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ON CONVERGENCE TOWARD AN EXTREME VALUE DISTRIBUTION IN $C[0,1]$

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The structure of extreme value distributions in infinite-dimensional space is well known. We characterize the domain of attraction of such extreme-value distributions in the framework of Giné Hahn and Vatan. We intend to use the result for statistical applications.

1. Introduction. The two northern provinces, Friesland and Groningen, of the Netherlands are almost completely below sea level. Since there are no natural coast defenses like sand dunes, the area is protected against inundation by a long dike. Since there is no subdivision of the area by dikes, a break in the dike at any place could lead to flooding of the entire area. This leads to the following mathematical problem.

Suppose we have a deterministic function f defined on $[0,1]$ (representing the top of the dike). Suppose we have i.i.d. random functions $\xi_1, \xi_2, \dots$ defined on $[0,1]$ (representing observations of high tide water levels monitored along the coast). The question is: how can we estimate

$$P\{\xi_i(t) \le f(t) \text{ for } i = 1, \dots, k, 0 \le t \le 1\} = P\Big\{\max_{1\le i\le k} \xi_i(t) \le f(t) \text{ for } 0 \le t \le 1\Big\} \tag{1.1}$$

on the basis of n observed independent realizations of the process ξ (n large)?

Now a typical feature of this kind of problem is that none of the observed processes ξ come even close to the boundary f that is, during the observation period there has not been any flooding-damage). This means that we have to extrapolate the distribution of ξ far into the tail. Since nonparametric methods cannot be used, we resort to a limit theory; that is we imagine that $n \to \infty$ but in doing so we wish to keep the essential feature that the observations are far from the boundary. This leads to the assumption that f is not a fixed function when $n \to \infty$ but that in fact f depends on n and moves to the upper boundary of the distribution of ξ when $n \to \infty$. Another way of expressing this is that we assume that the left-hand side in the second inequality has a limit distribution after normalization. So in order to answer this question, we need a limit theory for the pointwise maximum of i.i.d. random functions and this is the subject of the present paper. In fact, this theory of infinite-dimensional extremes is an extension of the corresponding theory in finite-dimensional space which is by now well understood. A short review

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of finite-dimensional results is useful at this point. For ease of writing we restrict ourselves to the two-dimensional case.

Suppose $(X, Y), (X_1, Y_1), (X_2, Y_2), \ldots$ are i.i.d. random vectors with distribution function F. Suppose also the marginal distribution functions are continuous.

The convergence $(n \to \infty)$ of

$$\left(\frac{\bigvee_{i=1}^n X_i - b_n}{a_n}, \frac{\bigvee_{i=1}^n Y_i - d_n}{c_n}\right) \tag{1.2}$$

in distribution (with norming constants $a_n, c_n > 0$, b_n, d_n) is equivalent to the convergence of

$$\left(\frac{\bigvee_{i=1}^n X_i - b_n}{a_n}\right), \tag{1.3}$$

$$\left(\frac{\bigvee_{i=1}^n Y_i - d_n}{c_n}\right) \tag{1.4}$$

and

$$\frac{1}{n}\left(\bigvee_{i=1}^n \frac{1}{1 - F_1(X_i)}, \bigvee_{i=1}^n \frac{1}{1 - F_2(Y_i)}\right) \tag{1.5}$$

in distribution, where F_1 and F_2 are the two marginal distribution functions of F. Note that $1/(1 - F_1(X))$ and $1/(1 - F_2(Y))$ both have the distribution function $1-(1/x)$, $x > 1$. So for the joint convergence it is sufficient to consider a standard or "simple" case.

The limit distributions of (1.3) and (1.4) are

$$\exp -(1 + \gamma_1 x)^{-1/\gamma_1} \quad \text{and} \quad \exp -(1 + \gamma_2 x)^{-1/\gamma_2},$$

respectively, so there are two real parameters: γ_1 for the first component, γ_2 for the second one. The limit distribution of (1.5) is

$$\exp\left\{-\int_0^{\pi/2}\left(\frac{1 \wedge \tan\theta}{x} \vee \frac{1 \wedge \cot\theta}{y}\right)\Phi(d\theta)\right\},$$

where Φ is the distribution function of a finite measure on $[0, \pi/2]$ with

$$\int_0^{\pi/2}(1 \wedge \tan\theta)\Phi(d\theta) = \int_0^{\pi/2}(1 \wedge \cot\theta)\Phi(d\theta) = 1$$

[de Haan and Resnick (1977), Deheuvels (1978), Pickands (1981)]. The limit distribution of (1.2) then becomes

$$\exp\left\{-\int_0^{\pi/2}\left(\frac{1 \wedge \tan\theta}{(1 + \gamma_1 x)^{1/\gamma_1}} \vee \frac{1 \wedge \cot\theta}{(1 + \gamma_2 x)^{1/\gamma_2}}\right)\Phi(d\theta)\right\}$$

depending on two real parameters and a finite measure on $[0, \pi/2]$.

It is useful to say a bit more about the origin of Φ. Write $U_i := (1 - F_1(X_i))^{-1}$ and $V_i := (1 - F_2(Y_i))^{-1}$, $i = 1, 2, \ldots$ and let F_0 be the joint distribution function. The limit relation

$$P\left\{\frac{1}{n}\bigvee_{i=1}^{n} U_i \le x, \frac{1}{n}\bigvee_{i=1}^{n} V_i \le y\right\} \to G(x, y) \tag{1.6}$$

is equivalent to

$$F_0^n(nx, ny) \to G(x, y) \tag{1.7}$$

and hence to

$$n\{1 - F_0(nx, ny)\} \to -\log G(x, y). \tag{1.8}$$

Define for $n = 1, 2, \ldots$ the measure ν_n by

$$\nu_n\{(s, t);\ s > x \text{ or } t > y\} := n\{1 - F_0(nx, ny)\}. \tag{1.9}$$

Then (1.8) says that the measures ν_n converge to a measure ν; that is,

$$nP\{n^{-1}(U_1, V_1) \in A\} \to \nu(A) \tag{1.10}$$

for any Borel set $A \subset [0, \infty)^2 \setminus \{(0, 0)\}$ with $\nu(\partial A) = 0$. Obviously ν is homogeneous, $\nu(\alpha A) = \alpha^{-1}\nu(A)$, hence for $r > 0$ and $0 \le \theta \le \pi/2$,

$$\nu\left\{(s, t)|s \vee t > r,\ \frac{t}{s} \le \tan\theta\right\} = r^{-1}\Phi(\theta)$$

with Φ as before. So Φ originates from a transformation very similar to the transformation to polar coordinates. In fact ($t \to \infty$),

$$tP\left\{U \vee V > t,\ \frac{V}{U} \le \tan\theta\right\} \to \Phi(\theta) \tag{1.11}$$

and convergence of (1.2) is equivalent to convergence of the two marginals and (1.10).

Note that we have discussed two topics: the characterization of the limit distribution and for each of those, the characterization of the domain of attraction. A somewhat more extensive review is contained in de Haan and de Ronde (1998). The mentioned results form the basis for statistical applications. These are reviewed in the same paper.

The most direct generalization to the infinite-dimensional case is by generalizing the concept of distribution function. Note that

$$\begin{aligned} &P\left\{\max_{1\le i\le n} \xi_i(t) \le f(t) \text{ for } 0 \le t \le 1\right\} \\ &\quad = P^n\left\{\xi(t) \le f(t) \text{ for } 0 \le t \le 1\right\}. \end{aligned} \tag{1.12}$$

That means that we can proceed as in the string of implications (1.6) $\Rightarrow$ (1.10) and beyond. But an equality like (1.12) is not valid for probabilities of the type

$$P\left\{\max_{1\le i\le n}\xi_i\in E\right\} \tag{1.13}$$

so that the connection with convergence of measures is much less obvious. That is actually the main problem in the extension to the infinite-dimensional situation. A theorem by Norberg (1984) states that the convergence of

$$P\left\{\max_{1\le i\le n}\xi_i(t)\le nf(t)\text{ for }0\le t\le 1\right\}$$

for all continuous functions f is equivalent to convergence of $n^{-1}\max_{1\le i\le n}\xi_i$ in distribution in the space of upper semicontinuous functions. So this set-up looks attractive, but there are several problems: we do not always get convergence of marginal distributions, it implies convergence of the probability of too few sets and it is difficult to communicate the result to nonmathematicians. So we decided not to use the framework of semicontinuous functions.

Before explaining the framework that we used, we review the two existing results on the characterization of the limit distributions.

First, note if η has the limit distribution of $(\bigvee_{i=1}^n\xi_i-b_n)/a_n$ where $a_n>0$ and b_n are norming functions, we have for $k=1,2,\dots,$

$$\left(\bigvee_{i=1}^k\eta_i-B_k\right)\bigg/A_k\stackrel{d}{=}\eta, \tag{1.14}$$

where $\eta_1,\eta_2,\dots$ are i.i.d. copies of η and $A_k>0$ and B_k norming functions. Here convergence could be in any metric space. In particular, if $b_n\equiv 0$ and $a_n\equiv n$, then

$$k^{-1}\bigvee_{i=1}^k\eta_i\stackrel{d}{=}\eta. \tag{1.15}$$

A process satisfying (1.14) is called max-stable and a process satisfying (1.15) is called simple max-stable.

All random functions are defined on [0, 1].

Proposition 1.1 [de Haan (1984), de Haan and Pickands (1986)]. *Suppose η is continuous in probability. The following are equivalent*:

(i) *η is simple max-stable.*

(ii) *There exists a collection of function $\{g_t\}_{t\in[0,1]}$ with $g_t\colon[0,1]\to R_+$, $g_t\in L_1$ for all t and $\{g_t\}_{t\in[0,1]}$ continuous in L_1 (i.e., $\|g_{t_n}-g_t\|_1\to 0$ if $t_n\to t$) such that for $0\le t_1<t_2<\cdots<t_k\le 1$ and $x_1,x_2,\dots,x_k>0$,*

$$P\{\eta(t_i)\le x_i,\ i=1,2,\dots,k\}=\exp\left\{-\int_0^1\bigvee_{i=1}^k\frac{g_{t_i}(s)}{x_i}\,ds\right\}. \tag{1.16}$$

PROPOSITION 1.2 [Resnick and Roy (1991)]. *Moreover, the process* η *has continuous sample paths if and only if* $\{g_t\}_{t\in[0,1]}$ *is continuous in* L_∞*-norm.*

The other result is the following.

PROPOSITION 1.3 [Giné, Hahn and Vatan (1990)]. *Suppose* η *is in* $C[0,1]$. *The following are equivalent*:

(i) η *is simple max-stable.*
(ii) *There exists a finite Borel measure* σ *on* $C_1^+ := \{f \in C[0,1];\ f > 0, \|f\|_\infty = 1\}$ *with* $\int_{C_1^+} f(t)\,d\sigma(f) = 1$ *for* $t \in [0,1]$ *such that for all* $f \in C[0,1],\ f > 0$,

$$P\{\eta < f\} = \exp\left\{-\int_{C_1^+[0,1]} \|g/f\|_\infty \,d\sigma(g)\right\}$$

or, equivalently, such that for all compact $K_1, K_2, \dots, K_m \subset [0,1]$ *and* $x_1, x_2, \dots, x_m$ *positive,*

$$P\left\{\sup_{t\in K_i} \eta(t) \le x_i,\ i = 1, 2, \dots, m\right\} = \int_{C_1^+[0,1]} \max_{1\le i\le m}\left(\frac{\sup_{t\in K_i} g(t)}{x_i}\right) d\sigma(g).$$

This characterizes the "simple" case.

COROLLARY 1.4 [Giné, Hahn and Vatan (1990)]. *A general max-stable process in* $C[0,1]$ [*i.e., a process satisfying* (1.14)] *can be represented as*

$$a(t)\frac{(\eta(t))^{\gamma(t)} - 1}{\gamma(t)} + b(t)$$

with $a, b, \gamma \in C[0,1]$ *and, a positive and with* ξ *as in Proposition* 1.3.

REMARK 1.5. The proposition and corollary are also true with $C[0,1]$ replaced by $D[0,1]$ throughout and $C_1^+[0,1]$ replaced by $D_1^+[0,1] := \{f \in D[0,1];\ \|f\|_\infty = 1, \inf_{t\in[0,1]} f(t) > 0\}$.

The set-up of Proposition 1.1 implies knowledge of (1.13) for very few sets E. Also the polar coordinate type transformation in Proposition 1.1 is less tractable than in Proposition 1.3. So we decided to proceed in the framework of random functions in $C[0,1]$ and $D[0,1]$.

2. Results. The following has been taken from Daley and Vere-Jones (1988).

Let X be a complete and separable metric space (CSMS).

DEFINITION 2.1. A Borel measure ν on a CSMS is boundedly finite if $\nu(A) < \infty$ for every bounded Borel set A.

We shall only consider such measures.

DEFINITION 2.2. A sequence of measures $\{\nu_k\}$ on a CSMS converges weakly to a measure ν if $\nu_k(A) \to \nu(A)$ for each bounded Borel set A with $\nu(\partial A) = 0$.

PROPOSITION 2.3. *The sequence $\{\nu_k\}$ converges to ν if and only if there exists a sequence $S^{(n)}$ of spheres, $S^{(n)} \uparrow X$, such that $\nu_k(A) \to \nu(A)$ for each n and each Borel set $A \subset S^{(n)}$ with $\nu(\partial A) = 0$.*

We shall consider measures on the spaces

$$C^+[0,1] := \{f \in C[0,1]; f > 0\},$$
$$D^+[0,1] := \{f \in D[0,1]; \inf_{t\in[0,1]} f(t) > 0\}.$$

By transform,

$$f \leftrightarrow \left(\|f\|_\infty, \frac{f}{\|f\|_\infty} \right),$$
$$C^+[0,1] = \mathscr{R}^+ \times C_1^+[0,1].$$

Note neither $\mathscr{R}^+$ nor $C_1^+[0,1]$ is CSMS. Define

$$\overline{C}_1^+[0,1] := \{f \in C[0,1];\ f \geq 0;\ \|f\|_\infty = 1\},$$

which is a CSMS and $(0,\infty]$ is a CSMS under the metric $\rho(x,y) = (1/x) - (1/y)$, $x, y \in (0,\infty]$. Hence

$$\overline{C}^+[0,1] = (0,\infty] \times \overline{C}_1^+[0,1] \text{ is a CSMS .}$$

We do the same to space D:

$$\overline{D}^+[0,1] = (0,\infty] \times \overline{D}_1^+[0,1] \quad \text{where}$$
$$\overline{D}_1^+[0,1] := \{f \in D[0,1];\ f \geq 0,\ \|f\|_\infty = 1\}.$$

THEOREM 2.4. *Suppose $\xi, \xi_1, \xi_2, \ldots$ are i.i.d. random elements of $D^+[0,1]$. Consider the following statements:*

(i) $(1/n)\bigvee_{i=1}^n \xi_i \xrightarrow{D} \eta$ *in* $D^+[0,1]$ (*and then* η *is simple max-stable*).

(ii) $\nu_n \xrightarrow{w} \nu$ *in the space of measures on* $\overline{D}^+[0,1]$ (*and then the measure* ν *is homogeneous of degree* -1) *with*

$$\nu_n(E) := nP\{n^{-1}\xi \in E\} \quad \text{for } E \in \mathscr{B}(\overline{D}^+[0,1]).$$

(iii) $N_n \xrightarrow{d} N$ *in the space of random measures on* $\overline{D}^+[0,1]$ *where*

$$N_n := \sum_{i=1}^n \varepsilon_{\{n^{-1}\xi_i\}}$$

and N is Poisson process.

We have the following implications: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)*; moreover* ν [*from* (ii)] *is the mean measure of the Poisson process in* (iii) *and, with* η *from* (i)*, for* $n = 1, 2, \ldots,$

$$P\{n \in A_{\overline{K}, \bar{x}}\} = \exp -\nu\Big(A^c_{\overline{K}, \bar{x}}\Big) \tag{2.1}$$

with, for $\overline{K} = (K_1, \ldots, K_m)$ *compact sets and* $\bar{x} = (x_1, \ldots, x_m)$*,*

$$A_{\overline{K}, \bar{x}} := \{f \in \overline{D}^{+}[0, 1];\ f(t) < x_i \text{ for } t \in K_i,\ i = 1, 2, \ldots, m\}.$$

Moreover,

$$\mathscr{L}(\eta) = \mathscr{L}\bigg(\bigvee_{i=1}^{\infty} \zeta_i\bigg), \tag{2.2}$$

where $\{\zeta_i\}_{i=1}^{\infty}$ *are the points of a realization of* N.

Finally, if $P\{\eta \in C[0, 1]\} = 1$ *for the process* η *from* (2.1)*, then* (ii) $\Rightarrow$ (i) *also holds.*

REMARK 2.5. The theorem also holds with D replaced by C everywhere. Hence for the space C in part (ii) of Theorem 2.4 it is sufficient to require

$$nP\{f_1 < n^{-1}\xi < f_2\} \to \nu\{f; f_1 < f < f_2\}$$

for arbitrary nonrandom functions f_1 and f_2 in $C^{+}[0, 1]$ and $\nu_n(S_\varepsilon) \to \nu(S_\varepsilon)$ for all $\varepsilon > 0$ [cf. Billingsley (1968), page 15, Corollary 2].

REMARK 2.6. If $P\{\xi \in \overline{C}^{+}[0, 1]\} = 1$, then (i) holds in the space $C[0, 1]$.

REMARK 2.7. This is the analogue of the equivalence of (1.6) and (1.8).

Theorem 2.4 characterizes convergence to a "simple" max-stable process. The general case is covered by the next result.

THEOREM 2.8. *Suppose* $\xi, \xi_1, \xi_2, \ldots$ *are i.i.d. random elements of* $C[0, 1]$*. Suppose* $F_t(x) := P\{\xi(t) \le x\}$ *is continuous with respect to* t *for each* x*. Define*

$$U_t(s) := F_t^{\leftarrow}(1 - 1/s), \qquad s > 0,\ 0 \le t \le 1.$$

The following three statements are equivalent:

(i)

$$\bigg(\bigvee_{i=1}^{n} \xi_i - b_n(t)\bigg)\Big/ a_n(t) \xrightarrow{w} \bar{\eta},$$

where $a_n(t) > 0$ *and* $b_n(t)$ *are continuous functions, chosen in such a way that for each* $t \in [0, 1]$*,*

$$P\{\bar{\eta}(t) \le x\} = \exp\big(-(1 + \gamma(t)x)^{-1/\gamma(t)}\big).$$

Then $\gamma(t)$ *is a continuous function.*

(ii)

$$\left(\bigvee_{i=1}^{n}\xi_i - U_t(n)\right)\Big/ a_n(t) \xrightarrow{w} \bar{\eta}.$$

(iiia)

$$\frac{1}{n}\bigvee_{i=1}^{n}\frac{1}{1-F_t(\xi_t(t))} \xrightarrow{w} (1+\gamma(t)\bar{\eta}(t))^{1/\gamma(t)}$$

and the limit is automatically simple max-stable.

(iiib)

$$\frac{U_t(ns)-U_t(n)}{a_n(t)} \to \frac{s^{\gamma(t)}-1}{\gamma(t)} \qquad n\to\infty$$

uniformly in t and locally uniformly in $s \in (0,\infty)$ with γ a continuous function (in $\mathscr{R}$).

(iv) *For each $f_1, f_2 \in C^+[0,1]$ and $f_1 < f_2$,*

$$nP\left(\frac{(f_1(t))^{\gamma(t)}-1}{\gamma(t)} < \frac{\xi - U_n(t)}{a_n(t)} < \frac{(f_2(t))^{\gamma(t)}-1}{\gamma(t)}\right) \tag{2.3}$$
$$\to \nu(f \in C^+[0,1];\ f_1 < f < f_2),$$

where ν is a measure on $C^+[0,1]$ and

$$P\{(1+\gamma(t)\bar{\eta}(t))^{1/\gamma(t)} \in A_{\overline{K},\bar{x}}\} = \exp -\nu\left(A^c_{\overline{K},\bar{x}}\right) \tag{2.4}$$

and for each $\varepsilon > 0$,

$$nP\left(\frac{\xi - U_t(n)}{a_n(t)} \not< f_\varepsilon(t)\right) \to \nu(S_\varepsilon), \tag{2.5}$$

where

$$f_\varepsilon(t) := \frac{\varepsilon^{\gamma(t)}-1}{\gamma(t)}$$

and $f_\varepsilon(t) = \log\varepsilon$ if $\gamma(t) = 0$.

REMARK 2.9. Part (iv) implies

$$nP\left\{\left(1+\gamma(t)\frac{\xi - U_n(t)}{a_n(t)}\right)^{1/\gamma(t)} \in \cdot\right\} \to \nu(\cdot)$$

in the space of weak convergence of finitely bounded measures on the space $C^+[0,1]$.

REMARK 2.10. We can reformulate statement (iiia) using Theorem 2.4.

REMARK 2.11. So, as in the finite-dimensional setting, convergence in the general case is equivalent to the (uniform) convergence of the marginals plus the convergence of a "simple" version.

REMARK 2.12. The theory goes through for random functions defined on any compact set S and not just the interval $[0,1]$.

Let ξ be a random function in $C[0,1]$. We say that ξ is in the domain of symmetric attraction of $\bar{\eta}$ if Theorem 2.8(i) is true with $a_n(t) \equiv cb_n(t) \equiv a_n$ (not depending on t) with $c > 0$.

COROLLARY 2.13. *$\xi \in \overline{C}^+[0,1]$ is in the domain of symmetric attraction of $\bar{\eta}$ iff for some $\alpha > 0$,*

$$\lim_{t\to\infty} \frac{P(\|\xi\|_\infty > tr)}{P(\|\xi\|_\infty > t)} = r^{-\alpha} \quad \text{for } r > 0 \tag{2.6}$$

and

$$\lim_{t\to\infty} P\left(\frac{\xi}{\|\xi\|_\infty} \in E | \|\xi\|_\infty > t\right) = \frac{\sigma(f; f^\alpha \in E)}{\sigma(C_1^+[0,1])} \tag{2.7}$$

for all Borel set $E \subset \overline{C}_1^+[0,1]$ with $\sigma(\partial E) = 0$, where σ is as in Proposition 1.3.

3. Proofs. We start with a proposition on simple max-stable processes.

LEMMA 3.1. *Let η be simple max-stable in $D[0,1]$.*

(i) $P\{\eta \in D^+[0,1]\} = 1$.

(ii) $P\{\bigcap_{i=1}^m \{\sup_{t\in K_i} \eta(t) < sx_i\}\} = P^{s^{-1}}\{\bigcap_{i=1}^m \{\sup_{t\in K_i} \eta(t) < x_i\}\}$ *for compact sets $K_1, K_2, \ldots, K_m \subset [0,1]$ and $x_1, x_2, \ldots, x_m \in \mathscr{R}^1$, $m = 1, 2, \ldots$ and $s > 0$.*

(iii) $P\{\eta < sf\} = P^{s^{-1}}\{\eta < f\}$ *for each $f \in D^+[0,1]$ and $s > 0$.*

PROOF. Statements (ii) and (iii) are obvious. For (i), note

$$D^+ = \{f \in D[0,1];\ f(t) > 0;\ f^-(t) := \lim_{s\uparrow t} f(s) > 0,\ t \in [0,1]\},$$

take $A := \{t; \eta(t) = 0\}$ and $B := \{t; \lim_{s\uparrow t} \eta(s) := \eta^-(t) = 0\}$. By definition (1.15) the random sets $\{t\colon (1/n) \vee_{i=1}^n \eta_i(t) = 0\} = \cap_{i=1}^n \{t\colon \eta_i(t) = 0\}$ have the same law as A. Moreover, $P\{t \in A\} = P\{\eta(t) = 0\} = 0$ for fixed t. Hence, Lemma 3.3(i) and Giné, Hahn and Vatan (1990), yields $P\{A = \phi\} = 1$. In a similar way we can get $P\{B = \phi\} = 1$. □

Next we isolate the most difficult part of the proof of Theorem 2.4.

LEMMA 3.2. *For $n = 1, 2, \dots$ let $\xi_n, \xi_{n,1}, \xi_{n,2}, \dots, \xi_{n,n}$ be* i.i.d. *random element of $D^+[0,1]$. Define for $n = 1, 2, \dots$,*

$$\nu_n(E) := nP\{\xi_n \in E\},$$
$$\nu_{n,\varepsilon}(E) := nP\{\xi_n \in E \cap S_\varepsilon\}$$

for all Borel sets E of $\overline{D}^+[0,1]$ where $\varepsilon > 0$ and

$$S_\varepsilon := \{f \in \overline{D}^+[0,1],\ \|f\|_\infty \geq \varepsilon\}.$$

If

$$M_n := \bigvee_{i=1}^{n} \xi_{n,i} \xrightarrow{D} \eta, \tag{3.1}$$

where η is simple max-stable, then for each positive ε the sequence $\{\nu_{n,\varepsilon}\}_{n=1}^\infty$ is relatively compact.

PROOF. We need to prove two things:

1. *The sequence $\nu_{n,\varepsilon}(\overline{D}^+[0,1])$, $n \geq 1$ is bounded.* First note that by (3.1) and simple max-stability,

$$\begin{aligned}\lim_{n\to\infty} n \log P\{\|\xi_n\|_\infty < \varepsilon\} &= \lim_{n\to\infty} \log P\left(\left\|\bigvee_{i=1}^{n} \xi_{n,i}\right\|_\infty < \varepsilon\right)\\ &= \log P\{\|\eta\|_\infty < \varepsilon\}\\ &= \varepsilon^{-1} \log P\{\|\eta\|_\infty < 1\},\end{aligned}$$

the last equality reflecting the fact that a simple max-stable random variable has distribution function $\exp\{-1/x\}$, $x > 0$. Hence

$$\begin{aligned}\lim_{n\to\infty} \nu_{n,\varepsilon}(\overline{D}^+[0,1]) &= \lim_{n\to\infty} nP\{\|\xi_n\|_\infty > \varepsilon\}\\ &= \lim_{n\to\infty} -n \log P\{\|\xi_n\|_\infty < \varepsilon\}\\ &= -\varepsilon^{-1} \log P(\|\eta\|_\infty < 1).\end{aligned} \tag{3.2}$$

2. $\{\nu_{n,\varepsilon}\}_{n=1}^\infty$ *is tight for each $\varepsilon > 0$.*

Note, since $\nu_{n,\varepsilon}(\overline{D}^+[0,1])$ has a finite limit as $n \to \infty$, we can check tightness for the sequence $\{\nu_{n,\varepsilon}\}$ as if it were a sequence of probability measures. According to Theorem 15.3a, Billingsley (1968), this is equivalent to the following:

(i) For each positive β there exists an $\alpha > 0$, such that

$$\nu_{n,\varepsilon}(S_\alpha) \leq \beta \quad \text{for all } n,$$

where

$$S_\varepsilon := \{f \in \overline{D}^+[0,1];\ \|f\|_\infty \geq \varepsilon\} \quad \text{for each } \varepsilon > 0.$$

(ii) For each positive β and α, there exists an δ, $0 < \delta < 1$, and an integer n_0, such that

(iia)

$$\nu_{n,\varepsilon}(\{f\colon \omega''_f(\delta) \geq \alpha\}) \leq \beta \quad \text{for } n \geq n_0$$

with

$$\omega''_f(\delta) := \sup_{\substack{t_1 \leq t \leq t_2 \\ |t_2 - t_1| \leq \delta}} \min(|f(t) - f(t_1)|, |f(t_2) - f(t)|);$$

(iib)

$$\nu_{n,\varepsilon}(\{f\colon \omega_f[0,\delta) \geq \alpha\}) \leq \beta \quad \text{for } n \geq n_0$$

with

$$\omega_f[0,\delta) := \sup_{0 \leq s,t < \delta} |f(s) - f(t)|;$$

(iic)

$$\nu_{n,\varepsilon}(\{f\colon \omega_f[1-\delta,1) \geq \alpha\}) \leq \beta \quad \text{for } n \geq n_0$$

with

$$\omega_f[1-\delta,1) := \sup_{1-\delta \leq s,t < 1} |f(s) - f(t)|.$$

Now (i) follows from the first part of the proof. Next we prove (iia); the other parts are similar. Relation (3.1) implies convergence in distribution, hence tightness, of $\{M_n \vee \alpha/2\}_{n=1}^\infty$. Consequently

$$P(\{\omega''_{M_n \vee \alpha/2}(\delta) \geq \alpha/2\}) \leq \beta^* \quad \text{for } n \geq n_0^*.$$

Define

$$Q_{n,\alpha} := (M_n \vee \alpha/2) I_{\{\|\eta_{n,i}\| \geq \alpha/2 \text{ for some } i,\ \|\eta_{n,j}\| < \alpha/2 \text{ for } j \neq i\}}.$$

Since $Q_{n,\alpha}$ is either 0 or $M_n \vee \alpha/2$, we have

$$P\{\omega''_{Q_{n,\alpha}}(\delta) \geq \alpha/2\} \leq P(\{\omega''_{M_n \vee \alpha/2}(\delta) \geq \alpha/2\}) \leq \beta^* \text{ for } n \geq n_0^*.$$

Hence by the definition of $Q_{n,\alpha}$,

$$\begin{aligned} nP^{n-1}\left\{\|\eta_n\|_\infty < \frac{\alpha}{2}\right\} P\left\{\omega''_{\eta_n \vee \alpha/2}(\delta) \geq \frac{\alpha}{2}\right\} \\ = P\left\{\omega''_{Q_{n,\alpha}}(\delta) \geq \frac{\alpha}{2}\right\} \leq \beta^* \quad \text{for } n \geq n_0^* \end{aligned} \tag{3.3}$$

Now

$$P^{n-1}\left\{\|\eta_n\|_\infty < \frac{\alpha}{2}\right\} = P^{(n-1)/n}\left\{\|M_n\|_\infty < \frac{\alpha}{2}\right\} = P\left\{\|\xi\|_\infty < \frac{\alpha}{2}\right\} =: d. \tag{3.4}$$

Hence by (3.2), (3.3) and the definition of ν_n,

$$\nu_n^*(\{f\colon \omega''_{f\vee\alpha/2}(\delta) \geq \alpha/2\}) \leq 2\beta^*/d =: \beta \quad \text{for } n \geq n_0.$$

Since

$$\omega''_f(\delta) \leq \omega''_{f\vee\alpha/2}(\delta) + \alpha/2,$$

we find

$$\nu_n^*(\{f; \omega''_f(\delta) \geq \alpha\}) \leq \beta \quad \text{for } n \geq n_0.$$

So in particular,

$$\nu_{n,\varepsilon}(\{f; \omega''_f(\delta) \geq \alpha\}) \leq \beta \quad \text{for } n \geq n_0. \qquad \square$$

PROOF OF THEOREM 2.4. (i) $\Rightarrow$ (ii). Note that S_ε is a sequence of closed spheres in $\overline{D}^+[0,1]$ and $S_\varepsilon \uparrow \overline{D}^+[0,1]$ as $\varepsilon \to 0$. Lemma 3.2 tells us that the sequence $\{\nu_{n,\varepsilon}\}_{n=1}^\infty$ is relatively compact for any $\varepsilon > 0$. Hence by Proposition A2.6.IV of Daley and Vere-Jones (1988), the sequence $\{\nu_n\}_{n=1}^\infty$ is relatively compact. Now

$$\begin{aligned}
&\lim_{n\to\infty} \nu_n\left(A^c_{\overline{K},\bar{x}}\right) \\
&\quad = \lim_{n\to\infty} -n \log P\{n^{-1}\xi \in A_{\overline{K},\bar{x}}\} \\
&\quad = \lim_{n\to\infty} -\log P\{n^{-1} \vee_{i=1}^n \xi_i \in A_{\overline{K},\bar{x}}\} \\
&\quad = -\log P\{\eta \in A_{\overline{K},\bar{x}}\} \\
&\quad =: \nu(A^c_{\overline{K},\bar{x}}).
\end{aligned} \tag{3.5}$$

In particular, $\nu_n\{f \in \overline{D}^+[0,1];\ \|f\|_\infty \geq \varepsilon\} \to \nu\{f \in \overline{D}^+[0,1];\ \|f\|_\infty \geq \varepsilon\}$. Note that the measure ν is determined by its values on $A^c_{\overline{K},\bar{x}}$ for any $\overline{K}$ and $\bar{x}$. Since the sequence $\{\nu_n\}$ is relative compact and any convergent subsequence has the same limit, the proof is complete.

(ii) $\Leftrightarrow$ (iii). By Daley and Vere-Jones [(1988), Lemma 9.1.IV], the statement in (iii) is equivalent to

$$\begin{aligned}
&(N_n(A_1), N_n(A_2), \dots, N_n(A_m)) \\
&\qquad \xrightarrow{d} (N(A_1), N(A_2), \dots, N(A_m))
\end{aligned} \tag{3.6}$$

for each $m \in \mathscr{N}$ and $A_1, A_2, \dots, A_m$ bounded disjoint N-continuity sets. The latter means that

$$P\{N(\partial A_i)\} = 0 \quad \Leftrightarrow \quad \nu(\partial A_i) = E(N(\partial A_i)) = 0 \Leftrightarrow A_i \text{ is a } \nu\text{-continuity set.}$$

Now (3.5) is equivalent to the convergence of

$$
\begin{aligned}
&E \exp\left\{-\sum_{i=1}^{m} \lambda_i N_n(A_i)\right\} \\
&\quad = \left[E \exp\left\{-\sum_{i=1}^{m} \lambda_i I_{\{n^{-1}\xi \in A_i\}}\right\}\right]^n \\
&\quad = \left[1 + \sum_{i=1}^{m} P\{n^{-1}\xi \in A_i\}(e^{-\lambda_i} - 1)\right]^n \\
&\quad = \exp\left(n \log\left\{1 + \sum_{i=1}^{m} P\{n^{-1}\xi \in A_i\}(e^{-\lambda_i} - 1)\right\}\right)
\end{aligned} \tag{3.7}
$$

to

$$
\begin{aligned}
&E \exp\left\{-\sum_{i=1}^{m} \lambda_i N(A_i)\right\} \\
&\quad = \exp\left\{\sum_{i=1}^{m} \nu(A_i)(e^{-\lambda_i} - 1)\right\}.
\end{aligned} \tag{3.8}
$$

Now clearly (3.6) converges if and only if

$$
\begin{aligned}
&\exp\left(\sum_{i=1}^{m} nP\{n^{-1}\eta \in A\}(e^{-\lambda_i} - 1)\right) \\
&\quad = \exp \sum_{i=1}^{m} \nu_n(A)(e^{-\lambda_i} - 1)
\end{aligned}
$$

converges to the same limit (3.7). And this convergence is equivalent to

$$\nu_n(A) \to \nu(A)$$

for all bounded Borel sets A of $D^+[0,1]$ with $\nu(\partial A) = 0$.

(ii) $\Rightarrow$ (i) under the extra condition $P\{\eta \in C[0,1]\} = 1$. The weak convergence $\nu_n \to \nu$ implies $\nu_{n,\varepsilon} \xrightarrow{w} \nu_\varepsilon$ with $\nu_{n,\varepsilon}$ from Lemma 3.2 and

$$\nu_\varepsilon(E) := \nu(E \cap S_\varepsilon)$$

for Borel sets E of $\overline{D}^+[0,1]$. Then by the previous part of the proof, we have

$$N_{n,\varepsilon} \to N_\varepsilon \quad \text{weakly}$$

with $N_{n,\varepsilon} := N_n I_{\{f \in S_\varepsilon\}}$ and $N_\varepsilon := N I_{\{f \in S_\varepsilon\}}$. Since the map

$$\sum_{i=1}^{m} \varepsilon_{\{f_i\}} \to \varepsilon_{\{\vee_{i=1}^m f_i\}}$$

is continuous for the point process in $C[0,1]$ (which is not true in space D), we have

$$\frac{1}{n} \bigvee_{i=1}^{n} \xi_i \vee \varepsilon \xrightarrow{D} \eta \vee \varepsilon[0,1]$$

for each $\varepsilon > 0$. Hence

$$\frac{1}{n}\bigvee_{i=1}^{n} \xi_i \xrightarrow{D} \eta. \qquad \square$$

PROOF OF REMARKS 2.5–2.7. Since condition of tightness in space $C[0, 1]$ is very similar to the condition of tightness in space $D[0, 1]$, the proof is similar if D is replaced by C everywhere. Hence if $P\{\xi \in \bar{C}^+[0, 1]\} = 1$, then $P\{\eta \in \bar{C}^+[0, 1]\} = 1$

Note that the measure ν is defined on $C^+[0, 1]$ and $S_\varepsilon \uparrow \bar{C}^+[0, 1]$.

The convergence $\nu_n \to \nu$ is equivalent to

$$\begin{aligned} &\bar{\nu}_n \to \nu \text{ where } \bar{\nu}_n \text{ is the restriction of } \nu_n \text{ in } C^+[0, 1] \\ &\quad \text{and } \nu_n(S_\varepsilon) \to \nu(S_\varepsilon). \end{aligned} \tag{3.9}$$

$\square$

REMARK 3.3. Note that in the more general situation of Lemma 3.2 the proof still applies. So the results of Theorem 2.4 and Remarks 2.5–2.7 still hold if we change ξ_i to $\xi_{n,i}$.

PROOF OF THEOREM 2.8. (i) $\Rightarrow$ (ii). By the convergence in space C and the representation of the remark after Theorem 1.2, we have

$$F_t^n(a_n(t)x + b_n(t)) \to \exp(-(1 + \gamma(t)x))^{-1/\gamma(t)} \tag{3.10}$$

uniformly in t and locally uniformly in x where

$$F_t(x) := P\{\xi(t) \le x\} \quad \text{for } t \in [0, 1].$$

It follows that

$$n\{1 - F_t(a_n(t)x + b_n(t))\} \to (1 + \gamma(t)x)^{-1/\gamma(t)} \tag{3.11}$$

uniformly in t and locally uniformly in x. Since convergence of a sequence of monotone function is equivalent to convergence of their inverses, we have

$$\frac{U_t(ns) - b_n(t)}{a_n(t)} \to \frac{s^{\gamma(t)} - 1}{\gamma(t)} \tag{3.12}$$

uniformly in t and locally uniformly in $s \in (0, \infty)$. Hence $(U_t(n) - b_n(t))/a_n(t) \to 0$ and (ii) follows.

(ii) $\Leftrightarrow$ (iii). Relation (iiib) follows immediately from (3.10). Further, (ii) and the uniformity in (3.9) imply

$$\begin{aligned} &\frac{1}{n}\bigvee_{i=1}^{n} 1\Big/\{1 - F_t(\xi_i(t))\} \\ &\quad = 1\bigg/\left\{1 - F_t\left(U_t(n) + \frac{(1/n)\vee_{i=1}^n \xi_i - U_t(n)}{a_t(n)} a_t(n)\right)\right\} \\ &\quad \xrightarrow{w} (1 + \gamma(t)\bar{\eta}(t))^{1/\gamma(t)} \quad \text{in } C[0, 1]. \end{aligned} \tag{3.13}$$

Since all elements are in $C^+[0,1]$ [cf. Lemma 3.1 and Giné, Hahn and Vatan (1990), Corollary 3.4], we also have convergence in $C^+[0,1]$. The converse is similar.

(ii) $\Leftrightarrow$ (iv). Note that $\hat{\eta} := (1+\gamma(t)\bar{\eta}(t))^{1/\gamma(t)}$ is simple max-stable. Hence,

$$P\{\bar{\eta} > f_{1/n}\} = P\left\{\frac{(\hat{\eta}(t))^{\gamma(t)}-1}{\gamma(t)} > \frac{n^{-\gamma(t)}-1}{\gamma(t)} \text{ for all } t\right\} \to 1, \qquad n\to\infty.$$

Hence (ii) is equivalent to

$$\frac{\bigvee_{i=1}^n \xi_i - U_t(n)}{a_n(t)} \vee f_{1/n}(t) \overset{w}{\to} \bar{\eta}. \tag{3.14}$$

Since $f(t) \to (1+\gamma(t)f(t))^{1/\gamma(t)}$ is a continuous map, (3.14) is equivalent to

$$\bigvee_{i=1}^n \hat{\xi}_{i,n} := \bigvee_{i=1}^n \left(1+\gamma(t)\frac{\xi_i - U_t(n)}{a_n(t)} \vee f_{1/n}(t)\right)^{1/\gamma(t)} \overset{w}{\to} (1+\gamma(t)\bar{\eta}(t))^{1/\gamma(t)} = \hat{\eta}.$$

Define measures $\hat{\nu}_n$ by

$$\hat{\nu}_n(E) := nP(\hat{\xi}_{i,n} \in E) \quad \text{for each Borel set } E \in \bar{C}^+[0,1].$$

According to Theorem 2.4, Remarks 2.5 and 3.3, it is equivalent to

$$nP\{f_1 < n^{-1}\hat{\xi}_{1,n} < f_2\} \to \nu\{f; f_1 < f < f_2\}$$

for arbitrary nonrandom functions f_1 and f_2 in $C^+[0,1]$ and $\hat{\nu}_n(S_\varepsilon) \to \nu(S_\varepsilon)$ for all $\varepsilon > 0$. This is equivalent to (iv). $\square$

PROOF OF REMARK 1.5(i) $\Leftrightarrow$ (ii). Theorem 2.4 tells us that

$$P\{\eta \in A_{\bar{K},\bar{x}}\} = \exp -\nu\{A^c_{\bar{K},\bar{x}}\} \tag{3.15}$$

and that for $a > 0$ and $E \in \mathscr{B}(D^+[0,1])$,

$$\nu(aE) = a^{-1}\nu(E). \tag{3.16}$$

Now with F an arbitrary set in $\mathscr{B}(D_1^+[0,1])$, apply this relation for $E_r = \{f; \|f\|_\infty > r \text{ and } f/\|f\|_\infty \in F\}$. Then

$$\nu(E_r) = r^{-1}\nu(E_1).$$

Define the finite measures σ on $\mathscr{B}(D_1^+[0,1])$ by

$$\sigma(F) = \nu(E_1).$$

Then with $f(t) = x_i$ for $t \in K_i$ and infinite for t outside $\cup_{i=1}^k K_i$,

$$\begin{aligned}\nu\left(A^c_{\bar{K},\bar{x}}\right) &= \nu\{h; h(t) > f(t) \text{ for some } t\}\\ &= \nu\left\{h; \|h\|_\infty > \inf_t \left(f\Big/\frac{h}{\|h\|_\infty}\right)\right\}\\ &= \int_{D_1^+[0,1]} \left\|\frac{g}{f}\right\|_\infty d\sigma(g).\end{aligned}$$

(ii) $\Rightarrow$ (i). The converse is easy. $\square$

Final comments. Considering the content of Theorem 2.8, we intend to use the results not in the way sketched in the Introduction but more directly in the following manner [cf. de Haan and Sinha (1999)]: one is interested in evaluating (1.1) for fixed k (corresponding to one year, e.g.), so we use part (ii) of Theorem 2.4,

$$\begin{aligned} &P\{\xi^{(k)}(t) := \max_{i\le k} \xi_i(t) \le f(t) \text{ for } 0 \le t \le 1\} \\ &\qquad = P\left\{\frac{k}{n}\frac{1}{1 - F_t^k(\xi^{(k)}(t))} \le \frac{1}{(n/k)\{1 - F_t^k(f(t))\}} f(t) \text{ for } 0 \le t \le 1\right\} \\ &\qquad \approx (k/n)\nu\left\{g; g(t) \le \frac{1}{(n/k)\{1 - F_t^k(f(t))\}} \text{ for } 0 \le t \le 1\right\}, \end{aligned}$$

where the right-hand side of the last inequality is asymptotically constant.

What we need in order to estimate the right-hand side is an estimator for the measure ν (possibly via the spectral measure σ from Proposition 1.3) and asymptotic estimation of $(n/k)\{1 - F_t^k(x)\}$ which can be done more or less via one-dimensional extreme-value results. The latter requires, for example, estimation of the function $\gamma(t)$ from Corollary 1.4 by a continuous function. Both are objects of present research.

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