ELSEVIER Stochastic Processes and their Applications 61 (1996) 147–161

stochastic processes and their applications

Tails of subordinated laws: The regularly varying case

J. L. Geluk *

Econometrics Institute, Erasmus University Rotterdam, P.O. Box 1738, NL-3000 DR Rotterdam, Netherlands

Received October 1994; revised June 1995

Abstract

Suppose $X_i, i = 1, 2, \dots$ are i.i.d. positive random variables with d.f. F. We assume the tail d.f. $\overline{F} = 1 - F$ to be regularly varying $(\overline{F}(tx)/\overline{F}(t) \to x^{-\beta}, x > 0, t \to \infty)$ with $0 < \beta < 1$. The asymptotic behaviour of $P(S_N > x)$ as $x \to \infty$ where $S_N = \sum_1^N X_i$ and $N, X_i (i \geqslant 1)$ independent with $\sum_{n=0}^{\infty} P(N = n) x^n$ analytic at $x = 1$ is studied under an additional smoothness condition on F. As an application we give the asymptotic behaviour of the expected population size of an age-dependent branching process.

Keywords: Convolution; Regular variation; Subexponential distributions; Branching processes.

1. Introduction

Let F be a distribution function (d.f.) satisfying $F(0+) = 0$ and $F(x) < 1$ for $x \in \mathbb{R}$. Let $\{p_n\}_{n \geqslant 0}$ denote a probability distribution on $\{0, 1, 2, \dots\}$. Consider the d.f. G subordinate to F with subordinator $\{p_n\}$, i.e. $G(x) = \sum_{n=0}^{\infty} p_n F^{*n}(x)$, where F^{*n} denotes the n-fold (Stieltjes) convolution of F and F^{*0} is the unit mass at zero. Many authors have studied the asymptotic relation between $\overline{F}(x) := 1 - F(x)$ and $\overline{G}(x)$ as $x \to \infty$. One of the early papers in this area is Stam's in which the function $\overline{F}$ is assumed to be regularly varying. In the sequel we write $\overline{F} \in RV_{-\beta}$ to denote $\lim_{t \to \infty} \overline{F}(tx)/\overline{F}(t) = x^{-\beta}$ for $x > 0$.

For the class of subexponential d.f.'s S it is shown by Embrechts, et al.(1979) that the statements $F \in S, G \in S$ and $\overline{G}(x) \sim EN\overline{F}(x) (x \to \infty)$ where N is a r.v. with distribution $\{p_n\}_{n \geqslant 0}$ are equivalent if $\varphi(x) = \sum_{n=0}^{\infty} p_n x^n$ is analytic in $x = 1$; See also Cline (1987).

The asymptotic behaviour of the difference $R(x) := \overline{G}(x) - EN\overline{F}(x)$ is obtained in Omey and Willekens (1986) under the assumption that F has a regularly varying density with index $-(1+\beta)$ and $0 \leqslant \beta \leqslant 1$. The density condition can be weakened. In Geluk(1992) it is shown that $R(x) \sim -E\binom{N}{2}\overline{F}(x)^2 \quad (x \to \infty)$ if and only if $F \in S^2$ (or $G \in S^2$), the class of second-order subexponential distributions. For such distributions

* Fax. 31104527746.

SSDI 0304-4149(95)00070-4

$\overline{F}$ is slowly varying, so their means are infinite and they are not attracted to any stable law. This extends the Omey and Willekens result for $\beta = 0$. In the present paper conditions are imposed ensuring F is attracted to a stable law with infinite mean $\mu := \int_0^\infty x\, dF(x)$; in particular, we assume $\overline{F} \in RV_{-\beta}, 0 < \beta < 1$. For related results the reader is referred to Grübel (1984), Omey (1994) and Omey and Willekens (1987).

In our second result (Theorem 2.2) we obtain the asymptotic behaviour of $R(x)$ with a remainder term. Here the essential assumption is a second-order regular variation of $\overline{F}$, i.e. we assume that

$$\lim_{t\to\infty} \left(\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-\beta}\right) /a(t) \tag{1.1}$$

exists for $x > 0$, where $a(t) \to 0(t \to \infty)$. For a discussion of second-order regular variation the reader is referred to de Haan and Stadtmüller (to appear). For convenience, we give an outline of the basic ideas in the proof of the main results (Theorems 2.1 and 2.2).

Let N denote a r.v. with distribution $\{p_n\}_{n\geqslant 0}$. As in Omey and Willekens (1986), let G_k $(k = 0, 1, \ldots)$ be defined as

$$G_k = \sum_{n=0}^{\infty} p_n^{(k)} F^{*n}, \tag{1.2}$$

where $p_n^{(0)} = p_n$ and $p_n^{(k)} = \sum_{i=n+1}^{\infty} p_i^{(k-1)} (k = 1, 2, \ldots)$. Then

$$R(x) = \int_0^x R_2(x-y) dG_2(y), \tag{1.3}$$

where $R_2(x) = \overline{F^{*2}}(x) - 2\overline{F}(x)$ see Omey and Willekens (1986). We use earlier results (see Geluk, 1992, Theorems 1 and 3) in order to evaluate R_2 and $G_2(\infty) - G_2(x)$ in terms of $\overline{F}$ as accurate as necessary. The asymptotic evaluation of $\overline{G_2}$ in terms of $\overline{F}$ is obtained using Lebesgue's dominated convergence theorem (using Corollaries 2.2 and 2.4). Finally, the integral for R can be approximated by a similar integral with R_2 and G_2 replaced by F (Lemmas 2.1 and 2.2) which is evaluated using earlier results (see Geluk, 1994).

2. Results

Theorem 2.1. *Suppose $\overline{F} \in RV_{-\beta}$ with $0 < \beta < 1$. Suppose for $\varepsilon > 0$ there exist constants $t_0, c > 0$, such that*

$$\frac{\overline{F}(tx)}{\overline{F}(t)} - 1 \leqslant c(x^{-\beta-\varepsilon} - 1) \quad \textit{for } 0 < x < 1, tx \geqslant t_0. \tag{2.1}$$

Define $G(x) = \sum_{n=0}^{\infty} p_n F^{*n}(x)$ and

$$R(x) = \overline{G}(x) - EN \cdot \overline{F}(x). \tag{2.2}$$

If the function $\varphi(x) = \sum_{n=0}^{\infty} p_n x^n$ is analytic at $x = 1$, then

$$R(x) = (c_\beta + o(1)) E\tbinom{N}{2} \overline{F}(x)^2 (x \to \infty), \tag{2.3}$$

where $c_\beta = -\Gamma(1-\beta)^2/\Gamma(1-2\beta)$.

Note that a sufficient condition for (2.1) is the existence of a density $f \in RV_{-\beta-1}$; see Geluk (1994, Corollary 1).

In the sequel we denote by H (or $H_i, i \geqslant 1$) a measure on $(0,\infty)$ with $m = H(0,\infty) < \infty$. The tail of H is denoted by $\overline{H}(x) = H(x,\infty)$ for $x > 0$. The following result is essential in the proof of Theorem 2.1.

Lemma 2.1. *Suppose for* $i = 1, 2$

$$\overline{H}_{i+2}(x) - k_i\overline{H}_i(x) = (d_i + o(1))\overline{H}_i(x)^\alpha \quad (x \to \infty) \tag{2.4}$$

and

$$\overline{H}_i(x-b) - \overline{H}_i(x) = o(\overline{H}_i(x)^\alpha) \quad (x \to \infty), \tag{2.5}$$

where $\alpha > 1$, $k_i \geqslant 0, b, d_i \in \mathbb{R}$. *Then as* $x \to \infty$

$$\begin{aligned}
&\overline{H_3 * H_4}(x) - m_3\overline{H}_4(x) - m_4\overline{H}_3(x)\\
&= k_1k_2(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x) - m_1\overline{H}_2(x))\\
&+ o\left(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x) - m_1\overline{H}_2(x)\right) + o\left(\sum_{i=1}^{2}\overline{H}_i(x)^{\alpha\wedge 2}\right),
\end{aligned} \tag{2.6}$$

where $m_i = H_i(0,\infty)$ and $a \wedge b$ denotes minimum (a,b).

It is somewhat surprising that the asymptotic behaviour in (2.6) does not depend on the constants d_1 and d_2.

Related first-order conditions in order to have the so called max-sum equivalence $\overline{H_1 * H_2} \sim m_2\overline{H}_1 + m_2\overline{H}_2$ are given in Embrechts and Goldie (1980) and generalized by Cline (1987). The present lemma can be seen as a refinement of the Basic Lemma 2.4b in Cline's paper.

It is well known that the class of subexponential distribution functions S for which $\overline{F^{*2}}(x) \sim 2\overline{F}(x)(x \to \infty)$ is closed under asymptotic tail equivalence (see Pakes, 1975; Teugels, 1975). The following result is an immediate consequence of lemma 2.1 and provides us with a closure property for the class of d.f.'s F satisfying $\overline{F^{*2}}(x) - 2\overline{F}(x) \sim c\overline{F}(x)^\alpha(x \to \infty)$.

Corollary 2.1. *If*

$$\overline{H}_1(x-b) - \overline{H}_1(x) = o(\overline{H}_1(x)^\alpha), \quad b \in \mathbb{R}$$

and

$$\overline{H}_2(x) - k\overline{H}_1(x) = (d + o(1))\overline{H}_1(x)^\alpha \quad \text{where } \alpha > 1, k \geqslant 0, d \in \mathbb{R},$$

then as $x \to \infty$,

$$\overline{H_2^{*2}}(x) - 2m_2\overline{H}_2(x) = (k^2 + o(1))(\overline{H_1^{*2}}(x) - 2m_1\overline{H}_1(x)) + o(\overline{H}_1(x)^{\alpha\wedge 2})$$

and

$$\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x) - m_1\overline{H}_2(x) = (k + o(1))(\overline{H_1^{*2}}(x) - 2m_1\overline{H}_1(x)) + o(\overline{H}_1(x)^{\alpha\wedge 2}).$$

Hence, if in addition,

$$\overline{H_1^{*2}}(x) - 2m_1\overline{H}_1(x) \sim c\overline{H}_1(x)^\alpha \quad (x \to \infty),$$

then

$$\overline{H_2^{*2}}(x) - 2m_2\overline{H}_2(x) = ck^2\overline{H}_1(x)^\alpha + \mathrm{o}(\overline{H}_1(x)^{\alpha\wedge 2}) \quad \textit{and}$$

$$\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x) - m_1\overline{H}_2(x) = kc\overline{H}_1(x)^\alpha + \mathrm{o}(\overline{H}_1(x)^{\alpha\wedge 2}).$$

It is well known (see Geluk, 1994, Theorem 1) that for distribution functions F with a regularly varying tail function $\overline{F}$ satisfying (2.1) we have $\overline{F^{*2}}(x) - 2\overline{F}(x) \sim c\overline{F}(x)^2$ as $x \to \infty$, where c is a constant. This explains the interest for the case $\alpha = 2$ in Lemma 2.1. For this case we need the following analogue of the so-called Kesten inequality (see e.g. Athreya,1972): if $F \in S$ then for every $\varepsilon > 0$ there exists a finite constant c_F (independent of n) such that $\overline{F^{*n}}(x)/\ \overline{F}(x) \leqslant c_F(1+\varepsilon)^n$ for $x > 0, n = 1, 2, \ldots$

Corollary 2.2. *If*

$$\overline{H^{*2}}(x) - 2m\overline{H}(x) = (c + \mathrm{o}(1))\overline{H}(x)^2$$

and

$$\overline{H}(x-b) - \overline{H}(x) = \mathrm{o}(\overline{H}(x)^2) \ (x \to \infty),$$

then as $x \to \infty$

$$\overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x) = cm^{n-2}\tbinom{n}{2}\overline{H}(x)^2 + \mathrm{o}(\overline{H}(x)^2). \tag{2.7}$$

Moreover, for $\varepsilon > 0$ *there exist constants* c_H *and* $x_0 = x_0(\varepsilon)$ *such that for* $n \geqslant 2$

$$\begin{aligned} &\sup_{x \geqslant x_0} \{\overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x)\}/\ \overline{H}(x)^2 \leqslant c_H(m+\varepsilon)^n, \\ &\inf_{x \geqslant x_0} \{\overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x)\}/\ \overline{H}(x)^2 \geqslant -c_H(m+\varepsilon)^n. \end{aligned} \tag{2.8}$$

In order to prove a more precise analogue of Lemma 2.1, relation (2.5) is replaced by second-order regular variation of $\overline{H}$ together with some smoothness conditions (see (2.10 and (2.11) below).

Lemma 2.2. *Suppose there exist positive functions* a_i *and constants* $c_{H_i}, \alpha_i, \beta_i$ *such that*

$$\left(\frac{\overline{H}_i(t\,x)}{\overline{H}_i(t)} - x^{-\beta_i}\right) \Big/ a_i(t) \to c_{H_i} x^{-\beta_i}\frac{x^{\alpha_i}-1}{\alpha_i}, \quad x > 0 \textit{ as } t \to \infty, \tag{2.9}$$

where $a_i(t) \to 0 (t \to \infty), a_i \in RV_{\alpha_i}$ *and*

$$0 \geqslant \alpha_i > 2\beta_i - 1 > -1 \qquad \text{for } i = 1, 2.$$

Suppose moreover for $\varepsilon > 0$ *there exist* $t_0, c > 0$ *such that*

$$\left|\frac{\overline{H}_i(tx)}{\overline{H}_i(t)} - x^{-\beta_i}\right| \leqslant cx^{-\beta_i}\frac{x^{-\varepsilon-\alpha_i}-1}{\varepsilon+\alpha_i}a_i(t) \tag{2.10}$$

and

$$\left|\frac{\overline{H}_i^{\leftarrow}(1/tx)}{\overline{H}_i^{\leftarrow}(1/t)} - x^{1/\beta_i}\right| \leqslant c x^{-1/\beta_i} \frac{x^{-\varepsilon+\alpha_i/\beta_i} - 1}{\varepsilon - \alpha_i/\beta_i} a_i(H_i^{\leftarrow}(1/t)) \tag{2.11}$$

for $tx > t_0, 0 < x < 1, i = 1,2$. *If*

$$\overline{H}_{i+2}(x) = k_i\overline{H}_i(x) + d_i\overline{H}_i(x)^2 + (e_i + \mathrm{o}(1))a_i(x)\overline{H}_i(x)^2 + (f_i + \mathrm{o}(1))\overline{H}_i(x)^3 \tag{2.12}$$

$(x \to \infty)$, $i = 1,2$, *then*

$$\begin{aligned}&\overline{H_3 * H_4}(x) - m_3\overline{H}_4(x) - m_4\overline{H}_3(x)\\&= k_1k_2\xi_{\beta_1,\beta_2}\overline{H}_1(x)\overline{H}_2(x)\\&\quad + k_1k_2\sum_{i=1}^{2}\tau_i a_i(x)\overline{H}_1(x)\overline{H}_2(x)\\&\quad + k_1d_2\xi_{\beta_1,2\beta_2}\overline{H}_1(x)\overline{H}_2(x)^2\\&\quad + k_2d_1\xi_{2\beta_1,\beta_2}\overline{H}_1(x)^2\overline{H}_2(x)\\&\quad + \mathrm{o}\left(\sum_{i=1}^{2}a_i(x)\sum_{i=1}^{2}\overline{H}_i(x)^2\right) + \mathrm{o}\left(\sum_{i=1}^{2}\overline{H}_i(x)^3\right)\end{aligned} \tag{2.13}$$

$(x \to \infty)$, *where*

$$\begin{aligned}\xi_{\beta_1,\beta_2} &= -\Gamma(1-\beta_1)\Gamma(1-\beta_2)/\Gamma(1-\beta_1-\beta_2),\\ \tau_1 &= -\frac{c_{H_1}}{\alpha_1}\Gamma(1-\beta_2)\left\{\frac{\Gamma(1-\beta_1+\alpha_1)}{\Gamma(1-\beta_1-\beta_2+\alpha_1)} - \frac{\Gamma(1-\beta_1)}{\Gamma(1-\beta_1-\beta_2)}\right\},\\ \tau_2 &= -\frac{c_{H_2}}{\alpha_2}\Gamma(1-\beta_1)\left\{\frac{\Gamma(1-\beta_2+\alpha_2)}{\Gamma(1-\beta_1-\beta_2+\alpha_2)} - \frac{\Gamma(1-\beta_2)}{\Gamma(1-\beta_1-\beta_2)}\right\}.\end{aligned} \tag{2.14}$$

The smoothness conditions (2.10) and (2.11) are satisfied for many regularly varying d.f. tails $\overline{F} \in RV_{-\beta}$. For example, if the slowly varying function $x^{\beta}\overline{F}(x)$ tends to infinity and has a -1-varying derivative, then (2.10) and (2.11) are satisfied. Other sufficient conditions are given in Geluk (1994, Corollary 2).

From the above result it follows that the asymptotic behaviour of $\overline{H_3 * H_4}$ does not depend on the constants e_i and f_i.

As in Corollary 2.2 the above lemma can be used in order to formulate the asymptotic behaviour of $\overline{H^{*n}}$ for $n > 2$. As shown in Geluk(1994, Theorem 3), for $n = 2$ the function ρ_2 defined in (2.15) below satisfies $\rho_2(x) \sim 2\tau a(x)\overline{H}(x)^2 (x \to \infty)$. Unless $\overline{H}(x) = \mathrm{o}(a(x))(c_0 = \infty$ in the result below) another term of order $\overline{H}(x)^3$ is of importance in the asymptotic behaviour of ρ_n for $n > 2$.

Corollary 2.3. *Suppose $H = H_i$ satisfies (2.9)–(2.11) with $0 \geqslant \alpha > 2\beta - 1 > -1, a(t) \to 0, a \in RV_\alpha$. Suppose $\lim_{t\to\infty} a(t)/\overline{H}(t) = c_0 \in [0,\infty]$. Define the function ρ_n by*

$$\rho_n(x) = \overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x) + \frac{\Gamma(1-\beta)^2}{\Gamma(1-2\beta)} m^{n-2}\tbinom{n}{2}\overline{H}(x)^2. \tag{2.15}$$

Then the asymptotic behaviour of $\rho_n (n \geqslant 2)$ as $x \to \infty$ is given by

$$\rho_n(x) \sim 2\tbinom{n}{2}\tau m^{n-2} a(x)\overline{H}(x)^2 + \frac{\Gamma(1-\beta)^3}{\Gamma(1-3\beta)} m^{n-3}\left\{\frac{n^3-n}{6} - \tbinom{n}{2}\right\}\overline{H}(x)^3, \tag{2.16}$$

where

$$\tau = -\frac{c_H}{\alpha}\Gamma(1-\beta)\left\{\frac{\Gamma(1-\beta+\alpha)}{\Gamma(1-2\beta+\alpha)} - \frac{\Gamma(1-\beta)}{\Gamma(1-2\beta)}\right\}.$$

Corollary 2.4. *Under the assumptions of Corollary 2.3 with $c_0 < \infty$ for $\varepsilon > 0$ there exists a constant c_H depending on H and $x_0 = x_0(\varepsilon)$, such that*

$$\left|\, \overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x) - \xi_{\beta,\beta} m^{n-2}\tbinom{n}{2}\overline{H}(x)^2 \,\right| / \ \overline{H}(x)^3 \leqslant c_H(m+\varepsilon)^n \tag{2.17}$$

for $x > x_0, n \geqslant 2$. In case $c_0 = \infty$ there exists a version of the function a such that a similar inequality holds with $\overline{H}(x)^3$ replaced by $a(x)\overline{H}(x)^2$.

Theorem 2.2. *Suppose $\overline{F} \in RV_{-\beta}, 0 < \beta < \frac{1}{2}$ satisfies*

$$\left(\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-\beta}\right) \Big/ a(t) \to c_F x^{-\beta}\frac{x^\alpha - 1}{\alpha}, \quad x > 0 \text{ as } x \to \infty$$

where $a(t) \to 0 (x \to \infty), a \in RV_\alpha$ with $0 \geqslant \alpha > 2\beta - 1 > -1$ and c_F is a constant. Suppose

$$\lim_{t\to\infty} a(t)/\overline{F}(t) = c_0 \in [0,\infty] \tag{2.18}$$

and for $\varepsilon > 0$ there exist t_0, $c > 0$ such that

$$\left|\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-\beta}\right| \leqslant c x^{-\beta}\frac{x^{-\varepsilon-\alpha} - 1}{\varepsilon + \alpha} a(t) \tag{2.19}$$

and

$$\left|\frac{\overline{F}^{\leftarrow}(1/tx)}{\overline{F}^{\leftarrow}(1/t)} - x^{1/\beta}\right| \leqslant c x^{1/\beta}\frac{x^{-\varepsilon+\alpha/\beta} - 1}{\varepsilon - \alpha/\beta} a(\overline{F}^{\leftarrow}(1/t)) \tag{2.20}$$

for $tx > t_0, 0 < x < 1$.

If the function $\varphi(x) = \sum_{n=0}^{\infty} p_n x^n$ is analytic at $x = 1$, then

$$R(x) = -E\tbinom{N}{2}\frac{\Gamma(1-\beta)^2}{\Gamma(1-2\beta)}\overline{F}(x)^2 + 2E\tbinom{N}{2}\tau a(x)\overline{F}(x)^2$$

$$+E\tbinom{N}{3}\frac{\Gamma(1-\beta)^3}{\Gamma(1-3\beta)}\overline{F}(x)^3 + o(a(x)\overline{F}(x)^2) + o(\overline{F}(x)^3), \tag{2.21}$$

where

$$\tau = -\frac{c_F}{\alpha}\Gamma(1-\beta)\left\{\frac{\Gamma(1-\beta+\alpha)}{\Gamma(1-2\beta+\alpha)} - \frac{\Gamma(1-\beta)}{\Gamma(1-2\beta)}\right\}.$$

3. Applications

Consider an age-dependent branching process with lifetime distribution F. Let $M(t)$ be the expected population size at time $t > 0$ of the process with one ancestor and a per capita mean number of offspring $m < 1$. It is well known that for F subexponential, in particular for $\overline{F} \in RV_{-\beta}(0 < \beta < 1)$, we have $M(t) \sim (1-m)^{-1}\overline{F}(t)$ as $t \to \infty$ (see Athreya,1972; Pakes, 1975). If $\overline{F} \in RV_{-\beta}(0 < \beta < 1)$ in addition satisfies the inequality (2.1) (as pointed out above this is the case e.g. if F has a regularly varying density with exponent $-\beta - 1$), then this estimate can be improved as follows.

It is well known that

$$M(t) := \sum_{k=0}^{\infty} m^k \overline{F^{*k+1}}(t) - \sum_{k=0}^{\infty} m^k \overline{F^{*k}}(t); \tag{3.1}$$

See e.g. Athreya (1972, Ch. IV, 3). Application of Theorem 2.1 for each term in (3.1) gives the more precise result

$$M(t) = \frac{1}{1-m}\overline{F}(t) + (c_\beta + o(1))\frac{m}{(1-m)^2}\overline{F}(t)^2 \qquad \text{as } t \to \infty.$$

This estimate for the mean number of offspring as $t \to \infty$ can be further improved under circumstances. In particular, under the conditions of Theorem 2.2, we have as $t \to \infty$

$$\begin{aligned} M(t) = {} & \frac{1}{1-m}\overline{F}(t) + \frac{c_\beta m}{(1-m)^2}\overline{F}(t)^2 + \frac{2m}{(1-m)^2}\tau a(t)\overline{F}(t)^2 \\ & + \frac{m^2}{(1-m)^3}\frac{\Gamma(1-\beta)^3}{\Gamma(1-3\beta)}\overline{F}(t)^3 + o(a(t)\overline{F}(t)^2) + o(\overline{F}(t)^3). \end{aligned} \tag{3.2}$$

For example, if the lifetime distribution is a stable distribution on $(0,\infty)$ of index $\beta < \frac{1}{2}$, then (3.2) is satisfied (with $0 < c_0 < \infty, \alpha = -\beta$).

In case $c_0 = \lim_{t\to\infty} a(t)/\overline{F}(t) = \infty$ it follows that for $t \to \infty$

$$\begin{aligned} M(t) = {} & (1-m)^{-1}\overline{F}(t) + \frac{c_\beta m}{(1-m)^2}\overline{F}(t)^2 + \\ & + \{2\tau\frac{m}{(1-m)^2} + o(1)\}a(t)\overline{F}(t)^2. \end{aligned} \tag{3.3}$$

An example with this behaviour is the following: if the lifetime distribution is $\exp(2V)$ with $V \sim \chi_k^2$ (then $\beta = \frac{1}{4}, \alpha = 0, c_0 = \infty$). In this case the lifetime distribution has a log-gamma law.

4. Proofs

Proof of Lemma 2.1. By assumption, for $\varepsilon > 0$ there exists $a > 0$ such that $\overline{H}_i(x) < \varepsilon, \overline{H}_{i+2}(x) - k_i\overline{H}_i(x) \leqslant (d_i + \varepsilon)\overline{H}_i(x)^\alpha$ for $x > a, i = 1, 2$. It follows that

$$\begin{aligned}
&\overline{H_3 * H_4}(x) - m_3\overline{H}_4(x) \\
&= \int_0^x \overline{H}_3(x-u)\,\mathrm{d}H_4(u) \\
&\leqslant \int_0^{x-a} \overline{H}_3(x-u)\,\mathrm{d}H_4(u) + m_3(\overline{H}_4(x-a) - \overline{H}_4(x)) \\
&\leqslant k_1 \int_0^{x-a} \overline{H}_1(x-u)\,\mathrm{d}H_4(u) + (d_1+\varepsilon)\int_0^{x-a} \overline{H}_1(x-u)^\alpha \mathrm{d}H_4(u) \\
&\qquad + m_3k_2(\overline{H}_2(x-a) - \overline{H}_2(x)) + \mathrm{o}(\overline{H}_2(x)^\alpha) \\
&=: k_1I_1 + (d_1+\varepsilon)I_2 + \mathrm{o}(\overline{H}_2(x)^\alpha) \quad (x \to \infty).
\end{aligned} \tag{4.1}$$

Now I_1 is estimated as follows:

$$\begin{aligned}
I_1 &= \int_0^{x-a} \overline{H}_1(x-u)\,\mathrm{d}H_4(u) = \int_0^x \overline{H}_1(x-u)\,\mathrm{d}H_4(u) + \mathrm{o}(\overline{H}_2(x)^\alpha) \\
&= \int_0^{x-a} \overline{H}_4(x-u)\,\mathrm{d}H_1(u) + m_4\overline{H}_1(x) - m_1\overline{H}_4(x) + \mathrm{o}\left(\sum_{i=1}^{2}\overline{H}_i(x)^\alpha\right) \\
&\leqslant k_2\int_0^{x-a} \overline{H}_2(x-u)\,\mathrm{d}H_1(u) + (d_2+\varepsilon)\int_0^{x-a} \overline{H}_2(x-u)^\alpha dH_1(u) \\
&\qquad + m_4\overline{H}_1(x) - m_1\overline{H}_4(x) + \mathrm{o}\left(\sum_{i=1}^{2}\overline{H}_i(x)^\alpha\right) \leqslant k_2(\overline{H_1 * H_2}(x) \\
&\qquad - m_2\overline{H}_1(x)) + (d_2+\varepsilon)\int_0^{x-a} \overline{H}_2(x-u)^\alpha dH_1(u) \\
&\qquad + m_4\overline{H}_1(x) - m_1\overline{H}_4(x) + \mathrm{o}\left(\sum_{i=1}^{2}\overline{H}_i(x)^\alpha\right) \quad (x \to \infty).
\end{aligned} \tag{4.2}$$

We estimate the last integral as follows: for $x > a$ and $\varepsilon > 0$ arbitrary

$$\begin{aligned}
0 &\leqslant \int_0^{x-a} (\overline{H}_2(x-u)^\alpha - \overline{H}_2(x)^\alpha)\,\mathrm{d}H_1(u) \\
&\leqslant \alpha\int_0^{x-a} \overline{H}_2(x-u)^{\alpha-1}(\overline{H}_2(x-u) - \overline{H}_2(x))\,\mathrm{d}H_1(u) \\
&\leqslant \alpha\varepsilon^{\alpha-1}\int_0^{x-a} (\overline{H}_2(x-u) - \overline{H}_2(x))\,\mathrm{d}H_1(u) \\
&\leqslant \alpha\varepsilon^{\alpha-1}\int_0^{x} (\overline{H}_2(x-u) - \overline{H}_2(x))\,\mathrm{d}H_1(u) \\
&= \alpha\varepsilon^{\alpha-1}(\overline{H_1 * H_2}(x) - m_1\overline{H}_2(x) - m_2\overline{H}_1(x) + \overline{H}_1(x)\overline{H}_2(x)).
\end{aligned} \tag{4.3}$$

Note that $\overline{H}_1(x)\overline{H}_2(x) \leqslant \frac{1}{2}\sum_{i=1}^2 \overline{H}_i(x)^2 = \mathrm{O}\left(\sum_{i=1}^2 \overline{H}_i(x)^{\alpha\wedge 2}\right)$. It follows that

$$\begin{aligned}
\int_0^{x-a} \overline{H}_2(x-u)^\alpha dH_1(u) &= m_1\overline{H}_2(x)^\alpha + \mathrm{o}(\overline{H_1 * H_2}(x) - m_1\overline{H}_2(x) - m_2\overline{H}_1(x)) \\
&\qquad + \mathrm{o}\left(\sum_{i=1}^{2}\overline{H}_i(x)^{\alpha\wedge 2}\right).
\end{aligned} \tag{4.4}$$

Since a lower inequality for I_1 can be proved similarly, combination with (4.2) gives

$$\begin{aligned}
I_1 &= k_2(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x)) + m_4\overline{H}_1(x) - m_1\overline{H}_4(x) + m_1d_2\overline{H}_2(x)^\alpha \\
&\qquad + \mathrm{o}(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x) - m_1\overline{H}_2(x)) + \mathrm{o}\left(\sum_{i=1}^{2}\overline{H}_i(x)^{\alpha\wedge 2}\right).
\end{aligned}$$

Since $\overline{H}_4(x) = k_2\overline{H}_2(x) + (d_2 + o(1))\overline{H}_2(x)^\alpha$ we find

$$I_1 = k_2(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x)) + m_4\overline{H}_1(x) - m_1k_2\overline{H}_2(x)$$
$$+o(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x) - m_1\overline{H}_2(x)) + o\left(\sum_{i=1}^{2}\overline{H}_i(x)^{\alpha\wedge 2}\right). \tag{4.5}$$

Next we estimate I_2. As in (4.3) we have

$$0 \leqslant \int_0^{x-a}(\overline{H}_1(x-u)^\alpha - \overline{H}_1(x)^\alpha)dH_4(u)$$
$$\leqslant \alpha\varepsilon^{\alpha-1}\int_0^{x-a}(\overline{H}_1(x-u) - \overline{H}_1(x))dH_4(u)$$
$$\leqslant \alpha\varepsilon^{\alpha-1}\int_0^{x}(\overline{H}_4(x-u) - \overline{H}_4(x))dH_1(u)$$
$$= \alpha\varepsilon^{\alpha-1}\int_0^{x-a}(\overline{H}_4(x-u) - \overline{H}_4(x))dH_1(u) + o(\overline{H}_1(x)^\alpha)$$
$$\leqslant \alpha\varepsilon^{\alpha-1}[k_2\int_0^{x-a}(\overline{H}_2(x-u) - \overline{H}_2(x))dH_1(u) +$$
$$+(d_2+\varepsilon)\int_0^{x-a}\overline{H}_2(x-u)^\alpha dH_1(u)$$
$$-(d_2-\varepsilon)\int_0^{x-a}\overline{H}_2(x)^\alpha dH_1(u)] + o\left(\sum_{i=1}^{2}\overline{H}_i(x)^\alpha\right).$$

Combination with (4.4) now gives

$$I_2 = \int_0^{x-a}\overline{H}_1(x-u)^\alpha dH_4(u) = m_4\overline{H}_1(x)^\alpha + o(\overline{H_1 * H_2}(x) - m_2\overline{H}_1(x)$$
$$-m_1\overline{H}_2(x)) + o\left(\sum_{i=1}^{2}\overline{H}_i(x)^{\alpha\wedge 2}\right). \tag{4.6}$$

Since a corresponding lower inequality for (4.1) can be proved similarly, combination of (4.1), (4.5) and (4.6) gives an expression for $\overline{H_3 * H_4}(x) - m_3\overline{H}_4(x)$. Subtracting $m_4\overline{H}_3(x) = m_4(k_1\overline{H}_1(x) + d_1\overline{H}_1(x)^\alpha + o(\overline{H}_1(x)^\alpha)$ then gives the required result. □

Proof of Corollary 2.1 Obvious from Lemma 2.1. □

Proof of Corollary 2.2 The proof of both parts is by induction. Suppose $\overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x) = (a_n+o(1))\overline{H}(x)^2$. Using Corollary 2.1 we find $\overline{H^{*n+1}}(x) = \overline{H * H^{*n}}(x) = m^n\overline{H}(x) + m\overline{H^{*n}}(x) + nm^{n-1}c\overline{H}(x)^2 + o(\overline{H}(x)^2) = m^n\overline{H}(x) + m(nm^{n-1}\overline{H}(x) + a_n\overline{H}(x)^2) + nm^{n-1}c\overline{H}(x)^2 + o(\overline{H}(x)^2)$, hence

$$\overline{H^{*n+1}}(x) - (n+1)m^n\overline{H}(x) = (ma_n + nm^{n-1}c)\overline{H}(x)^2 + o(\overline{H}(x)^2).$$

It follows that $a_{n+1} = ma_n + nm^{n-1}c$ implying the first statement, since $a_2 = c$. The proof of the second Statement is similar to the proof of Lemma 2 in Geluk and Pakes (1991). (Note that $\int_0^x \overline{H}(x-u)^2 m dH(u) \sim m\overline{H}(x)^2$ by Corollary 2.1.) □

Proof of Theorem 2.1 As mentioned in the introduction we have $R(x) = \overline{G}(x) - EN\overline{F}(x) = \int_0^x R_2(x-y)dG_2(y)$ where $R_2(x) = \int_0^x \overline{F}(x-y)dF(y) - \overline{F}(x) = \overline{F^{*2}}(x) - 2\overline{F}(x)$ and $G_2(x) = \sum_{n=0}^{\infty} p_n^{(2)} F^{*n}(x)$ with $p_n^{(k)} = \sum_{i=n+1}^{\infty} p_i^{(k-1)}$, $p_n^{(0)} = p_n$.

Note that $R_2(x) \sim c_\beta\overline{F}(x)^2$ by Theorem 1 in Geluk (1994). Since φ is analytic at 1, $EN^m < \infty$ and it is easily verified that $G_m(\infty) = \sum_{n=0}^{\infty} p_n^{(m)} = E\binom{N}{m}$ hence

$$\overline{G}_2(x) - E\binom{N}{3}\overline{F}(x) = \sum_{n=0}^{\infty} p_n^{(2)}(\overline{F^{*n}}(x) - n\overline{F}(x)), \tag{4.7}$$

where $\overline{G}_2(x) = G_2(\infty) - G_2(x)$. In view of Corollary 2.2 we may apply Lebesgue's dominated convergence theorem after division by $\overline{F}(x)^2$ in (4.7) to find $\overline{G}_2(x) - E\binom{N}{3}\overline{F}(x) \sim c_\beta \sum_{n=0}^{\infty}\binom{n}{2}p_n^{(2)}\overline{F}(x)^2 = c_\beta E\binom{N}{4}\overline{F}(x)^2$. Application of Lemma 2.1 twice gives

$$\begin{aligned} R(x) &= \int_0^x R_2(x-y)\,\mathrm{d}G_2(y) = \int_0^x \overline{F^{*2}}(x-y)\,\mathrm{d}G_2(y) - 2\int_0^x \overline{F}(x-y)\,dG_2(y) \\ &= \overline{G}_2(0)(\overline{F^{*2}}(x) - 2\overline{F}(x)) + \mathrm{o}(\overline{F}(x)^2) \\ &= E\binom{N}{2}(\overline{F^{*2}}(x) - 2\overline{F}(x)) + \mathrm{o}(\overline{F}(x)^2). \end{aligned} \tag{4.8}$$

Since $\overline{F^{*2}}(x) - 2\overline{F}(x) = (c_\beta + \mathrm{o}(1))\overline{F}(x)^2$ the proof is complete. □

Proof of Lemma 2.2 In the sequel we write $\overline{H_i}$ for $\overline{H_i}(x), a_i = a_i(x)$ and $\overline{H_i * H_j} = \overline{H_i * H_j}(x)$. As in the proof of Lemma 2.1 we have $\overline{H_i} < \varepsilon, \overline{H}_{i+2} - k_i\overline{H_i} - d_i\overline{H_i}^2 \leqslant (e_i + \varepsilon)a_i\overline{H_i}^2 + (f_i + \varepsilon)\overline{H_i}^3$ for $x > a, i = 1, 2$. It follows that for $x > a$ and $\varepsilon > 0$

$$\begin{aligned} \overline{H_3 * H_4}(x) - m_3\overline{H}_4(x) &\leqslant \int_0^{x-a}\overline{H}_3(x-u)dH_4(u) + m_3(\overline{H}_4(x-a) - \overline{H}_4) \\ &\leqslant k_1 I_{1,4} + d_1 J_{1,4} + (e_1 + \varepsilon)K_{1,4} + (f_1 + \varepsilon)L_{1,4} \\ &\quad + (m_3 k_2 + \mathrm{o}(1))(\overline{H}_2(x-a) - \overline{H}_2) \end{aligned} \tag{4.9}$$

where $I_{i,j} = \int_0^{x-a}\overline{H_i}(x-u)\mathrm{d}H_j(u), J_{i,j} = \int_0^{x-a}\overline{H_i}(x-u)^2\mathrm{d}H_j(u), K_{i,j} = \int_0^{x-a} a_i(x-u)\,\overline{H_i}(x-u)^2\mathrm{d}H_j(u), L_{i,j} = \int_0^{x-a}\overline{H_i}(x-u)^3\mathrm{d}H_j(u)$. By assumption (2.10) we have for $a > 0$, x sufficiently large and $i = 1, 2$

$$|\,\overline{H_i}(x-a)/\overline{H_i} - (1-a/x)^{-\beta_i}\,| \leqslant 2c\{(1-a/x)^{-\varepsilon-\alpha_i} - 1\}a_i/(\varepsilon + \alpha_i) = \mathrm{O}(a_i/x),$$

hence

$$\begin{aligned} \overline{H_i}(x-a) - \overline{H_i} &\leqslant \mathrm{O}(a_i\overline{H_i}/x) + \overline{H_i}[(1-a/x)^{-\beta_i} - 1] \\ &= \mathrm{O}(a_i\overline{H_i}/x) + \mathrm{O}(\overline{H_i}/x) = \mathrm{O}(\overline{H_i}/x) = \mathrm{o}(a_i\overline{H_i}^2), \end{aligned} \tag{4.10}$$

the last equality being true since $\overline{H_i} \in RV_{-\beta_i}, a_i \in RV_{\alpha_i}$, with $\alpha_i - \beta_i + 1 > 0$. Using the same arguments as for (4.2) we find

$$\begin{aligned} I_{1,4} &\leqslant k_2 I_{2,1} + d_2 J_{2,1} + (e_2 + \varepsilon)K_{2,1} + (f_2 + \varepsilon)L_{2,1} \\ &\quad + m_4\overline{H}_1 - m_1\overline{H}_4 + \mathrm{o}\left(\sum_{i=1}^{2} a_i\overline{H_i}^2\right), \end{aligned} \tag{4.11}$$

where $I_{2,1}, J_{2,1}, \ldots$ are defined as above.

Application of Theorem 3 in Geluk(1994) gives

$$\begin{aligned} I_{2,1} &= \overline{H_1 * H_2} - m_2\overline{H}_1 + \mathrm{o}(a_1\overline{H}_1^2) \\ &= m_1\overline{H}_2 + \xi_{\beta_1,\beta_2}\overline{H}_1\overline{H}_2 + \sum_{i=1}^{2}(\tau_i + \mathrm{o}(1))a_i\overline{H}_1\overline{H}_2 + \mathrm{o}(a_1\overline{H}_1^2) \\ &= m_1\overline{H}_2 + \xi_{\beta_1,\beta_2}\overline{H}_1\overline{H}_2 + \sum_{i=1}^{2}\tau_i a_i\overline{H}_1\overline{H}_2 + \mathrm{o}\left(\sum_{i=1}^{2} a_i \sum_{i=1}^{2}\overline{H_i}^2\right). \end{aligned} \tag{4.12}$$

It is easy to see that the assumptions of Theorem 3 in Geluk (1994) are satisfied with $F = H_1/m_1$ and $1 - G = \overline{H}_2^2/m_2^2$ (note that $0 \geqslant \alpha_2 > 2\beta_2 - 1 > -1$). It follows that

$$J_{2,1} = m_1\overline{H}_2^2 + \xi_{\beta_1,2\beta_2}\overline{H}_1\overline{H}_2^2 + \mathrm{o}\left(\sum_{i=1}^{2} a_i \sum_{i=1}^{2} \overline{H}_i^2\right). \tag{4.13}$$

Note that we may choose a function $a_2^*(x) \sim a_2(x)$ such that $a_2^*\overline{H}_2^2$ has a regularly varying derivative with exponent $\alpha_2 - 2\beta_2 - 1$ (see eg. Bingham et al.(1987) or Geluk and de Haan(1987)). It follows that we may assume w.l.o.g. that $a_2\overline{H}_2^2$ is smooth. Hence, the function $\overline{G}(x) := a_2\overline{H}_2^2/a_2(0)m_2^2$ satisfies (1.2b) in Geluk (1994) with $\gamma = \alpha_2 - 2\beta_2$ (see also Corollary 1 in Geluk (1994)). Application of theorem 1 in Geluk (1994) and (4.10) gives

$$K_{2,1} = m_1 a_2\overline{H}_2^2 + \mathrm{o}\left(\sum_{i=1}^{2} a_i \sum_{i=1}^{2} \overline{H}_i^2\right). \tag{4.14}$$

Similarly, we find

$$L_{2,1} = (m_1 + \mathrm{o}(1))\overline{H}_2^3.$$

Since a lower estimate in (4.11) is obtained similarly, combination of the above estimates shows that

$$\begin{aligned} I_{1,4} = {} & k_2(m_1\overline{H}_2 + \xi_{\beta_1,\beta_2}\overline{H}_1\overline{H}_2 + \sum_{i=1}^{2}\tau_i a_i\overline{H}_1\overline{H}_2) \\ & + \mathrm{o}\left(\sum_{i=1}^{2} a_i \sum_{i=1}^{2} \overline{H}_i^2\right) + d_2(m_1\overline{H}_2^2 + \xi_{\beta_1,2\beta_2}\overline{H}_1\overline{H}_2^2) + e_2 m_1 a_2\overline{H}_2^2 \\ & + f_2 m_1\overline{H}_2^3 + m_4\overline{H}_1 - m_1[k_2\overline{H}_2 + d_2\overline{H}_2^2 + e_2 a_2\overline{H}_2^2 + (f_2 + \mathrm{o}(1))\overline{H}_2^3], \end{aligned}$$

hence

$$\begin{aligned} I_{1,4} = {} & m_4\overline{H}_1 + k_2\xi_{\beta_1,\beta_2}\overline{H}_1\overline{H}_2 + k_2\sum_{i=1}^{2}\tau_i a_i\overline{H}_1\overline{H}_2 + d_2\xi_{\beta_1,2\beta_2}\overline{H}_1\overline{H}_2^2 \\ & + \mathrm{o}\left(\sum_{i=1}^{2} a_i \sum_{i=1}^{2} \overline{H}_i^2\right) + \mathrm{o}\left(\sum_{i=1}^{2} \overline{H}_i^3\right). \end{aligned} \tag{4.15}$$

In order to evaluate $J_{1,4}$ we introduce the measure H_0 with tail function $\overline{H}_0 = \overline{H}_0(x, \infty) = \overline{H}_1^2$. Note that

$$\begin{aligned} & J_{1,4} - \overline{H}_1^2(m_4 - \overline{H}_4) + \mathrm{o}(a_2\overline{H}_2^2) \\ & \quad = \int_0^x (\overline{H}_0(x-u) - \overline{H}_0)\mathrm{d}H_4(u) + \mathrm{o}(a_2\overline{H}_2^2) \\ & \quad = \int_0^x (\overline{H}_4(x-u) - \overline{H}_4)\mathrm{d}H_0(u) + \mathrm{o}(a_2\overline{H}_2^2) \\ & \quad = k_2\int_0^x (\overline{H}_2(x-u) - \overline{H}_2)\mathrm{d}H_0(u) + d_2\int_0^x (\overline{H}_2(x-u)^2 - \overline{H}_2^2)\mathrm{d}H_0(u) \\ & \qquad + (e_2 + \mathrm{o}(1))\int_0^x (a_2(x-u)\overline{H}_2(x-u)^2 - a_2\overline{H}_2^2)\mathrm{d}H_0(u) \\ & \qquad + (f_2 + \mathrm{o}(1))\int_0^x (\overline{H}_2(x-u)^3 - \overline{H}_2^3)\mathrm{d}H_0(u). \end{aligned} \tag{4.16}$$

Application of Theorem 1 in Geluk (1994) now shows that the first term on the right-hand side equals

$$k_2(\overline{H_0 * H_2} - m_1^2\overline{H}_2 - m_2\overline{H}_1^2 + \overline{H}_1^2\overline{H}_2) = k_2(\xi_{2\beta_1,\,\beta_2} + 1 + o(1))\overline{H}_1^2\overline{H}_2.$$

Since the other terms are of smaller order we have

$$\begin{aligned} J_{1,4} &= m_4\overline{H}_1^2 - \overline{H}_4\overline{H}_1^2 + k_2(\xi_{2\beta_1,\beta_2} + 1 + o(1))\overline{H}_1^2\,\overline{H}_2 \\ &= m_4\overline{H}_1^2 - k_2\overline{H}_2\,\overline{H}_1^2 + k_2(\xi_{2\beta_1,\beta_2} + 1 + o(1))\overline{H}_1^2\,\overline{H}_2 \\ &= m_4\overline{H}_1^2 + (k_2\xi_{2\beta_1,\beta_2} + o(1))\overline{H}_1^2\overline{H}_2. \end{aligned} \tag{4.17}$$

Finally, we evaluate $K_{1,4}$ and $L_{1,4}$. As in (4.14) we find $\int_0^x a_1(x-u)\overline{H}_1(x-u)^2\mathrm{d}H_2(u) \sim m_2a_1\overline{H}_1^2$. Since $\overline{H}_4 - k_2\overline{H}_2 \sim d_2\overline{H}_2^2$ we can apply Lemma 2.1 to find

$$K_{1,4} \sim m_4a_1\overline{H}_1^2. \tag{4.18}$$

Note that the analogue of (2.5) is satisfied for the function $a_1\overline{H}_1^2$ since $a_1(x-b)\overline{H}_1(x-b)^2 - a_1\overline{H}_1^2 = O(\mathrm{d}/\mathrm{d}x\ a_1\overline{H}_1^2) = o(a_1^2\overline{H}_1^4)$, the last equality being true since we may assume $\mathrm{d}/\mathrm{d}x\ a_i\overline{H}_1^2$ to be regularly varying with exponent

$$\alpha_1 - 2\beta_1 - 1 < 2\alpha_1 - 4\beta_1.$$

Similarly, it can be shown that

$$L_{1,4} \sim m_4\overline{H}_1^3. \tag{4.19}$$

The result of the lemma follows since (4.9), (4.10), (4.18) and (4.19) show that

$$\begin{aligned} \overline{H_3 * H_4} - m_3\overline{H}_4 &= k_1I_{1,4} + d_1J_{1,4} + e_1K_{1,4} + f_1L_{1,4} + o\left(\sum_{i=1}^{2}a_i\sum_{i=1}^{2}\overline{H}_i^2\right) \\ &\quad + o\left(\sum_{i=1}^{2}\overline{H}_i^3\right). \end{aligned}$$

Substitution of (4.15), (4.17), (4.18) and (4.19) on the right-hand side, together with the expression $\overline{H}_3 = k_1\overline{H}_1 + d_1\overline{H}_1^2 + (e_1 + o(1))a_1\overline{H}_1^2 + (f_1 + o(1))\overline{H}_1^3$ then gives the statement of the lemma. □

Proof of Corollary 2.3. First suppose $c_0 = 0$. The proof is by induction. Suppose

$$\overline{H^{*n}} - nm^{n-1}\overline{H} - \xi_{\beta,\beta}m^{n-2}\tbinom{n}{2}\overline{H}^2 = (b_n + o(1))\overline{H}^3. \tag{4.20}$$

Then $b_2 = 0$ by Theorem 3 in Geluk(1994). Using Lemma 2.2 and the above induction hypothesis we find

$$\begin{aligned} \overline{H^{*n+1}} &= \overline{H^{*n} * H} = m\overline{H^{*n}} + m^n\overline{H} + nm^{n-1}\xi_{\beta,\beta}\overline{H}^2 \\ &\quad + \xi_{\beta,\beta}m^{n-2}\tbinom{n}{2}\xi_{2\beta,\beta}\overline{H}^3 + o(\overline{H}^3) \\ &= (n+1)m^n\overline{H} + \xi_{\beta,\beta}m^{n-1}\tbinom{n+1}{2}\overline{H}^2 \\ &\quad + \{mb_n + \xi_{\beta,\beta}\xi_{2\beta,\beta}m^{n-2}\tbinom{n}{2}\}\overline{H}^3 + o(\overline{H}^3), \end{aligned}$$

hence $b_{n+1} = mb_n + \xi_{\beta,\beta}\xi_{2\beta,\beta}m^{n-2}\binom{n}{2}$ from which the statement follows.

In case $c_0 = \infty$ a similar argument applies. In case $0 < c_0 < \infty$ we find under the induction hypothesis (4.21) that the sequence b_n satisfies

$$b_{n+1} = mb_n + 2nm^{n-1}\tau c_0 + \xi_{\beta,\beta}\xi_{2\beta,\beta}m^{n-2}\binom{n}{2},$$

$b_2 = 2\tau c_0$ from which the statement follows. $\square$

Proof of Corollary 2.4. We only prove the upper inequality in case $c_0 < \infty$. The proof of the lower inequality and the case $c_0 = \infty$ are similar. Define

$$\theta_n(x_0) = \sup_{x > x_0}\{\overline{H^{*n}}(x) - nm^{n-1}\overline{H}(x) - \xi_{\beta,\beta}m^{n-2}\binom{n}{2}\overline{H}(x)^2\}/\overline{H}(x)^3, \quad n \geqslant 2$$

and $\theta_n := \theta_n(0)$. Note that by Corollary 2.3 the quotient on the right-hand side has a finite limit as $x \to \infty$. The proof is by induction. For $0 < x_0 < x$,

$$\begin{aligned}
&\overline{H^{*n+1}} - (n+1)m^n\overline{H} \\
&\quad = \int_0^x \overline{H^{*n}}(x-u)\mathrm{d}H(u) - nm^n\overline{H} \\
&\leqslant \int_0^{x-x_0}(nm^{n-1}\overline{H}(x-u) + \xi_{\beta,\beta}m^{n-2}\tbinom{n}{2}\overline{H}(x-u)^2 \\
&\quad +\theta_n(x_0)\overline{H}(x-u)^3)\mathrm{d}H(u) + \int_{x-x_0}^x \overline{H^{*n}}(x-u)\mathrm{d}H(u) - nm^n\overline{H} \\
&\leqslant nm^{n-1}(\overline{H^{*2}} - 2m\overline{H}) + \xi_{\beta,\beta}m^{n-2}\tbinom{n}{2}I_1 + \theta_n(x_0)I_2 + m^n(\overline{H}(x-x_0) - \overline{H}) \\
&\leqslant nm^{n-1}(\xi_{\beta,\beta}\overline{H}^2 + \theta_2\overline{H}^3) + \xi_{\beta,\beta}m^{n-2}\tbinom{n}{2}I_1 + \theta_n(x_0)I_2 \\
&\quad +m^n(\overline{H}(x-x_0) - \overline{H}),
\end{aligned} \tag{4.21}$$

where $I_1 = \int_0^x \overline{H}(x-u)^2\mathrm{d}H(u)$ and $I_2 = \int_0^x \overline{H}(x-u)^3\mathrm{d}H(u)$. As in (4.13) and (4.14) we find $I_1 = m\overline{H}^2 + \xi_{\beta,2\beta}\overline{H}^3 + \mathrm{o}(\overline{H}^3)$ and $I_2 \sim m\overline{H}^3$. Moreover, as in (4.10) it follows that $\overline{H}(x-x_0) - \overline{H} = \mathrm{O}(\overline{H}^3)$. Hence, for $\varepsilon > 0$ there exist constants $c_1 > 0$ and $x_0 = x_0(\varepsilon)$ such that $I_1 \leqslant m\overline{H}(x)^2 + c_1\overline{H}(x)^3, I_2 \leqslant (m+\varepsilon)\overline{H}(x)^3$ and $\overline{H}(x-x_0) - \overline{H} \leqslant c_1\overline{H}^3$ for $x > x_0$. Substituting this in (4.21) then gives

$$\theta_{n+1}(x_0) \leqslant \theta_2 nm^{n-1} + c_1\xi_{\beta,\beta}m^{n-2}\binom{n}{2} + (m+\varepsilon)\theta_n(x_0) + c_1m^n.$$

It follows that the sequence $\{\theta_n(x_0)\}$ satisfies

$$\begin{aligned}
\theta_{n+1}(x_0) &\leqslant c_2n^2m^n + (m+\varepsilon)\theta_n(x_0) \\
&\leqslant c_3(m+\varepsilon)^n + (m+\varepsilon)\theta_n(x_0)
\end{aligned}$$

for some constants c_2, c_3. The result follows by iteration. $\square$

Proof of Theorem 2.2. The proof is similar to the proof of Theorem 2.1. We give the proof for the case $c_0 = \infty$. The case $c_0 < \infty$ can be treated similarly. Replace (4.7) by

$$\overline{G}_2(x) - E\binom{N}{3}\overline{F}(x) - E\binom{N}{4}\xi_{\beta,\beta}\overline{F}(x)^2$$

$$= \sum_{n=0}^{\infty} p_n^{(2)}\left(\overline{F^{*n}}(x) - n\overline{F}(x) - \xi_{\beta,\beta}\binom{n}{2}\overline{F}(x)^2\right)$$

$$\sim 2\sum_{n=0}^{\infty} p_n^{(2)}\binom{n}{2}\tau a(x)\overline{F}(x)^2 = 2E\binom{N}{4}\tau a(x)\overline{F}(x)^2,$$

the last asympotic equality being justified by Corollaries 2.3 and 2.4. In order to evaluate the first integral in

$$R(x) = \int_0^x \overline{F^{*2}}(x-y)\mathrm{d}G_2(y) - 2\int_0^x \overline{F}(x-y)\mathrm{d}G_2(y)$$

note that $\overline{F^{*2}}(x) - 2\overline{F}(x) - \xi_{\beta,\beta}\overline{F}(x)^2 \sim 2\tau a(x)\overline{F}(x)^2$ by Theorem 3 in Geluk (1994). Application of Lemma 2.2 twice gives

$$\begin{aligned} R(x) &= \overline{G}_2(0)\overline{F^{*2}} + 2E\binom{N}{3}\xi_{\beta,\beta}\overline{F}^2 + 2E\binom{N}{3}2\tau a\overline{F}^2 \\ &\quad + 2E\binom{N}{4}\xi_{\beta,\beta}\xi_{\beta,2\beta}\overline{F}^3 + \xi_{\beta,\beta}E\binom{N}{3}\xi_{2\beta,\beta}\overline{F}^3 \\ &\quad -2[\overline{G}_2(0)\overline{F} + E\binom{N}{3}\xi_{\beta,\beta}\overline{F}^2 + E\binom{N}{3}2\tau a\overline{F}^2 \\ &\quad + E\binom{N}{4}\xi_{\beta,\beta}\xi_{2\beta,\beta}\overline{F}^3] + \mathrm{o}(a\overline{F}^2) + \mathrm{o}(\overline{F}^3) \\ &= E\binom{N}{2}\xi_{\beta,\beta}\overline{F}^2 + 2E\binom{N}{2}\tau a\overline{F}^2 \\ &\quad + \xi_{\beta,\beta}E\binom{N}{3}\xi_{2\beta,\beta}\overline{F}^3 + \mathrm{o}(a\overline{F}^2) + \mathrm{o}(\overline{F}^3) = E\binom{N}{2}\xi_{\beta,\beta}\overline{F}^2 \\ &\quad + 2E\binom{N}{2}\tau a\overline{F}^2 + \mathrm{o}(a\overline{F}^2). \end{aligned}$$

Note that the last equality is a consequence of the assumption $c_0 = \infty$, i.e. $\overline{F} = \mathrm{o}(a)$. □

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