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LOGARITHMIC RESIDUES IN BANACH ALGEBRAS

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Let f be an analytic Banach algebra valued function and suppose that the contour integral of the logarithmic derivative $f'f^{-1}$ around a Cauchy domain D vanishes. Does it follow that f takes invertible values on all of D? For important classes of Banach algebras, the answer is positive. In general, however, it is negative. The counterexample showing this involves a (nontrivial) zero sum of logarithmic residues (that are in fact idempotents). The analysis of such zero sums leads to results about the convex cone generated by the logarithmic residues.

1. INTRODUCTION

Let D be a bounded Cauchy domain in the complex plane $\mathbb{C}$ (cf. [TL]), and let f be a complex-valued function with the following properties: f is defined and analytic in an open neighbourhood of $\bar{D}$ and f does not vanish on ∂D, the (positively oriented) boundary of D. It is known from complex function theory that the logarithmic residue

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(\lambda)}{f(\lambda)}\,d\lambda$$

is equal to the number of zeros of f in D (counted according to multiplicity). In particular, the identity

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(\lambda)}{f(\lambda)}\,d\lambda = 0$$

implies that f does not vanish in D.

The issue in this paper is the extension of the latter result to the more general setting where f takes its values in a (complex) Banach algebra rather than in $\mathbb{C}$. Of course, the desired conclusion then takes the form: $f(\lambda)$ is invertible for all $\lambda \in D$. Several instances are known where such a generalization is valid indeed.

The most notable among these comes from spectral theory: If T is a bounded linear operator on a Banach space X and D is a bounded Cauchy domain such that ∂D is contained in the resolvent set $\rho(T)$ of T, then the identity

$$(1.1) \qquad \frac{1}{2\pi i}\int_{\partial D} (\lambda I_X - T)^{-1} d\lambda = 0$$

implies that D is contained in $\rho(T)$, i.e., $\lambda I_X - T$ is invertible for all $\lambda \in D$. The left hand side of (1.1) is the spectral or Riesz projection associated with T and D.

Another generalization involves the case when f takes its values in a commutative Banach algebra. This situation has been discussed in [B1]. The argument presented there is based on a simple application of standard Gelfand Theory. By the way, the result discussed in the previous paragraph fits into this framework. To see this, consider the closed subalgebra generated by T and the identity operator I_X.

Finally, we mention that a positive result exists for Fredholm operator valued functions. This follows from the material on multiplicities of operator valued functions developed by I.C. Gohberg and E.I. Sigal [GS]. In the present paper (Section 3) we shall give a new proof of this result using the systems theoretical methods from [BGK1].

How about the general case? As it turns out, the picture is mixed. On the one hand, using material from [BES1], we present a (nonexotic) counterexample showing that the desired generalization does not always hold true (Section 4). On the other hand, we demonstrate that the generalization is valid for a variety of important Banach algebras (also Section 4). For instance, a positive result is obtained for polynomial-identity Banach algebras. Another interesting example is the Banach subalgebra of $\mathcal{L}(L_2(\mathbb{T}))$ generated by all compact operators on $L_2(\mathbb{T})$, all operators on $L_2(\mathbb{T})$ of multiplication by piecewise-continuous functions, and the operator S of singular integration along $\mathbb{T}$. Here $\mathbb{T}$ is the unit circle in $\mathbb{C}$.

It is worthwhile to say something about the nature of the counterexample mentioned in the preceeding paragraph. Banach algebras for which the generalization that we are looking for is valid have the following property: A finite sum of idempotents can only vanish if all terms in the sum vanish individually. Now there do exist Banach algebras lacking this property. This has been established in [BES1], where the theme of zero sums of idempotents is taken up as a separate topic (see Section 6 for a brief summary). An example is provided by the Banach algebra of all bounded operators on infinite dimensional Hilbert space.

In the present paper, we pay attention to the more general phenomenon of zero sums of logarithmic residues (Section 5). The discussion leads to results on the convex cone generated by the logarithmic residues. One of these results (Theorem 5.2) can be viewed as a comment on an open problem stated by N. Krupnik ([K], Section 29, Problem 12).

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2. PRELIMINARIES ON LOGARITHMIC RESIDUES

Throughout this section, B will be a (complex) Banach algebra with unit element. In working with contour integrals, we shall employ bounded Cauchy domains (in $\mathbb{C}$) and their (positively oriented) boundaries. For a discussion of these notions, see, for instance, [TL].

Let D be a bounded Cauchy domain. The (positively oriented) boundary of D will be denoted by ∂D. We write $\mathcal{A}_\partial(D;B)$ for the set of all B-valued functions f with the following properties: f is defined and analytic on an open neighbourhood of the closure $\bar{D}$ $(=D\cup\partial D)$ of D and f takes invertible values on all of ∂D. For $f\in\mathcal{A}_\partial(D;B)$ one can then define

$$(2.1)\qquad \mathrm{LogRes}(f;D)=\frac{1}{2\pi i}\int_{\partial D} f'(\lambda)f(\lambda)^{-1}d\lambda$$

where f' stands for the derivative of f.

The elements of the form $\mathrm{LogRes}(f;D)$ are the logarithmic residues in B. More specifically we call $\mathrm{LogRes}(f;D)$ the *logarithmic residue of f with respect to D.*

Since B is not assumed to be commutative, a remark is in order here. The expression (2.1) involves the *left* logarithmic derivative $f'(\lambda)f(\lambda)^{-1}$of f. It is just as well possible to work with the *right* logarithmic derivative $f(\lambda)^{-1}f'(\lambda)$ or even with logarithmic derivatives of *mixed* type: $f(\lambda)^{-k}f'(\lambda)f(\lambda)^{k-1}$, where k is an arbitrary (but fixed) integer. Such an alternative approach leads to analogous results.

It is illuminating to give some examples. The first and third formalize things already outlined in the introduction. Along the way we shall fix additional notation and terminology.

EXAMPLE 2.1 Suppose $B=\mathbb{C}$, so we are dealing with scalar analytic functions. Then $\mathrm{LogRes}(f;D)$, provided it is well-defined, is equal to the number of zeros of f in D counted according to multiplicity. In particular, $\mathrm{LogRes}(f;D)=0$ if and only if f does not vanish in D. □

EXAMPLE 2.2 Let B be the Banach algebra of all upper triangular complex $n\times n$

matrices. It is easy to see that in this case too, $\operatorname{LogRes}(f;D)=0$ if and only if f takes invertible values on D.

To describe the logarithmic residues in B, we introduce some notation. By $\mathcal{R}$ we denote the set of all (upper triangular) complex $n \times n$ matrices R with the following property: There exists $j \in \{1,\ldots,n\}$ such that

1) the first $j-1$ colums of R are zero,
2) the last $n-j$ rows of R are zero,
3) the (j,j)-th entry of R (so the j-th diagonal entry of R) is a nonnegative integer larger than or equal to the rank of R.

Also, let $\mathcal{R}'$ be defined in the same way as $\mathcal{R}$ with the understanding that 3) is replaced by

3') the (j,j)-th entry of R and the rank of R are both equal to 1.

Note that $\mathcal{R}'$ consists of the $n \times n$ matrices of the form

$$\sum_{s=1}^{j} \sum_{t=j}^{n} \alpha_s \alpha_t e_s e_t^T$$

where $e_1,\ldots,e_n$ is the standard basis in $\mathbb{C}^n$, $\alpha_1,\ldots,\alpha_n$ are complex numbers, $\alpha_j=1$, and the superscript T signals the operation of taking the transpose. The integer j is allowed to range from 1 to n.

The logarithmic residues in B are precisely the finite sums of elements in $\mathcal{R}$. Clearly $\mathcal{R}'$ is a subset of $\mathcal{R}$ and it can be shown that each element of $\mathcal{R}$ is a finite sum of elements of $\mathcal{R}'$. So the logarithmic residues in B can also be characterized as the finite sums of elements in $\mathcal{R}'$, where, of course, the zero matrix corresponds to the empty sum. It is easy to see that $\mathcal{R}'$ is the set of all idempotent upper triangular rank one $n \times n$ matrices. Below we shall prove that each sum of idempotents is a logarithmic residue (Example 2.4 and Proposition 2.5). Thus the logarithmic residues in B are just the finite sums of idempotent upper triangular $n \times n$ matrices (of rank at most one).

Specializing to $n=2$, we have that the logarithmic residues in the Banach algebra of all upper triangular complex 2×2 matrices are the matrices of the form

$$\begin{pmatrix} m & \alpha \\ 0 & k \end{pmatrix}$$

where m and k are nonnegative integers, α is a complex number and α is zero whenever both m and k vanish. This is easy to prove. In the general case of arbitrary n, the verification of the facts presented above is somewhat more complicated. For details, see [BES2]. □

If Y is a (complex) Banach space, then $\mathcal{L}(Y)$ stands for the Banach algebra of all bounded linear operators on Y. The unit element in $\mathcal{L}(Y)$ is the identity operator I_Y on Y. The spectrum of an element $T\in\mathcal{L}(Y)$ will be denoted by $\sigma(T)$, the resolvent set by $\rho(T)$.

EXAMPLE 2.3 Suppose $B=\mathcal{L}(Y)$, where Y is a complex Banach space. Let $T\in\mathcal{L}(Y)$, and write $F_T(\lambda)=\lambda I_Y-T$. If D is a bounded Cauchy domain, then $\mathrm{LogRes}(F_T;D)$ is defined if and only if $\sigma(T)\cap\partial D=\emptyset$, and in that case $\mathrm{LogRes}(F_T;D)=P(T;D)$, where

$$P(T;D)=\frac{1}{2\pi i}\int_{\partial D}(\lambda I_Y-T)^{-1}d\lambda$$

is the *Riesz projection* associated with T and D. From spectral theory we know that $P(T;D)^2=P(T;D)$, so $P(T;D)$ is an idempotent in $\mathcal{L}(Y)$. Another important fact is that $P(T;D)=0$ if and only if $D\subset\rho(T)$, i.e., $F_T(\lambda)=\lambda I_Y-T$ is invertible for all $\lambda\in D$. For a generalization to the case where λI_Y-T is replaced by the more general pencil $\lambda S-T$, see [St] and [GGK].

EXAMPLE 2.4 Returning to an arbitrary Banach algebra B with unit element e, we note that each idempotent in B is a logarithmic residue in B. To see this, let p be an idempotent in B, write $f(\lambda)=e-p+\lambda p$, and note that $f'(\lambda)f(\lambda)^{-1}=\lambda^{-1}p$, $\lambda\neq 0$. □

In Example 2.4, the inverse $f(\lambda)^{-1}$ has a simple pole at the origin. In fact, a logarithmic residue corresponding to a simple pole of the inverse of an analytic function is always an idempotent. To see this, consider Laurent expansions and use the relations between their coefficients. A rather complicated and restrictive sufficient condition on f under which $\mathrm{LogRes}(f;D)$ is an idempotent has been given in [Mt].

We conclude this section with two simple observations.

PROPOSITION 2.5 *A (finite) sum of logarithmic residues in a Banach algebra B is again a logarithmic residue in B.*

PROOF It is sufficient to prove that the sum of two logarithmic residues in B is again a logarithmic residue in B. Let b_1 and b_2 be logarithmic residues in B. Write $b_1=\mathrm{LogRes}(f_1;D_1)$ and $b_2=\mathrm{LogRes}(f_2;D_2)$ with appropriately chosen f_1,D_1 and f_2,D_2. We may assume that $\bar{D}_1\cap\bar{D}_2=\emptyset$ (apply a simple translation). Put $D=D_1\cup D_2$, and let f be a function coinciding with f_1 on an open neighbourhood of $\bar{D}_1$ and with f_2 on an open neighbourhood of $\bar{D}_2$. Then $b_1+b_2=\mathrm{LogRes}(f;D)$. □

PROPOSITION 2.6 *Let b be a logarithmic residue in a Banach algebra B, and let s be an invertible element in B. Then $s^{-1}bs$ is again a logarithmic residue in B.*

PROOF Write $b=\mathrm{LogRes}(f;D)$ and put $g(\lambda)=s^{-1}f(\lambda)s$. Then $g'(\lambda)g(\lambda)^{-1}=s^{-1}f'(\lambda)f(\lambda)^{-1}s$, and hence $s^{-1}bs=\mathrm{LogRes}(g;D)$. □

3. LOGARITHMIC RESIDUES OF FREDHOLM OPERATOR VALUED FUNCTIONS

As we shall see in subsequent sections, logarithmic residues of matrix valued or, more generally, Fredholm operator valued functions are of special interest. For that reason this section is devoted to them. We begin by establishing some notation and terminology.

Let Y be a (complex) Banach space, and let $T \in \mathcal{L}(Y)$. The null space of T will be denoted by $\mathrm{Ker}T$, its image by $\mathrm{Im}T$. The operator T is called a *Fredholm* operator if $\mathrm{Ker}T$ is finite-dimensional and $\mathrm{Im}T$ has finite codimension in Y. The latter condition implies that $\mathrm{Im}T$ is closed (cf. [GGK], Chapter XI).

Now let D be a bounded Cauchy domain, let $F \in \mathcal{A}_{\partial}(D;\mathcal{L}(Y))$, and assume that $F(\lambda)$ is a Fredholm operator for all $\lambda \in D$. Write Γ for the set of all μ in D for which $F(\mu)$ is not invertible. From the literature on analytic Fredholm operator valued functions (see [GGK], and the references given there), it is known that Γ is a finite set and that the function $F(\lambda)^{-1}$ is *finite meromorphic* on D. The latter means that $F(\lambda)^{-1}$ is meromorphic on D and that the principal parts of the Laurent expansions around the points of D have finite rank coefficients only. These facts imply that

$$\mathrm{LogRes}(F;D) = \frac{1}{2\pi i} \int_{\partial D} F'(\lambda) F(\lambda)^{-1} d\lambda$$

is a finite sum of finite rank operators (one for each point of Γ), and hence a finite rank operator itself.

THEOREM 3.1 *Let $F \in \mathcal{A}_{\partial}(D;\mathcal{L}(Y))$, where Y is a Banach space and D is a bounded Cauchy domain. Assume $F(\lambda)$ is a Fredholm operator for each $\lambda \in D$. Then* $\mathrm{LogRes}(F;D)$ *is a finite rank operator whose trace is a nonnegative integer. Further, the following statements are equivalent:*

(*i*) $\mathrm{LogRes}(F;D) = 0$,

(*ii*) $\mathrm{trace}\left[\mathrm{LogRes}(F;D)\right] = 0$,

(*iii*) *F takes invertible values on all of D.*

A quick proof of the theorem can be obtained by using the material on multiplicities of analytic Fredholm operator valued functions developed by I.C. Gohberg and E.I. Sigal [GS]. The key point in this approach is that

$$\operatorname{trace}\left[\operatorname{LogRes}(F;D)\right] = \sum_{\mu\in\Gamma} M(F;\mu)$$

where Γ is as above and $M(F;\mu)$ is the algebraic multiplicity of F at μ (see also [GGK], and the references given there). We prefer, however, to give a proof based on the ideas involving systems theoretical concepts developed in [BGK1]. This new proof is another application of the so called State Space Method in analysis (cf. [BGK4] and [B2]).

PROOF Let F be defined and analytic on the open neighbourhood Ω of $\bar{D}$. Since D is bounded we may assume that Ω is bounded too. But then F admits a *realization* on Ω. This means that there exist a Banach space X and bounded linear operators $A:X\to X$, $B:Y\to X$ and $C:X\to Y$ such that

$$F(\lambda) = I_Y + C(\lambda I_X - A)^{-1}B, \qquad \lambda\in\Omega\subset\rho(A) \tag{3.1}$$

where $\rho(A)$ stands for the resolvent set of A. The space X and the operators A,B and C can be constructed explicity (cf. [Mg], [BGK1], and the references given there).

Consider the right hand side of the realization (3.1) which is well-defined for all $\lambda \in \rho(A)$. Invertibility aspects of this right hand side can be described in terms of the operator $A^{\times} = A - BC$. This fact is well-known and has been used many times before (cf., e.g., [BGK1] and [B2]). Restricting ourselves to values of λ in the subset Ω of $\rho(A)$, we have: $F(\lambda)$ is invertible if and only if $\lambda \in\rho(A^{\times})$, and in that case

$$F(\lambda)^{-1} = I_Y - C(\lambda I_X - A^{\times})^{-1}B \tag{3.2}$$

$$(\lambda I_X - A^{\times})^{-1} = (\lambda I_X - A)^{-1} - (\lambda I_X - A)^{-1}B\, F(\lambda)^{-1}\, C(\lambda I_X - A)^{-1}. \tag{3.3}$$

Verification can be carried out by direct computation taking into account the identity $BC = (\lambda I_X - A^{\times}) - (\lambda I_X - A)$ which gives rise to the necessary cancellations.

It follows that $\partial D \subset \rho(A^{\times})$ and $\Gamma = \sigma(A^{\times}) \cap D$, where Γ is the set of all μ in D such that $F(\mu)$ is not invertible. In particular $\sigma(A^{\times}) \cap D$ is a finite set. From (3.3) it is also clear that $(\lambda I_X - A^{\times})^{-1}$ is finite meromorphic on D. It is obvious now that the expressions

$$(3.4) \qquad \frac{1}{2\pi i}\int_{\partial D} -(\lambda I_X - A^\times)^{-1} BC(\lambda I_X - A)^{-1}\, d\lambda$$

$$(3.5) \qquad \frac{1}{2\pi i}\int_{\partial D} -C(\lambda I_X - A)^{-1}(\lambda I_X - A^\times)^{-1} B\, d\lambda$$

define finite rank operators on X and Y, respectively. These operators have the same trace. This can be seen by using the commutativity property of the trace and comparing the Laurent expansions of the integrals around the points of $\Gamma = \sigma(A^\times) \cap D$. Note in this context that the integrand in (3.5) is obtained from the integrand in (3.4) by changing the order of the factors $(\lambda I_X - A^\times)^{-1}B$ and $C(\lambda I_X - A)^{-1}$.

Now let us analyze (3.4). Replacing BC by $(\lambda I_X - A^\times) - (\lambda I_X - A)$, the expression (3.4) transforms into

$$\frac{1}{2\pi i}\int_{\partial D} \left[(\lambda I_X - A^\times)^{-1} - (\lambda I_X - A)^{-1}\right] d\lambda.$$

Recall that $(\lambda I_X - A)^{-1}$ is analytic on all of Ω. Thus (3.4) is equal to $P^\times$, where

$$P^\times = P(A^\times; D) = \frac{1}{2\pi i}\int_{\partial D} (\lambda I_X - A^\times)^{-1} d\lambda$$

is the Riesz projection of $A^\times$ corresponding to the bounded Cauchy domain D. Next we turn to (3.5). Combining (3.1) and (3.2), we see that for $\lambda \in \Omega \cap \rho(A^\times)$ the following identities hold true:

$$\begin{aligned} F'(\lambda)F(\lambda)^{-1} &= -C(\lambda I_X - A)^{-2}B\left[I_Y - C(\lambda I_X - A^\times)^{-1}B\right] \\ &= -C(\lambda I_X - A)^{-1}B + C(\lambda I_X - A)^{-2}BC(\lambda I_X - A^\times)^{-1}B \\ &= -C(\lambda I_X - A)^{-1}(\lambda I_X - A^\times)^{-1}B. \end{aligned}$$

The latter transition was again obtained by replacing BC by $(\lambda I_X - A^\times) - (\lambda I_X - A)$. It follows that (3.5) coincides with $\mathrm{LogRes}(F;D)$.

Recall that (3.4) and (3.5) have the same trace. Also $\mathrm{trace}\, P^\times = \mathrm{rank}\, P^\times$ since $P^\times$ is a projection (idempotent operator). Hence

$$(3.6) \qquad \mathrm{trace}\left[\mathrm{LogRes}(F;D)\right] = \mathrm{rank}\, P^\times.$$

In particular the trace of $\mathrm{LogRes}(F;\mathrm{D})$ is a nonnegative integer. It is zero if and only if $P^\times = 0$, and this implies that $\Gamma = \sigma(A^\times) \cap D = \emptyset$. Thus (*ii*) implies (*iii*). The implications $(i)\Rightarrow(ii)$ and $(iii)\Rightarrow(i)$ are obvious. □

There is one point in the above proof that deserves clarification: The realization (3.1) of F is not unique, and so, at first sight, the rank of $P^{\times}$ might vary with the choice of (3.1). The identity (3.6), however, shows that it does not. This is a special case of a more general phenomenon involving "spectral characteristics" associated with realizations which has been made transparant in [BGK3]; see also [BGK2].

Finally, note that in the matrix case $(Y=\mathbb{C}^n)$ things can be reduced to the scalar case (Example 2.1). For this, one employs the identity

$$\text{trace } F'(\lambda)F(\lambda)^{-1} = \frac{1}{\det F(\lambda)} \frac{d}{d\lambda} \det F(\lambda)$$

(see, for instance, [H]).

4. VANISHING LOGARITHMIC RESIDUES

Formulated in the language of Section 1, the issue in this paper is the following: *Does* $\mathrm{LogRes}(F;D)=0$ *imply that* f *takes invertible values on all of* D? We shall see that for a variety of important Banach algebras the answer is positive. In general, however, it is negative. An example will be given at the end of this section.

As before, the setting is a Banach algebra B with unit element, here to be denoted by e. By $\mathbb{C}^{n\times n}$ we mean the Banach algebra of all complex $n\times n$ matrices. The unit element in $\mathbb{C}^{n\times n}$ is the $n\times n$ identity matrix I_n. A mapping $\varphi:B\to\mathbb{C}^{n\times n}$ is called an *$n\times n$ matrix representation* of B if it is linear, continuous (bounded) and multiplicative. So a 1x1 matrix representation is just a multiplicative bounded linear functional on B. If φ is an $n\times n$ matrix representation of B, then $\varphi(e)$ is an idempotent $n\times n$ matrix. Hence $\varphi(e)$ is invertible if and only if $\varphi(e)=I_n$, and in that case $\varphi(b^{-1})=\varphi(b)^{-1}$ for each invertible $b\in B$. We call ψ a *matrix representation* of B if there exists n such that ψ is an $n\times n$ matrix representation for B. A nonempty set M of matrix representations of B is said to be a *sufficient family of matrix representations* for B if for each $b\in B$ the following holds: b is invertible if and only if $\det\psi(b)\neq 0$ for all $\psi\in M$.

This terminology is inspired by N. Krupnik [K]. In Section 29 of [K], the characterization of Banach algebras possessing a sufficient family of matrix representations is identified as an open problem. For our purposes here it is of interest to note that the class of Banach algebras possessing a sufficient family of matrix representations is large. It contains all matrix algebras $\mathbb{C}^{n\times n}$ (take $M=\{I_n\}$)

and all commutative Banach algebras with unit element (Gelfand Theory). More generally, each polynomial-identity Banach algebra possesses a sufficient family of matrix representations (cf. [K]). Recall that B is called a *polynomial-identity Banach algebra* if there exist a positive integer k and a polynomial $p(x_1,\dots,x_k)$ in k noncommuting variables $x_1,\dots,x_k$, $p\neq 0$, such that $p(b_1,\dots,b_k)=0$ for arbitrary $b_1,\dots,b_k\in B$. As an example of a polynomial-identity Banach algebra we mention $\mathbb{C}^{nxn}$. In that case one can take for p the so called standard polynomial involving 2n variables, i.e., the polynomial $\sum_{\sigma}(sgn\sigma)x_{\sigma(1)}\cdots x_{\sigma(2n)}$, where σ runs through the symmetric group S_{2n} (cf.[AL]). For a proof of this result and more examples, see [K].

THEOREM 4.1 *Suppose B is a Banach algebra possessing a sufficient family of matrix representations M, and let $f\in A_{\partial}(D;B)$, where D is a bounded Cauchy domain. The following statements are equivalent:*

(i) $\operatorname{LogRes}(f;D)=0$,

(ii) $\operatorname{trace}\left[\psi\left(\operatorname{LogRes}(f;D)\right)\right]=0$ *for all* $\psi\in M$,

(iii) *f takes invertible values on all of D.*

PROOF Obviously, (iii) implies (i) and (i) implies (ii). So it remains to prove that (ii) implies (iii). Take $\psi\in M$. Then

$$\psi\left(\operatorname{LogRes}(f;D)\right)=\operatorname{LogRes}(\psi\circ f;D). \tag{4.1}$$

Observe that $\psi\circ f$ is a matrix (hence Fredholm operator) valued function. Combining (4.1) and (i), we see that $\operatorname{trace}\left[\operatorname{LogRes}(\psi\circ f;D)\right]=0$. Now apply Theorem 3.1. This gives that $\psi(f(\lambda))$ is invertible for all $\lambda\in D$. Since ψ was taken arbitrarily from the sufficient family of matrix representations M, we may conclude that f takes invertible values on all of D. □

Our next result is about a type of Banach algebra that appears in the (numerical) work of S. Roch and B. Silbermann (cf. [Sm]). Banach algebras of this type need not possess a sufficient family of matrix representations (see Examples 4.4 and 4.5 below).

THEOREM 4.2 *Let B be a Banach algebra with unit element e and let Ω be a nonempty (index) set. Assume that for each $\omega\in\Omega$ there exists a Banach space X_ω, a continuous homomorphism W_ω: $B\to\mathcal{L}(X_\omega)$ and a twosided closed ideal J_ω in B such that*

(i) *$W_\omega(e)=I_\omega$, where I_ω is the identity operator on X_ω,*

(ii) *the restriction of W_ω to J_ω defines an isomorphism between J_w and the ideal of compact operators on X_ω.*

Suppose, in addition, that

$(iii)\quad J_{\omega_1} \subset \operatorname{Ker} W_{\omega_2}$ *for all* $\omega_1,\omega_2 \in \Omega,\ \omega_1 \neq \omega_2$.

Write J *for the smallest closed ideal in* B *containing all ideals* J_ω *and assume, finally, that the quotient space* B/J *possesses a sufficient family of matrix representations. Then, for each bounded Cauchy domain* D *and each* $f \in \mathcal{A}_\partial(D;B)$, *the identity*

$$\frac{1}{2\pi i}\int_{\partial D} f'(\lambda)f(\lambda)^{-1}d\lambda = 0$$

implies that f *takes invertible values on all of* D.

As will become clear in the proof, the condition on B/J can be relaxed to: for each bounded Cauchy domain D and each $\varphi \in \mathcal{A}_\partial(D;B/J)$, the identity

$$\frac{1}{2\pi i}\int_{\partial D} \varphi'(\lambda)\varphi(\lambda)^{-1}d\lambda = 0 \tag{4.2}$$

implies that φ takes invertible values on all of D. Note that condition (iii) in Theorem 4.2 is redundant when Ω consists of just one element (cf. Remark 4.3 below).

PROOF Let D be a bounded Cauchy domain, let $f \in \mathcal{A}_\partial(D;B)$, and suppose $\operatorname{LogRes}(f;D) = 0$. Write κ for the canonical homomorphism from B onto B/J, and put $\varphi = \kappa \circ f$. Then $\varphi \in \mathcal{A}_\partial(D;B/J)$ and (4.2) is satisfied. So our assumptions on B/J imply that φ takes invertible values on all of D. In other words $\kappa(f(\lambda))$ is invertible for all $\lambda \in D$.

By [Sm], Theorem 2, this implies that $W_\omega(f(\lambda))$ is Fredholm for all $\lambda \in D$ and all $\omega \in \Omega$. Take $\omega \in \Omega$ and write $F_\omega(\lambda) = W_\omega(f(\lambda))$. Then $F_\omega \in \mathcal{A}_\partial(D;L(X_\omega))$ and $F_\omega(\lambda)$ is a Fredholm operator for all $\lambda \in D$. Since W_ω is a continuous homomorphism, we have $\operatorname{LogRes}(F_w;D) = 0$. It follows from Theorem 3.1 that F_ω takes invertible values on all of D. In other words $W_\omega(f(\lambda))$ is invertible for all $\lambda \in D$.

From [Sm], Theorem 2, we know that an element $b \in B$ is invertible if and only if $\kappa(b)$ and all operators $W_\omega(b)$ are invertible. Take $\lambda \in D$. We have seen that $\kappa(f(\lambda))$ is invertible and that $W_\omega(f(\lambda))$ is invertible for all $\omega \in \Omega$. Hence $f(\lambda)$ is invertible. □

There is a variety of operator algebras to which Theorem 4.2 applies. The following remark illustrates this fact.

REMARK 4.3 Let Y be a complex Banach space and let $\mathcal{K}(Y)$ stand for the set of all compact operators on Y. Then $\mathcal{K}(Y)$ is a closed ideal in $\mathcal{L}(Y)$. Let

$\mathcal{C}(Y)=\mathcal{L}(Y)/\mathcal{K}(Y)$ be the Calkin algebra associated with Y, and let κ denote the canonical homomorphism from $\mathcal{L}(Y)$ onto $\mathcal{C}(Y)$. Suppose now that $\mathcal{A}$ is a closed subalgebra of $\mathcal{C}(Y)$ possessing a sufficient family of matrix representations. Then $B=\kappa^{-1}[\mathcal{A}]=\{T\in\mathcal{L}(Y)\,|\,\kappa(T)\in\mathcal{A}\}$ is a closed subalgebra of $\mathcal{L}(Y)$ to which Theorem 4.2 applies. Indeed, take $\Omega=\{0\}$, $X_0=Y$, $J_0=\mathcal{K}(Y)$ and let W_0 be the embedding of B into $\mathcal{L}(Y)$. □

We shall now discuss two instances of Banach algebras of the type considered in Theorem 4.2 that do not possess a sufficient family of matrix representations.

EXAMPLE 4.4 Consider the sequence space ℓ_2, and let B be the Banach subalgebra of $\mathcal{L}(\ell_2)$ consisting of all operators of the form $\alpha I+T$ where $\alpha\in\mathbb{C}$, I is the identity operator on ℓ_2 and $T\in\mathcal{K}(\ell_2)$, i.e., T is a compact operator on ℓ_2. Then B is a Banach algebra of the type discussed in Remark 4.3. To see this, take $Y=\ell_2$ and let $\mathcal{A}$ be the one-dimensional subalgebra of $\mathcal{C}(\ell_2)$ generated by $\kappa(I)$. In particular, B meets the requirements of Theorem 4.2. However, B does not possess a sufficient family of matrix representations. This can be proved as follows. It is known that $\mathcal{K}(\ell_2)$ is the unique (nontrivial) closed twosided ideal in B (cf. [K], Example 22.1). This implies that B has essentially only one matrix representation, namely the one which sends $\alpha I+T\in B$ into $\alpha\in\mathbb{C}$. This matrix representation obviously does not determine the invertibility of elements in B. □

EXAMPLE 4.5 Consider the space $L_2(\mathbb{T})$, where $\mathbb{T}=\{z\in\mathbb{C}\,|\,|z|=1\}$ is the unit circle, and let B be the Banach subalgebra of $\mathcal{L}(L_2(\mathbb{T}))$ generated by all compact operators on $L_2(\mathbb{T})$, all operators on $L_2(\mathbb{T})$ of multiplication by piecewise-continuous functions, and the operator S of singular integration along $\mathbb{T}$. This Banach algebra is studied extensively in [K], Chapter V. From the results obtained there, it can be deduced that the image $\mathcal{A}=\kappa[B]$ of B under the canonical homomorphism κ from $\mathcal{L}(L_2(\mathbb{T}))$ onto the Calkin algebra $\mathcal{C}(L_2(\mathbb{T}))$ possesses a sufficient family of matrix representations. It follows that $B=\kappa^{-1}[\mathcal{A}]$ is a Banach algebra to which Theorem 4.2 applies (cf. Remark 4.3). Again, B does not possess a sufficient family of matrix representations. To see this, we argue as follows. Let B_0 be the subalgebra of B consisting of all operators of the form $\alpha I+T$ where $\alpha\in\mathbb{C}$, I is the identity operator on $L_2(\mathbb{T})$ and T is compact. From Example 4.4 we know that B_0 does not possess a sufficient family of matrix representations. But then the same is true for B. Note here that the inverse of an operator in B_0, provided it exists, belongs to B_0. Hence B_0 is inverse closed, i.e., an element of B_0 is invertible in B_0 if and only if it is invertible in B. □

In the preceding example, the unit circle $\mathbb{T}$ can be replaced by suitable curves composed of Lyapunov arcs (cf. [GK1], [GK2] and [RS]). We now present an an example showing that, in general, the identity $\mathrm{LogRes}(f;D)=0$ does not imply that f takes invertible values on all of D.

EXAMPLE 4.6 Let H be an infinite-dimensional (separable) Hilbert space. The idempotents in $\mathcal{L}(H)$ are just the bounded projections on H. In [BES1], Section 3 we established that there exist five nonzero bounded projections on H such that $P_1+P_2+P_3+P_4+P_5=0$. Introduce

$$D=\bigcup_{j=1}^{5}\left\{z\in\mathbb{C}\ \middle|\ |z-j|<\tfrac{1}{4}\right\}.$$

Then D is a bounded Cauchy domain. Define $f\in\mathcal{A}_{\partial}(D;\mathcal{L}(H))$ by stipulating that

$$f(\lambda)=I_H-P_j+(\lambda-j)P_j,\quad |\lambda-j|<\tfrac{1}{3};\quad j=1,\dots,5.$$

Then $\mathrm{LogRes}(f;D)=P_1+P_2+P_3+P_4+P_5=0$. However, the function values $f(j)=I_H-P_j$ are not invertible in $\mathcal{L}(H)$. □

The function f appearing in the above example has a domain consisting of five components. Can one do with less? Using the idempotents $P_1,\dots,P_5$ appearing in [BES1], Example 3.1, we have been able to construct a counterexample involving a function defined on a domain consisting of three components. Of course, the ideal situation would be: just one component. For this, we tried

$$g(\lambda)=(I-P_1+\lambda P_1)(I-P_2+\lambda P_2)(I-P_3+\lambda P_3)(I-P_4+\lambda P_4)(I-P_5+\lambda P_5)\tag{4.3}$$

where, again, $P_1,\dots,P_5$ are as in [BES1], Example 3.1 and I is the appropriate identity operator. Unfortunately, the logarithmic residue of g (at the origin) turns out to be different from zero (see next paragraph). So this attempt to find a counterexample with a one component domain was unsuccessful.

If g is given by (4.3), where $P_1,\dots,P_5$ are (noncommuting) idempotents, then the logarithmic residue of g (at the origin) can be expressed as a polynomial in $P_1,\dots,P_5$. This polynomial consists of hundreds of monomials in $P_1,\dots,P_5$. The fact that the logarithmic residue does not vanish when $P_1,\dots,P_5$ are as in [BES1], Example 3.1 (so, in particular, $P_1+\dots+P_5=0$), was verified by using a computer package NCAlgebra developed by J.W. Helton and M. Stankus for simplifying algebraic expressions (noncommutative case). The results were rechecked on another computer package by J. Wavrik. NCAlgebra runs under Mathematica and is available from ncalg@osiris.ucsd.edu. The authors thank J.W. Helton and J.F. Kaashoek for their help in the computer aided analysis of the function g.

5. CONVEX CONES GENERATED BY LOGARITHMIC RESIDUES

In Example 4.6, we encountered a nontrivial zero sum of idempotents. An idempotent (in a Banach algebra with unit element) is always a logarithmic residue (see Example 2.4). This leads us to considering vanishing linear combinations of logarithmic residues involving positive coefficients only. We begin with a simple proposition that may serve as a background for Example 4.6.

PROPOSITION 5.1 *Let B be a Banach algebra with unit element. Assume that for each bounded Cauchy domain D and each $f \in \mathcal{A}_\partial(D;B)$, the identity*

$$\frac{1}{2\pi i}\int_{\partial D} f'(\lambda)f(\lambda)^{-1}d\lambda = 0$$

implies that f takes invertible values on all of D. Then a sum of logarithmic residues in B can vanish only if all terms in the sum vanish individually. In particular, a sum of idempotents in B can vanish only if these idempotents vanish individually.

PROOF For $j=1,\ldots,n$, let D_j be a bounded Cauchy domain and let $f_j \in \mathcal{A}_\partial(D_j;B)$. Assume

$$\sum_{j=1}^{n}\mathrm{LogRes}(f_j;D_j) = 0.$$

Applying appropriate translations, we can reach the situation where $\bar{D}_j \cap \bar{D}_k = \emptyset$, $j,k=1,\ldots,n$; $j \neq k$. The union $D = D_1 \cup \ldots \cup D_n$ then is a bounded Cauchy domain. Let $f \in \mathcal{A}_\partial(D;B)$ be a function which coincides with f_j on an open neighbourhood of $\bar{D}_j$, $j=1,\ldots,n$. Then

$$\mathrm{LogRes}(f;D) = \sum_{j=1}^{n}\mathrm{LogRes}(f_j;D_j)$$

and hence $\mathrm{LogRes}(f;D) = 0$. It follows that f takes invertible values on all of D. Thus, for $j=1,\ldots,n$, the function f_j takes invertible values on D_j and $\mathrm{LogRes}(f_j;D) = 0$. To get the statement about the zero sum of idempotents, note that each idempotent is a logarithmic residue (cf. Example 2.4). □

Consider the situation of Proposition 5.1. Let $b_1,\ldots,b_m$ be logarithmic residues in B, and suppose there exist positive *rational* numbers $\alpha_1,\ldots,\alpha_m$ such that $\alpha_1 b_1 + \ldots + \alpha_m b_m = 0$. By multiplying with an appropriate positive integer, we may assume that $\alpha_1,\ldots,\alpha_m$ are positive integers. But then Proposition 5.1 guarantees that $b_1 = \ldots = b_m = 0$.

For Banach algebras possessing a sufficient family of matrix representations one can do a little better. In fact, one can drop the rationality condition appearing above. We shall formulate the result in terms of cones. A cone C is said to be *pointed* if $C \cap (-C) = (0)$.

THEOREM 5.2 *Suppose B is a Banach algebra possessing a sufficient family of matrix representations. Then the convex cone generated by the logarithmic residues in B is pointed.*

Recall that the idempotents in B are logarithmic residues in B. So if the convex cone generated by the logarithmic residues in B is pointed, then so is the convex cone generated by the idempotents in B. Note that in general the convex cone generated by the idempotents in a Banach algebra is not pointed (cf. Section 3 of [BESi] and Example 4.6).

PROOF Let L be the convex cone generated by the logarithmic residues in B. By definition, L is the smallest convex cone in B that contains all logarithmic residues. It is easy to see that L consists of all finite linear combinations $\alpha_1 b_1 + \ldots + \alpha_m b_m$ where $b_1, \ldots, b_m$ are logarithmic residues and $\alpha_1, \ldots, \alpha_m$ are non-negative real numbers. So to prove the theorem, we need to establish the following result: *Suppose $b_1, \ldots, b_m$ are logarithmic residues in B and $\alpha_1, \ldots, \alpha_m$ are positive real numbers such that*

$$(5.1) \qquad \alpha_1 b_1 + \ldots + \alpha_m b_m = 0.$$

Then $b_1 = \ldots = b_m = 0$.

Let M be a sufficient family of matrix representations for B, and take $\psi \in M$. Applying ψ to both sides of (5.1), we get the matrix identity

$$(5.2) \qquad \alpha_1 \psi(b_1) + \ldots + \alpha_m \psi(b_m) = 0.$$

For $j = 1, \ldots, m$, the matrix $\psi(b_j)$ is the logarithmic residue of a matrix valued function (cf. formula (4.1)). Hence trace $\psi(b_j) \geqq 0$ by Theorem 3.1. But then it follows from (5.2) that trace $\psi(b_j) = 0$, $j = 1, \ldots, m$. The desired result is now clear from Theorem 4.1. □

Theorem 5.2 can be viewed as a comment on Problem 12 in [K], Section 29. However, it does not provide a full answer to the question posed there. This appears from the following remark (cf. Remark 4.3 and Examples 4.4 and 4.5).

REMARK 5.3 The conclusion of Theorem 5.2 is also true when B is a Banach algebra of the type considered in Theorem 4.2. This can be seen as follows. Suppose

$b_1,\ldots,b_m$ are logarithmic residues in B such that (5.1) is satisfied with $\alpha_1,\ldots,\alpha_m$ positive real numbers. Write $b_j = \mathrm{LogRes}(f_j;D_j)$ and apply the canonical homomorphism κ from B onto B/J. This gives

$$\sum_{j=1}^{m} \alpha_j \mathrm{LogRes}(\kappa o f_j;D_j) = \sum_{j=1}^{m} \alpha_j \kappa(b_j) = 0.$$

By assumption B/J possesses a sufficient family of matrix representations. So it follows from Theorems 5.2 and 4.1 that $\kappa o f_j$ takes invertible values on all of $D_j, j=1,\ldots,m$. Theorem 2 in [Sm] now guarantees that the functions $W_\omega o f_j$ are Fredholm valued. Clearly

$$\sum_{j=1}^{m} \alpha_j \mathrm{LogRes}(W_\omega o f_j;D_j) = \sum_{j=1}^{m} \alpha_j W_\omega(b_j) = 0.$$

Taking the trace and using that $\alpha_1,\ldots,\alpha_m$ are positive, we get

$$\mathrm{trace}[\mathrm{LogRes}(W_\omega o f_j;D_j)] = 0.$$

Here $j=1,\ldots,m$, $\omega \in \Omega$. Hence, by Theorem 3.1, the function $W_\omega o f_j$ takes invertible values on all of D_j. We have already seen that $\kappa o f_j$ takes invertible values on all of D_j too. So [Sm], Theorem 2 guarantees that $f_j(\lambda)$ is invertible for all $\lambda \in D_j$. But then $b_1 = \ldots = b_m = 0$. □

6. ZERO SUMS OF IDEMPOTENTS

At several points in this paper, we touched upon the issue of zero sums of idempotents. This issue is taken up as a separate topic in [BES1]. The problem treated there is the following. *Let $p_1,\ldots,p_k$ be idempotents in a Banach algebra B, and assume that $p_1+\ldots p_k=0$. Can one conclude that $p_j=0$, $j=1,\ldots,k$?*

The results of [BES1] can be summarized as follows. For important classes of Banach algebras the answer turns out to be positive; in general, however, it is negative. A counterexample is given involving five nonzero bounded projections on infinite-dimensional separable Hilbert space. The number five is critical here: in Banach algebras nontrivial zero sums of four idempotents are impossible. In a purely algebraic context (no norm), the situation is different. There the critical number is four. In a rather abstract setting, this was established in [Ma]; see also [Be]. In [BES1], an example is given involving a concrete algebra of linear operators.

REFERENCES

[AL] Amitsur, S.A., Levitski, J.: Minimal identities for algebras, *Proc. Amer. Math. Soc.* 1 (1950), 449–463.

[B1] Bart, H.: Spectral properties of locally holomorphic vector-valued functions, *Pacific J. Math.* 52 (1974), 321–329.

[B2] Bart, H.: Transfer functions and operator theory, *Linear Algebra Appl.* 84 (1986), 33–61.

[Be] Bergman, G.M.: The diamond lemma for ring theory, *Advances in Mathematics* 29 (1978), 178–218.

[BES1] Bart, H., Ehrhardt, T., Silbermann, B.: Zero sums of idempotents in Banach algebras, *Integral Equations and Operator Theory*, to appear.

[BES2] Bart, H., Ehrhardt, T., Silbermann, B.: *Logarithmic residues of matrix valued functions*, Report Econometric Institute, Erasmus University Rotterdam, in preparation.

[BGK1] Bart, H., Gohberg, I., Kaashoek, M.A.: *Minimal Factorization of Matrix and Operator Functions*, Operator Theory: Advances and Applications, Vol. 1, Birkhäuser, Basel, 1979.

[BGK2] Bart, H., Gohberg, I., Kaashoek, M.A.: Wiener–Hopf factorization and realization, in: *Lecture Notes in Control and Information Sciences* (Balakrishnan, A.V. and Thoma, M., eds.), Vol. 58, Springer–Verlag, Berlin, 1984, 42–62.

[BGK3] Bart, H., Gohberg, I., Kaashoek, M.A.: Invariants for Wiener–Hopf equivalence of analytic operator functions, in: *Constructive Methods of Wiener–Hopf Factorization* (Gohberg, I. and Kaashoek, M.A., eds.), Operator Theory: Advances and Applications, Vol. 21, Birkhäuser, Basel, 1986, 317–355.

[BGK4] Bart, H., Gohberg, I., Kaashoek, M.A.: The state space method in analysis, in: *Proceedings ICIAM* 87, *Paris–La Vilette* (Burgh, A.H.P. van der and Mattheij, R.M.M., eds.), Reidel, 1987, 1–16.

[GGK] Gohberg, I., Goldberg, S., Kaashoek, M.A.: *Classes of Linear Operators, Vol. I*, Operator Theory: Advances and Applications, Vol. 49, Birkhäuser, Basel, 1990.

[GK1] Gohberg, I., Krupnik, N.: *One–Dimensional Linear Singular Integral Equations, Vol. I*, Operator Theory: Advances and Applications, Vol. 53, Birkhäuser, Basel, 1992.

[GK2] Gohberg, I., Krupnik, N.: *One–Dimensional Linear Singular Integral Equations, Vol. II*, Operator Theory: Advances and Applications, Vol. 54, Birkhäuser, Basel, 1992.

[GS] Gohberg, I.C., Sigal E.I.: An operator generalization of the logarithmic residue theorem and the theorem of Rouché, *Mat. Sbornik* 84 (126) (1971), 607–629 (Russian); English. Transl. *Math. USSR Sbornik* 13 (1971), 603–625.

[H] Howland, J.S.: Analyticity of determinants of operators on a Banach space, *Proc. Amer. Math. Soc.* 28 (1971), 177–180.

[K] Krupnik, N. Ya.: *Banach Algebras with Symbol and Singular Integral Operators*, Operator Theory: Advances and Applications, Vol. 26, Birkhäuser, Basel, 1987.

[Ma] Mauldon, J.G.: Nonorthogonal idempotents whose sum is idempotent, *Amer. Math. Monthly* 71 (1964), 963–973.

[Mt] Mittenthal, L.: Operator valued analytic functions and generalizations of spectral theory, *Pacific J. Math.* 24 (1968), 119–132.

[Mg] Mitiagin, B.: Linearization of holomorphic operator functions, I,II., *Integral Equations and Operator Theory* 1 (1978), 114–361 and 226–249.

[RS] Roch, S., Silbermann, B.: The Calkin image of algebras of singular integral operators, *Integral Equations and Operator Theory* 12 (1989), 855–897.

[Sm] Silbermann, B.: Symbol constructions and numerical analysis, in: *Integral Equations and Inverse Problems* (Lazarov, R. and Petkov, V. eds), Pitman Research Notes in Mathematics, Vol. 235, 1991, 241–252.

[St] Stummel, F.: Diskreter Konvergenz linearer Operatoren, II, *Math. Zeitschr.* 120 (1971), 231–264.

[TL] Taylor, A.E., Lay, D.C.: *Introduction to Functional Analysis*, *Second Edition*, John Wiley & Sons, New York, 1980.

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