On the uniform limit condition for discrete-time infinite horizon problems

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Abstract. In this note, a simplified version of the four main results for discrete-time infinite horizon problems, theorems 4.2-4.5 from Stokey, Lucas and Prescott (1989) [SLP], is presented. A novel assumption on these problems is proposed—the *uniform limit condition*, which is formulated in terms of the *data* of the problem. It can be used for example before one has started to look for the optimal value function and for an optimal plan or if one cannot find them analytically: one verifies the uniform limit condition and then one disposes of *criteria* for optimality of the value function and a plan in terms of the functional equation and the boundedness condition. A comparison to [SLP] is made. The version in [SLP] requires one to verify whether a candidate optimal value function satisfies the boundedness condition; it is easier to check the uniform limit condition instead, as is demonstrated by examples. There is essentially no loss of strength or generality compared to [SLP]. The necessary and sufficient conditions for optimality coincide in the present paper but not in [SLP]. The proofs in the present paper are shorter than in [SLP]. An earlier attempt to simplify, in Acemoglu (2009)—here the limit condition is used rather than the uniform limit condition—is not correct.

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1 Introduction

The aim of this note is to give a simplified version of the four main theorems for stationary discrete-time infinite horizon optimization problems from Stokey, Lucas and Prescott (1989)

[1] I would like to express my great appreciation to Daron Acemoglu for his comments and suggestions.

[SLP], which contains the most complete treatment of this type of problems. These four theorems, theorems 4.2-4.5, concern the necessary and the sufficient conditions for optimality of value functions and plans.

We propose a novel assumption, the *uniform limit condition*, formulated in terms of the *data* of the problem, to wit in terms of the instantaneous payoff function. In examples it can be readily verified whether this condition holds. We recommend to start the analysis of any problem by checking the uniform limit condition. Once this is done, one has *criteria* for value functions and plans in terms of the functional equation and the boundedness condition. Having criteria is for example of interest if one cannot find the optimal value function and an optimal plan. Thus one has the rather desirable situation that necessary and sufficient conditions coincide and that these are valid under an assumption that is easy to verify. We formulate one theorem giving the criteria under the assumption and offer a short proof.

An earlier attempt to simplify, in Acemoglu (2009)—here the limit condition is used rather than the uniform limit condition—is not correct.

We compare the uniform limit condition approach that is proposed in the present paper to the theorems 4.2-4.5 in [SLP]. The analysis of a problem by means of the theorems 4.2-4.5 in [SLP] requires that one verifies whether the boundedness condition holds for some *candidate optimal value function*. It is easier to verify the uniform limit condition instead. The relation between the two conditions is that the uniform limit condition implies the boundedness condition *for the optimal value function*. In [SLP] the necessary conditions do not coincide with the sufficient conditions, in the uniform limit approach they do: then criteria are obtained. It is of course rather convenient to have *criteria* rather than a *gap* between necessary and sufficient conditions.

There is essentially no loss of strength in the transition to the simplified version. Indeed, the proof of our theorem reveals this. It consists of the verification of four implications: for the necessary and for the sufficient conditions for value functions and for plans; these four arguments prove essentially the four theorems 4.2-4.5 from [SLP] if one eliminates the uniform limit condition from the arguments. Moreover, it is hoped that as a rule the theorem stated in the present paper will suffice for applications.

The proof that is given of the theorem is shorter than the proofs of the theorems 4.2-4.5.

2 Optimality criteria and the uniform limit condition

Stationary discrete-time infinite horizon optimization problems, also called sequence problems (SP), are optimization problems of the following type

$$
\begin{aligned}
\text{(SP)} \qquad & \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\
\text{s.t.} \quad & x_{t+1} \in \Gamma(x_t), \quad t = 0, 1, 2, \ldots, \\
& x_0 \in X \text{ given.}
\end{aligned}
$$

Here $\beta \in (0,1)$ is the discount factor, X is a set, $\Gamma : X \rightrightarrows X$ is the constraint correspondence, and $F : X \times X \to \mathbb{R}$ is the instantaneous payoff function. To be precise, we consider this problem as a problem depending on a parameter x_0, the initial state, $(\text{SP})_{x_0}$, that is, we are considering a family of optimization problems $((\text{SP})_{x_0})_{x_0 \in X}$.

A feasible plan of problem $(\text{SP})_{x_0}$ is a sequence $\{x_t\}_{t=0}^{\infty}$ for which $x_{t+1} \in \Gamma(x_t), t = 0, 1, 2, \ldots$. Let $\Pi(x_0)$ be the set of all feasible plans for $(\text{SP})_{x_0}$. It is assumed that $\Gamma(x)$ is nonempty for all $x \in X$ and that for each $x_0 \in X$ and each $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$ the sum $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ converges in $\mathbb{R} \cup \{\pm\infty\}$. The *optimal value* of problem $(\text{SP})_{x_0}$ is denoted by $v^*(x_0) \in \mathbb{R} \cup \{\pm\infty\}$ for all $x_0 \in X$. This gives a function $v^* : X \to \mathbb{R} \cup \{\pm\infty\}$ called the *optimal value function* of the sequence problem (SP).

Corresponding to this problem, we have the functional equation in the unknown function $v : X \to \mathbb{R} \cup \{\pm\infty\}$

$$
\text{(FE)} \qquad v(x) = \sup_{y \in \Gamma(x)} [F(x,y) + \beta v(y)], \quad \text{all } x \in X, \tag{1}
$$

and the boundedness condition in the unknown function v at $x_0 \in X$

$$
\lim_{t \to \infty} \beta^t v(x_t) = 0 \tag{2}
$$

for each $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$.

The uniform limit condition. The *uniform limit condition* for x_0 is defined to be the assumption that the discounted sum of instantaneous payoffs, $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$, converges *uniformly* in $\mathbb{R}$ over all feasible plans $\{x_t\}_{t=0}^{\infty}$ of $(\text{SP})_{x_0}$. Now we recall in a formal style the usual definition of the uniformity of the limit process. Without uniformity, we would have the so-called *limit condition* for $x_0 \in X$,

$$\forall \underline{x} = \{x(t)\}_{t=0}^{\infty} \in \Pi(x_0) \forall \varepsilon > 0 \exists N = N_{\underline{x}} > 0 \forall n \geq N_{\underline{x}} \quad |\sum_{t=n}^{\infty} \beta^t F(x_t, x_{t+1})| < \varepsilon,$$

or, equivalently,

$$\forall \varepsilon > 0 \forall \underline{x} = \{x_t\}_{t=0}^{\infty} \in \Pi(x_0) \exists N = N_{\underline{x}} > 0 \forall n \geq N = N_{\underline{x}} \quad |\sum_{t=n}^{\infty} \beta^t F(x_t, x_{t+1})| < \varepsilon.$$

Now we interchange $\forall \underline{x} = \{x(t)\}_{t=0}^{\infty} \in \Pi(x_0)$ and $\exists N > 0$. This gives the uniform limit condition

$$\forall \varepsilon > 0 \exists N > 0 \forall \{x(t)\}_{t=n}^{\infty} \in \Pi(x_0) \forall n \geq N \quad |\sum_{t=n}^{\infty} \beta^t F(x_t, x_{t+1})| < \varepsilon.$$

That is, for each $\varepsilon > 0$ there exists an N that works for all feasible plans. Thus, the uniform limit condition is stronger than the limit condition.

Illustration. Consider the problem to maximize $\sum_{t=0}^{\infty} \beta^t u_t^{\alpha}$ subject to $x_{t+1} = c(x_t - u_t), u_t \geq 0, t = 0, 1, 2, \ldots$ and x_0 given. Here $0 < \alpha < 1, c > 1, 0 < \beta < 1$ and $0 < \beta c^{\alpha} < 1$. In terms of the problem (SP) we have $F(x, y) = (x_t - c^{-1} x_{t+1})^{\alpha}$ and $\Gamma(x) = [0, cx]$. Then for each $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$ one has $\beta^t F(x_t, x_{t+1}) \leq \beta^t x_t^{\alpha}$ and this is by repeated use of the inclusion constraint $\leq \beta^t (c^t x_0)^{\alpha} = (\beta c^{\alpha})^t x_0^t$. Thus the series $\sum_{t=0}^{\infty} \beta^t u_t^{\alpha}$ has been majorized termwise by a convergent geometric series that does not depend on the chosen element of $\Pi(x_0)$, which implies that the uniform limit condition holds.

Theorem. Optimality criteria under the uniform limit condition. *Consider the sequence problem (SP). Assume that the uniform limit condition holds for all $x_0 \in X$. Then*

1. *a function $v : X \to \mathbb{R} \cup \{\infty\}$ is the optimal value function v^* if and only if it satisfies the functional equation* (1) *and the boundedness condition* (2) $\forall x_0 \in X$*;*

2. *a feasible plan $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$ for a given $x_0 \in X$ is an optimal plan for (P_{x_0}) if and only if*

$$v^*(x_t) = F(x_t, x_{t+1}) + \beta v^*(x_{t+1})$$

for $t = 0, 1, \ldots$.

Remark 1. In particular, the uniform limit condition for all $x_0 \in X$ implies the boundedness condition for the optimal value function.

Remark 2. This result suggests the following procedure: 1) verify the uniform limit condition, 2) write down the characterization of the optimal value function and optimal plans in terms of the functional equation and the boundedness condition, 3) try to find them analytically: a) determine the optimal value function by means of the first criterion, b) determine the optimal plan by means of the second criterion.

Proof of the theorem.

Criterion 1.

'only if': $v^*(x_0)$ is defined as the supremum of $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ where $\{x_t\}_{t=0}^{\infty}$ runs over $\Pi(x_0)$. Taking the supremum, while holding some arbitrary $x_1 \in G(x_0)$ fixed, gives $F(x_0, x_1) + \beta v^*(x_1)$; taking after this the supremum of this expression over all $x_1 \in G(x_0)$ gives the result that v^* satisfies the functional equation (1). The expression $\beta^t v^*(x_t)$ equals, by the definitions, the supremum of the expressions $\sum_{s=t}^{\infty} \beta^s F(x_s, x_{s+1})$, where $\{x_s\}_{s=0}^{\infty}$ runs over $\Pi(x_0)$, and this tends for $t \to \infty$ to zero, by the uniform limit condition. This shows that v^* satisfies the boundedness condition (2) for each $x_0 \in X$.

'if': Assume $v : X \to \mathbb{R} \cup \{\infty\}$ satisfies (1) and (2) for x_0. Applying $n+1$ times that v is a solution of the functional equation, one gets

$$v(x_0) = \sup_{\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)} [\sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})] + \beta^{n+1} v(x_{n+1}).$$

Taking the limit $n \to \infty$ gives $v(x_0) = v^*(x_0)$, using that v satisfies (2) and using the definition of $v^*(x_0)$. As $x_0 \in X$ is arbitrary, it follows that $v = v^*$.

Criterion 2.

'only if': if $\{x_t\}_{t=0}^{\infty}$ is a solution of (P_{x_0}), then $v^*(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ and $\{x_t\}_{t=1}^{\infty}$ is a solution of (P_{x_1}) by the stationarity of the problem, so $v^*(x_1) = \sum_{t=1}^{\infty} \beta^t F(x_t, x_{t+1})$. It follows that $v^*(x_0) = F(x_0, x_1) + \beta v^*(x_1)$. Replacing for each $t = 0, 1, \ldots$ in this equality (x_0, x_1) by (x_t, x_{t+1}), as we may, we get the required equality $v^*(x_t) = F(x_t, x_{t+1}) + \beta v^*(x_{t+1})$.

'if': if $\{x_t\}_{t=0}^{\infty}$ satisfies the equation $v^*(x_t) = F(x_t, x_{t+1}) + \beta v^*(x_{t+1})$, then applying repeatedly—$n+1$ times—this equation gives $v^*(x_0) = \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} v^*(x_{n+1})$. Now we take the limit $n \to \infty$, using that $\beta^{n+1} v^*(x_{n+1})$ tends to zero for $n \to \infty$ by criterion 1. This gives $v^*(x_0) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$, and so, by the definition of $v^*(x_0)$, we get that $\{x_t\}_{t=0}^{\infty}$ is a solution of (P_{x_0}).

Remark 3. In order to simplify the comparison of the theorem to the theorems 4.2-4.5 in [S-L-P], we have conformed to [S-L-P] in our main text, working with plans rather than with policies. Now we show that using policies, a further simplification of the theorem is possible. A feasible *policy* is a mapping $\pi : X \to X$ for which $\pi(x) \in \Gamma(x)$ for all $x \in X$. A feasible policy π gives for given initial state x_0 a feasible plan $\{x_t\}_{t=0}^{\infty}$ by defining recursively $x_{t+1} = \pi(x_t), t \geq 0$. A feasible element of the sequence problem (SP) *in feedback form* (also called *closed loop form*) is a pair (v, π) consisting of a feasible policy $\pi : X \to X$ such that for all $x_0 \in X$ the value of the plan of π is $v(x_0)$. This is called a solution of the sequence problem (SP) if v is the optimal value function of the sequence problem (SP). Now we state the theorem above in terms of policies.

Theorem (in terms of policies). *Consider the sequence problem (SP) and assume the uniform limit condition. The following two conditions on a feasible element in feedback form (v, π) are equivalent:*

- *(v, π) is a solution of (SP)*
- *v is a solution of the functional equation (1) and the boundedness condition (2) and for each $x \in X$ the supremum in the functional equation (1) is assumed for $y = \pi(x)$.*

3 Comparison to Stokey, Lucas and Prescott

By essentially the same arguments as those used to prove the theorem in the previous section, one can prove the theorems 4.2-4.5 from SLP]. We display these theorems for convenience.

Theorem 4.2 [SLP]. *The function v^* satisfies (FE).*

Theorem 4.3 [SLP]. *If v is a solution to (FE) and satisfies*

$$\lim_{n\to\infty} \beta^n v(x_n) = 0, \quad all(x_0, x_1, \ldots) \in \Pi(x_0), all\ x_0 \in X,$$

then $v = v^*$.

Theorem 4.4. [SLP]. *Let $\underline{x}^* \in \Pi(x_0)$ be a feasible plan that attains the supremum in (SP) for initial state x_0, Then*

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*),\ t = 0, 1, 2, \ldots.$$

Theorem 4.5. [SLP]. *Let* $\underline{x}^* \in \Pi(x_0)$ *be a feasible plan from* x_0 *satisfying*

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*),\ t = 0, 1, 2, \ldots.$$

and with

$$\limsup_{t\to\infty} \beta^t v^*(x_t^*) \leq 0.$$

Then $\underline{x}^*$ *attains the supremum in (SP) for initial state* x_0.

These results suggest the following procedure: 1) determine the optimal value function by means of theorem 4.3, 2) determine the optimal plan by means of theorem 4.5 (note that the limsup condition in theorem 4.5 does not have to be veerified in this context; it holds automatically if the previous step has been carried out. If one cannot find the optimal value function and an optimal plan analytically, then the theorems 4.2-4.5 do not give criteria and therefore are of limited use. In such situations one can often verify the uniform limit condition and this gives a characterization of the optimal value function and an optimal plan.

One sees that theorem 4.5 requires an assumption on the solution, a form of the boundedness condition on the optimal value function. Moreover, there is a gap between necessary and sufficient conditions:

- the sufficient condition for the optimal value function contains a condition that is not contained in the necessary condition: $\lim_{n\to\infty} \beta^n v(x_n) = 0$, all $(x_0, x_1, \ldots) \in \Pi(x_0)$, all $x_0 \in X$.
- the sufficient condition for the optimal plan contains a condition that is not contained in the necessary condition: $\limsup_{t\to\infty} \beta^t v^*(x_t^*) \leq 0$.

4 Impossibility of using the limit condition

The limit condition for $x_0 \in X$ is defined as follows: the discounted sum of instantaneous payoffs, $\lim_{n\to\infty} \sum_{t=0}^n \beta^t F(x_t, x_{t+1})$, converges for all feasible plans $\{x_t\}_{t=0}^\infty \in \Pi(x_0)$ with initial state x_0. We are going to show by means of counterexamples—variants of the examples on page 74 and page 76 of Stokey, Lucas and Prescott (1989) of a consumer whose objective function is discounted consumption—that it is impossible to simplify theorems 1 and 2 by replacing the uniform limit condition by the limit condition. Actually we will do more: we will analyze for each theorem both implications, 'if' and 'only if'.

On the implication 'only if' in theorem 1. The optimal value function v^* always satisfies the functional equation (1), even without the limit condition. This follows from the proof of theorem 1. Now we give an example where the limit condition holds and the optimal value function v^* does not satisfy the boundedness condition (2).

$$I(\{c(t)\}_{t=0}^{\infty}, \{x_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t c(t)$$

subject to

$$c(t) = x_t - \beta x_{t+1}, 0 \leq c(t), x_t \geq 0, \ t = 0, 1, \ldots,$$

$$x(0) = x_0$$

with discount factor $\beta \in (0, 1)$.

If we take in the set-up of the previous section $\beta \in (0,1), F(x,y) = x - \beta y, G(x) = [0, \beta^{-1}x], X = [0, \infty)$, then we get this problem. We note that $G(x) = [0, \beta^{-1}x]$ is nonempty for all $x \in [0, \infty)$. Now we check that the limit condition holds for all $x_0 \in X$. The series $\sum_{t=0}^{\infty} \beta^t c(t)$ is convergent as all terms are nonnegative and the partial sums are bounded above: the sum of the terms from $t = 0$ till $t = T$ equals $\sum_{t=0}^{T} \beta^t (x_t - \beta x_{t+1}) = x_0 - \beta^{T+1} x_{t+1} \leq x_0$. The optimal value function of this family of problems is $v^*(x) = x$: we have already seen that x_0 is an upper-bound for the objective function of the problem (P_{x_0}), and this upper-bound can be achieved for example by choosing $c(0) = x_0, c(t) = 0, \forall t > 0$ and $x_t = 0, \forall t > 0$. Now we consider the feasible plan $x_t = \beta^{-t} x_0, t = 0, 1, \ldots$ ('always invest everything and never consume anything'). Then $\lim_{t\to\infty} \beta^t v^*(x_t) = \lim_{t\to\infty} \beta^t \beta^{-t} x_0 = x_0$, which is not zero if $x_0 \neq 0$. Therefore, Vv^* does not satisfy the boundedness condition (2) for $x_0 \neq 0$.

On the implication 'if' in theorem 1. This implication always holds, even without the limit condition. This follows from the proof of theorem 1. Now we give an example where the limit condition holds and where the functional equation (1) has an other solution than the optimal value function.

$$I(\{c(t)\}_{t=0}^{\infty}, \{x_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t c(t)$$

subject to

$$c(t) = x_t - \beta x_{t+1}, 0 \leq c(t) \leq 1, \ t = 0, 1, \ldots,$$

$$x(0) = x_0$$

with discount factor $\beta \in (0,1)$.

If we take in the set-up of the previous section $\beta \in (0,1), F(x,y) = x - \beta y, G(x) = [\beta^{-1}x - \beta^{-1}, \beta^{-1}x], X = \mathbb{R}$, then we get this problem. Note that $[\beta^{-1}x - \beta^{-1}, \beta^{-1}x]$ is nonempty for all $x \in \mathbb{R}$. The limit condition holds for all $x_0 \in X$, as the series $\sum_{t=0}^{\infty} \beta^t c(t)$ is convergent as $0 \leq c(t) \leq 1$ for all $t \geq 0$. The optimal value function of this family of problems is $v^*(x_0) = (1-\beta)^{-1}$ for all $x_0 \in X$. This can be seen by choosing $c(t) = 1$ for all $t \geq 0$, and then determining the time path of the wealth of the consumer recursively by $x(0) = x_0$, $1 = x_t - \beta x_{t+1}, t = 0, 1, \ldots$. This gives the feasible plan $x^*(t) = (x_0 - (1-\beta)^{-1})\beta^{-t} + (1-\beta)^{-1}$. Now we consider the functional equation (1) for this consumer problem:

$$v(x) = \sup_{y \in [\beta^{-1}x - \beta^{-1}, \beta^{-1}x]} [x - \beta y + \beta V v(y)].$$

This has more than one solution: besides the solution $v^*(x) = (1-\beta)^{-1}$, it has the solution $\tilde{v}(x) = x$.

On the implication 'only if' in theorem 2. This implication always holds, even without the limit condition. This follows from the proof of theorem 2.

On the implication 'if' in theorem 2. We give an example where the limit condition holds and a non-optimal feasible plan satisfies the functional equation (1). It is an example that was already considered above.

$$I(\{c(t)\}_{t=0}^{\infty}, \{x_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t c(t)$$

subject to

$$c(t) = x_t - \beta x_{t+1}, 0 \leq c(t), x_t \geq 0,\ t = 0, 1, \ldots,$$

$$x(0) = x_0$$

with discount factor $\beta \in (0,1)$.

We have checked above that the limit condition holds for all $x_0 \in X$. We consider again the feasible plan $x_t = \beta^{-t}x_0, t = 0, 1, \ldots$ ('always invest everything and never consume anything'). This satisfies the equation $v^*(x_t) = F(x_t, x_{t+1}) + \beta v^*(x_{t+1})$ for $t = 0, 1, \ldots$. Indeed, the left-hand side is $\beta^{-t}x_0$ and the right-hand side is $(x_t - \beta x_{t+1}) + \beta x_{t+1} = \beta^{-t}x_0$ as well. However, this feasible plan is not optimal if $x_0 > 0$: it gives value 0 as nothing is ever consumed.

On the possibility that v^* takes value ∞. Here is an example where the limit condition holds and where the optimal value function takes the value ∞.

We consider the example: $\beta \in (0,1), X = \mathbb{R}, G(x) = \{x\} \forall x \neq 1, G(1) = \{2, 3, \ldots\}, F(x,y) = y, \forall x, y \in \mathbb{R}$. Then the assumptions are satisfied: $G(x)$ is nonempty for all $x \in X$, and if $x \neq 1$, the only feasible plan is $(x, x, \ldots)$ and $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(x,x) = (1-\beta)^{-1}x$, so it exists and is finite; if $x = 1$, then all feasible plans are of the form $(1, k, k, k, \ldots)$ with $k \in \{2, 3, \ldots\}$ and $\lim_{n\to\infty}(F(1,k) + \sum_{t=1}^{n} F(k,k)) = (1-\beta)^{-1}k$, so it exists and is finite; thus the limit condition is verified. However, $v^*(1) = \sup_{k=2,3,\ldots}(1-\beta)^{-1}k = \infty$.

Now we discuss what went wrong in the proofs of Theorems 6.1 and 6.2 in Acemoglu (2009), which state that theorems 1 and 2 hold under the limit condition and that the boundedness condition (2) can be omitted in theorem 1.

On the proof of the equivalence of values result. The optimal value function v^* satisfies indeed the functional equation (1), and the proof of this fact that is given is correct. That conversely each solution of the functional equation (1) has to be equal to v^* is not true: a counterexample has been given above. The proof is split up in two parts.

The aim of the first part is to show that for a solution v of the functional equation (1), one has $v(x_0) \geq J(\{x_t\}_{t=0}^{\infty})$ for each $x_0 \in X$ and each $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$. Applying $n+1$ times that v is a solution of the functional equation, one gets

$$v(x_0) \geq [\sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})] + \beta^{n+1} v(x(n+1))$$

(the factor β^t is missing but the next line shows that this is a typo). This is suggestive, as the first term of the right hand side tends to $J((x_t)_{t=0}^{\infty})$ for $n \to \infty$. Therefore, it suffices to prove that $\lim_{n\to\infty} \beta^{n+1} v(x(n+1)) = 0$. This need not be true: a counterexample has been given in the previous section. It is stated that this is an immediate consequence of the formula

$$\lim_{n\to\infty} \beta^{n+1} v(x(n+1)) = \lim_{n\to\infty} [\beta^{n+1} \lim_{m\to\infty} \sum_{t=n}^{m} \beta^t F(x_t, x_{t+1})]$$

by the assumption that the series that defines $J(\{x_t\}_{t=0}^{\infty})$ converges to a real number ('the limit condition'). This implication holds indeed, but this formula cannot be justified (and as has been shown by counterexample, need not be true).

The aim of the second part of the proof is to show that for each $x_0 \in X$ and each $\varepsilon > 0$ there exists $\{x_t\}_{t=0}^{\infty} \in \Pi(x_0)$ such that $v(x_0) \leq J(\{x_t\}_{t=0}^{\infty}) + \varepsilon$. It uses $\lim_{n\to\infty} \beta^{n+1} v(x(n+1)) = 0$, which as we have seen is not justified (and need not be true).

On the proof of the principle of optimality. For each optimal plan $(x^*(t))_{t=0}^{\infty}$, one has

$$v^*(x_t) = F(x^*(t), x^*(t+1)) + \beta v^*(x^*(t+1))$$

for $t = 0, 1, \ldots$, and the proof of this fact that is given is correct.

That conversely each $\{x^*(t)\}_{t=0}^{\infty} \in \Pi(x_0)$ that satisfies the equation $v^*(x^*(t)) = F(x^*(t), x^*(t+1)) + \beta v^*(x^*(t+1))$ for $t = 0, 1, \ldots$, is optimal, is not true, as has been shown by counterexample in the previous section. The proof begins by applying repeatedly—$n+1$ times—this equation, and this gives

$$v^*(x_0) = \sum_{t=0}^{n} \beta^t F(x^*(t), x^*(t+1)) + \beta^{n+1} v^*(x^*(n+1)).$$

To finish the proof, it remains to show that $\lim_{n\to\infty} \beta^{n+1} v^*(x^*(n+1)) = 0$. This would be the case if v^* would be bounded. It is stated that the assumption that $J(\{x^*(t)\}_{t=0}^{\infty})$ is well-defined as a real number implies that v^* is bounded, but this implication is not justified. In fact, v^* is not bounded in the first counterexample given in the previous section.

On the role of infinite values of the optimal value function in the proofs. The possibility that the optimal value function v^* or the solution v of the functional equation can take the value ∞ is ignored; these possibilities require additional arguments in the proofs given in Stokey, Lucas and Prescott (1989).

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