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On All Strong Kleene Generalizations of Classical Logic

Abstract. By using the notions of exact truth ('true and not false') and exact falsity ('false and not true'), one can give 16 distinct definitions of classical consequence. This paper studies the class of relations that results from these definitions in settings that are paracomplete, paraconsistent or both and that are governed by the (extended) Strong Kleene schema. Besides familiar logics such as Strong Kleene logic (**K3**), the Logic of Paradox (**LP**) and First Degree Entailment (**FDE**), the resulting class of *all Strong Kleene generalizations of classical logic* also contains a host of unfamiliar logics. We first study the members of our class semantically, after which we present a uniform sequent calculus (the **SK** calculus) that is sound and complete with respect to all of them. Two further sequent calculi (the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus) will be considered, which serve the same purpose and which are obtained by applying general methods (due to Baaz et al.) to construct sequent calculi for many-valued logics. Rules and proofs in the **SK** calculus are much simpler and shorter than those of the $\mathbf{SK}^{\mathscr{P}}$ and the $\mathbf{SK}^{\mathscr{N}}$ calculus, which is one of the reasons to prefer the **SK** calculus over the latter two. Besides favourably comparing the **SK** calculus to both the $\mathbf{SK}^{\mathscr{P}}$ and the $\mathbf{SK}^{\mathscr{N}}$ calculus, we also hint at its philosophical significance.

Keywords: Classical logic, Strong Kleene Logic (K3), Logic of Paradox (LP), First Degree Entailment (FDE), Exactly True Logic, Uniform Sequent Calculus.

1. Introduction

1.1. Strong Kleene Generalizations of Classical Logic

According to classical logic, *truth* and *falsity* are mutually exclusive and jointly exhaustive. As a consequence, truth coincides with *non-falsity*. Moreover, in the classical setting truth coincides with *exact truth* (which we will also denote as truth*), where a sentence is exactly true just in case it is true and not false. Similarly, falsity coincides with *exact falsity* (falsity*), where a sentence is exactly false just in case it is false and not true, and so truth not only coincides with truth* and non-falsity, but also with non-falsity*.

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To sum up:

$$\text{Classical setting: truth} = \text{truth}^* = \text{non-falsity} = \text{non-falsity}^* \qquad (1)$$

A typical way to characterize a logical consequence relation is in terms of *the preservation of truth over a class of valuations* $\mathbf{V}$. According to this General Schema (GS) a premise set Γ is said to entail a conclusion φ just in case, in passing from Γ to φ, truth is preserved for each valuation $V \in \mathbf{V}$, i.e.

$$V(\gamma) \text{ is true for all } \gamma \in \Gamma \implies V(\varphi) \text{ is true}, \qquad \text{for all } V \in \mathbf{V} \qquad \text{(GS)}$$

When we quantify, in (GS), over the class of all *classical* valuations, we obtain a characterization of the classical consequence relation. However, (1) testifies that, when quantifying over all classical valuations, substituting 'true*', 'non-false' or 'non-false*' for any of the occurrences of 'true' in (GS) delivers another characterization of classical consequence. Indeed, with $x, y \in \{\text{true}, \text{true}^*, \text{non-false}, \text{non-false}^*\}$, the schema $\text{GS}(x, y)$ that results when we substitute x for the occurrence of 'true' on the left side of the implication of (GS) and y for the occurrence of 'true' on its right side, also defines—when we let $\mathbf{V}$ be the class of all classical valuations—classical consequence.

In this paper, we will study the relations that are defined by our 16 schemas $\text{GS}(x, y)$ in settings that are *paracomplete*, *paraconsistent* or both. A setting that is both paracomplete and paraconsistent acknowledges the possibility that a sentence is *neither* true nor false (hence the setting is paracomplete) as well as the possibility that a sentence is *both* true and false (hence the setting is paraconsistent). It readily follows that in such a setting, truth, truth*, non-falsity and non-falsity* are all distinct from one another and hence, that our 16 schemas potentially define 16 distinct relations. Of course, whether they actually do so depends on the class of valuations $\mathbf{V}$ with respect to which the schemas are evaluated, to which we now turn.

In this paper, we will be concerned with valuations for a propositional language $\mathscr{L}$ whose BNF form is as follows (where p comes from a countably infinite set of propositional atoms)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$$

The valuations of $\mathscr{L}$ that we will consider are those associated with the *extended Strong Kleene schema* (cf. Belnap [12,13] and Dunn [16]) as defined below.

DEFINITION 1. An *(extended) Strong Kleene valuation* is any function from $Sen(\mathscr{L})$ to $\mathbf{4} := \{\mathbf{T}, \mathbf{B}, \mathbf{N}, \mathbf{F}\}$ that respects the following truth tables for $\wedge$, $\vee$ and $\neg$.

$\wedge$	**T**	**B**	**N**	**F**
T	**T**	**B**	**N**	**F**
B	**B**	**B**	**F**	**F**
N	**N**	**F**	**N**	**F**
F	**F**	**F**	**F**	**F**

$\vee$	**T**	**B**	**N**	**F**
T	**T**	**T**	**T**	**T**
B	**T**	**B**	**T**	**B**
N	**T**	**T**	**N**	**N**
F	**T**	**B**	**N**	**F**

$\neg$	
T	**F**
B	**B**
N	**N**
F	**T**

We write $\mathbf{V_4}$ to denote the set of all (extended) Strong Kleene valuations. We use $\mathbf{V_{3n}}$ to denote the set of all valuations in $\mathbf{V_4}$ whose range is a subset of $\mathbf{3n} := \{\mathbf{T}, \mathbf{N}, \mathbf{F}\}$, $\mathbf{V_{3b}}$ to denote the set of all valuations in $\mathbf{V_4}$ whose range is a subset of $\mathbf{3b} := \{\mathbf{T}, \mathbf{B}, \mathbf{F}\}$, and $\mathbf{V_2}$ to denote the set of all (classical) valuations in $\mathbf{V_4}$ whose range is $\mathbf{2} := \{\mathbf{T}, \mathbf{F}\}$.

The canonical interpretation of the elements of $\mathbf{4}$ is an epistemological one: according to Belnap's ([12]) *told-interpretation*, a sentence is valuated as **B** just in case one is told both that the sentence is true and that it is false. In this paper however, the elements of $\mathbf{4}$ will, for sake of simplicity (and as nothing hinges on it), be interpreted ontologically; a sentence is either **T** (exactly true), **F** (exactly false), **B** (both true and false) or **N** (neither true nor false). Hence, the valuations of $\mathbf{V_4}$ are associated with a setting that is both paracomplete and paraconsistent. Likewise, the valuations of $\mathbf{V_{3n}}$ are associated with a setting that is (only) paracomplete and the valuations of $\mathbf{V_{3b}}$ with a setting that is (only) paraconsistent. Finally, the valuations of $\mathbf{V_2}$ are associated with the classical setting that is neither paracomplete nor paraconsistent.

It will be convenient to introduce a uniform notation for *the Strong Kleene Generalizations (of classical logic)*, i.e. for the relations[1] that are obtained when our 16 schemas are instantiated with $\mathbf{V_2}$, $\mathbf{V_{3n}}$, $\mathbf{V_{3b}}$ and $\mathbf{V_4}$ respectively.[2] To do so, we first introduce the following notation for subsets of $\mathbf{4}$:

$$\begin{array}{llll} \mathsf{1} := \{\mathbf{T}, \mathbf{B}\} & \mathsf{0} := \{\mathbf{F}, \mathbf{B}\} & \mathbf{t} := \{\mathbf{T}\} & \mathbf{f} := \{\mathbf{F}\} \\ \hat{\mathsf{1}} := \{\mathbf{F}, \mathbf{N}\} & \hat{\mathsf{0}} := \{\mathbf{T}, \mathbf{N}\} & \hat{\mathbf{t}} := \{\mathbf{F}, \mathbf{N}, \mathbf{B}\} & \hat{\mathbf{f}} := \{\mathbf{T}, \mathbf{N}, \mathbf{B}\} \end{array} \tag{2}$$

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[1]As will become apparent, some Strong Kleene Generalizations will equal the empty set (and one of the relations thus obtained will be non-transitive). It is awkward to call the empty set a consequence relation, and hence we will not refer to the class of all Strong Kleene Generalizations as a class of consequence relations.

[2]Note that the restrictions of (the truth functions denoted by) $\neg$, $\wedge$ and $\vee$ to $\mathbf{2}$ ($\mathbf{3b}$, $\mathbf{3n}$) are truth functions on $\mathbf{2}$ ($\mathbf{3b}$, $\mathbf{3n}$). Indeed, this ensures that we can approach the Strong Kleene Generalizations in a uniform and concise manner.

Thus, 1 codes for truth, $\mathbf{t}$ codes for truth*, $\hat{0}$ codes for non-falsity and $\hat{\mathbf{f}}$ for non-falsity*. Our uniform notation for the Strong Kleene Generalizations (of classical logic) is provided by the following definition.

Definition 2. Let $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$ and let $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}, \mathbf{3b}, \mathbf{4}\}$. The relation $\models^{\mathbf{z}}_{xy}$ between sets of sentences and sentences of $\mathscr{L}$ is defined as follows:

$$\Gamma \models^{\mathbf{z}}_{xy} \varphi \iff \text{ if } V(\gamma) \in x \cap \mathbf{z} \text{ for all } \gamma \in \Gamma \text{ then } V(\varphi) \in y \cap \mathbf{z} \quad \text{for all } V \in \mathbf{V_z} \tag{3}$$

A relation $\models^{\mathbf{z}}_{xy}$ as defined by (3) is called a *Strong Kleene Generalization* (of classical logic). When $x, y \in \{\mathbf{t}, \hat{\mathbf{f}}\}$, we say that $\models^{\mathbf{z}}_{xy}$ is *exact* . When $x, y \in \{1, \hat{0}\}$, we say that $\models^{\mathbf{z}}_{xy}$ is *regular*. When $\models^{\mathbf{z}}_{xy}$ is neither exact nor regular, it is *mixed.*

Quite some of the Strong Kleene Generalizations are well-known. To be sure, for the classical setting ($\mathbf{z} = \mathbf{2}$) all 16 instantiations of (3) define the classical consequence relation. But also for the considered paracomplete ($\mathbf{z} = \mathbf{3n}$) and paraconsistent setting ($\mathbf{z} = \mathbf{3b}$) all instantiations of (3) are well-known. It turns out that for $\mathbf{z} \in \{\mathbf{3n}, \mathbf{3b}\}$, the consequence relation $\models^{\mathbf{z}}_{xy}$ is either equal to

- $\models_{K3}$, the consequence relation of *strong Kleene Logic* **K3** (cf. Kleene [19]).
- $\models_{LP}$, the consequence relation of the *Logic of Paradox* **LP** (cf. Priest [24]).
- $\models_{CL}$, the consequence relation of classical logic **CL**.
- $\emptyset$, the empty set.

In particular, each of these 4 relations can be represented as an exact Strong Kleene Generalization, as for $\mathbf{z} \in \{\mathbf{3n}, \mathbf{3b}\}$, we have[3]

$$\models^{\mathbf{z}}_{\mathbf{tt}} = \models_{K3} \qquad \models^{\mathbf{z}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}} = \models_{LP} \qquad \models^{\mathbf{z}}_{\mathbf{t}\hat{\mathbf{f}}} = \models_{CL} \qquad \models^{\mathbf{z}}_{\hat{\mathbf{f}}\mathbf{t}} = \emptyset \tag{4}$$

Our presentation of **K3**, **LP**, **CL** as in (4) naturally raises the question what the "**4** equivalents" of these familiar consequence relations look like. More concretely, which logics are defined by $\models^{\mathbf{4}}_{\mathbf{tt}}$, $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$ and $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$? To the best of our knowledge, of these three relations only $\models^{\mathbf{4}}_{\mathbf{tt}}$ has been considered before in the literature. Per definition, we have

[3] Except for $\models^{\mathbf{z}}_{\mathbf{t}\hat{\mathbf{f}}} = \models_{CL}$, which we prove in Sect. 2, all equalities mentioned by (4) are immediate consequences of the involved definitions.

– $\models^{\mathbf{4}}_{\mathbf{tt}} = \models_{ETL}$, the consequence relation of *Exactly True Logic* **ETL** which was studied by Pietz and Rivieccio in [23].

In [23], Pietz and Rivieccio compare **ETL** with the logic of *First Degree Entailment* (**FDE**), a well-known logic that is studied in [12] and [13] and that is defined in terms of the preservation of truth or—what turns out to be equivalent—non-falsity over $\mathbf{V_4}$ valuations:

– $\models^{\mathbf{4}}_{\mathbf{11}} = \models^{\mathbf{4}}_{\hat{\mathbf{0}}\hat{\mathbf{0}}} = \models_{FDE}$, the consequence relation of **FDE**.

Pietz and Rivieccio motivate their study of **ETL** by observing that in a **4**-valued setting, it seems natural to define one's consequence relation in terms of the preservation of **T**:

> A curious feature of [...] **FDE** is that the overdetermined value **B** (both true and false) is treated as a designated value. Although there are good theoretical reasons for this, it seems prima facie more plausible to have only one of the four values designated, namely **T**. [23, p. 125]

Our motivation to study **ETL** in this paper is rather different. The main purpose of this paper is to study the Strong Kleene Generalizations in a uniform way. As **ETL** is a Strong Kleene Generalization, it belongs *a fortiori*, to the object of our study. Moreover, there is some interest in the relations $\models^{\mathbf{4}}_{\mathbf{tt}}$ (i.e. **ETL**), $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$ and $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$, as in light of (4), these relations are the natural generalizations of respectively **K3**, **LP** and **CL** to a **4**-valued setting. We will study the Strong Kleene Generalizations both semantically and syntactically—we will advocate one and present three uniform sequent calculi to capture the $\models^{\mathbf{z}}_{xy}$ relations—as explained in more detail below.

1.2. Structure of the Paper

In Sect. 2, we study the Strong Kleene Generalizations semantically. It turns out that the different instantiations of (3) define 8 distinct non-empty Strong Kleene Generalizations: **CL**, **K3**, **LP**, **FDE**, **ETL**, $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$, $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ (which were already mentioned) and $\models^{\mathbf{4}}_{\mathbf{1}\hat{\mathbf{f}}}$. Of the 8 relations just mentioned, **FDE** turns out to be the weakest, and **CL** the strongest one: whenever an argument is **FDE** valid, it is valid according to each of the 8 relations and when an argument is valid according to any of the 8 relations, it is **CL** valid. In Sect. 2.5 we present an exhaustive comparison of the strengths of all 8 relations. However, we first show, in Sect. 2.1, that for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$, (3) either defines $\emptyset$, **CL**, **K3**, or **LP**. Then we study the exact, regular and mixed

Strong Kleene Generalizations associated with $\mathbf{z} = \mathbf{4}$ in Sects. 2.2, 2.3 and 2.4 respectively. We will see that although quite some of the interrelations between **K3**, **LP** and **CL** carry over to their **4**-valued counterparts, these counterparts have rather unusual properties at the level of meta-inferences. For instance, $\models^{\mathbf{4}}_{1\hat{\mathsf{f}}}$ turns out to be a non-transitive relation and according to $\models^{\mathbf{4}}_{\hat{\mathsf{f}}\hat{\mathsf{f}}}$, a premise γ may entail both α and β without entailing $\alpha \wedge \beta$. In Sect. 2.6, we observe that none of the Strong Kleene Generalizations contains an *appropriate implication connective* in the sense of Arieli and Avron [2] and we show that and how such connectives can be added to our language.

In Sects. 3 and 4, we study the Strong Kleene Generalizations syntactically. In Sect. 3 we present the **SK** *calculus* (Strong Kleene calculus), which is a uniform sequent calculus that is sound and complete with respect to all the Strong Kleene Generalizations. The **SK** calculus recognizes four distinct notions of provability: showing that an argument is $\models^{\mathbf{z}}_{xy}$ valid comes down to showing that an appropriate sequent is $\mathbf{z}$-provable. The notions of $\mathbf{z}$-provability differ only in the initial sequent rules that are allowed to occur in a proof and in particular the operational sequent rules that can be used in a $\mathbf{z}$-proof are the same for each value of $\mathbf{z}$. Although the **SK** calculus is a cut-free calculus, admissible cut rules for the calculus are readily available, as we discus in Sect. 3.4. In Sect. 3.5, we illustrate a convenient property of the *tableau calculus* that is associated with the **SK** calculus: to check whether an argument is valid according to, respectively, **CL**, **K3**, **LP** or **FDE** requires the inspection of the $\mathbf{z}$-closure of a *single* tableau.

In Sect. 4, we consider two other uniform sequent calculi for the $\models^{\mathbf{z}}_{xy}$ relations—the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus which are based on the notions of $\mathscr{P}$(ositive) and $\mathscr{N}$(egative) validity respectively—that can be obtained by applying the general methods of Baaz et al. [6]. Although sequents in all three calculi are sets of signed sentences, in the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus signs are members of $\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ that code for the corresponding *elements* of **4**, whereas in the **SK** calculus signs are elements of $\{1, 0, \hat{1}, \hat{0}\}$ and code for *subsets* of **4**: the **SK** calculus reflects the fact that **T**, **B**, **N**, and **F** are best thought of as *combinations* of (regular) truth values and its signs capture the "underlying" values of (non-)truth and (non-)falsity. Doing so has advantageous consequences as compared to the other two calculi, the sequent rules and proofs of the **SK** calculus are much simpler and shorter.

In Sect. 5 we reflect on the results that were achieved in earlier sections. In Sect. 5.1 we favourably compare the **SK** calculus to both the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus. In Sect. 5.2 we confront the **SK** calculus with the fact that "more standard" 2-sided sequent calculi for some of the Strong Kleene Generalizations exist. Doesn't this deprive the **SK** calculus of its interest? No. Sect. 5.3 hints at the significance of the **SK** calculus for (i) (a generalization of) the *bilaterlalistic* account of meaning as developed by Restall [26] and (ii) the account of logical pluralism—called *intra-theoretic pluralism*—as developed by Hjortland [18].

Section 6 concludes.

2. A Taxonomy of the Strong Kleene Generalizations

2.1. Familiar Strong Kleene Generalizations: $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}, \mathbf{3b}\}$

For sake of completeness, the following proposition recalls that for $\mathbf{z} = \mathbf{2}$, all instantiations of (3) define classical logic.

PROPOSITION 1. $\models^{\mathbf{2}}_{xy} = \models_{CL}$ *for any* $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$

The next proposition will show that the relations $\models^{\mathbf{z}}_{xy}$ that are induced by letting $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}\}$ are either equal to **K3**, **LP**, **CL** or $\emptyset$. In order to prove it, we will need the following lemma.

LEMMA 1. *Let* Σ *be a set of sentences of* $\mathscr{L}$ *and let* $V \in \mathbf{V_{3n}}$ *be such that* $V(\sigma) \in \{\mathbf{T}, \mathbf{F}\}$ *for all* $\sigma \in \Sigma$*. Then there is a* $V' \in \mathbf{V_2}$ *which coincides with* V *on* Σ*.*

PROOF. The partial (information) order $\leq$ on **3n** is defined by stipulating that $\mathbf{N} \leq \mathbf{T}$ and $\mathbf{N} \leq \mathbf{F}$. This order on **3n** induces a partial order on $\mathbf{V_{3n}}$, where $V \leq V' \iff V(\sigma) \leq V'(\sigma)$ for all sentences σ. Let V be as indicated above. Let atomic valuation v be the restriction of V to the propositional atoms, define atomic valuation v' by stipulating that $v'(p) = \mathbf{T}$ if $v(p) = \mathbf{N}$ and that $v'(p) = v(p)$ otherwise, and let V' be the unique element of $\mathbf{V_2}$ whose restriction to the propositional atoms is v'. Observe that $v(p) \leq v'(p)$ for each propositional atom p and that, as the connectives of $\mathscr{L}$ define truth functions on **3n** that are monotonic with respect to $\leq$, this implies that $V \leq V'$, which establishes the lemma. ■

PROPOSITION 2. *The following relations hold.*

1*a*. $\models^{\mathbf{3n}}_{xy} = \models_{K3}$ *for* $x, y \in \{1, \mathbf{t}\}$

2*a*. $\models^{\mathbf{3n}}_{xy} = \models_{LP}$ *for* $x, y \in \{\hat{0}, \hat{\mathbf{f}}\}$

3*a*. $\models^{\mathbf{3n}}_{xy} = \models_{CL}$ *for* $\langle x, y\rangle \in \{1, \mathbf{t}\} \times \{\hat{0}, \hat{\mathbf{f}}\}$

4*a*. $\models^{\mathbf{3n}}_{xy} = \emptyset$ *for* $\langle x, y\rangle \in \{\hat{0}, \hat{\mathbf{f}}\} \times \{1, \mathbf{t}\}$

1*b*. $\models^{\mathbf{3b}}_{xy} = \models_{K3}$ *for* $x, y \in \{\hat{0}, \mathbf{t}\}$

2*b*. $\models^{\mathbf{3b}}_{xy} = \models_{LP}$ *for* $x, y \in \{1, \hat{\mathbf{f}}\}$

3*b*. $\models^{\mathbf{3b}}_{xy} = \models_{CL}$ *for* $\langle x, y\rangle \in \{\hat{0}, \mathbf{t}\} \times \{1, \hat{\mathbf{f}}\}$

4*b*. $\models^{\mathbf{3b}}_{xy} = \emptyset$ *for* $\langle x, y\rangle \in \{1, \hat{\mathbf{f}}\} \times \{\hat{0}, \mathbf{t}\}$

Proof. 1a, 1b, 2a and 2b all follow immediately from an inspection of the definitions (we suppose familiarity with the definition of the consequence relations of **K3** and **LP**). It is easily seen that 4a and 4b hold by observing that the valuation which assigns $\mathbf{N}$ to each sentence and the valuation which assigns $\mathbf{B}$ to each sentence are elements of $\mathbf{V_{3n}}$ and $\mathbf{V_{3b}}$ respectively. In order to prove 3a note that, as $\mathbf{V_2} \subseteq \mathbf{V_{3n}}$, it easily follows that $\models^{\mathbf{3n}}_{\mathbf{t\hat{f}}} \subseteq \models_{CL}$. For suppose that an argument is not classically valid. Then there is a $V \in \mathbf{V_2}$ that is a *counter model* to the classical validity of the argument: V valuates all premisses of the argument as $\mathbf{T}$ and the conclusion as $\mathbf{F}$. But then, as V is also an element of $\mathbf{V_{3n}}$, V is also a counter model to the $\models^{\mathbf{3n}}_{\mathbf{t\hat{f}}}$ $-$validity of the argument. It thus suffices to show that $\models_{CL} \subseteq \models^{\mathbf{3n}}_{\mathbf{t\hat{f}}}$. To do so, we reason again by contraposition. If an argument is not $\models^{\mathbf{3n}}_{\mathbf{t\hat{f}}}$ $-$valid, there is a counter model $V \in \mathbf{V_{3n}}$ such that V valuates all premisses of the argument as $\mathbf{T}$ and the conclusion as $\mathbf{F}$. According to Lemma 1, there is a $V' \in \mathbf{V_2}$ which valuates the premisses and conclusion of the argument just as V does. Hence, this V' is a counter model to the classical validity of the argument, which completes our proof of 3a. The proof of 3b is completely similar to the proof of 3a. ■

Propositions 1 and 2 jointly testify that for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}, \mathbf{3b}\}$ the relations $\models^{\mathbf{z}}_{xy}$ are all well-known. In the remainder of Sect. 2 we will study the Strong Kleene Generalizations that are associated with $\mathbf{z} = \mathbf{4}$, and the next section starts by considering the exact ones (cf. Definition 2) amongst them.

2.2. The Exact Strong Kleene Generalizations for $\mathbf{z} = \mathbf{4}$

This section studies the exact Strong Kleene Generalizations. These relations are particularly interesting as in virtue of (4), $\models^{\mathbf{4}}_{\mathbf{tt}}$, $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$ and $\models^{\mathbf{4}}_{\mathbf{t\hat{f}}}$ may be called 'the $\mathbf{4}$-equivalents of **K3**, **LP** and **CL**'. In particular, this section investigates to what extent this classification is justified. In order to do so, it is convenient to have the following definition of familiar notions.

Definition 3. Let $\models^{\mathbf{z}}_{xy}$ be a Strong Kleene Generalization. We say that φ is a *tautology* of $\models^{\mathbf{z}}_{xy}$ just in case, for all $V \in \mathbf{V_z}$, $V(\varphi) \in y$. We say that φ is an *anti-tautology* of $\models^{\mathbf{z}}_{xy}$ just in case, for all $V \in \mathbf{V_z}$, $V(\varphi) \notin x$. We will write $\mathsf{Tau}(\models^{\mathbf{z}}_{xy})$ to denote the set of all tautologies of $\models^{\mathbf{z}}_{xy}$ and $\mathsf{ATau}(\models^{\mathbf{z}}_{xy})$ to denote its set of anti-tautologies.

Thus, a tautology follows from any set of premisses whereas an anti-tautology implies any conclusion. **K3** has no tautologies and its anti-tautologies coincide with those of classical logic. For **LP** the situation is completely reversed: **LP** has no anti-tautologies and its tautologies coincide with those of classical logic. Moreover a sentence is an **LP** tautology just in case its negation is an anti-tautology of **K3**. The following proposition attests that these well-known interrelations between **K3**, **LP** and **CL** carry over to the **4**-valued case.

PROPOSITION 3. *Let* $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$*. The following holds.*

1*a*. $\mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{tt}}) = \emptyset$ 1*b*. $\mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{tt}}) = \mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{t\hat{f}}})$

2*a*. $\mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}}) = \mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{t\hat{f}}})$ 2*b*. $\mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}}) = \emptyset$

3. $\varphi \in \mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}}) \iff \neg\varphi \in \mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{tt}})$

PROOF. 1a and 2b follow from the observation that the valuation which assigns $\mathbf{N}$ to each sentence and the valuation which assigns $\mathbf{B}$ to each sentence are elements of $\mathbf{V_{3n}} \subseteq \mathbf{V_4}$ and $\mathbf{V_{3b}} \subseteq \mathbf{V_4}$ respectively. 1b, 2a and 3 follow immediately from the definitions. ■

Another well-known relation between **K3** and **LP** is that ψ implies φ according to **K3** just in case the negation of φ implies the negation of ψ according to **LP**. The following proposition attests that this relation also carries over to the **4**-valued case.

PROPOSITION 4. *Let* $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$*. The following holds.*

$$\neg\varphi \models^{\mathbf{z}}_{\mathbf{tt}} \neg\psi \iff \psi \models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}} \varphi$$

PROOF. By inspection of definitions. Left to the reader. ■

The following observation is an immediate corollary of Proposition 3.

COROLLARY 1. *Let* $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$*. The following holds.*

$$\mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{tt}}) \cup \mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}}) = \mathsf{Tau}(\models^{\mathbf{z}}_{\mathbf{t\hat{f}}}) \qquad \mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{tt}}) \cup \mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}}) = \mathsf{ATau}(\models^{\mathbf{z}}_{\mathbf{t\hat{f}}})$$

In particular, corollary 1 implies that the union of the tautologies, respectively anti-tautologies, of **K3** and **LP** gives us the tautologies, respectively anti-tautologies, of classical logic. This observation suggests that the union of **K3** and **LP** just is classical logic. This suggestion however is mistaken,[4] as with $\alpha := (p \wedge \neg p) \vee q$ and $\beta := (r \vee \neg r) \wedge q$, we have

$$\text{For } \mathbf{z} \in \{\mathbf{3n}, \mathbf{3b}\}: \qquad \alpha \models^{\mathbf{z}}_{\mathbf{t\hat{f}}} \beta, \qquad \alpha \not\models^{\mathbf{z}}_{\mathbf{tt}} \beta, \qquad \alpha \not\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}} \beta \tag{5}$$

Observe that (5) does not hold for $\mathbf{z} = \mathbf{4}$, as we do *not* have that

$$(p \wedge \neg p) \vee q \models^{\mathbf{4}}_{\mathbf{t\hat{f}}} (r \vee \neg r) \wedge q, \tag{6}$$

[4]However, the mistake is not uncommon. In a recent AJP paper, P. Allo explicitly asserts that **CL** is the union of **K3** and **LP** (cf. [1, pp. 80, 83]).

as revealed by a valuation V such that $V(p) = V(r) = \mathbf{B}$ and $V(q) = \mathbf{N}$. However, there are $\models^{\mathbf{4}}_{\mathbf{t\hat{f}}}$-valid arguments that are neither $\models^{\mathbf{4}}_{\mathbf{tt}}$- nor $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$-valid: with $\gamma := (p \wedge \neg p) \vee (q \wedge \neg q)$, we have

$$\gamma \models^{\mathbf{4}}_{\mathbf{t\hat{f}}} q \qquad \gamma \not\models^{\mathbf{4}}_{\mathbf{tt}} q \qquad \gamma \not\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}} q \tag{7}$$

The following proposition summarizes the previous observations and states that classical logic is a *proper* extension of the union of **K3** and **LP** and that this relation carries over to the **4**-valued case.

PROPOSITION 5. *For* $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$*:* $\models^{\mathbf{z}}_{\mathbf{tt}} \cup \models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}} \subset \models^{\mathbf{z}}_{\mathbf{t\hat{f}}}$

PROOF. Observe that any counter model V to the $\models^{\mathbf{z}}_{\mathbf{t\hat{f}}}$-validity of an argument is both a counter model to its $\models^{\mathbf{z}}_{\mathbf{tt}}$-validity as well as to its $\models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}}$-validity. Hence $\models^{\mathbf{z}}_{\mathbf{tt}} \cup \models^{\mathbf{z}}_{\mathbf{\hat{f}\hat{f}}} \subseteq \models^{\mathbf{z}}_{\mathbf{t\hat{f}}}$ for any $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$. That the inclusion is proper follows for $\mathbf{z} \in \{\mathbf{3b}, \mathbf{3n}\}$ from observation (5) and for $\mathbf{z} = \mathbf{4}$ from (7). ■

Remember that $\models^{\mathbf{4}}_{\mathbf{tt}}$ is called Exactly True Logic (**ETL**) by Pietz and Rivieccio [23]. **ETL** is *explosive* in the sense that a contradiction, i.e. a sentence of form $\varphi \wedge \neg\varphi$, implies any sentence whatsoever. Indeed, it is easily verified that a contradiction is never valuated as $\mathbf{T}$ which is to say that **ETL** has the contradictions amongst its anti-tautologies. As every **ETL** anti-tautology clearly is a classical anti-tautology and as contradictions are classical anti-tautologies, one wonders whether the anti-tautologies of **ETL** coincide with those of classical logic. They do not and in particular the disjunction of two distinct contradictions such as $(p \wedge \neg p) \vee (q \wedge \neg q)$ is not an anti-tautology of **ETL**, as a valuation according to which $V(p) = \mathbf{B}$ and $V(q) = \mathbf{N}$ testifies. It is interesting to note that these observations imply that the consequence relation of **ETL** has certain unusual properties on the level of *meta-inferences*. In particular the following meta-inference is not valid according to **ETL**.

$$\alpha \models \gamma, \quad \beta \models \gamma \implies \alpha \vee \beta \models \gamma \tag{8}$$

Indeed, taking $\alpha = p \wedge \neg p$, $\beta = q \wedge \neg q$ and $\gamma = r$ testifies that meta-inference (8) fails for **ETL**. And although (8) is valid according to $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$, Proposition 4 readily implies that a dual meta-inference must fail for $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$:

$$\gamma \models \alpha, \quad \gamma \models \beta \implies \gamma \models \alpha \wedge \beta \tag{9}$$

Taking $\alpha = p \vee \neg p$, $\beta = q \vee \neg q$ and $\gamma = r$ shows that (9) fails for $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$. Moreover, $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$ does not only have unusual properties at the level of meta-inferences but also at the level of inferences, as for instance it does *not* validate (10)

$$p, q \models p \wedge q \tag{10}$$

In a multiple conclusion setting,[5] we see that the failure of (10) for $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$ is mirrored by the failure of (11) for **ETL**:

$$p \vee q \models p, q \tag{11}$$

Interestingly, $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ satisfies both (10) and (in a multiple conclusion setting) (11) as is easily verified. On the other hand, it violates both (8) and (9), which is testified by the same examples that were considered above. In addition, $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ is *non-transitive*. That is, it invalidates the following *structural* meta-inference (a meta-inference is structural when it is expressible without referring to logical connectives):

$$\alpha \models \beta, \quad \beta \models \gamma \implies \alpha \models \gamma \tag{12}$$

To see that $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ is non-transitive, one takes $\alpha = (p \wedge \neg p) \vee (q \wedge \neg q)$, $\beta = q \wedge \neg q$ and $\gamma = r$. Together with reflexivity and monotonicity, transitivity is often taken to be an essential property of logical consequence. For a proponent of this view, $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ does not define a "genuine" consequence relation. Such a proponent may hold that although $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ results from a certain generalization of classical logic, its non-transitivity testifies that it does not result from a *proper* such generalization. Proper generalizations of classical logic, one may hold, are defined in terms of the *Preservation of Designated Value*, i.e. they are instantiations of the (PDV) schema for some set of designated values $\mathscr{D}$ and class of valuations $\mathbf{V}$

$$\Gamma \models \varphi \Leftrightarrow V(\gamma) \in \mathscr{D} \text{ for all } \gamma \in \Gamma \implies V(\varphi) \in \mathscr{D}, \quad \text{for all } V \in \mathbf{V} \tag{PDV}$$

It readily follows that any relation that is defined in terms of the preservation of designated values is transitive. However, not the other way around: $\models^{\mathbf{3n}}_{\mathbf{t}\hat{\mathbf{f}}}$ is not defined in terms of the preservation of designated value but coincides with the (transitive) consequence relation of classical logic. This paper is not the place to argue that transitivity is / is not central to logical consequence. It should be noted though that recently, non-transitive consequence relations have attracted quite some attention. Although their motivations differ widely, Weir [33], Zardini [35], Cobreros et al. [14] and Ripley [27] all advocate a non-transitive consequence relation.

Let us now turn to the regular Strong Kleene Generalizations associated with $\mathbf{z} = \mathbf{4}$.

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[5] In Sect. 2.5 we discuss the consequence of going multiple conclusion in some more detail.

2.3. The Regular Strong Kleene Generalizations for $\mathbf{z} = 4$

As the next proposition attests, the regular Strong Kleene Generalizations are also familiar. To establish the proposition, we first have the following lemma.

Lemma 2. *Let $V \in \mathbf{V_4}$. Then there is a $V' \in \mathbf{V_4}$, called the* **BN***-swap of V, such that*

$$V'(\varphi) = \mathbf{T} \Leftrightarrow V(\varphi) = \mathbf{T} \qquad V'(\varphi) = \mathbf{F} \Leftrightarrow V(\varphi) = \mathbf{F}$$
$$V'(\varphi) = \mathbf{B} \Leftrightarrow V(\varphi) = \mathbf{N} \qquad V'(\varphi) = \mathbf{N} \Leftrightarrow V(\varphi) = \mathbf{B}$$

Proof. Define the atomic valuation v' by letting $v'(p) = \mathbf{B}$ if $V(p) = \mathbf{N}$, $v'(p) = \mathbf{N}$ if $V(p) = \mathbf{B}$ and $v'(p) = V(p)$ otherwise.[6] Let V' be the recursive extension of v' in accordance with the truth tables of Definition 1. By induction on sentential complexity one shows that V' has the desired properties. This can safely be left to the reader. ■

Proposition 6. *The following relations hold.*

$$1a.\ \models^{\mathbf{4}}_{\mathbf{11}} = \models_{FDE} \qquad 1b.\ \models^{\mathbf{4}}_{\hat{\mathbf{0}}\hat{\mathbf{0}}} = \models_{FDE}$$
$$2a.\ \models^{\mathbf{4}}_{\mathbf{1}\hat{\mathbf{0}}} = \emptyset \qquad 2b.\ \models^{\mathbf{4}}_{\hat{\mathbf{0}}\mathbf{1}} = \emptyset$$

Proof. 1a. is just the definition of **FDE**. 1b immediately follows from Lemma 2. 2a and 2b follow from the observations that the valuations which assign respectively **B** to each sentence and **N** to each sentence are elements of $\mathbf{V_4}$. ■

The study of the mixed Strong Kleene Generalizations is taken up in the next section.

2.4. The Mixed Strong Kleene Generalizations for $\mathbf{z} = 4$

Quite some of the mixed Strong Kleene Generalizations equal the empty set, as shown by the following proposition.

Proposition 7. *We have:* $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\mathbf{1}} = \models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{0}}} = \models^{\mathbf{4}}_{\mathbf{1t}} = \models^{\mathbf{4}}_{\hat{\mathbf{0}}\mathbf{t}} = \emptyset$

Proof. From the observation that the valuations which assign, respectively, **N** to each sentence and **B** to each sentence, are elements of $\mathbf{V_4}$. ■

Besides contributing to our taxonomy of the mixed Strong Kleene Generalizations, the following proposition is interesting as it provides two alternative characterizations of **ETL**.

[6] Using Fitting's [17] conflation operator $-$, this can be expressed more concisely as $v'(p) = -V(p)$.

PROPOSITION 8. *We have:* $\models_{ETL} = \models^{4}_{\mathbf{tt}} = \models^{4}_{\mathbf{t1}} = \models^{4}_{\mathbf{t\hat{0}}}$

PROOF. Per definition, we have $\models^{4}_{\mathbf{tt}} \subseteq \models^{4}_{\mathbf{t1}}$. To show the reverse inclusion, suppose that an argument is not valid according to $\models^{4}_{\mathbf{tt}}$. Then there is a counter model V according to which all premisses are valuated as **T** and according to which the conclusion φ is valuated either as **B**, **N** or **F**. If $V(\varphi) = \mathbf{N}$ or if $V(\varphi) = \mathbf{F}$, then V is also a counter model to the $\models^{4}_{\mathbf{t1}}$ validity of the argument. If $V(\varphi) = \mathbf{B}$, then the **BN**-swap of V (cf. Lemma 2) is a counter model to the $\models^{4}_{\mathbf{t1}}$ validity of the argument. Hence $\models^{4}_{\mathbf{t1}} \subseteq \models^{4}_{\mathbf{tt}}$ and so $\models^{4}_{\mathbf{t1}} = \models^{4}_{\mathbf{tt}}$. Similarly, one shows that $\models^{4}_{\mathbf{tt}} = \models^{4}_{\mathbf{t\hat{0}}}$. ∎

In light of Proposition 8 it is natural to hypothesize that the remaining two mixed consequence relations coincide with (one another and with) $\models^{4}_{\mathbf{\hat{f}\hat{f}}}$. That hypothesis is almost correct, as the following proposition attests.

PROPOSITION 9. *The following relations hold.*

1. $\models^{4}_{\mathbf{1\hat{f}}} = \models^{4}_{\mathbf{\hat{0}\hat{f}}}$ 2. $\varphi \models^{4}_{\mathbf{\hat{f}\hat{f}}} \psi \iff \varphi \models^{4}_{\mathbf{1\hat{f}}} \psi$ 3. $\models^{4}_{\mathbf{\hat{f}\hat{f}}} \subseteq \models^{4}_{\mathbf{1\hat{f}}}$

PROOF. Both 1 and 2 follow by a by now familiar recipe: consider counter models and **BN**-swaps. 3 follows per definition. ∎

Thus, for single premise arguments, $\models^{4}_{\mathbf{1\hat{f}}}$ and $\models^{4}_{\mathbf{\hat{0}\hat{f}}}$ coincide with (one another and with) $\models^{4}_{\mathbf{\hat{f}\hat{f}}}$. This result cannot be strengthened to cover arbitrary arguments, as:

$$p, q \models^{4}_{\mathbf{1\hat{f}}} p \wedge q \qquad p, q \not\models^{4}_{\mathbf{\hat{f}\hat{f}}} p \wedge q$$

Dually, Proposition 8 crucially relies on the fact that we are working in a single conclusion setting, as $p \vee q \models p, q$ is valid according to (the multiple conclusion version of) $\models^{4}_{\mathbf{t1}}$ but not according to (the multiple conclusion version of) **ETL**.

The next section exploits the results obtained in Sects. 2.1, 2.2, 2.3 and 2.4 to exhaustively compare the strength of all the Strong Kleene Generalizations.

2.5. Comparing the Strength of All Strong Kleene Generalizations

In this section, we compare the strength of all the Strong Kleene Generalizations. For sake of completeness, we will also compare the *single premise* and *multiple conclusion* versions of these relations, for which we introduce the following notation.

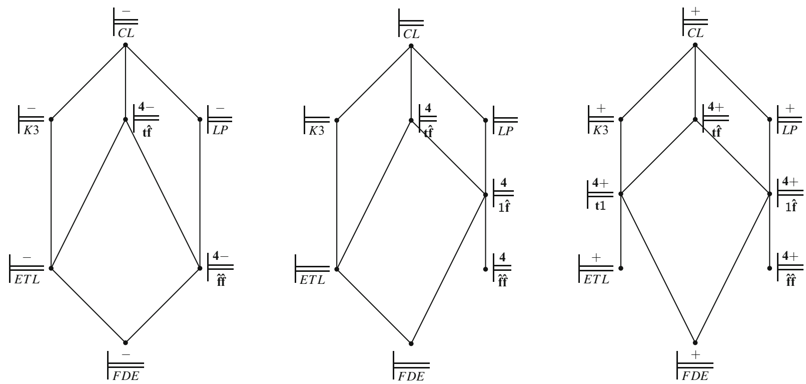

Figure 1. Comparing the Strong Kleene Generalizations in three settings: single premise, normal and multiple conclusion

Definition 4. Let $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$ and let $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}, \mathbf{3b}, \mathbf{4}\}$. The *single premise relation* $\models^{\mathbf{z}-}_{xy}$ is the restriction of $\models^{\mathbf{z}}_{xy}$ to singleton premise sets. The *multiple conclusion relation* $\models^{\mathbf{z}+}_{xy}$ between sets of sentences of $\mathscr{L}$ and sets of sentences of $\mathscr{L}$ is defined as follows:

$$\Gamma \models^{\mathbf{z}+}_{xy} \Delta \iff \text{if } V(\gamma) \in x \text{ for all } \gamma \in \Gamma \text{ then } V(\delta) \in y$$
$$\text{for some } \delta \in \Delta \quad \text{for all } V \in \mathbf{V_z}$$

We will also write $\models^{-}_{ETL}$ for the single premise version of **ETL**, $\models^{+}_{FDE}$ for the multiple conclusion version of **FDE** etc. Figure 1 contains Hasse diagrams of the partial orders on the three types (single premise, normal, multiple conclusion) of Strong Kleene Generalizations that are induced by $\subseteq$. For sake of brevity, Figure 1 picks a single representative definition for each consequence relation that is involved in the comparison. How the other definitions map onto the ones used in Figure 1 is, for $\mathbf{z} = \mathbf{4}$ displayed by Table 1 below.[7] Figure 1 and Table 1 jointly deliver an exhaustive comparison of all the Strong Kleene Generalizations. Besides the results that were established in earlier parts of Sect. 2, Figure 1 also exploits the following, easily established, proposition.

Proposition 10. *Let $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$ and $z \in \{\mathbf{3b}, \mathbf{3n}\}$. The following holds.*

$$\models^{\mathbf{4}-}_{xy} \subseteq \models^{\mathbf{z}-}_{xy} \subseteq \models^{\mathbf{2}-}_{xy} \qquad \models^{\mathbf{4}}_{xy} \subseteq \models^{\mathbf{z}}_{xy} \subseteq \models^{\mathbf{2}}_{xy} \qquad \models^{\mathbf{4}+}_{xy} \subseteq \models^{\mathbf{z}+}_{xy} \subseteq \models^{\mathbf{2}+}_{xy}$$

[7]For $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ this information is provided by propositions 1 and 2.

Table 1. The **4**-valued Strong Kleene Generalizations and their name in Figure 1

Single premise		Normal		Multiple conclusion	
Fig. 1	Equals	Fig. 1	Equals	Fig. 1	Equals
$\models^{-}_{FDE}$	$\models^{\mathbf{4}-}_{\mathbf{11}}$, $\models^{\mathbf{4}-}_{\hat{\mathbf{0}}\hat{\mathbf{0}}}$	$\models_{FDE}$	$\models^{\mathbf{4}}_{\mathbf{11}}$, $\models^{\mathbf{4}}_{\hat{\mathbf{0}}\hat{\mathbf{0}}}$	$\models^{+}_{FDE}$	$\models^{\mathbf{4}+}_{\mathbf{11}}$, $\models^{\mathbf{4}+}_{\hat{\mathbf{0}}\hat{\mathbf{0}}}$
$\models^{-}_{ETL}$	$\models^{\mathbf{4}-}_{\mathbf{tt}}$, $\models^{\mathbf{4}-}_{\mathbf{t1}}$, $\models^{\mathbf{4}-}_{\mathbf{t}\hat{\mathbf{0}}}$	$\models_{ETL}$	$\models^{\mathbf{4}}_{\mathbf{tt}}$, $\models^{\mathbf{4}}_{\mathbf{t1}}$, $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{0}}}$	$\models^{+}_{ETL}$	$\models^{\mathbf{4}+}_{\mathbf{tt}}$
$\models^{\mathbf{4}-}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}-}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$, $\models^{\mathbf{4}-}_{\mathbf{1}\hat{\mathbf{f}}}$, $\models^{\mathbf{4}-}_{\hat{\mathbf{0}}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}+}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}+}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$
$\models^{\mathbf{4}-}_{\mathbf{t}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}-}_{\mathbf{t}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}+}_{\mathbf{t}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}+}_{\mathbf{t}\hat{\mathbf{f}}}$
		$\models^{\mathbf{4}}_{\mathbf{1}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}}_{\mathbf{1}\hat{\mathbf{f}}}$, $\models^{\mathbf{4}}_{\hat{\mathbf{0}}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}+}_{\mathbf{1}\hat{\mathbf{f}}}$	$\models^{\mathbf{4}+}_{\mathbf{1}\hat{\mathbf{f}}}$, $\models^{\mathbf{4}+}_{\hat{\mathbf{0}}\hat{\mathbf{f}}}$
				$\models^{\mathbf{4}+}_{\mathbf{t1}}$	$\models^{\mathbf{4}+}_{\mathbf{t1}}$, $\models^{\mathbf{4}+}_{\mathbf{t}\hat{\mathbf{0}}}$
$\emptyset$	the 7 other $\models^{\mathbf{4}-}_{xy}$	$\emptyset$	the 7 other $\models^{\mathbf{4}}_{xy}$	$\emptyset$	the 7 other $\models^{\mathbf{4}+}_{xy}$

Proof. We show that $\models^{\mathbf{4}}_{xy} \subseteq \models^{\mathbf{3b}}_{xy}$. All other cases are similar and are left to the reader. Suppose that it is not the case that $\Gamma \models^{\mathbf{3b}}_{xy} \varphi$. Then, there is a valuation $V \in \mathbf{V_{3b}}$ such that $V(\gamma) \in x \cap \mathbf{z}$ for all $\gamma \in \Gamma$ and such that $V(\varphi) \notin y \cap \mathbf{z}$. Note that $V \in \mathbf{V_4}$ as $\mathbf{V_{3b}} \subseteq \mathbf{V_4}$ and hence it is not the case that $\Gamma \models^{\mathbf{4}}_{xy} \varphi$. ■

This ends our taxonomy of the Strong Kleene Generalizations as they are defined over the (classical propositional) language $\mathscr{L}$. In the next subsection, we briefly consider what becomes of the Strong Kleene Generalizations when they are defined over a propositional language that has more (truth-functional) expressive power than $\mathscr{L}$.

2.6. Implication Connectives, Expressivity and Functional Completeness

The vocabulary of our language $\mathscr{L}$ is quite restricted as it does not contain an implication connective. With respect to classical logic **CL**, this is not a genuine restriction: the implication $\rightarrow$, defined by stipulating that $\varphi \rightarrow \psi := \neg\varphi \vee \psi$, is what Arieli and Avron [2] call an *appropriate implication connective* for **CL** as it corresponds to **CL** entailment in the sense of (13).

$$\Gamma, \varphi \models_{CL} \psi \iff \Gamma \models_{CL} \varphi \rightarrow \psi \tag{13}$$

The definition of $\rightarrow$ in terms of $\neg$ and $\vee$ determines its truth-functional behaviour with respect to each $V \in \mathbf{V_4}$ and hence, one may study the inferential behaviour of $\rightarrow$ with respect to all Strong Kleene Generalizations. By doing so, one readily observes that $\rightarrow$ fails to be an appropriate implication connective for any (non-empty) Strong Kleene Generalization other than **CL**. For some Strong Kleene Generalizations, such as **K3**, only the left-to-right direction (expressing a *deduction theorem*) of the definition of an appropriate implication connective fails:

$$\Gamma, \varphi \models_{K3} \psi \not\Rightarrow \Gamma \models_{K3} \varphi \rightarrow \psi \qquad \Gamma, \varphi \models_{K3} \psi \Leftarrow \Gamma \models_{K3} \varphi \rightarrow \psi$$

For other Strong Kleene Generalizations, such as **LP**, only the right-to-left direction (expressing a *resolution theorem*) fails:

$$\Gamma, \varphi \models_{LP} \psi \Rightarrow \Gamma \models_{LP} \varphi \rightarrow \psi \qquad \Gamma, \varphi \models_{LP} \psi \not\Leftarrow \Gamma \models_{LP} \varphi \rightarrow \psi$$

And yet other Strong Kleene Generalizations such as **FDE** neither enjoy a deduction nor a resolution theorem (in terms of $\rightarrow$ or any $\mathscr{L}$ definable connective whatsoever).

The fact that $\mathscr{L}$ does not allow the Strong Kleene Generalizations to enjoy appropriate implication connectives provides a motivation to consider extensions of $\mathscr{L}$. Extensions of $\mathscr{L}$ with appropriate implication connectives for **K3**, **LP** and **FDE** haven been considered at various places in the literature (for example in Avron [4], Batens and de Clerq [8] or in Arieli and Avron [3]). It is in the spirit of this paper to consider appropriate implication connectives for all the Strong Kleene Generalizations and to add them to our language in one fell swoop. In order to do so, we will extend $\mathscr{L}$ with four implication connectives $\supset_x$, where $x \in \{\mathbf{1}, \hat{\mathbf{0}}, \mathbf{t}, \hat{\mathbf{f}}\}$, and consider language $\mathscr{L}^\star$ whose BNF form is as follows:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \supset_{\mathbf{1}} \varphi \mid \varphi \supset_{\hat{\mathbf{0}}} \varphi \mid \varphi \supset_{\mathbf{t}} \varphi \mid \varphi \supset_{\hat{\mathbf{f}}} \varphi$$

The semantics of the connectives, as well as the notion of an $\mathscr{L}^\star$ *valuation* and that of a *Strong Kleene** *Generalization*, are given by the following definition.

Definition 5. An $\mathscr{L}^\star$ *valuation* is a function V from the sentences of $\mathscr{L}^\star$ to $\mathbf{4}$ that respects the truth tables for $\wedge, \vee$ and $\neg$ as given by Definition 1 and which is such that, for each $x \in \{\mathbf{1}, \hat{\mathbf{0}}, \mathbf{t}, \hat{\mathbf{f}}\}$:

$$V(\varphi \supset_x \psi) \begin{cases} V(\psi) & \text{if } V(\varphi) \in x \\ \mathbf{T} & \text{if } V(\varphi) \notin x \end{cases}$$

We will use $\mathbf{V}^\star_{\mathbf{4}}$ to denote the set of all $\mathscr{L}^\star$ valuations and we will use $\mathbf{V}^\star_{\mathbf{2}}$, $\mathbf{V}^\star_{\mathbf{3b}}$ and$\mathbf{V}^\star_{\mathbf{3n}}$ to denote the sets of $\mathscr{L}^\star$ valuations whose range is, respectively, a subset of $\mathbf{2}$, $\mathbf{3b}$ and $\mathbf{3n}$. With $x \in \{\mathbf{1}, \hat{\mathbf{0}}, \mathbf{t}, \hat{\mathbf{f}}\}$ and $\mathbf{z} \in \{\mathbf{2}, \mathbf{3n}, \mathbf{3b}, \mathbf{4}\}$, the relation $\models^{\mathbf{z}\star}_{xy}$ is defined as expected:

$$\Gamma \models^{\mathbf{z}\star}_{xy} \varphi \iff \text{if } V(\gamma) \in x \cap \mathbf{z} \text{ for all } \gamma \in \Gamma \text{ then } V(\varphi) \in y \cap \mathbf{z} \quad \text{for all } V \in \mathbf{V}^\star_{\mathbf{z}} \tag{14}$$

The relations $\models^{\mathbf{z}\star}_{xy}$ as defined by (14) we call *Strong Kleene** *Generalizations*.

With $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$ we will write $\supset_x^{\mathbf{z}}$ to denote the restriction of $\supset_x$ to $\mathbf{z}$. As is readily verified, $\supset_x^{\mathbf{z}}$ defines a truth function on $\mathbf{z}$. The following proposition explains how the restricted implications are related.

PROPOSITION 11. *The following relations hold.*

- $\supset_1^{\mathbf{2}} = \supset_{\hat{0}}^{\mathbf{2}} = \supset_{\mathbf{t}}^{\mathbf{2}} = \supset_{\hat{\mathbf{f}}}^{\mathbf{2}}$ *(the material conditional of classical logic)*
- $\supset_1^{\mathbf{3b}} = \supset_{\hat{\mathbf{f}}}^{\mathbf{3b}}$, $\supset_{\mathbf{t}}^{\mathbf{3b}} = \supset_{\hat{0}}^{\mathbf{3b}}$, $\supset_1^{\mathbf{3n}} = \supset_{\mathbf{t}}^{\mathbf{3n}}$, $\supset_{\hat{\mathbf{f}}}^{\mathbf{3n}} = \supset_{\hat{0}}^{\mathbf{3n}}$

PROOF. It readily follows from the definition of the implication connectives that whenever $x \cap \mathbf{z} = y \cap \mathbf{z}$ we have that $\supset_x^{\mathbf{z}} = \supset_y^{\mathbf{z}}$. ■

In Avron [4], the extension of **K3** to the language $\mathscr{L}$ augmented with appropriate implication connective $\supset_{\mathbf{t}}^{\mathbf{3n}}$ is studied. Likewise, [4], studies the extension of **LP** to the language $\mathscr{L}$ augmented with $\supset_1^{\mathbf{3b}}$ and the same logic is also studied by, for instance Batens [7] and Batens and de Clerq [8]. Finally, the extension of **FDE** to the language $\mathscr{L}$ augmented with $\supset_1^{\mathbf{4}}$ is studied in, amongst others, Arieli and Avron [3]. In our uniform framework all these logics are available as Strong Kleene* Generalizations (restricted to the appropriate fragment of $\mathscr{L}^\star$). The following proposition attest that the introduction of the four connectives $\supset_x$ ensures that in fact all Strong Kleene* Generalizations have an appropriate implication connective.

PROPOSITION 12. *For all* $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$*:* $\Gamma, \varphi \models_{xy}^{\mathbf{z}\star} \psi \iff \Gamma \models_{xy}^{\mathbf{z}\star} \varphi \supset_x \psi$

PROOF. $\Rightarrow$ Assume that $\Gamma, \varphi \models_{xy}^{\mathbf{z}\star} \psi$. Let $V \in \mathbf{V}_{\mathbf{z}}^\star$ and suppose that $V(\gamma) \in x$ for all $\gamma \in \Gamma$. We need to show that $V(\varphi \supset_x \psi) \in x$. When $V(\varphi) \in x$, it follows that $V(\varphi \supset_x \psi) = V(\psi) \in y$ as $\Gamma, \varphi \models_{xy}^{\mathbf{z}\star} \psi$. When $V(\varphi) \notin x$, it follows that $V(\varphi \supset_x \psi) = \mathbf{T} \in y$ as $y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$.

$\Leftarrow$ Reason by contraposition and assume that $\Gamma, \varphi \not\models_{xy}^{\mathbf{z}\star} \psi$. Thus, for some $V \in \mathbf{V}_{\mathbf{z}}^\star$, we have that $V(\gamma) \in x$ for all $\gamma \in \Gamma$, $V(\varphi) \in x$ and $V(\psi) \notin y$. It thus follows that $V(\varphi \supset_x \psi) = V(\psi) \notin y$ and that $\Gamma \not\models_{xy}^{\mathbf{z}\star} \varphi \supset_x \psi$. ■

Thus, it is relatively straightforward to extend our basic language $\mathscr{L}$ with appropriate implication connectives, and to do so in a uniform way. It should be noted though, that most of the results that we established in Sect. 2 for the Strong Kleene Generalizations do *not* carry over to the Strong Kleene* Generalizations. As an example, whereas we have that $\models_{1\hat{0}}^{\mathbf{3n}} = \models_{1\hat{0}}^{\mathbf{2}}$ (cf. 3a of Proposition 2) we do *not* have that $\models_{1\hat{0}}^{\mathbf{3n}\star} = \models_{1\hat{0}}^{\mathbf{2}\star}$ as, for instance we have $\models_{1\hat{0}}^{\mathbf{2}\star} p \supset_1 q \vee \neg q$ but not $\models_{1\hat{0}}^{\mathbf{3n}\star} p \supset_1 q \vee \neg q$. More generally, the reader may observe that Lemma 2 is no longer valid when $\mathbf{V_4}$ is replaced with $\mathbf{V_4^\star}$ and that this lemma is used in the proof of Propositions 8 and 9. Likewise, the valuations which assign **B**, respectively **N** to every sentence are elements of $\mathbf{V_4}$ but not of $\mathbf{V_4^\star}$ and this fact ensures that (the counterparts of) some propositions of Sect. 2 do not hold for the Strong Kleene* Generalizations. To provide a detailed taxonomy of the Strong Kleene* Generalizations, however, is far beyond the scope of this paper.

Although the (extended) language $\mathscr{L}^\star$ contains appropriate implication connectives, it lacks, for instance, a connective expressing strong negation and so one may also consider adding such a connective, together with various others. Or, in order to ensure that all possible truth functions on **4** are expressible in one's language, one may consider a language that is functionally complete with respect to **4**. In the literature, various authors have studied the relation of **FDE** consequence on functional complete extensions of $\mathscr{L}$ (see e.g. Muskens [20], Arieli and Avron [3], Ruet [29], Pynko [25], or Omori and Sano [22]). In this paper, we will not consider functionally complete extensions of $\mathscr{L}$. The reason we do not, however, has not only to do with restrictions of scope and length. For, as the reader will have already observed (see also footnote 2) our uniform treatment of the Strong Kleene (and Strong Kleene*) Generalizations relies on the fact that the restrictions of the (truth functions denoted by the) connectives of $\mathscr{L}^\star$ and $\mathscr{L}$ to $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ are truth functions on **z**. When we consider a language that is functionally complete with respect to **4**, all truth functions on **4** are expressible and so in particular those whose restriction to $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ is not a truth function on **z**. As a consequence, there are no valuations of a functionally complete language whose range is a proper subset of **4** and hence, to study the "Strong Kleene Generalizations" for $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ is nonsensical in this case.

3. A Uniform Sequent Calculus for the Strong Kleene Generalizations

3.1. Sequent Calculi for Many-Valued Logics

The main goal of Sect. 3 is to present a *uniform* signed sequent calculus (which we call the **SK** calculus) for all the Strong Kleene Generalizations: a calculus that can be used to characterize each and every Strong Kleene Generalization. In Sect. 4, we then present two other such calculi (the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus) than can be obtained by applying the general methods of Baaz et al. [6] to construct sequent calculi for many-valued logics. We think that it is instructive to first briefly sketch the rationale of these general methods and to indicate in which sense our **SK** calculus differs from the calculi obtained by these general methods.

In a standard sequent calculus for classical logic, sequents are 2-sided objects of form $\Gamma \Rightarrow \Delta$, where Γ and Δ can be taken to be sets of sentences. Equivalently though, a 2-sided sequent $\Gamma \Rightarrow \Delta$ can be presented as a set $\{l : \gamma \mid \gamma \in \Gamma\} \cup \{r : \delta \mid \delta \in \Delta\}$ consisting of signed sentences, where the sign of a sentence indicates whether it occurs on the *l*eft side or on the *r*ight side of $\Rightarrow$. But one may also choose other signs than l and r that are more informative in the sense that they hint at the semantic interpretation of a provable sequent. Interestingly, there are two (equivalent but) distinct ways to interpret provable sequents, associated with the following two translations of a 2-sided sequent to a 2-signed one:

i Translate $\Gamma \Rightarrow \Delta$ as $\{\mathbf{F} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathbf{T} : \delta \mid \delta \in \Delta\}$. Provable sequents correspond to $\mathscr{P}$(ostive) valid ones: when $\{\mathbf{F} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathbf{T} : \delta \mid \delta \in \Delta\}$ is provable, in every $V \in \mathbf{V_2}$, $V(\gamma) = \mathbf{F}$ for some $\gamma \in \Gamma$ or $V(\delta) = \mathbf{T}$ for some $\delta \in \Delta$.

ii Translate $\Gamma \Rightarrow \Delta$ as $\{\mathbf{T} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathbf{F} : \delta \mid \delta \in \Delta\}$. Provable sequents correspond to $\mathscr{N}$(egative) valid ones: when $\{\mathbf{T} : \gamma \mid \gamma \in \Gamma\} \cup \{\mathbf{F} : \delta \mid \delta \in \Delta\}$ is provable, in every $V \in \mathbf{V_2}$, $V(\gamma) \neq \mathbf{T}$ for some $\gamma \in \Gamma$ or $V(\delta) \neq \mathbf{F}$ for some $\delta \in \Delta$.

Of course, the proof systems for classical logic that result from translation i and ii are, modulo an insignificant difference in signs, exactly alike. Interestingly though, Baaz et al [6] show how, for any n-valued logic, two *dual* n-signed sequent calculi can be given that correspond to the (generalized) notions of $\mathscr{P}$- validity and $\mathscr{N}$-validity respectively. The dual calculi that are obtained as such may differ substantially when $n > 2$ (as then not being valuated as $\mathbf{T}$ is not the same as being valuated as $\mathbf{F}$) which is vividly illustrated by the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus that we obtain by applying those methods in Sect. 4.

The signs exploited by the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus are members of $\{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ and code for the corresponding elements of **4**. In contrast, the signs of the **SK** calculus are elements of $\{1, 0, \hat{1}, \hat{0}\}$ and code for the corresponding *subsets* of **4** as indicated by (2). The **SK** calculus (which is based on a generalized notion of $\mathscr{N}$-(in)validity as will become apparent) reflects the fact that **T**, **B**, **N**, and **F** are best thought of as *combinations* of (regular) truth values and its signs capture the "underlying" values of (non-) truth and (non-) falsity. Doing so has formal and philosophical advantageous consequences as we point out in Sect. 5.

3.2. The SK Calculus

First some notational conventions. We introduce the *complement* operation com on the set $\{1, \hat{1}, 0, \hat{0}\}$ of signs of the **SK** calculus by stipulating that $\mathsf{com}(1) = \hat{1}$, $\mathsf{com}(\hat{1}) = 1$, $\mathsf{com}(0) = \hat{0}$ and $\mathsf{com}(\hat{0}) = 0$. Further, with x a sign and, with Γ a set of sentences, we write $x : \Gamma$ as shorthand for $\{x : \gamma \mid \gamma \in \Gamma\}$. Finally, it will be convenient to let $\mathbf{Z} := \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}, \mathbf{4}\}$.

Sequents of the **SK** calculus will be sets of signed sentences of $\mathscr{L}$, as indicated above. Provable sequents of our calculus will correspond to *unsatisfiable* sets of signed sentences. In fact, we will distinguish four kinds of unsatisfiability, as explained by the following definition.

Definition 6. Let Θ be a sequent let $V \in \mathbf{V_4}$ be a valuation. We say that V *satisfies* Θ iff every $x : \varphi \in \Theta$ is such that:

$$x = 1 \implies V(\varphi) \in \{\mathbf{T}, \mathbf{B}\} \qquad x = 0 \implies V(\varphi) \in \{\mathbf{F}, \mathbf{B}\}$$
$$x = \hat{1} \implies V(\varphi) \in \{\mathbf{F}, \mathbf{N}\} \qquad x = \hat{0} \implies V(\varphi) \in \{\mathbf{T}, \mathbf{N}\}$$

With $\mathbf{z} \in \mathbf{Z}$, we say that Θ is $\mathbf{z}$-*unsatisfiable* just in case no $V \in \mathbf{V_z}$ satisfies Θ.

Observe that all regular Strong Kleene Generalizations have a natural alternative definition in terms of $\mathbf{z}$-unsatisfiability. For, if $x, y \in \{1, \hat{0}\}$ we have that

$$\Gamma \models^{\mathbf{z}}_{xy} \varphi \iff x : \Gamma \ \cup \ \{\mathsf{com(y)} : \varphi\} \ \text{ is } \mathbf{z}\text{-unsatisfiable} \tag{15}$$

Corresponding to the four notions of $\mathbf{z}$-unsatisfiability, the **SK** calculus will distinguish four notions of $\mathbf{z}$-provability. The latter notions will differ only with respect to the *initial sequent rules* (or axioms) that may be used in a proof. To define the four sets of initial sequent rules associated with the notions of $\mathbf{z}$-provability, we consider the (initial) sequent rules of form (R_{xy}),

$$\frac{}{x : \varphi, \ y : \varphi} \, (R_{xy})$$

in terms of which we define four sets of initial sequent rules. **4**-provability will only allow initial sequent rules that occur in $R_\mathbf{4}$:

$$R_\mathbf{4} = \{(R_{xy}) \mid \langle x, y \rangle \in \{\langle 1, \hat{1} \rangle, \langle 0, \hat{0} \rangle\}\}$$

More generally, $\mathbf{z}$-provability will only allow initial sequent rules that occur in $R_\mathbf{z}$, where:

$$R_\mathbf{3b} = R_\mathbf{4} \cup \{(R_{\hat{1}\hat{0}})\} \qquad R_\mathbf{3n} = R_\mathbf{4} \cup \{(R_{10})\} \qquad R_\mathbf{2} = R_\mathbf{3b} \cup R_\mathbf{3n}$$

Indeed, for all $\mathbf{z} \in \mathbf{Z}$, we have that $R_{\mathbf{z}} \subseteq R_{\mathbf{2}}$. The structural rules of the **SK** calculus will consist of the initial sequent rules that are contained in $R_{\mathbf{2}}$, together with (W)eakening:

$$\frac{\Sigma}{\Sigma'}\,(W) \qquad \text{where } \Sigma \subseteq \Sigma'$$

The **SK** calculus augments these structural rules with operational rules for the logical connectives of $\mathscr{L}$ that are provided by the following definition.

Definition 7. (The **SK** calculus) The structural rules of the **SK** calculus are the initial sequent rules of $R_{\mathbf{2}}$, together with (W). The operational rules of the **SK** calculus are as follows:

$$\frac{\Sigma,\ y:\varphi}{\Sigma,\ x:\neg\varphi}\,(\neg_x)$$

if $\langle x, y\rangle \in \{\langle 1, 0\rangle, \langle \hat{1}, \hat{0}\rangle, \langle 0, 1\rangle, \langle \hat{0}, \hat{1}\rangle\}$

$$\frac{\Sigma,\ x:\varphi,\ x:\psi}{\Sigma,\ x:\varphi\wedge\psi}\,(\wedge_x)$$

if $x \in \{1, \hat{0}\}$

$$\frac{\Sigma,\ x:\varphi \qquad \Sigma,\ x:\psi}{\Sigma,\ x:\varphi\wedge\psi}\,(\wedge_x)$$

if $x \in \{\hat{1}, 0\}$

$$\frac{\Sigma,\ x:\varphi \qquad \Sigma,\ x:\psi}{\Sigma,\ x:\varphi\vee\psi}\,(\vee_x)$$

if $x \in \{1, \hat{0}\}$

$$\frac{\Sigma,\ x:\varphi,\ x:\psi}{\Sigma,\ x:\varphi\vee\psi}\,(\vee_x)$$

if $x \in \{\hat{1}, 0\}$

With $\mathbf{z} \in \mathbf{Z}$, the rules of the $\mathbf{SK}_{\mathbf{z}}$ *calculus* are (W), the initial sequent rules of $R_{\mathbf{z}}$ and all operational sequent rules. A sequent Θ is said to be $\mathbf{z}$-*provable* if some finite $\Theta_0 \subseteq \Theta$ has a proof tree respecting the rules of the $\mathbf{SK}_{\mathbf{z}}$ calculus.

It will turn out to be convenient to introduce a general form for our sequent rules. For this, we pick $T_1, \ldots, T_n/B$, where each T_i is a set of signed sentences, called a *top set* of the rule and where B is a set of signed sentences called the *bottom set* of the rule. For instance one instantiation of (R_{10}) could formally be written as $\emptyset/\Sigma \cup \{1:\varphi, 0:\varphi\}$ and one instantiation of $(\wedge_0)$ as $\Sigma \cup \{0:\varphi\}, \Sigma \cup \{0:\psi\}/\Sigma \cup \{0:\varphi\wedge\psi\}$. The following proposition exploits the general form a sequent rule and explains how the rules of the **SK** calculus can be interpreted in terms of satisfaction.

Proposition 13. *The following claims hold.*

1. *The bottom set of each initial sequent rule in* $R_{\mathbf{z}}$ *is* $\mathbf{z}$*-unsatisfiable.*

2. *A valuation* $V \in \mathbf{V_4}$ *satisfies the bottom set of an operational rule of the* **SK** *calculus iff it satisfies some top set of that rule.*
3. *A valuation* $V \in \mathbf{V_4}$ *dissatisfies (i.e. does not satisfy) the bottom set of an operational rule of the* **SK** *calculus iff it dissatisfies all top sets of that rule.*

Proof. By inspection. ■

Claims 1 and 3 of Proposition 13 hint at Theorem 1, which states that the **SK** calculus is sound and complete with respect to **z**-unsatisfiability.

Theorem 1. *With* $\mathbf{z} \in \mathbf{Z}$*, a sequent* Θ *is* **z***-provable if and only if* Θ *is* **z***-unsatisfiable.*

Proof. First, it should be noted that it follows from the general results on the compactness of propositional (finitely) many-valued logics due to Woodruff [34] that a sequent Θ is **z**-unsatisfiable iff some finite $\Theta_0 \subseteq \Theta$ is **z**-unsatisfiable. This, together with the definition of **z**-provability, ensures that it suffices to establish the theorem for finite sequents.

Let $\mathbf{z} = \mathbf{4}$. The $\Rightarrow$ direction follows easily by an induction on proof depth (of **4**-proofs) plus observation 3 of Proposition 13. For the $\Leftarrow$ direction, let Θ be a finite sequent that is not **4**-provable. We use induction on the total number n of connectives occurring in Θ. If $n = 0$, Θ is a set of signed propositional constants that is not a conclusion of the 2 rules in $R_{\mathbf{4}}$. This means that Θ does not contain a pair $1 : p$, $\hat{1} : p$, or a pair $0 : p$, $\hat{0} : p$. Consider the valuation $V \in \mathbf{V_4}$ which valuates the propositional atoms of $\mathscr{L}$ as follows:

$$V(p) = \begin{cases} \mathbf{T} & 1 : p \in \Theta,\ 0 : p \notin \Theta \\ \mathbf{B} & 1 : p \in \Theta,\ 0 : p \in \Theta \\ \mathbf{N} & 1 : p \notin \Theta,\ 0 : p \notin \Theta \\ \mathbf{F} & 1 : p \notin \Theta,\ 0 : p \in \Theta \end{cases}$$

Observe that this definition of V, together with the fact that Θ is **4**-provable implies that if Θ contains $1 : p$ then $V(p) \in \{\mathbf{T}, \mathbf{B}\}$; if Θ contains $0 : p$ then $V(p) \in \{\mathbf{F}, \mathbf{B}\}$; if Θ contains $\hat{1} : p$ then Θ does not contain $1 : p$ and so $V(p) \in \{\mathbf{F}, \mathbf{N}\}$; and if Θ contains $\hat{0} : p$ then Θ does not contain $0 : p$ and so $V(p) \in \{\mathbf{T}, \mathbf{N}\}$. Indeed, we just established that V satisfies Θ.

If $n > 0$, Θ can be written as a sequent Σ, θ, where θ is some signed sentence containing at least one connective and $\theta \notin \Sigma$. Inspection of the rules shows that in this case Θ follows from a sequent Θ_1 or from a pair of sequents Θ_1 and Θ_2, each containing fewer than n connectives. One of these top sequents must be **4**-unprovable and hence, by induction, satisfied by some valuation V. Observation 2 of Proposition 13 gives that Θ is satisfied by the same V. Thus, from the fact that Θ is not **4**-provable, it follows that there is a $V \in \mathbf{V_4}$ which satisfies Θ.

For $\mathbf{z} \in \{\mathbf{2}, \mathbf{3b}, \mathbf{3n}\}$ the proof is entirely similar, albeit that the definition of the valuation $V \in \mathbf{V_z}$ that is needed to establish the induction base is slightly different. ■

3.3. Capturing the Strong Kleene Generalizations Via the SK Calculus

With Theorem 1 at our disposal, we can now explain how the **SK** calculus captures the Strong Kleene Generalizations. In light of observation (15), our calculus straightforwardly captures the regular consequence relations, as attested by the following proposition.

PROPOSITION 14. *Let* $\mathbf{z} \in \mathbf{Z}$ *and let* $x, y \in \{1, \hat{0}\}$. *We have:*

$$\Gamma \models^{\mathbf{z}}_{xy} \varphi \iff x : \Gamma \cup \{\mathsf{com(y)} : \varphi\} \text{ is } \mathbf{z}\text{-provable}$$

PROOF. Immediate from Theorem 1 and observation (15). ■

According to Propositions 1, 2 and 6 the (non-empty) regular Strong Kleene Generalizations are either equal to **CL**, **K3**, **LP**, or **FDE** and so Proposition 14 attests that the **SK** calculus is tailor made to capture these four familiar consequence relations.

Indeed, the very signs of the **SK** calculus reflect that it is tailor made to capture the regular Strong Kleene Generalizations. As for any $\mathbf{z} \neq \mathbf{4}$, every (exact or mixed) relation $\models^{\mathbf{z}}_{xy}$ can (in virtue of propositions 1 and 2) be expressed as a regular consequence relation, our calculus can capture all these relations. However, as not all of the considered **4**-valued consequence relations can be expressed as regular consequence relations, the question arises whether our calculus can also be invoked to capture those. More concretely, this question comes down (cf. Table 1) to asking whether the **SK** calculus can capture $\models_{ETL}$, $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$, $\models^{\mathbf{4}}_{\mathbf{t}\hat{\mathbf{f}}}$ and $\models^{\mathbf{4}}_{\mathbf{1}\hat{\mathbf{f}}}$. Let us first consider $\models_{ETL}$ in terms of its canonical definition $\models^{\mathbf{4}}_{\mathbf{tt}}$ and observe that

$$\Gamma \models^{\mathbf{4}}_{\mathbf{tt}} \varphi \iff \begin{cases} 1 : \Gamma \cup \hat{0} : \Gamma \cup \{0 : \varphi\} \text{ is } \mathbf{4}\text{-unsatisfiable and} \\ 1 : \Gamma \cup \hat{0} : \Gamma \cup \{\hat{1} : \varphi\} \text{ is } \mathbf{4}\text{-unsatisfiable} \end{cases}$$

This observation naturally suggests to define the syntactic correlate of $\models_{ETL}$ in terms of *two* provable sequents of our calculus. However, as $\models_{ETL}$ can also be defined as $\models^{\mathbf{4}}_{\mathbf{t1}}$ (cf. Proposition 8), the following proposition attests that a single proof tree suffices to establish $\models_{ETL}$ consequence.

Proposition 15. *The following relations hold.*

$$\Gamma \models^{\mathbf{4}}_{\mathbf{t1}} \varphi \iff 1 : \Gamma \ \cup\ \hat{0} : \Gamma \ \cup\ \{\hat{1} : \varphi\} \text{ is } \mathbf{4}\text{-provable}$$

$$\Gamma \models^{\mathbf{4}}_{\mathbf{t\hat{f}}} \varphi \iff 1 : \Gamma \ \cup\ \hat{0} : \Gamma \ \cup\ \{\hat{1} : \varphi, 0 : \varphi\ \} \text{ is } \mathbf{4}\text{-provable}$$

$$\Gamma \models^{\mathbf{4}}_{\mathbf{1\hat{f}}} \varphi \iff 1 : \Gamma \ \cup\ \{\hat{1} : \varphi, 0 : \varphi\ \} \text{ is } \mathbf{4}\text{-provable}$$

Proof. By inspection of definitions, using Theorem 1. ■

The remaining relation on our list, $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$, can also be captured by the **SK** calculus, albeit in the rather artificial manner of the following proposition.

Proposition 16. *The following relation holds.*

$$\Gamma \models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}} \varphi \iff \forall f \in \{1, \hat{0}\}^{\Gamma} \quad \{f(\gamma) : \gamma \mid \gamma \in \Gamma\} \ \cup\ \{0 : \varphi, \hat{1} : \varphi\} \quad \text{is } \mathbf{4}\text{-provable},$$

where $\{1, \hat{0}\}^{\Gamma}$ *is the set of all functions from* Γ *to* $\{1, \hat{0}\}$.

Proof. In light of Theorem 1, it suffices to show that $\Gamma \models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}} \varphi$ iff for every $f \in \{1, \hat{0}\}^{\Gamma}$ the sequent $\Theta_f := \{f(\gamma) : \gamma \mid \gamma \in \Gamma\} \cup \{0 : \varphi, \hat{1} : \varphi\}$ is unsatisfiable. For the left-to-right direction, suppose that for some f, some $V \in \mathbf{V_4}$ satisfies Θ_f. Then V is such that $V(\gamma) \in \{\mathbf{T}, \mathbf{B}, \mathbf{N}\}$ for all $\gamma \in \Gamma$ while $V(\varphi) = \mathbf{F}$. Hence it is not the case that $\Gamma \models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}} \varphi$. For the right-to-left direction, suppose that there is a $V \in \mathbf{V_4}$ such that $V(\gamma) \in \{\mathbf{T}, \mathbf{B}, \mathbf{N}\}$ for all $\gamma \in \Gamma$ and $V(\varphi) = \mathbf{F}$. Let $\Gamma_1 := \{\gamma \in \Gamma \mid V(\gamma) \in \{\mathbf{T}, \mathbf{B}\}\}$ and let $f \in \{1, \hat{0}\}^{\Gamma}$ be such that $f(\gamma) = 1$ iff $\gamma \in \Gamma_1$. Then V satisifies Θ_f and hence Θ_f is not unsatisfiable. ■

And so although the **SK** calculus captures the $\models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}}$ relation, the way it does so is rather cumbersome: when Γ is a finite set of premises, a proof of $\Gamma \models^{\mathbf{4}}_{\mathbf{\hat{f}\hat{f}}} \varphi$ consists of $2^{|\Gamma|}$ distinct proof trees.

Together, Propositions 14, 15 and 16 testify that the **SK** calculus allows us to capture all the Strong Kleene Generalizations. The **SK** calculus is a cut-free calculus; in the next section we explain that and which cut rules can be added to the **SK** calculus.

3.4. A Note on Adding Cut-Rules to the SK Calculus

Although the structural rules of the **SK** calculus merely consist of Weakening and the initial sequent rules of $R_{\mathbf{2}}$, the usual structural rules of Contraction and Permutation are implicitly built into our calculus as our sequents are sets of signed sentences. Further, although **SK** is a cut-free sequent calculus, several (admissible) cut rules are readily available. To introduce them in a convenient manner, we introduce the following four auxiliary sets of signed sentences.

$$S_{\mathbf{T}}^{\varphi} = \{1 : \varphi, \hat{0} : \varphi\}, \qquad S_{\mathbf{B}}^{\varphi} = \{1 : \varphi, 0 : \varphi\}, \qquad S_{\mathbf{N}}^{\varphi} = \{\hat{1} : \varphi, \hat{0} : \varphi\},$$
$$S_{\mathbf{F}}^{\varphi} = \{\hat{1} : \varphi, 0 : \varphi\}$$

An auxiliary set S_x^{φ} is satisfied by a valuation V iff $V(\varphi) = x$. With $\mathbf{z} \in \mathbf{Z}$, the rule $(Cut_{\mathbf{z}})$ has, as its top sets, the sets $\Sigma \cup S_x^{\varphi}$ for $x \in \mathbf{z}$ and has, as its bottom set, Σ. As an example, consider the rule $(Cut_{\mathbf{4}})$:

$$\frac{\Sigma, 1 : \varphi, \hat{0} : \varphi \qquad \Sigma, 1 : \varphi, 0 : \varphi \qquad \Sigma, \hat{1} : \varphi, \hat{0} : \varphi \qquad \Sigma, \hat{1} : \varphi, 0 : \varphi}{\Sigma} \, (Cut_{\mathbf{4}})$$

The following proposition explain the sense in which adding a $(Cut_{\mathbf{z}})$ rule to the **SK** calculus is admissible.

PROPOSITION 17. *A sequent Θ is $\mathbf{z}$-provable if and only if it is provable according to the rules of the $\mathbf{SK_z}$ calculus with $(Cut_{\mathbf{z}})$ added to it.*

PROOF. A valuation $V \in \mathbf{V_z}$ dissatisfies (cf. Proposition 13.3) all top sets of $(Cut_{\mathbf{z}})$ just in case V dissatisfies its bottom set. And so, as the $\mathbf{SK_z}$ calculus is complete with respect to $\mathbf{z}$-unsatisfiable sets (cf. Theorem 1), the result follows. ■

Although adding the $(Cut_{\mathbf{z}})$ rules to the **SK** calculus is admissible in the sense of Proposition 17, adding those cut rules is unnatural in the following sense. Basically, $(Cut_{\mathbf{z}})$ states that a sentence always takes one of the truth values in $\mathbf{z}$. This insistence on values is in conflict with the spirit of the **SK** calculus, whose four signs code for *sets* of values rather than for values as such. More natural cut rules are available though. To define them, we first consider all cut rules of form (Cut_{xy}),

$$\frac{\Sigma, x : \varphi \qquad \Sigma, y : \varphi}{\Sigma} \, (Cut_{xy})$$

in terms of which we define, for each $\mathbf{z} \in \mathbf{Z}$, the set of cut rules $CUT_{\mathbf{z}}$, where:

$$CUT_{\mathbf{4}} = \{(Cut_{1\hat{1}}), (Cut_{\hat{0}0})\} \qquad CUT_{\mathbf{3b}} = CUT_{\mathbf{4}} \cup \{(Cut_{10})\}$$
$$CUT_{\mathbf{3n}} = CUT_{\mathbf{4}} \cup \{(Cut_{\hat{1}\hat{0}})\} \qquad CUT_{\mathbf{2}} = CUT_{\mathbf{3b}} \cup CUT_{\mathbf{3n}}$$

Observe that, when $(Cut_{xy}) \in CUT_{\mathbf{z}}$, a valuation $V \in V_{\mathbf{z}}$ dissatisfies the two top sets of (Cut_{xy}) just in case it dissatisfies its bottom set. Intuitively then, $(Cut_{xy}) \in CUT_{\mathbf{z}}$ may be interpreted as specifying that, for each valuation $V \in V_{\mathbf{z}}$ and sentence φ, we have that $V(\varphi) \in x$ or $V(\varphi) \in y$. Although we feel that it is more natural to add the rules of $CUT_{\mathbf{z}}$ to the **SK** calculus rather than ($Cut_{\mathbf{z}}$), the rules in $CUT_{\mathbf{z}}$ turn out to be *interderivable* with ($Cut_{\mathbf{z}}$), as follows from the following proposition.

PROPOSITION 18. *Let* $\mathbf{z} \in \mathbf{Z}$. *In the presence of* ($W$)*eakening, the* ($Cut_{\mathbf{z}}$) *rule can be derived from the rules in* $CUT_{\mathbf{z}}$ *and conversely, each rule in* $CUT_{\mathbf{z}}$ *can be derived from the* ($Cut_{\mathbf{z}}$) *rule.*

PROOF. We will first consider $\mathbf{z} = \mathbf{3b}$.
$\Rightarrow$ We show that the ($Cut_{\mathbf{3b}}$) rule can be derived from the rules in $CUT_{\mathbf{3b}}$. So suppose that we can derive the three top sets of the ($Cut_{\mathbf{3b}}$) rule, i.e. suppose that we can derive, for each for $x \in \mathbf{3b}$, the set $\Sigma \cup S^{\varphi}_{x}$. Given a derivation of $\Sigma \cup S^{\varphi}_{\mathbf{T}}$ and a derivation of $\Sigma \cup S^{\varphi}_{\mathbf{B}}$ we may derive, via $(Cut_{\hat{0}0}) \in CUT_{\mathbf{3b}}$, the sequent $\Sigma, \mathsf{1} : \varphi$. Given a derivation of $\Sigma \cup S^{\varphi}_{\mathbf{B}}$ and a derivation of $\Sigma \cup S^{\varphi}_{\mathbf{F}}$ we may derive, via $(Cut_{1\hat{1}}) \in CUT_{\mathbf{3b}}$ the sequent $\Sigma, \mathsf{0} : \varphi$. From the thus derived sequents $\Sigma, \mathsf{1} : \varphi$ and $\Sigma, \mathsf{0} : \varphi$ we may derive, via $(Cut_{10}) \in CUT_{\mathbf{3b}}$ the sequent Σ, i.e. the bottom set of ($Cut_{\mathbf{3b}}$), which is what we had to show.
$\Leftarrow$ Consider $(Cut_{10}) \in CUT_{\mathbf{3b}}$ and suppose that we have derived the two top sets $\Sigma, \mathsf{1} : \varphi$ and $\Sigma, \mathsf{0} : \varphi$ of this rule. Applying (W) to $\Sigma, \mathsf{1} : \varphi$ delivers $\Sigma \cup S^{\varphi}_{\mathbf{T}}$ and $\Sigma \cup S^{\varphi}_{\mathbf{B}}$ and applying (W) to $\Sigma, \mathsf{0} : \varphi$ yields $\Sigma \cup S^{\varphi}_{\mathbf{F}}$. We have thus obtained the three top sets of ($Cut_{\mathbf{3b}}$) and so we may derive Σ, i.e. the bottom set of (Cut_{10}), which is what we had to show. Likewise one obtains ($Cut_{1\hat{1}}$) and ($Cut_{\hat{0}0}$) from ($Cut_{\mathbf{3b}}$) and (W).

For $\mathbf{z} = \mathbf{3n}$ and $\mathbf{z} = \mathbf{4}$ the proof is entirely similar. For $\mathbf{z} = \mathbf{2}$, suppose that we can derive $\Sigma \cup S^{\varphi}_{\mathbf{T}}$ and $\Sigma \cup S^{\varphi}_{\mathbf{F}}$, i.e. the two top sets of the ($Cut_{\mathbf{2}}$) rule. By applying (W) we obtain $\Sigma \cup S^{\varphi}_{\mathbf{T}}, \mathsf{0} : \varphi$ and $\Sigma \cup S^{\varphi}_{\mathbf{F}}, \mathsf{1} : \varphi$ and when we apply $(Cut_{\hat{1}\hat{0}}) \in CUT_{\mathbf{2}}$ to these two sequents we obtain $\Sigma \cup S^{\varphi}_{\mathbf{B}}$. Hence, we can derive the three top sets of the ($Cut_{\mathbf{3b}}$) rule and so, as $CUT_{\mathbf{3b}} \subseteq CUT_{\mathbf{2}}$, we may derive Σ just as above. The proof of the converse direction is entirely similar to the proof for $\mathbf{z} = \mathbf{3b}$ and is left to the reader. ∎

3.5. Capturing K3, LP, FDE and CL Via a *single* SK Tableau

By turning the **SK** calculus upside-down, a tableau calculus is obtained, as explained in some more detail in Definition 8 below. Tableaux of the $\mathbf{SK}_{\mathsf{tab}}$ calculus that is thus obtained are sets of sets of signed sentences or, as we will also put it, sets of *branches*. The calculus recognizes four distinct closure conditions corresponding to the four notions of **z**-provability, as explained by the following definition.

DEFINITION 8. (The $\mathbf{SK}_{\mathsf{tab}}$ calculus) The tableau rules of the $\mathbf{SK}_{\mathsf{tab}}$ *calculus* are the bottom-up versions of the operational sequent rules of the **SK** calculus. With $\mathbf{z} \in \mathbf{Z}$, a branch $\mathscr{B}$ of a tableau is $\mathbf{z}$-*closed* if the bottom set of some initial sequent rule $(R_{xy}) \in R_{\mathbf{z}}$ is a subset of $\mathscr{B}$. A tableau is $\mathbf{z}$-closed if all its branches are $\mathbf{z}$-closed and a set of signed sentences Θ is said have a $\mathbf{z}$-closed tableau just in case some finite $\Theta_0 \subseteq \Theta$ has a $\mathbf{z}$-closed tableau.

The **SK** calculus and the $\mathbf{SK}_{\mathsf{tab}}$ calculus are really two sides of the same coin, as the following proposition attests.

PROPOSITION 19. *A set of signed sentences is* $\mathbf{z}$*-provable iff it has a* $\mathbf{z}$*-closed tableau.*

PROOF. Note that it suffices to consider finite sets of signed sentences. So let Θ be a finite set of signed sentences. We use induction on the total number n of connectives occurring in Θ.

$\Rightarrow$ Let $n = 0$ and suppose that Θ is $\mathbf{z}$-provable. Then Θ is a set of signed propositional atoms which is the conclusion of some rule in $R_{\mathbf{z}}$. As Θ only contains propositional atoms, the unique tableau of Θ has Θ as its sole branch and this tableau is $\mathbf{z}$-closed as Θ is $\mathbf{z}$-closed. Let $n > 0$ and suppose that Θ is $\mathbf{z}$-provable. As $n > 0$, Θ can be written as Σ, θ where θ is some signed sentence containing at least one connective and $\theta \notin \Sigma$. We illustrate how it can be shown that Θ has a $\mathbf{z}$-closed tableau by considering two representative cases. (i) Suppose that $\Theta = \Sigma, \mathsf{1} : \varphi \wedge \psi$. Note that the $\mathbf{z}$-provability of Θ implies that $\Sigma, \mathsf{1} : \varphi, \mathsf{1} : \psi$ is $\mathbf{z}$-provable and so, in virtue of the induction hypothesis, $\Sigma, \mathsf{1} : \varphi, \mathsf{1} : \psi$ has a $\mathbf{z}$-closed tableau $\mathscr{T}$. Define $\mathscr{T}' = \{B \cup \{\mathsf{1} : \varphi \wedge \psi\} \mid B \in \mathscr{T}\}$ and observe that $\mathscr{T}'$ is a $\mathbf{z}$-closed tableau of Θ. (ii) Suppose that $\Theta = \Sigma, \mathsf{0} : \varphi \wedge \psi$. Then, by the same argument as above, $\Sigma, \mathsf{0} : \varphi$ and $\Sigma, \mathsf{0} : \psi$ have $\mathbf{z}$-closed tableaux $\mathscr{T}$ and $\mathscr{U}$ respectively. Observe that $\mathscr{V} = \{B \cup \{\mathsf{0} : \varphi \wedge \psi\} \mid B \in \mathscr{T} \cup \mathscr{U}\}$ is a $\mathbf{z}$-closed tableau of Θ.

$\Leftarrow$ Let $n = 0$ and suppose that Θ has a $\mathbf{z}$-closed tableau $\mathscr{T}$. As Θ only contains propositional atoms, we have that $\mathscr{T} = \{\Theta\}$ and so, as $\mathscr{T}$ is $\mathbf{z}$-closed, so is Θ. As Θ is $\mathbf{z}$-closed, it is the conclusion of some rule in $R_{\mathbf{z}}$ and hence, Θ is $\mathbf{z}$-provable. Let $n > 0$ and suppose that Θ has a $\mathbf{z}$-closed tableau. As $n > 0$, Θ can be written as Σ, θ where θ is some signed sentence containing at least one connective and $\theta \notin \Sigma$. We illustrate how it can be shown that Θ is $\mathbf{z}$-provable by considering two representative cases. (i) Suppose that $\Theta = \Sigma, \mathsf{1} : \varphi \wedge \psi$. From the fact that Θ has a $\mathbf{z}$-closed tableau, it readily follows that $\Theta' := \Sigma, \mathsf{1} : \varphi, \mathsf{1} : \psi$ has a $\mathbf{z}$-closed tableau. As Θ' contains fewer than n connectives, it follows from the induction hypothesis that Θ' is $\mathbf{z}$-provable and as by applying rule $(\wedge_1)$ to Θ' we obtain Θ, it follows that Θ is $\mathbf{z}$-provable. (ii) Suppose that $\Theta = \Sigma, \mathsf{0} : \varphi \wedge \psi$. From the fact that Θ has a $\mathbf{z}$-closed tableau, it readily follows that both $\Theta' := \Sigma, 0 : \varphi$ and $\Theta'' := \Sigma, 0 : \psi$ have $\mathbf{z}$-closed tableaux. It follows from the induction hypothesis that Θ' and Θ'' are $\mathbf{z}$-provable and as by applying rule $(\wedge_0)$ to Θ' and Θ'' we obtain Θ, it follows that Θ is $\mathbf{z}$-provable. ■

A convenient property of the $\mathbf{SK}_{\mathsf{tab}}$ calculus is that, in order to check whether an argument is, respectively, **K3**, **LP**, **FDE** or **CL** valid, it suffices to check whether a *single* tableau is $\mathbf{z}$-closed (for variable $\mathbf{z}$). To see this, first observe that each of these familiar four consequence relations can be expressed as a regular Strong Kleene Generalization:

$$\models^{\mathbf{2}}_{11} = \models_{CL} \qquad \models^{\mathbf{3b}}_{11} = \models_{LP} \qquad \models^{\mathbf{3n}}_{11} = \models_{K3} \qquad \models^{\mathbf{4}}_{11} = \models_{FDE}$$

And so, it follows from Propositions 14 and 19 that in order to check, say, whether the argument $\neg p, p \vee q \models \neg r \vee (r \wedge q)$ is valid according to one of the familiar four logics, it suffices to check whether $\{1 : \neg p, 1 : p \vee q, \hat{1} : \neg r \vee (r \wedge q)\}$ has a $\mathbf{z}$-closed tableau for the appropriate value of $\mathbf{z}$. Consider Figure 2 below.

The reader may care to verify that the $\mathbf{z}$-closure of the four branches of the tableau of Figure 2 is as indicated in Table 2 below.

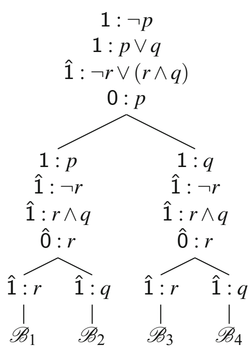

Figure 2. (Completed) tableau of $\{1 : \neg p, 1 : p \vee q, \hat{1} : \neg r \vee (r \wedge q)\}$

Table 2. $\mathbf{z}$-closure of the four branches of the tableau of Figure 2

	$\mathscr{B}_1$	$\mathscr{B}_2$	$\mathscr{B}_3$	$\mathscr{B}_4$
$\mathbf{z} = \mathbf{2}$	Closed	Closed	Closed	Closed
$\mathbf{z} = \mathbf{3n}$	Closed	Closed	Open	Closed
$\mathbf{z} = \mathbf{3b}$	Closed	Open	Closed	Closed
$\mathbf{z} = \mathbf{4}$	Open	Open	Open	Closed

Indeed, as $\{1 : \neg p, 1 : p \vee q, \hat{1} : \neg r \vee (r \wedge q)\}$ only has a $\mathbf{z}$-closed tableau for $\mathbf{z} = \mathbf{2}$, the argument $\neg p, p \vee q \models \neg r \vee (r \wedge q)$ is classically valid but neither valid according to **K3**, **LP** or **FDE**.

4. Two Other Sequent Calculi for the Strong Kleene Generalizations

In this section we will construct two other uniform sequent calculi for the $\models^{\mathbf{z}}_{xy}$ relations by applying the general methods of Baaz et al. [6] to construct sequent calculi for a many-valued logic. Sequents of the calculi that are obtained by these methods are sets of signed sentences of form $x : \varphi$, where a sign x is an element of $[\mathbf{4}] := \{\mathbf{t}, \mathbf{b}, \mathbf{n}, \mathbf{f}\}$ that codes for the corresponding value of $\mathbf{4}$. More generally, when $S \subseteq \mathbf{4}$, we will use $[S] \subseteq [\mathbf{4}]$ to denote the set of signs that code for the values of S.

4.1. Constructing Two Sequent Calculi by the Method of Baaz et al.

In [6], Baaz et al. show that for each m-valued logic, two *dual* sequent calculi can be given that are associated with two different notions of validity (and satisfaction); $\mathscr{P}$(ositive) and $\mathscr{N}$(egative) validity. The notions generalize the two ways (sketched in Sect. 3.1) in which the validity of a classical provable sequent can be interpreted, to many-valued logics. The species of $\mathscr{P}$- and $\mathscr{N}$-satisfaction relevant for this paper are defined below.

Definition 9. We say that $V \in \mathbf{V_4}$ *$\mathscr{P}$-satisfies* a sequent Θ just in case some $x : \varphi \in \Theta$ fulfils condition (16).

$$\begin{aligned} x = \mathbf{t} \implies V(\varphi) = \mathbf{T}, \quad x = \mathbf{f} \implies V(\varphi) = \mathbf{F} \\ x = \mathbf{b} \implies V(\varphi) = \mathbf{B}, \quad x = \mathbf{n} \implies V(\varphi) = \mathbf{N} \end{aligned} \tag{16}$$

And we say that V *$\mathscr{N}$-satisfies* Θ just in case some $x : \varphi \in \Theta$ *does not* fulfil condition (16). Further, with $\mathbf{z} \in \mathbf{Z}$, Θ is said to be *$\mathscr{P}_{\mathbf{z}}$-valid* (*$\mathscr{N}_{\mathbf{z}}$-valid*) just in case every $V \in \mathbf{V_z}$ $\mathscr{P}$-satisfies ($\mathscr{N}$-satisfies) Θ.

And so, when a valuation V *$\mathscr{N}$-dissatisfies* Θ, i.e. when V does not $\mathscr{N}$-satisfy Θ, *every* $x : \varphi$ in Θ fulfils condition (16). Hence, the notion of satisfaction that is at stake in the **SK** calculus (cf. Definition 6) may be described as a generalization of the notion of $\mathscr{N}$-dissatisfaction to a set-up where signs are allowed to code for sets of truth values rather than for values as such. Despite the similarity in their underlying notions of satisfaction, however, the **SK** calculus and the calculus associated with the notion of $\mathscr{N}$-(dis)satisfaction differ widely from one another, as we will see below.

In general, a $\mathscr{P}$*-calculus* ($\mathscr{N}$*-calculus*) for a many-valued logic is a sequent calculus that is sound and complete with respect to $\mathscr{P}$-validity ($\mathscr{N}$-validity). Baaz et al. show that for each many-valued logic, $\mathscr{P}$-and $\mathscr{N}$-calculi can be given. Their argument crucially relies on the notion of a $\mathscr{P}$*-admissible* ($\mathscr{N}$*-admissible*) *rule for a logical connective at a (sign for a) truth value*. In our case,[8] these notions are defined as follows.

Definition 10. With $\oplus$ an n-ary connective of $\mathscr{L}$ (so $n = 1$ or $n = 2$), a sequent rule of the following form:

$$\frac{\Sigma, \Delta_1 \quad \ldots \quad \Sigma,\ \Delta_i \quad \ldots \quad \Sigma, \Delta_m}{\Sigma,\ x : \oplus(\varphi_1, \ldots \varphi_n)} \qquad \text{where } y : \sigma \in \Delta_j \Rightarrow \sigma \in \{\varphi_1, \ldots \varphi_n\},$$

is called a *rule for* $\oplus$ *at* x. Such a rule is $\mathscr{P}$*-admissible* ($\mathscr{N}$*-admissible*) if, for every $V \in \mathbf{V_4}$, V $\mathscr{P}$-satisfies ($\mathscr{N}$-satisfies) all top sets of the rule if and only if V $\mathscr{P}$-satisfies ($\mathscr{N}$-satisfies) the bottom set of the rule.

As Baaz et al. show, $\mathscr{P}$-admissible ($\mathscr{N}$-admissible) rules for a connective at a truth value exist for each many-valued logic—they present a construction of such a rule in terms of the truth tables of a many-valued logic—but they are far from unique. Moreover, the authors show that, for $X \in \{\mathscr{P}, \mathscr{N}\}$, when a *complete* set of X-admissible rules—i.e. the set contains one X-admissible rule for each connective at each truth value—is augmented with a Weakening rule and with X*-initial sequent rules*, the resulting set of rules constitutes a cut-free X-calculus. The Weakening rule that is relevant for our purposes will be denoted by ($\mathscr{W}$):[9]

$$\frac{\Sigma}{\Sigma'}\ (\mathscr{W}) \quad \text{where } \Sigma \subseteq \Sigma'$$

The $\mathscr{P}$-initial sequent rule of a many-valued logic basically states that a sentence always has to take one of the truth values of the logic under

[8] The $\mathscr{P}$-calculi and $\mathscr{N}$-calculi that are considered by Baaz et al. have what Negri and von Plato ([21]) call *independent contexts* whereas the $\mathscr{P}$-calculi and $\mathscr{N}$-calculi as presented in this paper have what they call *shared contexts*. For instance, the $\mathscr{P}$-admissible rule for $\wedge$ at $\mathbf{t}$ of this paper's $\mathbf{SK}^{\mathscr{P}}$ calculus (in the appendix denoted as $(\wedge^{\mathscr{P}}_{\mathbf{t}})$) allows us to infer $\Sigma, \mathbf{t} : \varphi \wedge \psi$ from $\Sigma, \mathbf{t} : \varphi$ and $\Sigma, \mathbf{t} : \psi$. The sequents involved in the $(\wedge^{\mathscr{P}}_{\mathbf{t}})$ rule have the shared context Σ. When reformulated as a rule with independent contexts, the rule allows us to infer $\Sigma_1, \Sigma_2, \mathbf{t} : \varphi \wedge \psi$ from $\Sigma_1, \mathbf{t} : \varphi$ and $\Sigma_2, \mathbf{t} : \psi$. Induction on proof complexity readily reveals that the $\mathscr{P}$-calculi ($\mathscr{N}$-calculi) with independent contexts as considered by Baaz et al. prove the same sequents as the $\mathscr{P}$-calculi ($\mathscr{N}$-calculi) with shared contexts as presented in this paper.

[9] Indeed, the only difference between ($\mathscr{W}$) and (W) is the kind of sets of signed sentences that they manipulate.

consideration. The $\mathscr{P}$- initial sequent rules relevant for our purposes will be denoted as $(R^{\mathscr{P}}_{\mathbf{z}})$, where for each $\mathbf{z} \in \mathbf{Z}$:[10]

$$\frac{}{\{x : \varphi \mid x \in [\mathbf{z}]\}}\,(R^{\mathscr{P}}_{\mathbf{z}})$$

Indeed, the bottom set of each rule $(R^{\mathscr{P}}_{\mathbf{z}})$ is $\mathscr{P}_{\mathbf{z}}$-valid and basically states that according to a valuation in $\mathbf{V}_{\mathbf{z}}$, a sentence always has to take some value in $\mathbf{z}$.

Whereas a $\mathscr{P}$-initial sequent rule roughly states that a sentence has to take some truth value, the $\mathscr{N}$-initial sequent rules of a many-valued logic (jointly) state that a sentence cannot take two distinct ones. The sets of $\mathscr{N}$-initial sequent rules that are relevant for our purposes will contain rules of form $(R^{\mathscr{N}}_{xy})$:

$$\frac{}{x : \varphi,\ y : \varphi}\,(R^{\mathscr{N}}_{xy})$$

The first set of $\mathscr{N}$-initial sequent rules that we define is $R^{\mathscr{N}}_{\mathbf{4}}$:

$$R^{\mathscr{N}}_{\mathbf{4}} = \{(R^{\mathscr{N}}_{xy}) \mid x, y \in [\mathbf{4}], x \neq y\}$$

The other three sets are then defined as follows:

$$R^{\mathscr{N}}_{\mathbf{3b}} = R^{\mathscr{N}}_{\mathbf{4}} \cup \{(R^{\mathscr{N}}_{\mathbf{nn}})\} \qquad R^{\mathscr{N}}_{\mathbf{3n}} = R^{\mathscr{N}}_{\mathbf{4}} \cup \{(R^{\mathscr{N}}_{\mathbf{bb}})\} \qquad R^{\mathscr{N}}_{\mathbf{2}} = R^{\mathscr{N}}_{\mathbf{3b}} \cup R^{\mathscr{N}}_{\mathbf{3n}}$$

With $(\mathscr{W})$eakening and the initial sequent rules in place, what remains to be done—in order to apply the results of Baaz et al. to the case at hand—is the specification of $\mathscr{P}$- and $\mathscr{N}$-admissible rules[11] for each $\mathscr{L}$ connective at each sign $x \in [\mathbf{4}]$.

In order to find a $\mathscr{P}$-admissible rule for $\wedge$ at $\mathbf{f}$, we try to express $V(\varphi \wedge \psi) = \mathbf{F}$ as a conjunction of disjunctions of statements of form $V(\sigma) = \mathbf{X}$, with $\sigma \in \{\varphi, \psi\}$ and $\mathbf{X} \in \mathbf{4}$. As the reader may care to verify, expression (17) gives us what we want, as $V(\varphi \wedge \psi) = \mathbf{F}$ if and only if

$$\begin{aligned} &(V(\varphi) = \mathbf{F} \text{ or } V(\psi) = \mathbf{F} \text{ or } V(\varphi) = \mathbf{B} \text{ or } V(\psi) = \mathbf{B}) \quad \text{and} \\ &(V(\varphi) = \mathbf{F} \text{ or } V(\psi) = \mathbf{F} \text{ or } V(\varphi) = \mathbf{N} \text{ or } V(\psi) = \mathbf{N}) \end{aligned} \tag{17}$$

From (17), it readily follows that $(\wedge^{\mathscr{P}}_{\mathbf{f}})$ is a $\mathscr{P}$-admissible rule for $\wedge$ at $\mathbf{f}$.

$$\frac{\Sigma,\ \mathbf{f} : \varphi, \mathbf{f} : \psi, \mathbf{b} : \varphi, \mathbf{b} : \psi \qquad \Sigma,\ \mathbf{f} : \varphi, \mathbf{f} : \psi, \mathbf{n} : \varphi, \mathbf{n} : \psi}{\Sigma,\ \ \mathbf{f} : \varphi \wedge \psi}\,(\wedge^{\mathscr{P}}_{\mathbf{f}})$$

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[10] Remember that $[\mathbf{z}]$ is the set of signs coding for the values of $\mathbf{z}$.

[11] Although Baaz et al. show that such rules can always be constructed from the truth-tables of the many-valued logic under consideration, to actually carry out that construction typically results in complex and cumbersome sequent rules.

It is interesting to compare $(\wedge^{\mathscr{P}}_{\mathbf{f}})$ with $(\wedge^{\mathscr{N}}_{\mathbf{f}})$, an $\mathscr{N}$-admissible rule for $\wedge$ at $\mathbf{f}$ that is defined in accordance with the following rationale. First we seek to express $V(\varphi \wedge \psi) \neq \mathbf{F}$ as a conjunction of disjunctions of statements of form $V(\sigma) \neq \mathbf{X}$, with $\sigma \in \{\varphi, \psi\}$ and $\mathbf{X} \in \mathbf{4}$. Equivalently then, we may also express $V(\varphi \wedge \psi) = \mathbf{F}$ as a disjunction of conjunctions of statements of form $V(\sigma) = \mathbf{X}$ (with σ and $\mathbf{X}$ as before), which is more convenient. As the reader may care to verify, (18) gives us what we want, as

$$V(\varphi \wedge \psi) = \mathbf{F} \iff \begin{cases} V(\varphi) = \mathbf{F}, \text{ or} \\ V(\psi) = \mathbf{F}, \text{ or} \\ (V(\varphi) = \mathbf{B} \text{ and } V(\psi) = \mathbf{N}), \text{ or} \\ (V(\varphi) = \mathbf{B} \text{ and } V(\psi) = \mathbf{N}). \end{cases} \tag{18}$$

And so, by negating both sides of the equivalence of (18), we see that $(\wedge^{\mathscr{N}}_{\mathbf{f}})$ is an $\mathscr{N}$-admissible rule for $\wedge$ at $\mathbf{f}$.

$$\frac{\Sigma,\ \mathbf{f}:\varphi, \qquad \Sigma,\ \mathbf{f}:\psi \qquad \Sigma,\ \mathbf{b}:\varphi,\ \mathbf{n}:\psi \qquad \Sigma,\ \mathbf{n}:\varphi,\ \mathbf{b}:\psi}{\Sigma,\ \ \mathbf{f}:\varphi \wedge \psi}\,(\wedge^{\mathscr{N}}_{\mathbf{f}})$$

According to the same rationale we define, in the appendix, a $\mathscr{P}$-admissible rule $(\oplus^{\mathscr{P}}_{x})$ and $\mathscr{N}$-admissible rule $(\oplus^{\mathscr{N}}_{x})$ for each $\mathscr{L}$ connective $\oplus$ and each $x \in [\mathbf{4}]$. We have thus defined a complete set of $\mathscr{P}$-admissible ($\mathscr{N}$-admissible) rules and so we may apply the results of Baaz et al. to the case at hand, which we do in Theorem 2 below. But first, we explicitly define the $\mathbf{SK}^{\mathscr{P}}$calculus and the $\mathbf{SK}^{\mathscr{N}}$calculus.

DEFINITION 11. (The $\mathbf{SK}^{\mathscr{P}}$and $\mathbf{SK}^{\mathscr{N}}$calculus) The rules of the $\mathbf{SK}^{\mathscr{P}}$ *calculus* are as follows.

- The Weakening rule $(\mathscr{W})$.
- The initial sequent rules $(R^{\mathscr{P}}_{\mathbf{2}})$, $(R^{\mathscr{P}}_{\mathbf{3b}})$, $(R^{\mathscr{P}}_{\mathbf{3n}})$ and $(R^{\mathscr{P}}_{\mathbf{4}})$.
- The operational sequent rules $(\oplus^{\mathscr{P}}_{x})$ as defined in the appendix.

With $\mathbf{z} \in \mathbf{Z}$, the $\mathbf{SK}^{\mathscr{P}}_{\mathbf{z}}$ *calculus* has $(\mathscr{W})$, the rules $(\oplus^{\mathscr{P}}_{x})$ and $(R^{\mathscr{P}}_{\mathbf{z}})$ as its rules. A sequent Θ is said to be *$\mathscr{P}_{\mathbf{z}}$-provable* if some finite $\Theta_0 \subseteq \Theta$ has a proof tree respecting the rules of the $\mathbf{SK}^{\mathscr{P}}_{\mathbf{z}}$ calculus.

The rules of the $\mathbf{SK}^{\mathscr{N}}$ *calculus* are as follows.

- The Weakening rule $(\mathscr{W})$.
- The initial sequent rules contained in $R^{\mathscr{N}}_{\mathbf{2}}$.
- The operational sequent rules $(\oplus^{\mathscr{N}}_{x})$ as defined in the appendix.

With $\mathbf{z} \in \mathbf{Z}$, the $\mathbf{SK}^{\mathscr{N}}_{\mathbf{z}}$ *calculus* has $(\mathscr{W})$, the rules $(\oplus^{\mathscr{N}}_{x})$ and the rules in $R^{\mathscr{N}}_{\mathbf{z}}$ as its rules. A sequent Θ is said to be $\mathscr{N}_{\mathbf{z}}$*-provable* if some finite $\Theta_0 \subseteq \Theta$ has a proof tree respecting the rules of the $\mathbf{SK}^{\mathscr{N}}_{\mathbf{z}}$ calculus.

We have the following soundness and completeness result.

THEOREM 2. *A sequent Θ is $\mathscr{P}_{\mathbf{z}}$-provable ($\mathscr{N}_{\mathbf{z}}$-provable) iff Θ is $\mathscr{P}_{\mathbf{z}}$-valid ($\mathscr{N}_{\mathbf{z}}$-valid).*

PROOF. This follows from the main result of Baaz et al [6]. ■

Although the $\mathbf{SK}^{\mathscr{P}}$and $\mathbf{SK}^{\mathscr{N}}$calculus are cut-free calculi, it follows from the result of Baaz et al that admissible (in a sense resembling that of Proposition 17) cut-rules for both calculi can easily be added.[12] Further, the authors show that tableaux versions of the $\mathbf{SK}^{\mathscr{P}}$and $\mathbf{SK}^{\mathscr{N}}$calculus can be obtained by "turning them upside down".

In light of Theorem 2, to capture the Strong Kleene Generalizations syntactically in terms of the $\mathbf{SK}^{\mathscr{P}}$ calculus ($\mathbf{SK}^{\mathscr{N}}$ calculus) we must express these relations in terms of $\mathscr{P}_{\mathbf{z}}$-validity ($\mathscr{N}_{\mathbf{z}}$-validity). Propositions 20 and 21 show how this works out for the $\mathbf{SK}^{\mathscr{P}}$ calculus and $\mathbf{SK}^{\mathscr{N}}$ calculus respectively.

PROPOSITION 20. *With $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$ and $\mathbf{z} \in \mathbf{Z}$, $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ iff*

$$\bigcup_{s\in[\mathbf{z}-x]} s : \Gamma \ \cup \{t : \varphi \mid t \in [y \cap \mathbf{z}]\} \quad \textit{is } \mathscr{P}_{\mathbf{z}}\textit{-provable}$$

PROOF. $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ iff, for every $V \in \mathbf{V}_{\mathbf{z}}$, $V(\gamma) \in \mathbf{z} - x$ for some $\gamma \in \Gamma$ or $V(\varphi) \in y \cap \mathbf{z}$. Hence, $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ iff the sequent $\bigcup_{s\in[\mathbf{z}-x]} s : \Gamma \ \cup \{t : \varphi \mid t \in [y \cap \mathbf{z}]\}$ is $\mathscr{P}_{\mathbf{z}}$-valid which according to Theorem 2 holds iff the sequent is $\mathscr{P}_{\mathbf{z}}$ provable. ■

PROPOSITION 21. *With $x, y \in \{1, \hat{0}, \mathbf{t}, \hat{\mathbf{f}}\}$ and $\mathbf{z} \in \mathbf{Z}$, $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ iff*

$$\forall f \in [x \cap \mathbf{z}]^{\Gamma} \ \forall s \in [\mathbf{z} - y], \quad \{f(\gamma) : \gamma \mid \gamma \in \Gamma\} \cup \{s : \varphi\} \quad \textit{is } \mathscr{N}_{\mathbf{z}}\textit{-provable}$$

PROOF. $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ iff, for every $V \in V_{\mathbf{z}}$, $V(\gamma) \notin x \cap \mathbf{z}$ for some $\gamma \in \Gamma$ or $V(\varphi) \notin \mathbf{z} - y$. Thus, with f and s as above, $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ iff all sequents of form $\{f(\gamma) : \gamma \mid \gamma \in \Gamma\} \cup \{s : \varphi\}$ are $\mathscr{N}_{\mathbf{z}}$-valid iff (cf. Theorem 2) all these sequents are $\mathscr{N}_{\mathbf{z}}$-provable. ■

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[12]In fact, Baaz et al. consider $\mathscr{P}$- and $\mathscr{N}$-calculi with cut rule(s) and then show that any provable sequent in their calculi has a cut-free proof.

Proposition 21 tells us that, in the $\mathbf{SK}^{\mathscr{N}}$ calculus, a proof of $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ may consist of multiple proof trees. In fact, an immediate corollary of Proposition 21 is that, when Γ is a finite set of premisses, the number of proof trees that make up a proof of $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ is equal to $|\mathbf{z} - y| \cdot |x \cap \mathbf{z}|^{|\Gamma|}$. For example, an **FDE** proof of $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ consists of $2 \cdot 2^{|\Gamma|}$ proof trees, an **LP** proof of $2^{|\Gamma|}$ trees, an **ETL** proof of 3 trees and a **K3** proof of 2 proof trees.[13]

5. Reflecting on the SK Calculus

In this section we reflect on the results that were achieved in earlier sections. In Sect. 5.1 we favourably compare the **SK** calculus to both the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus. In Sect. 5.2 we compare the **SK** calculus with 2-sided sequent calculi that have been proposed for some of the more familiar Strong Kleene Generalizations. In Sect. 5.3 we hint at the philosophical significance of the **SK** calculus.

5.1. Comparing the SK Calculus with the $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ Calculus

A slight advantage of the $\mathbf{SK}^{\mathscr{P}}$ calculus over the **SK** calculus is that the former calculus requires a single proof tree whereas the latter require $2^{|\Gamma|}$ proof trees (cf. Proposition 16) to establish a proof of $\Gamma \models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}} \varphi$. However, for all relations $\models^{\mathbf{z}}_{xy}$ other than $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$, both calculi requires a single proof tree to establish a proof of $\Gamma \models^{\mathbf{z}}_{xy} \varphi$. Moreover, we do not think that the mentioned advantage of the $\mathbf{SK}^{\mathscr{P}}$ calculus is a genuine advantage. For the results in Sect. 2 testify (see e.g. (9) and (10)) that $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$ is (at least *prima facie*) a highly unattractive consequence relation. In fact, we feel that the results in Sect. 2 justify a similar judgement with respect to all "unfamiliar" Strong Kleene Generalizations, by which we mean all relations other than *the Familiar Four*: **CL**, **K3**, **LP**, and **FDE**. In the literature, none of the unfamiliar relations has been advocated for—Pietz and Rivieccio [23] investigate but do not advocate **ETL**—and the formal properties of these relations as explored in Sect. 2 strongly suggest that a (philosophical) defence of any of the unfamiliar relations is not expected to be forthcoming. On the other hand, **CL**,

[13] With $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and when $\{\gamma_1, \ldots, \gamma_n\} \models^{\mathbf{z}}_{xy} \varphi \iff \gamma_1 \wedge \cdots \wedge \gamma_n \models^{\mathbf{z}}_{xy} \varphi$, i.e. when the *Conjunction Equivalence* (CE) holds, we may, by proving the sequent associated with the r.h.s. of CE, reduce the number of proof trees required to establish a proof of $\Gamma \models^{\mathbf{z}}_{xy} \varphi$ in the $\mathbf{SK}^{\mathscr{N}}$ calculus to $|\mathbf{z} - y| \cdot |x \cap \mathbf{z}|$. With the exception of $\models^{\mathbf{4}}_{\hat{\mathbf{f}}\hat{\mathbf{f}}}$, CE holds for all the relations $\models^{\mathbf{z}}_{xy}$.

K3, **LP**, and (albeit to a lesser extent) **FDE**, have been advocated for and used by various authors in the literature. As Hjortland [18] puts it:

> And indeed, classical logic, **K3** and **LP** are systems with interesting applications in philosophy, e.g. for reasoning with semantic paradoxes, vagueness, and presuppositions. [18, p. 371]

We concur and submit that both as a relevance logic and as a logic that tells us 'how a computer that receives partial and inconsistent information should think', **FDE** also has interesting applications. Given the interest in the Familiar Four logics and the disinterest in the unfamiliar $\models^{\mathbf{z}}_{xy}$ relations, a relevant comparison of the **SK**, $\mathbf{SK}^{\mathscr{P}}$ and $\mathbf{SK}^{\mathscr{N}}$ calculus can safely neglect the unfamiliar relations. At any rate, we will compare the three calculi only with respect to **CL**, **K3**, **LP**, and **FDE**.

According to Proposition 20, to prove that an argument is valid according to one of the Familiar Four logics always takes a single proof tree in the $\mathbf{SK}^{\mathscr{P}}$ calculus, whereas it follows from Proposition 21 that such a proof typically consist of multiple proof trees in the $\mathbf{SK}^{\mathscr{N}}$ calculus. As the complexity of the rules of the $\mathbf{SK}^{\mathscr{P}}$ calculus and $\mathbf{SK}^{\mathscr{N}}$ calculus are by and large comparable, we take it that the $\mathbf{SK}^{\mathscr{P}}$ calculus is preferable to the $\mathbf{SK}^{\mathscr{N}}$ calculus as a calculus for the Familiar Four logics.

The most important (formal) advantage of the **SK** calculus over the $\mathbf{SK}^{\mathscr{P}}$ calculus is that its rules and accordingly its proof trees are much simpler. Whereas both calculi require a single proof tree to establish that an argument is valid according to one of the Familiar Four logics, the height and the number of branches of the required proof tree in the **SK** calculus will be far less than those of the required tree in the $\mathbf{SK}^{\mathscr{P}}$ calculus, as a comparison of the rules of the calculi reveals. For instance, in the $\mathbf{SK}^{\mathscr{P}}$ calculus the binary connectives are provided with sequent rules that either involve two or three distinct top sets whereas in the **SK** calculus the binary connectives are provided with sequent rules that either involve one or two distinct top sets. In this respect, the **SK** calculus resembles (a signed) sequent calculus for classical logic as given by e.g. Smullyan [32].

Another (formal) advantage of the **SK** calculus over the $\mathbf{SK}^{\mathscr{P}}$ calculus is best explained in terms of their associated tableau calculi. In Sect. 3.3, we saw that, in the tableau version of the **SK** calculus, a *single* tableau suffices to assess the validity of an argument in terms of the Familiar Four logics. As the reader may care to verify,[14] to assess the validity of an argument in

[14] For sake of definiteness, the tableau calculus associated with the $\mathbf{SK}^{\mathscr{P}}$ calculus, is given in Appendix.

terms of the Familiar Four logics via the tableau version of the $\mathbf{SK}^{\mathscr{P}}$ calculus requires the construction of 4 tableaux.

In Sect. 4, we observed that the notion of satisfaction associated with the **SK** calculus is a generalization—"from values to sets of values"—of the notion of $\mathscr{N}$-(dis)satisfaction that is constitutive for the $\mathbf{SK}^{\mathscr{N}}$ calculus. Given the sheer reduction in complexity of the calculus that is achieved by this generalization, a number of interesting question arise. What does a calculus for the Familiar Four logics that is obtained by a similar generalization of the notion of $\mathscr{P}$-satisfaction look like? Can we come up with general results, comparable to those of Baaz et al., that show that sequent calculi associated with the two envisaged notions of generalized of $\mathscr{P}$-satisfaction and $\mathscr{N}$-satisfaction always exist? Exploring such questions is left for further research.

5.2. Comparing the SK Calculus with 2-Sided Calculi

It is sometimes argued (e.g. Ripley [28] or Restall [26], see also Sect. 5.3) that the left and right introduction rule of a connective in a 2-sided sequent calculus for classical logic specify the meaning of that connective. The **SK** calculus is a 4-signed (equivalently, 4-sided) sequent calculus, but for **K3**, **LP** and **FDE** various 2-sided sequent calculi have been proposed in the literature (see e.g. Avron [4], Arieli and Avron [3] or Beall [10]). Is a characterization of, say **LP**, by a more standard 2-sided calculus not to be preferred[15] over a (4-sided) characterization by the **SK** calculus? Can't we argue that, just like in the classical case, the left and right introduction rule of a connective in a 2-sided sequent calculus for **LP** specify the meaning of that connective? No we cannot. At least not without much further argument. For not any introduction rule for a connective can be taken to bestow that connective with meaning. How then, do the introduction rules for a connective in a 2-sided sequent calculus for classical logic manage to do this; what is so attractive about these rules? Well, the rules (i) have the subformula property, (ii) they introduce exactly one occurrence of a connective in their conclusion and (iii) no other connective is mentioned anywhere else in their formulation. These properties ensure, amongst others, that the meaning of a connective can be understood independently from the meaning of other connectives which, arguably, is an attractive feature from a meaning-theoretic

[15]It should be noted that, to the best of my knowledge, none of the 2-sided sequent calculi that are proposed in the literature for *some* of the Strong Kleene Generalizations can be used to characterize them *all*.

perspective. However, the rules of none of the 2-sided sequent calculi for **LP** have these attractive properties. For instance, the "left introduction rule for negated conjunctions" of Beall's [9] 2-sided calculus for **LP** fails to have *any* of these properties:

$$\frac{\Gamma,\ \neg A \wedge \neg B \vdash \Delta}{\Gamma,\ \neg(A \wedge B) \vdash \Delta}\,(\neg \wedge \text{Left})$$

To say the least, it is far from clear that and how the rules of Beall's calculus specify the meaning of the logical connectives. And so, pending any further argument, it is not clear why a 2-sided calculus for **LP** would be preferable to the (characterization of **LP** by the) **SK** calculus.[16] Further, a 2-sided Gentzen calculus which has attractive properties (i), (ii) and (iii) is called a *canonical calculus* by Avron and Lev [5]. Although canonical calculi are (philosophically) rather attractive, *it is impossible to give a canonical calculus for any of the non-classical Strong Kleene Generalizations*. Such is an immediate consequence of the following theorem by Avron and Lev:

THEOREM 3. *Let* **G** *be a canonical calculus. Then either* **G** *is inconsistent [i.e. allows one to prove every sequent], or it defines a logic which is a fragment of classical logic, or it has no finite characteristic matrix.*

PROOF. See [5]. ∎

It is interesting to observe that the (operational) rules of the **SK** calculus, albeit in a 4-signed setting, do enjoy properties (i), (ii) and (iii). Moreover, in the next section I argue that (suitably interpreted) the rules of the SK calculus may serve as specifying the meaning of the logical connectives. Hence, although 2-sided sequent calculi are available for some of the Strong Kleene Generalizations, it is not clear (to say the least) that and in which sense they are preferable to the **SK** calculus.

5.3. The Philosophical Significance of the SK Calculus

We put forward two considerations that hint at the potential philosophical relevance of the **SK** calculus.

Our first consideration has to do with the justification of a logic from an inferentialist theory of meaning. In a nutshell, inferentialism is the view that

[16] For what it's worth, perhaps one can argue that a 2-sided calculus is preferable to a many-sided one because the 2 sides correspond to premises and conclusions and hence, that (only) 2-sided calculi resemble the structure of genuine arguments. Although I am not entirely sure of the value of such an argument, I should be noted that "genuine" arguments (arguably) involve a *single* conclusion.

meaning (in particular the meaning of the logical connectives) is determined by correctness of inference. Michael Dummett [15] famously argued that, when starting from such an inferentialist view on meaning, one can justify intuitionistic but not classical logic. However, various authors (e.g. Smiley [31], Rumfitt [30] and Restall [26]) have argued that Dummett's conclusion only follows because his inferentialism does not give *denial* its due. According to these authors, denial should not, *pace* Dummett be understood as the assertion of a negation but rather as a (primitive) speech act on its own, on the same footing as assertion. When one does so, one arrives at a species of inferentialism —called *bilaterlism*— which explains meaning in terms of constraints on assertion *and denial* and from which, so it proponents claim, one can justify classical logic. In particular Restall [26] provides such a justification by showing that the rules of a sequent calculus for classical logic can be interpreted in terms of constraints on assertion and denial. On this interpretation, a sequent $\Gamma \Rightarrow \Delta$ is read as stating that it is incorrect ("out of bounds") to assert all of Γ whilst denying all of Δ and so in particular, an initial sequent $\varphi \Rightarrow \varphi$ states that it is incorrect to both assert and deny the same sentence φ. Interestingly, the **SK** calculus allows us to extends Restall's *bi*lateral account of classical logic to a "quadlateral" account of all the Familiar Four logics as, by acknowledging two kinds of assertion and two corresponding kinds of denial, we may understand **CL**, **K3**, **LP**, and **FDE**, all in terms of (distinct) constraints on our four assertoric notions. To get an idea of what we have in mind, say that one may assert$_1$ a sentence φ if one knows φ to be true whereas one may assert$_{\hat{0}}$ φ if one knows φ to be not false. Dually, one may deny$_1$ φ if one knows φ to be not true whereas one may deny$_{\hat{0}}$ φ if one knows φ to be false. When we think of the signs 1, $\hat{0}$, $\hat{1}$, and 0, as coding for, respectively assertion$_1$, assertion$_{\hat{0}}$, denial$_1$ and denial$_{\hat{0}}$, the rules of the **SK** calculus can be understood in terms of constraints on assertion and denial and the **SK** calculus thus seems to give us the means to widen the scope of applicability of Restall's inferentialist account of meaning. Although clearly lots more need to be said to back up these sketchy remarks (which we will do in a future paper), we take it that at least the potential significance of the **SK** calculus with respect to providing a justification of the Familiar Four logics is clear.[17]

[17] On Restall's interpretation, an initial sequent $\varphi \Rightarrow \varphi$ states that it is incorrect to assert and deny the same sentence. Note that this is akin to an interpretation of sequents along the lines of the notion of $\mathscr{N}$-satisfaction. On the corresponding interpretation along the lines of $\mathscr{P}$-satisfaction, an initial sequent would state that it is always correct to assert or correct to deny a sentence. Interestingly, cases of potential ignorance suggest that the latter interpretation is problematic. Thus, the fact that the notion of satisfaction associated

Finally, we want to point out the potential significance of the **SK** calculus for the now in vogue position of *logical pluralism* by which, following Hjortland [18], we mean the view that there are at least two admissible *all-purpose* logics, i.e. at least two logics that are applicable to reasoning in all discourse domains and which are both correct[18] (or equally good with no better option). Beall and Restall [11] advocate a well-known version of logical pluralism which, so they claim, is independent of meaning-variance: distinct admissible all-purpose logics can assign the same meaning to the logical connectives (and differ only in the meaning they assign to 'valid'). In [18], Hjortland criticizes Beall and Restall's account of pluralism by claiming that its alleged independence of meaning-variance fails, after which he puts forward an account of logical pluralism that does not fall prey to meaning-variance:

> The main part of the paper develops a new notion of logical pluralism with the aim of showing [that there is a notion of logical pluralism that is independent of meaning-variance]. The overarching idea is to think of logical pluralism not as a plurality of logical theories, but as a plurality of consequence relations (and derivability relations) within one and the same logical theory. I call this *intra-theoretic pluralism*. I give an example of how such a theory can be formally achieved using a generalization of sequent calculus with n-sided sequents. [18, p. 356]

The "generalization of sequent calculus with n-sided sequents" that Hjortland is referring to is a three-sided $\mathscr{P}$-calculus (obtained by the methods of Baaz et al.) for **CL**, **K3**, and **LP**. The importance of the calculus for Hjortland's position is that it is a single logical theory (in our words: a uniform calculus) that can be used to characterize **CL**, **K3**, and **LP** by relying on the same operational sequent rules which means, so it is argued, that the three logics assign the same meaning to the logical connectives. As the **SK** calculus (or its $\mathbf{SK_{3b}}$ or $\mathbf{SK_{3n}}$ sub calculus) has these same properties however, it should be clear that this paper's formal work on the **SK** calculus, together with the observations in Sect. 5), is highly relevant for a proper assessment

Footnote 17 continued
with the **SK** calculus is a generalization of the notion of $\mathscr{N}$-(dis)satisfaction may turn out to be philosophically relevant.

[18] As Hjortland explains on page 357, correctness may either be understood according to a descriptive or to a normative standard.

of Hjortland's intra-theoretic pluralism. To provide this assessment in full detail is the topic of further work.

6. Conclusion

By using the notions of exact truth and exact (non)-falsity we defined the class of all Strong Kleene Generalizations (of classical logic). Amongst the Strong Kleene Generalizations are the Familiar Four logics of **CL**, **K3**, **LP** and **FDE**, but also a host of unfamiliar ones. The unfamiliar Strong Kleene Generalizations turn out to have quite some counter intuitive properties, especially at the level of meta-inferences, which makes it hard to imagine that they will be advocated for. However, studying the class of all Strong Kleene Generalizations (is not only interesting of itself but also) sheds novel light on the interrelations between the Familiar Four logics. We developed a uniform sequent calculus, the **SK** calculus, that is sound and complete with respect to all Strong Kleene Generalizations. Although sequent calculi that serve the same purpose can be obtained by applying the general methods of Baaz et al., we showed that the **SK** calculus is preferable to these calculi, as its rules are simpler, its proofs are shorter and as it is better suited to study the interrelation between the Familiar Four logics. Finally, we hinted at the philosophical significance of the **SK** calculus by indicating the role that it can play in debates on *bilateralism* and on *logical pluralism*.

Appendix: The $\mathbf{SK}^{\mathscr{P}}$, the $\mathbf{SK}^{\mathscr{N}}$, and the $\mathbf{SK}^{\mathscr{P}}_{\mathsf{tab}}$ Calculus

The operational sequent rules of the $\mathbf{SK}^{\mathscr{P}}$calculus are defined as follows.
$\mathscr{P}$-admissible rules for $\wedge$:

$$\frac{\Sigma,\ \mathbf{t}:\varphi \qquad \Sigma,\ \mathbf{t}:\psi}{\Sigma,\ \mathbf{t}:\varphi\wedge\psi}\,(\wedge^{\mathscr{P}}_{\mathbf{t}}) \qquad \frac{\Sigma,\ \mathbf{f}:\varphi,\mathbf{f}:\psi,\mathbf{b}:\varphi,\mathbf{b}:\psi \qquad \Sigma,\ \mathbf{f}:\varphi,\mathbf{f}:\psi,\mathbf{n}:\varphi,\mathbf{n}:\psi}{\Sigma,\quad \mathbf{f}:\varphi\wedge\psi}\,(\wedge^{\mathscr{P}}_{\mathbf{f}})$$

$$\frac{\Sigma,\ x:\varphi,x:\psi \qquad \Sigma,\ x:\varphi,\mathbf{t}:\varphi \qquad \Sigma,\ x:\psi,\mathbf{t}:\psi}{\Sigma,\quad x:\varphi\wedge\psi}\,(\wedge^{\mathscr{P}}_{x}) \quad \text{if } x\in\{\mathbf{b},\mathbf{n}\}$$

$\mathscr{P}$-admissible rules for $\vee$:

$$\frac{\Sigma,\ \mathbf{t}:\varphi,\mathbf{t}:\psi,\mathbf{b}:\varphi,\mathbf{b}:\psi \qquad \Sigma,\ \mathbf{t}:\varphi,\mathbf{t}:\psi,\mathbf{n}:\varphi,\mathbf{n}:\psi}{\Sigma,\quad \mathbf{t}:\varphi\vee\psi}\,(\vee^{\mathscr{P}}_{\mathbf{t}}) \qquad \frac{\Sigma,\ \mathbf{f}:\varphi \qquad \Sigma,\ \mathbf{f}:\psi}{\Sigma,\ \mathbf{f}:\varphi\vee\psi}\,(\vee^{\mathscr{P}}_{\mathbf{f}})$$

$$\frac{\Sigma,\ x:\varphi,x:\psi \qquad \Sigma,\ x:\varphi,\mathbf{f}:\varphi \qquad \Sigma,\ x:\psi,\mathbf{f}:\psi}{\Sigma,\quad x:\varphi\vee\psi}\,(\vee^{\mathscr{P}}_{x}) \quad \text{if } x\in\{\mathbf{b},\mathbf{n}\}$$

$\mathscr{P}$-admissible rules for $\neg$:

$$\frac{\Sigma,\ y:\varphi}{\Sigma,\ x:\neg\varphi}\,(\neg^{\mathscr{P}}_x)\qquad \text{if } \langle x,y\rangle \in \{\langle \mathbf{t},\mathbf{f}\rangle, \langle \mathbf{f},\mathbf{t}\rangle, \langle \mathbf{b},\mathbf{b}\rangle, \langle \mathbf{n},\mathbf{n}\rangle\}$$

The operational sequent rules of the $\mathbf{SK}^{\mathscr{N}}$ calculus are defined as follows.
$\mathscr{N}$-admissible rules for $\wedge$:

$$\frac{\Sigma,\ \mathbf{t}:\varphi,\ \mathbf{t}:\psi}{\Sigma,\ \mathbf{t}:\varphi\wedge\psi}\,(\wedge^{\mathscr{N}}_{\mathbf{t}})\quad \frac{\Sigma,\ \mathbf{f}:\varphi \quad \Sigma,\ \mathbf{f}:\psi \quad \Sigma,\ \mathbf{b}:\varphi,\ \mathbf{n}:\psi \quad \Sigma,\ \mathbf{n}:\varphi,\ \mathbf{b}:\psi}{\Sigma,\ \mathbf{f}:\varphi\wedge\psi}\,(\wedge^{\mathscr{N}}_{\mathbf{f}})$$

$$\frac{\Sigma,\ x:\varphi, x:\psi \quad \Sigma,\ x:\varphi, \mathbf{t}:\psi \quad \Sigma, \mathbf{t}:\varphi, x:\psi}{\Sigma,\ x:\varphi\wedge\psi}\,(\wedge^{\mathscr{N}}_x)\qquad \text{if } x\in\{\mathbf{b},\mathbf{n}\}$$

$\mathscr{N}$-admissible rules for $\vee$:

$$\frac{\Sigma,\ \mathbf{t}:\varphi \quad \Sigma,\ \mathbf{t}:\psi \quad \Sigma,\ \mathbf{b}:\varphi,\ \mathbf{n}:\psi \quad \Sigma,\ \mathbf{n}:\varphi,\ \mathbf{b}:\psi}{\Sigma,\ \mathbf{t}:\varphi\vee\psi}\,(\vee^{\mathscr{N}}_{\mathbf{t}})\quad \frac{\Sigma,\ \mathbf{f}:\varphi,\ \mathbf{f}:\psi}{\Sigma,\ \mathbf{f}:\varphi\vee\psi}\,(\vee^{\mathscr{N}}_{\mathbf{f}})$$

$$\frac{\Sigma,\ x:\varphi, x:\psi \quad \Sigma,\ x:\varphi, \mathbf{f}:\psi \quad \Sigma, \mathbf{f}:\varphi, x:\psi}{\Sigma,\ x:\varphi\vee\psi}\,(\vee^{\mathscr{N}}_x)\qquad \text{if } x\in\{\mathbf{b},\mathbf{n}\}$$

$\mathscr{N}$-admissible rules for $\neg$:

$$\frac{\Sigma,\ y:\varphi}{\Sigma,\ x:\neg\varphi}\,(\neg^{\mathscr{N}}_x)\qquad \text{if } \langle x,y\rangle \in \{\langle \mathbf{t},\mathbf{f}\rangle, \langle \mathbf{f},\mathbf{t}\rangle, \langle \mathbf{b},\mathbf{b}\rangle, \langle \mathbf{n},\mathbf{n}\rangle\}$$

The tableau calculus associated with the $\mathbf{SK}^{\mathscr{P}}$ calculus is defined as follows.

DEFINITION 12. (The $\mathbf{SK}^{\mathscr{P}}_{\mathsf{tab}}$ calculus) The tableau rules of the $\mathbf{SK}^{\mathscr{P}}_{\mathsf{tab}}$ *calculus* are the bottom-up versions of the operational sequent rules of the $\mathbf{SK}^{\mathscr{P}}$ calculus. With $\mathbf{z}\in\mathbf{Z}$, a branch $\mathscr{B}$ of a tableau is *$\mathscr{P}_{\mathbf{z}}$-closed* if the bottom set of $(R^{\mathscr{P}}_{\mathbf{z}})$ is a subset of $\mathscr{B}$. A tableau is $\mathscr{P}_{\mathbf{z}}$-closed if all its branches are $\mathscr{P}_{\mathbf{z}}$-closed and a set of signed sentences Θ is said have a $\mathscr{P}_{\mathbf{z}}$-closed tableau just in case some finite $\Theta_0\subseteq\Theta$ has a $\mathscr{P}_{\mathbf{z}}$-closed tableau.

The following proposition will not come as a surprise.

PROPOSITION 22. *A sequent Θ has a $\mathscr{P}_{\mathbf{z}}$-closed tableau if and only if Θ is $\mathscr{P}_{\mathbf{z}}$-provable.*

PROOF. See [6]. ■

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