least in certain cases for our last two relations). In Sec. VI we show explicitly how all previously derived (nontight) relations of Refs. [6–

8,11] follow from ours. We finish in Sec. VII with some considerations on the case of mixed states, and then conclude.

II. APPROXIMATE JOINT MEASUREMENTS OF INCOMPATIBLE OBSERVABLES: NOTATIONS AND DEFINITIONS

A. Two incompatible observables to be measured on a given quantum state

In this paper we consider a quantum state $|\psi\rangle$ in some finite-dimensional Hilbert space \mathcal{H} together with two observables (Hermitian operators) A and B acting on \mathcal{H} , and we wish to quantify how well a joint measurement of A and B on $|\psi\rangle$ can be approximated.

The given observables A and B and the state $|\psi\rangle$ define the standard deviations¹ $\Delta A = \langle\psi|(A - \langle A\rangle)^2|\psi\rangle^{1/2}$ and $\Delta B = \langle\psi|(B - \langle B\rangle)^2|\psi\rangle^{1/2}$, as well as the parameter

$$C_{AB} = \frac{1}{2i} \langle\psi|[A, B]|\psi\rangle, \quad (1)$$

which depends on the commutator $[A, B] = AB - BA$ and quantifies the incompatibility of A and B —as we shall see, when $C_{AB} \neq 0$ their perfect joint measurement on $|\psi\rangle$ is impossible. With these notations, Robertson's well-known uncertainty relation [4] (which, again, quantifies the preparation uncertainty aspect of the uncertainty principle and not its measurement uncertainty aspect) writes

$$\Delta A \Delta B \geq |C_{AB}|. \quad (2)$$

We shall assume throughout the paper that² $\Delta A, \Delta B > 0$. It will be convenient to define the reduced observables $\tilde{A}_0 = [A - \langle A\rangle]/\Delta A$, $\tilde{B}_0 = [B - \langle B\rangle]/\Delta B$, such that $\langle\tilde{A}_0\rangle = \langle\tilde{B}_0\rangle = 0$ and $\Delta\tilde{A}_0 = \Delta\tilde{B}_0 = 1$, and the reduced parameter

$$\tilde{C}_{AB} = \frac{C_{AB}}{\Delta A \Delta B} = \text{Im}\langle\tilde{A}_0\tilde{B}_0\rangle \quad (3)$$

(where Im denotes an imaginary part). Note that $\tilde{A}_0|\psi\rangle$ and $\tilde{B}_0|\psi\rangle$ are unit ket vectors, and hence $|\langle\tilde{A}_0\tilde{B}_0\rangle| \leq 1$ and $\tilde{C}_{AB} = \text{Im}\langle\tilde{A}_0\tilde{B}_0\rangle \in [-1, 1]$ —as implied also by Robertson's relation above.

B. Approximate joint measurements

A general strategy for approximating the joint measurement of A and B is to actually measure two compatible—i.e., *commuting*—observables \mathcal{A} and \mathcal{B} , which are taken to approximate A and B , respectively. In full generality these can be measured on an extended Hilbert space, i.e., on the

state $|\psi, \xi\rangle = |\psi\rangle \otimes |\xi\rangle \in \mathcal{H} \otimes \mathcal{K}$, where $|\xi\rangle$ is the state of an ancillary system in some ancillary Hilbert space³ \mathcal{K} . Following Ozawa [6,7,32], we quantify here the *inaccuracies* in the approximations of A and B by the *root-mean-square errors*

$$\epsilon_{\mathcal{A}} = \langle\psi, \xi|(\mathcal{A} - A)^2|\psi, \xi\rangle^{1/2}, \quad (4)$$

$$\epsilon_{\mathcal{B}} = \langle\psi, \xi|(\mathcal{B} - B)^2|\psi, \xi\rangle^{1/2}, \quad (5)$$

where for simplicity A and B are used as shorthand notations for $A \otimes \mathbb{1}_{\mathcal{K}}$ and $B \otimes \mathbb{1}_{\mathcal{K}}$, respectively.

The approximations \mathcal{A} and \mathcal{B} also define (together with $|\xi\rangle$) the standard deviations $\Delta\mathcal{A} = \langle\psi, \xi|(\mathcal{A} - \langle\mathcal{A}\rangle)^2|\psi, \xi\rangle^{1/2}$ and $\Delta\mathcal{B} = \langle\psi, \xi|(\mathcal{B} - \langle\mathcal{B}\rangle)^2|\psi, \xi\rangle^{1/2}$, and the (first moment) biases $\delta_{\mathcal{A}} = \langle\mathcal{A} - A\rangle$ and $\delta_{\mathcal{B}} = \langle\mathcal{B} - B\rangle$. These quantities are related through

$$\begin{aligned} \epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2 &= \Delta A^2 + \Delta\mathcal{A}^2 - 2 \text{Re}\langle(A - \langle A\rangle)(\mathcal{A} - \langle\mathcal{A}\rangle)\rangle, \\ &= \Delta A^2 + \Delta\mathcal{A}^2 - 2\Delta A \text{Re}\langle\tilde{A}_0\tilde{\mathcal{A}}_0\rangle, \end{aligned} \quad (6)$$

$$\begin{aligned} \epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2 &= \Delta B^2 + \Delta\mathcal{B}^2 - 2 \text{Re}\langle(B - \langle B\rangle)(\mathcal{B} - \langle\mathcal{B}\rangle)\rangle, \\ &= \Delta B^2 + \Delta\mathcal{B}^2 - 2\Delta B \text{Re}\langle\tilde{B}_0\tilde{\mathcal{B}}_0\rangle, \end{aligned} \quad (7)$$

where Re indicates a real part. Using the fact that (from the Cauchy-Schwartz inequality) $|\langle(A - \langle A\rangle)(\mathcal{A} - \langle\mathcal{A}\rangle)\rangle| \leq \Delta A \Delta\mathcal{A}$, and similarly for B and \mathcal{B} , it follows in particular that⁴

$$(\Delta A - \Delta\mathcal{A})^2 \leq \epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2 \leq (\Delta A + \Delta\mathcal{A})^2, \quad (8)$$

$$(\Delta B - \Delta\mathcal{B})^2 \leq \epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2 \leq (\Delta B + \Delta\mathcal{B})^2. \quad (9)$$

Note that while the above framework is presented for the scenario of approximate joint measurements, the case of measurement disturbance—where an approximate measurement of A disturbs a subsequent measurement of B —can also be cast into the same framework [6,32]. In that case the *inaccuracy* (root-mean-square error) $\epsilon_{\mathcal{B}}$ is interpreted as the *disturbance* $\eta_{\mathcal{B}}$ on B , with the same definition (5). All the error-trade-off relations we shall derive, which bound the possible values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$, hence also hold as a particular case for $(\epsilon_{\mathcal{A}}, \eta_{\mathcal{B}})$ and can thus also be interpreted as error-disturbance relations. An important difference between the two scenarios, however, is that in the measurement-disturbance case \mathcal{B} is forced to have the same spectrum as B ; this may in general impose stronger constraints, leading to stronger error-disturbance relations [16] (see Sec. III E).

C. Fixed versus variable parameters

As defined above, the parameters $\Delta A, \Delta B, \langle A\rangle, \langle B\rangle$, and \tilde{C}_{AB} depend on the given state $|\psi\rangle$ and the given observables A and B under consideration. These will be considered fixed throughout the paper.

¹Here and throughout the paper, the notation $\langle\cdot\rangle$ stands for $\langle\psi|\cdot|\psi\rangle$ or $\langle\psi, \xi|\cdot|\psi, \xi\rangle$, depending on the context (or $\text{Tr}[\cdot\rho]$, $\text{Tr}[\cdot(\rho \otimes |\xi\rangle\langle\xi|)]$ in Sec. VII). Furthermore, for simplicity of notation, $A - \langle A\rangle$ for instance stands for $A - \langle A\rangle\mathbb{1}_{\mathcal{H}}$, where $\mathbb{1}_{\mathcal{H}}$ is the identity operator on \mathcal{H} .

²If $\Delta A \Delta B = 0$ our problem becomes trivial, as one of the two observables then always gives the same outcome for $|\psi\rangle$, and does not need to be actually measured; a perfect joint “measurement” of both observables on $|\psi\rangle$ can then be performed by simply outputting the expected outcome for that observable, and measuring the other one. Note that, in such a case, $C_{AB} = 0$.

³Another way to present the general case is to consider a positive operator valued measure (POVM) instead of projective measurements \mathcal{A} and \mathcal{B} ; see e.g. the supporting information of Ref. [16] for a detailed treatment from this perspective.

⁴Noting that $\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2 = \Delta[\mathcal{A} - A]^2$ and $\epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2 = \Delta[\mathcal{B} - B]^2$, Eqs. (8) and (9) are in fact nothing but triangle inequalities for variances.

On the other hand, the parameters $\Delta\mathcal{A}, \Delta\mathcal{B}, \delta_{\mathcal{A}}, \delta_{\mathcal{B}}$ (or equivalently $\langle\mathcal{A}\rangle, \langle\mathcal{B}\rangle, \epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$) depend on the particular choice of approximation strategy—namely, on \mathcal{A} and \mathcal{B} . These will be considered as variable parameters. Depending on the situation under study, we shall consider cases where their values are specified—e.g., the values of $\Delta\mathcal{A}$ and $\Delta\mathcal{B}$ are specified in Sec. III B below, and the error-trade-off relation derived there holds when the approximations indeed give the specified values; similarly, $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ are assumed to be given in Sec. III C—or we shall let them free—in which case these can be optimized to minimize for instance the values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$.

III. ERROR-TRADE-OFF RELATIONS

We derive in this section different error-trade-off relations that restrict the allowed values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$, as a function of different parameters—namely, whether or not one may specify the values of $\Delta\mathcal{A}, \Delta\mathcal{B}$ and/or $\delta_{\mathcal{A}}, \delta_{\mathcal{B}}$ —and whether or not an additional assumption—the “same-spectrum assumption”; see Sec. III E below—is imposed. All these relations will be shown to follow from our first relation (12) below.

A. Error-trade-off relation for specified values of $\Delta\mathcal{A}, \delta_{\mathcal{A}}, \Delta\mathcal{B}$, and $\delta_{\mathcal{B}}$

The first relation we derive holds for the case where the standard deviations $\Delta\mathcal{A}, \Delta\mathcal{B}$ and the biases $\delta_{\mathcal{A}}, \delta_{\mathcal{B}}$ are specified. We assume that $\Delta\mathcal{A}, \Delta\mathcal{B} \neq 0$, and define

$$f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}) = \sqrt{1 - \left(\frac{\Delta\mathcal{A}^2 + \Delta\mathcal{A}^2 - (\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2)}{2\Delta\mathcal{A}\Delta\mathcal{A}} \right)^2}, \quad (10)$$

$$f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}) = \sqrt{1 - \left(\frac{\Delta\mathcal{B}^2 + \Delta\mathcal{B}^2 - (\epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2)}{2\Delta\mathcal{B}\Delta\mathcal{B}} \right)^2}. \quad (11)$$

Note that Eqs. (8) and (9) ensure for instance that $|\frac{\Delta\mathcal{A}^2 + \Delta\mathcal{A}^2 - (\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2)}{2\Delta\mathcal{A}\Delta\mathcal{A}}| \leq 1$, and that the above square roots are real.

Our first error-trade-off relation states that for any given values of $\Delta\mathcal{A} > 0, \delta_{\mathcal{A}}, \Delta\mathcal{B} > 0$, and $\delta_{\mathcal{B}}$ (and whatever the values of $\Delta\mathcal{A}, \Delta\mathcal{B}$ and \tilde{C}_{AB} , which—we recall—are fixed parameters in our study once A, B and $|\psi\rangle$ are fixed), the inaccuracies $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ are bound to satisfy

$$f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}})^2 + f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}})^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}) f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}) \geq \tilde{C}_{AB}^2. \quad (12)$$

The proof follows similar lines as the proofs of Ref. [16]. It makes use of the following geometric Lemma, introduced and proven previously in [16].

Lemma: Let \hat{a}, \hat{b} be two unit vectors of a Euclidean space \mathcal{E} , and let us denote by $\chi = \hat{a} \cdot \hat{b}$ their scalar product. For any two orthogonal unit vectors \hat{x} and \hat{y} of \mathcal{E} , defining $a_{\perp} = \sqrt{1 - (\hat{a} \cdot \hat{x})^2}$ and $b_{\perp} = \sqrt{1 - (\hat{b} \cdot \hat{y})^2}$, one has

$$a_{\perp}^2 + b_{\perp}^2 + 2\sqrt{1 - \chi^2} a_{\perp} b_{\perp} \geq \chi^2. \quad (13)$$

[Note furthermore that a necessary condition for inequality (13) to be saturated is that $\hat{a}, \hat{b}, \hat{x}$, and \hat{y} are coplanar [16].]

Proof of Eq. (12). Assuming $\Delta\mathcal{A}, \Delta\mathcal{B} > 0$, let us define $\tilde{\mathcal{A}}_0 = [\mathcal{A} - \langle\mathcal{A}\rangle]/\Delta\mathcal{A}$, $\tilde{\mathcal{B}}_0 = [\mathcal{B} - \langle\mathcal{B}\rangle]/\Delta\mathcal{B}$, and the (normal-

ized) ket vectors

$$\begin{aligned} |a\rangle &= \tilde{\mathcal{A}}_0 |\psi, \xi\rangle, & |b\rangle &= \tilde{\mathcal{B}}_0 |\psi, \xi\rangle, \\ |x\rangle &= \tilde{\mathcal{A}}_0 |\psi, \xi\rangle, & |y\rangle &= \tilde{\mathcal{B}}_0 |\psi, \xi\rangle \end{aligned}$$

(where, as previously, $\tilde{\mathcal{A}}_0$ and $\tilde{\mathcal{B}}_0$ are used as shorthand notations for $\tilde{\mathcal{A}}_0 \otimes \mathbb{1}_{\mathcal{K}}$ and $\tilde{\mathcal{B}}_0 \otimes \mathbb{1}_{\mathcal{K}}$). By writing these vectors in any orthonormal basis of $\mathcal{H} \otimes \mathcal{K}$ (e.g., the common eigenbasis of \mathcal{A} and \mathcal{B}), one can define the following real vectors:

$$\begin{aligned} \hat{a} &= \begin{pmatrix} \text{Re}|a\rangle \\ \text{Im}|a\rangle \end{pmatrix}, & \hat{b} &= \begin{pmatrix} \text{Im}|b\rangle \\ -\text{Re}|b\rangle \end{pmatrix}, \\ \hat{x} &= \begin{pmatrix} \text{Re}|x\rangle \\ \text{Im}|x\rangle \end{pmatrix}, & \hat{y} &= \begin{pmatrix} \text{Im}|y\rangle \\ -\text{Re}|y\rangle \end{pmatrix}. \end{aligned}$$

One then has

$$\begin{aligned} \|\hat{a}\|^2 &= (\text{Re}|a\rangle)^{\top} \cdot (\text{Re}|a\rangle) + (\text{Im}|a\rangle)^{\top} \cdot (\text{Im}|a\rangle) = \langle a|a\rangle = 1, \\ \|\hat{b}\|^2 &= \|\hat{x}\|^2 = \|\hat{y}\|^2 = 1, \end{aligned}$$

$$\begin{aligned} \hat{a} \cdot \hat{b} &= (\text{Re}|a\rangle)^{\top} \cdot (\text{Im}|b\rangle) - (\text{Im}|a\rangle)^{\top} \cdot (\text{Re}|b\rangle) = \text{Im}\langle a|b\rangle \\ &= \text{Im}\langle \tilde{\mathcal{A}}_0 \tilde{\mathcal{B}}_0 \rangle = \tilde{C}_{AB}, \end{aligned}$$

$$\hat{x} \cdot \hat{y} = \text{Im}\langle \tilde{\mathcal{A}}_0 \tilde{\mathcal{B}}_0 \rangle = \frac{1}{2i} \frac{\langle \psi, \xi | [\mathcal{A}, \mathcal{B}] | \psi, \xi \rangle}{\Delta\mathcal{A}\Delta\mathcal{B}} = 0.$$

Hence the real vectors $\hat{a}, \hat{b}, \hat{x}, \hat{y}$ satisfy the assumptions of the geometric Lemma above, with $\chi = \tilde{C}_{AB}$. Furthermore,

$$\begin{aligned} \hat{a} \cdot \hat{x} &= (\text{Re}|a\rangle)^{\top} \cdot (\text{Re}|x\rangle) + (\text{Im}|a\rangle)^{\top} \cdot (\text{Im}|x\rangle) = \text{Re}\langle a|x\rangle \\ &= \frac{\text{Re}(\langle A - \langle A \rangle \rangle (\mathcal{A} - \langle \mathcal{A} \rangle))}{\Delta\mathcal{A}\Delta\mathcal{A}} = \frac{\Delta\mathcal{A}^2 + \Delta\mathcal{A}^2 - (\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2)}{2\Delta\mathcal{A}\Delta\mathcal{A}}, \end{aligned}$$

where the last equality follows from Eq. (6), and thus

$$a_{\perp} = \sqrt{1 - (\hat{a} \cdot \hat{x})^2} = f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}).$$

Similarly, one finds that $b_{\perp} = f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}})$. Equation (12) then directly follows from inequality (13) of the Lemma. ■

Relation (12) lower bounds the possible values of $(f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}), f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}))$ for some specified values of $\Delta\mathcal{A}, \delta_{\mathcal{A}}, \Delta\mathcal{B}$, and $\delta_{\mathcal{B}}$. Note that $f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}})$ increases with $\epsilon_{\mathcal{A}}$ for $\epsilon_{\mathcal{A}}^2 \leq \Delta\mathcal{A}^2 + \Delta\mathcal{A}^2 + \delta_{\mathcal{A}}^2$ and decreases for $\epsilon_{\mathcal{A}}^2 \geq \Delta\mathcal{A}^2 + \Delta\mathcal{A}^2 + \delta_{\mathcal{A}}^2$ [and similarly for $f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}})$]. Hence, relation (12) bounds $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$ from both below and above. Note furthermore that due to (8) and (9), there are absolute lower and upper bounds on $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$; for instance, if $\Delta\mathcal{A}$ and $\delta_{\mathcal{A}}$ are such that $(\Delta\mathcal{A} - \Delta\mathcal{A})^2 + \delta_{\mathcal{A}}^2 > 0$, then whatever $\epsilon_{\mathcal{B}}, \epsilon_{\mathcal{A}}$ cannot be zero.

In the case where $\Delta\mathcal{A} = 0$, one finds [e.g., from (8)] that $\epsilon_{\mathcal{A}}^2 = \Delta\mathcal{A}^2 + \delta_{\mathcal{A}}^2$ is fixed, and $\epsilon_{\mathcal{B}}$ is then only⁵ bounded by Eq. (9). The constraint on $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ thus writes

$$\begin{aligned} \epsilon_{\mathcal{A}}^2 &= \Delta\mathcal{A}^2 + \delta_{\mathcal{A}}^2, \\ (\Delta\mathcal{B} - \Delta\mathcal{B})^2 + \delta_{\mathcal{B}}^2 &\leq \epsilon_{\mathcal{B}}^2 \leq (\Delta\mathcal{B} + \Delta\mathcal{B})^2 + \delta_{\mathcal{B}}^2. \end{aligned} \quad (14)$$

⁵Note that when $\Delta\mathcal{A} = 0$, \mathcal{A} can be taken to be proportional to the identity (it then does not actually need to be measured), which commutes with any observable \mathcal{B} ; there is thus no additional constraint on \mathcal{B} .

Similarly, in the case where $\Delta B = 0$,

$$\epsilon_B^2 = \Delta B^2 + \delta_B^2$$

$$(\Delta A - \Delta \mathcal{A})^2 + \delta_A^2 \leq \epsilon_A^2 \leq (\Delta A + \Delta \mathcal{A})^2 + \delta_A^2. \quad (15)$$

These constraints indeed correspond to the limits of the constraint (12) when $\Delta \mathcal{A} \rightarrow 0$ and $\Delta B \rightarrow 0$, respectively.

B. Error-trade-off relation for specified values of $\Delta \mathcal{A}$ and ΔB

None of the previously derived error-trade-off relations [6–8,11,16] involved the biases δ_A or δ_B explicitly. When these parameters are not specified, one can optimize them and derive, from (12), error-trade-off relations which do not include them, as follows.

Note that, when δ_A and δ_B are left free, the only constraints on ϵ_A and ϵ_B that follow from (8) and (9) are $\epsilon_A \geq |\Delta A - \Delta \mathcal{A}|$ and $\epsilon_B \geq |\Delta B - \Delta \mathcal{B}|$ (which ensure in particular that $\frac{\Delta A^2 + \Delta \mathcal{A}^2 - \epsilon_A^2}{2\Delta A \Delta \mathcal{A}} \leq 1$ and $\frac{\Delta B^2 + \Delta \mathcal{B}^2 - \epsilon_B^2}{2\Delta B \Delta \mathcal{B}} \leq 1$): ϵ_A and ϵ_B still have an absolute lower bound, but are not upper bounded any more.

Using the fact that $|\Delta A^2 + \Delta \mathcal{A}^2 - (\epsilon_A^2 - \delta_A^2)| \geq \max[\Delta A^2 + \Delta \mathcal{A}^2 - \epsilon_A^2, 0] \geq 0$, it follows that for all $\Delta \mathcal{A} > 0$, δ_A and ϵ_A satisfying (8),

$$f_{\Delta A, \delta_A}(\epsilon_A) \leq g_{\Delta A}(\epsilon_A), \quad \text{with} \quad (16)$$

$$g_{\Delta A}(\epsilon_A) = \sqrt{1 - \max\left[\frac{\Delta A^2 + \Delta \mathcal{A}^2 - \epsilon_A^2}{2\Delta A \Delta \mathcal{A}}, 0\right]}, \quad (17)$$

and with equality in (16) for $\delta_A^2 = \max[0, \epsilon_A^2 - (\Delta A^2 + \Delta \mathcal{A}^2)]$. Similarly, one can show that for all $\Delta B > 0$, δ_B and ϵ_B satisfying (9),

$$f_{\Delta B, \delta_B}(\epsilon_B) \leq g_{\Delta B}(\epsilon_B), \quad \text{with} \quad (18)$$

$$g_{\Delta B}(\epsilon_B) = \sqrt{1 - \max\left[\frac{\Delta B^2 + \Delta \mathcal{B}^2 - \epsilon_B^2}{2\Delta B \Delta \mathcal{B}}, 0\right]}, \quad (19)$$

with equality in (18) for $\delta_B^2 = \max[0, \epsilon_B^2 - (\Delta B^2 + \Delta \mathcal{B}^2)]$.

Using Eqs. (16) and (18), it follows from our previous relation (12) that for any given values of $\Delta \mathcal{A} > 0$ and $\Delta B > 0$,

$$g_{\Delta A}(\epsilon_A)^2 + g_{\Delta B}(\epsilon_B)^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} g_{\Delta A}(\epsilon_A) g_{\Delta B}(\epsilon_B) \geq \tilde{C}_{AB}^2. \quad (20)$$

This relation lower bounds the possible values of $(g_{\Delta A}(\epsilon_A), g_{\Delta B}(\epsilon_B))$, for some specified values of $\Delta \mathcal{A}$ and ΔB . Now, $g_{\Delta A}(\epsilon_A)$ and $g_{\Delta B}(\epsilon_B)$ only increase with ϵ_A and ϵ_B . Hence the above relation also only bounds the allowed values of (ϵ_A, ϵ_B) from below (with absolute lower bounds of $|\Delta A - \Delta \mathcal{A}|$ and $|\Delta B - \Delta \mathcal{B}|$); as emphasized above, these are no longer bounded from above. This relation can be compared to those of Refs. [8,11], which also involve the parameters $\Delta \mathcal{A}$ and ΔB (see Secs. VIB and VIC below).

In the case where $\Delta \mathcal{A} = 0$ or $\Delta B = 0$, the constraint on (ϵ_A, ϵ_B) is simply

$$\epsilon_A^2 \geq \Delta A^2, \quad \text{or} \quad \epsilon_A^2 \geq (\Delta A - \Delta \mathcal{A})^2, \quad (21)$$

$$\epsilon_B^2 \geq (\Delta B - \Delta \mathcal{B})^2, \quad \text{or} \quad \epsilon_B^2 \geq \Delta B^2,$$

which indeed correspond to the limits of the constraint (20) when $\Delta \mathcal{A} \rightarrow 0$ or $\Delta B \rightarrow 0$, respectively.

C. Error-trade-off relation for specified values of δ_A and δ_B

Instead of relaxing and letting the values of δ_A, δ_B free as above, one may relax the values of $\Delta \mathcal{A}, \Delta B$ and optimize over these to derive, from (12), error-trade-off relations which do not include the latter. Note that when $\Delta \mathcal{A}$ and ΔB are left free, the only constraints on ϵ_A and ϵ_B that follow from (8) and (9) are $\epsilon_A \geq \delta_A$ and $\epsilon_B \geq \delta_B$; as before, ϵ_A and ϵ_B still have an absolute lower bound, but are not upper bounded any more.

By looking for the maximum of $f_{\Delta \mathcal{A}, \delta_A}(\epsilon_A)$ as $\Delta \mathcal{A}$ varies⁶ between $|\Delta A - \sqrt{\epsilon_A^2 - \delta_A^2}|$ and $\Delta A + \sqrt{\epsilon_A^2 - \delta_A^2}$ [so as to satisfy (8)], while δ_A and ϵ_A are kept fixed, one can show that for all $\Delta \mathcal{A} > 0$, δ_A and ϵ_A satisfying (8),

$$f_{\Delta \mathcal{A}, \delta_A}(\epsilon_A) \leq h_{\delta_A}(\epsilon_A), \quad \text{with} \quad (22)$$

$$h_{\delta_A}(\epsilon_A) = \frac{\sqrt{\epsilon_A^2 - \delta_A^2}}{\Delta A}, \quad (23)$$

with equality in (22) if $\epsilon_A^2 - \delta_A^2 \leq \Delta A^2$ and $\Delta \mathcal{A} = \sqrt{\Delta A^2 - (\epsilon_A^2 - \delta_A^2)}$. Note that contrary to $f_{\Delta \mathcal{A}, \delta_A}(\epsilon_A)$ and $g_{\Delta A}(\epsilon_A)$, $h_{\delta_A}(\epsilon_A)$ is well defined when $\Delta \mathcal{A} = 0$, in which case it takes the value 1 [as from (8), $\Delta \mathcal{A} = 0$ implies $\epsilon_A^2 - \delta_A^2 = \Delta A^2$].

Similarly, one can show that for all $\Delta B > 0$, δ_B and ϵ_B satisfying (9),

$$f_{\Delta B, \delta_B}(\epsilon_B) \leq h_{\delta_B}(\epsilon_B), \quad \text{with} \quad (24)$$

$$h_{\delta_B}(\epsilon_B) = \frac{\sqrt{\epsilon_B^2 - \delta_B^2}}{\Delta B}, \quad (25)$$

with equality in (24) if $\epsilon_B^2 - \delta_B^2 \leq \Delta B^2$ and $\Delta \mathcal{B} = \sqrt{\Delta B^2 - (\epsilon_B^2 - \delta_B^2)}$. Again, $h_{\delta_B}(\epsilon_B)$ is well defined when $\Delta \mathcal{B} = 0$, in which case it takes the value 1.

Using Eqs. (22) and (24), it follows from our previous relation (12) that for any given values of δ_A and δ_B ,

$$h_{\delta_A}(\epsilon_A)^2 + h_{\delta_B}(\epsilon_B)^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} h_{\delta_A}(\epsilon_A) h_{\delta_B}(\epsilon_B) \geq \tilde{C}_{AB}^2. \quad (26)$$

(Strictly speaking, this relation has so far been proven in the case where $\Delta \mathcal{A}, \Delta B > 0$; given the above remarks, when $\Delta \mathcal{A} = 0$ or $\Delta B = 0$ the relation also trivially holds.)

Relation (26) lower bounds the possible values of $(h_{\delta_A}(\epsilon_A), h_{\delta_B}(\epsilon_B))$, for some specified values of δ_A and δ_B .

⁶Formally, one could also look for the maximum of $f_{\Delta \mathcal{A}, \delta_A}(\epsilon_A)$ as ΔA varies and of $f_{\Delta B, \delta_B}(\epsilon_B)$ as ΔB varies, and derive a similar relation to (26), with ΔA and ΔB replaced by $\Delta \mathcal{A}$ and $\Delta \mathcal{B}$ in the definitions of $h_{\delta_A}(\epsilon_A)$ and $h_{\delta_B}(\epsilon_B)$ (but not in the definition of \tilde{C}_{AB})—and a similar relation to (29) as well, with $\tilde{\epsilon}_A$ and $\tilde{\epsilon}_B$ replaced by $\epsilon_A/\Delta \mathcal{A}$ and $\epsilon_B/\Delta \mathcal{B}$. However, as explained in Sec. IIC, this is not in the spirit of our approach to vary ΔA and ΔB , and the error-trade-off relation thus obtained would in general not be tight.

Now, as it was the case with $g_{\Delta\mathcal{A}}(\epsilon_{\mathcal{A}})$ and $g_{\Delta\mathcal{B}}(\epsilon_{\mathcal{B}})$, $h_{\delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}})$ and $h_{\delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}})$ only increase with $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$. Hence the above relation also only bounds the allowed values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ from below (with absolute lower bounds $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$); as emphasized above, these are again not bounded from above.

D. General error-trade-off relation for any values of $\Delta\mathcal{A}$, $\delta_{\mathcal{A}}$, $\Delta\mathcal{B}$, or $\delta_{\mathcal{B}}$

From (26), one can now easily derive an error-trade-off relation that involves neither $\Delta\mathcal{A}$, $\Delta\mathcal{B}$, nor $\delta_{\mathcal{A}}$, $\delta_{\mathcal{B}}$, as, e.g., that of Ozawa [6].

It is indeed trivial to note that for all $\delta_{\mathcal{A}}$, $\epsilon_{\mathcal{A}}(\geq \delta_{\mathcal{A}})$, $\delta_{\mathcal{B}}$ and $\epsilon_{\mathcal{B}}(\geq \delta_{\mathcal{B}})$,

$$h_{\delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}) \leq \frac{\epsilon_{\mathcal{A}}}{\Delta\mathcal{A}} \quad \text{and} \quad h_{\delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}) \leq \frac{\epsilon_{\mathcal{B}}}{\Delta\mathcal{B}}, \quad (27)$$

with equality for $\delta_{\mathcal{A}} = 0$ and $\delta_{\mathcal{B}} = 0$. Defining

$$\tilde{\epsilon}_{\mathcal{A}} = \frac{\epsilon_{\mathcal{A}}}{\Delta\mathcal{A}} \quad \text{and} \quad \tilde{\epsilon}_{\mathcal{B}} = \frac{\epsilon_{\mathcal{B}}}{\Delta\mathcal{B}}, \quad (28)$$

it then follows from (26)⁷ that

$$\tilde{\epsilon}_{\mathcal{A}}^2 + \tilde{\epsilon}_{\mathcal{B}}^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} \tilde{\epsilon}_{\mathcal{A}} \tilde{\epsilon}_{\mathcal{B}} \geq \tilde{C}_{AB}^2, \quad (29)$$

which is precisely the error-trade-off relation recently introduced in Ref. [16],⁸ and experimentally tested in [14,15]. This relation lower bounds the possible values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ as a function of the fixed parameters $\Delta\mathcal{A}$, $\Delta\mathcal{B}$, and \tilde{C}_{AB} of the problem, whatever the values of the (variable) parameters $\Delta\mathcal{A}$, $\delta_{\mathcal{A}}$, $\Delta\mathcal{B}$, and $\delta_{\mathcal{B}}$.

Note that contrary to the previous cases, when $\Delta\mathcal{A}$, $\delta_{\mathcal{A}}$, $\Delta\mathcal{B}$, and $\delta_{\mathcal{B}}$ are free to take any values, there are no absolute lower bounds on $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$, which can (individually) take any non-negative value (there are no upper bounds either); for instance, $\epsilon_{\mathcal{A}}$ can always be zero if $\epsilon_{\mathcal{B}}$ is large enough.

E. With binary, ± 1 -valued observables and the same-spectrum assumption

Our previous relations (12), (20), (26), and (29) are valid (and tight, as we will show in Sec. V and in the Appendix) for any state $|\psi\rangle$ of any (finite-dimensional) Hilbert space, any pair of observables A and B , and any commuting approximate observables \mathcal{A} and \mathcal{B} —in particular, we did not assume that \mathcal{A} and \mathcal{B} should have the same spectrum as A and B .

We now consider a particular case, where A and B are assumed to be dichotomic, ± 1 -valued observables, and where the approximations \mathcal{A} and \mathcal{B} are also constrained to be ± 1 valued. Let us recall that this same-spectrum assumption is a natural one for \mathcal{B} in a measurement-disturbance scenario⁹ [6,16]. As

⁷Note that (29) can also be proven to follow from (20), after showing that (for $\Delta\mathcal{A}, \Delta\mathcal{B} > 0$) $g_{\Delta\mathcal{A}}(\epsilon_{\mathcal{A}}) \leq \tilde{\epsilon}_{\mathcal{A}}$ and $g_{\Delta\mathcal{B}}(\epsilon_{\mathcal{B}}) \leq \tilde{\epsilon}_{\mathcal{B}}$.

⁸The proof of relation (29) given in [16] is based on a corollary of the geometric Lemma of Sec. III A and, although similar, is somewhat more direct than the present derivation. However, our approach here allows us to explicitly show how all our relations follow from one another.

⁹By symmetry we also impose the same-spectrum assumption on \mathcal{A} here; it is straightforward to adapt our study to cases where it is imposed on \mathcal{B} only.

mentioned before, this assumption further restricts the possible approximation strategies, and hence the allowed values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ [or of $(\epsilon_{\mathcal{A}}, \eta_{\mathcal{B}})$ in a measurement-disturbance scenario] [16]. We shall indeed derive below stronger error-trade-off (or error-disturbance) relations for ± 1 -valued observables, when the same-spectrum assumption is imposed.

Note that in the case considered here, where $A^2 = \mathcal{A}^2 = B^2 = \mathcal{B}^2 = \mathbb{1}_{\mathcal{H}}$, one has $\Delta A^2 = 1 - \langle A \rangle^2$, $\Delta B^2 = 1 - \langle B \rangle^2$, $\Delta \mathcal{A}^2 = 1 - \langle \mathcal{A} \rangle^2$, $\Delta \mathcal{B}^2 = 1 - \langle \mathcal{B} \rangle^2$ (with $|\langle A \rangle|, |\langle B \rangle|, |\langle \mathcal{A} \rangle|, |\langle \mathcal{B} \rangle| \leq 1$), and Eqs. (8) and (9) imply

$$1 - \langle A \rangle \langle \mathcal{A} \rangle - \Delta A \sqrt{1 - \langle \mathcal{A} \rangle^2} \leq \epsilon_{\mathcal{A}}^2/2 \leq 1 - \langle A \rangle \langle \mathcal{A} \rangle + \Delta A \sqrt{1 - \langle \mathcal{A} \rangle^2}, \quad (30)$$

$$1 - \langle B \rangle \langle \mathcal{B} \rangle - \Delta B \sqrt{1 - \langle \mathcal{B} \rangle^2} \leq \epsilon_{\mathcal{B}}^2/2 \leq 1 - \langle B \rangle \langle \mathcal{B} \rangle + \Delta B \sqrt{1 - \langle \mathcal{B} \rangle^2}. \quad (31)$$

Moreover, the lower bounds above are further lower bounded by zero (attained when $\langle \mathcal{A} \rangle = \langle A \rangle$ and $\langle \mathcal{B} \rangle = \langle B \rangle$), while the upper bounds are further upper bounded by 2 (attained when $\langle \mathcal{A} \rangle = -\langle A \rangle$ and $\langle \mathcal{B} \rangle = -\langle B \rangle$)—which implies in particular that there is here an absolute upper bound of 2 on $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$. Note also that just like ΔA and ΔB , $\langle A \rangle$ and $\langle B \rangle$ are fixed parameters of our problem; the variable parameters in Eqs. (30) and (31), which depend on the particular approximations \mathcal{A} and \mathcal{B} , are $\langle \mathcal{A} \rangle$, $\epsilon_{\mathcal{A}}$, $\langle \mathcal{B} \rangle$, and $\epsilon_{\mathcal{B}}$ (see the discussion in Sec. II C).

1. Relation for ± 1 -valued observables A , B , \mathcal{A} , and \mathcal{B} , and for specified values of $\langle \mathcal{A} \rangle$ and $\langle \mathcal{B} \rangle$

Assuming $\Delta\mathcal{A}, \Delta\mathcal{B} > 0$ —i.e., $|\langle \mathcal{A} \rangle|, |\langle \mathcal{B} \rangle| < 1$ —one has

$$\frac{\Delta A^2 + \Delta \mathcal{A}^2 - (\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2)}{2\Delta A \Delta \mathcal{A}} = \frac{1 - \langle A \rangle \langle \mathcal{A} \rangle - \epsilon_{\mathcal{A}}^2/2}{\Delta A \sqrt{1 - \langle \mathcal{A} \rangle^2}}, \quad (32)$$

$$\frac{\Delta B^2 + \Delta \mathcal{B}^2 - (\epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2)}{2\Delta B \Delta \mathcal{B}} = \frac{1 - \langle B \rangle \langle \mathcal{B} \rangle - \epsilon_{\mathcal{B}}^2/2}{\Delta B \sqrt{1 - \langle \mathcal{B} \rangle^2}}, \quad (33)$$

and $f_{\Delta\mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}) = f_{\langle \mathcal{A} \rangle}(\epsilon_{\mathcal{A}})$, $f_{\Delta\mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}) = f_{\langle \mathcal{B} \rangle}(\epsilon_{\mathcal{B}})$, with

$$f_{\langle \mathcal{A} \rangle}(\epsilon_{\mathcal{A}}) = \sqrt{1 - \left(\frac{1 - \langle A \rangle \langle \mathcal{A} \rangle - \epsilon_{\mathcal{A}}^2/2}{\Delta A \sqrt{1 - \langle \mathcal{A} \rangle^2}} \right)^2}, \quad (34)$$

$$f_{\langle \mathcal{B} \rangle}(\epsilon_{\mathcal{B}}) = \sqrt{1 - \left(\frac{1 - \langle B \rangle \langle \mathcal{B} \rangle - \epsilon_{\mathcal{B}}^2/2}{\Delta B \sqrt{1 - \langle \mathcal{B} \rangle^2}} \right)^2}. \quad (35)$$

[Note that Eqs. (30) and (31) ensure that the square roots above are real.]

Thus our relation (12) implies here that when A and B are ± 1 -valued observables, and \mathcal{A} and \mathcal{B} are restricted to have the same spectrum (± 1) as A and B , the possible values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ are bound to satisfy, for any specified values of $|\langle \mathcal{A} \rangle|, |\langle \mathcal{B} \rangle| < 1$,

$$f_{\langle \mathcal{A} \rangle}(\epsilon_{\mathcal{A}})^2 + f_{\langle \mathcal{B} \rangle}(\epsilon_{\mathcal{B}})^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} f_{\langle \mathcal{A} \rangle}(\epsilon_{\mathcal{A}}) f_{\langle \mathcal{B} \rangle}(\epsilon_{\mathcal{B}}) \geq \tilde{C}_{AB}^2. \quad (36)$$

This relation lower bounds the possible values of $(f_{\langle \mathcal{A} \rangle}(\epsilon_{\mathcal{A}}), f_{\langle \mathcal{B} \rangle}(\epsilon_{\mathcal{B}}))$. As $f_{\langle \mathcal{A} \rangle}(\epsilon_{\mathcal{A}})$ increases with $\epsilon_{\mathcal{A}}$ for

$\epsilon_A^2/2 \leq 1 - \langle A \rangle \langle \mathcal{A} \rangle$ and then decreases for $\epsilon_A^2/2 \geq 1 - \langle A \rangle \langle \mathcal{A} \rangle$ [and similarly for $f_{\langle B \rangle}(\epsilon_B)$], the above relation bounds ϵ_A and ϵ_B from both below and above—with absolute lower and upper bounds given by (30) and (31).

In the case where $\langle \mathcal{A} \rangle = \pm 1$, $\epsilon_A^2/2 = 1 \mp \langle A \rangle$ is fixed and ϵ_B is then only bounded by Eq. (31). Similarly, in the case where $\langle \mathcal{B} \rangle = \pm 1$, $\epsilon_B^2/2 = 1 \mp \langle B \rangle$ is fixed and ϵ_A is only bounded by Eq. (30). These constraints indeed correspond to the limits of the constraint (36) when $\langle \mathcal{A} \rangle \rightarrow \pm 1$ and $\langle \mathcal{B} \rangle \rightarrow \pm 1$, respectively.

2. Relation for ± 1 -valued observables A , B , \mathcal{A} , and \mathcal{B} , valid for any values of $\langle \mathcal{A} \rangle$ and $\langle \mathcal{B} \rangle$

As was done before, we can derive, from Eq. (36), an error-trade-off relation that does not involve $\langle \mathcal{A} \rangle$ and $\langle \mathcal{B} \rangle$.

Optimizing over $\langle \mathcal{A} \rangle$ (with $-1 < \langle \mathcal{A} \rangle < 1$), one can indeed prove that

$$1 - \left(\frac{1 - \langle A \rangle \langle \mathcal{A} \rangle - \epsilon_A^2/2}{\Delta A \sqrt{1 - \langle \mathcal{A} \rangle^2}} \right)^2 \leq \frac{1 - (1 - \epsilon_A^2/2)^2}{\Delta A^2}, \quad (37)$$

so that for all $\langle \mathcal{A} \rangle \neq \pm 1$ and ϵ_A satisfying (30),

$$f_{\langle \mathcal{A} \rangle}(\epsilon_A) \leq k_A(\epsilon_A), \quad \text{with} \quad (38)$$

$$k_A(\epsilon_A) = \frac{\sqrt{1 - (1 - \epsilon_A^2/2)^2}}{\Delta A}, \quad (39)$$

with equality in (38) if $|\langle \mathcal{A} \rangle| < |1 - \epsilon_A^2/2|$ and $\langle \mathcal{A} \rangle = \frac{\langle A \rangle}{1 - \epsilon_A^2/2}$. Contrary to $f_{\langle \mathcal{A} \rangle}(\epsilon_A)$, $k_A(\epsilon_A)$ is well defined for $\langle \mathcal{A} \rangle = \pm 1$, in which case, as $\epsilon_A^2/2 = 1 \mp \langle A \rangle$, it takes the value 1.

Similarly, one finds that for all $\langle \mathcal{B} \rangle \neq \pm 1$ and ϵ_B satisfying (31),

$$f_{\langle \mathcal{B} \rangle}(\epsilon_B) \leq k_B(\epsilon_B), \quad \text{with} \quad (40)$$

$$k_B(\epsilon_B) = \frac{\sqrt{1 - (1 - \epsilon_B^2/2)^2}}{\Delta B}, \quad (41)$$

with equality in (40) if $|\langle \mathcal{B} \rangle| < |1 - \epsilon_B^2/2|$ and $\langle \mathcal{B} \rangle = \frac{\langle B \rangle}{1 - \epsilon_B^2/2}$. Again, $k_B(\epsilon_B)$ is also well defined for $\langle \mathcal{B} \rangle = \pm 1$, in which case it takes the value 1.

Using Eqs. (38) and (40), it follows from our previous relation (36) that when A and B are ± 1 -valued observables, and \mathcal{A} and \mathcal{B} are restricted to have the same spectrum (± 1) as A and B , the following relation must be satisfied:

$$k_A(\epsilon_A)^2 + k_B(\epsilon_B)^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} k_A(\epsilon_A) k_B(\epsilon_B) \geq \tilde{C}_{AB}^2 \quad (42)$$

(which, strictly speaking, has so far been proven in the case where $|\langle \mathcal{A} \rangle|, |\langle \mathcal{B} \rangle| < 1$, but also trivially holds for $|\langle \mathcal{A} \rangle|, |\langle \mathcal{B} \rangle| = 1$). This relation lower bounds the possible values of $(k_A(\epsilon_A), k_B(\epsilon_B))$. As $k_A(\epsilon_A)$ increases with ϵ_A for $\epsilon_A^2/2 \leq 1$ and then decreases [and similarly for $k_B(\epsilon_B)$], the above relation bounds ϵ_A and ϵ_B from both below and above—with a trivial absolute lower bound of zero, and an absolute upper bound of 2 [see the discussion after Eqs. (30) and (31)]. Note that Eq. (42) is strictly more restrictive than

relation (29), for which the same-spectrum assumption was not imposed.

Relation (42) was introduced and experimentally tested in [14], and generalizes a relation first introduced in Ref. [16] for the particular case where $\langle A \rangle = \langle B \rangle = 0$; our approach here allows us to prove it in a more general case, without this restriction.

To finish this section, let us mention that the different cases and assumptions considered in the derivation of all our relations above do not need to be the same for A, \mathcal{A} and B, \mathcal{B} . One can indeed derive “hybrid” relations involving for instance (say) $f_{\Delta A, \delta A}(\epsilon_A)$ and $h_{\delta B}(\epsilon_B)$, or where the same-spectrum assumption is imposed on one observable only—which is indeed relevant in the measurement-disturbance scenario; note, e.g., that the relation involving $\tilde{\epsilon}_A$ and $k_B(\epsilon_B)$ has also been considered and tested experimentally in Ref. [14].

IV. ALTERNATIVE FORM FOR OUR ERROR-TRADE-OFF RELATIONS

Another, equivalent form for the error-trade-off relations derived in the previous section can be given, which may be convenient to use in certain cases. In this alternative form, the parameter C_{AB} (or \tilde{C}_{AB}) related to the commutator of A and B only appears in the right-hand side, as a lower bound—as it is the case for instance in Robertson’s relation (2) [4], or in the error-trade-off relations previously derived by Ozawa [6,7], Hall [8], and Weston *et al.* [11] [Eqs. (89), (92), and (97) in Sec. VI below].

Note first that all our above relations (12), (20), (26), (29), (36), and (42) are of the general form

$$u_A^2 + u_B^2 + 2\sqrt{1 - \tilde{C}_{AB}^2} u_A u_B \geq \tilde{C}_{AB}^2, \quad (43)$$

with $u_A, u_B \geq 0$ (and $\tilde{C}_{AB}^2 \in [0, 1]$). As we show below, this turns out to be equivalent to¹⁰

$$\begin{aligned} u_A^2 + u_B^2 &\geq 1 \quad \text{or} \\ u_A^2 + u_B^2 &\leq 1 \quad \text{and} \\ u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2} &\geq |\tilde{C}_{AB}|. \end{aligned} \quad (44)$$

(Furthermore, the case where Eq. (43) is saturated is equivalent to $[u_A^2 + u_B^2 \leq 1$ and $u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2} = |\tilde{C}_{AB}|$].)

Proof. Consider the following three cases.

(i) If $u_A^2 + u_B^2 \geq 1$, then both Eqs. (43) and (44) trivially hold.

(ii) If $\tilde{C}_{AB}^2 \leq u_A^2 + u_B^2 \leq 1$, then Eq. (43) still trivially holds; on the other hand, note that, for $u_A^2 + u_B^2 \leq 1$ (which implies in particular that both $u_A^2 \leq 1$ and $u_B^2 \leq 1$), one has

¹⁰Note that one can similarly prove that (43) and (44) are also equivalent to

$$\begin{aligned} u_A^2 + u_B^2 &\geq \tilde{C}_{AB}^2 \quad \text{or} \\ u_A^2 + u_B^2 &\leq \tilde{C}_{AB}^2 \quad \text{and} \quad u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2} \geq |\tilde{C}_{AB}| \end{aligned}$$

(and that the case of equality in (43) is also equivalent to $[u_A^2 + u_B^2 \leq \tilde{C}_{AB}^2$ and $u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2} = |\tilde{C}_{AB}|$].)

$\sqrt{1 - u_A^2} \sqrt{1 - u_B^2} \geq u_A u_B$ and hence

$$\begin{aligned} & (u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2})^2 \\ &= u_A^2 + u_B^2 + 2u_A u_B (\sqrt{1 - u_A^2} \sqrt{1 - u_B^2} - u_A u_B) \\ &\geq u_A^2 + u_B^2 \geq \tilde{C}_{AB}^2. \end{aligned}$$

Taking the square root leads to the last inequality in (44).

(iii) If $u_A^2 + u_B^2 < \tilde{C}_{AB}^2$, then Eq. (43) is equivalent to

$$(\tilde{C}_{AB}^2 - u_A^2 - u_B^2)^2 - 4(1 - \tilde{C}_{AB}^2)u_A^2 u_B^2 \leq 0,$$

which can be written as

$$\begin{aligned} & [\tilde{C}_{AB}^2 - (u_A \sqrt{1 - u_B^2} - u_B \sqrt{1 - u_A^2})^2] \\ & \times [\tilde{C}_{AB}^2 - (u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2})^2] \leq 0. \end{aligned}$$

Noting that

$$\begin{aligned} & (u_A \sqrt{1 - u_B^2} - u_B \sqrt{1 - u_A^2})^2 \\ &= u_A^2 + u_B^2 - 2u_A u_B (u_A u_B + \sqrt{1 - u_A^2} \sqrt{1 - u_B^2}) \\ &\leq u_A^2 + u_B^2 < \tilde{C}_{AB}^2, \end{aligned}$$

we conclude from the previous equation that $C_{AB}^2 \leq (u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2})^2$, the square root of which gives again the last inequality in (44).

Taken together, the study of the three cases above shows that Eqs. (43) and (44) are indeed equivalent. [The equality case in Eq. (43) can be analyzed along similar lines as above.] ■

The equivalence between Eqs. (43) and (44) allows one to write all the error-trade-off relations of the previous section in a different form. For instance, our general relation (29) is equivalent to

$$\begin{aligned} & \tilde{\epsilon}_A^2 + \tilde{\epsilon}_B^2 \geq 1 \quad \text{or} \\ & \tilde{\epsilon}_A^2 + \tilde{\epsilon}_B^2 \leq 1 \quad \text{and} \\ & \tilde{\epsilon}_A \sqrt{1 - \tilde{\epsilon}_B^2} + \tilde{\epsilon}_B \sqrt{1 - \tilde{\epsilon}_A^2} \geq |\tilde{C}_{AB}|. \end{aligned} \quad (45)$$

Note that all the alternative relations thus obtained from Eqs. (12), (20), (26), (29), (36), and (42) could also be derived from one another as in the previous section, starting from an alternative formulation of the geometric Lemma used in the proof of Eq. (12)—obtained by replacing (13) by a relation of the form of (44)—and using the fact that when $u_A^2 + u_B^2 \leq 1$, $u_A \sqrt{1 - u_B^2} + u_B \sqrt{1 - u_A^2}$ increases with both u_A and u_B .

V. TIGHTNESS OF OUR ERROR-TRADE-OFF RELATIONS

In this section we prove the tightness of the error-trade-off relations of Sec. III [only in some particular cases for Eqs. (36) and (42)]—i.e., we show that any values of (ϵ_A, ϵ_B) that saturate these relations can be obtained, for some possible choice of \mathcal{A} and \mathcal{B} . For simplicity we present here the proofs for the

case where¹¹ $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$. For our first four relations, the case where $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| < 1$ can be studied along similar lines; as it involves more tedious calculations, we present it in the Appendix.

As emphasized above all these relations are of the general form (43) [or (44), equivalently]. Defining $\phi = \arg \langle \tilde{A}_0 \tilde{B}_0 \rangle \in [-\pi, \pi]$ —such that, in the case here where $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$, $\langle \tilde{A}_0 \tilde{B}_0 \rangle = e^{i\phi}$, and $\tilde{C}_{AB} = \text{Im} \langle \tilde{A}_0 \tilde{B}_0 \rangle = \sin \phi$ —the values of (u_A, u_B) that saturate the relation (43) can be parametrized by

$$(u_A = \left| \sin \left(\frac{\varphi + \phi}{2} \right) \right|, \quad u_B = \left| \sin \left(\frac{\varphi - \phi}{2} \right) \right|), \quad (46)$$

for a varying value of $\varphi \in [-|\phi|, |\phi|]$ if $\cos \phi \geq 0$, or $\varphi \in [|\phi|, 2\pi - |\phi|]$ if $\cos \phi \leq 0$.

In order to show the tightness of our error-trade-off relations, we will prove that the corresponding functions (u_A, u_B) can indeed take these values. In the case where u_A and u_B only increase with ϵ_A and ϵ_B , the corresponding relations [Eqs. (20), (26), and (29)] only lower bound the values of (ϵ_A, ϵ_B) , and proving that the values of Eq. (46) can be reached is sufficient to show that the lower bound is tight. On the other hand, in the case where u_A and u_B increase and then decrease with ϵ_A and ϵ_B , the corresponding relations [Eqs. (12), (36), and (42)] both lower and upper bound the values of (ϵ_A, ϵ_B) ; we will then verify that the desired values of (u_A, u_B) above can be obtained from both a lower and a greater value of ϵ_A and ϵ_B , so as to show the tightness of both the lower and the upper bounds on (ϵ_A, ϵ_B) .

A. Parametrization of \mathcal{A} and \mathcal{B}

We first introduce a particular choice¹² for the approximations \mathcal{A} and \mathcal{B} . It will be sufficient here to use measurements \mathcal{A} and \mathcal{B} acting on \mathcal{H} only, without introducing any ancillary system.

In the case where $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$, $\tilde{A}_0 |\psi\rangle$ and $\tilde{B}_0 |\psi\rangle$ are, up to a phase, the same unit vector, orthogonal to $|\psi\rangle$. Let us define

$$|v_1\rangle = |\psi\rangle, \quad |v_2\rangle = e^{i\phi/2} \tilde{A}_0 |\psi\rangle = e^{-i\phi/2} \tilde{B}_0 |\psi\rangle, \quad (47)$$

so that $\{|v_1\rangle, |v_2\rangle\}$ forms an orthonormal basis of (the two-dimensional) $\text{span}\{|\psi\rangle, \tilde{A}_0 |\psi\rangle, \tilde{B}_0 |\psi\rangle\}$.

For a given $\varphi \in [-|\phi|, |\phi|]$ if $\cos \phi \geq 0$ or $\varphi \in [|\phi|, 2\pi - |\phi|]$ if $\cos \phi \leq 0$, and for a parameter $\theta \in \mathbb{R}$ such that $\cos \theta \sin \theta \neq 0$ we define, with the shorthand notations $c_\theta = \cos \theta$ and $s_\theta = \sin \theta$,

$$|m_1\rangle = c_\theta |v_1\rangle + e^{i\varphi/2} s_\theta |v_2\rangle, \quad (48)$$

$$|m_2\rangle = s_\theta |v_1\rangle - e^{i\varphi/2} c_\theta |v_2\rangle, \quad (49)$$

so that $\{|m_1\rangle, |m_2\rangle\}$ also forms an orthonormal basis of $\text{span}\{|\psi\rangle, \tilde{A}_0 |\psi\rangle, \tilde{B}_0 |\psi\rangle\}$. If the dimension of \mathcal{H} is larger than

¹¹Recall that $\tilde{A}_0 = [A - \langle A \rangle] / \Delta A$, $\tilde{B}_0 = [B - \langle B \rangle] / \Delta B$, that $\tilde{A}_0 |\psi\rangle$ and $\tilde{B}_0 |\psi\rangle$ are unit vectors, and hence that $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| \leq 1$ in general. As noted in [16], $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$ always holds in particular for the case of pure qubit states, where $\mathcal{H} = \mathbb{C}^2$ is two dimensional.

¹²Our parametrization here is somewhat clearer, but equivalent to that used in [16].

2, we complete that basis with other vectors $|m_{k \geq 3}\rangle$ orthogonal to $|m_1\rangle$ and $|m_2\rangle$ (which are hence orthogonal to $|\psi\rangle$).

The basis $\{|m_k\rangle\}$ is chosen to define the common eigenbasis of \mathcal{A} and \mathcal{B} ; denoting by α_k and β_k their eigenvalues, we thus define

$$\mathcal{A} = \sum_k \alpha_k |m_k\rangle \langle m_k|, \quad \mathcal{B} = \sum_k \beta_k |m_k\rangle \langle m_k|. \quad (50)$$

With these definitions, we get

$$\Delta \mathcal{A}^2 = (\alpha_1 - \alpha_2)^2 c_\theta^2 s_\theta^2, \quad \Delta \mathcal{B}^2 = (\beta_1 - \beta_2)^2 c_\theta^2 s_\theta^2, \quad (51)$$

$$\delta_{\mathcal{A}} = \alpha_1 c_\theta^2 + \alpha_2 s_\theta^2 - \langle A \rangle, \quad \delta_{\mathcal{B}} = \beta_1 c_\theta^2 + \beta_2 s_\theta^2 - \langle B \rangle, \quad (52)$$

and

$$\begin{aligned} \text{Re}\langle \tilde{A}_0 \mathcal{A} \rangle &= \sum \alpha_k \text{Re}\langle \psi | \tilde{A}_0 | m_k \rangle \langle m_k | \psi \rangle \\ &= (\alpha_1 - \alpha_2) c_\theta s_\theta \cos\left(\frac{\varphi + \phi}{2}\right), \end{aligned} \quad (53)$$

$$\text{Re}\langle \tilde{B}_0 \mathcal{B} \rangle = (\beta_1 - \beta_2) c_\theta s_\theta \cos\left(\frac{\varphi - \phi}{2}\right), \quad (54)$$

so that Eqs. (6) and (7) give

$$\begin{aligned} \epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2 &= \Delta \mathcal{A}^2 + \Delta \mathcal{A}^2 \\ &\quad - 2\Delta \mathcal{A} (\alpha_1 - \alpha_2) c_\theta s_\theta \cos\left(\frac{\varphi + \phi}{2}\right), \end{aligned} \quad (55)$$

$$\begin{aligned} \epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2 &= \Delta \mathcal{B}^2 + \Delta \mathcal{B}^2 \\ &\quad - 2\Delta \mathcal{B} (\beta_1 - \beta_2) c_\theta s_\theta \cos\left(\frac{\varphi - \phi}{2}\right). \end{aligned} \quad (56)$$

B. Tightness of relation (12)

Relation (12) bounds the possible values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ when $\Delta \mathcal{A}$, $\delta_{\mathcal{A}}$, $\Delta \mathcal{B}$, and $\delta_{\mathcal{B}}$ are specified. Let us define, for these specified values and for some choice of $\tau_\alpha = \pm 1$ and $\tau_\beta = \pm 1$, the eigenvalues of \mathcal{A} and \mathcal{B} (corresponding to the eigenstates $|m_k\rangle$ parametrized above) to be

$$\alpha_1 = \langle A \rangle + \delta_{\mathcal{A}} + \tau_\alpha \frac{s_\theta}{c_\theta} \Delta \mathcal{A}, \quad (57)$$

$$\alpha_2 = \langle A \rangle + \delta_{\mathcal{A}} - \tau_\alpha \frac{c_\theta}{s_\theta} \Delta \mathcal{A}, \quad (58)$$

$$\beta_1 = \langle B \rangle + \delta_{\mathcal{B}} + \tau_\beta \frac{s_\theta}{c_\theta} \Delta \mathcal{B}, \quad (59)$$

$$\beta_2 = \langle B \rangle + \delta_{\mathcal{B}} - \tau_\beta \frac{c_\theta}{s_\theta} \Delta \mathcal{B}. \quad (60)$$

With these definitions, \mathcal{A} and \mathcal{B} obtained as in (50) indeed give the desired values of $\Delta \mathcal{A}$, $\delta_{\mathcal{A}}$, $\Delta \mathcal{B}$, and $\delta_{\mathcal{B}}$ [see Eqs. (51) and (52)].

From Eqs. (55) and (56) we then have, for $\Delta \mathcal{A}, \Delta \mathcal{B} > 0$,

$$\frac{\Delta \mathcal{A}^2 + \Delta \mathcal{A}^2 - (\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2)}{2\Delta \mathcal{A} \Delta \mathcal{A}} = \tau_\alpha \cos\left(\frac{\varphi + \phi}{2}\right), \quad (61)$$

$$\frac{\Delta \mathcal{B}^2 + \Delta \mathcal{B}^2 - (\epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2)}{2\Delta \mathcal{B} \Delta \mathcal{B}} = \tau_\beta \cos\left(\frac{\varphi - \phi}{2}\right) \quad (62)$$

(independent of θ), and from the definitions (10) and (11) we get, as in (46),

$$f_{\Delta \mathcal{A}, \delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}) = \left| \sin\left(\frac{\varphi + \phi}{2}\right) \right|, \quad (63)$$

$$f_{\Delta \mathcal{B}, \delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}) = \left| \sin\left(\frac{\varphi - \phi}{2}\right) \right|, \quad (64)$$

which saturate relation (12). From Eqs. (61) and (62) it appears clearly that the choice of $\tau_\alpha, \tau_\beta = \pm 1$ allows one to saturate either the lower (for $\tau_\alpha = +\text{sgn}[\cos(\frac{\varphi+\phi}{2})]$, $\tau_\beta = +\text{sgn}[\cos(\frac{\varphi-\phi}{2})]$) or the upper (for $\tau_\alpha = -\text{sgn}[\cos(\frac{\varphi+\phi}{2})]$, $\tau_\beta = -\text{sgn}[\cos(\frac{\varphi-\phi}{2})]$) bounds on $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$. This proves the tightness of both the lower and upper bounds imposed by relation (12), for any specified values of $\Delta \mathcal{A}, \Delta \mathcal{B} > 0$ and of $\delta_{\mathcal{A}}, \delta_{\mathcal{B}} \in \mathbb{R}$.

Note that in the case where $\Delta \mathcal{A} = 0$ ($\Delta \mathcal{B} = 0$), the choice $\varphi = \phi$ ($\varphi = -\phi$) in the parametrization of $\{|m_k\rangle\}$, together with the choice of eigenvalues given in (57)–(60), also allows one to saturate the constraints of Eq. (14) [(15)].

C. Tightness of relation (20)

Relation (20) considers the case where $\Delta \mathcal{A}$ and $\Delta \mathcal{B}$ are specified. Let us now define $\tau_\alpha = \text{sgn}[\cos(\frac{\varphi+\phi}{2})]$, $\tau_\beta = \text{sgn}[\cos(\frac{\varphi-\phi}{2})]$ and, for the specified values of $\Delta \mathcal{A}$ and $\Delta \mathcal{B}$,

$$\alpha_1 = \langle A \rangle + \tau_\alpha \frac{s_\theta}{c_\theta} \Delta \mathcal{A}, \quad \alpha_2 = \langle A \rangle - \tau_\alpha \frac{c_\theta}{s_\theta} \Delta \mathcal{A}, \quad (65)$$

$$\beta_1 = \langle B \rangle + \tau_\beta \frac{s_\theta}{c_\theta} \Delta \mathcal{B}, \quad \beta_2 = \langle B \rangle - \tau_\beta \frac{c_\theta}{s_\theta} \Delta \mathcal{B}. \quad (66)$$

That is, we just set $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ to zero (which is their optimal value here) and τ_α, τ_β to $\text{sgn}[\cos(\frac{\varphi \pm \phi}{2})]$ in the previous definitions (57)–(60). With these definitions, \mathcal{A} and \mathcal{B} indeed give the desired values of $\Delta \mathcal{A}$ and $\Delta \mathcal{B}$.

For this choice of eigenvalues, Eqs. (61) and (62) become, in the case $\Delta \mathcal{A}, \Delta \mathcal{B} > 0$,

$$\frac{\Delta \mathcal{A}^2 + \Delta \mathcal{A}^2 - \epsilon_{\mathcal{A}}^2}{2\Delta \mathcal{A} \Delta \mathcal{A}} = \left| \cos\left(\frac{\varphi + \phi}{2}\right) \right|, \quad (67)$$

$$\frac{\Delta \mathcal{B}^2 + \Delta \mathcal{B}^2 - \epsilon_{\mathcal{B}}^2}{2\Delta \mathcal{B} \Delta \mathcal{B}} = \left| \cos\left(\frac{\varphi - \phi}{2}\right) \right| \quad (68)$$

(again, independent of θ), and from the definitions (17) and (19) we get, again as in (46),

$$g_{\Delta \mathcal{A}}(\epsilon_{\mathcal{A}}) = \left| \sin\left(\frac{\varphi + \phi}{2}\right) \right|, \quad g_{\Delta \mathcal{B}}(\epsilon_{\mathcal{B}}) = \left| \sin\left(\frac{\varphi - \phi}{2}\right) \right|, \quad (69)$$

which saturate relation (20). This proves the tightness of the lower bound on $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ imposed by relation (20), for any specified values of $\Delta \mathcal{A}, \Delta \mathcal{B} > 0$. [Recall that relation (20) does not impose any upper bound on $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$.]

Note that in the case where $\Delta \mathcal{A} = 0$ or $\Delta \mathcal{B} = 0$, the choice $\varphi = \phi$ or $-\phi$, together with the definitions (65) and (66), also allows one to saturate the constraints of Eq. (21).

D. Tightness of relation (26)

Instead of specifying $\Delta\mathcal{A}$ and $\Delta\mathcal{B}$, relation (26) considers the case where $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ are specified. For these given values, let us now choose

$$\alpha_1 = \langle A \rangle + \delta_{\mathcal{A}} + \Delta A \frac{s_\theta}{c_\theta} \cos\left(\frac{\varphi + \phi}{2}\right), \quad (70)$$

$$\alpha_2 = \langle A \rangle + \delta_{\mathcal{A}} - \Delta A \frac{c_\theta}{s_\theta} \cos\left(\frac{\varphi + \phi}{2}\right), \quad (71)$$

$$\beta_1 = \langle B \rangle + \delta_{\mathcal{B}} + \Delta B \frac{s_\theta}{c_\theta} \cos\left(\frac{\varphi - \phi}{2}\right), \quad (72)$$

$$\beta_2 = \langle B \rangle + \delta_{\mathcal{B}} - \Delta B \frac{c_\theta}{s_\theta} \cos\left(\frac{\varphi - \phi}{2}\right). \quad (73)$$

That is, we set $\Delta\mathcal{A} = \Delta A |\cos(\frac{\varphi+\phi}{2})|$, $\Delta\mathcal{B} = \Delta B |\cos(\frac{\varphi-\phi}{2})|$ (which are their optimal values) and $\tau_\alpha = \text{sgn}[\cos(\frac{\varphi+\phi}{2})]$, $\tau_\beta = \text{sgn}[\cos(\frac{\varphi-\phi}{2})]$ in the definitions (57)–(60). With these definitions, \mathcal{A} and \mathcal{B} indeed give the desired values of $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$.

Using Eqs. (55) and (56) we then find, for the above values,

$$\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2 = \Delta A^2 \sin^2\left(\frac{\varphi + \phi}{2}\right), \quad (74)$$

$$\epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2 = \Delta B^2 \sin^2\left(\frac{\varphi - \phi}{2}\right) \quad (75)$$

(independent of θ), and from the definitions (23) and (25) we get, again as in (46),

$$h_{\delta_{\mathcal{A}}}(\epsilon_{\mathcal{A}}) = \left| \sin\left(\frac{\varphi + \phi}{2}\right) \right|, \quad h_{\delta_{\mathcal{B}}}(\epsilon_{\mathcal{B}}) = \left| \sin\left(\frac{\varphi - \phi}{2}\right) \right|, \quad (76)$$

which saturate relation (26). This proves the tightness of the lower bound on $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ imposed by relation (26), for any specified values of $\delta_{\mathcal{A}}, \delta_{\mathcal{B}}$. [Recall again that relation (26) does not impose any upper bound on $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$.]

E. Tightness of relation (29)

When no values of $\Delta\mathcal{A}$, $\delta_{\mathcal{A}}$, $\Delta\mathcal{B}$, or $\delta_{\mathcal{B}}$ are specified *a priori*, let us define the eigenvalues

$$\alpha_1 = \langle A \rangle + \Delta A \frac{s_\theta}{c_\theta} \cos\left(\frac{\varphi + \phi}{2}\right), \quad (77)$$

$$\alpha_2 = \langle A \rangle - \Delta A \frac{c_\theta}{s_\theta} \cos\left(\frac{\varphi + \phi}{2}\right), \quad (78)$$

$$\beta_1 = \langle B \rangle + \Delta B \frac{s_\theta}{c_\theta} \cos\left(\frac{\varphi - \phi}{2}\right), \quad (79)$$

$$\beta_2 = \langle B \rangle - \Delta B \frac{c_\theta}{s_\theta} \cos\left(\frac{\varphi - \phi}{2}\right). \quad (80)$$

That is, we just set $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$ to zero (which is again their optimal value here) in the previous definitions (70) and (73). Equations (74) and (75) then simply give

$$\tilde{\epsilon}_{\mathcal{A}} = \frac{\epsilon_{\mathcal{A}}}{\Delta A} = \left| \sin\left(\frac{\varphi + \phi}{2}\right) \right|, \quad \tilde{\epsilon}_{\mathcal{B}} = \frac{\epsilon_{\mathcal{B}}}{\Delta B} = \left| \sin\left(\frac{\varphi - \phi}{2}\right) \right| \quad (81)$$

(independent of θ), which saturate relation (29). This proves the tightness of our general error-trade-off relation (29) (which had already been proven in [16]).

Note that, in this case, the above-chosen values of α_k and β_k [Eqs. (77)–(80)] are equal to

$$\alpha_k = \langle A \rangle + \Delta A \text{Re} \frac{\langle m_k | \tilde{A}_0 | \psi \rangle}{\langle m_k | \psi \rangle} = \text{Re} \frac{\langle m_k | A | \psi \rangle}{\langle m_k | \psi \rangle}$$

and

$$\beta_k = \langle B \rangle + \Delta B \text{Re} \frac{\langle m_k | \tilde{B}_0 | \psi \rangle}{\langle m_k | \psi \rangle} = \text{Re} \frac{\langle m_k | B | \psi \rangle}{\langle m_k | \psi \rangle}.$$

These correspond to the “weak values” of A and B in the preselected state $|\psi\rangle$ and the postselected state $|m_k\rangle$ [33], which are indeed optimal to minimize, for a given projection eigenbasis $\{|m_k\rangle\}$, the root-mean-square errors $\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}}$ [8,16]. Furthermore, these optimal choices for the approximate observables \mathcal{A} and \mathcal{B} satisfy $\Delta\mathcal{A}^2 = \Delta A^2 - \epsilon_{\mathcal{A}}^2$ and $\Delta\mathcal{B}^2 = \Delta B^2 - \epsilon_{\mathcal{B}}^2$, as noted in Ref. [8].

F. Tightness of relations (36) and (42) for $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle = 0$ and $\langle A \rangle = \langle B \rangle = 0$ (and $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$)

We now consider the case where A, B, \mathcal{A} , and \mathcal{B} are all ± 1 -valued observables, which are the assumptions under which relations (36) and (42) hold.

Note that our choice of eigenbasis $\{|m_k\rangle\}$ introduced above allows for only two possible projection outcomes when measuring $|\psi\rangle$: $|m_1\rangle$ or $|m_2\rangle$. As the corresponding eigenvalues of \mathcal{A} and \mathcal{B} are assumed here to be ± 1 , this forces the values of $\langle \mathcal{A} \rangle$ and $\langle \mathcal{B} \rangle$, with our choice of $\{|m_k\rangle\}$, to satisfy either $|\langle \mathcal{A} \rangle| = 1$, $|\langle \mathcal{B} \rangle| = 1$, or $|\langle \mathcal{A} \rangle| = |\langle \mathcal{B} \rangle| = |\langle m_1 | \psi \rangle|^2 - |\langle m_2 | \psi \rangle|^2 = |c_\theta^2 - s_\theta^2|$. This restricts the cases that our construction allows us to consider. In fact, we shall now prove the tightness of relations (36) and (42) in even further restricted cases, namely when $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle = 0$ for relation (36), and when $\langle A \rangle = \langle B \rangle = 0$ for relation (42).

Note that, as before, the proof here is presented for the case $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$. Contrary to the previous relations however, we do not provide in the Appendix a similar proof for the case $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| < 1$, and leave that (together with the cases where $\langle \mathcal{A} \rangle, \langle \mathcal{B} \rangle \neq 0$ or $\langle A \rangle, \langle B \rangle \neq 0$) as an open problem.

1. Tightness of relation (36) for $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle = 0$

Let us use the value $\theta = \frac{\pi}{4}$ in the definition of our projection eigenbasis $\{|m_k\rangle\}$ above, and let us define the eigenvalues of \mathcal{A} and \mathcal{B} to be

$$\alpha_1 = -\alpha_2 = \tau_\alpha = \pm 1, \quad \beta_1 = -\beta_2 = \tau_\beta = \pm 1. \quad (82)$$

These give the desired average values $\langle \mathcal{A} \rangle = \langle \mathcal{B} \rangle = 0$ (and $\Delta\mathcal{A} = \Delta\mathcal{B} = 1$). Using Eqs. (55) and (56) (with $\Delta A^2 = 1 - \langle A \rangle^2$, $\Delta B^2 = 1 - \langle B \rangle^2$, and $c_\theta s_\theta = \frac{1}{2}$), we then find

$$\epsilon_{\mathcal{A}}^2/2 = 1 - \tau_\alpha \Delta A \cos\left(\frac{\varphi + \phi}{2}\right), \quad (83)$$

$$\epsilon_{\mathcal{B}}^2/2 = 1 - \tau_\beta \Delta B \cos\left(\frac{\varphi - \phi}{2}\right), \quad (84)$$

and from Eqs. (34) and (35) we get, again as in (46),

$$f_{\langle A \rangle=0}(\epsilon_A) = \sqrt{1 - \left(\frac{1 - \epsilon_A^2/2}{\Delta A}\right)^2} = \left| \sin\left(\frac{\varphi + \phi}{2}\right) \right|, \quad (85)$$

$$f_{\langle B \rangle=0}(\epsilon_B) = \sqrt{1 - \left(\frac{1 - \epsilon_B^2/2}{\Delta B}\right)^2} = \left| \sin\left(\frac{\varphi - \phi}{2}\right) \right|, \quad (86)$$

which saturate relation (36). From Eqs. (83) and (84), it appears clearly that the choice of $\tau_\alpha, \tau_\beta = \pm \text{sgn}[\cos(\frac{\varphi \pm \phi}{2})]$ allows one to saturate either the corresponding lower or the corresponding upper bounds on ϵ_A and ϵ_B . This proves the tightness of relation (36), for the case where $\langle A \rangle = \langle B \rangle = 0$ and $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$. Proving its tightness in the general case,¹³ or otherwise deriving a tighter relation, is left as an open problem.

2. Tightness of relation (42) for $\langle A \rangle = \langle B \rangle = 0$

Assuming now that $\langle A \rangle = \langle B \rangle = 0$, and hence $\Delta A = \Delta B = 1$, the previous choice of α_k and β_k (i.e., of \mathcal{A} and \mathcal{B}) gives, for the definitions of Eqs. (39) and (41) [and using (83) and (84)],

$$k_{A, \Delta A=1}(\epsilon_A) = \sqrt{1 - (1 - \epsilon_A^2/2)^2} = \left| \sin\left(\frac{\varphi + \phi}{2}\right) \right|, \quad (87)$$

$$k_{B, \Delta B=1}(\epsilon_B) = \sqrt{1 - (1 - \epsilon_B^2/2)^2} = \left| \sin\left(\frac{\varphi - \phi}{2}\right) \right|, \quad (88)$$

which saturate relation (42). Again, the choice of $\tau_\alpha, \tau_\beta = \pm \text{sgn}[\cos(\frac{\varphi \pm \phi}{2})]$ allows one to saturate either the corresponding lower or upper bounds on ϵ_A and ϵ_B . This proves the tightness of our relation (42), for the case where $\langle A \rangle = \langle B \rangle = 0$ and $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$ (which had already been proven in [16]). As before, proving its tightness in the general case [13], or otherwise deriving a tighter relation, is left as an open problem.

VI. OZAWA'S [7], HALL'S [8], AND WESTON *et al.*'S [11] ERROR-TRADE-OFF RELATIONS ALL FOLLOW FROM OUR TIGHT RELATIONS

Ozawa [7], Hall [8], and Weston *et al.* [11] have previously also derived universally valid error-trade-off relations that restrain the possible values of (ϵ_A, ϵ_B) for all possible approximate joint measurements. We now review these and show explicitly that they follow from the relations we derived in Sec. III—which are thus stronger than these (nontight) previous relations.

A. Ozawa's relation [7]

Ozawa's error-trade-off relation [7] (first derived in terms of error-disturbance trade-offs [6]) states that

$$\epsilon_A \epsilon_B + \Delta B \epsilon_A + \Delta A \epsilon_B \geq |C_{AB}|, \quad (89)$$

¹³From the discussion above, this would require one to use a different projection eigenbasis for $\{|m_k\rangle\}$ —one that is not restricted to the two-dimensional span $\{|\psi\rangle, \tilde{A}_0|\psi\rangle, \tilde{B}_0|\psi\rangle\}$ —and to introduce in general an ancillary system (or equivalently to consider general POVMs).

or equivalently (when $\Delta A \Delta B > 0$, with $\tilde{\epsilon}_A = \frac{\epsilon_A}{\Delta A}$ and $\tilde{\epsilon}_B = \frac{\epsilon_B}{\Delta B}$ as before)

$$\tilde{\epsilon}_A \tilde{\epsilon}_B + \tilde{\epsilon}_A + \tilde{\epsilon}_B \geq |\tilde{C}_{AB}|. \quad (90)$$

We already showed in Ref. [16] that this relation follows directly from our relation (29). This can also easily be verified using the alternative (equivalent) form of Eq. (45):

$$\begin{aligned} \text{if } \tilde{\epsilon}_A^2 + \tilde{\epsilon}_B^2 \geq 1 : \\ \tilde{\epsilon}_A \tilde{\epsilon}_B + \tilde{\epsilon}_A + \tilde{\epsilon}_B &\geq \tilde{\epsilon}_A + \tilde{\epsilon}_B \geq \sqrt{\tilde{\epsilon}_A^2 + \tilde{\epsilon}_B^2} \geq 1 \geq |\tilde{C}_{AB}|, \\ \text{if } \tilde{\epsilon}_A^2 + \tilde{\epsilon}_B^2 \leq 1 : \\ \tilde{\epsilon}_A \tilde{\epsilon}_B + \tilde{\epsilon}_A + \tilde{\epsilon}_B &\geq \tilde{\epsilon}_A + \tilde{\epsilon}_B \\ &\geq \tilde{\epsilon}_A \sqrt{1 - \tilde{\epsilon}_B^2} + \tilde{\epsilon}_B \sqrt{1 - \tilde{\epsilon}_A^2} \geq |\tilde{C}_{AB}|. \end{aligned} \quad (91)$$

As can be seen above (and as already noted in [16]), Ozawa's relation actually remains valid if one drops the first product term $\tilde{\epsilon}_A \tilde{\epsilon}_B$. Furthermore, his relation (even without the first product term) can only be saturated in the very specific case where $\tilde{\epsilon}_A = 0$ or $\tilde{\epsilon}_B = 0$. On the contrary, as shown in the previous section our relation (29) [or (45), equivalently] can always be saturated, and is thus strictly stronger than Ozawa's.

B. Hall's relation [8]

Independently from Ozawa, Hall [8] derived the relation

$$\epsilon_A \epsilon_B + \Delta B \epsilon_A + \Delta A \epsilon_B \geq |C_{AB}|. \quad (92)$$

Although Hall's relation looks quite similar to Ozawa's, it is in fact fundamentally different. Note indeed that instead of the (fixed) standard deviations $\Delta A, \Delta B$ of the observables A and B , it involves the standard deviations $\Delta \mathcal{A}, \Delta \mathcal{B}$ of the approximations \mathcal{A} and \mathcal{B} . In particular, contrary to Ozawa's relation, one cannot drop the first product term $\epsilon_A \epsilon_B$ in Hall's relation (when $|C_{AB}| > 0$), as it is for instance always possible to choose approximate observables \mathcal{A}, \mathcal{B} such that $\Delta \mathcal{A} = \Delta \mathcal{B} = 0$, and hence $\Delta \mathcal{A} \epsilon_B + \Delta \mathcal{B} \epsilon_A$ cannot be lower bounded by any strictly positive term.

Because Hall's relation involves the parameters $\Delta \mathcal{A}, \Delta \mathcal{B}$ that depend on the particular choice of \mathcal{A} and \mathcal{B} , it cannot—contrary to Ozawa's—be derived from (29). We show below that it follows instead from our relation (20) (which precisely also involves $\Delta \mathcal{A}, \Delta \mathcal{B}$). For that, let us first show more generally that from any relation of the form (43)—or (44), equivalently—with $u_A, u_B \leq 1$, it follows that, for any $X, Y \geq 0$:

$$\begin{aligned} &\sqrt{u_A^2 + (\sqrt{1 - u_A^2} - X)^2} \sqrt{u_B^2 + (\sqrt{1 - u_B^2} - Y)^2} \\ &+ Y \sqrt{u_A^2 + (\sqrt{1 - u_A^2} - X)^2} + X \sqrt{u_B^2 + (\sqrt{1 - u_B^2} - Y)^2} \\ &\geq |\tilde{C}_{AB}|. \end{aligned} \quad (93)$$

Proof. Similarly to Hall's derivation [8], our proof is based on the decomposition of a scalar product $\hat{a} \cdot \hat{b}$ in the form $\hat{a} \cdot \hat{b} = (\hat{a} - \vec{x}) \cdot (\hat{b} - \vec{y}) + \hat{a} \cdot \vec{y} + \vec{x} \cdot \hat{b}$, where \vec{x}, \vec{y} are two orthogonal vectors ($\vec{x} \cdot \vec{y} = 0$), and on the application of the Cauchy-Schwartz (CS) inequality to the three scalar products of the decomposition. Let us indeed consider the following two cases.

(i) Assume first that $u_A^2 + u_B^2 \leq 1$. From the CS inequality, we have

$$\begin{aligned} & \left(\frac{\sqrt{1-u_A^2}-x}{u_A} \right) \left(\frac{u_B}{\sqrt{1-u_B^2}-Y} \right) \\ & \leq \sqrt{u_A^2+(\sqrt{1-u_A^2}-x)^2} \sqrt{u_B^2+(\sqrt{1-u_B^2}-Y)^2}, \\ & \left(\frac{\sqrt{1-u_A^2}-x}{u_A} \right) \begin{pmatrix} 0 \\ Y \end{pmatrix} \leq Y \sqrt{u_A^2+(\sqrt{1-u_A^2}-x)^2}, \\ & \begin{pmatrix} X \\ 0 \end{pmatrix} \left(\frac{u_B}{\sqrt{1-u_B^2}-Y} \right) \leq X \sqrt{u_B^2+(\sqrt{1-u_B^2}-Y)^2}. \end{aligned}$$

By summing the right-hand side (RHS) of the three inequalities above, we get the left-hand side (LHS) of Eq. (93). On the other hand, the sum of the LHS of the three inequalities above gives

$$\left(\frac{\sqrt{1-u_A^2}}{u_A} \right) \left(\frac{u_B}{\sqrt{1-u_B^2}} \right) = u_A \sqrt{1-u_B^2} + u_B \sqrt{1-u_A^2}$$

which, by assumption [from Eq. (44), in the case where $u_A^2 + u_B^2 \leq 1$], is lower bounded by $|\tilde{C}_{AB}|$. Summing the three inequalities above thus leads to Eq. (93).

(ii) Assume now that $u_A^2 + u_B^2 \geq 1$. Following similar calculations as in the previous case, but replacing u_B by $\sqrt{1-u_A^2}$, leads to a similar inequality as Eq. (93), where $u_B^2+(\sqrt{1-u_B^2}-Y)^2$ is replaced by $1-u_A^2+(u_A-Y)^2$ and the RHS is 1 ($\geq |\tilde{C}_{AB}|$). Now, one can easily check that, when $u_A^2 + u_B^2 \geq 1$, $u_B^2+(\sqrt{1-u_B^2}-Y)^2 \geq 1-u_A^2+(u_A-Y)^2$, which allows one to conclude that Eq. (93) still holds. ■

In the case where $\Delta\mathcal{A}, \Delta\mathcal{B} > 0$, relation (20) thus implies Eq. (93) with $u_A = g_{\Delta\mathcal{A}}(\epsilon_A)$ and $u_B = g_{\Delta\mathcal{B}}(\epsilon_B)$; taking $X = \frac{\Delta\mathcal{A}}{\Delta\mathcal{A}}$ and $Y = \frac{\Delta\mathcal{B}}{\Delta\mathcal{B}}$, this leads, after simplification and multiplication by $\Delta\mathcal{A}\Delta\mathcal{B}$, to

$$\epsilon_A^{(\min)} \epsilon_B^{(\min)} + \Delta\mathcal{B} \epsilon_A^{(\min)} + \Delta\mathcal{A} \epsilon_B^{(\min)} \geq |\mathcal{C}_{AB}|, \quad (94)$$

with $\epsilon_A^{(\min)} = \min[\epsilon_A, \sqrt{\Delta\mathcal{A}^2 + \Delta\mathcal{A}^2}]$ and $\epsilon_B^{(\min)} = \min[\epsilon_B, \sqrt{\Delta\mathcal{B}^2 + \Delta\mathcal{B}^2}]$. Using $\epsilon_A \geq \epsilon_A^{(\min)}$ and $\epsilon_B \geq \epsilon_B^{(\min)}$, this in turn implies Hall's relation (92). In the case where $\Delta\mathcal{A} = 0$, using Eq. (21, left) we directly find

$$\begin{aligned} \epsilon_A \epsilon_B + \Delta\mathcal{B} \epsilon_A & \geq \Delta\mathcal{A} (|\Delta\mathcal{B} - \Delta\mathcal{B}| + \Delta\mathcal{B}) \\ & \geq \Delta\mathcal{A} \Delta\mathcal{B} \geq |\mathcal{C}_{AB}|, \end{aligned} \quad (95)$$

which gives again Hall's relation (with $\Delta\mathcal{A} = 0$); the case where $\Delta\mathcal{B} = 0$ is obtained similarly.

The independent use of three CS inequalities in the proof above, which cannot in general all be saturated simultaneously, explains the nonoptimality of Hall's relation, as opposed to our relation (20). Looking at the saturation conditions for the three CS inequalities, one can see that Hall's relation can only be saturated if $\epsilon_A = 0$ or $\epsilon_B = 0$, or if $\Delta\mathcal{A}\Delta\mathcal{B} = 0$, $\epsilon_A = \Delta\mathcal{A} - \Delta\mathcal{A}$, $\epsilon_B = \Delta\mathcal{B} - \Delta\mathcal{B}$, and A, B , and $|\psi\rangle$ saturate Robertson's relation (i.e., $\Delta\mathcal{A}\Delta\mathcal{B} = |\mathcal{C}_{AB}|$, or equivalently $|\tilde{C}_{AB}| = 1$).

Note that other (nontight) inequalities can be derived from our relations of Sec. III and by using Eq. (93), for different values of X and Y . For instance, from Eq. (29)—more precisely,

from a similar (in fact, equivalent) relation to Eq. (29) where $\tilde{\epsilon}_A$ and $\tilde{\epsilon}_B$ are replaced by $\tilde{\epsilon}_A = \min[\tilde{\epsilon}_A, 1]$ and $\tilde{\epsilon}_B = \min[\tilde{\epsilon}_B, 1]$ —and with the choice $X = \sqrt{1-\tilde{\epsilon}_A^2}$ and $Y = \sqrt{1-\tilde{\epsilon}_B^2}$, one obtains

$$\tilde{\epsilon}_A \tilde{\epsilon}_B + \sqrt{1-\tilde{\epsilon}_B^2} \tilde{\epsilon}_A + \sqrt{1-\tilde{\epsilon}_A^2} \tilde{\epsilon}_B \geq |\tilde{\mathcal{C}}_{AB}|, \quad (96)$$

which in turn implies Ozawa's relation (89). The nonoptimality of Ozawa's relation—and in particular its first, unnecessary product term—thus also appears to be due to the independent use of three CS inequalities in its proof [6,7], which cannot in general be saturated simultaneously.

C. Weston *et al.*'s relation [11]

A new error-trade-off relation was recently derived by Weston *et al.* [11], which reads

$$\frac{\Delta\mathcal{B} + \Delta\mathcal{B}}{2} \epsilon_A + \frac{\Delta\mathcal{A} + \Delta\mathcal{A}}{2} \epsilon_B \geq |\mathcal{C}_{AB}|. \quad (97)$$

We will show in a similar manner that this relation can be derived from (29). Let us here first show more generally that from any relation of the form (43)—or (44), equivalently—with $u_A, u_B \leq 1$, it follows that, for any $X, Y \geq 0$,

$$\begin{aligned} & \frac{1+Y}{2} \sqrt{u_A^2+(\sqrt{1-u_A^2}-X)^2} \\ & + \frac{1+X}{2} \sqrt{u_B^2+(\sqrt{1-u_B^2}-Y)^2} \geq |\tilde{\mathcal{C}}_{AB}|. \end{aligned} \quad (98)$$

Proof. The proof here uses a similar trick as the proof of Eq. (93) above, using now (as in Weston *et al.*'s derivation [11]) the decomposition of a scalar product $\hat{a} \cdot \hat{b}$ in the form $\hat{a} \cdot \hat{b} = \frac{1}{2}[(\hat{a}-\vec{x}) \cdot \hat{b} + \hat{a} \cdot (\hat{b}-\vec{y})] + (\hat{a}-\vec{x}) \cdot \vec{y} + \vec{x} \cdot (\hat{b}-\vec{y})$, where \vec{x}, \vec{y} are two orthogonal vectors ($\vec{x} \cdot \vec{y} = 0$), and applying the CS inequality to the four scalar products of the decomposition. We consider again the following two cases.

(i) Assume first that $u_A^2 + u_B^2 \leq 1$. From the CS inequality, we have

$$\begin{aligned} & \left(\frac{\sqrt{1-u_A^2}-x}{u_A} \right) \left(\frac{u_B}{\sqrt{1-u_B^2}} \right) \leq \sqrt{u_A^2+(\sqrt{1-u_A^2}-x)^2}, \\ & \left(\frac{\sqrt{1-u_A^2}}{u_A} \right) \left(\frac{u_B}{\sqrt{1-u_B^2}-Y} \right) \leq \sqrt{u_B^2+(\sqrt{1-u_B^2}-Y)^2}, \\ & \begin{pmatrix} X \\ 0 \end{pmatrix} \left(\frac{u_B}{\sqrt{1-u_B^2}-Y} \right) \leq X \sqrt{u_B^2+(\sqrt{1-u_B^2}-Y)^2}, \\ & \left(\frac{\sqrt{1-u_A^2}-x}{u_A} \right) \begin{pmatrix} 0 \\ Y \end{pmatrix} \leq Y \sqrt{u_A^2+(\sqrt{1-u_A^2}-x)^2}. \end{aligned}$$

By summing the RHS of the four inequalities above and dividing by 2, we get the LHS of Eq. (98). On the other hand, the sum of the LHS of the four inequalities above, divided by 2, gives again

$$\left(\frac{\sqrt{1-u_A^2}}{u_A} \right) \left(\frac{u_B}{\sqrt{1-u_B^2}} \right) = u_A \sqrt{1-u_B^2} + u_B \sqrt{1-u_A^2}$$

which, by assumption [from Eq. (44), in the case where $u_A^2 + u_B^2 \leq 1$], is lower bounded by $|\tilde{C}_{AB}|$. Summing the four inequalities above thus leads to Eq. (98).

(ii) Assume now that $u_A^2 + u_B^2 \geq 1$. Following again similar calculations as in the previous case, but replacing u_B by $\sqrt{1 - u_A^2}$, leads to a similar inequality as Eq. (98), where $u_B^2 + (\sqrt{1 - u_B^2} - Y)^2$ is replaced by $1 - u_A^2 + (u_A - Y)^2$ and the RHS is 1 ($\geq |\tilde{C}_{AB}|$). As previously, using the fact that when $u_A^2 + u_B^2 \geq 1$, $u_B^2 + (\sqrt{1 - u_B^2} - Y)^2 \geq 1 - u_A^2 + (u_A - Y)^2$ allows one to conclude that Eq. (98) still holds. ■

Similar to the previous subsection, in the case where $\Delta A, \Delta B > 0$, relation (20) also implies Eq. (98) with $u_A = g_{\Delta A}(\epsilon_A)$ and $u_B = g_{\Delta B}(\epsilon_B)$; taking again $x = \frac{\Delta A}{\Delta A}$ and $Y = \frac{\Delta B}{\Delta B}$, this leads, after simplification and multiplication by $\Delta A \Delta B$, to

$$\frac{\Delta B + \Delta B}{2} \epsilon_A^{(\min)} + \frac{\Delta A + \Delta A}{2} \epsilon_A^{(\min)} \geq |C_{AB}| \quad (99)$$

(with again $\epsilon_A^{(\min)} = \min[\epsilon_A, \sqrt{\Delta A^2 + \Delta A^2}]$ and $\epsilon_B^{(\min)} = \min[\epsilon_B, \sqrt{\Delta B^2 + \Delta B^2}]$), which in turn implies Weston *et al.*'s relation (97). In the case where $\Delta A = 0$, Eq. (21, left) directly implies

$$\frac{\Delta B + \Delta B}{2} \epsilon_A + \frac{\Delta A}{2} \epsilon_B \geq \frac{\Delta B + \Delta B}{2} \Delta A + \frac{\Delta A}{2} |\Delta B - \Delta B| \geq \Delta A \Delta B \geq |C_{AB}|, \quad (100)$$

which gives again Weston *et al.*'s relation (with $\Delta A = 0$); the case where $\Delta B = 0$ is obtained similarly.

Once again, the independent use of four CS inequalities in the proof above explains the nonoptimality of Weston *et al.*'s relation, as opposed to our relation (20). As before, one can check that Weston *et al.*'s relation can only be saturated if $\epsilon_A = 0$ or $\epsilon_B = 0$, or if $\Delta A \Delta B = 0$, $\epsilon_A = \Delta A - \Delta A$, $\epsilon_B = \Delta B - \Delta B$, and A, B , and $|\psi\rangle$ saturate Robertson's relation.

Note again that one could also derive other (nontight) inequalities from our relations of Sec. III, by using Eq. (98) with different values of x and Y .

VII. CASE OF MIXED STATES

So far we only considered the case where the state $|\psi\rangle \in \mathcal{H}$ under study is pure. In this final section we extend our analysis to mixed states and thus consider instead a density matrix $\rho \in L(\mathcal{H})$ [where $L(\mathcal{H})$ is the space of linear operators acting on the Hilbert space \mathcal{H}], on which the joint measurement of two (incompatible) observables A and B is to be approximated.

All the definitions introduced in Sec. II easily generalize; for instance, we now have $\Delta A = \text{Tr}[(A - \langle A \rangle)^2 \rho]^{1/2}$, $\Delta B = \text{Tr}[(B - \langle B \rangle)^2 \rho]^{1/2}$, $C_{AB} = \frac{1}{2i} \text{Tr}([A, B] \rho)$, and, still assuming $\Delta A, \Delta B > 0$, $\tilde{C}_{AB} = \frac{C_{AB}}{\Delta A \Delta B}$. Moreover, Ozawa's framework for approximate joint measurements also generalizes easily: a joint measurement of A and B can be approximated by the measurement of two compatible observables \mathcal{A} and \mathcal{B} on the system $\rho \otimes |\xi\rangle\langle\xi|$ composed on the state ρ and of an ancillary state $|\xi\rangle\langle\xi| \in L(\mathcal{K})$. Ozawa's inaccuracies

(root-mean-square errors) ϵ_A, ϵ_B are now given by

$$\epsilon_A^2 = \text{Tr}[(\mathcal{A} - A)^2 (\rho \otimes |\xi\rangle\langle\xi|)], \quad (101)$$

$$\epsilon_B^2 = \text{Tr}[(\mathcal{B} - B)^2 (\rho \otimes |\xi\rangle\langle\xi|)] \quad (102)$$

(where, again, A and B stand for simplicity for $A \otimes \mathbb{1}_{\mathcal{K}}$ and $B \otimes \mathbb{1}_{\mathcal{K}}$). Note in particular that the squared inaccuracies ϵ_A^2 and ϵ_B^2 are linear in ρ . The definitions of ΔA , ΔB , δ_A , and δ_B are also trivial to generalize.

It is straightforward to check that all error-trade-off relations derived in Sec. III *still hold* in the case of mixed states: indeed, $\rho \in L(\mathcal{H})$ can always be thought of as the partial trace of a pure state $|\psi\rangle$ in some extended Hilbert space $\mathcal{H} \otimes \mathcal{H}'$. By extending the observables A, B, \mathcal{A} , and \mathcal{B} appropriately in the form $A' = A \otimes \mathbb{1}_{\mathcal{H}'}$, $B' = B \otimes \mathbb{1}_{\mathcal{H}'}$, $\mathcal{A}' = \mathcal{A} \otimes \mathbb{1}_{\mathcal{H}'}$, and $\mathcal{B}' = \mathcal{B} \otimes \mathbb{1}_{\mathcal{H}'}$, we have (with obvious notations)

$$\begin{aligned} \Delta A' &= \Delta A, & \Delta B' &= \Delta B, & \tilde{C}_{A'B'} &= \tilde{C}_{AB}, \\ \epsilon_{A'} &= \epsilon_A, & \epsilon_{B'} &= \epsilon_B, \\ \Delta \mathcal{A}' &= \Delta \mathcal{A}, & \Delta \mathcal{B}' &= \Delta \mathcal{B}, & \delta_{A'} &= \delta_A, & \delta_{B'} &= \delta_B. \end{aligned} \quad (103)$$

Now, the relations of Sec. III, proven for pure states, necessarily hold for the primed quantities above; as the nonprimed quantities are the same, the relations also hold for the latter.

However, the main difference with the case of pure states is that our relations are in general *not tight* for mixed states. The reason why the proofs of Sec. V and of the Appendix fail here is that these assumed that the approximate observables \mathcal{A} and \mathcal{B} could access the whole Hilbert space containing the pure state $|\psi\rangle$ (or, in fact, at least the whole span $\{|\psi\rangle, A_0|\psi\rangle, \tilde{B}_0|\psi\rangle\}$). Now, if the mixed state ρ is the partial trace of a pure state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}'$, and one can only access and measure the state ρ , the possible approximate measurements \mathcal{A} and \mathcal{B} are restricted to act only on \mathcal{H} (and possibly on the ancillary space \mathcal{K}), rather than on $\mathcal{H} \otimes \mathcal{H}'$. As the constructions for \mathcal{A} and \mathcal{B} used in Sec. V and in the Appendix would not be restricted here to the subspace \mathcal{H} (but would act on the space $\mathcal{H} \otimes \mathcal{H}'$ containing the purification $|\psi\rangle$), these cannot in general be used in the case of mixed states.

A. Simple example

Let us illustrate these considerations with the following example. Consider the fully mixed qubit state $\rho = \mathbb{1}_{\mathcal{H}}/2 = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$ (with $\mathcal{H} = \mathbb{C}^2$), and choose the two observables A and B to be the Pauli operators $A = \sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$ and $B = \sigma_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$. For this state and this pair of observables, we find (writing from now on the dependency on the state explicitly)

$$\Delta A(\rho) = \Delta B(\rho) = 1, \quad C_{AB}(\rho) = 0, \quad (104)$$

and hence our error-trade-off relation (29) writes

$$\epsilon_A(\rho)^2 + \epsilon_B(\rho)^2 + 2\epsilon_A(\rho)\epsilon_B(\rho) \geq 0, \quad (105)$$

which is trivial and does not impose any restriction on the possible values of $(\epsilon_A(\rho), \epsilon_B(\rho))$.

Note however that for the states $|0\rangle$ and $|1\rangle$, one still gets $\Delta A(|0\rangle) = \Delta A(|1\rangle) = \Delta B(|0\rangle) = \Delta B(|1\rangle) = 1$, but now $C_{AB}(|0\rangle) = -C_{AB}(|1\rangle) = 1$, and our relation (29) imposes that

$$\epsilon_A(|0\rangle)^2 + \epsilon_B(|0\rangle)^2 \geq 1, \quad \epsilon_A(|1\rangle)^2 + \epsilon_B(|1\rangle)^2 \geq 1. \quad (106)$$

Now, by the linearity of ϵ_A^2 and ϵ_B^2 in the density matrix, and from the decomposition $\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$, we have

$$\epsilon_A(\rho)^2 = \frac{1}{2}[\epsilon_A(|0\rangle)^2 + \epsilon_A(|1\rangle)^2], \quad (107)$$

$$\epsilon_B(\rho)^2 = \frac{1}{2}[\epsilon_B(|0\rangle)^2 + \epsilon_B(|1\rangle)^2], \quad (108)$$

and by summing these relations we find, using (106),

$$\epsilon_A(\rho)^2 + \epsilon_B(\rho)^2 \geq 1. \quad (109)$$

This tells us that the possible values of $(\epsilon_A(\rho), \epsilon_B(\rho))$ are in fact restricted¹⁴—which our relation (29) failed to detect when applied directly [as we first did above in Eq. (105)]. As a matter of fact, the relation (109) is tight: taking, for instance, $\mathcal{A} = \cos \varphi (\cos \varphi \sigma_x + \sin \varphi \sigma_y)$ and $\mathcal{B} = \sin \varphi (\cos \varphi \sigma_x + \sin \varphi \sigma_y)$, for any $\varphi \in \mathbb{R}$, gives $\epsilon_A(\rho)^2 = \sin^2 \varphi$ and $\epsilon_B(\rho)^2 = \cos^2 \varphi$, which saturate (109).

It is instructive to see in this example why the tightness of the (then trivial) relation (29) for any purification of ρ does not imply its tightness for the mixed state (partial trace) ρ . The fully mixed state $\rho = \mathbb{1}_{\mathcal{H}}/2$ can be purified to a maximally entangled state, say $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathcal{H} \otimes \mathcal{H}'$, with $\mathcal{H}' = \mathcal{H} = \mathbb{C}^2$. As described above, the observables $A = \sigma_x$ and $B = \sigma_y$ can be extended to $A' = \sigma_x \otimes \mathbb{1}_{\mathcal{H}'}$ and $B' = \sigma_y \otimes \mathbb{1}_{\mathcal{H}'}$, to be measured on $|\Phi^+\rangle$. According to (29), and as in (105), there is in general no restriction on $(\epsilon_{\mathcal{A}}(|\Phi^+\rangle), \epsilon_{\mathcal{B}}(|\Phi^+\rangle))$ if the approximations have access to the extended Hilbert space; one can indeed get $\epsilon_{\mathcal{A}}(|\Phi^+\rangle) = \epsilon_{\mathcal{B}}(|\Phi^+\rangle) = 0$ by choosing, for instance, $\mathcal{A} = |\Phi^+\rangle\langle\Psi^+| + |\Psi^+\rangle\langle\Phi^+| + |\Psi^-\rangle\langle\Psi^-|$ and $\mathcal{B} = i(|\Phi^+\rangle\langle\Psi^-| - |\Psi^-\rangle\langle\Phi^+| + |\Psi^+\rangle\langle\Psi^-| - |\Psi^-\rangle\langle\Psi^+|)$ [with $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$]. However, the values $\epsilon_{\mathcal{A}}(|\Phi^+\rangle) = \epsilon_{\mathcal{B}}(|\Phi^+\rangle) = 0$ cannot be obtained with any observables \mathcal{A}, \mathcal{B} of the form $\mathcal{A}_{\mathcal{H} \otimes \mathcal{K}} \otimes \mathbb{1}_{\mathcal{H}'}, \mathcal{B}_{\mathcal{H} \otimes \mathcal{K}} \otimes \mathbb{1}_{\mathcal{H}'}$ (and neither can any values such that $\epsilon_{\mathcal{A}}^2 + \epsilon_{\mathcal{B}}^2 < 1$)—as this would otherwise violate the relation (109).

B. Strengthening our error-trade-off relations for mixed states

The previous example shows that although our error-trade-off relations of Sec. III—e.g. (29)—are in general not tight (and may even be trivial) for mixed states, one can still obtain stronger relations by combining them in appropriate ways.

Consider, more generally, any decomposition $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ (with $p_i \geq 0$, $\sum_i p_i = 1$) of an arbitrary mixed state ρ . By linearity of ϵ_A^2 and ϵ_B^2 , one has $\epsilon_A(\rho)^2 = \sum_i p_i \epsilon_A(|\psi_i\rangle)^2$ and $\epsilon_B(\rho)^2 = \sum_i p_i \epsilon_B(|\psi_i\rangle)^2$. Now, the values of each pair $(\epsilon_{\mathcal{A}}(|\psi_i\rangle), \epsilon_{\mathcal{B}}(|\psi_i\rangle))$ are restricted to satisfy in particular the (tight) relation (29); let us denote the set of possible values $(\epsilon_{\mathcal{A}}(|\psi_i\rangle), \epsilon_{\mathcal{B}}(|\psi_i\rangle))$ by $\mathcal{S}_{\mathcal{A}, \mathcal{B}}(|\psi_i\rangle)$. This allows us to state the following necessary condition that the values of $(\epsilon_{\mathcal{A}}(\rho), \epsilon_{\mathcal{B}}(\rho))$ must satisfy, for any such decomposition

of ρ ,

$$\begin{aligned} \forall i, \exists (\epsilon_{\mathcal{A}}^{(i)}, \epsilon_{\mathcal{B}}^{(i)}) \in \mathcal{S}_{\mathcal{A}, \mathcal{B}}(|\psi_i\rangle), \\ \text{s.t. } \epsilon_{\mathcal{A}}(\rho)^2 = \sum_i p_i (\epsilon_{\mathcal{A}}^{(i)})^2, \quad \epsilon_{\mathcal{B}}(\rho)^2 = \sum_i p_i (\epsilon_{\mathcal{B}}^{(i)})^2. \end{aligned} \quad (110)$$

We leave open the problem of finding a systematic way to translate this constraint into a simple relation, and to systematically find the optimal decomposition that leads to a tight relation for $(\epsilon_{\mathcal{A}}(\rho), \epsilon_{\mathcal{B}}(\rho))$ —as we did above in the particular case of the one-qubit fully mixed state.

To finish with, let us mention that the other error-trade-off relations of Sec. III can also be strengthened for mixed states, using similar ideas as above. However, that may in general require one to solve even more tedious constrained optimization problems.

VIII. CONCLUSION

We have derived in this paper a number of error-trade-off relations—Eqs. (12), (20), (26), (29), (36), and (42)—bounding Ozawa's inaccuracies in the approximate joint measurement of two incompatible observables A and B , and bounding the inaccuracy and the disturbance in a measurement-disturbance scenario. These relations are adapted to different cases, where one may specify certain properties of the approximations \mathcal{A} and \mathcal{B} . They were shown to all follow from our first relation (12), which holds for some specified values of the standard deviations $\Delta\mathcal{A}$, $\Delta\mathcal{B}$, and of the biases $\delta_{\mathcal{A}}$, $\delta_{\mathcal{B}}$. As our relations are derived in the general framework of Ozawa, they could be tested with the same experimental setups as those used in Refs. [9–15].

We showed that our relations Eqs. (12), (20), (26), and (29) are all *tight* for pure states, and so are the relations (36) and (42) in some particular cases. In particular, they are stronger than the previously known relations of Ozawa [6,7], Hall [8], and Weston *et al.* [11], which could be explicitly derived from our relations directly. Our study has allowed us to clarify the difference between the similar-looking relations of Ozawa and Hall, as well as with Weston *et al.*'s relation, and to pinpoint precisely why their derivations are nonoptimal. The question of the tightness of our relations for mixed states has also been addressed; we have shown how to combine our (no longer tight) relations and still obtain possibly tight constraints for mixed states. The question of the tightness of error-trade-off or error-disturbance relations is quite relevant indeed: tight relations give not only negative results (by quantifying “what *cannot* be done quantum mechanically”), but also positive results (by specifying “what *can* be done”). In our proofs that our relations are tight, we constructed explicit measurement schemes allowing one to obtain any values of $(\epsilon_{\mathcal{A}}, \epsilon_{\mathcal{B}})$ that saturate our relations.

Our work has been focused on Ozawa's framework and definitions for measurement inaccuracies. As mentioned in the Introduction, however, other approaches have been (and certainly will be) proposed to quantify the measurement uncertainty aspect of Heisenberg's uncertainty principle. In the prospect of possible applications in quantum information science, a promising direction of research is for instance

¹⁴This relation strengthens significantly one given in [27], where it is claimed that no restrictions on $(\epsilon_{\mathcal{A}}(\rho), \epsilon_{\mathcal{B}}(\rho))$, for $\rho = \mathbb{1}_{\mathcal{H}}/2$, could be obtained from previously known error-trade-off relations [including (29)]. We show here that such restrictions can in fact be obtained, indirectly.

the study of *entropic* relations, as recently considered in Refs. [27,31]. It will be interesting to see if any of the techniques used in this paper (e.g., the cascaded derivation of error-trade-off relations from one another, the study of their tightness) could be adapted and used in other contexts, where another approach to quantifying Heisenberg's uncertainty principle is studied.

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APPENDIX: TIGHTNESS OF OUR ERROR-TRADE-OFF RELATIONS (12), (20), (26), AND (29) FOR $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| < 1$

In this appendix we prove that our relations (12), (20), (26), and (29) are tight when $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| < 1$, thus completing the proof of Sec. V in which the case $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| = 1$ was considered. The proof here follows very similar lines to that of the main text, and just involves slightly more tedious calculations. For simplicity we shall not repeat all the details of the proof, but insist on what is different from Sec. V.

Let us again define $\phi = \arg(\tilde{A}_0 \tilde{B}_0) \in [-\pi, \pi]$, and let us now also introduce $r = |\langle \tilde{A}_0 \tilde{B}_0 \rangle|$, with $0 \leq r < 1$ in the case considered here, and $\phi' = \arcsin \tilde{C}_{AB} \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ —such that $\langle \tilde{A}_0 \tilde{B}_0 \rangle = r e^{i\phi}$ and $\tilde{C}_{AB} = \text{Im}(\tilde{A}_0 \tilde{B}_0) = r \sin \phi = \sin \phi'$. The values of (u_A, u_B) that saturate the general relation (43) [or (44), equivalently] can then be parametrized by

$$\left(u_A = \left| \sin \left(\frac{\varphi + \phi'}{2} \right) \right|, \quad u_B = \left| \sin \left(\frac{\varphi - \phi'}{2} \right) \right| \right), \quad (\text{A1})$$

for a varying value of $\varphi \in [-|\phi'|, |\phi'|]$. As in Sec. V, in order to show the tightness of our error-trade-off relations, we will show that the corresponding functions (u_A, u_B) can indeed take these values.

It will be convenient below to use the notations $c_{\phi^{(i)}} = \cos \phi^{(i)}$ and $s_{\phi^{(i)}} = \sin \phi^{(i)}$ (such that $r s_{\phi} = s_{\phi'}$), and to define $q = \frac{c_{\phi}}{c_{\phi'}}$ (note that $c_{\phi'} \geq |c_{\phi}|$ and $c_{\phi'} > 0$, so that $|q| \leq 1$).

1. Parametrization of \mathcal{A} and \mathcal{B}

Let us first introduce a different parametrization¹⁵ for the approximate observables \mathcal{A} and \mathcal{B} , acting on (the now three-dimensional) $\text{span}\{|\psi\rangle, \tilde{A}_0|\psi\rangle, \tilde{B}_0|\psi\rangle\}$.

We first define

$$|v_1\rangle = |\psi\rangle, \quad (\text{A2})$$

$$|v_2\rangle = \frac{e^{i\phi/2} \tilde{A}_0 + e^{-i\phi/2} \tilde{B}_0}{\sqrt{2+2r}} |\psi\rangle, \quad (\text{A3})$$

$$|v_3\rangle = \frac{e^{i\phi/2} \tilde{A}_0 - e^{-i\phi/2} \tilde{B}_0}{\sqrt{2-2r}} |\psi\rangle, \quad (\text{A4})$$

so that $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$ forms an orthonormal basis of $\text{span}\{|\psi\rangle, \tilde{A}_0|\psi\rangle, \tilde{B}_0|\psi\rangle\}$. Note that $\tilde{A}_0|\psi\rangle$ and $\tilde{B}_0|\psi\rangle$ are then given by

$$\tilde{A}_0|\psi\rangle = e^{-i\phi/2} \left(\sqrt{\frac{1+r}{2}} |v_2\rangle + \sqrt{\frac{1-r}{2}} |v_3\rangle \right), \quad (\text{A5})$$

$$\tilde{B}_0|\psi\rangle = e^{i\phi/2} \left(\sqrt{\frac{1+r}{2}} |v_2\rangle - \sqrt{\frac{1-r}{2}} |v_3\rangle \right). \quad (\text{A6})$$

For a given $\varphi \in [-|\phi'|, |\phi'|]$, and for any three parameters $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ such that $\cos \theta_1 \sin \theta_1 \cos \theta_2 \sin \theta_2 \neq 0$, let us define (with $c_j = \cos \theta_j$ and $s_j = \sin \theta_j$ for $j = 1, 2, 3$) the complex parameters

$$\gamma_1 = \sqrt{\frac{1+q}{2}} c_3 e^{i\varphi/2} - i \text{sgn}(s_{\phi}) \sqrt{\frac{1-q}{2}} s_3 e^{-i\varphi/2}, \quad (\text{A7})$$

$$\gamma_2 = \sqrt{\frac{1+q}{2}} s_3 e^{-i\varphi/2} - i \text{sgn}(s_{\phi}) \sqrt{\frac{1-q}{2}} c_3 e^{i\varphi/2}, \quad (\text{A8})$$

such that $|\gamma_1|^2 + |\gamma_2|^2 = 1$, and

$$|m_1\rangle = c_1 |v_1\rangle - s_1 \gamma_1 |v_2\rangle - s_1 \gamma_2 |v_3\rangle,$$

$$|m_2\rangle = s_1 c_2 |v_1\rangle + (c_1 c_2 \gamma_1 + s_2 \gamma_2^*) |v_2\rangle + (c_1 c_2 \gamma_2 - s_2 \gamma_1^*) |v_3\rangle,$$

$$|m_3\rangle = s_1 s_2 |v_1\rangle + (c_1 s_2 \gamma_1 - c_2 \gamma_2^*) |v_2\rangle + (c_1 s_2 \gamma_2 + c_2 \gamma_1^*) |v_3\rangle \quad (\text{A9})$$

(where γ_j^* denotes the complex conjugate of γ_j), so that $\{|m_1\rangle, |m_2\rangle, |m_3\rangle\}$ also forms an orthonormal basis of $\text{span}\{|\psi\rangle, \tilde{A}_0|\psi\rangle, \tilde{B}_0|\psi\rangle\}$. If the dimension of \mathcal{H} is larger than 3, we complete that basis with other vectors $|m_{k \geq 4}\rangle$ orthogonal to $|m_1\rangle, |m_2\rangle$, and $|m_3\rangle$ (which are hence orthogonal to $|\psi\rangle$).

As in Sec. V, the basis $\{|m_k\rangle\}$ is chosen to be the common eigenbasis of \mathcal{A} and \mathcal{B} , defined again through Eq. (50). With this parametrization, we get

$$\begin{aligned} \langle \mathcal{A} \rangle &= \alpha_1 c_1^2 + \alpha_2 s_1^2 c_2^2 + \alpha_3 s_1^2 s_2^2, \quad \delta_{\mathcal{A}} = \langle \mathcal{A} \rangle - \langle A \rangle, \\ \Delta \mathcal{A}^2 &= (\alpha_1 - \langle \mathcal{A} \rangle)^2 c_1^2 + (\alpha_2 - \langle \mathcal{A} \rangle)^2 s_1^2 c_2^2 + (\alpha_3 - \langle \mathcal{A} \rangle)^2 s_1^2 s_2^2, \end{aligned} \quad (\text{A10})$$

$$\begin{aligned} \langle \mathcal{B} \rangle &= \beta_1 c_1^2 + \beta_2 s_1^2 c_2^2 + \beta_3 s_1^2 s_2^2, \quad \delta_{\mathcal{B}} = \langle \mathcal{B} \rangle - \langle B \rangle, \\ \Delta \mathcal{B}^2 &= (\beta_1 - \langle \mathcal{B} \rangle)^2 c_1^2 + (\beta_2 - \langle \mathcal{B} \rangle)^2 s_1^2 c_2^2 + (\beta_3 - \langle \mathcal{B} \rangle)^2 s_1^2 s_2^2. \end{aligned} \quad (\text{A11})$$

2. Calculation of $\epsilon_{\mathcal{A}}$ and $\epsilon_{\mathcal{B}}$

Defining

$$\begin{aligned} F_{\pm} &= \sqrt{\frac{1+q}{2}} \sqrt{\frac{1 \pm r}{2}} \cos \left(\frac{\varphi \pm \phi}{2} \right) \\ &\quad \pm \text{sgn}(s_{\phi}) \sqrt{\frac{1-q}{2}} \sqrt{\frac{1 \mp r}{2}} \sin \left(\frac{\varphi \pm \phi}{2} \right), \end{aligned} \quad (\text{A12})$$

$$\begin{aligned} G_{\pm} &= \pm \sqrt{\frac{1+q}{2}} \sqrt{\frac{1 \pm r}{2}} \cos \left(\frac{\varphi \mp \phi}{2} \right) \\ &\quad - \text{sgn}(s_{\phi}) \sqrt{\frac{1-q}{2}} \sqrt{\frac{1 \mp r}{2}} \sin \left(\frac{\varphi \mp \phi}{2} \right), \end{aligned} \quad (\text{A13})$$

¹⁵Note that the parametrization introduced in Sec. V cannot be used in the case $|\langle \tilde{A}_0 \tilde{B}_0 \rangle| < 1$, as the unit vectors $\tilde{A}_0|\psi\rangle$ and $\tilde{B}_0|\psi\rangle$ are no longer the same (up to a phase). Our parametrization here is again somewhat clearer, but equivalent to that used in [16].

we find, after some calculations,

$$\text{Re}\langle\psi|\tilde{A}_0|m_1\rangle\langle m_1|\psi\rangle = -c_1s_1(c_3F_++s_3F_-), \quad (\text{A14})$$

$$\begin{aligned} \text{Re}\langle\psi|\tilde{A}_0|m_2\rangle\langle m_2|\psi\rangle &= s_1c_2[c_1c_2(c_3F_++s_3F_-) \\ &\quad + s_2(s_3F_+-c_3F_-)], \quad (\text{A15}) \end{aligned}$$

$$\begin{aligned} \text{Re}\langle\psi|\tilde{A}_0|m_3\rangle\langle m_3|\psi\rangle &= s_1s_2[c_1s_2(c_3F_++s_3F_-) \\ &\quad - c_2(s_3F_+-c_3F_-)], \quad (\text{A16}) \end{aligned}$$

and similarly for $\text{Re}\langle\psi|\tilde{B}_0|m_k\rangle\langle m_k|\psi\rangle$, where F_{\pm} are replaced by G_{\pm} .

Note further that

$$\begin{aligned} &(c_3F_++s_3F_-)^2 + (s_3F_+-c_3F_-)^2 \\ &= F_+^2 + F_-^2 \\ &= \frac{1}{2}[1 + (qc_{\phi} + \sqrt{1-q^2}\sqrt{1-r^2}|s_{\phi}|) \cos\varphi - rs_{\phi} \sin\varphi] \\ &= \frac{1}{2}\left[1 + \left(\frac{c_{\phi}^2}{c_{\phi'}} + \sqrt{\frac{c_{\phi'}^2 - c_{\phi}^2}{c_{\phi'}^2}} \sqrt{s_{\phi}^2 - s_{\phi'}^2}\right) \cos\varphi - s_{\phi'} \sin\varphi\right] \\ &= \frac{1}{2}(1 + c_{\phi'} \cos\varphi - s_{\phi'} \sin\varphi) = \cos^2\left(\frac{\varphi + \phi'}{2}\right), \quad (\text{A17}) \end{aligned}$$

and, similarly,

$$(c_3G_++s_3G_-)^2 + (s_3G_+-c_3G_-)^2 = \cos^2\left(\frac{\varphi - \phi'}{2}\right). \quad (\text{A18})$$

This implies that there exist two real parameters θ_a, θ_b such that, with $c_a = \cos\theta_a$, $s_a = \sin\theta_a$, $c_b = \cos\theta_b$, and $s_b = \sin\theta_b$,

$$c_3F_+ + s_3F_- = c_a \cos\left(\frac{\varphi + \phi'}{2}\right), \quad (\text{A19})$$

$$s_3F_+ - c_3F_- = s_a \cos\left(\frac{\varphi + \phi'}{2}\right), \quad (\text{A20})$$

$$c_3G_+ + s_3G_- = c_b \cos\left(\frac{\varphi - \phi'}{2}\right), \quad (\text{A21})$$

$$s_3G_+ - c_3G_- = s_b \cos\left(\frac{\varphi - \phi'}{2}\right). \quad (\text{A22})$$

Equations (6) and (7) (with $\text{Re}\langle\tilde{A}_0\mathcal{A}\rangle = \sum \alpha_k \text{Re}\langle\psi|\tilde{A}_0|m_k\rangle\langle m_k|\psi\rangle$ and $\text{Re}\langle\tilde{B}_0\mathcal{B}\rangle = \sum \beta_k \text{Re}\langle\psi|\tilde{B}_0|m_k\rangle\langle m_k|\psi\rangle$), together with Eqs. (A14)–(A16), then imply

$$\epsilon_{\mathcal{A}}^2 - \delta_{\mathcal{A}}^2 = \Delta A^2 + \Delta \mathcal{A}^2 - 2\Delta A X_{\alpha} \cos\left(\frac{\varphi + \phi'}{2}\right), \quad (\text{A23})$$

$$\epsilon_{\mathcal{B}}^2 - \delta_{\mathcal{B}}^2 = \Delta B^2 + \Delta \mathcal{B}^2 - 2\Delta B Y_{\beta} \cos\left(\frac{\varphi - \phi'}{2}\right), \quad (\text{A24})$$

with

$$\begin{aligned} X_{\alpha} &= s_1[(-\alpha_1 + \alpha_2c_2^2 + \alpha_3s_2^2)c_1c_a + (\alpha_2 - \alpha_3)c_2s_2s_a], \\ Y_{\beta} &= s_1[(-\beta_1 + \beta_2c_2^2 + \beta_3s_2^2)c_1c_b + (\beta_2 - \beta_3)c_2s_2s_b]. \quad (\text{A25}) \end{aligned}$$

3. Tightness of relation (12)

We shall now only sketch the calculations that allow us to prove the tightness of our relations (12), (20), (26), and (29), and refer to Sec. V of the main text for more detailed explanations.

Relation (12) holds for some specified values of $\Delta\mathcal{A}$, $\Delta\mathcal{B}$, $\delta_{\mathcal{A}}$, and $\delta_{\mathcal{B}}$. For these values and for some choice of $\tau_{\alpha}, \tau_{\beta} = \pm 1$, we define the eigenvalues of \mathcal{A} and \mathcal{B} to be

$$\alpha_1 = \langle A \rangle + \delta_{\mathcal{A}} - \tau_{\alpha} \frac{s_1c_a}{c_1} \Delta\mathcal{A}, \quad (\text{A26})$$

$$\alpha_2 = \langle A \rangle + \delta_{\mathcal{A}} + \tau_{\alpha} \frac{c_1c_2c_a + s_2s_a}{s_1c_2} \Delta\mathcal{A}, \quad (\text{A27})$$

$$\alpha_3 = \langle A \rangle + \delta_{\mathcal{A}} + \tau_{\alpha} \frac{c_1s_2c_a - c_2s_a}{s_1s_2} \Delta\mathcal{A}, \quad (\text{A28})$$

$$\beta_1 = \langle B \rangle + \delta_{\mathcal{B}} - \tau_{\beta} \frac{s_1c_b}{c_1} \Delta\mathcal{B}, \quad (\text{A29})$$

$$\beta_2 = \langle B \rangle + \delta_{\mathcal{B}} + \tau_{\beta} \frac{c_1c_2c_b + s_2s_b}{s_1c_2} \Delta\mathcal{B}, \quad (\text{A30})$$

$$\beta_3 = \langle B \rangle + \delta_{\mathcal{B}} + \tau_{\beta} \frac{c_1s_2c_b - c_2s_b}{s_1s_2} \Delta\mathcal{B}. \quad (\text{A31})$$

Note that these are defined such that \mathcal{A} and \mathcal{B} indeed give the desired values of $\Delta\mathcal{A}$, $\Delta\mathcal{B}$, $\delta_{\mathcal{A}}$, $\delta_{\mathcal{B}}$ [see (A10) and (A11)], and furthermore satisfy

$$X_{\alpha} = \tau_{\alpha} \Delta\mathcal{A}, \quad Y_{\beta} = \tau_{\beta} \Delta\mathcal{B}. \quad (\text{A32})$$

Equations (A23) and (A24) then imply that Eqs. (61) and (62) hold again for $\Delta\mathcal{A}, \Delta\mathcal{B} > 0$ (with ϕ replaced by ϕ' , and independent of $\theta_1, \theta_2, \theta_3$), which, as in Sec. VB, leads to the conclusion that relation (12) is tight. In the case where $\Delta\mathcal{A} = 0$ or $\Delta\mathcal{B} = 0$, the above choice of eigenvalues (A26)–(A31) also allows one to saturate the constraints of Eq. (14) or (15).

4. Tightness of relation (20)

In order to prove the tightness of relation (20), which holds for specified values of $\Delta\mathcal{A}$ and $\Delta\mathcal{B}$, one can as in Sec. VC simply set $\delta_{\mathcal{A}} = \delta_{\mathcal{B}} = 0$ and $\tau_{\alpha} = \text{sgn}[\cos(\frac{\varphi + \phi'}{2})]$, $\tau_{\beta} = \text{sgn}[\cos(\frac{\varphi - \phi'}{2})]$ in the previous definitions (A26)–(A31).

These still give the desired values for $\Delta\mathcal{A}$ and $\Delta\mathcal{B}$, and lead now, for $\Delta\mathcal{A}, \Delta\mathcal{B} > 0$, to Eqs. (67) and (68) (with $\phi \rightarrow \phi'$)—which, as in Sec. VC, leads to the conclusion that relation (20) is tight. In the case where $\Delta\mathcal{A} = 0$ or $\Delta\mathcal{B} = 0$, the given choice of eigenvalues also allows one to saturate the constraints of Eq. (21).

5. Tightness of relation (26)

To prove the tightness of relation (26), which holds for specified values of $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$, one can now, as in Sec. VD, set $\Delta\mathcal{A} = \Delta A |\cos(\frac{\varphi + \phi'}{2})|$, $\Delta\mathcal{B} = \Delta B |\cos(\frac{\varphi - \phi'}{2})|$ and $\tau_{\alpha} = \text{sgn}[\cos(\frac{\varphi + \phi'}{2})]$, $\tau_{\beta} = \text{sgn}[\cos(\frac{\varphi - \phi'}{2})]$ in the definitions (A26)–(A31).

With these definitions, \mathcal{A} and \mathcal{B} indeed give the desired values of $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{B}}$, and we get

$$X_{\alpha} = \Delta A \cos\left(\frac{\varphi + \phi'}{2}\right), \quad Y_{\beta} = \Delta B \cos\left(\frac{\varphi - \phi'}{2}\right), \quad (\text{A33})$$

so that Eqs. (A23) and (A24) imply that Eqs. (74) and (75) hold again (with $\phi \rightarrow \phi'$)—which, as in Sec. VD, leads to the conclusion that relation (26) is tight.

6. Tightness of relation (29)

To finally prove the tightness of relation (29), one can set $\delta_{\mathcal{A}} = \delta_{\mathcal{B}} = 0$, $\Delta_{\mathcal{A}} = \Delta A |\cos(\frac{\varphi+\phi'}{2})|$, $\Delta_{\mathcal{B}} = \Delta B |\cos(\frac{\varphi-\phi'}{2})|$, and $\tau_{\alpha} = \text{sgn}[\cos(\frac{\varphi+\phi'}{2})]$, $\tau_{\beta} = \text{sgn}[\cos(\frac{\varphi-\phi'}{2})]$ in the choice of

eigenvalues (A26)–(A31). Note that this amounts to defining $\alpha_k = \text{Re} \frac{\langle m_k | A | \psi \rangle}{\langle m_k | \psi \rangle}$ and $\beta_k = \text{Re} \frac{\langle m_k | B | \psi \rangle}{\langle m_k | \psi \rangle}$, as already noted in Sec. V D. We find that (81) still holds (with $\phi \rightarrow \phi'$), which proves the tightness of relation (29).

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