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ORIGINAL PAPER

Inner and outer approximation of convex sets using alignment

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Abstract We show that there exists, for each closed bounded convex set C in the Euclidean plane with nonempty interior, a quadrangle Q having the following two properties. Its sides support C at the vertices of a rectangle r and at least three of the vertices of Q lie on the boundary of a rectangle R that is a dilation of r with ratio 2. We will prove that this implies that quadrangle Q is contained in rectangle R and that, consequently, the inner approximation r of C has an area of at least half the area of the outer approximation Q of C. The proof makes use of *alignment* or *Schüttelung*, an operation on convex sets.

Keywords Convex figure · Schuettelung · Approximation · Rectangle · Theorem of moreau-rockafellar · Separation of convex sets

1 Introduction

In 1952 Radziszewski ([5,6]) proposed the following result on a *convex figure* C, that is, a closed bounded convex set in the Euclidean plane with nonempty interior.

There exists a rectangle r having its vertices on the boundary of C and having an area of at least half the area of C.

This is sometimes considered a folklore result, as the original proof is rather sketchy. It makes use of an interesting operation on convex figures, called *Schüttelung* (Blaschke) or *déplacement* (Radziszewski), and for which we propose the term *alignment*. In the present paper we prove the following stronger result (see Fig. 1).

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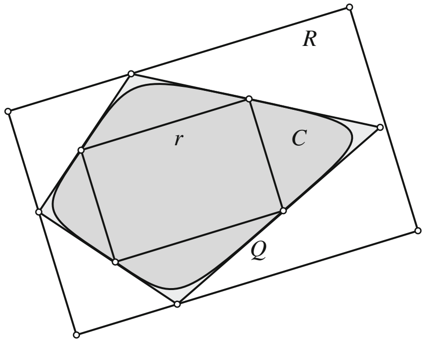

Fig. 1 The theorem

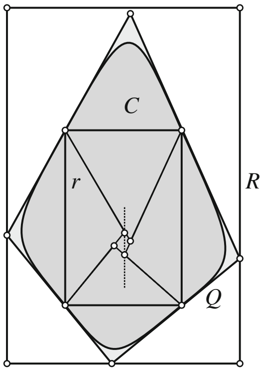

Fig. 2 A proof by folding triangles

Theorem *For each convex figure C there exists a quadrangle Q having the following two properties:*

1. *its sides support C at the vertices of a rectangle r,*
2. *at least three of its vertices lie on the boundary of a rectangle R that is a dilation of r with ratio equal to 2.*

The theorem has the following consequences.

Corollary 1. *The quadrangle Q is contained in the rectangle R,*
2. *the area of the quadrangle Q is at most half the area of the rectangle R,*
3. *the area of the rectangle r is at least half the area of the quadrangle Q.*

Now we give a short derivation of the corollary from the theorem, illustrated by Fig. 2. We assume without loss of generality that the rectangles r and R are upright and that the bottom, left and right hand vertices lie on the boundary of R.

Fold the part of Q that does not belong to the rectangle r—this consists of four triangles—over the sides of r (see Fig. 2). After folding, the four triangles do not overlap. Indeed, to begin with, each pair of adjacent triangles has no overlap after folding as the sum of the angles of these two triangles at their common vertex is equal to the angle of the rectangle r at this point, that is, $\frac{1}{2}\pi$. The left and right hand vertex of Q lie after folding on a vertical line as they lie before folding on the vertical sides of

R and R is a dilation of r with ratio 2. In particular, the left and right hand triangle do not overlap after folding and—using the verticality property—it readily follows that the top and bottom triangle do not overlap after folding (see Fig. 2). The sum of the areas of the four triangles is at most the area of r, as there is no overlap. The total area of the left and right hand triangle equals half the area of r by the verticality property and by the area formulas for triangles and rectangles. It follows that the total area of the top and bottom triangle is less than half the area of r. This readily implies that the top vertex of Q lies inside R, using also that the bottom vertex of Q lies on the boundary of R, that R is a dilation of r with ratio 2 and using again the area formulas for triangles and rectangles. This proves the first statement.

The second and third statements are equivalent as R is a dilation of r with dilation ratio 2 and so the ratio of the areas of R and r is 4. The difference of the areas of Q and r is equal to the sum of the areas of the four triangles and this is at most the area of r, by the non-overlapping property, and this proves the third statement.

The theorem and its corollary describe the close relations between the outer and inner approximations Q, R and r of the convex figure C. The theorem implies the result by Radziszewski. Moreover, it implies a stronger result of Schwarzkopf et al. [8,9] and of Lassak [3]: the existence of a pair of dilated rectangles r, R with $r \subseteq C \subseteq R$ and dilation ratio at most 2. In the present paper the usual outer approximation, the rectangle R, is replaced by a new and tighter one, the quadrangle Q; this gives an improvement of the quotient of the areas of outer and inner approximation of the convex figure by a factor of at least 2 (by the last statement of the corollary).

Fleischer et al. [1] have applied inner and outer approximations of a convex figure C to obtain certificates for the impossibility or possibility of obstacle-avoiding motions of C. For example, if the outer approximation can be moved along a path without hitting a given set of obstacles, then this is also possible for C.

The core of the proof of the theorem is the analysis of a certain optimization problem, the search for a so-called v-maximal parallelogram. After a reformulation of this problem, which involves pushing a convex set into a corner by means of alignment operations, a geometric characterization of the unique solution is given. This information is then pulled back by means of alignment operations to the original problem using the separation theorem for convex sets and a consequence, the theorem of Moreau–Rockafellar, which is the convex calculus rule for the subdifferential of the sum of two convex functions. Some subtlety is involved in this pull back. This involves in each case special arguments; the few arguments that are used repeatedly are collected in a section that precedes the proof of the theorem. An illustration of the subtlety of the situation is given after the proof of the theorem.

At the end of the paper, we make some comments and discuss similar results in the current literature.

2 Alignment and its properties

In this section, alignment operations on convex figures are introduced, and the basic properties of alignment are given. These operation and their properties will be used frequently in the proof of the theorem.

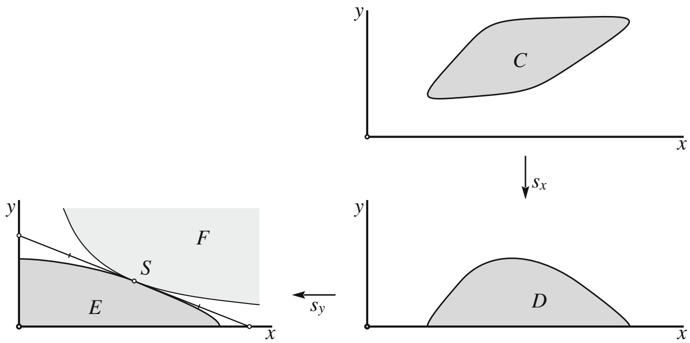

Fig. 3 Alignment from C via D to E

2.1 Definition of alignment

Suppose that the Euclidean plane has been turned into the Cartesian plane by the choice of coordinates. We will need alignment of a convex figure C against the x-axis and against the y-axis. We assume without loss of generality that C lies in the first quadrant. *Alignment* of C against the x-axis gives the set $D = a_x(C)$ for which each vertical line intersects C and D in segments of equal length, and where for D this segment has its lowest point lying on the x-axis (see the right-hand side of Fig. 3). We verify below that D is again a convex figure. Thus alignment against the x-axis transforms C by pushing it downwards in vertical direction against the x-axis. Alignment preserves the area, clearly. Moreover, alignment of an upright rectangle gives a congruent upright rectangle. Furthermore, alignment preserves inclusion: $C_1 \subseteq C_2 \Rightarrow a_x(C_1) \subseteq a_x(C_2)$.

Alignment against the y-axis is defined in an analogous way giving $a_y(C)$.

2.2 Alignment in terms of functions

Now we describe alignment in terms of functions. To this end, we view a convex figure C as the solution set of $f(x) \le y \le g(x), a \le x \le b$ for two numbers $a, b \in \mathbb{R}$ with $a \le b$ and two functions $f, g : [a, b] \to \mathbb{R}$ with f convex and lower semicontinuous (that is, the epigraph $\{(x, r)|r \ge f(x)\}$ is a closed set) and g concave and upper semicontinuous (that is, the hypograph $\{(x, r)|r \le g(x)\}$ is a closed set), and $f \le g, f \ne g$. Then alignment of C against the x-axis gives the solution set $D = a_x(C)$ of $a \le x \le b, 0 \le y \le g(x) - f(x)$. This description shows that D is again a convex figure, as the function $g - f$ is concave, upper semicontinuous, nonnegative and not identically zero.

2.3 Alignment and supporting lines

In the proof of the theorem, we will need the proposition below. This is a novel result that will allow the pullback of supporting lines under alignment. We call the region in the Cartesian plane that consists of all points that lie between two half-lines with a common endpoint an *angle-region*. For each convex set A in the Euclidean plane that has a finite, closed intersection with each vertical line its *generalized alignment* $a_x(A)$ against the x-axis is the set in the Cartesian plane for which each vertical line intersects A and $a_x(A)$ in segments of equal length, and where for $a_x(A)$ this segment has its lowest point lying on the x-axis. We call the region in the Cartesian plane that consists of all points that lie between two half-lines with a common endpoint an *angle-region*.

Proposition *Let C be a convex figure and let B be an angle-region with one leg on the x-axis such that the convex figure $D = a_x(C)$, the result of generalized alignment of C against the x-axis, is inscribed in B. Then there exists an angle-region A such that*

- *C is inscribed in A,*
- *generalized alignment of A against the x-axis gives B, that is, $a_x(A) = B$.*

Now we give a reformulation of the proposition in terms of functions. Let the convex figure C be described by functions $f, g : [a, b] \to \mathbb{R}$ as in the previous section and let l be a nonzero affine (= linear plus constant) function of a real variable that supports the concave function $g - f$ at the point $(u, g(u) - f(u))$. Then there exist affine functions m, n of a real variable such that the following three requirements are satisfied:

1. $l = n - m$,
2. the graph of m is a supporting line at the point $(u, m(u)) = (u, f(u))$ to the convex figure C,
3. the graph of n is a supporting line at the point $(u, n(u)) = (u, g(u))$ to the convex figure C.

The connection between the two formulations is given by letting B be the solution set of $0 \le y \le l(x)$ and A the solution set of $m(x) \le y \le n(x)$.

The proof of this proposition makes use of the convex calculus rule for the subdifferential of the sum of two convex functions, called the theorem of Moreau–Rockafellar (see [7]). Recall that for a convex function φ of one real variable that is defined in a two-sided or one-sided neighborhood U of a number u, the subdifferential of φ in u is defined to be the nonempty set of real numbers $\partial\varphi(u) = \{\eta \in \mathbb{R} | \varphi(\bar{u}) \ge \varphi(u) + \eta(\bar{u} - u) \, \forall \bar{u} \in U\}$. That is, $\eta \in \partial\varphi(u)$ means that the line with slope η that runs through the point on the graph of φ above u, $(u, \varphi(u))$, is a supporting line to the convex function φ at u: this line lies nowhere above the graph of the convex function φ. This can also be expressed by saying that this line is a supporting line at the point $(u, \varphi(u))$ to the epigraph of φ—which is a convex set. If φ is differentiable at u, then the subdifferential consists of the derivative, $\partial\varphi(u) = \{\varphi'(u)\}$; however we are also interested in convex functions φ that are defined in a two-sided neighborhood of u but that are not differentiable at u.

Convex calculus rule (Moreau–Rockafellar). *For two convex functions f_1, f_2 of one real variable that are both defined in the same two-sided or one sided neighborhood of u we have*

$$\partial(f_1 + f_2)(u) = \partial f_1(u) + \partial f_2(u).$$

Here the addition on the left is addition of functions and the addition on the right is addition of subsets of $\mathbb{R}$.

Now we are ready to prove the proposition.

Proof of the proposition Let C and B be described by functions as in the previous section. Let α be the slope of the graph of l. Then $-\alpha \in \partial(f-g)(u)$; the function $f-g$ is convex as $f, -g$ are convex. Therefore, by Moreau–Rockafellar there exist $\beta \in \partial f$ and $\gamma \in \partial(-g)$ such that $\beta + \gamma = -\alpha$. Take the affine function m for which the slope is β and for which $m(u) = f(u)$ and take the affine function n for which the slope is $-\gamma$ and for which $n(u) = g(u)$. These choices satisfy the requirements (1)–(3) on m and n above by the definition of the subdifferential and its interpretation in terms of supporting lines to the epigraph of a convex function.

3 Proof of the theorem

3.1 Outline of the proof

Choose *a unit vector* v and turn the Euclidean plane into the Cartesian plane such that v points vertically upwards. Without loss of generality, we assume that C lies in the first quadrant. The proof consists of eight steps. The vector v and the choice of coordinates will remain fixed during the first five steps of the proof.

In the first four steps, we will construct explicit sets $p = p_C$, $Q = Q_C$, $P = P_C$ in the Euclidean plane depending on v that would satisfy the properties in the theorem for the convex figure C if the word "rectangle" were to be replaced with "parallelogram with two sides vertical" and with p playing the role of r, and P the role of R Moreover we show that p has an appropriate maximality property. To be more precise we will check that the constructed sets have the following properties.

- The set p is a parallelogram that has the so-called *v-maximality property*, which means that it is a parallelogram of maximum area among the parallelograms that are contained in C and that have two vertical sides (that is, sides that are parallel to the fixed unit vector v).

We claim that such a parallelogram is inscribed in C. Indeed, suppose that one of the vertices of the parallelogram p does not lie on the boundary of C. Then one can increase the area of the parallelogram by first carrying out a translation of the vertical side on which this vertex lies—keeping the opposite side fixed—in the following way. One translates this side slightly in its direction in such a way that the other endpoint becomes an interior point of C. Then one moves this side slightly in horizontal direction, away from the opposite side; then the new parallelogram would have a larger area; but this would contradict the maximality property of the original parallelogram. Thus the claim is established.

- The set Q is a quadrangle the sides of which support C at the vertices of the parallelogram p.
- The set P is a parallelogram that is a dilation of p with ratio 2 with all sides except possibly the top side containing a vertex of Q (this implies that Q is contained in R; the proof is essentially the same as the proof of the first statement of the corollary).

In the fifth step we deal with the non-uniqueness of the parallelogram p. Then, in the remaining three steps we prove that there exists a choice of the unit vector v for which the parallelogram p can be chosen to be a rectangle. This will conclude the proof of the theorem.

3.2 Details of the proof

Step 1: Pushing C into a corner by means of repeated alignment

Let $D = a_x(C)$ and let $E = a_y(D) = a_y(a_x(C))$ (see Fig. 3). For later use, we give the description of C, D and E in terms of functions. Let C be the solution set of $a \leq x \leq b, f(x) \leq y \leq g(x)$ for suitable functions $f, g : [a, b] \to \mathbb{R}$. Then D is the solution set of $a \leq x \leq b, 0 \leq y \leq g(x) - f(x)$. Now view D as the solution set of $h(y) \leq x \leq k(y), 0 \leq y \leq c$ for a number $c \in (0, \infty)$ and for two functions $h, k : [0, c] \to \mathbb{R}$ with h non-decreasing, convex and lower semicontinuous, and k non-increasing, concave and upper semicontinuous, and such that $h \leq k, h \neq k$. We write $q = k - h$. The function q is non-increasing, upper-semicontinuous and concave with $q \geq 0, q \neq 0$. Then E is the solution set of $0 \leq y \leq c, 0 \leq x \leq q(y)$.

Step 2: The theorem for E by means of explicit sets r_E, Q_E and R_E

We construct explicit sets r_E, Q_E, R_E—with the rectangles r_E, R_E upright—that satisfy the properties in the theorem for the convex figure E and for which r_E is the unique upright inscribed rectangle in E of maximum area. Each upright rectangle inside E is contained in a rectangle of the type $\{(x, y)|0 \leq x \leq s_1, 0 \leq y \leq s_2\}$ for some point $S = (s_1, s_2)$ on the graph of q, and the area of this rectangle is $s_1 s_2$. Here we use the fact that the function $x = q(y)$ is non-increasing. Therefore, the problem of maximizing the area of upright rectangles in E is equivalent to that of finding a point S on the graph of q for which the product $s_1 s_2$ is maximized.

Existence of a maximum point S follows from the theorem of Weierstrass that a continuous function on a nonempty closed bounded set assumes its maximum. One must have $s_1, s_2 > 0$ as the convex figure E has nonempty interior. Uniqueness of a maximum point S follows from the fact that if $s_1 s_2 = t_1 t_2$ for $s_1, s_2, t_1, t_2 > 0$, then $\frac{s_1+t_1}{2} \frac{s_2+t_2}{2} > s_1 s_2$.

It remains to find the maximum point S. Consider the set F of solutions of the inequalities $xy > s_1 s_2, x \geq 0, y \geq 0$ (see Fig. 3). There exists a straight line m that separates the convex sets E and F. This follows from the separation theorem of convex sets that two convex sets in the plane that have no point in common can be separated by a straight line. That E and F have no point in common is a reformulation of the maximality property of the point S. A separating line must pass through S, as S lies in E and on the boundary of F. That is, it supports the convex sets E and F at S.

The tangent line to the hyperbola $xy = s_1 s_2$ at the point S is clearly the unique line through the point S that supports the convex set F at the point S. It follows that this tangent line supports the convex figure E. The equation of this tangent line is $s_2 x + s_1 y = 2 s_1 s_2$. It intersects the coordinate axes at $(0, 2s_2)$ and $(2s_1, 0)$, so S

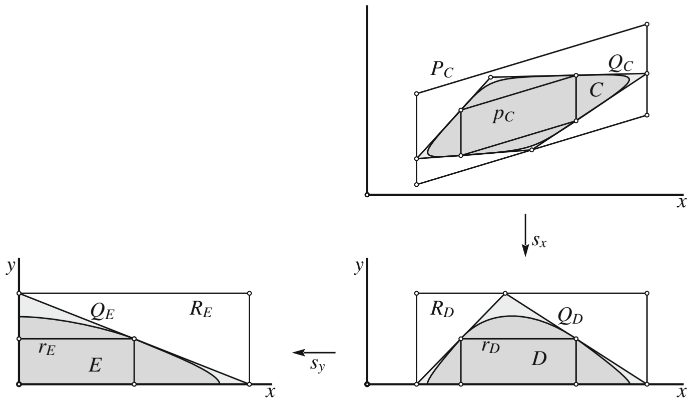

Fig. 4 Transfer of structure from E to C

lies exactly half-way between these two points. Note that each point S in the positive quadrant is the midpoint of a unique segment with endpoints on the coordinate axes: the segment with endpoints $(0, 2s_2)$ and $(2s_1, 0)$. This gives the following conclusion (see the bottom left-hand side of Fig. 3).

The maximum point S is the unique point in the positive quadrant for which the convex figure E is supported at S by the segment that has midpoint S and endpoints on the coordinate axes.

We choose the following sets (see the bottom left-hand side of Fig. 4):

- r_E is the upright rectangle for which O and S are a pair of opposite vertices—the other two vertices are $(s_1, 0)$ and $(0, s_2)$;
- Q_E is the right-angled triangle with vertices O, $(2s_1, 0)$, $(0, 2s_2)$; one can view Q_E as a degenerate quadrangle by counting the vertex at the origin double.
- R_E is the upright rectangle for which O and $2S$ are a pair of opposite vertices—the other two vertices are $(2s_1, 0)$ and $(0, 2s_2)$.

Then the sides of the right-angled triangle Q_E support the convex figure E at the vertices of the rectangle r_E: for the vertex S, this holds as the hypotenuse of Q_E lies on the separating line m, and this line supports E; for the other vertices of r_E this is trivial. The three vertices of the triangle Q_E are vertices of the rectangle R_E. Moreover, R_E is a dilation of r_E with ratio 2. We conclude that r_E, Q_E, R_E satisfy the properties in the theorem for E and that r_E is the unique upright inscribed rectangle in E of maximum area.

Step 3: The theorem for D by means of explicit sets r_D, Q_D and R_D

We construct explicit sets r_D, Q_D, R_D for which alignment against the y-axis gives r_E, Q_E, R_E—with the rectangles r_D, R_D upright—that satisfy the properties in the theorem for the convex figure D and for which r_D is the unique upright inscribed rectangle in D of maximum area. The constructions and all verifications are based on one idea: transfer of structure from E to D, using alignment against the y-axis. We will only display details of some of the verifications.

We choose the following sets (see the bottom right-hand side of Fig. 4):

- r_D is the unique upright rectangle in D for which alignment against the y-axis gives the upright rectangle r_E, that is, $r_E = a_y(r_D)$.

We check that this is well-defined. The top side of the upright rectangle r_E is precisely the intersection of the convex figure E and the horizontal line through the point S, and this has the same length as the intersection of the convex figure D with this line, as $a_y(D) = E$. Therefore, there exists a unique upright rectangle r_D for which the top side is contained in the convex figure D for which $a_y(r_D) = r_E$. Rectangle r_D lies entirely in the convex figure D, as D is a region bounded from above by the graph of a function, to wit $g - f$, and from below by the x-axis.
Explicitly, r_D is the upright rectangle with top side the intersection of the convex figure D and the horizontal line through the point S and with bottom side on the x-axis. The rectangle r_D is seen to be inscribed in D.

- Q_D is chosen to be a triangle containing D for which alignment against the y-axis gives Q_E, that is, $a_y(Q_D) = Q_E$. The existence of such a triangle follows from the proposition. The triangle Q_D can be seen as a degenerate quadrangle by viewing some point $(d, 0)$ with $d \in (h(s_2), k(s_2))$ as a vertex. Then the four sides of Q_D support D at the vertices of r_D. By transfer from E to D we get that the top vertex of the triangle Q_D lies on the horizontal line through $2S$.
- R_D is the unique upright rectangle containing Q_D for which alignment against the y-axis gives R_E, that is, $a_y(R_D) = R_E$. This rectangle is well-defined and it is a dilation of r_D with ratio 2: this follows by transfer from E to D. Explicitly, R_D is the rectangle with bottom side the intersection of Q_D with the x-axis and top side lying on the horizontal line through the point $2S$.

We claim that the rectangle r_D is the unique upright rectangle in D of maximum area. Indeed, choose arbitrarily an upright rectangle in D other than r_D. Alignment of this rectangle against the y-axis gives a rectangle in $E = a_y(D)$ other than $r_E = a_y(r_D)$; here we use that r_D is the unique upright rectangle in D for which alignment against the y-axis gives r_E. The area of the chosen rectangle is preserved under alignment against the y-axis, and this area is smaller than the area of r_E by what we have proved in step 2. Finally, $r_E = a_y(r_D)$, so r_D and r_E have the same area. Thus we get that the chosen rectangle has area smaller than the area of r_D. This concludes the proof of the claim.

We conclude that r_D, Q_D, R_D satisfy the properties in the theorem and that r_D is the unique upright rectangle in D with maximum area. Note that Q_D has an additional property: it is inscribed in R_D; in particular, the top vertex of Q_D lies on the top side of the upright rectangle R_D.

Step 4: The parallelogram version of the theorem for C by explicit sets p, Q and P

We are ready to construct explicit sets p, Q, P that satisfy the properties promised above.

As before, the constructions and all verifications are based on one idea. This time the idea is transfer of structure from D to C using alignment. As before, we will display only part of the verifications in detail.

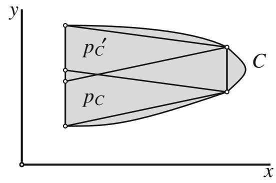

Fig. 5 Nonuniqueness of the parallelogram p_C

We choose the following sets (see the top right-hand side of Fig. 4).

- p is, to begin with, taken to be a parallelogram in C for which two of its sides are vertical and for which alignment against the x-axis gives the upright rectangle r_D, that is $a_x(p) = r_D$.

This parallelogram is not unique if one of its vertical sides is lying on the boundary of C and can be moved slightly upward or downward without leaving C (see Fig. 5). Observe that this cannot be the case for both vertical sides, by the maximality property of r_D and by $r_D = a_x(p)$. Therefore, in order to achieve uniqueness, we demand moreover that the two lower vertices of p are lying on the graph of the function f—which describes the lower part of the boundary of C.
By transfer from D to C we get that p has the v-maximality property.

- Q will be constructed out of Q_D by means of the alignment operation, but the construction requires an additional device: we have to work with angle-regions. Q is a quadrangle for which its sides support C at the vertices of p and for which $Q = A_1 \cap A_2$ for the two angle-regions A_i, $i = 1, 2$ for which $Q_D = a_x(A_1) \cap a_x(A_2)$. Existence of such a quadrangle follows by applying twice the proposition.
- P is the unique parallelogram that is a dilation of p with ratio 2 and with all sides except possibly the top side containing a vertex of Q.

We conclude that p, Q, P satisfy the promised properties.

For each parallelogram with one pair of sides vertical we single out one of its two angles: the one at the bottom right-hand vertex (or, what is the same, the one at the top left-hand vertex). This will be called the main angle of the parallelogram. Angles are taken between 0 and π.

Step 5: The set of all main angles of parallelograms with the v-maximality property is a finite closed non-empty interval

This step is illustrated by Fig. 5. The parallelogram p has been made unique by an artificial additional condition on top of the v-maximality property: its bottom vertices are made to lie on the graph of f. Now we omit this condition. Then $p = p(v)$ is not uniquely determined by the v-maximality property if and only if the left-hand vertical side or the right-hand vertical side of p can be moved upward in vertical direction without leaving C. This implies that this vertical side is part of the boundary of C. Then each upward move for which the side does not leave C gives an inscribed parallelogram in C with two vertical sides having maximum area. In this way one gets all of these parallelograms.

In case of the left-hand vertical side, the largest main angle of such a parallelogram is obtained by taking the parallelogram p. The smallest main angle is obtained by taking the parallelogram where the left-hand side is moved up as high as possible (without leaving C). All intermediate angles are assumed by such parallelograms. In case of the right-hand vertical side, the smallest main angle of such a parallelogram is obtained by taking the parallelogram p and the largest main angle is obtained by taking the parallelogram where the right-hand side is moved up as high as possible (without leaving C). In all, we conclude that the set of main angles of parallelograms with the maximality property is a finite closed non-empty interval.

Preamble to the last three steps

In the first five steps of the proof, the unit vector v has been kept fixed. Now we want to vary v. We emphasize that p, Q, P depend on the choice of the unit vector v: $p = p(v)$, $Q = Q(v)$, $P = P(v)$. Observe that for a parallelogram p with two sides parallel to v, the concept of main angle of p with respect to v is defined (it means the main angle of p when coordinates are chosen such that the direction of v is the direction of the positive y-axis). In the remaining three steps of the proof, we will show that there exists a unit vector v for which there exists a parallelogram p with the v-maximality property for which its main angle with respect to v is $\frac{1}{2}\pi$. This means that the parallelogram is a *rectangle*. This will establish the theorem. For this, a 'continuity' argument will be used. Some care is needed here as for some unit vectors v the parallelogram $p(v)$ and so its main angle is not uniquely determined by the v-maximality property.

Step 6: 'Continuous' dependence on v for the set of main angles of the parallelograms with the v-maximality property

For each parallelogram p in C with two sides parallel to a unit vector v and each $\varepsilon > 0$, there is $\delta > 0$ such that for each unit vector w for which the euclidean distance $\|v - w\|$ between v and w is smaller than δ, there exists a parallelogram contained in p with two sides parallel to w and with area at least $\text{area}(p) - \varepsilon$. This property is readily seen to hold true: 'shrink and rotate p'. This implies 'continuous' dependence on v for the set of main angles of the parallelograms with the v-maximality property in the following precise sense.

For each sequence of unit vectors v_i that converges to v and for each sequence of parallelograms p_i with the v_i-maximality property, the sequence of main angles of the parallelograms p_i with respect to v_i has a convergent subsequence with limit the main angle with respect to v of a parallelogram p with the v-maximality property.

Step 7: Consideration of a parallelogram of maximum area with vertices on the boundary of C

We are going to show that there exists a unit vector u and a parallelogram having the u-maximality property such that its main angle with respect to u is $\leq \frac{1}{2}\pi$ and such that there exists a unit vector w and a parallelogram having the w-maximality

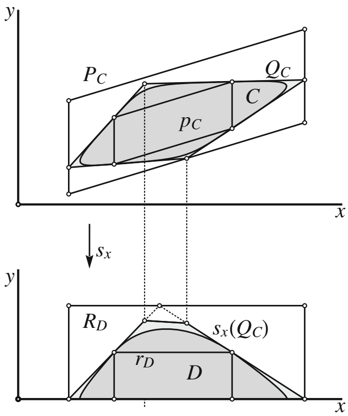

Fig. 6 Alignment of the quadrangle Q_C

property such that its main angle with respect to w is $\geq \frac{1}{2}\pi$. To this end, we consider a parallelogram Π contained in C that has maximum area; its existence is assured by the extreme value theorem of Weierstrass.

Let u and w be two unit vectors that are parallel to two nonparallel sides of Π. Then the parallelogram Π has both the u-maximality property and the w-maximality property, clearly. In particular, Π is inscribed in C. Let ϕ_u respectively ϕ_w be the main angle of Π with respect to u respectively with respect to w. Then $\phi_u + \phi_w = \pi$, as ϕ_u and ϕ_w are adjacent angles of the parallelogram Π. Therefore, one of the angles ϕ_u and ϕ_w is $\leq \frac{1}{2}\pi$ and the other one is $\geq \frac{1}{2}\pi$. We may assume that $\phi_u \leq \frac{1}{2}\pi$ and $\phi_w \geq \frac{1}{2}\pi$ by the symmetry in u and w. We conclude in particular that there exist unit vectors u, w such that $\phi_u \leq \frac{1}{2}\pi$ and $\phi_w \geq \frac{1}{2}\pi$, as required.

Step 8: Conclusion of the proof

By the result of step 7 and by the ‘continuity’ property established in step 6, there exists a choice of unit vector w for which the main angle of parallelograms with the w-maximality property assumes values $\leq \frac{1}{2}\pi$ as well as values $\geq \frac{1}{2}\pi$. As the set of all values that are assumed for one v is a closed interval by the result of step 5, it follows that $\frac{1}{2}\pi$ is assumed as well. This concludes the proof of the theorem.

The triangle Q_D and alignment of the quadrangle Q

Some subtlety is required in the construction of the sets in the proof of the theorem. For example, *applying alignment against the x-axis to Q in general does not give Q_D*, but it gives a convex set having the shape of the triangle Q_D that has been truncated near the top by a straight cut (see Fig. 6). Indeed, it is readily seen that the function that describes the upper side of the convex figure $a_x(Q)$ is concave, piecewise linear with two kinks and that it agrees at the beginning and end of the interval $[a, b]$ with the function that describes the upper side of Q_D.

4 Comments

Radziszewski proceeds as follows in [5]. The given convex figure C is transformed in two steps to the corner of the positive quadrant, giving a convex figure that can be handled more easily. Then, by means of a geometric construction, the obtained information on the displaced convex figure is transferred to the original convex figure. A continuity argument concludes the proof. An alternative proof of the result by Radziszewski is given in Süss [11].

Pólya and Szegő [4] prove that each convex figure can be sandwiched between dilated rectangles r, R of ratio ≤ 3 and they pose the problem of improving this result. Schwarzkopf et al. [8,9] and Lassak [3] find the best possible improvement, to ≤ 2. This result implies the result by Radziszewski. The present paper gives a new solution to the problem of Pólya and Szegő. The method in [3,8,9] is in some sense dual to the method of the present paper, starting with the construction of an outer approximation R satisfying a minimality property; in the present paper, we start with the construction of an inner approximation r satisfying a maximality property.

At this point we note a subtlety: a rectangle in a convex figure of maximum area need not have all its vertices on the boundary of the convex figure. For example, the rectangle of maximum area in the quadrangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1 + \varepsilon, 1 + \varepsilon)$ for a small positive number $\varepsilon > 0$, is the rectangle with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$; one of its vertices, $(1, 1)$, does not lie on the boundary of this quadrangle.

The quadrangle Q in the present paper is a novel feature; this much tighter outer approximation is suggested by the alignment approach. The resulting outer rectangle R in [3,8,9] is different from the one obtained in this paper: for example, the rectangle R obtained in [3,8,9] is circumscribed to C, so there is no room in it for a quadrangle Q that contains C and that has at most half the area of R.

There is an extensive literature on related questions; see for example the recent paper Knauer et al. [2] and its list of references. Algorithmic questions are also considered in the literature.

A crucial role is played in the proof by the 'continuous' dependence of the set of parallelograms that have the v-maximality property on the unit vector v. This is a special case of an interesting general principle, the Maximum Theorem of Berge (see, for example, Stokey et al. [10]). This general result gives—under weak assumptions—the 'continuous' dependence of the solution set of an optimization problem on a parameter in the problem. In the present paper a direct proof has been given for the required continuity property, in order to make the proof self-contained.

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