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THE LOCAL INVARIANT FACTORS OF A PRODUCT OF HOLOMORPHIC MATRIX FUNCTIONS: THE ORDER 4 CASE.

G. Philip A. Thijsse

Let $\lambda^{\alpha_n}|\lambda^{\alpha_{n-1}}|\dots|\lambda^{\alpha_2}|\lambda^{\alpha_1}$, resp. $\lambda^{\beta_n}|\lambda^{\beta_{n-1}}|\dots|\lambda^{\beta_2}|\lambda^{\beta_1}$ be the (given) invariant factors of the square matrices A, resp. B of order n over the ring of germs of holomorphic functions in 0 such that $\det A(\lambda)B(\lambda)\neq 0$, $\lambda\neq 0$. A description of all possible invariant factors $\lambda^{\gamma_n}|\lambda^{\gamma_{n-1}}|\dots|\lambda^{\gamma_2}|\lambda^{\gamma_1}$ of the product $C=AB$ is given in the following cases: (i) β_1 (or $\alpha_1)\leq 2$; (ii) $\beta_3=0$ (or $\alpha_3=0$); (iii) $\alpha_1-\alpha_\ell$, $\beta_1-\beta_m\leq 1$, $\alpha_{\ell+1}=\beta_{m+1}=0$. These results, which hold for arbitrary n, are complemented with a few results leading to the description of all possible exponents $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ for arbitrary $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, $\beta_1,\beta_2,\beta_3,\beta_4$ in the case where the order $n\leq 4$.

INTRODUCTION

In [2] I. Gohberg and M.A. Kaashoek raised the question of describing the (local) invariant factors of the product $C=AB$ of (monic) matrix polynomials A,B in terms of the invariant factors of A,B. At that time such a description was only known for the case where $1,\dots,1$, $\lambda,\dots,\lambda$ are the invariant factors of B (cf. [7]), but soon the description was extended to the case where $1,\dots,1,\lambda^{\beta_1}$ are the invariant factors of B (see [6], Theorem 6 and [8], Proposition 6) and the –much more complicated– case where $1,\dots,1,\lambda^{\alpha_\ell},\dots,\lambda^{\alpha_1}$, $\alpha_1-\alpha_\ell\leq 1$ are the invariant factors of A and $1,\dots,1,\lambda^{\beta_m},\dots,\lambda^{\beta_1}$, $\beta_1-\beta_m\leq 1$ are the invariant factors of B (see [6], Theorem 7). In [8] there was also a complete analysis of the case where the order of the matrices A,B was less than or equal to 3. In [9] the case where $1,\lambda,\lambda^2$ are the only possible invariant factors of B and the case where $1,\dots,1,\lambda^{\beta_2},\lambda^{\beta_1}$ are the invariant factors of B were dealt with, the latter result also covering the order 3 case.
Remarkably enough, a complete description of the exponents $\gamma_1,\dots,\gamma_n$ of the invariant

factors of $C=AB$ in terms of those of the factors A,B –that is, of $\alpha_1,\dots,\alpha_n$, resp. $\beta_1,\dots,\beta_n$– was already obtained, in a quite different setting, by T. Klein in [3],[4]. There necessary and sufficient conditions on the sequence $\gamma_1,\dots,\gamma_n$ in order that $\lambda^{\gamma_n}|\dots|\lambda^{\gamma_1}$ are the invariant factors of $C=AB$ are phrased in terms of the existence of a certain Young Tableau involving the sequences $\alpha_1,\dots,\alpha_n$, resp. $\beta_1,\dots,\beta_n$ of exponents of the invariant factors of A, resp. B. This result will be presented, with a proof adapted to the present setting, in Section 2. Another complete description of the exponents $\gamma_1,\dots,\gamma_n$ –to be called the *partial multiplicities* of $C=AB$– was indicated in [2], and confirmed in [6]: Choose fixed nilpotent matrices $N(A)$, resp. $N(B)$ such that $\lambda^{\alpha_n}|\dots|\lambda^{\alpha_1}$, resp. $\lambda^{\beta_n}|\dots|\lambda^{\beta_1}$ are the invariant factors of $\lambda I-N(A)$, resp. $\lambda I-N(B)$, then all possible sequences $\lambda^{\gamma_n}|\dots|\lambda^{\gamma_1}$ of invariant factors turn up as the invariant factors of

$$\lambda I-\begin{pmatrix} N(A) & X \\ 0 & N(B)\end{pmatrix},$$

where X ranges over all matrices of the appropriate size. This result was used extensively in [6].

A different approach has been tried in [8], [9]: It had already been observed quite early (see, e.g. [7]) that there exist divisibility relations involving the invariant factors $\lambda^{\gamma_i},\lambda^{\alpha_i},\lambda^{\beta_i}$, which –using the partial multiplicities, that is, the exponents of the invariant factors– can be expressed as inequalities, for example,

$$\gamma_{r_1}+\gamma_{r_2}+\dots+\gamma_{r_m}\le\alpha_1+\alpha_2+\dots+\alpha_m+\beta_{r_1}+\beta_{r_2}+\dots+\beta_{r_m},\quad 1\le r_1<r_2<\dots<r_m\le n$$

(see [7]) holds for each product $C=AB$. Now the following approach has been suggested by R.C. Thompson, see [14]:

(a) find a description of all index sets $(r_1,\dots,r_k,\ \ s_1,\dots,s_k,\ \ t_1,\dots,t_k)$ (to be called *index triplets*) such that the inequality

$$\gamma_{r_1}+\gamma_{r_2}+\dots+\gamma_{r_k}\le\alpha_{s_1}+\alpha_{s_2}+\dots+\alpha_{s_k}+\beta_{t_1}+\beta_{t_2}+\dots\beta_{t_k}$$

holds for each product $C=AB$; such inequalities will be called *rules*.

(b) prove that each triplet $(\gamma_1,\dots,\gamma_n,\ \ \alpha_1,\dots,\alpha_n,\ \ \beta_1,\dots,\beta_n)$ of partial multiplicities with $\gamma_1+\dots+\gamma_n=\alpha_1+\dots+\alpha_n+\beta_1+\dots+\beta_n$ and such that all inequalities derived in (a) are met, can be realized as the partial multiplicities of a product $C=AB$.

Of course, in many cases it suffices already that a few of the inequalities from (a) are met in order that $(\gamma_1,\ldots,\gamma_n,\ \alpha_1,\ldots,\alpha_n,\ \beta_1,\ldots,\beta_n)$ can be realized by a product $C=AB$.

In [13] R.C. Thompson obtained certain conditions in terms of Young tableaux in order that an index triplet $(r_1,\ldots,r_k,\ s_1,\ldots s_k,\ t_1,\ldots,t_k)$ generates a rule. Apart from a certain minimality condition it seems that these are, in fact, all triplets generating rules. In [10],[11] this conjecture was confirmed for $k\leq 3$, for $r_k-k\leq 3$ and thus, effectively, for $n\leq 7$. Also in [10],[11] several important "systems" of rules were derived which confirm that the necessary and sufficient conditions which give a full description in the cases $\beta_1\leq 2$, $\beta_3=0$, and $\alpha_1-\alpha_\ell$, $\beta_1-\beta_m\leq 1$, $\alpha_{\ell+1}=\beta_{m+1}=0$ can be derived from the inequalities of type (a) obtained so far.

Since C and its transpose C^T have the same invariant factors it is clear that one may interchange the roles of $(\alpha_1,\ldots,\alpha_n)$ and $(\beta_1,\ldots,\beta_n)$ in all results. Further symmetry properties of the present problem can be derived from the following observation: If $\lambda^{\alpha_n}|\ldots|\lambda^{\alpha_1}$ are the invariant factors of A, then, given $a\geq\alpha_1$, the invariant factors of $\lambda^a A(\lambda)^{-1}$ are $\lambda^{a-\alpha_1}|\ldots|\lambda^{a-\alpha_n}$. Using that $C=AB$ if and only if $\lambda^a B=(\lambda^a A^{-1})C$ one can relate the partial multiplicity sequences $(\gamma_1,\ldots,\gamma_n)$, $(\alpha_1,\ldots,\alpha_n)$, $(\beta_1,\ldots,\beta_n)$ to the sequences $(a+\beta_1,\ldots,a+\beta_n)$, $(a-\alpha_n,\ldots,a-\alpha_1)$, $(\gamma_1,\ldots,\gamma_n)$ and $(a+b-\gamma_n,\ldots,a+b-\gamma_1)$, $(a-\alpha_n,\ldots,a-\alpha_1)$, $(b-\beta_n,\ldots,b-\beta_1)$, where $b\geq\beta_1$. These symmetry results, which are derived in Section 1 after the necessary introductory definitions, are quite useful in order to obtain new theorems from existing ones (e.g., from the description in the case where $\alpha_1-\alpha_\ell$, $\beta_1-\beta_m\leq 1$, $\alpha_{\ell+1}=\beta_{m+1}=0$ one easily obtains a description for the case where $\alpha_1=\alpha_\ell$, $\beta_1=\beta_m$, $\alpha_{\ell+1}$, $\beta_{m+1}\leq 1$) and for limiting the number of special cases in need of proof.
In Section 2 we state and prove the above-mentioned theorem of T. Klein, and we apply it to obtain new proofs for the cases $\beta_1\leq 2$ and $\alpha_1-\alpha_\ell$, $\beta_1-\beta_m\leq 1$, $\alpha_{\ell+1}=\beta_{m+1}=0$. The case $\beta_3=0$ is dealt with in Section 3, where the proof is based on that in [9], Section V.2. In the final Section 4 we combine the results obtained in the previous sections with a reduction technique (which might also be of interest for matrices of higher orders) in order to obtain a full description of the case where A, B and $C=AB$ are of order $n\leq 4$.

Throughout the text the symbol ■ will stand for "end of proof" or "end of example".

1. DEFINITIONS AND AUXILIARY RESULTS

Let the $n\times n$ matrix function $A(\lambda)$ be analytic in a neighbourhood of $0\in\mathbb{C}$, such that $\det A(\lambda)\neq 0$, $\lambda\neq 0$. There exist $n\times n$ matrix functions E,F, analytic in a neighbourhood of 0 such that $\det E(0)\neq 0$, $\det F(0)\neq 0$ and

$$(1.1) \qquad A(\lambda) = E(\lambda)D(\lambda)F(\lambda) = E(\lambda)\begin{pmatrix} \lambda^{\alpha_1} & & & & \\ & \lambda^{\alpha_2} & 0 & & \\ & & \cdot & & \\ & 0 & & \cdot & \\ & & & & \lambda^{\alpha_n} \end{pmatrix} F(\lambda), \ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$$

The matrix function $D(\lambda) = \text{diag}\ (\lambda^{\alpha_1}, \lambda^{\alpha_2}, \ldots, \lambda^{\alpha_n})$ is called the *local Smith-form* of A (at 0), and $\lambda^{\alpha_1}, \lambda^{\alpha_2}, \ldots, \lambda^{\alpha_n}$ are the (local) *invariant factors* of A. The nonnegative integers $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_n \geq 0$ are called the *partial multiplicities* of A (at 0), and sometimes $m_0(A) := \alpha_1 + \alpha_2 + \ldots + \alpha_n$ is called the *total* (*zero*) *multiplicity* of A at 0. In this paper we consider the following problem: Given two sequences $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \geq 0$ of nonnegative integers, what sequences $\gamma_1 \geq \gamma_2 \geq \ldots \gamma_n \geq 0$ can appear as partial multiplicity sequences of a product $C = AB$, where $\alpha_1, \ldots, \alpha_n$, resp. $\beta_1, \ldots, \beta_n$ are the partial multiplicity sequences of A, resp. B?

We introduce some notation: with $\mathcal{B}_n$ we denote the set of all (germs of) $n \times n$-matrix functions that are analytic on a neighbourhood of 0 such that $\det A(\lambda) \neq 0$ for $\lambda \neq 0$. Given two finite sequences $\alpha = (\alpha_i)_{i=1}^n$, $\beta = (\beta_i)_{i=1}^n$ of nonnegative integers such that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$, called *multiplicity sequences* (*of order* n), one defines

$$\Delta_n(\alpha,\beta) = \left\{ \gamma = (\gamma_i)_{i=1}^n \;\middle|\; \begin{array}{l} \gamma \text{ partial multiplicity sequence of } C = AB, \\ \text{where } A, B \in \mathcal{B}_n \text{ have partial multiplicity} \\ \text{sequence } \alpha, \text{ resp. } \beta \end{array} \right\}.$$

Our main problem can thus be summarized as follows:

Given multiplicity sequences α,β of order n, describe $\Delta_n(\alpha,\beta)$

Obviously, the matrices $C, C^T \in \mathcal{B}_n$ have the same partial multiplicities; since $(AB)^T = B^T A^T$ this implies that $\Delta_n(\alpha,\beta) = \Delta_n(\beta,\alpha,)$. If $\alpha_1 \geq \ldots \geq \alpha_n$ are the partial multiplicities of $A \in \mathcal{B}_n$, then for given $a \geq \alpha_1$ the sequence $(a - \alpha)$, defined by $a - \alpha_n \geq a - \alpha_{n-1} \geq \ldots \geq a - \alpha_1$ is the sequence of partial multiplicities of $\lambda^a A(\lambda)^{-1} \in \mathcal{B}_n$; since $\lambda^a B(\lambda) = \lambda^a A(\lambda)^{-1} C(\lambda)$ if $C = AB$, $A, B \in \mathcal{B}_n$, one has that $a + \beta \in \Delta_n(a - \alpha, \gamma)$ if and only if $\gamma \in \Delta_n(\alpha,\beta)$ (here $a + \beta = (a + \beta_i)_{i=1}^n$ is the partial multiplicity sequence of $\lambda^a B(\lambda)$). If α,β are the partial multiplicity sequences of $A, B \in \mathcal{B}_n$, then, choosing $a \geq \alpha_1$, $b \geq \beta_1$, one has $\lambda^{a+b} C(\lambda)^{-1} = \lambda^b B(\lambda)^{-1} \lambda^a A(\lambda)^{-1}$ if $C = AB$; further, if $x \leq \alpha_n$, $y \leq \beta_n$ then $\lambda^{-x} A(\lambda)$, $\lambda^{-y} B(\lambda) \in \mathcal{B}_n$, and $\lambda^{-(x+y)} C(\lambda) = (\lambda^{-x} A(\lambda))(\lambda^{-y} B(\lambda))$ if $C = AB$. Using these and similar relations one obtains

PROPOSITION 1.1. *Let α,β and γ be multiplicity sequences of order n with $c\geq\gamma_1$, $a\geq\alpha_1$, $b\geq\beta_1$. The following statements are equivalent:*

(i)	$\gamma\in\Delta_n(\alpha,\beta)$;	(i)′	$\gamma\in\Delta_n(\beta,\alpha)$
(ii)	$a+\beta\in\Delta_n(\gamma,a-\alpha)$;	(ii)′	$a+\beta\in\Delta_n(a-\alpha,\gamma)$
(ii)″	$b+\alpha\in\Delta_n(\gamma,b-\beta)$;	(ii)‴	$b+\alpha\in\Delta_n(b-\beta,\gamma)$
(iii)	$a+b-\gamma\in\Delta_n(a-\alpha,b-\beta)$;	(iii)′	$a+b-\gamma\in\Delta_n(b-\beta,\ a-\alpha)$
(iv)	$c-\beta\in\Delta_n(c-\gamma,\alpha)$;	(iv)′	$c-\beta\in\Delta_n(\alpha,c-\gamma)$
(iv)″	$c-\alpha\in\Delta_n(c-\gamma,\beta)$;	(iv)‴	$c-\alpha\in\Delta_n(\beta,c-\gamma)$

It is not difficult to find some necessary conditions in order that $\gamma\in\Delta_n(\alpha,\beta)$. If $A\in\mathcal{B}_n$ has the partial multiplicities $\alpha_1\geq\alpha_2\geq\ldots\geq\alpha_n$, then $\lambda^{\alpha_{n+1-k}}\lambda^{\alpha_{n+2-k}}\ldots\lambda^{\alpha_n}$ is the greatest common divisor of all nonzero $k\times k$-minors of $\det A(\lambda)$. This implies that for $k=1,2,\ldots,n$.

$$\alpha_{n+1-k}+\alpha_{n+2-k}+\ldots+\alpha_n=n(\gcd\{|A_k|\,\big|\,|A_k|\neq 0\ k\times k\text{-minor of }\det A(\lambda)\}),\tag{1.2}$$

where $n(f)$ denotes the zero order at 0 of a scalar function f which is analytic at 0. Hence for $\gamma\in\Delta_n(\alpha,\beta)$ one has

$$\gamma_1+\gamma_2+\ldots+\gamma_n=n(\det A(\lambda)B(\lambda))=\alpha_1+\alpha_2+\ldots+\alpha_n+\beta_1+\beta_2+\ldots+\beta_n,\tag{1.3}$$

whereas it follows from the Cauchy–Binet formula that

$$\gamma_{\ell+1}+\gamma_{\ell+2}+\ldots+\gamma_n\geq\alpha_{\ell+1}+\alpha_{\ell+2}+\ldots+\alpha_n+\beta_{\ell+1}+\beta_{\ell+2}+\ldots+\beta_n\tag{1.4}$$

for $\ell=1,2,\ldots,n-1$. Combining (1.3) and (1.4) one obtains

$$\gamma_1+\gamma_2+\ldots+\gamma_\ell\leq\alpha_1+\alpha_2+\ldots+\alpha_\ell+\beta_1+\beta_2+\ldots+\beta_\ell,\qquad \ell=1,2,\ldots,n-1.$$

which is the most obvious of a large class of inequalities of the type

$$\gamma_{r_1}+\gamma_{r_2}+\ldots+\gamma_{r_m}\leq\alpha_{s_1}+\alpha_{s_2}+\ldots+\alpha_{s_m}+\beta_{t_1}+\beta_{t_2}+\ldots+\beta_{t_m}\tag{1.5}$$

which hold for certain index sets $r=\{r_1,\ldots,r_m\}$, $s=\{s_1,\ldots,s_m\}$, $t=\{t_1,\ldots,t_m\}$ with $1\leq r_i,s_i,t_i\leq n$ and $r_i<r_{i+1}$, $s_i<s_{i+1}$, $t_i<t_{i+1}$. We shall call $r\|s|t$ an *index triplet* of

order m, and we say that $r\|s|t$ *generates a rule* if (1.5) holds for each $\gamma\in\Delta_n(\alpha,\beta)$. We define

$$\mathrm{Rul}_m(n)=\left\{r\|s|t \;\middle|\; \begin{array}{l} r\|s|t \text{ is an index triplet of} \\ \text{order } m \text{ which generates a rule} \end{array}\right\}$$

and we set $\mathrm{Rul}(n) = \bigcup_{m=0}^{n}\mathrm{Rul}_m(n)$. Important examples of such rules are the *standard rules* (or *standard inequalities*), generated by $r\|s|t$ with $i+r_i=s_i+t_i$, $i=1,2,\ldots,m$. These rules have been obtained independently by several authors, see e.g. [12], [9] and [11] Example 1.6.(i).

The symmetry properties of $\Delta_n(\alpha,\beta)$ give rise to similar symmetry properties for $\mathrm{Rul}(n)$ (see [11], Proposition 1.1). In order to describe these, we associate with the index set $r=\{r_1,\ldots,r_m\}\subseteq\{1,\ldots,n\}$, $r_i<r_{i+1}$, three further index sets, namely $r^c=\{r_1^c,\ldots,r_{n-m}^c\}=\{1,\ldots,n\}\backslash r$, $r_j^c<r_{j+1}^c$, called the *complement* of r, $n+1-r=\{n+1-r_m,\ldots,n+1-r_1\}$, called the *reflexion* of r, and $\rho=\{\rho_1,\ldots,\rho_{n-m}\}=(n+1-r)^c=n+1-r^c$, called the *inversion* of r. Using these new index sets one can associate with the index triplet $r\|s|t$ eleven other index triplets each of which generates a rule if and only if $r\|s|t$ does so. An essential selection of these is provided in

PROPOSITION 1.2. *Let $r\|s|t$ be an index triplet of order m. Equivalent are:* (i) $r\|s|t\in\mathrm{Rul}_m(n)$; (ii) $r\|t|s\in\mathrm{Rul}_m(n)$; (iii) $t^c\|\sigma|r^c\in\mathrm{Rul}_{n-m}(n)$; (iv) $(n+1-s)\|(n+1-r)|t\in\mathrm{Rul}_m(n)$; (v) $\rho\|\sigma|\tau\in\mathrm{Rul}_{n-m}(n)$, *where σ,τ denote the inversions of s, resp. t.*

If $r\|s|t$, $r'\|s'|t'$ are index triplets of order m, and $r_i'\le r_i$, $s_i'\ge s_i$, $t_i'\ge t_i$, $i=1,2,\ldots,m$ then we write $r\|s|t\ge r'\|s'|t'$, and $r\|s|t$ generates a rule if $r'\|s'|t'$ does so. Of course we should concentrate on those index triplets in $\mathrm{Rul}(n)$ which are minimal with respect to the partial ordering $\le$; we set $\mathrm{Rul}_m^*(n)=\{r\|s|t\in\mathrm{Rul}_m(n)\,|\,r\|s|t$ is minimal with respect to $\le\}$ and $\mathrm{Rul}^*(n)=\bigcup_{m=0}^{n}\mathrm{Rul}_m^*(n)$. The correct condition for minimality seems to be that the *deviation* $d(r\|s|t)=\sum_{i=1}^{m}(i+r_i-s_i-t_i)=0$, and, indeed, all known minimal triplets in $\mathrm{Rul}(n)$ have zero deviation, and no triplets with negative deviation have been found in $\mathrm{Rul}(n)$ for any n. An important class of rules of zero deviation was described by R.C. Thompson, [13], Theorem 2. These rules are characterized by the existence of certain Young Tableaux, and we shall refer to this class as $\mathrm{Tab}_m(n)$. In [14] the class Tab is shown to be, in a sense, self-recursive, and in [11] it is proved that $\mathrm{Tab}_m(n)=\mathrm{Rul}_m^*(n)$ if $n\le 7$, if $m\le 3$ or if $n-m\le 3$.

Now consider arbitrary multiplicity sequences γ,α,β of order n such that $\gamma_1+\ldots+\gamma_n=\alpha_1+\ldots+\alpha_n+\beta_1+\ldots+\beta_n$. In order that $\gamma\in\Delta_n(\alpha,\beta)$ it is a necessary condition that

$$\sum_{i=1}^{m}\gamma_{r_i} \le \sum_{i=1}^{m}(\alpha_{s_i} + \beta_{t_i})$$

holds for each triplet $r\|s|t\in\mathrm{Rul}_m(n)$, for each m, but it is not clear, whether this condition is sufficient. For given multiplicity sequences α,β of order n we define

$$R_n(\alpha,\beta)=\left\{\gamma=(\gamma_i)_{i=1}^{n}\ \middle|\ \begin{array}{l}\gamma \text{ is a multiplicity sequence of order } n\\ \text{with } \alpha_1+\ldots+\alpha_n+\beta_1+\ldots+\beta_n=\gamma_1+\ldots+\gamma_n \text{ such}\\ \text{that (1.5) holds for each } r\|s|t\in\mathrm{Rul}(n)\end{array}\right\}$$

Since $r\|t|s\in\mathrm{Rul}(n)$ if and only if $r\|s|t\in\mathrm{Rul}(n)$ it is clear that $R_n(\alpha,\beta)=R_n(\beta,\alpha)$. Further, if $a\ge\alpha_1$ and $\gamma\in R_n(\alpha,\beta)$ then $a+\beta\in R_n(a-\alpha,\gamma)$: Indeed, if $r\|s|t\in\mathrm{Rul}(n)$, then

$$\sum_{i=1}^{m}(a+\beta)_{r_i}=am+\sum_{i=1}^{m}\beta_{r_i}\le\sum_{i=1}^{m}((a-\alpha)_{s_i}+\gamma_{t_i})=am+\sum_{i=1}^{m}(\gamma_{t_i}-\alpha_{n+1-s_i})$$

if and only if $\sum_{i=1}^{m}\gamma_{t_i}\ge\sum_{i=1}^{m}(\beta_{r_i}+\alpha_{n+1-s_i})$, that is, if and only if

$$\sum_{j=1}^{n-m}\gamma_{t_j^c}\le\sum_{j=1}^{n-m}(\beta_{r_j^c}+\alpha_{\sigma_j})$$

where σ denotes the inversion of s. But $t^c\|\sigma|r^c\in\mathrm{Rul}_{n-m}(n)$, so the final inequality holds, and hence the initial inequality is correct. In this way one proves

PROPOSITION 1.3. *The conclusion of Proposition* 1.1 *remains valid if in each statement* Δ_n *is replaced by* R_n.

If S is some subset of $\mathrm{Rul}(n)$ and $S(\alpha,\beta)=\{\gamma|\gamma$ multiplicity sequence of order n, $\sum_{i=1}^{n}\gamma_i=\sum_{i=1}^{n}(\alpha_i+\beta_i)$, (1.5) holds for each $r\|s|t\in S\}\subseteq\Delta_n(\alpha,\beta)$, then $\Delta_n(\alpha,\beta)=R_n(\alpha,\beta)=S(\alpha,\beta)$, as always $\Delta_n(\alpha,\beta)\subseteq R_n(\alpha,\beta)\subseteq S(\alpha,\beta)$. In many cases for given α,β a relatively small subset S of $\mathrm{Rul}(n)$ suffices to obtain $S(\alpha,\beta)\subseteq\Delta_n(\alpha,\beta)$. This leads to

CONJECTURE 1.4. *For each choice of multiplicity sequences* α,β *of order* n *one has that* $\Delta_n(\alpha,\beta)=R_n(\alpha,\beta)$.

Observe that $R_n(\alpha,\beta)$ does not change if one replaces Rul by Rul* in the definition, since $r\|s|t\ge r'\|s'|t'$ implies that $\sum_{i=1}^{m}\gamma_{r_i}\le\sum_{i=1}^{m}\gamma_{r'_i}$ and $\sum_{i=1}^{m}(\alpha_{s_i}+\beta_{t_i})\ge\ \ge\sum_{i=1}^{m}(\alpha_{s'_i}+\beta_{t'_i})$.

Quite often it is convenient to replace the fixed order n of the matrices involved by arbitrary orders $\ell\ge n$. To this end one defines $\alpha_i=0, i>n$ for the multiplicity sequence $(\alpha_i)_{i=1}^{n}$ of order n. The multiplicity sequence $\alpha=(\alpha_i)_{i=1}^{\infty}$ can be associated with $A(\lambda)\oplus I_{\ell-n}\in\mathcal{B}_\ell$, $\ell\ge n$, if $(\alpha_i)_{i=1}^{n}$ is the sequence of partial multiplicities of $A\in\mathcal{B}_n$; if $\alpha=(\alpha_i)_{i=1}^{\infty}$, $\beta=(\beta_i)_{i=1}^{\infty}$ are multiplicity sequences with $\alpha_{\ell+1}=\beta_{m+1}=0$, then $\gamma_{\ell+m+1}=0$ for each $\gamma\in\Delta_k(\alpha,\beta)$, $k\geq\max\{\ell,m\}$. Thus one can define $\Delta(\alpha,\beta)=\Delta_{\ell+m}(\alpha,\beta)$, and $\Delta_n(\alpha,\beta)=\{\gamma\in\Delta(\alpha,\beta)\,|\,\gamma_{n+1}=0\}$ if $n\geq\max\{\ell,m\}$. By setting $\mathrm{Rul}^{(*)}=\bigcup_{n=0}^{\infty}\mathrm{Rul}^{(*)}(n)$, $\mathrm{Rul}_m^{(*)}=\bigcup_{n=m}^{\infty}\mathrm{Rul}_m^{(*)}(n)$ as in [10], [11] one can also define $R(\alpha,\beta)$ independently of the order n.

2. THE KLEIN THEOREM WITH SOME APPLICATIONS

In this section we provide a proof (taken from [10]) for the necessary and sufficient (Young-tableau) conditions for $\gamma\in\Delta_n(\alpha,\beta)$ which were mentioned in the introduction and which were derived in [3], [4] in a more ring-theoretical setting. Recently O. Azenhaz and E. Marquez de Sa provided a constructive proof for matrices over principal ideal domains (cf. [1]). As an application we provide necessary and sufficient conditions for $\gamma\in\Delta(\alpha,\beta)$ if $\beta_1\leq 2$ and if $\alpha_{\ell+1}=\beta_{m+1}=0$, $\alpha_1-\alpha_\ell$, $\beta_1-\beta_m\leq 1$. Further, we observe that these conditions can be derived from certain rules in Rul, which implies that $\Delta(\alpha,\beta)=R(\alpha,\beta)$ in each of the cases mentioned above. The proof of this observation derives from [9], Sections V.1 and V.3. We conclude with the description of some results which can be obtained by application of the symmetry properties from the Propositions 1.1 and 1.2. to the above-mentioned cases.

THEOREM 2.1. *Let $\alpha=(\alpha_i)_{i=1}^n$, $\beta=(\beta_i)_{i=1}^n$ be multiplicity sequences. In order that a multiplicity sequence $\gamma=(\gamma_i)_{i=1}^n$ belongs to $\Delta_n(\alpha,\beta)$ it is necessary and sufficient that there exist multiplicity sequences $\alpha=\sigma^0,\sigma^1,\ldots,\sigma^r=\gamma$ such that (with $\beta_{n+1}=0$)*

$$(2.1)\qquad \begin{cases} \text{(a)}\quad 0\leq\sigma_j^i-\sigma_j^{i-1}\leq 1, \quad i=1,2,\ldots,r, \qquad j=1,2,\ldots,n \\ \text{(b)}\quad \sum_{j=\ell}^{n}(\sigma_j^{i+1}-\sigma_j^i)\ \leq\ \sum_{j=\ell}^{n}(\sigma_j^i-\sigma_j^{i-1}) \quad \ell=1,2,\ldots,n, \qquad i=1,2,\ldots,r-1 \\ \text{(c)}\quad \beta_t-\beta_{t+1}=\#\{i\,|\,\sum_{j=1}^{n}(\sigma_j^{i+1}-\sigma_j^i)=t\}, \quad t=1,2,\ldots,n \end{cases}$$

Excluding that $\sigma^i=\sigma^{i+1}$ for some i, one has, in particular, that $r=\beta_1$.

Below we shall provide a direct proof of this result; in the proof of the sufficiency part we shall also obtain a construction algorithm, yielding a sequence $S_1=A_1T_1$, $S_2=A_2T_2,\ldots,S_r=A_rT_r=AB=C$ such that α is the multiplicity sequence of $A_1\sim A_2\sim A_3\sim\ldots\sim A_r$, σ^k is the multiplicity sequence of S_k, and the multiplicity sequence τ^k of T_k is given by $\tau_{n+1}^k=0$ and

$$(2.2)\qquad \tau_t^k-\tau_{t+1}^k=\#\{i+1\leq k\,|\,\sum_{j=1}^{n}(\sigma_j^{i+1}-\sigma_j^i)=t\} \qquad t=1,2,\ldots,n.$$

Thus, $B=T_r$ will have the multiplicity sequence β.

The conditions (2.1) were introduced by D.E. Littlewood and A.R. Richardson, [5], Theorem III as rules for compounding Young tableaux C from given tableaux A,B. The partial multiplicity sequences α,β,γ correspond to the columns lengths in the tableaux A,B,C, and the column lengths of the intermediate tableaux correspond to the sequences $\sigma^1,\ldots,\sigma^{r-1}$. In [5] the construction is phrased in terms of row lengths, thus it deals, in fact, with the *conjugates* (or *dual multiplicities*, see (2.4) below) of α,β,γ.

PROOF 2.2. *of the necessity of* (2.1) *in Theorem* 2.1. Let $C=AB$ be *a* product with the multiplicity sequences γ,α,β; write the product in the $T_*=T_*D^+$–form, (see [1], Lemma 2.3) i.e., $B=\mathrm{diag}(\lambda^{\beta_i})_{i=1}^n=D_1D_2\ldots D_r$, where

$$(2.3)\qquad D_k = \begin{pmatrix} \lambda & & & & 0 \\ & \ddots & & & \\ & & \lambda & & \\ & & & 1 & \\ 0 & & & & \ddots \\ & & & & & 1 \end{pmatrix} = \mathrm{diag}\,(\lambda)_{i=1}^{\beta_k^*} \oplus I_{n-\beta_k^*+1},$$

with β_j^* (the *dual multiplicities*) defined by

$$(2.4)\qquad \beta_k^*:=\#\{i\,|\,\beta_i\geq k\},$$

so $\beta_1^*\geq\beta_2^*\geq\ldots\geq\beta_r^*>0=\beta_{r+1}^*$. Set $S_k=AD_1\ldots D_k$, and let σ^k denote the multiplicity sequence of S_k. Since $S_i=S_{i-1}D_i$, $i=1,2,\ldots,r$, the condition (2.1)(a) is met (cf. [7], Section 2,1, or [8], Proposition 3). Further, $\beta_i^*=\sum_{j=1}^n(\sigma_j^i-\sigma_j^{i-1})$ by construction, which proves (2.1)(c), since $\beta_t=\beta_t^{**}=\#\{i\,|\,\beta_i^*\geq t\}$.

In order to prove (2.1)(b) we consider an $(n+1-\ell)$–minor M_i in det S_i such that $n(M_i)=\sum_{j=\ell}^n\sigma_j^i$; let M_{i-1}, M_{i+1} denote the corresponding minors in det S_{i-1}, det S_{i+1}. Then $n(M_{i-1})=(\sum_{j=1}^n\sigma_j^i)-\delta\geq\sum_{j=\ell}^n\sigma_j^{i-1}$, where δ is the number of column indices in M_i, occuring in the set $\{1,2,\ldots,\beta_i^*\}$. Further, $n(M_{i+1})=(\sum_{j=\ell}^n\sigma_j^i)+\delta'\geq\sum_{j=\ell}^n\sigma_j^{i+1}$, where $\delta'\leq\delta$ is the number of column indices in M_i occuring in the set $\{1,2,\ldots,\beta_{i+1}^*\}$. Thus

$$\sum_{j=\ell}^n(\sigma_j^{i+1}-\sigma_j^i)\leq\delta'\leq\delta\leq\sum_{j=\ell}^n(\sigma_j^i-\sigma_j^{i-1}).\ \blacksquare$$

In order to prove the sufficiency of (2.1) we observe that, assuming $\sigma^i\neq\sigma^{i+1}$ for each i, the sequences $\tau^1,\ldots,\tau^r$ have the following property: If $\tau_j^k<k$ then $\tau_j^k=\tau_j^{k+1}=\ldots=\tau_j^r=\beta_j$ (i.e., for each fixed j the sequence $\tau_j^1,\ldots,\tau_j^r$ is strictly increasing until $\tau_j^i=\beta_j$ – this happens for $i=\beta_j$ – whereafter it remains constant).

PROOF 2.3. *of the sufficiency of* (2.1) *in Theorem* 2.1. Let $\sigma^i \neq \sigma^{i+1}$, $i=0,1,\ldots,r-1$, and define $\tau^1,\ldots,\tau^r$ by (2.2). We construct a sequence $S_1=A_1T_1$, $S_2=A_2T_2,\ldots,S_r=A_rT_r$ such that for $k=1,2,\ldots,r$

$$(2.5)\quad \begin{cases} \text{(a)} & \sigma^k,\tau^k \text{ and } \alpha \text{ are the multiplicity sequences of } S_k,\ T_k \text{ and } A_k, \text{ respectively,} \\ \text{(b)} & S_k=\operatorname{diag}(\lambda^{\sigma_j^k})_{j=1}^n,\ A_k \text{ and } T_k=(t_{ij}^k)_{i,j=1}^n \text{ are lower triangular,} \\ \text{(c)} & n(t_{11}^k),\ldots,n(t_{nn}^k) \text{ is a reordering of } \tau_1^k,\ldots,\tau_n^k, \text{ whereas} \\ & \text{for } i<j \text{ one has } n(t_{ii}^k)>n(t_{ij}^k)\geq n(t_{jj}^k) \text{ or } t_{ij}^k=0, \\ \text{(d)} & n(t_{jj}^k)=k \Leftrightarrow \sigma_j^k>\sigma_j^{k-1} \text{ (where } \sigma^0=\alpha). \end{cases}$$

Taking $S_1=A_1T_1=\operatorname{diag}\ (\lambda^{\alpha_j})_{j=1}^n\ \operatorname{diag}\left(\lambda^{\sigma_j^1-\alpha_j}\right)_{j=1}^n$ the desired product is constructed for $k=1$. Assume that $S_\ell=A_\ell T_\ell$ has been constructed for $\ell=1,2,\ldots,k<r$. We shall construct a product $S_{k+1}=A_{k+1}T_{k+1}$ such that (2.5) holds for $k+1$. By induction, we obtain a product $C=S_r=A_rT_r=AB$ such that (2.5) holds for $k=r$. As $\gamma=\sigma^r$, $\beta=\tau^r$ this will prove the sufficiency of the conditions (2.1).

We proceed as follows: Define $J_1:=\{j|\sigma_j^{k+1}-\sigma_j^k=1,\ \ n(t_{jj}^k)=k\}$, $J_2:=\{j|\sigma_j^{k+1}-\sigma_j^k=1,$ $n(t_{jj}^k)<k\}$ and $J_3:=\{j|\sigma_j^{k+1}=\sigma_j^k,\ n(t_{jj}^k)=k\}$. By 2.5(c) one has $t_{ij}^k=0$ for $j\in J_2\cup J_3$, $i\neq j$. According to (2.1)(b) one has $\#J_2\leq\#J_3$. Further, there exists a monotonically increasing injection $\pi:J_2\to J_3$ such that $\pi(j)>j$, $j\in J_2$; indeed, if $J_2=\{j_1,\ldots,j_y\}$, $j_1<j_2<\ldots<j_y$, then (2.1)(b) implies that $\#\{j\in J_3|j>j_x\}\geq y+1-x$, $x=1,2,\ldots,y$. Observe that π is such, that $\sigma_{\pi(j)}^k\leq\sigma_j^k$, $j\in J_2$.

Now construct the product $S_{k+1}'=A_kT_{k+1}'$ by adding in the product $S_k=A_kT_k$ each column with index $\pi(j)$ to the column with index j (in S_k and T_k) and multiplying both sides of the ensueing product on the right by diag $(\lambda^{\varepsilon_i})_{i=1}^n$, where $\varepsilon_i=1$, $i\in J_1\cup\pi(J_2)$, $\varepsilon_i=0$ otherwise. Then $T_{k+1}'=(t_{xy}')_{x,y=1}^n$ has the multiplicity sequence τ^{k+1}, since

$$n(\det(t_{ij}')_{i,j\notin J_1\cup\pi(J_2)}) = n(\det(t_{ij}^k)_{i,j\notin J_1\cup\pi(J_2)}) =$$

$$=\Sigma\{\tau_j^k|\tau_j^k<k\} + k\#(J_3\backslash\pi(J_2)),$$

and $n(t_{jj}')=k+1=\tau_j^k+1$ for $j\in J_1\cup\pi(J_2)$. Thus the final $n-\#J_1-\#J_2$ partial multiplicities have not increased at the transition of T_k to T_{k+1}', and the initial $\#J_1+\#J_2$ partial multiplicities have increased by 1.

Next, perform some elementary column and row operations in the product $S_{k+1}'=A_kT_{k+1}'$, constructing an equivalent product $S_{k+1}=A_{k+1}'T_{k+1}''$: For $j\in J_2$, subtract λ times the j^{th} column form the $\pi(j)^{\text{th}}$ column in S_{k+1}',T_{k+1}', and interchange these columns. In the resulting product $S''_{k+1}=A_kT''_{k+1}=(s''_{xy})_{x,y=1}^n$ one has $s''_{jj}=-\lambda^{\sigma_j^k+1}$, $s''_{\pi(j),\pi(j)}=\lambda^{\sigma_{\pi(j)}^k}$ and $s''_{j,\pi(j)}=\lambda^{\sigma_j^k}$, $j\in J_2$. Observe that $\sigma_j^k+1=\sigma_j^{k+1}$, $\sigma_{\pi(j)}^{k+1}=\sigma_{\pi(j)}^k\le\sigma_j^k$, $j\in J_2$. Now for $j\in J_2$ subtract in S''_{k+1} and A_k the $\pi(j)^{\text{th}}$ row times $\lambda^{(\sigma_j^k-\sigma_{\pi(j)}^k)}$ from the j^{th} row, and multiply the j^{th} row by (-1). The ensueing product is $S_{k+1}=\operatorname{diag}(\lambda^{\sigma_i^{k+1}})_{i=1}^n=A'_{k+1}T''_{k+1}$. Unfortunately, the triangular structure of T'_{k+1} (and A_k) is lost. However, the special structure of T_k allows to restore it.

Start with T''_{k+1}, using row operations (to be compensated by column operations in A'_{k+1} leaving S_{k+1} unchanged), and additions of multiples of columns to columns with a higher index in S_{k+1} and T''_{k+1}; the latter type of elementary operations allows us to retrieve S_{k+1} by row operations in S_{k+1} and A'_{k+1}.

First construct a product $S_{k+1}=A''_{k+1}T'''_{k+1}$, $T'''_{k+1}=(t'''_{xy})_{x,y=1}^n$ lower triangular, $(n(t'''_{11}),n(t'''_{22}),\ldots,n(t'''_{nn}))$ a reordering of $(\tau_1^{k+1},\tau_2^{k+1},\ldots,\tau_n^{k+1})$ with $n(t'''_{jj})=k+1$ for $j\in J_1\cup J_2$. Fix $j\in J_2$. For $T''_{k+1}=(t''_{xy})_{x,y=1}^n$ one has that $t''_{\pi(j),\pi(j)}=\lambda^k$, $t''_{\pi(j),i}=0$, $i\neq\pi(j)$ (since $t^k_{\pi(j),i}=0$ because of the structure of T_k, and $t''_{\pi(j),j}$ was made 0 again at the transition from T'_{k+1} to T''_{k+1}). Let $p_j=n(t^k_{jj})<k$. Then $t''_{j,\pi(j)}=\lambda^{p_j}$, $t''_{jj}=-\lambda^{p_j+1}$ and for certain $j<i<\pi(j)$ one might have $b_{ij}:=t''_{i,\pi(j)}=t^k_{ij}\neq0$, namely, where $p_j>n(t^k_{ij})\ge n(t^k_{ii})$ in T^k. This implies that such $i\notin J_1\cup J_3$. If $b_{ij}\neq0$ for $i\notin J_2$ then $t''_{i,j}=-\lambda b_{ij}$. Now interchange the rows with indices j and $\pi(j)$ for each $j\in J_2$ (so $t''_{i,\pi(j)}=b_{ij}$ is replaced by 0 if $i\in J_2$, as $\pi(i)>\pi(j)$), and subtract λ^{k-p_j} times the new $\pi(j)^{\text{th}}$ row (with entries $-\lambda^{p_j+1}$, λ^{p_j} on the places j, $\pi(j)$, with zeros in between) from the new j^{th} row (with zero everywhere, except for λ^k in the $\pi(j)^{\text{th}}$ place, now made 0). The resulting matrix has λ^{k+1} in the (j,j)-position, 0 in the (j,i)-positions for $i>j$ and some entries $-\lambda^{k-p_j}t''_{ji}$ in the (j,i)-positions, $i<j$, where $n(t''_{ji})\ge p_j$, as $n(t^k_{ii})>n(t^k_{ji})\ge p_j$ if $t^k_{ji}\neq0$, $j>i$. Possible remaining entries $t''_{i,\pi(j)}=b_{ij}\neq0$, $j\in J_2$, $i\notin J_1\cup J_2\cup J_3$ can be removed by adding appropriate multiples of the i^{th} column to the $\pi(j)^{\text{th}}$ column, starting with the lowest occurring i: Indeed, $n(t''_{ii})\le n(b_{ij})$. If $t''_{i',i}\neq0$ for some $i<i'<\pi(j)$, then $n(t''_{i',i})\ge n(t''_{i',i'})$, and any contribution of a multiple of this entry to $b_{i'j}$ can be removed when dealing with the i'-th row (this includes the possibility that $i'\in J_2$ and some entry $-\lambda^{k-p_{i'}}t''_{i',i}$ is present, as $n(-\lambda^{k-p_{i'}}t''_{i',i})>p_{i'}$). One thus obtains $S_{k+1}=A''_{k+1}T'''_{k+1}$ and, indeed, $n(t'''_{jj})=k+1$ for $j\in J_1\cup J_2$, as required, whereas the multiplicity $\tau_j^k=p_j<k$ has been shifted to the $\pi(j)$-location if $j\in J_2$. The fact that T'''_{k+1} is lower triangular and that its diagonal is a reordering of its invariant factors implies that $t'''_{ij}=0$ or $n(t'''_{ij})\ge n(t'''_{ii})$ if $j<i$. The remaining requirement in (2.5)(c) can be met by subtracting (in the order $i=n,\ldots,i=2$) multiples of the the j^{th} row, $j<1$, $n(t'''_{ij})\ge n(t'''_{jj})$ from the i^{th} row (again in decreasing order with respect to j). Calling the resulting matrix

T_{k+1}, the product $S_{k+1}=A_{k+1}T_{k+1}$ has been constructed. ■

A schematic description of the proof can be found in [10], 5.3. Observe that because of (2.5)(d) the sets J_1,J_2,J_3 are determined *a priori* by the given Young Tableau (2.1). The choice of $\pi:J_2\to J_3$ is thus a special case of the Steps 1 and 2 in Algorithm 3.4 in [1].

As an application of Theorem 2.1 we present a proof of Theorem V.1.1 in [9], obtained there by a different argument.

THEOREM 2.4. *Let* $\alpha=(\alpha_i)_{i=1}^{\infty}$, $\beta=(\beta_i)_{i=1}^{\infty}$ *be two multiplicity sequences with* $\beta_1\leq 2$. *Then* $\gamma=(\gamma_i)_{i=1}^{\infty}\in\Delta(\alpha,\beta)$ *if and only if the sequence* $(\delta_i)_{i=1}^{\infty}=(\gamma_i-\alpha_i)_{i=1}^{\infty}$ *meets the following conditions*:

(i) $0\leq\delta_i\leq 2$ *for all* i;

(ii) $\sum_{i=1}^{\infty}\delta_i = \sum_{i=1}^{\infty}\beta_i$;

(iii) $p:=\#\{i\,|\,\delta_i=2\}\leq m'=sup\{i\,|\,\beta_i=2\}$ (*with* $sup\ \phi=0$)

(iv) *the set* $I=\{i\,|\,\delta_i=1\}$ *contains two disjoint subsets* I_1,I_2 *such that* $\#I_1=\#I_2=m'-p$, *and there exists a bijection* $\tau:I_1\to I_2$ *such that* $\alpha_{\tau(i)}<\alpha_i$ *for each* $i\in I_1$.

PROOF. *Sufficiency*. Set $J:=\{i\,|\,\delta_i=2\}$. If $\alpha_i=\alpha_{i+1}$ for some $i\in I$, then $i+1\notin J$, as $\gamma_{i+1}\leq\gamma_i$. Given sets I_1,I_2 and the bijection τ as in (iv), one can assume that $\alpha_i=\alpha_{i+1}$ for $i\in I_1$ implies that $i+1\notin I\backslash I_1$. To see this, assume that $\alpha_i=\alpha_{i+1}$ for some $i\in I_1$, $i+1\in I\backslash I_1$. Apply the following algorithmic procedure: If $\alpha_i=\alpha_{i+1}$, $i\in I_1$, $i+1\in I\backslash I_1$, and $i+1\notin I_2$, then replace i by $i+1$ in I_1, leaving $\tau(i)$ unchanged; if $\alpha_i=\alpha_{i+1}$, $i\in I_1$, $i+1=\tau(i')\in I_2$, then replace i by $i+1$ in I_1, $i+1=\tau(i')$ by $\tau(i')=i$ in I_2. Then $\alpha_{\tau(i')}$ remains unchanged. After this step I_1,I_2,τ still have the properties as described in (iv); application of this procedure makes $\sum\{i\,|\,i\in I_1\}$ strictly increase, so the algorithm must terminate.

Now $\sigma^0=\alpha$, $\sigma^2=\gamma$; define $\sigma=\sigma^1$ by $\sigma_j=\alpha_j+1$ for all $j\in J\cup(I\backslash I_1)$, $\sigma_j=\alpha_j$ otherwise. Thus $\sigma_j=\gamma_j-1$, $j\in J\cup I_1$, $\sigma_j=\gamma_j$ otherwise. Observe that $\sigma_i\geq\sigma_{i+1}$ for each i: if $\sigma_i<\sigma_{i+1}$, then $\sigma_i=\alpha_i=\alpha_{i+1}<\sigma_{i+1}$, whereas $\gamma_i\geq\gamma_{i+1}$, and thus $\sigma_i=\gamma_i-1$, that is, $i\in I_1$, $i+1\in J\cup(I\backslash I_1)$, a contradiction. It is clear, that Condition (2.1)(a) is met; (2.1)(b) holds, as

$$\sum_{j=\ell}^{\infty}(\gamma_j-\sigma_j)=\#(J\cup I_1)\cap\{\ell,\ell+1,\ldots,\})\leq\#((J\cup\tau(I_1)\cap\{\ell,\ell+1,\ldots,\})\leq$$

$$\leq\#((J\cup I\backslash I_1)\cap\{\ell,\ell+1,\ldots,\}) = \sum_{j=\ell}^{\infty}(\sigma_j-\alpha_j).$$

Set $m=\#\{i\,|\,\beta_i\neq 0\}=\max\{i\,|\,\beta_i\neq 0\}$. If $m=m'$, then $I=\phi$ and $\sum_{j=1}^{\infty}\sigma_j-\alpha_j=\sum_{j=1}^{\infty}\gamma_j-\sigma_j=m=m'$, and (2.1)(c) holds, as $\beta_1=\beta_m=2$, $\beta_{m+1}=0$. If $m>m'$ we have $\beta_{m+1}=0$, $\beta_m=1$, $\beta_{m'}-\beta_{m'+1}=1$, and

(2.1)(c) holds, since $m=\sum_{i=1}^{\infty}\beta_i-m'=\#(J\cup I)\backslash I_1=\sum_{j=1}^{\infty}\sigma_j-\alpha_j, m'=\#(J\cup I_1)=\sum_{j=1}^{\infty}\gamma_j-\sigma_j$.

Necessity. In order that $\gamma:=\sigma^2\in\Delta(\alpha,\beta)$ it is necessary that $\sigma=\sigma^1$ can be defined such that (2.1) holds with $\sigma^0=\alpha$. The validity of the conditions (i), (ii) of the theorem is then evident; if $p>m'$, then $\sum_{j=1}^{\infty}\gamma_j-\sigma_j>m'$, $\sum_{j=1}^{\infty}\sigma_j-\alpha_j>m'$, and $\beta_{m'+1}=2$, contradicting the definition of m', so (iii) holds. Finally, defining $I_1=\{i|\alpha_i=\sigma_i<\gamma_i\}$ one can use (2.1)(b) for defining an injective mapping $\tau:I_1\to I\backslash I_1$ such that $\tau(i)>i$. Since $\alpha_i=\sigma_i\geq\sigma_{\tau(i)}=\alpha_{\tau(i)}+1$ one thus has $\alpha_i>\alpha_{\tau(i)}$, $i\in I_1$, and (iv) holds. ■

REMARK 2.5. The sufficiency proof in [9] is easier, using direct sums of blocks

$$C_i=A_iB_i=\begin{pmatrix}\lambda^{\alpha_i} & 0\\ 0 & \lambda^{\alpha_{\tau(i)}}\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda & \lambda^2\end{pmatrix}=\begin{pmatrix}\lambda^{\alpha_i} & 0\\ \lambda^{\alpha_{\tau(i)}+1} & \lambda^{\alpha_{\tau(i)}+2}\end{pmatrix},\ i\in I_1$$

(with partial multiplicities α_i+1, $\alpha_{\tau(i)}+1$), $C_j=A_jB_j=(\lambda^{\alpha_j})(\lambda^2)$, $j\in J$, $C_k=A_kB_k=(\lambda^{\alpha_k})(\lambda)$, $k\in I\backslash(I_1\cup I_2)$, $C_\ell=A_\ell B_\ell=(\lambda^{\alpha_\ell})(1)$, $\ell\notin I\cup J$. This indicates that the construction of a product $C=AB$ by means of the "intermediate" multiplicity sequences $\sigma^1,\ldots,\sigma^{r-1}$ from (2.1) may be a rather cumbersome approach; for example, we have been unable to find a proof based on (2.1) for the sufficiency part of Theorem 3.1 below.

The next result in this section was originally proved by L. Rodman and M. Schaps, [6], Theorem 7. An alternative proof, using the (I,J)-rules, described in Example 2.9.(ii) of [11], was given in [9], Theorem V.3.1 Here we present an outline of a proof based on Theorem 2.1.

THEOREM 2.6. *Let $\alpha=(\alpha_i)_{i=1}^{\infty}$, $\beta=(\beta_i)_{i=1}^{\infty}$ be multiplicity sequences with $\alpha_{\ell+1}=\beta_{m+1}=0$, $\alpha_\ell,\beta_m\neq 0$, $\alpha_1-\alpha_\ell\leq 1$, $\beta_1-\beta_m\leq 1$. Then $\gamma=(\gamma_i)_{i=1}^{\infty}\in\Delta(\alpha,\beta)$ if and only if $\gamma_{m+\ell+1}=0$ and*

(2.6) *There exist index sets $I_0=\{i_1,\ldots,i_r\}\subseteq\{1,\ldots,\ell\}$, $J_0=\{j_1,\ldots,j_r\}\subseteq\{1,\ldots,m\}$ such that $\#I_0=\#J_0=r$, $r\geq 0$, and integers $0\leq\delta_x\leq\beta_{j_x}\leq\alpha_{i_x}+\delta_x, 1\leq x\leq r$, such that $(\gamma_i)_{i=1}^{\ell+m}$ is the ordered representation of*
(*) $\{\alpha_{i_1}+\delta_1,\ldots,\alpha_{i_r}+\delta_r\}\cup\{\beta_{j_1}-\delta_1,\ldots,\beta_{j_r}-\delta_r\}\cup$
$\cup\{\alpha_i|i\leq\ell,\ i\notin I_0\}\cup\{\beta_j|j\leq m,\ j\notin J_0\}$

The sufficiency of the conditions in this theorem is clear even for arbitrary α,β with $\alpha_\ell>0=\alpha_{\ell+1},\beta_m>0=\beta_{m+1}$:

LEMMA 2.7. *Let α,β be multiplicity sequences with α_ℓ, $\beta_m>0$, $\alpha_{\ell+1}=\beta_{m+1}=0$.* If *$\gamma=(\gamma_i)_{i=1}^{\infty}$ is a multiplicity sequence such that $\gamma_{m+\ell+1}=0$ and condition* (2.6) *holds, then $\gamma\in\Delta(\alpha,\beta)$.*

PROOF. Construct the product $C=AB$ as a direct sum of blocks of the form

$$C_x=A_xB_x = \begin{pmatrix}\lambda^{\alpha_{i_x}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \lambda^{\beta_{j_x}-\delta_x} & \lambda^{\beta_{j_x}}\end{pmatrix} = \begin{pmatrix}\lambda^{\alpha_{i_x}} & 0\\ \lambda^{\beta_{j_x}-\delta_x} & \lambda^{\beta_{j_x}}\end{pmatrix},\ 1\le x\le r,$$

where C_x has the partial multiplicities $(\alpha_{i_x}+\delta_x,\ \beta_{j_x}-\delta_x)$ as $\beta_{j_x}-\delta_x\le\alpha_{i_x}$, and of order 1 blocks $(\lambda^{\alpha_i})(1)$, $i\notin I_0$, $1\le\ell$, resp. $(1)(\lambda^{\beta_j})$, $j\notin J_0$, $j\le m$. ■

Without further assumptions the converse of Lemma 2.7 not true:

EXAMPLE 2.8. Let $\alpha=(3,1,0,\dots)$, $\beta=(2,0,0,\dots)$, $\gamma=(4,2,0\dots)$. Then $\gamma\in\Delta(\alpha,\beta)$, according to Theorem 2.4 (cf. Remark 2.5), but Condition (2.6) is not met. ■

PROOF 2.9 *of the necessity part in Theorem* 2.6. Set $\alpha_1:=a$, $\beta_1:=b$, $n=\ell+m$. It suffices to consider the finite sequences $\alpha=(\alpha_i)_{i=1}^n$, $\beta=(\beta_i)_{i=1}^n$. Let $\gamma\in\Delta_n(\alpha,\beta)$. There exist $\alpha=\sigma^0,\sigma^1,\dots,\sigma^{b-1},\sigma^b=\gamma$ such that (2.1) holds. Define $\varepsilon_{ij}=\sigma_j^{i+1}-\sigma_j^i$, $i=1,2,\dots,b$. Then $\varepsilon_{ij}\in\{0,1\}$. Because of (2.1)(b),(c) one has that $\sum_{j=1}^n\varepsilon_{ij}=m$, $1\le i\le b-1$, whereas $\sum_{j=1}^n\varepsilon_{bj}=m'$ with $1\le m'=\max\{j\,|\,\beta_j=b\}\le m$.
From the condition (2.1)(b) one has that for each $1\le i\le b$ the following holds:

$$\text{if } \varepsilon_{ix}=0,\ \ell<z\le x\le n,\ \text{then } \varepsilon_{i'x}=0,\quad i'\ge i,\ x\ge z,$$

and using that $\sigma^1,\dots,\sigma^b$ are multiplicity sequences one has:

$$\text{(2.7)}\qquad \text{if } \varepsilon_{ix}=0,\ x>\ell,\ \text{then } \varepsilon_{i'y}=0 \text{ for all } x\le y\le n,\ \text{all } i\le i'\le b.$$

Using that $\sum_{j=1}^n\varepsilon_{ij}=m$, $1\le i\le b-1$ one has from (2.1)(b) for fixed z, fixed $i\le b-1$:

$$\text{if } \varepsilon_{ix}=1,\ 1\le x\le z\le\ell \text{ then } \varepsilon_{i'x}=1,\ 1\le x\le z,\ i\le i'\le b-1,$$

and using that $\sigma^1,\dots,\sigma^{b-1}$ are multiplicity sequences one has:

$$\text{(2.8)}\qquad \text{if } \varepsilon_{ix}=1,\ \sigma_x^{i-1}\ge a,\ x\le\ell,\ i\le b-1,\quad \text{then } \varepsilon_{i'y}=1 \text{ for all } 1\le y\le x, \text{ for all } i\le i'\le b-1.$$

The conclusion in (2.8) may be incorrect if $\sigma_x^{i-1}=a-1$ (which means that $x>\ell'=\max\{j\le\ell\,|\,\alpha_j=a\}$ and $\varepsilon_{i'x}=0$, $i'\le i-1$). Defining $h_x=\min\{i\,|\,\varepsilon_{ix}=1\}$ for $\ell'<x\le\ell$ with $\sigma_x^{b-1}=a$ one has that $h_x\le h_{x+1}$ if $\sigma_{x+1}^{b-1}\ge a$.
Now we make the additional assumption that $\alpha_1=\alpha_\ell$, $\beta_1=\beta_m$; then the latter phenomenon cannot occur, as $\sigma_x^{i-1}\ge a$ for each $x\le\ell$, and the conclusion of (2.8) holds for $i'=b$ as well. Then (2.8) implies that there exist $k_1\le k_2\le\dots\le k_b\le\ell$, $\ell<k_1'\le k_2'\le\dots\le k_b'\le n$ such that

$$\varepsilon_{ij}=1,\quad 1\le j\le k_i,\quad \ell<j\le k_i',\quad \varepsilon_{ij}=0 \text{ otherwise},\ i=1,2,\ldots,b.$$

Since $k_i+k_i'=m+\ell$ it is not difficult to see that for $1\le j\le \min\{\ell,m\}$ one has that $\max\{i|\varepsilon_{ij}=0\}=\max\{i|\varepsilon_{i,n+1-j}=1\}=\gamma_{n+1-j}$, and hence

$$\delta_j=\gamma_j-\alpha_j=\sum_{i=1}^{b}\varepsilon_{ij}=b-\max\{i|\varepsilon_{ij}=0\}=b-\gamma_{n+1-j}=\beta_j-\gamma_{n+1-j},$$

which proves the desired result for $\alpha_1=\alpha_\ell$, $\beta_1=\beta_m$, taking $r=\max\{j|\delta_j>0\}$. Observe that $\delta_1\ge\delta_2\ge\ldots\ge\delta_r>0$ in this case.

Next we consider the case where $\alpha_1=\alpha_\ell=a$, but $\beta_1=b=\beta_m+1$. Set $\tilde{\gamma}=\sigma^{b-1}$, $\tilde{\beta}=(\tilde{\beta}_j)_{j=1}^n$ defined by $\tilde{\beta}=\ldots\tilde{\beta}_m=b-1$, $\tilde{\beta}_{m+1}=0$. Then $\tilde{\gamma}=\tilde{\Delta}(\alpha,\tilde{\beta})$ and we can use the previous case to find $\tilde{\delta}_1\ge\tilde{\delta}_2\ge\ldots\ge\delta_{\tilde{r}}>0$ such that $\tilde{\gamma}_j=\alpha_j+\tilde{\delta}_j=a+\tilde{\delta}_j\ge b-1$, $\tilde{\gamma}_{n+1-j}=b-1-\tilde{\delta}_j$, whereas $\tilde{\gamma}_i=a$, $\tilde{r}<i\le\ell$, $\tilde{\gamma}_i=b-1$, $\ell<i\le n-\tilde{r}$. Let $I'=\{j|\varepsilon_{bj}=1\}$. Then $I'\subseteq\{1,\ldots,n-\tilde{r}\}$ because of (2.7) and with $s=\#\{j\in I'|\tilde{r}<j\le\ell\}$, $t=\#\{j\in I'|\ell<j\le n-\tilde{r}\}$ one has $s+t\le n-\tilde{r}-\ell$ because of (2.1)(b), as $\varepsilon_{b-1,j}=0$ for $\tilde{r}<j\le\ell$ and for $j>n-\tilde{r}$. For $1\le j\le\tilde{r}$, we define $\delta_j:=\tilde{\delta}_j+1$, $i\in I$, $\delta_j=\tilde{\delta}_j$ otherwise. Observe that $b-1-\tilde{\delta}_i=b-(\tilde{\delta}_i+1)=b-\delta_i$ in the first case. We set $r:=\tilde{r}+s$ and for $\tilde{r}<i\le r$ we have $\gamma_i=a+1=\alpha_i+1$. We set $\delta_i=1$ in this event. Since $s+t\le n-\tilde{r}-\ell$ we can write $\gamma_{n+1-j}=b-1=b-\delta_j$, $\tilde{r}<j\le r$. Finally, we have $\gamma_j=b$ for $\ell<j\le\ell+t$. Observe that $\#\{j\in I'|1\le j\le\tilde{r}\}+t+s=m'$.

In Theorem 2.6 we can always interchange the role of α and β, setting $\eta_x=\alpha_{i_x}+\delta_x-\ \beta_{j_x}$ as $0\le\eta_x\le\alpha_{i_x}\le\beta_{j_x}+\eta_x$, and thus the desired result also holds for $\alpha_1=a=\alpha_\ell+1$, $\beta_1=\beta_m=b$.

Finally, we consider the case where $\alpha_1-\alpha_\ell=\beta_1-\beta_m=1$. Again, we set $\tilde{\gamma}=\sigma^{b-1}$, $\tilde{\beta}=(\tilde{\beta}_j)_{j=1}^n$, $\tilde{\beta}_1=\ldots\tilde{\beta}_m=b-1$, $\tilde{\beta}_{m+1}=0$. For $\tilde{\gamma}=\Delta(\alpha,\tilde{\beta})$ there exist a set $\tilde{I}_0=\{i_1,\ldots,i_{\tilde{r}}\}\subseteq\{1,\ldots,\ell\}$, $\#\tilde{I}_0=\tilde{r}$, and $\tilde{\delta}_1,\ldots,\tilde{\delta}_{\tilde{r}}$ such that $0\le\tilde{\delta}_x\le b-1\le\alpha_{i_x}+\tilde{\delta}_x$ and $\tilde{\gamma}$ is a reordering of $\{\alpha_{i_1}+\tilde{\delta}_1,b-1-\tilde{\delta}_1,\ldots,\alpha_{i_{\tilde{r}}}+\tilde{\delta}_{\tilde{r}},b-1-\tilde{\delta}_{\tilde{r}}\}\cup\{\alpha_i|i\notin\tilde{I}_0\}\cup\{b-1|1\le j\le m-\tilde{r}\}$. We can assume that $\tilde{\delta}_1\ge\tilde{\delta}_2\ge\ldots\ge\tilde{\delta}_{\tilde{r}}>0$ and that $\tilde{\gamma}_j=\alpha_{i_j}+\tilde{\delta}_j$, $1\le j\le\tilde{r}$. Then $\tilde{\gamma}_{n+1-j}=b-1-\tilde{\delta}_j$. Again, let $I'=\{j|\varepsilon_{bj}=1\}$. If $\varepsilon_{bj}=0$ for each $\tilde{r}<j\le\ell$, then we apply the same argument as in the case where $\alpha_1=\alpha_\ell$. Otherwise one must have $\varepsilon_{b-1,j}=1$ for some $\tilde{r}<j\le\ell$ as well, because of (2.1)(b). So let $\varepsilon_{b-1,j}=1$ for $\tilde{r}<r'+1\le j\le r'+d$, $\varepsilon_{b-1,j}=0$ for $\tilde{r}<j\le r'$, $r'+d<j\le\ell$. Evidently $\tilde{\gamma}_j\ge a$ for $\tilde{r}<j\le r'+d$; on the other hand, $\tilde{\gamma}_j=\alpha_{j'}, j'\in\{1,\ldots,\ell\}\setminus\tilde{I}_0$ for these j, and hence $\tilde{\gamma}_j=a$, $\tilde{r}<j\le r'+d$. Further, $\varepsilon_{b-1,j}=0$ for $j>n-\tilde{r}$ and hence $\varepsilon_{b-1,j}=1$, $1\le j\le\tilde{r}-d$, $\varepsilon_{b-1,j}=0$, $\tilde{r}+1-d\le j\le\tilde{r}$ as explained after (2.8). According to (2.8) we must have $\tilde{\gamma}_j=a$ for $\tilde{r}-d<j\le r'+d$, as $\tilde{\gamma}_{\tilde{r}-d+1}>a$ would contradict $\varepsilon_{b-1,\tilde{r}-d+1}=0$. So $\#\{j\in I'|\tilde{r}-d<j\le n-\tilde{r}\}\le n-\tilde{r}+d$, whereas $\#\{j\in I'|r'+d<j\le n-\tilde{r}\}\le n-\tilde{r}$.

Thus, if $\#\{j\in I'|\tilde{r}'-d<j\le n-\tilde{r}\}>n-\tilde{r}$, then $\varepsilon_{bj}=1$ for $\tilde{r}-d<j\le\tilde{r}$, as $\tilde{\gamma}_{\tilde{r}}=a+\varepsilon_{b\tilde{r}}\ge\gamma_j=a+\varepsilon_{bj}$, $r<j\le r'+d$. Hence $\#\{j\in I'|\tilde{r}<j\le\ell\}+\#\{j\in I'|\ell<j\le n-\tilde{r}\}\le n-\tilde{r}$ in all events. Now we set $\delta_j=\tilde{\delta}+\varepsilon_{bj}$, $1\le j\le\tilde{r}$, observing that $\gamma_{n+1-j}=b-1-\tilde{\delta}_j=b-(\tilde{\delta}_j+1)$; with $r=\tilde{r}+\#\{j\in I'|\tilde{r}<j\le\ell\}$ we set $\delta_x=1$, $\tilde{r}<x\le r$, and writing $\{i'_{\tilde{r}+1},\ldots,i'_r\}=\{j\in I'|\tilde{r}<j\le\ell\}$ there exists for each $\tilde{r}<x\le r$ an $i_x\in\{1,\ldots,\ell\}\setminus\tilde{I}_0$ such that $\gamma_{i'_x}=\alpha_{i_x}+1=\alpha_{i_x}+\delta_x$, whereas $\gamma_{n+1-x}=b-1=b-\delta_x$. Setting $I_0=\tilde{I}_0\cup\{i_{\tilde{r}+1},\ldots,i_r\}$ the proof is complete. ■

Next we show that the necessary conditions in the Theorems 2.4 and 2.6 can be derived from the inequalities described by Rul, thus proving that $R(\alpha,\beta)=\Delta(\alpha,\beta)$ in each case.

PROPOSITION 2.10. *Let* α,β *be given multiplicity sequences and assume that* $\gamma\in R(\alpha,\beta)$

(a) *If* $\beta_3=0$, *then the conditions* (*i*), (*ii*), (*iii*), (*iv*), *of Theorem* 2.4 *are met for the sequence* $(\delta_i)_{i=1}^{\infty}=(\gamma_i-\alpha_i)_{i=1}^{\infty}$.

(b) *If* $\alpha_{\ell+1}=\beta_{m+1}=0$ *and* $\alpha_1-\alpha_\ell$, $\beta_1-\beta_m\le 1$, *then* $\gamma_{\ell+m+1}=0$ *and the condition* (2.6) *of Theorem* 2.6 *is met.*

PROOF. In both cases the proof consist in using the identity $\sum_{i=1}^{\infty}\gamma_i=\sum_{i=1}^{\infty}(\alpha_i+\beta_i)$ and inequalities $\sum_{i=1}^{m}\gamma_{r_i}\le\sum_{i=1}^{m}(\alpha_{s_i}+\beta_{t_i})$ for certain $r\|s|t\in$Rul. We specify the index triplets which are used, and give a reference, if necessary, in order to verify that they do indeed generate rules.

(a) Condition (i) follows from $\alpha_i\le\gamma_i\le\alpha_i+\beta_1\le\alpha_i+2$, and (ii) is the identity $\sum_{i=1}^{\infty}(\gamma_i-\alpha_i-\beta_i)=0$; if condition (iii) is not met, i.e., if $\#J=p>m'$ for the set $J=\{i|\delta_i=2\}$, then $\sum_{i\in J}\gamma_i=\sum_{i\in J}\alpha_i+2p>\sum_{i\in J}\alpha_i+\sum_{i=1}^{p}\beta_i\ge\sum_{i\in J}\gamma_i$, a contradiction ($r\|s|t$, with $s=r=J, t=\{1,2,\dots,p\}$). In order to derive (iv) set $m=\#\{i|\beta_i\ne 0\}$. Then $m+m'=\sum_{i=1}^{\infty}\beta_i=\sum_{i=1}^{\infty}\delta_i=2p+\#I$, where $I=\{i|\delta_i=1\}$. Thus $s=\#I\ge 2(m'-p)$. Set $s'=s-(m'-p)$, and write $I=\{i_1,\dots,i_s\}$, $i_j<i_{j+1}$. Define $I_1=\{i_1,\dots,i_{m'-p}\}$, $I_2=\{i_{s'+1},\dots,i_s\}$, $\tau:I_1\to I_2$ by $\tau(j)=i_{s'+j}$. Then $\alpha_{\tau(i)}\le\alpha_i$, $i\in I_1$: Assume that $\alpha_{i_k}=\alpha_{\tau(i_k)}=a$ for some $i_k\in I_1$. Set $i_0=\min\{i\in I|\alpha_i=a\}$, $j_0=\max\{i\in I|\alpha_i=a\}$. Then $j_0-i_0\ge\tau(i_k)-i_k=i_{s'+k}-i_k\ge s'$, and $\{i|i_0\le i\le j_0\}\subseteq I$. So for each $j\in J$ one has $j<i_0$ or $j>j_0$. Thus, with $J'=\{j\in J|j<J_0\}$, $J''=J\backslash J'$ one has

$$\sum_{j\in J'}\gamma_j+\gamma_{j_0}+\sum_{j\in J''}\gamma_j\le(\sum_{j\in J'}\alpha_j)+\alpha_{i_0}+\sum_{j\in J''}\alpha_j+(\sum_{j=1}^{p}\beta_j)+\beta_{j_0+p+1-i_0} \tag{2.9}$$

($r=J'\cup\{j_0\}\cup J'', s=J'\cup\{i_0\}\cup J'', t=\{1,\dots,p\}\cup\{j_0+p+1-i_0\}$; then $r\|s|t$ is weakly zero-reducible: Apply [11], Proposition 1.5(i),(ii) in order to achieve $J'=\phi$, then (iii) in order to obtain $J''=\phi$, $p=0$). Now $j_0-i_0+p+1\ge s'+p+1=m+1$, so $\beta_{j_0+p+1-i_0}=0$, and (2.9) implies that $\alpha_{i_0}\ge\gamma_{j_0}=a+1=\alpha_{i_0}+1$. Contradiction, so $\alpha_{\tau(i)}<\alpha_i$, $i\in I_1$.

(b) Clearly, $\gamma_{\ell+m+1}\le\alpha_{\ell+1}+\beta_{m+1}=0$. Set $n=\ell+m$, and assume that $\ell\ge m$ for definiteness. As in the proof of Theorem 2.6 we write $a=\alpha_1$, $b=b_1$, $\ell'=\max\{i|\alpha_i=a\}$, $m'=\max\{i|\beta_i=b\}$; set $a'=a-1$, $b'=b-1$. For $1\le j\le\ell-m$ we have $a'\le\alpha_{m+j}\le\gamma_{m+j}\le\alpha_1+\beta_{m+j}=a$; set $j_0=\sup\{1\le j\le\ell-m|\gamma_{m+j}=a\}$ (with 0=sup ϕ). Using

$$\sum_{j\neq i,\, n+1-i}\gamma_j \;\le\; \sum_{j=1}^{n-1}(\alpha_j+\beta_j)-\alpha_\ell-\beta_m \;\le\; \sum_{j=1}^{n}\gamma_j-a'-b', \qquad 1\le i\le m$$

($r\|s|t$ a rule of coorder 2, cf. [11], Example 2.5.(ii)) and the rule of order 2 generated by $r=(i,n+1-i)$, $s=(1,\ell+1)$, $t=(1,m+1)$, see [11], Proposition 2.6, one has

$$a'+b' \;\le\; \gamma_i+\gamma_{n+1-i} \;\le\; \alpha_1+\beta_1+\alpha_{\ell+1}+\beta_{m+1}=a+b, \qquad 1\le i\le m.$$

Define $k=\max\{i\le m\,|\,\gamma_i\ge a\}$. As $\gamma_{\ell+j}\le\beta_1=b$ one has $b'\le\gamma_{n+1-i}\le b$ for $k+1\le i\le m$. Set $j_1=\sup\{1\le j\le m-k\,|\,\gamma_{\ell+j}=b\}$ (observe that $j_1=0$ if $m=k$ and that $j_0=0$ if $m>k$). Define the (disjoint) index sets $I_1,I_2,I_3\subseteq\{1,2,\dots,k\}$ by

$$I_1=\{1\le i\le k\,|\,\gamma_i+\gamma_{n+1-i}=a+b\};\; I_2=\{1\le i\le k\,|\,\gamma_i+\gamma_{n+1-i}=a+b'=a'+b\};$$
$$I_3=\{1\le i\le k\,|\,\gamma_i+\gamma_{n+1-i}=a'+b'\},$$

and set $x=\#I_1$, $y=\#I_2$, $z=\#I_3$. Then $x\le\min\{\ell'-j_0,m'-j_1\}$. Indeed, assume that $x>\ell'-j_0$; application of the so–called (I,J)-rule (see [11], Example 2.9 with $r=I_1\cup\{m+1,\dots,m+j_0\}\cup n+1-I_1$, $s=\{1,\dots,x,x+1,\dots,x+j_0\}\cup\{\ell+1,\dots,\ell+x\}$, $t=\{1,\dots,x\}\cup\{m+1,\dots,m+x+j_0\}$ (where $\{m+1,\dots,m+j_0\}=\phi$ if $j_0=0$) then yields

$$(x+j_0)a+xb=\sum_r\gamma_j\le\sum_s\alpha_j+\sum_t\beta_j=\ell'a+(x+j_0-\ell')a'+\sum_{i=1}^{x}\beta_i+0$$
$$\le(x+j_0)a-(x+j_0-\ell')+xb,$$

a contradiction. The other inequality follows in the same way, with $r=I_1\cup\{\ell+1,\dots,\ell+j_1\}\cup n+1-I_1$, $s=\{1,\dots,x\}\cup\{\ell+1,\dots,\ell+x+j_1\}$, $t=\{1,\dots,x,x+1,\dots,j_1\}\cup\{m+1,\dots,m+x\}$. (Here $\{\ell+1,\dots,\ell+j_1\}=\phi$ if $j_1=0$), yielding the contradiction $xa+(x+j_1)b\le xa+(x+j_1)b-(x+j_1-m')$ for $x>m'-j_1$.

Observe that $x+y+z=k$; this yields the identity

$$ka'+kb'+2x+y=x(a+b)+y(a+b-1)+z(a'+b') \;=\; \sum(\gamma_i+\gamma_{n+1-i})=$$
$$=\sum_{i=1}^{\ell}\alpha_i+\sum_{i=1}^{m}\beta_i-\sum_{i=k+1}^{n-k}\gamma_i=(\ell a'+\ell'+mb'+m')-((\ell-m)a'+j_0+(m-k)(a'+b')+j_1)=$$
$$=\; ka'+\ell'-j_0+kb'+m'-j_1.$$

Hence $x+y\ge\max\{\ell'-j_0,m'-j_1\}$, and it is possible to decompose $I_2=I_2'\cup I_2''$, $I_2'\cap I_2''=\phi$, with $\#I_2'=\ell'-j_0-x$. Then $\#I_2''=y-(\ell'-j_0-x)=m'-j_1-x$. Further, $z+\#I_2'=k-(\ell'-j_0)$, $z+\#I_2''=k-(m'-j_1)$. We can interpret this result as: $\gamma_i+\gamma_{n+1-i}=a+b$, $i\in I_1$, $\gamma_i+\gamma_{n+1-i}=a+b'$, $i\in I_2'$, $\gamma_i+\gamma_{n+1-i}=a'+b$, $i\in I_2''$, $\gamma_i+\gamma_{n+1-i}=a'+b'$, $i\in I_3$. Setting $\delta_i=\gamma_i-a$, $i\in I_1\cup I_2'$, $\delta_i=\gamma_i-a'$, $i\in I_3\cup I_2''$ and using bijections from $\{1,\dots\ell'-j_0\}$ onto $I_1\cup I_2'$, from $\{\ell'+1,\dots,k+j_0\}$ onto $I_3\cup I_2''$ one forms the set I_0 as in (2.6), and J_0 is defined analogously. This completes the proof. ■

COROLLARY 2.11. *Let* $\gamma \in R_n(\alpha,\beta)$. *Then* $\gamma \in \Delta_n(\alpha,\beta)$ *if one of the following conditions is met*: (i) $\alpha_1-\alpha_n \le 2$; (ii) $\beta_1-\beta_n \le 2$; (iii) $\gamma_1-\gamma_n \le 2$; (iv) $\alpha_1-\alpha_\ell \le 1$, $\alpha_{\ell+1}=\alpha_n$, $\beta_1-\beta_m \le 1$, $\beta_{m+1}=\beta_n$; (v) $\gamma_1-\gamma_k \le 1$; $\gamma_{k+1}=\gamma_n$, $\alpha_1=\alpha_s$, $\alpha_{s+1}=\alpha_n \le 1$, (vi) $\gamma_1-\gamma_k \le 1$, $\gamma_{k+1}=\gamma_n$, $\beta_1=\beta_t$, $\beta_{t+1}=\beta_n \le 1$, (vii) $\alpha_1=\alpha_s$, $\alpha_{s+1}-\alpha_n \le 1$, $\beta_1=\beta_t$, $\beta_{t+1}=\beta_n \le 1$, (viii) $\gamma_1=\gamma_r$, $\gamma_{r+1}-\gamma_n \le 1$, $\alpha_1-\alpha_\ell \le 1$, $\alpha_{\ell+1}=\alpha_n$; (ix) $\gamma_1=\gamma_r$, $\gamma_{r+1}-\gamma_n \le 1$, $\beta_1-\beta_m \le 1$, $\beta_{m+1}=\beta_n$.

This follows from Proposition 1.1 and Proposition 1.3. Only the condition (vii) leads to the description of $\Delta(\alpha,\beta)$ for certain α,β.

THEOREM 2.12. *Let* α,β *be multiplicity sequences with* $\alpha_1=\alpha_s>1$, $\beta_1=\beta_t=b>1$, $\alpha_{s+1}=\alpha_\ell=\beta_{t+1}=\beta_m=1$, $\alpha_{\ell+1}=\beta_{m+1}=0$ (*taking* $1\le s\le t$ *for definiteness*). *Then* $\gamma\in\Delta(\alpha,\beta)$ *if and only if* $\gamma_{\ell+m+1}=0$ *and* $(\gamma_i)_{i=1}^{\ell+m}$ *is the ordered representation of some set*

$$(2.10)\quad \begin{cases} \{a+\delta_i+\varepsilon_i,\ b-\delta_i+\eta_i \mid 1\le i\le x\} \cup \{a+\varepsilon_i \mid x+1\le i\le s\} \cup \\ \cup\ \{b+\eta_i \mid x+1\le i\le t\} \cup \{\varepsilon_{s+i}+\eta_{t+i} \mid 1\le i\le n-(s+t)\}, \\ \text{where } x\le s,\ \delta_i,\dots,\delta_s,\ \eta_i,\dots,\eta_{n-s},\ \varepsilon_i,\dots,\varepsilon_{n-t}\ge 0,\ \varepsilon_i,\eta_i\ \in\{0,1\}, \\ \sum_{i=1}^{n-s}\eta_i=\ell-s,\ \sum_{i=1}^{n-t}\varepsilon_i=m-t,\ b-a+\eta_i-\varepsilon_i\le\delta_i\le b-\varepsilon_i. \end{cases}$$

This result can be derived from Theorem 2.6, by applying the description provided by (2.6) to $(a-\alpha)=(\tilde{\alpha}_i)_{i=1}^{\ell+m}$, $(b-\beta)=(\tilde{\beta}_i)_{i=1}^{\ell+m}$, and selecting $\tilde{\gamma}=(\tilde{\gamma}_i)_{i=1}^{\infty}$, such that $\tilde{\gamma}_{\ell+m+1}=0$. Then $\gamma=(a+b-\tilde{\gamma}_{\ell+m+1-i})_{i=1}^{\ell+m}$ has the form described by (2.10).

3. THE CASE $\beta_3=0$.

If α,β,γ are multiplicity sequences, and $\alpha^*,\beta^*,\gamma^*$ denote their duals (or conjugates), defined according to (2.4), then known results suggest that $\gamma\in\Delta(\alpha,\beta)$ if and only of $\gamma^*\in\Delta(\alpha^*,\beta^*)$ (see, e.g., [15], where a conjecture of *E.* Marques de Sa is mentioned), but no formal proof seems to have been published. However, combining the Theorems III and II from [5] an indirect proof might be available: If A,B,C are Young tableaux for α,β,γ (that is, with columns lengths α,β,γ) then Theorem III states that the number of ways in which C can, according to specific rules, be compounded from A and B is the coefficient c_γ of the Schur function $\{\gamma^*\}$ in the product $\{\alpha^*\}\{\beta^*\}$ of the Schur functions associated with α^*,β^*. In particular, $c_\gamma\ne 0$ iff $\gamma\in\Delta(\alpha,\beta)$. But Theorem II states a necessary and sufficient condition for $c_\gamma\ne 0$ in terms of a different set of rules for constructing C from A and B. This latter set is completely symmetric with respect to rows and columns in the tableaux (and with respect to A and B), thus proving that $c_\gamma\ne 0$ iff $c_{\gamma^*}\ne 0$, where c_{γ^*} is the coefficient of $\{\gamma\}$ in the product $\{\alpha\}\{\beta\}$ (it also shows that $\Delta(\alpha,\beta)=\Delta(\beta,\alpha)$ but that is not remarkable in the present context). Unfortunately, the proof of Theorem II in [5] is completely unrelated to the present setting, and Theorem III is verified only for the case where $\beta_3^*=0$, that is, where $\beta_1\leq 2$ (a complete proof of Theorem III of [5] is presented by I.G. Macdonald in his book *Symmetric Functions and Hall Polynomials*, Oxford, 1979, where A. Lascoux and M.P. Schützenberger and, independently, G.P. Thomas, are credited for the first full proof; there is no mention of Theorem II of [5]).

Using this relationship between $\Delta(\alpha,\beta)$ and $\Delta(\alpha^*,\beta^*)$ it is not difficult to find a description for $\Delta(\alpha,\beta)$ if $\beta_1^*\leq 2$, that is, if $\beta_3=0$: just apply Theorem 2.4 to $\Delta(\alpha^*,\beta^*)$. Other examples of the analogy between $\Delta(\alpha,\beta)$ and $\Delta(\alpha^*,\beta^*)$ can be found in [9], Appendix.

THEOREM 3.1. *Let* $\alpha=(\alpha_i)_{i=1}^{\infty}$, $\beta=(\beta_i)_{i=1}^{\infty}$ *be multiplicity sequences with* $\beta_3=0$. *Then* $\gamma=(\gamma_i)_{i=1}^{\infty}\in\Delta(\alpha,\beta)=R(\alpha,\beta)$ *if and only if the following conditions are met:*

(i) $\gamma_i\leq\alpha_i+\beta_1,\ \gamma_{i+1}\leq\alpha_i+\beta_2,\ i=1,2,\ldots,;$

(ii) $\gamma_{i+2}\leq\alpha_i\leq\gamma_i,\ i=1,2,\ldots;$

(iii) *the sequences* $(\varepsilon_i)_{i=1}^{\infty}$, $(\eta_i)_{i=1}^{\infty}$ *defined by* $\varepsilon_1=0$, $\varepsilon_i=max\{0,\gamma_i-\alpha_{i-1}\}$, $i\geq 2$, $\eta_i=\gamma_i-(\alpha_i+\varepsilon_i+\varepsilon_{i+1})$, $i\geq 1$ *meet the following conditions*:

(a) $p:=\sum_{i=1}^{\infty}\varepsilon_i\ \leq\ \beta_2$; (b) $\eta_i\ \leq\ \beta_1-p,\ i=1,2,\ldots$

The proof is, with minor alterations, taken from [9], Section V.2. In the necessity part we show that $\gamma\in R(\alpha,\beta)$, $\beta_3=0$ implies (i), (ii), (iii); in the sufficiency part $\gamma\in\Delta(\alpha,\beta)$ is shown by constructing a product $C=AB$ with the desired partial multiplicity sequences. We have not been able to describe the Young tableau which would allow the application of Theorem 2.1.

PROOF 3.2 *of the necessity of the conditions* (i), (ii), (iii) *in Theorem* 3.1. Let $\gamma\in R(\alpha,\beta)$ and $\beta_3=0$. The necessity part of the conditions (i), (ii) is clear, as they follow from well-known standard inequalities. In order to prove (iii)(a) let $I=\{i\geq 2|\gamma_i-\alpha_{i-1}\geq 0\}$. The index triplet $r\|s|t$ with $r=I, s=I-1$, $t=\{2,\ldots,\#I+1\}$ generates a standard rule, so $\sum_{i\in I}\gamma_i\leq\sum_{i\in I}\alpha_{i-1}+\beta_2$. To show the necessity of (iii)(b) we assume that $\eta_k>\beta_1+p$ for some k. As $\gamma_k=\alpha_k+\eta_k+\varepsilon_k+\varepsilon_{k+1}$, this implies $\gamma_k>\alpha_k+\beta_1-((\sum_{i=2}^{\infty}\varepsilon_i)-(\varepsilon_k+\varepsilon_{k+1}))$. Let $J=I\backslash\{k,k+1\}$, I as above, and set $r=J\cup\{k\}$, $s=(J-1)\cup\{k\}$, $t_1=1$, $t_{i+1}=i+2$, $1\leq i\leq\#J$. Then $r\|s|t\in$Rul, as it is strongly zero-reducible in the sense of [11], Theorem 2.7 (remove k from r,s, 1 from t according to Theorem 2.7.(i); the reduced triplet generates an elementary standard rule). Thus

$$\gamma_k+\sum_{j\in J}\gamma_i\leq\alpha_k+\beta_1+\sum_{j\in J}\alpha_{j-1}=\alpha_k+\beta_1+\sum_{j\in J}(\gamma_j-\varepsilon_i)$$

a contradiction. ■

PROOF 3.3 *of the sufficiency of the conditions* (i), (ii), (iii) *in Theorem* 3.1. Given multiplicity sequences γ,α,β such that $\beta_3=0$ we construct a product $C=AB$, γ, resp. α, resp. β the sequence of partial multiplicities of C, resp. A, resp. B. Let $n=\sup\{i|\alpha_i\neq 0\}$. Then $\gamma_{n+3}=0$, and we can carry out the construction in $(n+2)\times(n+2)$-matrix functions. Let $(\gamma_i')_{i=1}^{n+1}$ be a multiplicity sequence with $\gamma_i'\geq\alpha_i\geq\gamma_{i+1}'$, $\sum_{i=1}^{n+1}(\gamma_i'-\alpha_i)=\beta_2$. Set $x_{n+1}=\gamma_{n+1}'$, $x_k=\sum_{i=k}^{n+1}(\gamma_i'-\alpha_i)$. Let $\mathrm{row}(d_i)_{i=1}^k$ denote a row $(d_1d_2\ldots d_k)$ in a matrix, and $\mathrm{diag}(f_i)_{i=1}^k$ the $k\times k$- matrix $(\delta_{ij}f_i)_{i,j=1}^k$. It is not difficult to see that the matrix function $E_{n+1}\in\mathcal{B}_{n+1}$,

$$(3.1)\qquad E_{n+1}=\begin{pmatrix}-\mathrm{diag}\left(\lambda^{\alpha_{n+1-i}}\right)_{i=1}^{n} & \vdots & 0\\ \cdots\cdots & & \\ \mathrm{row}\left(\lambda^{x_{n+2-i}}\right)_{i=1}^{n} & & \lambda^{\beta_2}\end{pmatrix}$$

is equivalent to $\mathrm{diag}\left(\lambda^{\gamma_{n+2-i}'}\right)_{i=1}^{n+1}$. Here we shall prove that one can choose $(\gamma_i')_{i=1}^{n+1}$ in such a way, that for appropriate $c_1,\ldots,c_{n+1}$ with $n(c_1)\geq\beta_2$ the matrix

$$(3.2)\qquad C=\begin{pmatrix}-\mathrm{diag}\left(\lambda^{\alpha_{n+1-i}}\right)_{i=1}^{n} & \vdots & 0 & 0\\ \cdots\cdots & & & \\ \mathrm{row}\left(\lambda^{x_{n+2-i}}\right)_{i=1}^{n} & & \lambda^{\beta_2} & 0\\ \mathrm{row}\left(c_{n+3-i}\right)_{i=2}^{n+1} & & c_1 & \lambda^{\beta_1}\end{pmatrix}=\begin{pmatrix} E_{n+1} & \vdots & 0\\ \cdots\cdots & & \\ \mathrm{row}\left(c_{n+2-i}\right)_{i=1}^{n+1} & & \lambda^{\beta_1}\end{pmatrix}$$

is equivalent to

$$(3.3)\qquad \tilde{C}=\begin{pmatrix}\mathrm{diag}\left(\lambda^{\gamma_{n+2-i}'}\right)_{i=1}^{n+1} & \vdots & 0\\ \cdots\cdots & & \\ \mathrm{row}\left(\lambda^{y_{n+3-i}}\right)_{i=1}^{n+1} & & \lambda^{\beta_1}\end{pmatrix},$$

where $y_{n+2}=\gamma_{n+2}$, $y_k=\sum_{i=k}^{n+2}(\gamma_i-\gamma_i')$. As $\tilde{C}$ is equivalent to $\mathrm{diag}\left(\lambda^{\gamma_{n+3-i}}\right)_{i=1}^{n+2}$, this would complete the proof, as

$$C=AB=\begin{pmatrix}-\mathrm{diag}\left(\lambda^{\alpha_{n+1-i}}\right)_{i=1}^{n} & \vdots & 0 & 0\\ \cdots\cdots & & & \\ \mathrm{row}\left(\lambda^{x_{n+2-i}}\right)_{i=1}^{n} & & 1 & 0\\ \mathrm{row}\left(c_{n+3-i}\right)_{i=2}^{n+1} & & c' & 1\end{pmatrix}\cdot\mathrm{diag}\,(1)_{i=1}^{n}\oplus\begin{pmatrix}\lambda^{\beta_2} & 0\\ 0 & \lambda^{\beta_1}\end{pmatrix},$$

where $c' = \lambda^{-\beta_2} c_1$ is holomorphic in 0, as $n(c_1) \geq \beta_2$.
Now we construct the sequence $(\gamma_i')_{i=1}^{n+1}$ and the functions $c_1, \ldots, c_{n+1}$.
Let $H_i = \{(i,j) \mid 1 \leq j \leq \eta_i\}$ if $\eta_i \neq 0, H_i = \phi$ if $\eta_i = 0$, and set

$$H = \bigcup_{i=1}^{n+1} H_i.$$

Then $\# H = \beta_1 + \beta_2 - 2p$. Because of condition (3)(ii) there exist subsets K_1, K_2 of H with $\# K_1 = \# K_2 = \beta_2 - p$, $K_1 \cap K_2 = \phi$ and a bijection $\tau: K_1 \to K_2$ such that $j < i$ if $(j,y) = \tau(i,x) \in K_2$. We define

$$\eta_i' = \# H_i \cap K_2, \quad \tilde{\eta}_i = \eta_i - \eta_i'.$$

Then $\tilde{\eta}_i \geq \# H_i \cap K_1$, $\eta_{n+1}' = 0$, and for each k one has

$$\sum_{i=k+1}^{n+1} \tilde{\eta}_i - \sum_{i=k}^{n} \eta_i' \geq 0. \tag{3.4}$$

Now define $\gamma_i' = \alpha_i + \eta_i' + \varepsilon_{i+1} \leq \alpha_i + \eta_i + \varepsilon_{i+1} + \varepsilon_i = \gamma_i$. Then $\gamma_i' \geq \alpha_i + \varepsilon_{i+1} \geq \gamma_{i+1}$; further,

$$\left\{ \begin{aligned} &\alpha_i - \gamma_{i+1}' - \gamma_i' + \gamma_{i+1} = \alpha_i - \alpha_{i+1} - \eta_{i+1}' - \varepsilon_{i+2} - \alpha_i + \eta_i' - \varepsilon_{i+1} + \alpha_{i+1} + \\ &\qquad + \varepsilon_{i+2} + \eta_{i+1} + \varepsilon_{i+1} = \eta_{i+1} - \eta_{i+1}' - \eta_i' = \tilde{\eta}_{i+1} - \eta_i'. \end{aligned} \right. \tag{3.5}$$

Clearly, $\sum_{i=1}^{n+1} \gamma_i' = \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \eta_i' + \sum_{i=1}^{n+1} \varepsilon_{i+1} = \sum_{i=1}^{n} \alpha_i + \beta_2 - p + p = \sum_{i=1}^{n} \alpha_i + \beta_2$. Put $x_k^{(0)} = x_k$, as defined above; then $x_1^{(0)} = \beta_2$. Consider the matrix E_{n+1} defined in (3.1). Interchange in E_{n+1} the first and last row, and then make the first row equal to $(\lambda^{\gamma_{n+1}'} \; 0 \ldots 0)$ by subtracting multiples of the first column; finally make the first entry of the last row zero by subtracting a multiple of the first row. Then E_{n+1} is replaced by the equivalent matrix function

$$F = \left(\lambda^{\gamma_{n+1}'} \right) \oplus E_n$$

where the $n \times n$ matrix function E_n is of the same type as E_{n+1}, but with $-\text{diag}(\lambda^{\alpha_{n-i}})_{i=1}^{n-1}$ instead of $-\text{diag}(\lambda^{\alpha_{n+1-i}})_{i=1}^{n}$ and with $\text{row}(\lambda^{x_{n+1-i}'})_{i=1}^{n}$ instead of $\text{row}(\lambda^{x_{n+2-i}})_{i=1}^{n+1}$; here $x_k^{(1)} = x_k' = x_k^{(0)} - x_{n+1}^{(0)} - \alpha_n = \gamma_n' + \sum_{i=k}^{n-1} (\gamma_i' - \alpha_i)$. Repeating this procedure in E_n, one arrives in n steps at the equivalent matrix function $(\lambda^{\gamma_{n+2-i}'})_{i=1}^{n+1}$. Performing the reduction of E_{n+1} in C, defined by (3.2), one obtains after the first step the matrix

$$C' = \begin{pmatrix} F & \vdots & 0 \\ \cdots\cdots & & \\ \operatorname{row}\left(c'_{n+2-i}\right)_{i=1}^{n+1} & & \lambda^{\beta_1} \end{pmatrix}$$

where $c'_{n+1} = c_{n+1}$, $c'_i = c_i + c_{n+1}\cdot\lambda^{\alpha_n + x_i - \gamma'_{n+1}}$. Clearly, in order to obtain the matrix function $\tilde{C}$, defined in (3.3), after n steps, we must take $c_{n+1} = \lambda^{y_{n+2}} = \lambda^{\gamma_{n+2}}$, and c'_n must be λ^{y_n}. To achieve this, we define

$$x_i^{(k+1)} = \alpha_{n-k} + x_i^{(k)} - x_{n+1-k}^{(k)} = \alpha_{n-k} + \left(x_i^{(k)} - \gamma'_{n+1-k}\right) =$$

$$= \gamma'_{n-k} + \sum_{j=1}^{n-k-1} (\gamma'_j - \alpha_j), \quad (i = 1,2,\ldots,n-k)$$

(note that $x_i^{(0)}, x_i^{(1)}$ have already been defined). Obviously, $x_{n+1-j}^{(j)} = \gamma'_{n+1-j}$. For $k = 0,1,\ldots,n-1$ we define

$$c_{n-k} = \lambda^{y_{n-k+1}} + \sum_{i=0}^{k} \lambda^{(y_{n+2-1} + x_{n-k}^{(i)} - \gamma'_{n+1-i})}.$$

With this definition the transition from (3.2) to (3.3) is guaranteed, if one performs in C the transformations which replace E_{n+1} by $\operatorname{diag}(\lambda^{\gamma'_{n+2-i}})_{i=1}^{n+1}$. So it suffices to prove that $\beta_2 \le n(c_1)$. To this end we show that β_2 is not greater than the zero order of any of the summands in

$$c_1 = \lambda^{y_2} + \sum_{i=0}^{n-1} \lambda^{(y_{n+2-i} + x_1^{(i)} - \gamma'_{n+1-i})}.$$

Set $t_j = y_j + x_1^{(n+2-j)} - \gamma'_{j-1}$. Then, with (3.4) and (3.5) one has

$$t_j = \sum_{i=j}^{n+2} (\gamma_i - \gamma'_i) + \sum_{i=1}^{j-2} (\gamma'_i - \alpha_i) = \beta_2 - \sum_{i=j-1}^{n+1} (\gamma'_i - \alpha_i) + \sum_{i=j}^{n+2} (\gamma_i - \gamma'_i) =$$

$$= \beta_2 + \sum_{i=j-1}^{n+1} (\gamma_{i+1} - \gamma'_{i+1} - \gamma'_i + \alpha_i) = \beta_2 + \sum_{i=j-1}^{n+1} (\tilde{\eta}_{i+1} - \eta'_i) \ge \beta_2$$

and

$$y_2 = \sum_{i=2}^{n+2} (\gamma_i - \gamma'_i) = \beta_2 + \sum_{i=1}^{n+1} \left((\gamma_{i+1} - \gamma'_{i+1}) - (\gamma'_i + \alpha'_i) \right) = \beta_2 + \sum_{i=1}^{n+1} (\tilde{\eta}_{i+1} - \eta'_i) \ge \beta_2.$$

This completes the proof. ■

If one works within the setting of multiplicity sequences (and matrices) of a fixed order n, then Theorem 3.1 can be rephrased to provide a description of

$\Delta_n(\alpha,\beta) = R_n(\alpha,\beta)$ when $\beta_3 = \beta_n$. Using the Propositions 1.1 and 1.3 one has

PROPOSITION 3.4. *Let α,β,γ be multiplicity sequences of order n, and let $\gamma \in R_n(\alpha,\beta)$. Then $\gamma \in \Delta_n(\alpha,\beta)$ if one of the following conditions is met:*
(i) $\alpha_3 = \alpha_n$; (ii) $\beta_3 = \beta_n$; (iii) $\gamma_3 = \gamma_n$; (iv) $\alpha_1 = \alpha_{n-2}$; (v) $\beta_1 = \beta_{n-2}$; (vi) $\gamma_1 = \gamma_{n-2}$.

If $\beta = (\beta_i)_{i=1}^n$ is such that $b = \beta_1 = \beta_{n-2}$, and $\alpha = (\alpha_i)_{i=1}^n$ is given, then, using the conditions of Theorem 3.1 for $\alpha_1 + b - \gamma \in \Delta_n(\alpha_1 - \alpha, b - \beta)$ one obtains

THEOREM 3.5. *Let $\alpha = (\alpha_i)_{i=1}^n$, $\beta = (\beta_i)_{i=1}^n$ be multiplicity sequences of order n with $b = \beta_1 = \beta_{n-2}$. Then $\gamma = (\gamma_i)_{i=1}^n \in \Delta_n(\alpha,\beta) = R_n(\alpha,\beta)$ if and only if thefollowing conditions are met:*

(i) $\gamma_j \geq \alpha_j + \beta_n$, $\gamma_j \geq \alpha_{j+1} + \beta_{n-1}$, $j = 1,2,\dots,n$

(ii) $\gamma_j \leq \alpha_j + b \leq \gamma_{j-2}$, $j = 3,\dots,n$

(iii) *the sequences $(\varepsilon_j')_{j=1}^{n+1}$, $(\eta_j')_{j=1}^n$, defined by $\varepsilon_1' = \varepsilon_{n+1}' = 0$, $\varepsilon_j' = \max\{0, \alpha_{j+1} + b - \gamma_j\}$, $2 \leq j \leq n$, $\eta_j' = \alpha_j + b - \gamma_j - \varepsilon_j' - \varepsilon_{j+1}'$, $1 \leq j \leq n$ meet the following conditions:*
(a) $p = \sum \varepsilon_j' \leq b - \beta_{n-1}$; (b) $\eta_j' \leq b - \beta_n - p$.

Since the restriction "$b = \beta_1 = \beta_{n-2}$" explicitly contains the order n of the multiplicity sequences involved in this theorem, it cannot be extended to $\Delta(\alpha,\beta)$.

4. THE CASE $\Delta_4(\alpha,\beta)$.

In this section we shall prove that for given multiplicity sequences $\alpha = (\alpha_i)_{i=1}^4$, $\beta = (\beta_i)_{i=1}^4$ one has $\Delta_4(\alpha,\beta) = R_4(\alpha,\beta)$, i.e., $\gamma = (\gamma_i)_{i=1}^4 \in \Delta_4(\alpha,\beta)$ if and only if $\sum_{i=1}^4 \gamma_i = \sum_{i=1}^4 (\alpha_i + \beta_i)$ and $\sum_r \gamma_i \leq \sum_s \alpha_i + \sum_t \beta_i$ for all index triplets $r\|s|t \in \mathrm{Rul}^*(4)$. The equivalent result has already been obtained for order $n \leq 3$ (see [8]) and Corollary 2.11 and Proposition 3.2 cover many other cases: evidently, the desired result only needs to be proved under the additional assumptions

- $\alpha_4 = \beta_4 = 0 < \alpha_3, \beta_3, \gamma_4$
- $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$, $\gamma_1 > \gamma_2$, $\gamma_3 > \gamma_4$
- no two of the following conditions are met simultaneously:
  (i) $\alpha_1 - \alpha_3 \leq 1$, (ii) $\beta_1 - \beta_3 \leq 1$, (iii) $\gamma_2 - \gamma_4 \leq 1$,
- no two of the following conditions are met simultaneously:
  (i)' $\alpha_2 \leq 1$, (ii)' $\beta_2 \leq 1$, (iii)' $\gamma_1 - \gamma_3 \leq 1$
- $\alpha_1 > 2$, $\beta_1 > 2$, $\gamma_1 > \gamma_4 + 2$.

The remaining cases for $\gamma \in R_4(\alpha,\beta)$ will be dealt with by means of a reduction technique which might work for arbitrary orders n as well: If $\gamma \in R_n(\alpha,\beta)$, and $\sum_r \gamma_i = \sum_s \alpha_i + \sum_t \beta_i$ for some (minimal) index triplet $r\|s|t \in \mathrm{Rul}_m^*(n)$, $1 \leq m \leq n-1$ then one can try to "split" the problem, by considering $\tilde{\gamma}=(\gamma_{r_i})_{i=1}^m$, $\tilde{\alpha}=(\alpha_{s_i})_{i=1}^m$, $\tilde{\beta}=(\beta_{t_i})_{i=1}^m$ and $\gamma\backslash\tilde{\gamma}$, $\alpha\backslash\tilde{\alpha}$, $\beta\backslash\tilde{\beta}$. To this end one must answer the question whether $\tilde{\gamma}\in R_m(\tilde{\alpha},\tilde{\beta})$, $\gamma\backslash\tilde{\gamma}\in R_{n-m}(\alpha\backslash\tilde{\alpha},\ \beta\backslash\tilde{\beta})$. If all inequalities generated by $r\|s|t\in \mathrm{Rul}_m^*(n)$, $1\le m\le n-1$ are strict for $\gamma\in R_n(\alpha,\beta)$ (that is, $\sum_r \gamma_i<\sum_s \alpha_i+\sum_t \beta_i$ for each $r\|s|t\in\mathrm{Rul}_m^*(n)$, $1\le m\le n-1$), then one can replace γ by γ', β by β' (or α by α') such that $\gamma'\in R_n(\alpha,\beta')$, $\gamma_i'\le\gamma_i$, $\beta_i'\le\beta_i$, $\sum_{i=1}^n\gamma_i'<\sum_{i=1}^n\gamma_i$, and try to prove that $\gamma'\in\Delta_n(\alpha,\beta')$ implies that $\gamma\in\Delta_n(\alpha,\beta)$. In view of the additional assumptions $\beta_1>\beta_2$, $\gamma_1>\gamma_2$ in the case $n=4$ a good candidate for this type of reduction is $\gamma_1'=\gamma_1-1$, $\beta_1'=\beta_1-1$, $\gamma_i'=\gamma_i$, $\beta_i'=\beta_i$, $i\ge 2$. In that case $\gamma'\in\Delta(\alpha,\beta')$ would imply $\gamma\in\Delta(\alpha,\beta)$:

LEMMA 4.1. *Let γ,α,β be multiplicity sequences of order n such that $\gamma\in\Delta_n(\alpha,\beta)$ (resp. $\gamma\in R_n(\alpha,\beta)$). Define $\hat{\gamma},\hat{\beta}$ by $\hat{\gamma}_1=\gamma_1+1$, $\hat{\beta}_1=\beta_1+1$, $\hat{\gamma}_i=\gamma_i$, $\hat{\beta}_i=\beta_i$, $2\le i\le n$. Then $\hat{\gamma}\in\Delta_n(\alpha,\hat{\beta})$ (resp. $\hat{\gamma}\in R_n(\alpha,\hat{\beta})$)*

PROOF. Since there are no triplets $r\|s|t\in\mathrm{Rul}(n)$ with $r_1=1$, $t_1>1$, the statement is evidently true with respect to R_n. So we assume that $\gamma\in\Delta_n(\alpha,\beta)$ and that $\alpha=\sigma^0,\sigma^1,\ldots,\sigma^r=\gamma$, $r=\beta_1$ be multiplicity sequences such that the conditions (2.1) of Theorem 2.1 are met for α,β,γ. Define $\sigma^{r+1}=\hat{\gamma}$, i.e, $\sigma_i^{r+1}-\sigma_i^r=0$, $i\ge 2$, $\sigma_1^{r+1}-\sigma_1^r=1$. Then the sequence $\sigma^0,\sigma^1,\ldots,\sigma^r,\sigma^{r+1}$ also meets the conditions (2.1)(a), (b), as $\sum_{j=\ell}^n(\sigma_j^{r+1}-\sigma_j^r)=0$, $\ell\ge 2$, $\sum_{j=1}^n(\sigma_j^{r+1}-\sigma_j^r)=1\le\sum_{j=1}^n(\sigma_j^r-\sigma_j^{r-1})$ (since $r=\beta_1$), and condition (2.1)(c) holds for $\hat{\beta}$, as $\#\{1\le i\le r\mid \sum_{j=1}^n(\sigma_j^{i+1}-\sigma_j^i)=t\}=\delta_{1t}+\#\{1\le i\le r-1\mid \sum_{j=1}^n(\sigma_j^{i+1}-\sigma_j^i)=t\}$. Thus $\hat{\gamma}\in\Delta_n(\alpha,\beta)$. ■

The applicability of the "splitting" principle relies on the verification of the following

CONJECTURE 4.2. *If $\gamma\in R_n(\alpha,\beta)$ and $r\|s|t\in\mathrm{Rul}_m^*(n)$, $1\le m\le n-1$ are such that $\sum_{i=1}^m\gamma_{r_i}=\sum_{i=1}^m(\alpha_{s_i}+\beta_{t_i})$, then $\gamma_r\in R_m(\alpha_s,\beta_t)$, $\gamma_{r^c}\in R_{n-m}(\alpha_{s^c},\beta_{t^c})$, where $\gamma_r=(\gamma_{r_i})_{i=1}^m$, $\alpha_s=(\alpha_{s_i})_{i=1}^m$, $\beta_t=(\beta_{t_i})_{i=1}^m$, and $\gamma_{r^c}=\gamma\backslash\gamma_r$, $\alpha_{s^c}=\alpha\backslash\alpha_s$, $\beta_{t^c}=\beta\backslash\beta_t$.*

If this conjecture can be verified for a fixed index triplet $r\|s|t\in\mathrm{Rul}_m^*(n)$, for each $\gamma\in R_n(\alpha,\beta)$, then we say that $r\|s|t$ *admits splitting.* That this "splitting property" needs only to be considered for minimal index triplets in Rul follows from the observation.

LEMMA 4.3 *If $r\|s|t$, $r'\|s'|t'\in\mathrm{Rul}_m(n)$, $r\|s|t\ge r'\|s'|t'$ and $\gamma\in R_n(\alpha,\beta)$ is such that $\sum_{i=1}^m\gamma_{r_i}=\sum_{i=1}^m(\alpha_{s_i}+\beta_{t_i})$, then $\sum_{i=1}^m\gamma_{r_i'}=\sum_{i=1}^m(\alpha_{s_i'}+\beta_{t_i'})$ as well.*

Indeed, $\sum_{i=1}^{m}\gamma_{r_i}\le\sum_{i=1}^{m}\gamma_{r'_i}\le\sum_{i=1}^{m}(\alpha_{s'_i}+\beta_{t'_i})\le\sum_{i=1}^{m}(\alpha_{s_i}+\beta_{t_i})$ if $r\|s|t\ge r'\|s'|t'\in\mathrm{Rul}_m$.

Next we describe a class of index triplets which admit splitting, and we extend this class by proving that the splitting property is compatible with the symmetry properties of [11], Proposition 1.1.

LEMMA 4.4. *Let* $r_i=s_i$, $t_i=i$, $i=1,2,\dots,m$ for $r\|s|t\in\mathrm{Rul}_m^*(n)$. *Then* $r\|s|t$ *admits splitting.*

PROOF. (i) If $x\|y|z\in\mathrm{Rul}_\ell(m)$, then we define $x'\|y'|z'$ by $x'_i=r_{x_i}$, $y'_i=s_{y_i}$, $z'_i=t_{z_i}=z_i$, $i=1,2,\dots,m$. Then $x'\|y'|z'\in\mathrm{Rul}_\ell(n)$: Indeed, since $r_j=s_j$, $t_j=j$, one can apply Theorem 3.8.(ii) in [9] in order to reduce the verification to the case where $r=s=t=\{1,\dots,m\}$ and hence $x'=x$, $y'=y$, $z'=z$.
(ii) If $x\|y|z\in\mathrm{Rul}_\ell(n-m)$, then we define $x'\|y'|z'$ as follows: $x'_j=r_{i-j}$ if $x_j+(i-j-1)<r_{i-j}<x_{j+1}+(i-j)$, $x'_i=i-j+x_j$ if $r_{i-j}<i-j+x_j<r_{i-j+1}$, y'_i is defined in the same way, replacing r by s, x,x' by y,y', and $z'_i=i$, $i\le m$, $z'_i=z_{i-m}+m$, $i>m$ ($x'\|y'|z'$ is constructed by "inserting" the index triplet $r_x^c\|s_y^c|t_z^c$ into $r\|s|t$). Then $x'\|y'|z'\in\mathrm{Rul}_{\ell+m}(n)$: Indeed, we can use induction on m. For $m=1$ the statement is identical to the statement of Theorem 2.7(i) in [11]. If the statement has been proved for $m-1$, then we consider $r'\|s'|t'\in\mathrm{Rul}_{m-1}^*(n-1)$ defined by $r'_i=s'_i=r_{i+1}=s_{i+1}$, $t'_i=i$, $1\le i\le m-1$, and construct $x''\|y''|z''$ by inserting $r_x'^c\|s_y'^c|z_z'^c$ into $r'\|s'|t'$. Then $x''\|y''|z''\in\mathrm{Rul}_{\ell+m-1}(n-1)$, and the statement follows from Theorem 2.7(i) in [11] through removing $x'_k=r_1=s_1=y'_k$, $1=z'_1$ from $x'\|y'|z'$.
(iii) Let $\gamma\in R_n(\alpha,\beta)$ and $\sum_{i=1}^{m}\gamma_{r_i}=\sum_{i=1}^{m}(\alpha_{s_i}+\beta_{t_i})$. According to (i) one has $\sum_x\gamma_{r_i}\le\sum_y\alpha_{s_i}+\sum_z\beta_{t_i}$ for $x\|y|z\in\mathrm{Rul}_\ell(m)$, and $\sum_x\gamma_{r_i^c}=\sum_{x'}\gamma_j-\sum_r\gamma_j\le(\sum_{y'}\alpha_j+\sum_{z'}\beta_j)-(\sum_s\alpha_j+\sum_t\beta_j)=\sum_y\alpha_{s_i^c}+\sum_z\beta_{t_i^c}$ for $x\|y|z\in\mathrm{Rul}_\ell(n-m)$, according to (ii). Hence one has $\gamma_r\in R_m(\alpha_s,\beta_t)$ and $\gamma\backslash\gamma_r\in R_{n-m}(\alpha\backslash\alpha_s,\ \beta\backslash\beta_t)$. ■

LEMMA 4.5. *Assume that the index triplet* $r\|s|t\in\mathrm{Rul}_m^*(n)$, $1\le m\le n$ *admits splitting. Then* $r\|t|s$ *admits splitting, and the inversion and both reflexions and complements of* $r\|s|t$, $r\|t|s$ *admit splitting.*

PROOF. Since $r\|s|t\in\mathrm{Rul}$ if and only if $r\|t|s\in\mathrm{Rul}$, and $R(\alpha,\beta)=R(\beta,\alpha)$ the first statement is, evidently, correct. In order to prove the second statement one observes that for a given multiplicity sequence $\alpha=(\alpha_i)_{i=1}^n$, $a\ge\alpha_1$, one has that $a-\alpha_s=(a-\alpha)_{n+1-s}$, $a-\alpha_{s^c}=(a-\alpha)_\sigma$. Consider the complement $t^c\|\sigma|r^c$ of $r\|s|t$. Assume that for some $\gamma\in R_n(\alpha,\beta)$ one has $\sum_{t^c}\gamma_j=\sum_\sigma\alpha_j+\sum_{r^c}\beta_j$ and choose $a\ge\alpha_1$. Then $\sum_r(a+\beta_j)=\sum_{\sigma^c}(a-\alpha_j)+\sum_t\gamma_j=\sum_{n+1-s}(a-\alpha_j)+\sum_t\gamma_j=\sum_s(a-\alpha)_j+\sum_t\gamma_j$. As $r\|s|t$ admits splitting this implies that $(a+\beta)_r\in R_m((a-\alpha)_s,\gamma_t)$, $(a+\beta)_{r^c}\in R_{n-m}((a-\alpha)_{s^c},\gamma_{t^c})$. Hence

$\gamma_t \in R_m(a-(a-\alpha)_s,\beta_r) = R_m(\alpha_{n+1-s},\beta_r) = R_m(\alpha_{\sigma^c},\beta_r)$, $\gamma_{t^c} \in R_{n-m}(a-(a-\alpha)_{s^c},\beta_{r^c}) = R_{n-m}(\alpha_\sigma,\beta_{r^c})$. This proves that $t^c\|\sigma|r^c$ admits splitting. Since the inversion and the reflexions of $r\|s|t$ can be obtained by repeated transition to a complement in combination with of the symmetry property from in the first statement, the other claims follow as well. ■

Next, consider $\gamma = (\gamma_i)_{i=1}^4 \in R_4(\alpha,\beta)$. Set $m(\gamma) = \gamma_1+\gamma_2+\gamma_3+\gamma_4$. If $m(\gamma) \le 10$ then at least one of the additional assumptions mentioned in the beginning of this section is not met, and $\gamma \in \Delta_4(\alpha,\beta)$. We proceed by induction. Assume that we have proved that $\gamma \in R_4(\alpha,\beta)$ implies $\gamma \in \Delta_4(\alpha,\beta)$ if $m(\gamma) \le k$. Take $\gamma \in R_4(\alpha,\beta)$ with $m(\gamma) = k+1$. If $\gamma_1 = \gamma_2$ or $\beta_1 = \beta_2$ then $\gamma \in \Delta_4(\alpha,\beta)$. So assume that $\gamma_1 > \gamma_2$, $\beta_1 > \beta_2$. If $\sum_r \gamma_j = \sum_s \alpha_j + \sum_t \beta_j$ for some index triplet $r\|s|t \in \mathrm{Rul}_m^*(4)$ which admits splitting, then $\gamma_r \in R_m(\alpha_s,\beta_t) = \Delta_m(\alpha_s,\beta_t)$, $\gamma_{r^c} \in R_{4-m}(\alpha_{s^c},\beta_{t^c})$, and hence $\gamma = \gamma_r \cup \gamma_{r^c} \in \Delta_4(\alpha,\beta)$. Now observe that each index triplet $r\|s|t \in \mathrm{Rul}^*(4)$ with $1 \notin r$, $1 \in t$ admits splitting: For $x\|x|1$, $x = 2,3,4$ and for $23\|23|12$, $24\|24|12$, $34\|34|12$ this follows from Lemma 4.4, and inversion and complementation yield all relevant order 3 triplets and $34\|14|14$, $34\|24|13$. So we need to consider only $23\|13|13$, $24\|23|13$ and $24\|13|14$, where the second, resp. third triplet is a reflexion, resp. complementation of the first, and it suffices to prove that $23\|13|13$ admits splitting. To this end, let $\gamma' \in R_4(\alpha',\beta')$ and $\gamma_2'+\gamma_3' = \alpha_1'+\alpha_3'+\beta_1'+\beta_3'$. Then $(\gamma_2',\gamma_3') \in R_2((\alpha_1',\alpha_3'),(\beta_1',\beta_3'))$, as $\gamma_2' \le \alpha_1'+\beta_1'$, $\gamma_3' \le \alpha_1'+\beta_3',\alpha_3'+\beta_1'$, and $(\gamma_1',\gamma_4') \in R_2((\alpha_2',\alpha_4'),(\beta_2',\beta_4'))$: From $\gamma_1'+\gamma_2'+\gamma_3' \le \alpha_1'+\alpha_2'+\alpha_3'+\beta_1'+\beta_2'+\beta_3'$ one has $\gamma_1' \le \alpha_2'+\beta_2'$, whereas $\gamma_4' \le \alpha_2'+\beta_4'$ follows from $\gamma_2'+\gamma_3'+\gamma_4' \le \alpha_1'+\alpha_2'+\alpha_3'+\beta_2'+\beta_3'+\beta_4' \le \alpha_1'+\alpha_2'+\alpha_3'+\beta_1'+\beta_3'+\beta_4'$ and $\gamma_4' \le \alpha_4'+\beta_2'$ is derived in the same way. Now assume that $\sum_r \gamma_j < \sum_s \alpha_j + \sum_t \beta_j$ for each $r\|s|t \in \mathrm{Rul}^*(4)$ with $1 \notin r$, $1 \in t$. Then $\tilde{\gamma} \in R_4(\alpha,\tilde{\beta})$, where $\tilde{\gamma}_1 = \gamma_1 - 1$, $\tilde{\beta}_1 = \beta_1 - 1$, $\tilde{\beta}_i = \beta_i$, $\tilde{\gamma}_i = \gamma_i$, $i = 2,3,4$. As $m(\tilde{\gamma}) = k$ this implies that $\tilde{\gamma} \in \Delta_4(\alpha,\tilde{\beta})$, and Lemma 4.1 yields that $\gamma \in \Delta_4(\alpha,\beta)$. Hence we have shown that $\gamma \in \Delta_4(\alpha,\beta)$ for each $\gamma \in R_4(\alpha,\beta)$ with $m(\gamma) = k+1$. By induction we thus have proved the following result:

THEOREM 4.6. *Let $\alpha = (\alpha_i)_{i=1}^4$, $\beta = (\beta_i)_{i=1}^4$ be multiplicity sequences of order 4. In order that $\gamma = (\gamma_i)_{i=1}^4 \in \Delta_4(\alpha,\beta)$ it is necessary and sufficient that $\gamma \in R_4(\alpha,\beta)$ i.e., that $\sum_{i=1}^4 \gamma_i = \sum_{i=1}^4 (\alpha_i+\beta_i)$ and $\sum_r \gamma_i \le \sum_s \alpha_i + \sum_t \beta_i$ for each $r\|s|t \in \mathrm{Rul}^*(4)$.*

Added in proof: Since the completion of the present paper a formal proof for the equivalence $\gamma \in \Delta(\alpha,\beta) \Leftrightarrow \gamma^* \in \Delta(\alpha^*,\beta^*)$ (see the introduction of Section 3) was presented by Ion Zaballa, cf. [16]. A private communication of the same author provided the additional information that this equivalence can also be obtained as an easy consequence of some of the results in Ch. I of I.G. Macdonalds book *Symmetric Functions and Hall Polynomials*.

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Econometrisch Instituut
Erasmus Universiteit
Postbus 1738
3000 DR ROTTERDAM
The Netherlands

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