The connective K theory of semidihedral groups

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To the memories of my mother... **Siembouy Saeoung** ....

Contents

Abstract

The real connective K-homology of finite groups $ko^*(BG)$, plays a big role in the Gromov-Lawson-Rosenberg (GLR) conjecture. In order to compute them, we can calculate complex connective K-cohomology, $ku^*(BG)$, first and then follow by computing complex connective K-homology, $ku_*(BG)$, or by real connective K-cohomology, $ko^*(BG)$. After we apply the η-Bockstein spectral sequence to $ku^*(BG)$ or the Greenlees spectral sequence to $ko^*(BG)$, we shall get $ko^*(BG)$. In this thesis, we compute all of them algebraically and explicitly to reduce the difficulties of geometric construction for GLR, especially for semidehedral group of order 16, SD_{16}, by using the methods developed by Prof.R.R. Bruner and Prof. J.P.C. Greenlees. We also calculate some relations at the stage of connective K-theory between SD_{16} and its maximal subgroup, (dihedral groups, quaternion groups and cyclic group of order 8).

Introduction

The connective K-theory of finite groups plays evidently a big role in the investigation of Gromov-Lawson-Rosenberg (GLR) conjecture. This conjecture involves the existence of a positive scalar curvature metric. That is; if $n \geq 5$, M is a spin n-manifold with $\pi_1(M) = G$, (we restrict our attention to the case when G is finite, because of the results of T.Schick, [31]), then M admits positive scalar curvature metric if and only if

$$[M] \in \ker(\Omega_n^{Spin}(BG) \longrightarrow ko_n(BG) \longrightarrow KO_n(BG) \longrightarrow KO_n^G),$$

[30]. There is no known counter example for finite groups yet, see more discussions in this direction in section 2.7 of [13], especially lemma 2.7.1 in [13]. In that, we will see that the E_∞-page of Greenlees spectral sequence for $ko_*(BG)$ is suitable for GLR. This is actually the principal motivation. In order to minimize the difficulties of geometric construction for GLR conjecture, we mainly concentrate to the algebraic calculations as much as possible.

The main stuff in this thesis is the calculation of four types of connective K-theory of finite groups which are $ku^*(BG)$, $ku_*(BG)$, $ko^*(BG)$, $ko_*(BG)$ for semidihedral groups G, by using the methods developed by Prof.R.R. Bruner and Prof. J.P.C. Greenlees, [14] and [13], which the author calls *Bruner-Greenlees methods.* The reasons for choosing semidihedral groups,

$$SD_{2^n} = \{s, t | s^{2^{n-1}} = t^2 = 1, tst = s^{2^{n-2}-1}\}, \quad \text{for } n \geq 4,$$

to be a case study is firstly because there is no explicit answer for them yet, secondly, because of their structures, e.g. $SD_{2^{n-1}}$ is not a subgroup of SD_{2^n} which contrasts with those of dihedral groups, quaternion groups and cyclic group, and finally, because of the first hoping that this groups might be the counter example of GLR, but so far, it does not, by [29].

The strategies of the calculation on four types of the connective K-theory, for finite group G, due to Bruner-Greenlees methods can be displayed as in the diagram below;

$$\begin{tikzcd}
 & ku^*(BG) \arrow[r, "\text{GSS}"] \arrow[d, "\text{BSS}"] & ku_*(BG) \arrow[d, "\text{BSS}"] \\
H^*(BG;\mathbb{F}_2) \arrow[ur, "\text{ASS}"] & ko^*(BG) \arrow[r, "\text{GSS}"] & ko_*(BG),
\end{tikzcd}$$

where ASS refer to Adams spectral sequence, BSS refer to η-Bockstein spectral sequence and GSS refer to Greenlees spectral sequence and all arrows in this diagram are not maps but they merely refer to the methods of the calculation which input and output lie in the tails and heads of the arrows respectively. From this diagram, to obtain $ko_*(BG)$ via Bruner-Greenlees methods with the input $H^*(BG;\mathbb{F}_2)$, we can do possibly in two ways, i.e. $ASS \longrightarrow BSS \longrightarrow GSS$ and $ASS \longrightarrow GSS \longrightarrow BSS$. However, from our experience, combining of both ways gives us more information.

§ 0.1 Outline

Note at first that, the prerequisites for this thesis is the standard basics from algebraic topology, K-theory, representation theory, commutative algebra and homological algebra, especially on the topic *spectral sequence.*

Since all four types of connective K-theories we consider are infinite dimensional CW-complex (classifying space BG) and are in the complete world (I-adic completion of equivariant connective K-theory, [19]), we collect some basic facts about direct limit, inverse limit, completion and p-adic integer in the Appendix. In preliminaries chapter we include some basic background on periodic and connective K-theory and recall the general methods of the calculation $ku^*(BG)$ by using Adams spectral sequence from [14]. Since all calculations by using Bruner-Greenlees methods are mainly based on representation theory, we collect and record some facts about them. The technique about exact sequence is often used, thus we collect some long induced exact sequences concerning our calculation at the end of this chapter as well.

We start to calculate $ku^*(BG)$, for semidihedral group G, by using Adams spectral sequence in Chapter 2. Since the representation theory (character theory) and cohomology ring are important tools in Bruner-Greenlees methods, we provide the explicit calculation of the character table for semidihedral groups at the start of the chapter. For cohomology rings, we investigate the results from [16] and try to make the explicit relations with Chern classes or Stiefel-Whitney classes (some Bockstein operations are included). After that, we calculate $H^*(BSD_{2^n};\mathbb{F}_2)$ as a module over the exterior algebra $E(1)$ followed by the calculation of E_2-page which we also provide the general structures for the E_2-page of Adams spectral sequence for $ku^*(BG)$. The calculation of differentials is not too hard and can be done by using the properties of Chern classes, Bockstein operation and the theorem of May and Milgram, [25]. Since the Adams spectral sequence for $ku^*(BG)$ is strongly convergent (see this discussion in section 1.2.3), the additive structure can be read from E_∞-page together with the character table and cohomology ring. Again, the representation theory and cohomology ring give us the multiplicative structure. Note that we make explicit the structure of $ku^*(BG)$ only for $G = SD_{16}$ and for general semidihedral group we merely say something about the relation with its periodic K-theory and cohomology ring. We end up this chapter by comparing our result $ku^*(BSD_{16})$ with $ku^*(BG)$ for dihedral groups, quaternion groups and cyclic groups G.

In Chapter 3, we calculate $ku_*(BSD_{16})$ as a module over $ku^*(BSD_{16})$ by using the Greenlees spectral sequence. To do that, we recall some background about local cohomology based on the Koszul complex (which is suitable to our purpose) at the beginning. The main strategy for calculating $ku_*(BSD_{16})$ is common with those in [14]. That is by considering short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BSD_{16}) \longrightarrow QU \longrightarrow 0,$$

where TU is v-torsion and QU is the image of $ku^*(BSD_{16})$ in periodic K-theory. This induces a long exact sequence of local cohomology and hence instead of doing calculation on $H_I^*(ku^*(BSD_{16}))$, we can do that on the TU part and the QU part which is evidently more comfortable. For the TU part, this is embedded in the cohomology ring and thus the calculation is not hard in local cohomology. For the QU part, we work on the character table to make explicit the reduction to a principal ideal and in that we try to link the generators of QU part to the Modified Rees ring, JU_i, (these generators will be used widely in this thesis). After that, we calculate $H_I^*(QU)$ which is the bulk of this chapter (we provide a great deal of detail about the calculations to avoid mistrust) by using row-column matrix operations. After finishing this calculation, we record all results in the $E_{1\frac{1}{2}}$ page. To get the E_2-page, we need to identify the differentials coming from the long exact sequence, which can be done by using the module structure of local cohomology and the connective property of ku. It turns out that the E_2-page is the E_∞-page and there is no any extension problem. Therefore, the results is an immediately result from E_∞-page and the actions of $ku^*(BSD_{16})$ can be read from the module structure of local cohomology. We end up this chapter by comparing our result $ku_*(BSD_{16})$ with $ku_*(BG)$ for dihedral groups, quaternion groups and cyclic groups G.

In Chapter 4, we calculate $ko^*(BSD_{16})$ by using the η-Bockstein spectral sequence with initial input $ku^*(BSD_{16})$. For the E_1-page, this is simple by just copying the input along the diagonal. For the E_2-page, we use the same strategies as in [13], i.e. by considering short exact sequence of the input

$$0 \longrightarrow TU \longrightarrow ku^*(BSD_{16}) \longrightarrow QU \longrightarrow 0,$$

as before. Next, we calculate homology of the part TU and QU where the differentials on the two parts are given by Sq^2 and $1 \pm \tau$, respectively. We calculate and display the kernel and homology (for homology we use Steenrod algebra and row-column matrix operations) of both parts explicitly based on cohomology ring and character table (in term of the generator of JU_i), which take a number of page for this. The calculations of connecting differentials and d_2, d_3 are examined in the same time with the help of representation theory and the properties of η-Bockstein spectral sequence, i.e. the fact that the spectral sequence collapses at E_4-page (since $\eta^3 = 0$). After we reach to the E_∞-page, there are some extension problems to consider. We solve that problem by using the structure of the E_∞-page together with the help of the η-Bockstein spectral sequence for mod 2 coefficients and the mod 2-Bockstein spectral sequence. Again, we end up this chapter by comparing our result $ko^*(BSD_{16})$ with $ko^*(BG)$ for dihedral groups, quaternion groups and cyclic groups G.

In Chapter 5, we calculate $ko_*(BSD_{16})$ as a module over $ko^*(BSD_{16})$ by using Greenlees spectral sequence with initial input $ko^*(BSD_{16})$. We proceed with the calculations in the same way as in Chapter 3 but this one has more structure. The strategy is closely the same as before but we need to consider two short exact sequences for $ko^*(BSD_{16})$. That is, for $ko_*(BSD_{16})$;

$$0 \longrightarrow T \longrightarrow ko^*(BSD_{16}) \xrightarrow{\pi^u} \overline{QO} \longrightarrow 0, \quad \text{and}$$

$$0 \longrightarrow \tau \longrightarrow T \longrightarrow TO \longrightarrow 0,$$

where $\overline{QO}$ is the image of $ko_*(BSD_{16})$ in $KU^*(BSD_{16})$, T is the kernel of π^u, and TO is concentrated in the zero line of the η-Bockstein spectral sequence which is a submodule of $H^*(BSD_{16};\mathbb{F}_2)$ and τ is the η-multiples of $ko_*(BSD_{16})$. To get the E_2-page, we need to calculate $H_I^*(ko^*(BSD_{16}))$ but, by the induced long exact sequence from the first short exact sequence, we can calculate $H_I^*(T)$, $H_I^*(\overline{QO})$ and their connecting differentials instead. For $H_I^*(T)$, we use the long exact sequence induced from the second short exact sequence, i.e. it is enough to find $H_I^*(TO)$, $H_I^*(\tau)$ and their connecting differentials. We take a number of page to find the radical ideal, $\sqrt{I}$, for all of the module τ, TO and $\overline{QO}$ over $ko^*(BSD_{16})$. The local cohomology calculation for TO, τ and $\overline{QO}$ is not too hard but there is a problem concerning the connecting differentials coming from the first short exact sequence. And also there are some extension problems when we reach to E_∞-page. We take the bulk of this chapter to solve them. However, some extension problems still remain and also some η-multiple elements of the module $ko_*(BSD_{16})$ are hindered by this methods. We write down some additive structure of $ko_*(BSD_{16})$ at the end of this chapter and postpone the remaining problems (precisely, in degree $8k$ and degree $8k+1$) and η-multiple structures to Chapter 6.

In Chapter 6, we calculate $ko_*(BSD_{16})$ as a module over $ko^*(BSD_{16})$ by using the η-Bockstein spectral sequence with initial input $ku_*(BSD_{16})$. The strategy is similar to Chapter 4; we consider homology calculation on even and odd degree part of $ku_*(BSD_{16})$ separately. For the even degree part, we do some explicit calculations to embed the second column of the E_∞-page of $ku_*(BSD_{16})$ in Chapter 3 to $H_*(BSD_{16};\mathbb{F}_2)$ and then calculate differential d_1 on this part by using the dual of Sq^2 operation, namely $d_1 = (Sq^2)^\vee$. For the odd degree part, this is similar to the QU part of Chapter 3, (i.e. by using character table and local ring for the calculation) where the differentials are given by $d_1 = 1 \pm \tau$. This part needs careful work which we take a large number of page to deal with in order to prevent mistrust. To find the differentials, it is easy now by referring to the results in Chapter 5. It turns out that the E_∞-page contains more extension problems than the GSS and most of them are non-trivial extension problems. By good fortune, in degree 8, it is a trivial extension problem and this resolves the extension problem in degree $8k$ from the last chapter. This also reveals the η-structure for $ko_*(BSD_{16})$. However, both methods, BSS and GSS, still leave the extension problem in degree $8k+1$. Fortunately, by the results of D.Bayen [7], we can complete the calculation.

§ 0.2 Main results and Conclusions

Main results:

From our calculation, the main results for $ku^*(BSD_{16})$ are;

- The generators and relations of $ku^*(BSD_{16})$ are shown explicitly for both additive and multiplicative structure (Theorem 2.5.5 and Theorem 2.6.1).
- Comparing with the result of $ku^*(BG)$ where G is finite groups of p-rank ≥ 2, e.g. dihedral groups and elementary abelian 2-groups, the complex connective K cohomology of them contains v-torsion in codegree 6 whereas $ku^6(BSD_{16})$ has no v-torsion (Lemma 2.5.2).
- $ku^*(BSD_{16})$ is embedded in $H^*(BSD_{16};\mathbb{F}_2) \oplus KU^*(BSD_{16})$ (Lemma 2.4.4(4)).
- $ku^*(BSD_{16})$ is embedded in $ku^*(BD_8) \oplus ku^*(BQ_8) \oplus ku^*(BC_8)$ (Theorem 2.7.2).
- $ku^*(BSD_{16})$ is generated by Chern classes and then the image in periodic K-theory is a Modified Rees ring based on the definition 2.3 in [19] by R.R. Bruner and J.P.C. Greenlees (Theorem 2.5.5).
- $ku^*(BSD_{2^n})$ is embedded in $ku^*(BD_{2^{n-1}}) \oplus ku^*(BQ_{2^{n-1}}) \oplus ku^*(BC_{2^{n-1}})$ (Theorem 2.7.3).

The main results for $ku_*(BSD_{16})$ are;

- The generators and relations of $ku_*(BSD_{16})$ are shown explicitly (Theorem 3.5.1) and the action of $ku^*(BSD_{16})$ on $ku_*(BSD_{16})$ can be read from the table 3.4 and table 3.5 in Chapter 3.
- There is an explicit map from $ku_*(BSD_{16})$ to $ku_*(BD_8) \oplus ku_*(BQ_8) \oplus ku_*(BC_8)$ (Proposition 3.6.2, 3.6.4, Subsection 3.6.3).

The main results for $ko^*(BSD_{16})$ are;

- The generators and relations of $ko^*(BSD_{16})$ are shown explicitly (Theorem 4.4.1).

- There is a natural injective map from $ko^*(BSD_{16})$ to $H^*(BSD_{16};\mathbb{F}_2)\oplus KO^*(BSD_{16})$ (Corollary 4.4.2).
- There is an explicit map from $ko^*(BSD_{16})$ to $ko^*(BD_8)\oplus ko^*(BQ_8)\oplus ko^*(BC_8)$ which is not monomorphism (Proposition 4.5.2, 4.5.4, 4.5.6).

The main results for $ko_*(BSD_{16})$ are;

- The generators and relations of $ko_*(BSD_{16})$ are shown explicitly (Theorem 6.5.2).
- There are elements of the second column of GSS detected in $H_*(BSD_{16};\mathbb{F}_2)$, namely $\frac{\widetilde{\eta}[\overline{u}_4]}{(\overline{u}_4)^{k+2}} \in ko_{8k+8}(BSD_{16})$ are detected by $(uyP^{2k+1})^\vee \in H_{8k+8}(BSD_{16};\mathbb{F}_2)$ (Theorem 6.5.2).

The conclusions:

The Bruner-Greenlees methods for the calculation of connective K-theory for finite groups is a powerful tool. This machine reduces the work in homotopy theory to algebra and can be attacked by representation theory and cohomology theory. It is no exaggeration to say that the combining of ASS, BSS and GSS gives an excellent and standard way to make explicit the structure of connective K-theory for finite groups. All in all from our calculations, we can conclude that even if the methods that uses to calculate connective K-theory are different, all of them still require representation theory to determine their differentials, and surprisingly, they give the same answer.

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Chapter 1

Preliminaries

We collect some meaning and properties of complex connective K-theory and the strategy to calculate $ku^*(BSD_{16})$ by using Adams spectral sequence from [14] in the first section. The second section, we provide some basic knowledge of representation theory involving to our calculation. And in the last section, we investigate some long exact sequences concerning both real and complex connective K-theory via killing homotopy groups.

§ 1.1 Periodic K-theory and connective K-theory

In this section, we collect some facts of complex periodic K-cohomology theory and complex connective K-cohomology theory which are relevant to our purpose, i.e. for calculation by using representation theory.

1.1.1 What is $KU^*(BG) = K^*(BG)$?

A useful way to think about the periodic K-theory of classifying space BG, for finite groups G is *equivariant K theory.* That is, by definition in [32], for a compact G-space X,

$$K_G(X) = \mathbb{Z}\{[\eta]\}/([\eta_1 \oplus \eta_2] = [\eta_1] + [\eta_2]), \tag{1.1}$$

where η is a *complex G-equivariant vector bundle over X*; precisely, it is a G-map $\pi : \xi \longrightarrow X$ so that for each $x \in X$ the fibre $\xi_x := \pi^{-1}(x)$ is a complex vector space and for each $g \in G$, the translation $g : \xi_x \longrightarrow \xi_{gx}$ is linear and furthermore ξ is locally trivial, [18]. Note that the tensor product makes equivariant K-theory, $K_G(X)$, to be a ring.

This definition can extend to locally compact G-space X, i.e.,

$$K^0_G(X) = \widetilde{K}_G(X^+) = \ker(K_G(X^+) \longrightarrow K_G(pt))$$

and

$$K^1_G(X) = \widetilde{K}_G(S^1 \wedge X^+),$$

where X^+ means one point compactification of X. This theory has the main properties that ([32])

1 $K_G(pt) = R(G)$, representation ring and by Bott periodicity, $K_G^n \cong K_G^{n+2}$, we get that
$$K_G^*(pt) \cong K_*^G(pt) = R(G)[v, v^{-1}],$$
where $v \in KU_2 = KU^{-2} = KU^0(S^2)$ is the Bott element,

2 $K_G(X) \cong K(X/G)$ if G acts freely on X.

The main theorem which relates to our purpose is the theorem of Atiyah and Segal;

Theorem 1.1.1. *([3]) The equivariant K-theory of EG is*
$$K_G^0(EG) = R(G)_J^\wedge \quad \textit{and} \quad K_G^1(EG) = 0,$$
where $J = \ker(R(G) \longrightarrow R(1) = \mathbb{Z})$.

Note that EG is a terminal free G-space in the homotopy category, i.e, for any free G-space X, there is a G-map $\nu_X : X \longrightarrow EG$, unique up to homotopy. In fact EG is free and non-equivariantly contractible and $BG = (EG)/G)$. Thus, by this fact, properties of K_G above, theorem 1.1.1 and Bott-periodicity we have;
$$KU^*(BG) = K^*(BG) = K^*(EG/G) = K_G^*(EG) = R(G)_J^\wedge[v, v^{-1}]. \tag{1.2}$$

Moreover, there is another useful theorem by Atiyah and D.O.Tall, ([5], III.1.1) which is suitable for our calculation (i.e. connective K-theory of finite p-groups), namely if G is a p-group, the J-adic and (p)-adic topology coincide on J so that
$$R(G)_J^\wedge \cong \mathbb{Z} \oplus J_{(p)}^\wedge, \tag{1.3}$$
where $J = \ker(R(G) \longrightarrow R(1) = \mathbb{Z})$, [18]. Furthermore, [18], for finite group G, J-adic completion $R(G) \longrightarrow R(G)_J^\wedge$ is injective if and only if G is a p-group, [3] and [5].

1.1.2 What is $ku^*(BG)$?

Roughly speaking, complex connective K-cohomology theory is an associated cohomology of the spectrum ku which is the fibre of the killing homotopy group of the spectrum[1] KU,
$$KU_0^\uparrow := KU < 0 > \longrightarrow KU \longrightarrow KU_\downarrow^{-1},$$
i.e. $ku := KU_0^\uparrow := KU < 0 >$ and more precisely
$$\pi_n(ku) = \begin{cases} \mathbb{Z}, & \text{if } n \text{ is even and } \geq 0; \\ 0, & \text{otherwise.} \end{cases} .$$

[1] *Note:* spectrum KU contains a sequence of spaces $\underline{KU}_i, i \in \mathbb{Z}$ s.t. $\underline{KU}_{2k} = \mathbb{Z} \times BU$, $\underline{KU}_{2k+1} = U$, by Bott periodicity theorem.

Also, by Proposition 16.6 in J.F.Adams book [2],

$$H^*(ku;\mathbb{F}_2) \cong A \otimes_{E(1)} \mathbb{F}_2, \tag{1.4}$$

where $E(1) = \bigwedge_{\mathbb{F}_2}(Q_0, Q_1)$, $Q_0 = Sq^1, Q_1 = Sq^1Sq^2 + Sq^2Sq^1$ and A is mod 2 Steenrod algebra.

Moreover, there is the assertion from, for example, [14], [13] and [19] that; $ku^*(-)$ is a complex orientable cohomology theory, $ku^*(BU(n)) = ku^*[[c_1, c_2, ..., c_n]]$, where $ku^* = ku^*(pt) = ku_*(pt) = \mathbb{Z}[v]$; and for finite group G, $ku^*(BG)$ is a Noetherian ring. Furthermore, there is a relation between equivariant and non-equivariant complex connective K-theory in [19] that; for any compact Lie group G,

$$ku^*(BG) \cong (ku^*_G)^\wedge_I, \tag{1.5}$$

where $I = \ker(ku^*_G \longrightarrow ku^*)$ is the augmentation ideal.

The relations between connective and periodic K-theory or connective K-theory and ordinary cohomology theory are evidently useful for our calculation which we can investigate them in [14], for example:

- There is a cofibre sequence $\Sigma^2 ku \longrightarrow ku \longrightarrow H\mathbb{Z}$, and there is an equivalence $KU \simeq ku[\frac{1}{v}]$.
- By lemma 1.1.1 in [14], for any space X,
$$ku^*(X)[\frac{1}{v}] \cong KU^*(X).$$
- For finite groups G, $\widetilde{ku}^i(BG^{(n)})$ is finite if $n > i$, where $BG^{(n)}$ is the n-skeleton of BG, thus the inverse system $\{ku^*(BG^{(n)})\}$ is Mittag-Leffler and hence (page 31, [14])
$$ku^*(BG) = \underset{\longleftarrow n}{\lim}\, ku^*(BG^{(n)}).$$
- For any representation V of G, the natural map
$$\begin{array}{ccccc} H^*(BG;\mathbb{F}_p) & \longleftarrow & ku^*(BG) & \longrightarrow & KU^0(BG) = R(G)^\wedge_J \quad \text{sends} \\ \overline{c}_i^H(V) & \longleftarrow & c_i^{ku}(V) & \longrightarrow & c_i^{KU}(V) = v^{-i}c_i^R(V), \end{array}$$
where $c_i^R(V) = \sum_{j=0}^{i}(-1)^j \begin{pmatrix} n-j \\ n-i \end{pmatrix} \wedge^j (V)$ and $c_i^E(V)$ is the i^{th} Chern class in cohomology theory E.

Remark 1.1.2. *To calculate $\wedge^j(\chi)$, where χ is an character irreducible representation V for some V (dimension n) of a groups G, we use Newton recurrence relation and Adams operations Ψ^k, namely, by using proposition 7.4 in [12],*

$$\Psi^k(\chi)(g) = \chi(g^k),$$

and the recurrence relation [5], $\Psi^k(\chi) - \Psi^{k-1}(\chi)\wedge^1(\chi) + ... + (-1)^{k-1}\Psi^1(\chi)\wedge^{k-1}(\chi) + (-1)^k k \wedge^k(\chi) = 0$*, or in other words,*

$$\wedge^k(\chi) = \frac{(-1)^k}{k}[\wedge^{k-1}(\chi)\Psi^1(\chi) - \wedge^{k-2}(\chi)\Psi^2(\chi) + ... + (-1)^{k-1}\Psi^k(\chi)].$$

Note also that $\wedge^1(\chi) = \chi$ *and* $\wedge^n(V) = \det(V)$.

To obtain $ku^*(BG)$, we can use Atiyah-Hirzebruch Spectral sequence;

$$E_2^{p,q} = H^p(X;\pi_q(ku)) \Rightarrow ku^{p+q}(X),$$

or Adams Spectral sequence, [14];

$$E_2^{*,*} = \mathrm{Ext}_A^{*,*}(H^*(ku;\mathbb{F}_p), H^*(BG;\mathbb{F}_p)) \Rightarrow ku^*(BG)_p^\wedge. \tag{1.6}$$

In this thesis, by Bruner-Greenlees methods, we use Adams spectral sequence.

1.1.3 How to calculate $ku^*(BG)$ by the Adams spectral sequence?

By (1.4), (1.6) and the standard change of ring argument (i.e., for algebra A (flat as a module) over a ring R, $\mathrm{Ext}_A(A\otimes_R N, M) \cong \mathrm{Ext}_R(N, M)$, where M, N are R-modules), the E_2-page of the Adams spectral sequence is reduced to

$$E_2^{*,*} = \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_p, H^*(BG;\mathbb{F}_p)),$$

where $E(1) = \bigwedge_{\mathbb{F}_2}(Q_0, Q_1)$, $Q_0 = Sq^1, Q_1 = Sq^1Sq^2 + Sq^2Sq^1$ and A is mod 2 Steenrod algebra. Moreover, if G is a discrete group, then (Theorem 2.4.11, [10]) $BG \simeq K(G,1)$, Eilenberg-MacLane space. Also by theorem 2.2.3 in [10], $H^*(G;\mathbb{F}_p) \cong H^*(K(G,1);\mathbb{F}_p)$. These facts lead the Adams spectral sequence for $ku^*(BG)$, where G is finite groups, to be

$$E_2^{*,*} = \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, H^*(G;\mathbb{F}_2)) \Rightarrow ku^*(BG)_2^\wedge. \tag{1.7}$$

The calculation of E_2-page is all homological algebra. The standard way is firstly calculate $H^*(G;\mathbb{F}_2)$ as a module over $E(1)$, i.e. calculate the actions of Q_0 and Q_1 on $H^*(G;\mathbb{F}_2)$. Secondly, take a projective resolution of $\mathbb{F}_2$ or an injective resolution of $H^*(G;\mathbb{F}_2)$. Thirdly, take $\mathrm{Hom}^*_{E(1)}(-,\mathbb{F}_2)$ or $\mathrm{Hom}^*_{E(1)}(\mathbb{F}_2,-)$ to the projective or the injective resolution and get a long exact sequence. Finally, calculate homology of the long exact sequence and then E_2-page follows. However, the module structure of E_2-page over $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2,\mathbb{F}_2)$ will play a big role for differential calculation, thus it is worth to find out such structure, see more details in the section 2.3.1 and 2.3.2.

The convergence of Adams spectral sequence for $ku^*(BG)_2^\wedge$ is *strong convergence.* This means that

$$ku^*(BG)_2^\wedge \cong \varprojlim_s (R/F^sR),$$

where $R = ku^*(BG)_2^\wedge = F^0R \supseteq F^1R \supseteq F^2R \supseteq ...$ in which $F^sR/F^{s+1}R = E_\infty^{s,n+s}$. In fact, the strong convergence is guaranteed by J.F.Adams, [2], and J.M. Boardman, [11].

Precisely, theorem 15.1 in [2] (applying for $Y = ku$, $E = H\mathbb{F}_2$, $X = BG$, $Y^E = Y_2^\wedge$, which all assumptions in the theorem are satisfied) yields the *conditional convergence* (theorem 15.1(iii)) and theorem 7.1 in [11] gives the strong convergence.

For the calculation of Adams differentials, the theorem of May and Milgram, [25], and representation theory are powerful tools. Knowing the additive structure from stable splitting, (i.e. if $BG \simeq X \vee Y$, then we have $E_2^{*,*}(X \vee Y) \cong E_2^{*,*}(X) \oplus E_2^{*,*}(X) \Longrightarrow ku^*(X) \oplus ku^*(Y)$ and the calculations of differentials can be done separately), is also helpful (see, for example, the calculation of $ku^*(BQ_{2^n})$ in [14]). Moreover, knowing, if we are lucky, the generator of cohomology ring as a characteristic class, will be very helpful to determine differentials as well (see, for example, the calculation of $ku^*(BD_{2^n})$ in [14]).

After we reach to E_∞-page, it remains to find additive and multiplicative structure. To do this the comparison of representation theory (by Atiyah and Segal theorem) and cohomology ring will play an important role (see more details in Chapter 2). However, some relations in $ku^*(BG)$ are obtained immediately from Lemma 1.3.4 in [14], i.e. for one dimensional representation α, β,

$$e^{ku}(\alpha\beta) = e^{ku}(\alpha) + e^{ku}(\beta) - ve^{ku}(\alpha)e^{ku}(\beta)$$

and Lemma 2.1.1 in [14], that if ρ is induced up from the trivial subgroup (e.g. regular representation), it will be annihilated by Euler classes. Note also that the target of the Adams spectral sequence is $ku^*(BG)_2^\wedge$ but we actually need to calculate $ku^*(BG)$. However, for p-groups G, two this things are nearly the same, i.e. $ku^*(BG) = \mathbb{Z}[v] \oplus \widetilde{ku}^*(BG)_2^\wedge$. In other words, for p-groups G,

$$\widetilde{ku}^*(BG)_p^\wedge \cong \widetilde{ku}^*(BG). \tag{1.8}$$

This fact, (1.8), follows from (1.5) which asserts that $ku^*(BG)$ is an I-adic completion ($I = \ker(ku_G^* \longrightarrow ku^*)$, the augmentation ideal) and the Atiyah theorem, [5], which asserts that, for p-groups G, J-adic topology and p-adic topology coincide.

1.1.4 Theorem of May and Milgram for connective K-theory

The relation between the Adams spectral sequence for $ku^*(BG)$ and the Bockstein spectral sequence for $H^*(BG;\mathbb{Z})$ come from the inclusion $E(0) \xrightarrow{i} E(1)$. This inclusion induces homomorphism

$$\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2)) \xrightarrow{i^*} \mathrm{Ext}_{E(0)}^{*,*}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2)).$$

Note also that Bockstein spectral sequence for calculation $H^*(BG;\mathbb{Z})$ can adapt to be;

$$E_2\text{-page} = \mathrm{Ext}_{E(0)}^{*,*}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2)) \Longrightarrow H^*(BG;\mathbb{Z}) = [BG, H\mathbb{Z}].$$

This yields the diagram below;

$$\begin{tikzcd}
\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2)) \Longrightarrow \arrow[d, "i^*"] & ku^*(BG) \arrow[d] \\
\mathrm{Ext}^{*,*}_{E(0)}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2)) \Longrightarrow & H^*(BG;\mathbb{Z}).
\end{tikzcd}$$

Hence, these two spectral sequences are related. Some relations are found by May and Milgram, theorem of May and Milgram [25], which guarantee that the towers in each E_r-page of both spectral sequences correspond and under this correspondence their differentials agree.

§ 1.2 Representation theory for connective K-theory

In this section we collect some facts of representation theory from [13] and [1] that are involved in our calculation. Since Chern classes and Stiefel-Whitney classes will play a role in, at least, the calculation of differentials in Adams spectral sequence, we record some facts about their relations as well.

1.2.1 Representation theory and real K-theory

An excellent source of representation theory for the calculation of connective K-theory is Real connective K-theory book, [13]. However, to be easier in looking back, we record some of definitions and propositions that concern our calculations (in Chapter 4) as below.

Definition 1.2.1. *([13]) A* real representation *of G is a representation V of G over $\mathbb{C}$ with a conjugate linear map $J : V \longrightarrow V$ with $J^2 = 1$. A* quaternionic representation *of G is a representation V of G over $\mathbb{C}$ with a conjugate linear map $J : V \longrightarrow V$ with $J^2 = -1$. A* complex representation *of G is the same as a representation over $\mathbb{C}$.*

The criterion to separate real, complex and quaternionic representation is;

Lemma 1.2.2. *([1]) Let V be a complex irreducible representation of a compact group G. Then*

$$\int_{g\in G} \chi_V(g^2) = \begin{cases} 1, & \text{if } V \text{ is real;} \\ 0, & \text{if } V \text{ is not self-conjugate;} \\ -1, & \text{if } V \text{ is quaternionic.} \end{cases}$$

In particular, for finite group G, $\int_{g\in G} \chi_V(g^2)$ is replaced by $\frac{1}{|G|}\sum_{g\in G} \chi_V(g^2)$.

Proof. See [1] page 70. □

The Grothendieck groups of finite dimensional real, complex and quaternionic representations of a groups G are denoted by $RO(G), RU(G)$ and $RSp(G)$ respectively.

There are natural transformations (forgetting the structure map J)

$$RO(G) \xrightarrow{c} RU(G) \xleftarrow[\widetilde{c}]{} RSp(G)\ ,$$

which are both called complexification and

$$RO(G) \xleftarrow[r]{} RU(G) \xrightarrow{q} RSp(G)\ ,$$

where r is realification and q is quaternionification (Chapter 2, [13]). There is also a conjugation map

$$\tau : RU(G) \longrightarrow RU(G).$$

The composites of these maps have some properties which are given by;

Lemma 1.2.3. *(Lemma 2.1.5, [13])*
(i) $rc = 2, cr = 1 + \tau$.
(ii) $q\widetilde{c} = 2, \widetilde{c}q = 1 + \tau$.
(iii) Every real or quaternionic representation is self-conjugate: $\tau c = c$ *and* $\tau\widetilde{c} = \widetilde{c}$.
Also, $r\tau = r$ *and* $q\tau = q$.

Proof. See, for example, [1], proposition 3.6 page 27. □

The calculation of representation theory is more comfortable if we calculate on character tables. The character table has meaning on complex representation. However, we can use character table on both real and quaternionic representation by working on their complexification. To do that, knowing the additive generator of $RO(G), RU(G)$ and $RSp(G)$ is useful. Suppose $\{U_i\}_{i\in I}$ is the list of simple real representations, $\{V_j, \tau V_j\}_{j\in J}$ is the list of simple complex representations, and $\{W_k\}_{k\in K}$ is the list of simple quaternionic representations. By the assertions from Chapter 2 (page 18-19) in [13] again, we have the additive basis of the representation groups as;

- $RO(G) = \mathbb{Z}\{U_i, rV_j, r\widetilde{c}W_k \quad | \quad i \in I, j \in J, k \in K\}$,
- $RU(G) = \mathbb{Z}\{cU_i, V_j, \tau V_j, \widetilde{c}W_k \quad | \quad i \in I, j \in J, k \in K\}$,
- $RSp(G) = \mathbb{Z}\{qcU_i, qV_j, W_k \quad | \quad i \in I, j \in J, k \in K\}$.

Thus, by these lists and Lemma 1.2.3, the calculation on character table is possible.

It is well known that the periodic real K-theory is an associated cohomology theory of the spectrum $KO = \mathbb{Z} \times BO$ which has period 8. Real connective K-cohomology theory is an associated cohomology theory of the connective cover of KO. In particular,

$$KO^*(pt) \cong \mathbb{Z}[\eta, \alpha, \beta, \beta^{-1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta),$$

with $\eta \in KO^{-1}(pt)$, $\alpha \in KO^{-4}(pt)$ and $\beta \in KO^{-8}(pt)$. And

$$ko^*(pt) \cong \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta).$$

Moreover, for non negative degrees, $ko^*(BG) \cong KO^*(BG)$ and in general, we have evidently $ko^*(BG)[\beta^{-1}] \cong KO^*(BG)$, see more details about this discussion in [13].

There is also the relations between complex and real K-theory in lemma 2.2.11 in [13] which is;

Lemma 1.2.4. *(Lemma 2.1.11 in [13])*
Complexification $KO^* \xrightarrow{c} KU^*$ *is the ring homomorphism given by* $c(\eta) = 0, c(\alpha) = 2v^2$ *and* $c(\beta) = v^4$. *Realification* $KU^* \xrightarrow{r} KO^*$ *is the* KO^**-module homomorphism given by* $r(1) = 2, r(v) = \eta^2, r(v^2) = \alpha$ *and* $r(v^3) = 0$.

The correspondence between representation theory and periodic real K-theory is given by M.F.Atiyah and G.B.Segal in [4] which we record as;

Theorem 1.2.5. *(cf.[13]) For compact groups* G,

$$KO^*(BG) \cong RO^{\epsilon}(G)^{\wedge}_{J}[\beta, \beta^{-1}],$$

such that ($\epsilon = 0, -1, ..., -7$)

$$\begin{array}{lll}
RO^0(G) \cong RO(G) & , & RO^{-1}(G) \cong RO(G)/rRU(G), \\
RO^{-2}(G) \cong RU(G)/\tilde{c}RSp(G) & , & RO^{-3}(G) = 0, \\
RO^{-4}(G) \cong RSp(G) & , & RO^{-5}(G) \cong RSp(G)/qRU(G), \\
RO^{-6}(G) \cong RU(G)/cRO(G) & , & RO^{-7}(G) = 0,
\end{array}$$

where J *is the augmentation ideal of* $RO(G)$, β *is Bott element in* $KO^8(pt)$. *If, moreover,* G *is a* p*-groups, then* J*-adic topology coincides with* p*-adic topology.*

Proof. See [5] and [4]. □

1.2.2 Chern classes and Stiefel-Whitney classes

The relation between Stiefel-Whitney class of real representation V of G and reduction mod 2 of Chern class for complexification of V, say $V_{\mathbb{C}}$, is given by;[2]

Proposition 1.2.6. *For an* n*-dimensional real representation* V *and its complexification* $V_{\mathbb{C}}$,

$$\overline{c_i(V_{\mathbb{C}})} = [w_i(V)]^2 \quad \text{in } H^*(BG; \mathbb{F}_2).$$

Proof. Let V be n-dimension real representation of a group G which is represented by $\rho_V : G \longrightarrow O(n)$. So, $V_{\mathbb{C}}$ can be represented by $\rho_{V_{\mathbb{C}}} : G \longrightarrow U(n)$ which $\rho_{V_{\mathbb{C}}} = c \circ \rho_V$, where $c : O(n) \longrightarrow U(n)$ is complexification. This induces

$$BG \xrightarrow{B\rho_V} BO(n) \xrightarrow{Bc} BU(n),$$

[2] $\overline{a}$ means reduction mod 2

and

$$H^*(BU(n);\mathbb{F}_2)=\mathbb{F}_2[\overline{c}_1,\overline{c}_2,...,\overline{c}_n]\to H^*(BO(n);\mathbb{F}_2)=\mathbb{F}_2[w_1,w_2,...,w_n]\to H^*(BG;\mathbb{F}_2),$$

such that $(B\rho_V)^*(w_i)=w_i(V)$ and $(B\rho_{V_{\mathbb{C}}})^*(\overline{c}_i)=\overline{c}_i(V_{\mathbb{C}})$. So, it remains to show that

$$(Bc)^*(\overline{c}_i)=w_i^2.$$

For $n=1$, we consider maps $O(n)\xrightarrow{det}O(1)$ and $U(n)\xrightarrow{det}U(1)$, which induces

$$BO(n)\xrightarrow{Bdet=w_1}BO(1)\ =K(\mathbb{Z}/2,1)=\mathbb{R}P^\infty,$$

and

$$BU(n)\xrightarrow{Bdet=c_1}BU(1)\ =K(\mathbb{Z},2)=\mathbb{C}P^\infty.$$

And thus, we have a commutative diagram;

$$\begin{tikzcd}
H^*(BU(1);\mathbb{F}_2) \arrow[r, "(Bc)_1^*"] \arrow[d, "c_1^*"] & H^*(BO(1);\mathbb{F}_2) \arrow[d, "w_1^*"] \\
H^*(BU(n);\mathbb{F}_2) \arrow[r, "(Bc)_n^*"] & H^*(BO(n);\mathbb{F}_2).
\end{tikzcd}$$

We see that the image of $\overline{c}_1\in H^*(BU(1);\mathbb{F}_2)$ via $(Bc)_n^*\circ c_1^*$ is $(Bc)^*(\overline{c}_1)$, because $c_1^*(\overline{c}_1)=\overline{c}_1\in H^*(BU(n);\mathbb{F}_2)$, which is also the image of $(Bc)_1^*(\overline{c}_1)=\lambda w_1^2$ (degree of $\overline{c}_1$ is 2) under w_1^*. We need to check that the image of $\overline{c}_1$ is not zero under $(Bc)_1^*$ i.e., need to check that $\lambda=1$. To do this, we use Serre spectral sequence for fibre sequence

$$BO(1)\longrightarrow\ BU(1)\xrightarrow{B2}BU(1)\ ,$$

which is simple to see that $\overline{c}_1$ is detected by w_1^2.

For $n>1$, we use the splitting principle. Note that $\overline{c}_i\in H^*(BU(n);\mathbb{F}_2)$ is the i^{th}-symmetric function on generator $\overline{x}_1,\overline{x}_2,...,\overline{x}_n$, where $\overline{x}_i=\overline{c}_1(z_i)$ for some 1-dimensional complex representation z_i. By case $n=1$, $\overline{x}_i\longmapsto t_i^2=[w_1(\xi_i)]^2$ for some 1-dimensional real representation ξ_i and hence

$$\overline{c}_i=\sigma_i(\overline{x}_1,\overline{x}_2,...,\overline{x}_n)\longmapsto\sigma_i(t_1^2,t_2^2,...,t_n^2)=[\sigma_i(t_1,t_2,...,t_n)]^2=w_i^2,$$

which completes the proof. □

For Stiefel-Whitney classes of n-dimension complex representation W of a group G and its relation with Chern classes, we have that $w_{2i+1}(W)=0$ and $w_{2i}(W)$ is the image of $c_i(W)$ under the coefficient homomorphism $H^{2i}(BG;\mathbb{Z})\longrightarrow H^{2i}(BG;\mathbb{F}_2)$ see, for example, proposition 3.8, page 83 in [23]. Similarly, for Pontryagin classes, see, for example, page 94 in [23], there is an assertion that

$$\overline{c_{2i}(V_{\mathbb{C}})}=\overline{p_i(V)}. \tag{1.9}$$

§ 1.3 Killing homotopy groups

Here, we investigate some long exact sequences involving to the relations of cohomology ring and real and complex connective K-theory. Let X be an spectrum. One can construct $X^n_\downarrow := X(-\infty, n]$ which is the spectrum X such that the homotopy above degree n are killed, i.e. $X^n_\downarrow = X \cup e^{n+2} \cup e^{n+3} \cup \ldots$. Then we have a natural map $X \xrightarrow{k} X^n_\downarrow$ and thus this forms a fibration

$$Fibre(k) \xrightarrow{l} X \xrightarrow{k} X^n_\downarrow ,$$

where $Fibre(k) = X^\uparrow_{n+1} = X < n+1 >$ is n*-connected cover of* X or $n+1$*-connective cover of* X, which yields that $\pi_i(k)$ is an isomorphism for $i \leq n$ and $\pi_i(l)$ is an isomorphism for $i \geq n+1$. By definition, for the Eilenberg-Maclane spectrum HA, we have

$$\pi_i(HA) = \begin{cases} A, & \text{if } i = 0; \\ 0, & \text{if } i \neq 0, \end{cases}$$

and it is well know that

$$\text{if } \pi_i(X) = \begin{cases} A, & \text{if } i = n; \\ 0, & \text{if } i \neq n, \end{cases} \quad \text{then } X \simeq \Sigma^n HA.$$

By these facts, we have, for example;

1 Cofibre sequence

$$ku = KU^\uparrow_0 \longrightarrow KU \longrightarrow KU^{-1}_\downarrow,$$

i.e. ku is a connective cover of KU,

2 Cofibre sequence

$$ko < 1 >= (ko^\uparrow_1) \longrightarrow ko \longrightarrow (ko^0_\downarrow) = H\mathbb{Z},$$

3 Cofibre sequence

$$ko < 2 > \longrightarrow ko < 1 > \longrightarrow \Sigma H\mathbb{F}_2,$$

because $\pi_* ko = (\mathbb{Z}, \underbrace{2,2,0,\mathbb{Z},0,0,0,\mathbb{Z},2,2,0,\mathbb{Z},0,0,0,\ldots}_{\pi_* ko<1>})$, (periodicity is 8).

By applying $\wedge X$ and π_* to the cofibre sequence 2, we obtain a long exact sequence;

$$\cdots \longrightarrow \pi_1(ko < 1 > \wedge X) \longrightarrow \pi_1(ko \wedge X) \longrightarrow \pi_1(H\mathbb{Z} \wedge X) \longrightarrow \pi_0(ko < 1 > \wedge X) \longrightarrow \cdots, \tag{1.10}$$

and clearly $\pi_0(ko < 1 > \wedge X) = 0$. By applying $\wedge X$ (connective) and π_* to the cofibre sequence 3, we obtain a long exact sequence

$$\cdots \pi_1(ko < 2 > \wedge X) \longrightarrow \pi_1(ko < 1 > \wedge X) \xrightarrow{\cong} \pi_1(\Sigma H\mathbb{F}_2 \wedge X) \longrightarrow 0 \longrightarrow 0, \tag{1.11}$$

such that $\pi_1(ko < 2 > \wedge X)$ is obviously zero and hence $\pi_1(ko < 1 > \wedge X) = H_0(X;\mathbb{F}_2)$. Therefore, by (1.10), (1.11) and $\pi_n(E \wedge X) := E_n(X)$ (definition in [2]), there is a natural long exact sequence;

$$\cdots \longrightarrow H_2(X;\mathbb{Z}) \longrightarrow H_0(X;\mathbb{F}_2) \longrightarrow ko_1(X) \longrightarrow H_1(X;\mathbb{Z}) \longrightarrow 0. \tag{1.12}$$

Note also further that the cofibre sequence

$$\Sigma ko \xrightarrow{\eta} ko \longrightarrow ku,$$

in [13], yields the induced long exact sequence

$$\longrightarrow ko_{n-1}(BG) \longrightarrow ko_n(BG) \longrightarrow ku_n(BG) \longrightarrow \tag{1.13}$$

and in particular;

$$\begin{gathered}\cdots \longrightarrow ku_3(BG) \longrightarrow ko_1(BG) \longrightarrow ko_2(BG) \longrightarrow ku_2(BG) \longrightarrow ko_0(BG) \longrightarrow \\ ko_1(BG) \longrightarrow ku_1(BG) \longrightarrow ko_{-1}(BG) \longrightarrow ko_0(BG) \longrightarrow ku_0(BG) \longrightarrow 0.\end{gathered}$$

Since $ko_{-1}(BG) = 0$, this exact sequence splits as

$$ko_0(BG) \cong ku_0(BG), \tag{1.14}$$

and

$$\begin{gathered}\cdots \longrightarrow ku_3(BG) \longrightarrow ko_1(BG) \longrightarrow ko_2(BG) \longrightarrow ku_2(BG) \longrightarrow ko_0(BG) \longrightarrow \\ ko_1(BG) \longrightarrow ku_1(BG) \longrightarrow 0.\end{gathered}$$

Remark 1.3.1. *It is well known that;*

- $ko^0(BG) = KO^0(BG) = RO(G)^\wedge_J$.
- $ko_0(BG) = H_0(BG;\mathbb{Z}) = \mathbb{Z}$, *and in fact* $ko_0(X) = H_0(BG;\mathbb{Z}) = \mathbb{Z}$, *for any space* X.
- $ku^0(BG) = KU^0(BG) = RU(G)^\wedge_J$.
- $KU_0(BG) = H^0_J(R(G)) = \mathbb{Z}$ *and* $KU_1(BG) = H^1_J(R(G))$ *for finite groups* G.
- $H_0(BG;\mathbb{F}_p) = \mathbb{F}_p$, *for prime* p.
- $H_1(BG;\mathbb{Z}) \cong G^{ab}$, *e.g.* $H_1(BSD_{16};\mathbb{Z}) \cong SD_{16}^{ab} = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Chapter 2

Complex connective K-cohomology

In this chapter, we will calculate complex connective K-cohomology for semi-dihedral group as a ring by using Adams spectral sequence with initial input $H^*(BSD_{2^n};\mathbb{F}_2)$ or equally, $H^*(SD_{2^n};\mathbb{F}_2)$. Actually, the Adams spectral sequence for calculation $ku^*(BG)$, where G is finite group, is given by;

$$E_2^{s,t} = \mathrm{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2)) \Longrightarrow ku^{-(t-s)}(BG)_2^{\wedge},$$

where $E(1)$ denotes the exterior algebra on the Milnor generators Q_0 and Q_1 ([14] page 28). The experiences from the book of R.R.Bruner and J.P.C. Greenlees , see [14], suggest that representation theory is fruitful to determine differentials, additive generator and multiplicative structure. Therefore, it is reasonable to calculate the character table as the first step and then followed by the calculation of E_2-page, their differentials, the additive structure and finally the multiplicative structure.

§ 2.1 Character table of semi-dihedral group

Presentation for semi-dihedral group of order 2^n, for $n \geq 4$, is given by

$$SD_{2^n} = \{s, t | s^{2^{n-1}} = t^2 = 1, tst = s^{2^{n-2}-1}\}.$$

In this section, we need to find the character table of them explicitly. To gain the general (order 2^n for $n \geq 4$) tables, it is natural to deal with the case $n = 4$ first.

2.1.1 Character table of SD_{16}

In this case, the structure of SD_{16} is represented by

$$\begin{aligned} SD_{16} &= < s, t | s^8 = t^2 = 1, tst = s^3 > \\ &= \{1, t, s, s^2, s^3, s^4, s^5, s^6, s^7, ts, ts^2, ts^3, ts^4, ts^5, ts^6, ts^7\} \end{aligned}$$

Note that any complex representation does not change under the same conjugacy class and can identify with its character. So, it is make sense to find its conjugacy class first. Recall that for any group G and $a \in G$, the conjugacy class of a, say $[a]$, consists of $b \in G$ such that $gag^{-1} = b$ for some $g \in G$. In other words, $[a] = \{gag^{-1} | g \in G\}$. Thus the conjugacy classes of s^m and ts^m, for each $m \in \{1, 2, 3, ..., 7\}$, can be found as below.

Lemma 2.1.1. *The conjugacy classes of SD_{16} consist of 7 classes which are* $[1] = \{1\}$, $[s] = \{s, s^3\}$, $[s^2] = \{s^2, s^6\}$, $[s^4] = \{s^4\}$, $[s^5] = \{s^5, s^7\}$, $[t] = \{t, ts^2, ts^4, ts^6\}$ *and* $[ts] = \{ts, ts^3, ts^5, ts^7\}$.

Proof. From the relation in SD_{16}, we get that:

$$\begin{aligned} ts^k &= s^{3k}t \\ (s^k)^{-1} &= s^{8-k} \\ (ts^k)^{-1} &= ts^{5k} \end{aligned}$$

for each $k \geq 0$. Thus, we have $(s^k)(s^m)(s^k)^{-1} = s^m$ and

$$\begin{aligned} (ts^k)(s^m)(ts^k)^{-1} &= (ts^k)(s^m)(ts^{5k}) = (ts^{m+k})(ts^{5k}) \\ &= (s^{3(m+k)}t)(ts^{5k}) = s^{3m} \end{aligned}$$

Then the conjugacy classes of s^m is $\{s^m, s^{3m}\}$ for each $m \in \{1, 2, 3, ..., 7\}$. Similarly, for conjugacy classes of ts^m,

$$\begin{aligned} (s^k)(ts^m)(s^k)^{-1} &= (s^k)(ts^m)(s^{8-k}) = (s^k)(s^{3m}t)(s^{8-k}) \\ &= (s^{3m+k})(ts^{8-k}) = (s^{3m+k})(s^{3(8-k)}t) \\ &= (s^{-2k})(ts^m), \text{ for all } k \in \{1, 2, 3, ..., 7\} \end{aligned}$$

and

$$\begin{aligned} (ts^k)(ts^m)(ts^k)^{-1} &= (s^{3k}t)(ts^m)(ts^{5k}) = (s^{3k+m})(ts^{5k}) \\ &= (s^{3k+m})(s^{15k}t) \\ &= s^{2k+m}t, \text{ for all } k \in \{1, 2, 3, ..., 7\}. \end{aligned}$$

So, the conjugacy classes of ts^m is $\{(s^{-2k})(ts^m), s^{2k+m}t \mid k \in \{1, 2, 3, ..., 7\}\}$ for each $m \in \{1, 2, 3, ..., 7\}$. ☐

Before doing further calculation, let us recall and collect some properties of linear representation of finite groups involving to our computation .

Proposition 2.1.2. *Let V be a complex vector space of dimension n and G be a finite group. If χ is the character of a representation ρ ($\rho : G \to GL(V)$) of degree n i.e. $\chi_\rho(s) = Tr(\rho(s))$ for each $s \in G$, we have:*

(1) $\chi(1) = n$, *degree of* ρ.

(2) $\chi(s^{-1}) = \overline{\chi(s)}$, *conjugate of complex number, for all* $s \in G$.

(3) $\chi(tst^{-1}) = \chi(s)$ *for all* $s \in G$.

(4) *If* ϕ *is the character of a representation* V, *then* (ϕ, ϕ) *is a positive integer and we have* $(\phi, \phi) = 1$ *if and only if* V *is irreducible, where*

$$(\phi, \phi) = \frac{1}{|G|} \sum_{s \in G} \phi(s)\overline{\phi(s)}.$$

(5) *Two representations with the same character are isomorphic.*
(***Note:*** $\rho \cong \rho' \Leftrightarrow TR_s = R'_s T$ *for some invertible matrix* T *and for all* $s \in G$, *where* R_s *and* R'_s *are the representation matrixes of* $\rho(s)$ *and* $\rho'(s)$ *respectively.)*

(6) *The number of irreducible representations of* G *(up to isomorphism) is equal to the number of conjugacy classes of* G.

(7) *The degree of the irreducible representation of* G *divide the order of* G. *Furthermore, it also divides* $(G : C)$ *where* C *is the centre of* G.

(8) *The character* r_G *of the regular representation is given by:* $r_G(1) = |G|$ *and* $r_G(s) = 0$ *if* $s \neq 1$.

(9) *If the irreducible characters of* G *are* $\chi_1, \chi_2, ..., \chi_h$ *then* $|G| = \sum_{i=1}^{h} n_i^2$ *where* $n_i = \chi_i(1)$ *and if* $s \in G$ *is different from 1, then we have* $\sum_{i=1}^{h} n_i \chi_i(s) = 0$.

Proof. See the book of J.P. Serre (Linear Representations of Finite Groups), [33]. □

Here, from lemma 2.1.1, SD_{16} has 7 conjugate classes. So, it is easy to see that this group has only 4 irreducible representations of dimension one. Thus, by the above proposition, it must have 3 irreducible representations of dimension 2. We define the representation ρ^h of SD_{16} by setting:

$$\rho^h(s^m) = \begin{pmatrix} w^{hm} & 0 \\ 0 & w^{3hm} \end{pmatrix} \text{ and } \rho^h(ts^m) = \begin{pmatrix} 0 & w^{3hm} \\ w^{hm} & 0 \end{pmatrix},$$

where $w = e^{\frac{\pi i}{4}}$ and $h, m \in \{0, 1, 2, 3, ..., 7\}$. It is not hard to check that ρ^h is representation for each $h \in \{0, 1, 2, 3, ..., 7\}$.

Moreover, we can see that $\rho^1 \cong \rho^3$, $\rho^2 \cong \rho^6$ and $\rho^5 \cong \rho^7$ because they have the same character. We claim that ρ^1, ρ^2 and ρ^5 are irreducible representation. This claim can be verified easily by direct calculation as follows.

$$\begin{aligned}
(\chi_{\rho^5}, \chi_{\rho^5}) &= \frac{1}{16}\sum_{x\in G} \chi_{\rho^5}\overline{\chi_{\rho^5}} \\
&= \frac{1}{16}\sum_{m=0}^{7} \chi_{\rho^5}(s^m)\overline{\chi_{\rho^5}(s^m)}, (\text{since } \chi_{\rho^5}(ts^m) = 0, \forall m \in \{0,1,2,3,...,7\}) \\
&= \frac{1}{16}\sum_{m=0}^{7} (w^{5m} + w^{15m})\overline{(w^{5m} + w^{15m})} \\
&= \frac{1}{16}\sum_{m=0}^{7} [2 + (w^{10m} + w^{-10m})] \\
&= \frac{1}{16}\sum_{m=0}^{7} [2 + 2\cos\frac{5m\pi}{2}] = 1
\end{aligned}$$

So, ρ^5 is irreducible and by the same calculation, the conclusion for ρ^1 and ρ^2 follows. Furthermore, we also see that ρ^4 is not irreducible representation because $(\chi_{\rho^4}, \chi_{\rho^4}) \neq 1$.

Whence, we obtain the character table of SD_{16} as below.

SD_{16}	$[1]$	$[s^4]$	$[s]$	$[s^2]$	$[s^5]$	$[t]$	$[ts]$
1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	-1	-1
χ_3	1	1	-1	1	-1	1	-1
χ_4	1	1	-1	1	-1	-1	1
χ_ρ	2	-2	$\sqrt{2}i$	0	$-\sqrt{2}i$	0	0
χ_{ρ^2}	2	2	0	-2	0	0	0
χ_{ρ^5}	2	-2	$-\sqrt{2}i$	0	$\sqrt{2}i$	0	0

Table 2.1: *The character table of* SD_{16}

Hence, from the table, we can see relations and then get the representation ring of semi-dihedral group of order 16 as (by setting $\sigma_1 =$ *representation with character* χ_ρ, $\sigma_2 =$ *representation with character* χ_{ρ^2}, $\sigma_3 =$ *representation with character* χ_{ρ^5})[1]:

$$R(SD_{16}) = Z[\chi_2, \chi_3, \chi_4, \sigma_1, \sigma_2, \sigma_3]/R \tag{2.1}$$

$R = (\chi_2^2 = \chi_3^2 = \chi_4^2 = 1,\ \chi_2\chi_3 = \chi_4,\ \sigma_1^2 = \sigma_3^2 = \sigma_2 + \chi_3 + \chi_4,\ \sigma_2^2 = 1 + \chi_2 + \chi_3 + \chi_4,$ $\sigma_1\sigma_2 = \sigma_2\sigma_3 = \sigma_1 + \sigma_3,\ \sigma_1\sigma_3 = \sigma_2 + 1 + \chi_2,\ \chi_2\sigma_1 = \sigma_1,\ \chi_2\sigma_2 = \sigma_2,\ \chi_2\sigma_3 = \sigma_3,$ $\chi_2\sigma_2 = \chi_3\sigma_2 = \chi_4\sigma_2 = \sigma_2,\ \chi_3\sigma_1 = \chi_4\sigma_1 = \sigma_3,\ \chi_3\sigma_3 = \chi_4\sigma_3 = \sigma_1)$

[1]We change the notation to avoid the double power of ρ^i.

2.1.2 Character table of SD_{2^n} for $n \geq 5$

We proceed this calculation as the case $n = 4$ starting with finding conjugacy classes of SD_{2^n} first.

Lemma 2.1.3. *The conjugacy classes of SD_{2^n} consist of $2^{n-2} + 3$ classes which are $[1]$, $[t]$, $[ts]$ and $[s^m]$ for each $m \in C' := \{1, 2, 3, ..., 2^{n-3}, 2^{n-3}+2, 2^{n-3}+4, ..., 2^{n-2}, 2^{n-2}+1, 2^{n-2} + 3, ..., 2^{n-2} + (2^{n-3} - 1)\}$.*

Proof. In SD_{2^n}, we have

$$ts^k = s^{(2^{n-2}-1)k}t,$$
$$s^k t = ts^{(2^{n-2}-1)k},$$
$$(ts^k)^{-1} = ts^{(2^{n-2}+1)k}.$$

By the same process as lemma 2.1.1, the conjugacy classes of SD_{2^n} are $\{1\}$, $\{s^m, s^{(2^{n-2}-1)m}\}$ for each $m \in \{1, 2, 3, ..., 2^{n-1} - 1\}$, $\{ts^{2k} \mid k \in \{0, 1, 2, ..., 2^{n-2} - 1\}\}$ and $\{ts^{2k+1} \mid k \in \{0, 1, 2, ..., 2^{n-2} - 1\}\}$. To be more precise, we need to explicit the collection of the different conjugacy classes coming from the part of s^m, say C^*. Note in SD_{2^n} that $s^{2^{n-1}} = 1$ then

$$s^{r_1} = s^{r_2} \iff r_1 \equiv r_2 \bmod 2^{n-1}$$

and also for each $i \in C = \{1, 2, 3, ..., 2^{n-1} - 1\}$,

$$i \equiv i(2^{n-2} - 1) \bmod 2^{n-1} \iff i = 2^{n-2}.$$

Let $C_1 = \{1, 2, 3, ..., 2^{n-3}\}$, $C_2 = \{2^{n-3} + i : i \in C_1\}$, $C_3 = \{2^{n-2} + i : i \in C_1\}$ and $C_4 = \{2^{n-2} + 2^{n-3} + i : i \in C_1 - \{2^{n-3}\}\}$. By direct calculation, from the set $C = C_1 \cup C_2 \cup C_3 \cup C_4$, we see that: for each $i \in C_1$, if i is odd, then

$$\begin{aligned} (2^{n-2} - 1)i &\equiv (2^{n-2} - i) \bmod 2^{n-1} \\ (2^{n-2} - 1)(2^{n-2} + i) &\equiv (2^{n-1} - i) \bmod 2^{n-1} \end{aligned}$$

and if i is even, then

$$\begin{aligned} (2^{n-2} - 1)i &\equiv (2^{n-1} - i) \bmod 2^{n-1} \\ (2^{n-2} - 1)(2^{n-3} + i) &\equiv (2^{n-2} + (2^{n-3} - i)) \bmod 2^{n-1}. \end{aligned}$$

This means that C^* consists of

$$[s^i] = \{s^i, s^{2^{n-2}-i}\} \text{ for each odd element } i \text{ of } C_1,$$
$$[s^i] = \{s^i, s^{2^{n-1}-i}\} \text{ for each even element } i \text{ of } C_1,$$
$$[s^{2^{n-3}+i}] = \{s^{2^{n-3}+i}, s^{2^{n-2}+(2^{n-3}-i)}\} \text{ for each even element } i \text{ of } C_1,$$
$$\text{and } [s^{2^{n-2}+i}] = \{s^{2^{n-2}+i}, s^{2^{n-1}-i}\} \text{ for each odd element } i \text{ of } C_1,$$

which completes the proof. $\square$

In this case, there are four 1-dimensional representations as in the case $n = 4$. For 2-dimensional representations, we define

$$\rho^h(s^m) = \begin{pmatrix} w^{hm} & 0 \\ 0 & w^{(2^{n-2}-1)hm} \end{pmatrix} \text{and } \rho^h(ts^m) = \begin{pmatrix} 0 & w^{(2^{n-2}-1)hm} \\ w^{hm} & 0 \end{pmatrix},$$

where $w = e^{\frac{\pi i}{(2^{n-2})}}$, h and m are in $\{1, 2, 3, ..., 2^{n-1}\}$. But, by the equality of character, $\rho^{h_1} \cong \rho^{h_2}$ if h_1 and h_2 are the power of s in the same conjugacy class, i.e. $[s^{h_1}] = \{s^{h_1}, s^{h_2}\}$, so we can say instead that $h \in C'$.

Lemma 2.1.4. *All ρ^h where $h \in C' - \{2^{n-2}\}$ are irreducible representations, where C' is the same set as in lemma 2.1.3.*

Proof. Let $h \in C'$. Consider

$$\begin{aligned}
(\chi_{\rho^h}, \chi_{\rho^h}) &= \frac{1}{2^n} \sum_{x \in G} \chi_{\rho^h}(x)\overline{\chi_{\rho^h}(x)} \\
&= \frac{1}{2^n} \sum_{m=1}^{2^{n-1}} \chi_{\rho^h}(s^m)\overline{\chi_{\rho^h}(s^m)}, (\text{since } \chi_{\rho^h}(ts^m) = 0, \forall m \in \{1, 2, 3, ..., 2^{n-1}\}) \\
&= \frac{1}{2^n} \sum_{m=1}^{2^{n-1}} (w^{hm} + w^{(2^{n-2}-1)hm})\overline{(w^{hm} + w^{(2^{n-2}-1)hm})} \\
&= \frac{1}{2^n} \sum_{m=1}^{2^{n-1}} [2 + (w^{(2^{n-2})hm} + w^{-(2^{n-2})hm})]
\end{aligned}$$

Thus,

$$(\chi_{\rho^h}, \chi_{\rho^h}) = \frac{1}{2^n} \sum_{m=1}^{2^{n-1}} [2 + 2\cos\frac{(2^{n-2}-2)hm\pi}{2^{n-2}}].$$

By using the formula $1 + 2\cos x + 2\cos 2x + 2\cos 3x + ... + 2\cos nx = \frac{\sin((n+\frac{1}{2})x)}{\sin\frac{x}{2}}$ and $\sin(2\pi + \theta) = \sin(\theta)$, we obtain that

$$\frac{1}{2^n} \sum_{m=1}^{2^{n-1}} 2\cos\frac{(2^{n-2}-2)hm\pi}{2^{n-2}} = 0$$

i.e. $(\chi_{\rho^h}, \chi_{\rho^h}) = 1, \forall h \in C' - \{2^{n-2}\}$. In particular, it is easy to see that $(\chi_{\rho^{2^{n-2}}}, \chi_{\rho^{2^{n-2}}}) \neq 1$, then $\rho^{2^{n-2}}$ is not irreducible which completes the proof. □

Finally, we reach to the character table of SD_{2^n} where $n \geq 5$. This table includes 4 one-dimensional irreducible characters and $2^{n-2} - 1$ two-dimensional irreducible characters which consists of χ_{ρ^h} for each $h \in C' - \{2^{n-2}\}$. For the filling any entry in the table, it is very useful to know that (which is easy to prove by using basic identity of trigonometry): for each $n \in \mathbb{N}$, natural number,

$$w^n + w^{(2^{n-2}-1)n} = \begin{cases} 2i\sin\frac{n\pi}{2^{n-2}}, & \text{if } n \text{ is odd;} \\ 2\cos\frac{n\pi}{2^{n-2}}, & \text{if } n \text{ is even.} \end{cases}$$

So,

$$* = \chi_{\rho^h}(s^m) = w^{hm} + w^{(2^{n-2}-1)hm}$$

$$= \begin{cases} 2i\sin\frac{hm\pi}{2^{n-2}}, & \text{if } hm \text{ is odd;} \\ 2\cos\frac{hm\pi}{2^{n-2}}, & \text{if } hm \text{ is even.} \end{cases}$$

Whence, we obtain the character table of $SD_{2^n}, n \geq 5, m \in C'$ and $h \in C' - \{2^{n-2}\}$ as below.

SD_{2^n}	$[1]$	$[s^{\alpha_0}]$	$[s]$	$[s^2]$	$[s^m]$	$[s^{\alpha_1}]$	$[t]$	$[ts]$
1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	-1	-1
χ_3	1	1	-1	1	$(-1)^m$	-1	1	-1
χ_4	1	1	-1	1	$(-1)^m$	-1	-1	1
χ_ρ	2	-2	$2i\sin\frac{\pi}{2^{n-2}}$	$2\cos\frac{\pi}{2^{n-3}}$	$*$	$-2i\sin\frac{\pi}{2^{n-2}}$	0	0
χ_{ρ^2}	2	2	$2\cos\frac{\pi}{2^{n-3}}$	$2\cos\frac{\pi}{2^{n-4}}$	α	$-2\cos\frac{\pi}{2^{n-3}}$	0	0
χ_{ρ^3}	2	-2	$2i\sin\frac{3\pi}{2^{n-2}}$	$2\cos\frac{3\pi}{2^{n-3}}$	$*$	$2i\sin\frac{3\pi}{2^{n-2}}$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$
χ_{ρ^h}	2	± 2	$*$	$2\cos\frac{h\pi}{2^{n-3}}$	$*$	$*$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$	$\vdots$
$\chi_{\rho^{\alpha_1}}$	2	-2	$-2i\sin\frac{\pi}{2^{n-2}}$	$-2\cos\frac{\pi}{2^{n-3}}$	$*$	$-2i\sin\frac{\pi}{2^{n-2}}$	0	0

Table 2.2: *The character table of* $SD_{2^n}, n \geq 5$ *where* $\alpha_0 = 2^{n-2}$, $\alpha_1 = 3(2^{n-3}) - 1$ *and* $\alpha = 2\cos\frac{m\pi}{2^{n-3}}$.

§ 2.2 mod 2 cohomology ring of semi-dihedral group

2.2.1 Cohomology ring of semi-dihedral group

L.Evens and S.Priddy, [16], they calculated the cohomology of semi-dihedral groups, both with integer and mod 2 coefficients. Their method for the calculation of mod 2 coefficient is to compare with the cohomology of dihedral and quaternion groups.

Specifically, using names for generators corresponding to those for the semi-dihedral group, $SD_{2^n} = Gp < s, t : s^{2^{n-1}} = t^2 = 1, tst = s^{2^{n-2}-1} >$, we take

$$D_{2^{n-1}} = Gp < \overline{s}, t \mid \overline{s}^{2^{n-2}} = 1 = t^2; t\overline{s}t = \overline{s}^{-1} >, \text{ for } n \geq 4,$$

which is a quotient of SD_{2^n} and

$$D_8 = Gp < \overline{s}^4 = 1 = t^2; t\overline{s}t = \overline{s}^{-1} > .$$

Similarly

$$Q_8 = Gp < \sigma, \tau \mid \sigma^4 = 1, \tau^2 = \sigma^2; \tau\sigma\tau^{-1} = \sigma^{-1} > .$$

Thus we may view D_8, Q_8 as subgroups of SD_{2^n} under the inclusions

$$\Phi : D_8 \longrightarrow SD_{2^n}; \overline{s} \mapsto s^{2^{n-3}}, t \mapsto t, \tag{2.2}$$

$$\Psi : Q_8 \longrightarrow SD_{2^n}; \sigma \mapsto s^{2^{n-3}}, \tau \mapsto st. \tag{2.3}$$

The cohomology of these groups is known [17]. In fact, we have

$$H^*(D_{2^{n-1}}; \mathbb{Z}/2) = \mathbb{Z}/2[x, y, w_2]/(x^2 + xy), \tag{2.4}$$

where x, y are one-dimensional classes defined by $< x, \overline{s} >= 1, < x, t >= 0$, $< y, t >= 1, < y, \overline{s} >= 0$. The class w_2 is the second Stiefel-Whitney class of the representation of $D_{2^{n-1}}$ on the plane (its first Stiefel-Whitney class is y). Similarly,

$$H^*(D_8; \mathbb{Z}/2) = \mathbb{Z}/2[\overline{x}, \overline{y}, \overline{w}_2]/(\overline{x}^2 + \overline{xy}), \tag{2.5}$$

where $< x, \overline{s} >= 1, < x, t >= 0$, $< y, t >= 1, < y, \overline{s} >= 0$ and $\overline{w}_2$ restricts to $\overline{z}^2$. Here, the class $\overline{z}$ is 1-dimensional class in the cohomology group of the cyclic subgroup generated by $\overline{s}^2$. The cohomology of the quaternion group of order 8 is given by

$$H^*(Q_8; \mathbb{Z}/2) = \mathbb{Z}/2[\widetilde{x}, \widetilde{y}, \widetilde{P}]/(\widetilde{x}^2 + \widetilde{x}\widetilde{y} + \widetilde{y}^2, \widetilde{x}^2\widetilde{y} + \widetilde{x}\widetilde{y}^2), \tag{2.6}$$

where $\widetilde{x}$ and $\widetilde{y}$ are one dimensional classes defined by $< \widetilde{x}, \sigma >= 1, < \widetilde{x}, \tau >= 0$ and $< \widetilde{y}, \tau >= 1, < \widetilde{y}, \sigma >= 0$. The class $\widetilde{P}$ is the mod 2 reduction of the first Pontryagin class of the natural representation of Q_8 on the quaternions; it restricts to $\widetilde{z}^4$ where $\widetilde{z}$ is the one dimensional class in the cohomology of the cyclic subgroup generated by σ^2, [17].

Now Evens and Priddy approach the cohomology of SD_{2^n} by considering the Lyndon-Hochschild-Serre spectral sequence of the central extension

$$(A) : \mathbb{Z}/2 < s^{2^{n-2}} > \longrightarrow SD_{2^n} \longrightarrow D_{2^{n-1}}.$$

To gain control of it they compare it with the central extensions

$$(B) : \mathbb{Z}/2 < s^2 > \longrightarrow D_8 \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 < \overline{s}, t >$$

and

$$(C) : \mathbb{Z}/2 < \sigma^2 > \longrightarrow Q_8 \longrightarrow \mathbb{Z}/2 \times \mathbb{Z}/2 < \sigma, \tau >,$$

which are already understood.

Since we need some of the details we will run through the proof.

Proposition 2.2.1. *[16] The cohomology of the semi-dihedral groups is given by the formula*

$$H^*(SD_{2^n}; \mathbb{Z}/2) = \mathbb{Z}/2[x, y, u, P]/(x^2 + xy, xu, x^3, u^2 + (x^2 + y^2)P),$$

where $|x| = |y| = 1, |u| = 3$ *and* $|P| = 4$. *Here,* $\Phi^* : H^*(SD_{2^n}; \mathbb{Z}/2) \longrightarrow H^*(D_8; \mathbb{Z}/2)$ *sends* x, y, u, P *to* $0, \overline{y}, \overline{w}_2\overline{y}, \overline{w}_2^2$ *and* $\Psi^* : H^*(SD_{2^n}; \mathbb{Z}/2) \longrightarrow H^*(Q_8; \mathbb{Z}/2)$ *sends* x, y, u, P *to* $\widetilde{y}, \widetilde{y}, 0, \widetilde{P}$ *respectively.*

We begin by identifying some key differentials in the spectral sequence of the extension (A).

Lemma 2.2.2. *[16] In the spectral sequence for (A),*

$$d_2 z = w + x^2, d_3 z^2 = x^3, d_5 z^4 = 0,$$

where z is the one dimensional class in the fiber of the extension (A).

Proof. The known facts from the extension(B) in [[17], prop.VI3.1, 3.2] is $d_2 \overline{z} = \overline{x}^2 + \overline{xy}$ and

$$\phi^*(x) = 0, \phi^*(y) = \overline{y}, \phi^*(w_2) = \overline{x}^2 + \overline{xy},$$

where $\phi : \mathbb{Z}/2 \times \mathbb{Z}/2 < \overline{s}, t > \hookrightarrow D_{2^{n-1}}$ induced by Φ. By naturality of spectral sequences between extension (A) and (B), $d_2 z = w_2 + \alpha x^2$ for some $\alpha \in \mathbb{Z}/2$. On the other hand, by [[34];5.2], for the spectral of (C), $d_2 \widetilde{z} = \widetilde{x}^2 + \widetilde{x}\widetilde{y} + \widetilde{y}^2$ and

$$\psi^*(x) = \widetilde{y}, \psi^*(y) = \widetilde{y}, \psi^*(w_2) = \widetilde{x}^2 + \widetilde{x}\widetilde{y},$$

where $\psi : \mathbb{Z}/2 \times \mathbb{Z}/2 < \sigma, \tau > \hookrightarrow D_{2^{n-1}}$ induced by Ψ. Again, by naturality of spectral sequences between extension (A) and (C), $d_2 z = w_2 + x^2$ or $d_2 z = w_2 + y^2$. Thus $d_2 z = w_2 + x^2$.

Now, d_3 and d_5 follow from d_2 and corollary 6.9 (page 189) in [27], i.e.,

$$d_3 z^2 = d_3 Sq^1 z = Sq^1 d_2 z = Sq^1(w_2 + x^2) = w_2 y = x^2 y \equiv x^3 \quad \text{(in the } E_3\text{-page)}$$

and

$$d_5 z^4 = d_5 Sq^2 z^2 = Sq^2 d_3 z^2 = Sq^2 x^3 = x^5 \equiv 0 \quad \text{(in the } E_5\text{-page)}.$$

□

Proof of Proposition 2.2.1:

Proof. By using Lemma 2.2.2 and noting that $w_2 + x^2$ is not zero-divisor, we see that

$$E_3(A) = \mathbb{Z}/2[x, y]/(x^2 + xy) \otimes \mathbb{Z}/2[z^2].$$

Since $d_3 z^2 = x^3$ and $d_5 z^4 = 0$, it is not hard now to see that the spectral sequence collapses at E_4. Accordingly,

$$E_\infty(A) = E_4(A) = \mathbb{Z}/2[x, y, u, P]/(R),$$

where $P = z^4, u = z^2(x + y)$ and R is the ideal generated by $x^2 - xy, xu, x^3, u^2 - (x^2 + y^2)P$. That is, a basis for E_∞ is given by

$$\{P^s x^\epsilon, P^s y^l, P^s u y^l : s, l \geq 0; \epsilon = 1, 2\}.$$

Also, by using ϕ^* and ψ^* in the proof of lemma 2.2.2, $E_\infty\Phi^*$ sends $P^s x^\epsilon, P^s y^l, P^s u y^l$ to $0, \overline{z}^{4s}\overline{y}^l, \overline{z}^{4s+2}\overline{y}^{l+1}$ and $E_\infty\Psi^*$ sends them to $\widetilde{z}^{4s}\widetilde{y}^\epsilon, \widetilde{z}^{4s}\widetilde{y}^l, 0$ respectively. Then $\ker E_\infty\Phi^* \cap \ker E_\infty\Psi^* = \{0\}$ and thus

$$E_\infty\Phi^* \oplus E_\infty\Psi^* : E_\infty(A) \longrightarrow E_\infty(B) \oplus E_\infty(C)$$

is injective. This also implies that $\Phi^* \oplus \Psi^*$ is injective. The relations in $H^*(SD_{2^n};\mathbb{Z}/2)$ can be deduced simply by using the injective property of $\Phi^* \oplus \Psi^*$. In this case, $x, y \in H^*(SD_{2^n};\mathbb{Z}/2)$ are uniquely determined i.e.,

$$\Phi^*(y) = \overline{y},\ \Phi^*(x) = 0 \text{ and } \Psi^*(x) = \Psi^*(y) = \widetilde{y}\ .$$

For u, P, one (Evens and Priddy) chooses them to be classes in $H^*(SD_{2^n};\mathbb{Z}/2)$ reducing to u and P in $E_\infty(A)$ which satisfy

$$\Phi^*(u) = \overline{w}_2\overline{y} \bmod < \overline{x}^3 >,\ \Phi^*(P) = \overline{w}_2^2 \bmod < \overline{w}_2\overline{x}^2, \overline{x}^4 >, \text{ and } \Psi^*(u) = 0 \bmod \overline{x}^3,$$

and the relation $\Psi^*(P) = \widetilde{P}$ is uniquely determined because $H^4(Q_8;\mathbb{Z}/2)$ has only one generator. From here, the relations in $H^*(SD_{2^n};\mathbb{Z}/2)$ are easy to verify and all are the same relations in $E_\infty(A)$ or, in other words, $H^*(SD_{2^n};\mathbb{Z}/2) \cong E_\infty(A)$ as an algebra. □

2.2.2 Characteristic classes in cohomology ring of SD_{2^n}

From now on, to emphasize the ring structure, we intend to use $\mathbb{F}_2$ instead of $\mathbb{Z}/2$. Let $\overline{a}$ denote a reduction modulo by 2 and $c_i(c), w_i(r)$ be the i^{th} Chern and Stiefel-Whitney characteristic classes of complex representation c and real representation r respectively. We have;

Lemma 2.2.3. *In $H^*(SD_{16};\mathbb{F}_2) = \mathbb{F}_2[x, y, u, P]/(x^2 + xy, xu, x^3, u^2 + (x^2 + y^2)P)$ and the character table of SD_{16}, we have $x = w_1(\chi_3), y = w_1(\chi_2)$ and $P = \overline{c_2(\sigma_1)} = \overline{c_2(\sigma_3)}$.*

Proof. From the above inclusion maps (2.2) and (2.3), as the subgroup of SD_{16}, $D_8 = Gp < s^2, t >$ and $Q_8 = Gp < s^2, ts^3 >$, we have the character table of these groups as follows.

D_8	$[1]$	$[s^4]$	$[s^2]$	$[t]$	$[ts^2]$
1	1	1	1	1	1
ψ_2	1	1	-1	-1	1
ψ_3	1	1	-1	1	-1
ψ_4	1	1	1	-1	-1
σ	2	-2	0	0	0

Table 2.3: *The character table of* D_8

where $[s^4] = \{s^4\}$, $[s^2] = \{s^2, s^6\}, [t] = \{t, ts^4\}$ and $[ts^2] = \{ts^2, ts^6\}$.

Q_8	$[1]$	$[s^4]$	$[s^2]$	$[ts^3]$	$[ts]$
1	1	1	1	1	1
ρ_2	1	1	-1	-1	1
ρ_3	1	1	-1	1	-1
ρ_4	1	1	1	-1	-1
υ	2	-2	0	0	0

Table 2.4: *The character table of* Q_8

where $[s^4] = \{s^4\}$, $[s^2] = \{s^2, s^6\}$, $[ts] = \{ts, ts^5\}$ and $[ts^3] = \{ts^3, ts^7\}$. So, we have explicit restriction on representation rings

$$\Phi^! : Rep(SD_{16}) \longrightarrow Rep(D_8)$$

which sends

$$1 \mapsto 1,\ \chi_2 \mapsto \psi_4,\ \chi_3 \mapsto 1,\ \chi_4 \mapsto \psi_4,\ \sigma_1 \mapsto \sigma,\ \sigma_2 \mapsto \psi_2 + \psi_3 \text{ and } \sigma_3 \mapsto \sigma,$$

by considering Table 2.1 and Table 2.3. Similarly, by Table 2.1 and Table 2.4,

$$\Psi^! : Rep(SD_{16}) \longrightarrow Rep(Q_8)$$

sends

$$1 \mapsto 1,\ \chi_2 \mapsto \rho_4,\ \chi_3 \mapsto \rho_4,\ \chi_4 \mapsto 1,\ \sigma_1 \mapsto \upsilon,\ \sigma_2 \mapsto \rho_2 + \rho_3 \text{ and } \sigma_3 \mapsto \upsilon.$$

We note that all elements in $Rep_1(SD_{16})$ are one dimensional real representations and that $w_1 : Rep_1(SD_{16}) \longrightarrow H^1(SD_{16}; \mathbb{F}_2)$ is natural isomorphism. It is clear that in $H^1(SD_{16}; \mathbb{F}_2)$ has only 3 distinct non-trivial elements, namely $x, y, x+y$. So, these elements must match with $w_1(\chi_2)$, $w_1(\chi_3)$ and $w_1(\chi_4)$ in some order. Furthermore, by the nature of Stiefel-Whitney classes that they commutes with natural pull-back maps in representation theory and cohomology, we get that;

$$\begin{aligned}
\Phi^*(w_1(\chi_2)) &= w_1(\Phi^!(\chi_2)) = w_1(\psi_4) \neq 0\\
\Phi^*(w_1(\chi_4)) &= w_1(\Phi^!(\chi_4)) = w_1(\psi_4) \neq 0\\
\Phi^*(w_1(\chi_3)) &= w_1(\Phi^!(\chi_3)) = w_1(1) = 0.
\end{aligned}$$

Combining this results and Proposition 2.2.1 ($\Phi^*(x) = 0$), it turns out that $x = w_1(\chi_3)$. Similarly on the side of Q_8, there is only $w_1(\chi_2)$ and $w_1(\chi_3)$ which is non zero in the image of Ψ^*. Since $\Psi^*(y) \neq 0$ and $H^*(SD_{16}; \mathbb{F}_2)$ remains just one candidate, namely $w_1(\chi_2)$, $y = w_1(\chi_2)$. Moreover $w_1(\chi_4) = w_1(\chi_2 \otimes \chi_3) = w_1(\chi_2) + w_1(\chi_3) = x + y$.

Since we have P restricts to $z^4 \in H^*(A; \mathbb{F}_2)$, $A = \mathbb{Z}/2 < s^4 >$, it is useful to consider commutative diagram below.

$$\begin{tikzcd}
H^*(SD_{16};\mathbb{Z}) \arrow[r, "i^*"] \arrow[d, "mod2"'] & H^*(A;\mathbb{Z}) \arrow[d] \\
H^*(SD_{16};\mathbb{F}_2) \arrow[r] & H^*(A;\mathbb{F}_2)
\end{tikzcd}$$

It is easy to see that $Rep(A) = \{1, \alpha\}$ with $z = w_1(\alpha)$. By character table, only σ_1 and σ_3 restrict to $\alpha \oplus \alpha$ and hence $i^*(c_2(\sigma_1)) = i^*(c_2(\sigma_3))$. We have $c_2(\sigma_1) \in H^4(SD_{16};\mathbb{Z})$ restricts to $c_1(\alpha)^2 \in H^4(A;\mathbb{Z})$ and then reduces mod 2 to $w_1(\alpha)^4 = z^4$. On the other hand, $c_2(\sigma_1)$ reduces to $\overline{c_2(\sigma_1)} \in H^4(SD_{16};\mathbb{F}_2)$ and similarly for σ_3. We also note that in $H^4(SD_{16};\mathbb{F}_2)$, P is the only one of the 3 generators (i.e. y^4, yu, P) that restricts to $z^4 = w_1(\alpha)^4$. This means that the candidates for $\overline{c_2(\sigma_1)}$ or $\overline{c_2(\sigma_3)}$ are $P + \lambda yu + \mu y^4$ for some $\lambda, \mu \in \{0, 1\}$.

To specify it, we request the injection of $\Phi^* \oplus \Psi^* : H^*(SD_{2^n};\mathbb{F}_2) \rightarrowtail H^*(D_8;\mathbb{F}_2) \oplus H^*(Q_8;\mathbb{F}_2)$ in Proposition 2.2.1. We see that, for $\epsilon = 1, 3$,

$$\begin{aligned}
\Phi^* \oplus \Psi^*(\overline{c_2(\sigma_\epsilon)}) &= (\Phi^*(\overline{c_2(\sigma_\epsilon)}), \Psi^*(\overline{c_2(\sigma_\epsilon)})) \\
&= (\overline{c_2(\Phi^!(\sigma_\epsilon))}, \overline{c_2(\Psi^!(\sigma_\epsilon))}) \\
&= (\overline{c_2(\sigma)}, \overline{c_2(\upsilon)}) \\
&= ([w_2(\sigma)]^2, \overline{P_1(\upsilon)}) \\
&= (\overline{w}_2^2, \widetilde{P}), \text{(by (2.5) and (2.6))} \\
&= \Phi^* \oplus \Psi^*(P),
\end{aligned}$$

which completes the proof. □

Remark 2.2.4. *With the same notation as in the lemma above, we have;*

1. *By this lemma and complexification of real representation, we obtain further that* $e_H(\chi_2) = \overline{c_1(\chi_2)} = [w_1(\chi_2)]^2 = y^2$ *and similarly* $e_H(\chi_3) = x^2$ *and* $e_H(\sigma_\epsilon) = \overline{c_2(\sigma_\epsilon)} = P$, *where* $\epsilon = 1, 2$ *and* $e_H(\alpha)$ *is the Euler class of* α *for cohomology ring.*

2. *Generally, in* $H^*(SD_{2^n};\mathbb{F}_2)$ *for* $n \geq 5$, *we can show in the same way as this lemma that* x, y *and* P *are also the first Stiefel-Whitney classes of* χ_3^n, χ_2^n *and second Chern class of* σ_{odd}^n *resp, where* χ_3^n, χ_2^n *and* σ_{odd}^n *are the representations of* SD_{2^n} *with character* χ_3, χ_2 *and* $\chi_{\rho^{odd}}$ *in the table* 2.2 *respectively.*

2.2.3 $H^*(BSD_{2^n};\mathbb{F}_2)$ as a module over $E(1)$

In order to compute E_2 page of Adam spectral sequence, we need to calculate module structure over the exterior algebra $E(1) = \bigwedge_{\mathbb{F}_2}(Q_0, Q_1)$, where $Q_0 = Sq^1, Q_1 = Sq^1Sq^2 + Sq^2Sq^1$. The Bockstein operations also play a role in the calculation of differential of such spectral sequence (by theorem of May and Milgram, [25]). Here, the Steenrod actions on $H^*(BSD_{2^n};\mathbb{F}_2)$ are obtained by such things on $H^*(D_8;\mathbb{Z}_2)$ and $H^*(Q_8;\mathbb{Z}_2)$ provided by $Sq^1(\overline{w_2}) = \overline{w_2 y}$ and $Sq^i(\widetilde{P}) = 0$ for each $i = 1, 2, 3$. Precisely, we have:

Proposition 2.2.5. *Steenrod action on* $H^*(SD_{2^n};\mathbb{F}_2)$ *is given by* $Sq^1(x) = x^2$, $Sq^1(y) = y^2$, $Sq^1(u) = Sq^1(P) = 0$, $Sq^2(u) = Px + Py + uy^2$, $Sq^2(P) = u^2$ *and the Bockstein operation is given by* $\beta_{n-1}(u) = P$.

Proof. The action Sq^1 on generator x, y is obvious from the Steenrod axiom (dimension). The remaining follows from the Proposition 2.2.1 above. More precisely, we compute Sq^1, Sq^2 on the image of $\Phi^* \oplus \Psi^*$ and then use the injectivity property. For instant, $\Phi^* \oplus \Psi^*(Sq^2(u)) = (Sq^2(\Phi^*(u)), Sq^2(\Psi^*(u))) = (Sq^2(\overline{w_2 y}), 0) = (\overline{w_2}^2\overline{y} + \overline{w_2 y}^3, 0)$ which is the image of $Px + Py + uy^2 \in H^5(SD_{2^n}; \mathbb{F}_2)$. By injectivity of $\Phi^* \oplus \Psi^*$, $Sq^2(u) = Px + Py + uy^2$.

For the Bockstein operation, we obtain it from the calculation of 2-Bockstein spectral sequence for $H^*(SD_{2^n}; \mathbb{Z})$, where 2 has bidegree $(0,1)$ detected by h_0 in Adams spectral sequence and detected by 2 in $H^*(SD_{2^n}; \mathbb{Z})$, ([14], page 19,31). Namely,

$$E_1^{*,*} = H^*(SD_{2^n}; \mathbb{F}_2)[2] \Longrightarrow H^*(SD_{2^n}; \mathbb{Z}),$$

which differential d_1 is given by Sq^1 and d_r is given by β_r, the higher Bockstein operation. We found that E_2-page has infinite tower on the generator $\{1\}$, $\{uP^n \mid n \geq 0\}$ and $\{P^n \mid n \geq 1\}$. But, corollary 1.4.9 in [14] confirms that in such Bockstein spectral sequence $E_N^{*,*} = E_\infty^{*,*}$ and $E_\infty^{s,*} = 0$ for $s \geq N$ where N is the exponent of $|G|$. This means that the differential $\beta_n = 0$ and β_{n-i} must be detected by some generators for some $1 \leq i \leq n-2$. By Theorem A in L.Evens and S.Priddy paper, [16], we have

$$H^*(SD_{2^n}; \mathbb{Z}) = \mathbb{Z}[\beta, \xi, \zeta, \gamma]/(2\beta, 2\xi, 2\gamma, 2^{n-1}\zeta, \xi^2, \beta\xi, \xi\gamma, \gamma^2 - \beta^3\xi),$$

where $|\beta| = |\xi| = 2, |\zeta| = 4, |\gamma| = 5$. Hence our results, i.e. $i = 1$, follow from the element ζ which has order 2^{n-1} and degree 4. $\square$

As stated above, we need to compute $H^*(SD_{2^n}; \mathbb{F}_2) = \mathbb{F}_2[x, y, u, P]/(x^2 + xy, xu, x^3, u^2 + (x^2 + y^2)P)$ as a module over $E(1)$. It is helpful to consider its structure first. By its relations, as abelian group, $H^*(SD_{2^n}; \mathbb{F}_2)$ contains additive generators which can be written explicitly as;

$$[x] \cup [y] \cup [u] \cup [P] \cup [uy] \cup [uP] \cup [xP] \cup [yP] \cup [uyP],$$

where

1.$[x] = \{x, x^2\}$	2.$[y] = \{y^k \vert k \in \mathbb{N}\}$	3.$[u] = \{u\}$
4.$[P] = \{P^m \vert m \in \mathbb{N}\}$	5.$[uy] = \{uy^m \vert m \in \mathbb{N}\}$	
6.$[uP] = \{uP^m \vert m \in \mathbb{N}\}$	7.$[xP] = \{xP^m, x^2P^m \vert m \in \mathbb{N}\}$	
8.$[yP] = \{y^kP^m \vert k, m \in \mathbb{N}\}$	9.$[uyP] = \{uy^kP^m \vert k, m \in \mathbb{N}\}$.	

The identity $Q_0(\alpha\beta) = \alpha Q_0(\beta) + \beta Q_0(\alpha)$ and $Q_1(\alpha\beta) = \alpha Q_1(\beta) + \beta Q_1(\alpha)$ are easy to verify and are so useful for computation. By using Proposition 2.2.5 and Steenrod axiom, especially Cartan formula, we get $E(1)$ action as below.

Generator	**Image of Q_0**	**Image of Q_1**
x^k	$\begin{cases} x^2 & ,k=1 \\ 0 & ,k=2 \end{cases}$	$0\ \forall k=1,2$
y^k	$\begin{cases} y^{k+1} & ,odd(k) \\ 0 & ,even(k) \end{cases}$	$\begin{cases} y^{k+3} & ,odd(k) \\ 0 & ,even(k) \end{cases}$
u	0	$x^2P+y^2P=u^2$
P^m	$0,\forall m$	$0,\forall m$
uy^k	$\begin{cases} uy^{k+1} & ,odd(k) \\ 0 & ,even(k) \end{cases}$	$\begin{cases} u^2y^k+uy^{k+3} & ,odd(k) \\ u^2y^k & ,even(k) \end{cases}$
uP^m	$0,\forall m$	$x^2P^{m+1}+y^2P^{m+1},\forall m$
x^kP^m	$\begin{cases} x^2P^m & ,k=1 \\ 0 & ,k=2 \end{cases}$	$0,\forall k,m$
y^kP^m	$\begin{cases} y^{k+1}P^m & ,odd(k) \\ 0 & ,even(k) \end{cases}$	$\begin{cases} y^{k+3}P^m & ,odd(k) \\ 0 & ,even(k) \end{cases}$
uy^kP^m	$\begin{cases} uy^{k+1}P^m & ,odd(k) \\ 0 & ,even(k) \end{cases}$	$\begin{cases} uy^{k+3}P^m+y^{k+2}P^{m+1} & ,odd(k) \\ y^{k+2}P^{m+1} & ,even(k) \end{cases}$

Table 2.5: *The action of $E(1)$ on $H^*(SD_{2^n};\mathbb{F}_2)$*

Therefore, as an $E(1)$-module,

$$M := H^*(SD_{2^n};\mathbb{F}_2) = M_{(1)} \oplus M_{(2)} \oplus M_{(3)} \oplus M_{(4)} \oplus M_{(5)}, \tag{2.7}$$

where;

- $M_{(1)} :=$ $H^*(BC_2;\mathbb{F}_2)$ generated by $[y]$.
- $M_{(2)} :=$ the direct sum of trivial module $\{P^m\}$ for each $m\in\mathbb{N}$.
- $M_{(3)} :=$ the direct sum of free module generated by $uy^{2k-1}P^{m-1}$, for each $k,m\in\mathbb{N}$ ($\{uy^{2k-1}P^{m-1}, uy^{2k}P^{m-1}, uy^{2k+2}P^{m-1}+y^{2k+1}P^m, y^{2k+2}P^m\}$).
- $M_{(4)} :=$ the direct sum of module $L_k=\{xP^k, x^2P^k\}$ for each $k\geq 0$.
- $M_{(5)} :=$ the direct sum of augmentation ideal, $I^kE(1)=\{uP^{k-1}, uy^2P^{k-1}+(x+y)P^k, (x^2+y^2)P^k\}$ for each $k\in\mathbb{N}$.

§ 2.3 E_2 page of Adams spectral sequence for SD_{2^n}

Recall that E_2 page of Adams spectral sequence for the calculation of $ku^*(BG)$ is reduced to $E_2^{s,t}(M) = \mathrm{Ext}^{s,t}_{E(1)}(\mathbb{F}_2, M)$, with degree s and degree t, where M is $H^*(BG;\mathbb{F}_2)$ viewed as $E(1)$- module and $E(1)$ acts trivially on $\mathbb{F}_2$. So, the main tasks in this section is to calculate the functor Ext which we will recall about this functor in subsection 2.3.1 below.

2.3.1 Some properties of the functor Ext

First note that in the Adams spectral conventions, (category of graded left Γ-modules, ${}_{\Gamma}\mathbf{Mod}$), for graded module M, $\Sigma^t M = \Sigma_{-t} M$ means decreasing codegree of module M by t (increasing degree by t) i.e. $(\Sigma^t M)_i = M_{i-t}$, (cf. [27], page 377). The graded version of the Hom-functor of the graded module A, B over graded algebra Γ is given by

$$\mathrm{Hom}^t_{\Gamma}(A,B) = \mathrm{Hom}^0_{\Gamma}(A,\Sigma^t B) = \mathrm{Hom}^0_{\Gamma}(\Sigma^{-t}A,B),$$

which can be though of as a group of homomorphisms from A to B that shift codegree down by t (cf. [27], page 376-377). Note also that $\mathrm{Hom}^0_{\Gamma}(A,\Sigma^t B) = [\mathrm{Hom}_{\Gamma}(A,B)]_{-t}$, and

$$\mathrm{Hom}_{\Gamma}(A,B) = \bigoplus_{t\in\mathbb{Z}} \mathrm{Hom}^t_{\Gamma}(A,B) = \bigoplus_{t\in\mathbb{Z}} [\mathrm{Hom}_{\Gamma}(A,B)]_{-t}.$$

In general, for any graded module M, N over a graded algebra Γ (Γ be a graded algebra over a field k with unit $\varepsilon : k \longrightarrow \Gamma$ and augmentation $\eta : \Gamma \longrightarrow k$), it is well known that $\mathrm{Ext}^{s,t}_{\Gamma}(M,N)$ can be calculated dualistically in two ways. The first way starts by taking projective resolution on M, say $P_\bullet$, applying $\mathrm{Hom}^t_{\Gamma}(P_\bullet, N)$ and then ends with taking homology, $H^s(\mathrm{Hom}^t_{\Gamma}(P_\bullet, N))$. The second way starts by taking injective resolution on N, say $I_\bullet$, applying $\mathrm{Hom}^t_{\Gamma}(M, I_\bullet)$ and then ends with taking homology, $H^s(\mathrm{Hom}^t_{\Gamma}(M, I_\bullet))$.

It is also well known from homological algebra that this functor is independent of the choices of projective or injective resolution. So, we can take resolutions to be minimal (actually, canonical free resolution). Here, by definition, a homomorphism, $f : M \longrightarrow N$ of left Γ-modules is said to be *minimal* if $f(M) \subseteq I(\Gamma)\cdot N$, where $I(\Gamma) = \ker(\eta : \Gamma \longrightarrow k)$. A projective resolution of a module M is said to be a *minimal resolution* if every mapping in the resolution is minimal (definition from [27], page 379).

The consequence of taking projective minimal resolution is useful when $N = k$. This is actually suitable to our trivial case, i.e., with $M, N = k$. Precisely, if $P_\bullet$ is minimal projective resolution, then

$$\mathrm{Ext}^{s,t}_{\Gamma}(M,k) \cong \mathrm{Hom}^t_{\Gamma}(P_s,k), \tag{2.8}$$

which the proof is straightforward or see, for example, Proposition 9.4 in [27].

As usual, for a short exact sequence of Γ-module, $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$, there is the induced long exact sequence;

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_{\Gamma}(C'',N) \longrightarrow \mathrm{Hom}_{\Gamma}(C,N) \longrightarrow \mathrm{Hom}_{\Gamma}(C',N) \longrightarrow \mathrm{Ext}^1_{\Gamma}(C'',N) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Ext}^n_{\Gamma}(C'',N) \longrightarrow \mathrm{Ext}^n_{\Gamma}(C,N) \longrightarrow \mathrm{Ext}^n_{\Gamma}(C',N) \longrightarrow \mathrm{Ext}^{n+1}_{\Gamma}(C'',N) \longrightarrow \cdots \end{gathered} \tag{2.9}$$

and

$$\begin{gathered} 0 \longrightarrow \mathrm{Hom}_{\Gamma}(M,C') \longrightarrow \mathrm{Hom}_{\Gamma}(M,C) \longrightarrow \mathrm{Hom}_{\Gamma}(M,C'') \longrightarrow \mathrm{Ext}^1_{\Gamma}(M,C') \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{Ext}^n_{\Gamma}(M,C') \longrightarrow \mathrm{Ext}^n_{\Gamma}(M,C) \longrightarrow \mathrm{Ext}^n_{\Gamma}(M,C'') \longrightarrow \mathrm{Ext}^{n+1}_{\Gamma}(M,C') \longrightarrow \cdots \end{gathered} \tag{2.10}$$

Note further that E_2-page of Adams spectral sequence for our interest is equipped with multiplicative structure, namely *composition product*,

$$\circ : \mathrm{Ext}_\Gamma^{p,t}(L, M) \otimes \mathrm{Ext}_\Gamma^{q,t'}(M, N) \longrightarrow \mathrm{Ext}_\Gamma^{p+q,t+t'}(L, N), \tag{2.11}$$

which defines for all $p, q, t, t' \geq 0$. For $[f] \in \mathrm{Ext}_\Gamma^{p,t}(L, M)$ and $[g] \in Ext_\Gamma^{q,t'}(M, N)$, the composition product for $[f]$ and $[g]$, say $[f] \circ [g]$ is defined as follows. First, note that $[f]$ and $[g]$ are represented by $f : P_p \longrightarrow \Sigma^t M$ and $g : Q_q \longrightarrow \Sigma^{t'} N$ for some projective resolutions $0 \longleftarrow L \longleftarrow P_\bullet$ and $0 \longleftarrow M \longleftarrow Q_\bullet$. Next, using the defining property of projective modules, lift f up the resolution to $f_q : P_{p+q} \longrightarrow \Sigma^t Q_q$. Finally, suspends g to $\Sigma^t g : \Sigma^t Q_q \longrightarrow \Sigma^{t+t'} N$ and let

$$[f] \circ [g] = [\Sigma^t g \circ f_q],$$

see details in Theorem 9.5 in [27] page 380.

In particular, for our case $L = M = N = \mathbb{F}_2$, we get that $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ is a graded ring via this product. Also, by applying $L = M = \mathbb{F}_2$ and $N = H^*(BG; \mathbb{F}_2)$ to (2.11), we have

$$E_2^{*,*}\text{-page} = \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, H^*(BG; \mathbb{F}_2)) \text{ is a graded module over } \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \tag{2.12}$$

via the composition product. This is a very useful fact for calculation of differentials in Adams spectral sequence.

We now return to our case of interest, $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, M)$, where $M = H^*(SD_{2^n}; \mathbb{F}_2)$. The fact that the functor Ext commutes with direct sums makes us comfortable to calculate E_2 page, i.e. we can calculate E_2 page for M separately by fixing calculation to each submodule $M_{(i)}$, $i \in \{1, 2, 3, 4, 5\}$. We see further that all $M_{(i)}$'s are the compositions of small modules, i.e., trivial module $\mathbb{F}_2$, free module $E(1)$, augmentation ideal module $IE(1)$, module $M_{Q_0} = E(1)/(Q_0)$, module $M_{Q_1} = E(1)/(Q_1)$, module $M_{Q_0Q_1} = E(1)/(Q_0Q_1)$ and string module $\widetilde{M}_{(1)} = \widetilde{H}^*(\mathbb{R}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[y] - \{1\}$, which are displayed as figure below. We will calculate $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, -)$ of them as modules over $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ in the next subsection.

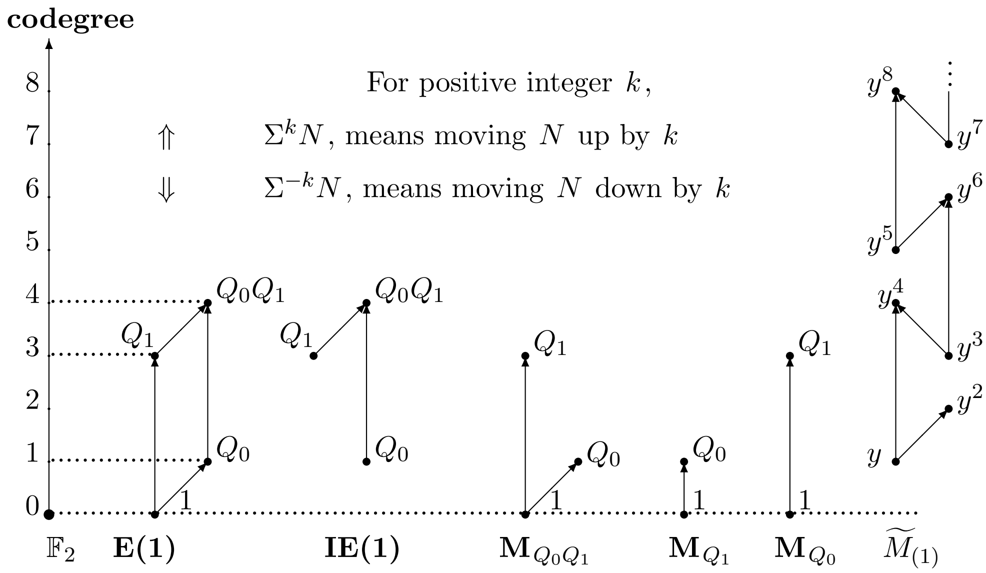


Figure 2.6: Position and structure of $\mathbb{F}_2, E(1)$, $IE(1)$, $M_{Q_0Q_1}$, M_{Q_1}, M_{Q_0} and $\widetilde{M}_{(1)}$.

2.3.2 Calculation of E_2-page

In order to calculate $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, H^*(SD_{2^n};\mathbb{F}_2))$ as a module over $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)$, it is enough to focus the calculation to small modules as in Figure 2.6 above. So, the first task is showing that $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)$ is a ring under composition product explicitly. Then all remaining tasks will follow by helps of the induced long exact sequence (2.10).

We now start with $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)$ by taking minimal projective resolution as;

$$P_\bullet : 0 \longleftarrow \mathbb{F}_2 \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3 \longleftarrow ...,$$

where, for $s \geq 0$,

$$P_s = \bigoplus_{i=0}^{i=s} \Sigma^{s+2i} E(1).$$

Since this is a minimal resolution,

$$\begin{aligned}
E_2^{s,t}(\mathbb{F}_2) &= \mathrm{Ext}^{s,t}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2)\\
&= H^s(\mathrm{Hom}^t_{E(1)}(P_\bullet,\mathbb{F}_2))\\
&= \mathrm{Hom}^t_{E(1)}(P_s,\mathbb{F}_2)\\
&= \bigoplus_{i=0}^{i=s} \mathrm{Hom}^t_{E(1)}(\Sigma^{s+2i}E(1),\mathbb{F}_2)\\
&= \begin{cases} \mathbb{F}_2, & \text{if } t = s+2i \text{ where } i = 0,1,...,s;\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Let h_0 and v be non-zero generators in $E_2^{1,1}(\mathbb{F}_2)$ and $E_2^{1,3}(\mathbb{F}_2)$ respectively. To see that $E_2^{s,t}(\mathbb{F}_2) \cong \mathbb{F}_2[h_0, v]$ as a ring under composition product, it suffices to show that multiplying by h_0 or v is not zero. This is guaranteed by diagram below;

Multiplied by h_0

$$\begin{tikzcd}
0 & \mathbb{F}_2 \arrow[l] & E(1) \arrow[l, "\varepsilon"'] & \Sigma^1E(1) \oplus \Sigma^3E(1) \arrow[l, "(Q_0\ Q_1)"'] \arrow[dl, "{(\varepsilon,0)=h_0}"'] \arrow[d, "h_{0,0}"', "{(1,0)}"] \arrow[dl, phantom, shift right=6, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & & \Sigma^2E(1) \oplus \Sigma^4E(1)\Sigma^6E(1) \arrow[ll, "{\begin{pmatrix} Q_0 & Q_1 & 0 \\ 0 & Q_0 & Q_1 \end{pmatrix}}"'] \arrow[d, "h_{0,1}"', "{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}"] & \arrow[l] \\
 & & \Sigma^1\mathbb{F}_2 & \Sigma^1E(1) \arrow[l, "\varepsilon"'] & & \Sigma^2E(1) \oplus \Sigma^4E(1) \arrow[ll, "(Q_0\ Q_1)"'] & \arrow[l]
\end{tikzcd}$$

Multiplied by v

$$\begin{tikzcd}
0 & \mathbb{F}_2 \arrow[l] & E(1) \arrow[l, "\varepsilon"'] & \Sigma^1E(1) \oplus \Sigma^3E(1) \arrow[l, "(Q_0\ Q_1)"'] \arrow[dl, "{(0,\varepsilon)=v}"'] \arrow[d, "v_{0,0}"', "{(0,1)}"] \arrow[dl, phantom, shift right=6, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & & \Sigma^2E(1) \oplus \Sigma^4E(1)\Sigma^6E(1) \arrow[ll, "{\begin{pmatrix} Q_0 & Q_1 & 0 \\ 0 & Q_0 & Q_1 \end{pmatrix}}"'] \arrow[d, "v_{0,1}"', "{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}"] & \arrow[l] \\
 & & \Sigma^3\mathbb{F}_2 & \Sigma^3E(1) \arrow[l, "\varepsilon"'] & & \Sigma^4E(1) \oplus \Sigma^6E(1) \arrow[ll, "(Q_0\ Q_1)"'] & \arrow[l]
\end{tikzcd}$$

Now, it is not hard to see that $h_0^i v^j \in \mathrm{Ext}_{E(1)}^{i+j,i+3j}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2$ is not zero for all $i, j \geq 0$ and hence (cf.[14]),

$$\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, v], \tag{2.13}$$

where $h_0 \in \mathrm{Ext}_{E(1)}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$ and $v \in \mathrm{Ext}_{E(1)}^{1,3}(\mathbb{F}_2, \mathbb{F}_2)$.

Next, for $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, M_{Q_1})$, we consider short exact sequence;

$$0 \longrightarrow \Sigma^1\mathbb{F}_2 \longrightarrow M_{Q_1} \longrightarrow \mathbb{F}_2 \longrightarrow 0.$$

This induces long exact sequence, (2.10), for each $t \in \mathbb{Z}$,

$$\begin{array}{llll}
0 \longrightarrow \mathrm{Hom}_{E(1)}^{0,t}(\mathbb{F}_2, \Sigma^1\mathbb{F}_2) \longrightarrow & \mathrm{Hom}_{E(1)}^{0,t}(\mathbb{F}_2, M_{Q_1}) \longrightarrow & \mathrm{Hom}_{E(1)}^{0,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \\
\xrightarrow{\ \delta\ } \mathrm{Ext}_{E(1)}^{1,t}(\mathbb{F}_2, \Sigma^1\mathbb{F}_2) \longrightarrow & \mathrm{Ext}_{E(1)}^{1,t}(\mathbb{F}_2, M_{Q_1}) \longrightarrow & \mathrm{Ext}_{E(1)}^{1,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \\
\xrightarrow{\ \delta\ } \mathrm{Ext}_{E(1)}^{2,t}(\mathbb{F}_2, \Sigma^1\mathbb{F}_2) \longrightarrow & \mathrm{Ext}_{E(1)}^{2,t}(\mathbb{F}_2, M_{Q_1}) \longrightarrow & \mathrm{Ext}_{E(1)}^{2,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \cdots
\end{array}$$

For the third column we have $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2[h_0, v]$ and for the first column we have $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \Sigma^1\mathbb{F}_2) \cong \Sigma^{-1}\mathbb{F}_2[h_0, v]$. Note here that we are using diagram of E_2-page under coordinate $(s, t-s)$ and thus $\Sigma^{-1}\mathbb{F}_2[h_0, v]$ means moving $\mathbb{F}_2[h_0, v]$ to the left one unit (i.e. by $(0, -1)$).

Once we determine differential $\delta : \mathrm{Ext}_{E(1)}^{s,t-s}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{E(1)}^{s+1,(t-s)-1}(\mathbb{F}_2, \Sigma^1\mathbb{F}_2)$, we will obtain $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, M_{Q_1})$ as

$$0 \longrightarrow \Sigma^{-1}\mathbb{F}_2[h_0, v]/im(\delta) \longrightarrow \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, M_{Q_1}) \longrightarrow \ker(\delta) \longrightarrow 0.$$

Since we have diagram;

$$\begin{tikzcd}
\mathbb{F}_2 & E(1) \arrow[l, "\varepsilon"'] \arrow[d] \arrow[dl, "1"] & \Sigma^1 E(1) \oplus \Sigma^3 E(1) \arrow[l] \arrow[d, "h_0=(\varepsilon,0)"] & \cdots \arrow[l] \\
\mathbb{F}_2 & M_{Q_1} \arrow[l] & \Sigma^1 \mathbb{F}_2 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

then $\delta(1) = \Sigma^{-1}h_0$ and hence

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_{Q_1}) \cong \Sigma^{-1}\mathbb{F}_2[v]. \tag{2.14}$$

Similarly, by considering on short exact sequence $0 \longrightarrow \Sigma^3\mathbb{F}_2 \longrightarrow M_{Q_0} \longrightarrow \mathbb{F}_2 \longrightarrow 0$, we get that $\delta(1) = \Sigma^{-3}v$ and

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_{Q_0}) \cong \Sigma^{-3}\mathbb{F}_2[h_0]. \tag{2.15}$$

Next, for $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_{Q_0Q_1})$, we first consider short exact sequence

$$0 \longrightarrow \Sigma^3\mathbb{F}_2 \longrightarrow M_{Q_0Q_1} \longrightarrow M_{Q_1} \longrightarrow 0.$$

The first column and the third column of the induced long exact sequence have been done. They are $\Sigma^{-3}\mathbb{F}_2[h_0, v] \cong \mathbb{F}_2[h_0, v] < g_2 >$ and $\Sigma^{-1}\mathbb{F}_2[v] \cong \mathbb{F}_2[v] < g_1 >$, where $0 \neq g_1 \in \mathrm{Ext}^{0,-1}_{E(1)}(\mathbb{F}_2, M_{Q_1})$ and $0 \neq g_2 \in \mathrm{Ext}^{0,-3}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$ (in coordinate $(s, t-s)$), respectively. It is clear that, by coordinate reason, differential $\delta = 0$ and hence

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_{Q_0Q_1}) \cong \mathbb{F}_2[h_0, v] < g_2 > \oplus \mathbb{F}_2[v] < g_1 > .$$

It remains to determine whether $vg_2 = h_0g_1$. To do this, we next consider another short exact sequence, i.e.,

$$0 \longrightarrow \Sigma^1\mathbb{F}_2 \longrightarrow M_{Q_0Q_1} \longrightarrow M_{Q_0} \longrightarrow 0.$$

By similar process as above, we have;

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_{Q_0Q_1}) \cong \mathbb{F}_2[h_0, v] < g_1' > \oplus \mathbb{F}_2[h_0] < g_2' >,$$

where $0 \neq g_2' \in \mathrm{Ext}^{0,-3}_{E(1)}(\mathbb{F}_2, M_{Q_1})$ and $0 \neq g_1' \in \mathrm{Ext}^{0,-1}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$.

Since g_1 and g_1' are both non-zero elements in $\mathrm{Ext}^{0,-1}_{E(1)}(\mathbb{F}_2, M_{Q_0Q_1}) \cong \mathbb{F}_2$, $g_1 = g_1'$ and similarly we can conclude that $g_2 = g_2'$. The consequence is that $vg_2 = h_0g_1$ and thus,

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_{Q_0Q_1}) \cong \mathbb{F}_2[h_0, v] < g_1, g_2 > /(vg_2 - h_0g_1), \tag{2.16}$$

where $|g_1| = (0, -1)$ and $|g_2| = (0, -3)$ in coordinate $(s, t-s)$.

Next, for $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, IE(1))$, we consider short exact sequence

$$0 \longrightarrow \Sigma^1 M_{Q_0} \longrightarrow IE(1) \longrightarrow \Sigma^3\mathbb{F}_2 \longrightarrow 0.$$

Again, we need to determine differential in the induced long exact sequence

$$\delta : \Sigma^{-3}\mathbb{F}_2[h_0, v] \longrightarrow \Sigma^{-4}\mathbb{F}_2[h_0].$$

It is not hard to see that $\delta(\Sigma^{-3}1) = \Sigma^{-4}h_0$ and hence

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, IE(1)) \cong \Sigma^{-4}\mathbb{F}_2[h_0]/im(\delta) \oplus \ker(\delta).$$

That is

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, IE(1)) \cong \Sigma^{-4}\mathbb{F}_2 \oplus \Sigma^{1,-1}\mathbb{F}_2[h_0, v], \tag{2.17}$$

where $\Sigma^{1,-1}\mathbb{F}_2[h_0, v]$ means shifting $\mathbb{F}_2[h_0, v]$ to the left one unit and to above one unit in coordinate $(s, t-s)$.

Finally, for $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \widetilde{M}_{(1)})$, we filtrate $\widetilde{M}_{(1)}$ as

$$\widetilde{M}_{(1)} = F_0 \supseteq F_1 \supseteq F_2 \supseteq ...,$$

so that

$$\begin{array}{lll}
F_0/F_1 \cong \Sigma^1 M_{Q_0Q_1} & := M_1, & \\
F_0/F_2 \cong M_2 & \text{in which} & 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \Sigma^3 M_{Q_0} \longrightarrow 0, \\
F_0/F_3 \cong M_3 & \text{in which} & 0 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \Sigma^5 M_{Q_0} \longrightarrow 0, \\
\cdots & \cdots & \cdots \\
F_0/F_n \cong M_n & \text{in which} & 0 \longrightarrow M_{n-1} \longrightarrow M_n \longrightarrow \Sigma^{2n-1} M_{Q_0} \longrightarrow 0, \\
\cdots & \cdots & \cdots,
\end{array}$$

and $\lim\limits_{n\to\infty} F_0/F_n \cong \widetilde{M}_{(1)}$. Before doing further, note that, generator of $\mathrm{Ext}^{0,*}_{E(1)}(\mathbb{F}_2, M)$ is any element in M annihilated by Q_0 and Q_1. This is because we take minimal projective resolution for $\mathbb{F}_2$ to be

$$0 \longleftarrow \mathbb{F}_2 \longleftarrow E(1) \xleftarrow[(Q_0\ Q_1)]{} \Sigma^1 E(1) \oplus \Sigma^3 E(1) \longleftarrow \cdots,$$

which yields

$$0 \longrightarrow M \xrightarrow{(Q_0\ Q_1)} \Sigma^{-1} M \oplus \Sigma^{-3} M \longrightarrow \cdots,$$

and $\mathrm{Ext}^{0,*}_{E(1)}(\mathbb{F}_2, M)$ is the kernel of $(Q_0\ Q_1)$ map. This fact and (2.16) imply that

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_1) \cong \mathbb{F}_2[h_0, v][\overline{y}]/(\overline{y}^3, v\overline{y}^2 - h_0\overline{y}),$$

where $\overline{y} = y^2 \in \widetilde{H}^2(\mathbb{R}P^\infty; \mathbb{F}_2)$.

To calculate $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_2)$, we use the same technique as (2.16), i.e., by considering $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \Sigma^3 M_{Q_0} \longrightarrow 0$ and $0 \longrightarrow \Sigma^1 M_{Q_1} \longrightarrow M_2 \longrightarrow \Sigma^2 M_1 \longrightarrow 0$. It is not hard to see that

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_2) \cong \mathbb{F}_2[h_0, v][\overline{y}]/(\overline{y}^4, v\overline{y}^2 - h_0\overline{y}).$$

By the same process and induction on n, we can conclude that

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, M_n) \cong \mathbb{F}_2[h_0, v][\overline{y}]/(\overline{y}^{n+1}, v\overline{y}^2 - h_0\overline{y}). \tag{2.18}$$

To conclude that (2.18) is true for all n, we need to consider short exact sequences;

$$0 \longrightarrow \widetilde{M}_{(1)} \longrightarrow L \longrightarrow \Sigma^{-1}\mathbb{F}_2 \longrightarrow 0,$$

and

$$0 \longrightarrow \Sigma^1 M_{Q_1} \longrightarrow \widetilde{M}_{(1)} \longrightarrow \Sigma^2\widetilde{M}_{(1)} \longrightarrow 0,$$

where L is the module $\widetilde{M}_{(1)}$ with extra generator in degree -1. That is $L = \varprojlim(L_k)$, where $L_k = L/\Sigma^{2k}L$. By using the same technique as above and taking inverse limit, we conclude that

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \widetilde{M}_{(1)}) \cong \mathbb{F}_2[h_0, v][\overline{y}]/(v\overline{y}^2 - h_0\overline{y}). \tag{2.19}$$

To be comfortable, we recollect results of (2.13)-(2.19) as;

Proposition 2.3.1. *In E_2-page of Adams spectral sequence for $ku^*(BG)$, we have (in coordinate $(s, t-s)$):*

- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, E(1)) \cong \Sigma^4\mathbb{F}_2$.
- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, E(1)/(Q_0, Q_1)) \cong \mathbb{F}_2[h_0, v]$, *where $|h_0| = (1, 0)$ and $|v| = (1, 2)$.*
- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, E(1)/(Q_1)) \cong \Sigma^{(0,-1)}\mathbb{F}_2[v]$.
- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, E(1)/(Q_0)) \cong \Sigma^{(0,-3)}\mathbb{F}_2[h_0]$.
- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, IE(1)) \cong \Sigma^{(0,-4)}\mathbb{F}_2 \oplus \Sigma^{(1,-1)}\mathbb{F}_2[h_0, v]$.
- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, E(1)/(Q_0Q_1)) \cong \mathbb{F}_2[h_0, v] < g_1, g_2 > /(vg_2 - h_0g_1)$, *where $|g_1| = (0, -1)$ and $|g_2| = (0, -3)$.*
- $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \widetilde{H}^*(\mathbb{R}P^\infty; \mathbb{F}_2)) \cong \mathbb{F}_2[h_0, v, \overline{y}]/(v\overline{y}^2 - h_0\overline{y})$, *where $\overline{y} = y^2 \in \widetilde{H}^2(\mathbb{R}P^\infty; \mathbb{F}_2)$.*

As an immediate result of this proposition, we have;

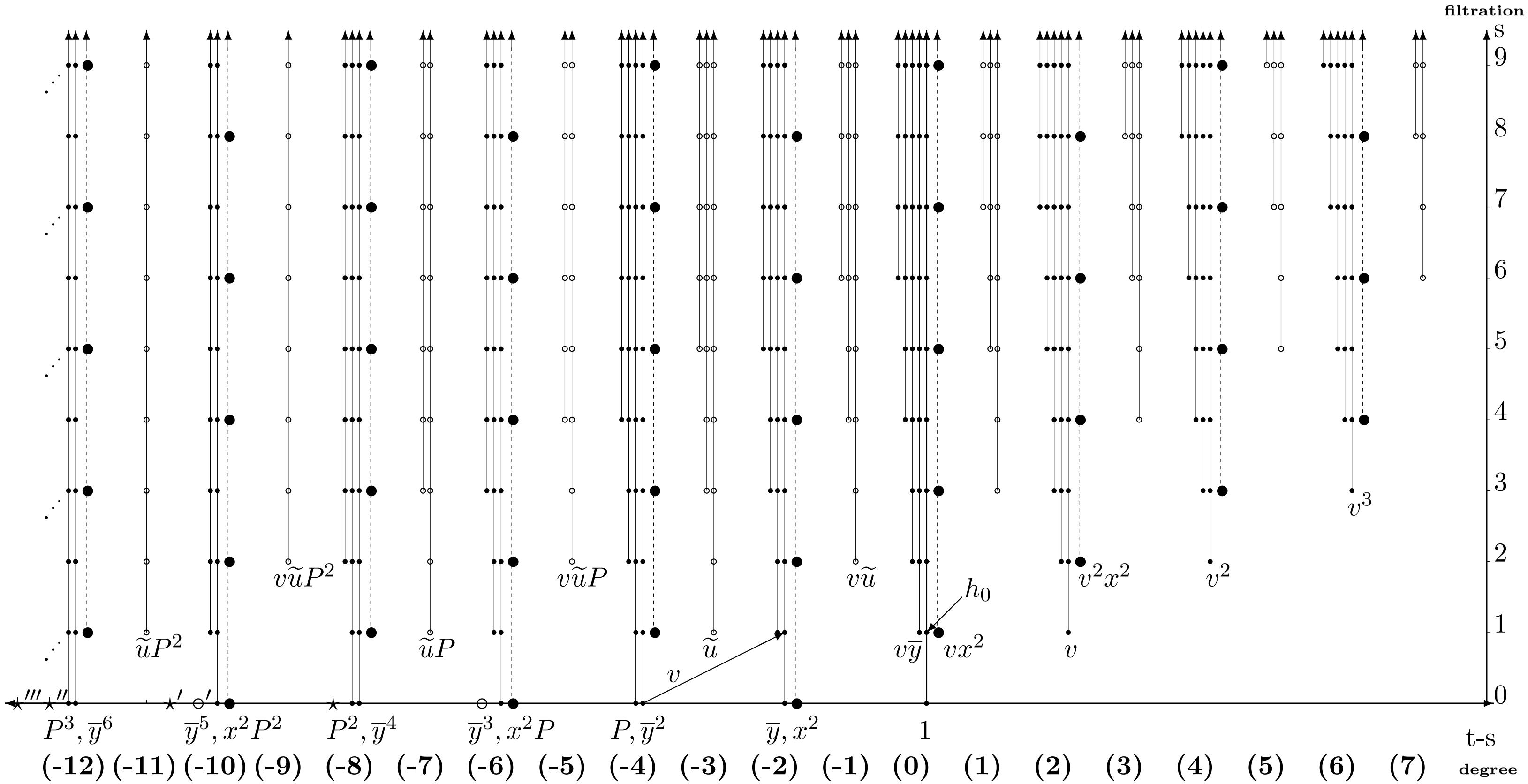
Lemma 2.3.2. *E_2-page of Adams spectral sequence for $ku^*(BSD_{2^n})$ is given by (with coordinate $(s, t-s)$ and the same notation as in (2.7));*

- $E_2^{*,*}(M_{(1)}) \cong \mathbb{F}_2[h_0, v] \oplus \mathbb{F}_2[h_0, v, \overline{y}]/(v\overline{y}^2 - h_0\overline{y})$.
- $E_2^{*,*}(M_{(2)}) = \bigoplus_{m\geq 1} \Sigma^{4m}(\mathbb{F}_2[h_0, v]) \cong \mathbb{F}_2[h_0, v][P]$, *a direct sum of free modules over $\mathbb{F}_2[h_0, v]$ generated by $P^m \in H^{4m}(SD_{2^n}; \mathbb{F}_2)$, for each $m \geq 1$.*

- $E_2^{*,*}(M_{(3)}) = E_2^{0,*}(M_{(3)}) = E_2^{0,*}(\bigoplus_{i,j\in\mathbb{N}} M_{(3)}^{i,j}) = \bigoplus_{i,j\in\mathbb{N}} E_2^{0,*}(M_{(3)}^{i,j})$, *which is an* $\mathbb{F}_2$*-vector space spanned by* $y^{2k+2}P^m \in H^{4m+4k+2}(SD_{2^n};\mathbb{F}_2)$ *for each* $k \geq 1, m \geq 1$.
- $E_2^{*,*}(M_{(4)}) = \bigoplus_{k\geq 0} E_2^{*,*}(L_k)$, *which is a direct sum of free modules over* $\mathbb{F}_2[v]$ *generated by* $x^2P^k \in H^{4k+2}(SD_{2^n};\mathbb{F}_2)$ *for each* $k \geq 0$.
- $E_2^{*,*}(M_{(5)}) = \bigoplus_{k\geq 1} E_2^{*,*}(I^k E(1))$, *which is a direct sum of* $\bigoplus_{k\geq 1}\mathbb{F}_2 < (x^2+y^2)P^k >$ *and* $\bigoplus_{k\geq 1}\mathbb{F}_2[h_0,v] < \widetilde{u}P^{k-1} >$, *where* $|\widetilde{u}P^{k-1}| = (1, -(4k-3))$.

Here, as in page 54 of [14], we use $\widetilde{u}$ for the non-zero element in $\mathrm{Ext}_{E(1)}^{1,-1}(\mathbb{F}_2, M)$. This is because u is the element of mod 2 cohomology that generates the tower of Bockstein spectral sequence (see the proof of proposition 2.2.5) which that tower corresponds to the tower generated by $\widetilde{u}$ of Adams spectral sequence here, by theorem of May and Milgram (version over $E(1)$; see this discussion in [14] page 31-32), and hence this will support differential.

We summarize the diagram of E_2-page in coordinate $(s, t-s)$ of $H^*(SD_{2^n};\mathbb{F}_2)$ in the next page.



Notation: $\overline{y} = y^2$, $\circ = (x^2+y^2)P$, $\circ' = (x^2+y^2)P^2$, $\star = y^4P$, $\star' = y^6P$, $\star'' = y^8P$, $\star''' = y^4P^2$, i.e. $\circ$'s from $M_{(5)}$ and $\star$'s from $M_{(2)}$.

Figure 2.7: Diagram $E_2^{s,t-s}(H^*(SD_{2^n};\mathbb{F}_2))$ page of Adams spectral sequence.

Remark 2.3.3. *1. Each tower is multiplied by h_0 except the tower generated by $v^sx^2P^k$, for each $k, s \geq 0$. 2. $v\overline{y}^2 = h_0\overline{y}$.*

§ 2.4 Differentials and E_∞-page

We record at first from the last section that, in $E_2^{s,t-s}$ term, v multiplies each tower of codegree l to the tower of codegree $l+2$ for all $l \in \mathbb{Z}$. Also v is not zero divisor in the filtration which is greater than 0 and is not zero divisor for all filtrations if it multiplies on codegree which is less than or equal to 4. In other words,

v acts monomorphically on positive filtration of $E_2^{s,t-s}$- page for $ku^*(BSD_{2^n})$.

Since relations between connective K theory, cohomology ring and representation theory are useful for the calculation of differentials, let us investigate some facts about them first.

Here, we define the Euler classes in connective K cohomology for SD_{16} by;

$$\begin{aligned} a &:= e_{ku}(\chi_3) \in ku^2 BSD_{16} \\ b &:= e_{ku}(\chi_2) \in ku^2 BSD_{16} \\ d &:= e_{ku}(\sigma) \in ku^4 BSD_{16} \\ d_2 &:= e_{ku}(\sigma_2) \in ku^4 BSD_{16} \\ d_3 &:= e_{ku}(\sigma_3) \in ku^4 BSD_{16}. \end{aligned}$$

Similarly, for SD_{2^n}, we define $a := e_{ku}(\chi_3^n), b := e_{ku}(\chi_2^n)$ and $d_i = e_{ku}(\sigma_i^n)$, see remark 2.2.4. Note that in this thesis, we intend to calculate explicitly only on SD_{16}. However, for SD_{2^n}, its calculation is similar to SD_{16} until E_∞-page because the initial input for Adams spectral sequence of both SD_{16} and SD_{2^n} are the same i.e., their E_2-page are the same, but it is different in the differential calculations. Precisely, we will see that the Adams spectral sequence for $ku^*(BSD_{16})$ collapses at E_4-page and it is not hard to see that the Adams spectral sequence for $ku^*(BSD_{2^n})$ collapses at E_n-page (this is because the higher Bockstein operation $\beta_{n-1}(u) = P$ in Proposition 2.2.5 and the theorem of May and Milgram). Furthermore, when n varies, the calculation in representation theory for $ku^*(BSD_{2^n})$ also varies (the number of generators and relations in $ku^*(BSD_{2^n})$). Henceforth, we mainly focus on the case SD_{16}.

The relations between connective K theory and cohomology ring come from the natural transformation, ([14] page 15-16),

$$ku^*(BSD_{16}) \longrightarrow H^*(BSD_{16};\mathbb{Z}) \longrightarrow H^*(BSD_{16};\mathbb{F}_2),$$

which we have;

$$\begin{aligned} a &\longmapsto c_1^{H\mathbb{Z}}(\chi_3) \longmapsto (w_1^{H\mathbb{F}_2}(\chi_3))^2 = x^2, \\ b &\longmapsto c_1^{H\mathbb{Z}}(\chi_2) \longmapsto (w_1^{H\mathbb{F}_2}(\chi_2))^2 = y^2 = \overline{y}, \\ d, d_3 &\longmapsto c_2^{H\mathbb{Z}}(\sigma_\epsilon) \longmapsto \overline{c_2^{H\mathbb{F}_2}(\sigma_\epsilon)} = P, \\ d_2 &\longmapsto c_2^{H\mathbb{Z}}(\sigma_2) \longmapsto \overline{c_2^{H\mathbb{F}_2}(\sigma_2)} = 0. \end{aligned}$$

On the other hand, the relations between connective K theory and representation theory come from the Atiyah completion theorem, [3], i.e., for finite group G,

$KU^*(BG) \cong R(G)^{\wedge}_J[v, v^{-1}]$, and natural transformation, ([14] page 15-16),

$$ku^*(BG) \xrightarrow{\rho_K} KU^*(BG)\ .$$

From here, the Euler classes in the representation theory is calculable easily by using lemma 1.3.3, 1.3.6 in [14]. Namely, for an n-dimensional complex representation V,

$$v^n e_{K_G}(V) = e^R(V) = \lambda(V) = 1 - V + \lambda^2 V - \cdots + (-1)^n \lambda^n(V).$$

Thus, in our case, the relations of Euler classes between ku theory and representation theory is given by:

$$\begin{aligned} A &:= va = 1 - \chi_3 \\ B &:= vb = 1 - \chi_2 \\ D &:= v^2 d = 1 - \sigma_1 + \det \sigma_1 = 1 - \sigma_1 + \chi_4 \\ D_2 &:= v^2 d_2 = 1 - \sigma_2 + \det \sigma_2 = 1 - \sigma_2 + \chi_2 \\ D_3 &:= v^2 d_3 = 1 - \sigma_3 + \det \sigma_3 = 1 - \sigma_3 + \chi_4\,. \end{aligned}$$

In computation, it is very useful to have the character table of these classes.

$R(SD_{16})$	$[1]$	$[s^4]$	$[s]$	$[s^2]$	$[s^5]$	$[t]$	$[ts]$
A	0	0	2	0	2	0	2
B	0	0	0	0	0	2	2
D	0	4	$-\sqrt{2}i$	2	$\sqrt{2}i$	0	2
D_2	0	0	2	4	2	0	0
D_3	0	4	$\sqrt{2}i$	2	$-\sqrt{2}i$	0	2

Table 2.8: *The character table of the Euler classes in* $R(SD_{16})$

We now turn to the calculation of Adams differentials. Note that differentials of Adams spectral sequence are compatible with natural maps. For any complex representation V on G, $\rho_V : G \longrightarrow U(n)$, one has $B\rho_V : BG \longrightarrow BU(n)$ and $(B\rho_V)^* : ku^*(BU(n)) \longrightarrow ku^*(BG)$ in which $c_n(V)$ is defined via this maps as $c_n(V) := (B\rho_V)^*(c_n)$, because ku is a complex oriented theory. Then all Chern classes are infinite cycles in Adams spectral sequence, since there is no differentials in $ku^*(BU(n))$. This implies that $x^2, \overline{y}$ and P are in the kernel of any differential. In other words, there is no non-zero differential departing from even codegree in Adams spectral sequence for $ku^*(BSD_{2^n})$.

By Proposition 2.2.5 , $\beta_3(u) = P$, and by the correspondent between tower of Adams and Bockstein spectral sequence (theorem of May and Milgram), we get that E_2-page $=E_3$-page and also there is evidently non-zero differential d_3 detected by $\widetilde{u}$ in codegree 3. It must take the form

$$d_3(\widetilde{u}) = a_0 h_0^4 P + vx$$

for some $a_0 \in \mathbb{F}_2$ and some vx in codegree 4 and filtration 4.

Lemma 2.4.1. *a_0 is not zero.*

Proof. By restriction to the 4 skeleton, i.e. $BSD_{16} \hookrightarrow BSD_{16}^{(4)}$, there is no any element lies in codegree greater than 4. So, $d_3(\widetilde{u}) = a_0 h_0^4 P$. Suppose that $d_3(\widetilde{u}) = 0$, then $h_0^4 P$ is an infinite cycle and hence $16d \neq 0$ in $ku^4(BSD_{16}^{(4)})$. This contradicts lemma 2.4.2 below and the fact that $H^n(BSD_{16}^{(n)};\mathbb{Z})$ is annihilated by 16, lemma 1.4.8 [14]. □

Lemma 2.4.2. *$ku^n(X^{(n)}) \cong H^n(X^{(n)};\mathbb{Z})$, where $X^{(n)}$ is an n skeleton of CW-complexs X.*

Proof. There is a cofibre sequence $\Sigma^2 ku \xrightarrow{v} ku \longrightarrow H\mathbb{Z}$ (Bott periodicity). Hence, for any paracompact space Y, there is a long exact sequence

$$\cdots \longrightarrow [Y, \Sigma^{2+n} ku] \longrightarrow [Y, \Sigma^n ku] \longrightarrow [Y, \Sigma^n H\mathbb{Z}] \longrightarrow [Y, \Sigma^{3+n} ku] \longrightarrow \cdots.$$

Since $\Sigma^{2+n} ku$ and $\Sigma^{3+n} ku$ are $n+1$ and $n+2$ connected space, respectively, the result follows by lemma below. □

Lemma 2.4.3. *Let Y be d-dimensional CW-complex and Z be n-connected space, if $d \leq n$ then $[Y, Z] = 0$.*

Proof. By induction on d, it is clear for the first step, i.e. Y is 0-dimensional. Suppose this holds for dimension d and $d+1 \leq n$. We consider cofibre sequence $Y^{(d)} \longrightarrow Y^{(d+1)} \longrightarrow \bigvee_i S^{d+1}$. Thus, there is a long exact sequence

$$\cdots \longleftarrow [\textstyle\bigvee_i S^d, Z] \longleftarrow [Y^{(d)}, Z] \longleftarrow [Y^{(d+1)}, Z] \longleftarrow [\textstyle\bigvee_i S^{d+1}, Z] \longleftarrow \cdots$$

Since Z is n-connected space and $d+1 \leq n$, $[\bigvee_i S^d, Z] = \prod_i \pi_d(Z) = 0$ and $[\bigvee_i S^{d+1}, Z]$ $= \prod_i \pi_{d+1}(Z) = 0$. This yields that $[Y^{(d)}, Z] \cong [Y^{(d+1)}, Z]$. By induction step, $[Y^{(d+1)}, Z] \cong [Y^{(d)}, Z] = 0$ and hence the result follows. □

Note further that the generators of the tower lying on odd codegree in E_2-page are of the form $v^\epsilon \widetilde{u} P^k$ for all $\epsilon \in \{0,1\}$ and $k \geq 0$. Since d_3 is derivation and d_3 vanishes on $v^\epsilon P^k$ for all $\epsilon \in \{0,1\}$ and $k \geq 0$,

$$d_3(v^\epsilon \widetilde{u} P^k) = v^\epsilon P^k d_3(\widetilde{u}) + \widetilde{u} d_3(v^\epsilon P^k) = (v^\epsilon P^k)(h_0^4 P + vx) \neq 0,$$

by E_2-page (precisely, v is monomorphism above zero line). Thus E_4 concentrates in even degree and hence $E_4 = E_\infty$.

Now, we have:

Lemma 2.4.4. *In the Adams spectral sequence*

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, H^*(BSD_{16};\mathbb{F}_2)) \Longrightarrow ku^*(BSD_{16})^\wedge_2,$$

(1) $d_3(\widetilde{u}) = h_0^4 P + v^2 h_0^2 P^2$,

(2) *$E_4 = E_\infty$ is generated over $\mathbb{F}_2[h_0, v]$ by the filtration zero classes x^2, $\overline{y}$, P, detecting a, b and d respectively,*

(3) *multiplication by v is a monomorphism in positive Adams filtration, and this holds for all filtration if v acts on elements in codegree being less than or equal to 4, and*

(4) *the natural map $ku^*(BSD_{16}) \longrightarrow H^*(BSD_{16};\mathbb{F}_2) \oplus K^*(BSD_{16})$ is a monomorphism.*

Proof. We have just proved (2). For (3), we assume for contradiction that this is a false statement. So, there exists $0 \neq [r] \in E_\infty$ which $v[r] = 0$. Thus $vr = d_3(\overline{z})$ for some $0 \neq \overline{z} \in E_2 = E_3$ and then

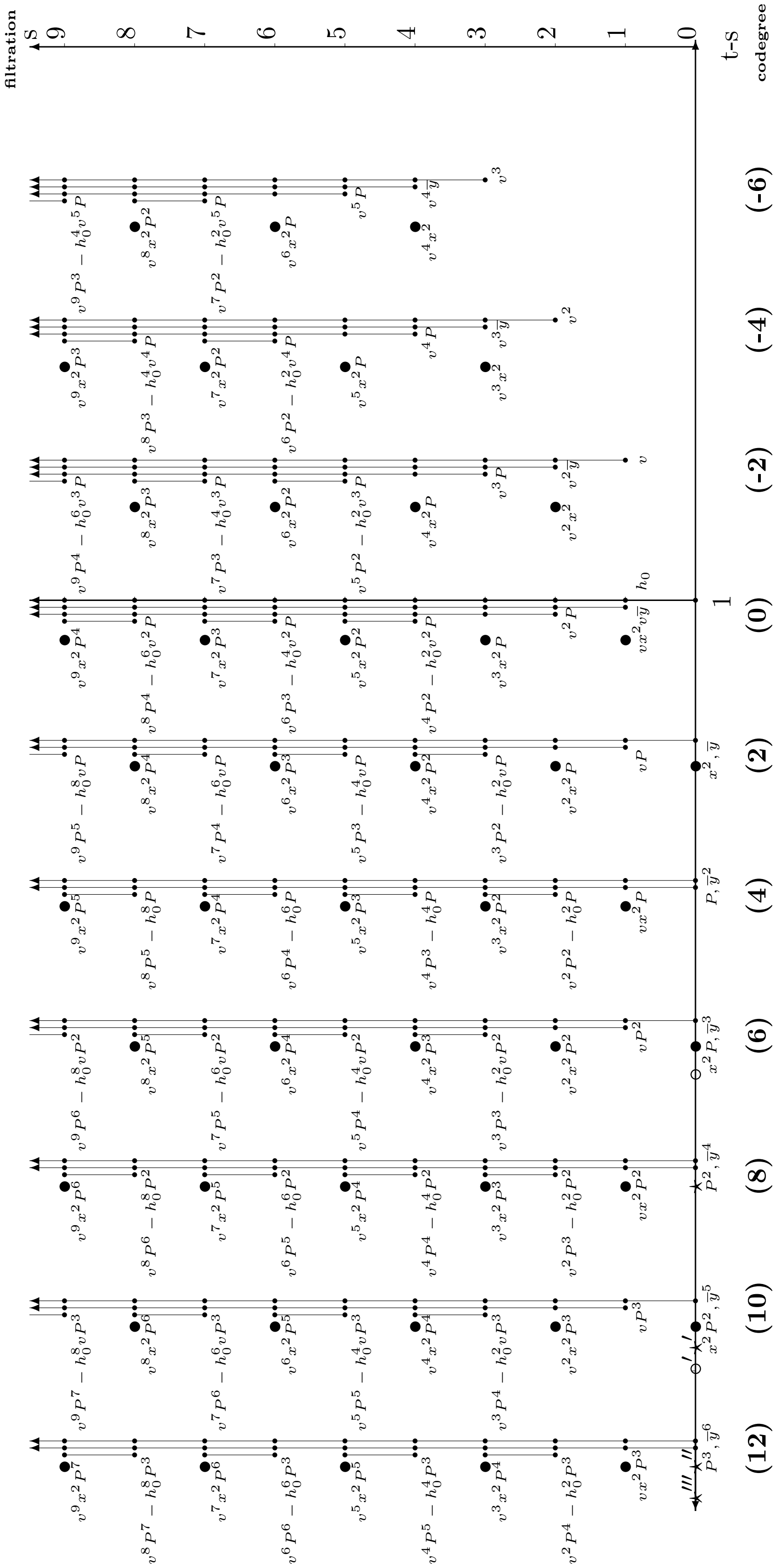
$$vr = z(h_0^4 P + vx) \tag{2.20}$$

for some $0 \neq z \in E_2 = E_3$. That is $v(r - xz) = z(h_0^4 P)$. Since $h_0^4 P : E_2^{(s\geq 1,*)}/(v) \longrightarrow E_2^{(s\geq 1,*)}/(v)$ is monomorphism, $z = vk$ for some $0 \neq k \in E_2 = E_3$ with $d_3(k) = 0$, because it is in even codegree. Substituting this result in equation (2.20), we obtain that $vr = vk(h_0^4 P + vx)$. Since v is monomorphism on E_2 page above zero lines or all lines if it acts on codegree being less than or equal to 4, $r = k(h_0^4 P + vx) = d_3(k\widetilde{u})$. Hence $[r] = 0$, which is a contradiction.

For (4), suppose $x \in ku^*BSD_{16}$ has image $(0, 0)$. Let $F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$, be the filtration of $F_0 := ku^*(BSD_{16})^\wedge_2$. Then $x \mapsto 0 \in H^*(BSD_{16};\mathbb{F}_2) \supseteq F_0/F_1$ and hence $x \in F_1$. From (3), $F_1 \overset{v}{\rightarrowtail} K^* = \mathrm{Colim}(ku^* \xrightarrow{v} ku^* \xrightarrow{v} ku^* \xrightarrow{v} \cdots)$. Therefore $x = 0$.

For(1), from the table 2.8, $16D - 12D^2 + 10D^3 - 6D^4 + D^5 = 0$ in representation theory. This is equivalent to say that $v^2(16d - 12v^2d^2 + 10v^4d^3 - 6v^6d^4 + v^8d^5) = 0$. Since v^2 acts monomorphically on $ku^4(BSD_{16})$, $r := 16d - 12v^2d^2 + 10v^4d^3 - 6v^6d^4 + v^8d^5 = 0$. It follows that $[r] = h_0^4 P + v^2 h_0^2 P^2 = 0$ in filtration 4 and even codegree of $E_4 = E_\infty$ page. So, it must be detected by some elements in codegree 3 and filtration 1 which has only one generator, i.e., $\widetilde{u}$. $\square$

Now, we reach to the main objective of this section, i.e. E_∞-page shown as in figure below.



Notation: $\overline{y} = y^2$, $\circ = (x^2+y^2)P$, $\circ' = (x^2+y^2)P^2$, $\star = y^4P$, $\star' = y^6P$, $\star'' = y^8P$, $\star''' = y^4P^2$ and a vertical line represents multiplication by h_0.

Figure 2.9: The $E_\infty^{s,t-s}$-page of Adams spectral sequence for $ku^*(BSD_{16})^\wedge_2$.

§ 2.5 The additive structure of $ku^*(BSD_{16})$

2.5.1 Generating set of $ku^*(BSD_{16})^\wedge_2$

Note that (see this discussion in the subsection 1.2.3) the convergence of Adams spectral sequence for $ku^*(BG)$, finite group G,

$$\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, H^*(BG;\mathbb{F}_2) \Longrightarrow ku^*(BG)^\wedge_2,$$

is strongly convergent ([2], [11]). This means that $\widehat{E} := ku^*(BG)^\wedge_2$ has filtration $\widehat{E} = F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots$ s.t.$\bigcap_s F_s = 0$ and $Gr(\widehat{E}) := \bigoplus_{s\geq 0} F_s/F_{s+1} = E^s_\infty$ where $(F_s/F_{s+1})_n = E^{s,n+s}_\infty$ in coordinate (s,t) and will be $E^{s,n}_\infty$ if we use coordinate $(s, t-s)$. In other words, for finite group G,

$$ku^*(BG)^\wedge_2 \cong \varprojlim \widehat{E}/F_s. \tag{2.21}$$

Furthermore, E_∞-page does give the generating set for $ku^*(BG)^\wedge_2$ since we have:

Proposition 2.5.1. *For finite group G, if $\{x_\alpha\}$ is a set of elements in $ku^n(BG)^\wedge_2$ such that $\overline{B} := \{x_\alpha + F_{f(x_\alpha)}\}$ is an additive generating set, as an $\mathbb{F}_2[h_0]$-module, for $E^{*,n}_\infty$, then $B := \{x_\alpha\}$ is an additive generating set for $ku^n(BG)^\wedge_2$ as an $\mathbb{Z}^\wedge_2$-module, where $f(x_\alpha)$ is the maximum filtration of x_α plus 1, i.e., $x_\alpha \in F_{f(x_\alpha)-1}$ but $x_\alpha \notin F_{f(x_\alpha)}$.*

Proof. Let $x \in ku^n(BG)^\wedge_2 = \widehat{E}$. Then $x + F_{f(x)} \in F_{f(x)-1}/F_{f(x)} = E^{f(x)-1,n}_\infty$. So,

$$x + F_{f(x)} = (\sum_{i=1}^{n_0} c_i x_{\alpha_i}) + F_{f(x)},$$

for some $c_i \in \mathbb{Z}^\wedge_2$, $x_{\alpha_i} \in B$ and $f(x_{\alpha_i}) \leq f(x)$ with $f(c_i x_{\alpha_i}) = f(x)$. Let $x_1 = \sum_{i=1}^{n_0} c_i x_{\alpha_i}$, then $x'_1 = x - x_1 \in F_{f(x)}$ which means that $x = x_1 \in \widehat{E}/F_{f(x)}$. We do the same process starting with x'_1, we get x_2 which is in term of elements in B and $x = x_1 + x_2 \in \widehat{E}/F_{f(x'_1)}$, where $f(x_2) > f(x_1)$. By induction on $f(x)$, we can write $x = x_1 + x_2 + x_3 + ... + x_n \in \widehat{E}/F_{f(x'_{n-1})}$ such that x_i's are written in term of elements in B, where $f(x_{i+1}) > f(x_i)$. Set $a_n = \sum_{i=1}^{i=n} x_i$, we get that this sequence, (a_n) converges uniquely to x, $(a_n) \longrightarrow x$, in the topology given by neighborhood (2-adic topology), since $\bigcap_s F_s = 0$, which completes the proof. □

From the last section, in our case, the additive generator, as an $\mathbb{F}_2[h_0]$- module, of $E^{*,*}_\infty$-page are;

$$\begin{gathered}\{v^i x^2 P^j, v^k P^l | i,j,k,l \geq 0\} \cup \{(x^2+y^2)P^i, \overline{y}^j P^k | i,k \geq 1, j > 1\} \\ \cup \{v^i, \overline{y}^j, v^k \overline{y} | i \geq 0, j,k \geq 1\}.\end{gathered}$$

This implies, by proposition 2.5.1 above, that these elements correspond to the additive generating set of $ku^*(BSD_{16})^\wedge_2$. That is, explicitly,

$$\{v^i ad^j, v^k d^l | i,j,k,l \geq 0\} \cup \{(a+b)d^i, b^j d^k | i,k \geq 1, j > 1\} \cup \{v^i, b^j, v^k b | i \geq 0, j,k \geq 1\},$$

generates $ku^*(BSD_{16})_2^\wedge$ as a $\mathbb{Z}_2^\wedge$- module.

Next, to determine its structure and additive extension problem precisely, equally, we need to find its basis generator over $\mathbb{Z}_2^\wedge$ for each codegree. A good way to do this is to compare the generator of connective K theory with cohomology ring theory, periodic K theory and character theory, i.e. using the injectivity of the natural homomorphism in lemma 2.4.4(4)

$$ku^*(BSD_{16}) \longrightarrow H^*(BSD_{16};\mathbb{F}_2) \oplus K^*(BSD_{16}),$$

which we start to do this in the next subsection.

2.5.2 Additive structure of $ku^*(BSD_{16})$

It is evident from E_∞-page and lemma 2.4.4 that, for codegree being less than or equal to 2, multiplying by v gives an isomorphism $\widetilde{ku}^{2n}(BSD_{16}) \cong \widetilde{ku}^{2(n-1)}(BSD_{16})$. So, it suffices to find additive basis for the generating set on codegree which is greater than or equal to 2.

We now first consider in codegree grater than 2. Let $k \geq 1$.

In codegree $4k$ the generator over $\mathbb{Z}_2^\wedge$ and their images are;

$ku^*(BSD_{16})$	$H^*(BSD_{16};\mathbb{F}_2)$	$K^*(BSD_{16})$	
b^{2k}	y^{4k}	B^{2k}	
d^{k+1}	P^{k+1}	D^{k+1}	
$v^{2i}d^{k+1+i}$	0	D^{k+i}	$i = 1,2,3,...$
$b^{2i+2}d^{k-1-i}$	$y^{4i+4}P^{k-1-i}$	$B^{2i+2}D^{k-1-i}$,	$i = 0,1,2,3,...k-2, (k \geq 2)$
$v^{2i+1}ad^{k+i}$	0	AD^{k+i},	$i = 0,1,2,3,...$

In codegree $4k+2$ the generator over $\mathbb{Z}_2^\wedge$ and their images are;

$ku^*(BSD_{16})$	$H^*(BSD_{16};\mathbb{F}_2)$	$K^*(BSD_{16})$	
b^{2k+1}	y^{4k+2}	B^{2k+1}	
$v^{2i-1}d^{k+i}$	0	D^{k+i}	$i = 1,2,3,...$
$b^{2i+1}d^{k-i}$	$y^{4i+2}P^{k-i}$	$B^{2i+1}D^{k-i}$,	$i = 1,2,3,...k-1, (k \geq 2)$
ad^k	x^2P^k	AD^k,	
$v^{2i}ad^{k+i}$	0	AD^{k+i},	$i = 1,2,3,...$
$(a+b)d^k$	$(x^2+y^2)P^k$	$(A+B)D^k$	

In table 2.8, we see that in codegree $4k$, $k \geq 2$ and $i \in \{0,1,2,3,...k-2\}$

$$B^{2i+2}D^{k-1-i} = 2^{k-1+i}BD = AB^{2i+1}D^{k-1-i}$$

which means that

$$b^{2i+2}d^{k-1-i} \mapsto (y^{4i+4}P^{k-1-i}, 2^{k-1+i}BD) \text{ and } ab^{2i+1}d^{k-1-i} \mapsto (0, 2^{k-1+i}BD)$$

have the same image in $K^*(BSD_{16})$. It follows that $b^{2i+2}d^{k-1-i} - ab^{2i+1}d^{k-1-i}$ has image $(y^{4i+4}P^{k-1-i}, 0)$ which is not zero. Thus, this element is annihilated by 2 and v and hence

$$b^{2i+2}d^{k-1-i} - ab^{2i+1}d^{k-1-i} \tag{2.22}$$

does not generate $\mathbb{Z}_2^\wedge$ part of $ku^{4k}(BSD_{16})$ but it generates $\mathbb{F}_2$ part instead. Similarly, in codegree $4k+2$, $k \geq 2$ and $i \in \{1, 2, 3, ...k-1\}$,

$$b^{2i+1}d^{k-i} - ab^{2i}d^{k-i} \tag{2.23}$$

is annihilated by 2 and v and hence generates $\mathbb{F}_2$ part of $ku^{4k+2}(BSD_{16})$. Note here that both of $ab^{2i}d^{k-i}$ and $ab^{2i+1}d^{k-1-i}$ are actually a combination of elements in the generating set for $ku^{4k+2}(BSD_{16})$ and $ku^{4k}(BSD_{16})$ respectively, which we will see clearly soon (after lemma 2.5.2 and lemma 2.5.3 below).

What next we have to concern with is the element $(a+b)d^k \in ku^{4k+2}(BSD_{16})$ for each $k > 0$. The image of this element is $((x^2+y^2)P^k, (A+B)D^k)$ which is not zero in both $H^*(BSD_{16}; \mathbb{F}_2)$ and $K^*(BSD_{16})$. Furthermore, since we have

$$2(A+B)D^k = \frac{1}{9}D^{k+4} + 4AD^k - \frac{16}{9}D^{k+1} - \frac{20}{9}D^{k+2} - \frac{8}{9}D^{k+3} \neq 0$$

in character table, thus

$$2(a+b)d^k = \frac{1}{9}v^7d^{k+4} + 4ad^k - \frac{16}{9}vd^{k+1} - \frac{20}{9}v^3d^{k+2} - \frac{8}{9}v^5d^{k+3} \tag{2.24}$$

is not zero in positive filtration and hence $(a+b)d^k$ is not 2-torsion or v-torsion.

To guarantee that the combination of $(a+b)d^k$ and other elements in $ku^{4k+2}(BSD_{16})$ is not 2 torsion or v-torsion, we need to check that the image of $(a+b)d^k \in ku^{4k+2}(BSD_{16})$ is not the images of some generating elements of $ku^{4k+2}(BSD_{16})$ in $K^*(BSD_{16})$ which lie in positive filtration. This is confirmed by:

Lemma 2.5.2. *Let $k > 0$. All generating elements of $ku^{4k+2}(BSD_{16})$ in positive filtration can be written as a $\mathbb{Z}_2^\wedge$- combination of elements in*

$$[B_{4k+2}] := \{b^{2k+1}, ad^k, 2(a+b)d^k, vd^{k+1}, v^3d^{k+2}, v^5d^{k+3}\}.$$

Proof. We will calculate on its image. Let x be a generating element of $ku^{4k+2}BSD_{16}$ in positive filtration which is not in $[B_{4k+2}]$ (otherwise it is obvious). Then $im(x)$ will be zero on $H^*(BSD_{16}; \mathbb{F}_2)$ and we will write its image as

$$(0, X) = (0, [x_1, x_2, x_3, x_4, x_5, x_6]),$$

where $x_1 = X([s^4])$, $x_2 = X([s])$, $x_3 = X([s^2])$, $x_4 = X([s^5])$, $x_5 = X([t])$, $x_6 = X([ts])$. (Note that we omit $x_0 = X([1])$ because this is zero for every x that we consider). Now, we have:

$$
\begin{array}{llllllll}
im(b^{2k+1}) & = (\overline{y}^{2k+1}, & [0 & 0 & 0 & 0 & 2^{2k+1} & 2^{2k+1}]) \\
im(ad^k) & = (x^2P^k, & [0 & 2(-\sqrt{2}i)^k & 0 & 2(\sqrt{2}i)^k & 0 & 2^{k+1}]) \\
im(2(a+b)d^k) & = (0, & [0 & 4(-\sqrt{2}i)^k & 0 & 4(\sqrt{2}i)^k & 0 & 2^{k+3}]) \\
im(vd^{k+1}) & = (0, & [4^{k+1} & (-\sqrt{2}i)^{k+1} & 2^{k+1} & (\sqrt{2}i)^{k+1} & 0 & 2^{k+1}]) \\
im(v^3d^{k+2}) & = (0, & [4^{k+2} & (-\sqrt{2}i)^{k+2} & 2^{k+2} & (\sqrt{2}i)^{k+2} & 0 & 2^{k+2}]) \\
im(v^5d^{k+3}) & = (0, & [4^{k+3} & (-\sqrt{2}i)^{k+3} & 2^{k+3} & (\sqrt{2}i)^{k+3} & 0 & 2^{k+3}]) \\
im(x) & = (0, & [x_1 & x_2 & x_3 & x_4 & x_5 & x_6])
\end{array}
$$

Thus x can be written as the combination of elements in $[B_{4k+2}]$ if

$$x = n_1(b^{2k+1}) + n_2(ad^k) + n_3(2(a+b)d^k) + n_4(vd^{k+1}) + n_5(v^3d^{k+2}) + n_6(v^5d^{k+3}),$$

where $n_i \in \mathbb{Z}_2^\wedge$ for each $i = 1, 2, 3, ..., 6$ s.t. $\sum_{i=1}^{i=6} n_i^2 \neq 0$ and n_1, n_2 are both even. This is equivalent to say that

$$
\begin{pmatrix}
0 & 0 & 0 & 4^{k+1} & 4^{k+2} & 4^{k+3} \\
0 & 2(-c)^k & 4(-c)^k & (-c)^{k+1} & (-c)^{k+2} & (-c)^{k+3} \\
0 & 0 & 0 & 2^{k+1} & 2^{k+2} & 2^{k+3} \\
0 & 2(c)^k & 4(c)^k & (c)^{k+1} & (c)^{k+2} & (c)^{k+3} \\
2^{2k+1} & 0 & 0 & 0 & 0 & 0 \\
2^{2k+1} & 2^{k+1} & 2^{k+3} & 2^{k+1} & 2^{k+2} & 2^{k+3}
\end{pmatrix}
\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix}
=
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}
$$

where $c = \sqrt{2}i$. We find the solution of n_i's by using row-reduced matrix, i.e.,

$$
\begin{pmatrix}
0 & 0 & 0 & 4^{k+1} & 4^{k+2} & 4^{k+3} & |x_1 \\
0 & 2(-c)^k & 4(-c)^k & (-c)^{k+1} & (-c)^{k+2} & (-c)^{k+3} & |x_2 \\
0 & 0 & 0 & 2^{k+1} & 2^{k+2} & 2^{k+3} & |x_3 \\
0 & 2(c)^k & 4(c)^k & (c)^{k+1} & (c)^{k+2} & (c)^{k+3} & |x_4 \\
2^{2k+1} & 0 & 0 & 0 & 0 & 0 & |x_5 \\
2^{2k+1} & 2^{k+1} & 2^{k+3} & 2^{k+1} & 2^{k+2} & 2^{k+3} & |x_6
\end{pmatrix}
$$

$$
\sim
\begin{pmatrix}
0 & 0 & 0 & 0 & 2 \cdot 4^{k+1} & 3 \cdot 4^{k+2} & |x_1 - 2^{k+1}x_3 = x_1' \\
0 & 0 & 0 & 2(-c)^{k+1} & 0 & 2(-c)^{k+3} & |x_2 - (-1)^k x_4 = x_2' \\
0 & 0 & 0 & 2^{k+1} & 2^{k+2} & 2^{k+3} & |x_3 \\
0 & 2(c)^k & 4(c)^k & (c)^{k+1} & (c)^{k+2} & (c)^{k+3} & |x_4 \\
2^{2k+1} & 0 & 0 & 0 & 0 & 0 & |x_5 \\
0 & 2^{k+1} & 2^{k+3} & 0 & 0 & 0 & |x_6 - x_5 - x_3 = x_6'
\end{pmatrix}
$$

$$
\sim
\begin{pmatrix}
0 & 0 & 0 & 0 & 2 \cdot 4^{k+1} & 3 \cdot 4^{k+2} & |x_1' \\
0 & 0 & 0 & 2(-c)^{k+1} & 0 & 2(-c)^{k+3} & |x_2' \\
0 & 0 & 0 & 0 & 2^{k+2} & 3 \cdot 2^{k+2} & |x_3 - (\frac{c^{k+1}}{2})x_2' = x_3' \\
0 & 2(c)^k & 4(c)^k & (c)^{k+1} & (c)^{k+2} & (c)^{k+3} & |x_4 \\
2^{2k+1} & 0 & 0 & 0 & 0 & 0 & |x_5 \\
0 & 2^{k+1} & 2^{k+3} & 0 & 0 & 0 & |x_6'
\end{pmatrix}
$$

$$\sim \left(\begin{array}{cccccc|l} 0 & 0 & 0 & 0 & 0 & 6\cdot 4^{k+1} & x_1' - 2^{k+1}x_3' = x_1'' \\ 0 & 0 & 0 & 2(-c)^{k+1} & 0 & 0 & x_2' - (\frac{(-c)^{k+3}}{3\cdot 4^{k+1}})x_1'' = x_2'' \\ 0 & 0 & 0 & 0 & 2^{k+2} & 0 & x_3' - (\frac{1}{2^{k+1}})x_1'' = x_3'' \\ 0 & 2(c)^k & 4(c)^k & 0 & 0 & 0 & x_4' \\ 2^{2k+1} & 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & -2^{k+1} & 0 & 0 & 0 & 0 & x_6' - 2(-c)^k x_4' = x_6'' \end{array} \right)$$

$$\sim \left(\begin{array}{cccccc|l} 0 & 0 & 0 & 0 & 0 & 6\cdot 4^{k+1} & x_1'' \\ 0 & 0 & 0 & 2(-c)^{k+1} & 0 & 0 & x_2'' \\ 0 & 0 & 0 & 0 & 2^{k+2} & 0 & x_3'' \\ 0 & 0 & 4(c)^k & 0 & 0 & 0 & x_4' + (\frac{1}{(-c)^k})x_6'' = x_4'' \\ 2^{2k+1} & 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & -2^{k+1} & 0 & 0 & 0 & 0 & x_6'' \end{array} \right),$$

where

$$\begin{aligned} x_4' &= x_4 - \frac{(-1)^{k+1}}{2}x_2'' - \frac{1}{(-c)^{k+2}}x_3'' - \frac{c^{k+3}}{6\cdot 4^{k+1}}x_1'' \\ &= \frac{1}{2}[x_4 + (-1)^k x_2] - (\frac{1}{(-c)^{k+2}})x_3''. \end{aligned}$$

Hence,

$$n_1 = \tfrac{x_5}{2^{2k+1}},\ n_2 = \tfrac{x_6''}{-2^{k+1}},\ n_3 = \tfrac{x_4''}{4(c)^k},\ n_4 = \tfrac{x_2''}{2(-c)^{k+1}},\ n_5 = \tfrac{x_3''}{2^{k+2}},\ n_6 = \tfrac{x_1''}{6\cdot 4^{k+1}}.$$

Recall that the images of generating elements in positive filtration of codegree $4k+2$ which is not in $[B_{4k+2}]$ are in the form AD^s, where $s \geq k+1$ and D^s where $s \geq k+4$. We are ready to check them now by starting with AD^s, where $s \geq k+1$, first.

case $AD^s = [x_1, x_2, x_3, x_4, x_5, x_6] = [0, 2(-c)^s, 0, 2(c)^s, 0, 2^{s+1}],\ s \geq k+1$

We need to check that $n_i \in \mathbb{Z}_2^\wedge$ for each $i = 1, 2, 3, ..., 6$ s.t. $\sum_{i=1}^{i=6} n_i^2 \neq 0$ and n_1, n_2 are both even. In this case we have $x_5 = 0$ (i.e. $n_1 = 0$), $x_1' = 0$, $x_2' = 2(c)^s[(-1)^s - (-1)^k]$, $x_3' = -(c)^{k+s+1}((-1)^s - (-1)^k)$, $x_6' = 2^{s+1}$. This implies that

$$\begin{aligned} x_1'' &= 2^{k+1}(c)^{k+s+1}[(-1)^s - (-1)^k] \\ &= \begin{cases} \pm 2^{k+2}(c)^{k+s+1}, & \text{if } k+s \text{ is odd;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

That is

$$n_6 = \frac{x_1''}{6\cdot 4^{k+1}} = \begin{cases} \pm\frac{2^{\frac{s-k-1}{2}}}{3}, & \text{if } k+s \text{ is odd;} \\ 0, & \text{otherwise,} \end{cases}$$

So, $n_6 \in \mathbb{Z}_2^\wedge$ because $s \geq k+1$.

Next, since we have

$$\begin{aligned} x_2'' &= x_2' - (\frac{(-c)^{k+3}}{3\cdot 4^{k+1}})x_1'' \\ &= 2(c)^s[(-1)^s-(-1)^k] - (\frac{(-c)^{k+3}}{3\cdot 4^{k+1}})2^{k+1}(c)^{k+s+1}[(-1)^s-(-1)^k] \\ &= 2(c)^s[(-1)^s-(-1)^k] - (-2\frac{(c)^s}{3}[(-1)^s-(-1)^k]) \\ &= \frac{8}{3}(c)^s[(-1)^s-(-1)^k], \end{aligned}$$

then

$$\begin{aligned} n_4 = \frac{x_2''}{2(-c)^{k+1}} &= \frac{(-1)^{k+1}\cdot 4}{3}c^{s-k-1}((-1)^s-(-1)^k) \\ &= \begin{cases} \pm 4\frac{2^{\frac{s-k+1}{2}}}{3}, & \text{if } k+s \text{ is odd;} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and hence $n_4 \in \mathbb{Z}_2^\wedge$. Next,

$$\begin{aligned} n_5 = \frac{x_3''}{2^{k+2}} &= (x_3' - (\frac{1}{2^{k+1}})x_1'')/2^{k+2} \\ &= [-2c^{s-k-1}((-1)^s-(-1)^k)]/2^{k+2} \\ &= \begin{cases} \pm 2^{\frac{s-k+1}{2}}, & \text{if } k+s \text{ is odd ;} \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

which yields that $n_5 \in \mathbb{Z}$.

Before doing analysis on n_2 and n_3, we need to find x_4' first. By direct calculation with $x_4' = \frac{1}{2}(x_4 + (-1)^k x_2) - (\frac{1}{(-c)^{k+2}})x_3''$, we have

$$x_4' = \begin{cases} 2c^s, & \text{if } k+s \text{ is even;} \\ -4c^{s-1}, & \text{if } k+s \text{ is odd.} \end{cases}$$

Consequently,

$$\begin{aligned} n_2 = \frac{x_6''}{-2^{k+1}} &= (x_6' - 2(-c)^k x_4')/-2^{k+1} \\ &= \begin{cases} -2^{s-k} + (-1)^{\frac{3k+s}{2}}2^{\frac{k+s}{2}+1}, & \text{if } k+s \text{ is even;} \\ -2^{s-k} - (-1)^{\frac{3k+s-1}{2}}2^{\frac{k+s-1}{2}+2}, & \text{if } k+s \text{ is odd.} \end{cases} \end{aligned}$$

Hence n_2 is an even integer. For $n_3 = \frac{x_4''}{4(c)^k}$, we have that

$$\begin{aligned} x_4'' = x_4' + (\frac{1}{(-c)^k})x_6'' &= (\frac{1}{(-c)^k})x_6' - x_4' \\ &= \begin{cases} \frac{2^{s+1}}{(-c)^k} - 2c^s, & \text{if } k+s \text{ is even;} \\ \frac{2^{s+1}}{(-c)^k} + 4c^{s-1}, & \text{if } k+s \text{ is odd.} \end{cases} \end{aligned}$$

It is now not hard to conclude that $n_3 \in \mathbb{Z}$, which complete the proof for this case.

case $D^s = [x_1, x_2, x_3, x_4, x_5, x_6] = [4^s, (-c)^s, 2^s, (c)^s, 0, 2^s]$, $s \geq k+4$

In this case we have $x_5 = 0$, $x_1' = 4^s - 2^{k+s+1}$, $x_2' = c^s[(-1)^s - (-1)^k]$, $x_3' = 2^s - \frac{c^{k+s+1}}{2}((-1)^s - (-1)^k)$ and $x_6' = 0$. So, $n_1 = 0$. Also, $x_1'' = 4^s - 2^{k+s+2} + 2^k \cdot c^{k+s+1}[(-1)^s - (-1)^k]$ gives

$$n_6 = \frac{x_1''}{6 \cdot 4^{k+1}} = \frac{4^s}{6 \cdot 4^{k+1}} - \frac{2^{k+s+2}}{6 \cdot 4^{k+1}} + \frac{2^k \cdot c^{k+s+1}}{6 \cdot 4^{k+1}}[(-1)^s - (-1)^k].$$

Since $s \geq k+4$, $\frac{4^s}{6 \cdot 4^{k+1}}$ and $\frac{2^{k+s+2}}{6 \cdot 4^{k+1}}$ are in the form $(\frac{1}{3})\alpha_1$ and $(\frac{1}{3})\alpha_2$ where α_1, α_2 are even integer. In the last part, it is zero if $k+s$ is an even and for $k+s$ is odd,

$$\frac{2^k \cdot c^{k+s+1}}{6 \cdot 4^{k+1}}[(-1)^s - (-1)^k] = \pm\frac{c^{k+s+1}}{6 \cdot 2^{k+1}} = (\pm\frac{1}{3})2^{\frac{s-k-3}{2}}.$$

Thus, $n_6 = (\pm\frac{1}{3})\alpha_3 \in \mathbb{Z}_2^\wedge$ for some even integer α_3.

By the results of n_6 above and

$$n_4 = \frac{x_2''}{2(-c)^{k+1}} = \frac{x_2'}{2(-c)^{k+1}} - \frac{(-c)^{k+3}}{2 \cdot (-c)^{k+1}} \cdot \frac{x_1''}{3 \cdot 4^{k+1}} = \frac{(-1)^{k+1}}{2} c^{s-k-1}[(-1)^s - (-1)^k] + 2n_6,$$

then $n_4 \in \mathbb{Z}_2^\wedge$. It is not hard to check that $\frac{x_3'}{2^{k+2}} = 2^{s-k-2} - \frac{c^{k+s+1}}{2^{k+3}}[(-1)^s - (-1)^k]$ is an even integer. This follows that

$$n_5 = \frac{x_3''}{2^{k+2}} = \frac{x_3'}{2^{k+2}} - \frac{x_1''}{2 \cdot 4^{k+1}} = \frac{x_3'}{2^{k+2}} - 3n_6 = \frac{x_3'}{2^{k+2}} - \alpha_3,$$

is an even integer.

Furthermore, we have $x_6'' = -2(-c)^k x_4'$ which yields $x_4'' = -x_4'$ and $x_4' = \frac{c^s}{2}[1 + (-1)^{k+s}] - \frac{x_3''}{(-c)^{k+2}}$, then

$$n_3 = \frac{x_4''}{4(c)^k} = \frac{-x_4'}{4(c)^k} = -\frac{c^{s-k}}{8}(1 + (-1)^{k+s}) - \frac{x_3''}{2^{k+3}} = \begin{cases} (\frac{1}{2})n_5, & \text{if } k+s \text{ is odd;} \\ -\frac{c^{s-k}}{4} + (\frac{1}{2})n_5, & \text{if } k+s \text{ is even.} \end{cases}$$

Then $n_3 \in \mathbb{Z}$ because n_5 is an even integer and $s \geq k+4$. Finally, we have $n_2 = \frac{x_6''}{-2^{k+1}} = (2(-c)^k)\frac{x_4'}{2^{k+1}} = -4(\frac{-x_4'}{4 \cdot c^k}) = -4n_3$ which completes the proof. □

Now, it is clear that $\{b^{2k+1}, ad^k, (a+b)d^k, vd^{k+1}, v^3d^{k+2}, v^5d^{k+3}\}$ is linearly independent set and spans all generating elements of $ku^{4k+2}(BSD_{16})$ over $\mathbb{Z}_2^\wedge$. In particular, as we stated above, for $i \geq 1$, $ab^{2i}d^{k-i} = 2^i(a+b)d^k - 2^i ad^k$ which can be written in term of generating set for $ku^{4k+2}(BSD_{16})$ by using 2.24. Moreover, we can say that v acts monomorphically on $ku^i(BSD_{16})$ for all $i \leq 6$.

Next, we need to identify the additive basis for $ku^{4k}(BSD_{16})$. To do this, we use the same method as in the lemma above. Since $v(a+b)d^k \neq 0$ and $2v(a+b)d^k \neq 0$ in $ku^{4k}(BSD_{16})$, $v(a+b)d^k$ will not lie in $\mathbb{F}_2$ parts. Actually, this will generate $\mathbb{Z}_2^\wedge$ parts instead.

Lemma 2.5.3. *Let $k>0$. All generating elements of $ku^{4k}(BSD_{16})$ in positive filtration can be written as a $\mathbb{Z}_2^\wedge$ combination of*

$$[B_{4k}] := \{b^{2k}, vad^k, v(a+b)d^k, d^k, v^2d^{k+1}, v^4d^{k+2}\}.$$

Proof. We will calculate on its image as the previous lemma. Let x be the generator of $ku^{4k}(BSD_{16})$ on positive filtration but not in $[B_{4k}]$. Then we have:

$$\begin{array}{llllllll}
im(b^{2k}) & = (\overline{y}^{2k}, & [0 & 0 & 0 & 0 & 2^{2k} & 2^{2k}]) \\
im(vad^k) & = (0, & [0 & 2(-\sqrt{2}i)^k & 0 & 2(\sqrt{2}i)^k & 0 & 2^{k+1}]) \\
im(v(a+b)d^k) & = (0, & [0 & 2(-\sqrt{2}i)^k & 0 & 2(\sqrt{2}i)^k & 0 & 2^{k+2}]) \\
im(d^k) & = (P^k, & [4^k & (-\sqrt{2}i)^k & 2^k & (\sqrt{2}i)^k & 0 & 2^k]) \\
im(v^2d^{k+1}) & = (0, & [4^{k+1} & (-\sqrt{2}i)^{k+1} & 2^{k+1} & (\sqrt{2}i)^{k+1} & 0 & 2^{k+1}]) \\
im(v^4d^{k+2}) & = (0, & [4^{k+2} & (-\sqrt{2}i)^{k+2} & 2^{k+2} & (\sqrt{2}i)^{k+2} & 0 & 2^{k+2}]) \\
im(x) & = (0, & [x_1 & x_2 & x_3 & x_4 & x_5 & x_6])
\end{array}$$

Thus x can be written as the combination of $[B_{4k}]$ if

$$x = n_1(b^{2k}) + n_2(vad^k) + n_3(v(a+b)d^k) + n_4(d^k) + n_5(v^2d^{k+1}) + n_6(v^4d^{k+2}),$$

where $n_i \in \mathbb{Z}_2^\wedge$ for each $i = 1,2,3,...,6$ s.t. $\sum_{i=1}^{i=6} n_i^2 \neq 0$ and n_1, n_4 are both even. This is equivalent to say that

$$\begin{pmatrix}
0 & 0 & 0 & 4^k & 4^{k+1} & 4^{k+2} \\
0 & 2(-c)^k & 2(-c)^k & (-c)^k & (-c)^{k+1} & (-c)^{k+2} \\
0 & 0 & 0 & 2^k & 2^{k+1} & 2^{k+2} \\
0 & 2(c)^k & 2(c)^k & (c)^k & (c)^{k+1} & (c)^{k+2} \\
2^{2k} & 0 & 0 & 0 & 0 & 0 \\
2^{2k} & 2^{k+1} & 2^{k+2} & 2^k & 2^{k+1} & 2^{k+2}
\end{pmatrix}
\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix}
=
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

where $c = \sqrt{2}i$. We find the solution n_i by using row-reduced matrix, i.e.,

$$\begin{pmatrix}
0 & 0 & 0 & 4^k & 4^{k+1} & 4^{k+2} & |x_1 \\
0 & 2(-c)^k & 2(-c)^k & (-c)^k & (-c)^{k+1} & (-c)^{k+2} & |x_2 \\
0 & 0 & 0 & 2^k & 2^{k+1} & 2^{k+2} & |x_3 \\
0 & 2(c)^k & 2(c)^k & (c)^k & (c)^{k+1} & (c)^{k+2} & |x_4 \\
2^{2k} & 0 & 0 & 0 & 0 & 0 & |x_5 \\
2^{2k} & 2^{k+1} & 2^{k+2} & 2^k & 2^{k+1} & 2^{k+2} & |x_6
\end{pmatrix}$$

$$\sim \begin{pmatrix}
0 & 0 & 0 & 0 & 2\cdot 4^k & 3\cdot 4^{k+1} & |x_1 - 2^k x_3 = x_1' \\
0 & 0 & 0 & 0 & 2(-c)^{k+1} & 0 & |x_2 - (-1)^k x_4 = x_2' \\
0 & 0 & 0 & 2^k & 2^{k+1} & 2^{k+2} & |x_3 \\
0 & 2(c)^k & 2(c)^k & (c)^k & (c)^{k+1} & (c)^{k+2} & |x_4 \\
2^{2k} & 0 & 0 & 0 & 0 & 0 & |x_5 \\
0 & 2^{k+1} & 2^{k+2} & 0 & 0 & 0 & |x_6 - x_5 - x_3 = x_6'
\end{pmatrix}$$

$$\sim \left(\begin{array}{cccccc|l} 0 & 0 & 0 & 0 & 0 & 3\cdot 4^{k+1} & x_1' + (2^k \cdot c^{k-1})x_2' = x_1'' \\ 0 & 0 & 0 & 0 & 2(-c)^{k+1} & 0 & x_2' \\ 0 & 0 & 0 & 2^k & 0 & 0 & x_3 - \frac{c^{k+1}}{2}x_2' - \frac{1}{3\cdot 2^k}x_1'' = x_3' \\ 0 & 2(c)^k & 2(c)^k & 0 & 0 & 0 & x_4' \\ 2^{2k} & 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 2^{k+1} & 0 & 0 & 0 & x_6' - (-c)^k x_4' = x_6'' \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|l} 0 & 0 & 0 & 0 & 0 & 3\cdot 4^{k+1} & x_1'' \\ 0 & 0 & 0 & 0 & 2(-c)^{k+1} & 0 & x_2' \\ 0 & 0 & 0 & 2^k & 0 & 0 & x_3' \\ 0 & 2(c)^k & 0 & 0 & 0 & 0 & x_4' - \frac{1}{(-c)^k}x_6'' = x_4'' \\ 2^{2k} & 0 & 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 2^{k+1} & 0 & 0 & 0 & x_6'' \end{array}\right)$$

where $x_4' = x_4 - \frac{1}{(-c)^k}x_3' - \frac{(-1)^{k+1}}{2}x_2' - \frac{c^{k+2}}{3\cdot 4^{k+1}}x_1''$.
Hence, $n_1 = \frac{x_5}{2^{2k}}$, $n_2 = \frac{x_4''}{2(c)^k}$, $n_3 = \frac{x_6''}{2^{k+1}}$, $n_4 = \frac{x_3'}{2^k}$, $n_5 = \frac{x_2'}{2(-c)^{k+1}}$, $n_6 = \frac{x_1''}{3\cdot 4^{k+1}}$.

Before doing further, we simplify x_3' and x_4' as

$$\begin{aligned} x_3' &= x_3 - \frac{c^{k+1}}{2}x_2' - \frac{1}{3\cdot 2^k}x_1'' \\ &= x_3 - 2^{k+1}\cdot n_5 - 2^{k+2}\cdot n_6 \end{aligned}$$

and $x_4' = \frac{1}{2}[x_4 - (-1)^{k+1}x_2] - (c^k)n_4 - (c^{k+2})n_6$, i.e.

$$\frac{x_4'}{c^k} = \frac{1}{2\cdot c^k}[x_4 - (-1)^{k+1}x_2] - n_4 + 2n_6.$$

Consequently, $n_4 = \frac{x_3}{2^k} - 2n_5 - 4n_6$, $n_2 = \frac{x_4''}{2(c)^k} = \frac{x_4'}{c^k} - \frac{x_6'}{2^{k+1}}$ and $n_3 = \frac{x_6''}{2^{k+1}} = (\frac{1}{2})\frac{x_4'}{c^k} - n_2$.

Recall that the images of generating element in positive filtration of codegree $4k$ are in the form AD^s, where $s \geq k+1$ and D^s where $s \geq k+3$.

case $AD^s = [x_1, x_2, x_3, x_4, x_5, x_6] = [0, 2(-c)^s, 0, 2(c)^s, 0, 2^{s+1}]$, $s \geq k+1$

We need to show that $n_i \in \mathbb{Z}_2^\wedge$ for each $i = 1, 2, 3, ..., 6$ s.t. $\sum_{i=1}^{i=6} n_i^2 \neq 0$ and n_1, n_4 are both even. In this case we have $x_5 = 0$ (i.e. $n_1 = 0$), $x_1' = 0$, $x_2' = 2(c)^s((-1)^s - (-1)^k)$. This implies that $x_1'' = 2^{k+1}(c)^{k+s-1}((-1)^s - (-1)^k)$ and hence

$$n_6 = \frac{x_1''}{3\cdot 4^{k+1}} = \begin{cases} (\pm\frac{1}{3})2^{\frac{s-k-1}{2}}, & \text{if } s+k \text{ is even;} \\ 0, & \text{if } s+k \text{ is odd.} \end{cases}$$

Thus $n_6 \in \mathbb{Z}_2^\wedge$. Similarly, $n_5 = \frac{x_2'}{2(-c)^{k+1}} \in \mathbb{Z}$ i.e. 0 or $\pm 2^{\frac{s-k+1}{2}}$. Since $x_3 = 0$, n_4 is immediately an even integer from relation above.

To justify n_2 and n_3, it remains to show that $\frac{x_4'}{c^k}$ is an even integer because $\frac{x_6'}{2^{k+1}} = \frac{2^{s+1}}{2^{k+1}} \in \mathbb{Z}$. Since $\frac{x_4'}{c^k} = \frac{1}{2\cdot c^k}[x_4 - (-1)^{k+1}x_2] - n_4 + 2n_6$, it reduces to check whether that $\frac{1}{2\cdot c^k}[x_4 - (-1)^{k+1}x_2]$ is an even integer. The result follows because $\frac{1}{2\cdot c^k}[x_4 - (-1)^{k+1}x_2] = c^{s-k}[1 - (-1)^{k+s+1}]$.

By using above relation, previous method and the relation $s \geq k+3$, we can verify that $v^{2(s-k)}d^s$, for each $s \geq k+3$, can be written in term of $[B_{4k}]$ over $\mathbb{Z}_2^\wedge$. □

From this lemma, we see that $ab^{2i+1}d^{k-i-1} = 2^i v(a+b)d^k - 2^i vad^k$ and its explicit combination in term of the generating set of $ku^{4k}(BSD_{16})$ follows by the relation $v^6d^{k+3} = 18vad^k - 9v(a+b)d^k - 8d^k - 2v^2d^{k+1} + 5v^4d^{k+2}$ or in other words,

$$v(a+b)d^k = 2vad^k - \frac{8}{9}d^k - \frac{2}{9}v^2d^{k+1} + \frac{5}{9}v^4d^{k+2} - \frac{1}{9}v^6d^{k+3} \tag{2.25}$$

Lemma 2.5.3 and this relation imply that an additive basis for $\mathbb{Z}_2^\wedge$ part of $ku^{4k}(BSD_{16})$, for $k \geq 1$, is $[B_{4k}]$ or, by changing basis, $\{b^{2k}, vad^k, d^k, v^2d^{k+1}, v^4d^{k+2}, v^6d^{k+3}\}$.

Finally,we use the same method to find an additive basis for $ku^2(BSD_{16})$. This is also enough to find the additive basis of $ku^{2k}(BSD_{16})$ for all $k \leq 0$ because $v : \widetilde{ku}^{2k}(BSD_{16}) \longrightarrow \widetilde{ku}^{2(k-1)}(BSD_{16})$ is an isomorphism for each $k \leq 1$.

Lemma 2.5.4. *All generating elements of $ku^2(BSD_{16})$ in positive filtration can be written as a $\mathbb{Z}_2^\wedge$ combination of $[B_2] := \{b, a, v^2(a+b)d, vd, v^3d^2, v^5d^3\}$.*

Proof. We will calculate on its image as before. Let x be a generating element of $ku^2(BSD_{16})$ in positive filtration but not in $[B_2]$. Then we have:

$$\begin{array}{lllllllll}
im(b) & = (y^2, & [0 & 0 & 0 & 0 & 2 & 2]) \\
im(a) & = (x^2, & [0 & 2 & 0 & 2 & 0 & 2]) \\
im(v^2(a+b)d) & = (0, & [0 & 2(-\sqrt{2}i) & 0 & 2(\sqrt{2}i) & 0 & 8]) \\
im(vd) & = (0, & [4 & -\sqrt{2}i & 2 & \sqrt{2}i & 0 & 2]) \\
im(v^3d^2) & = (0, & [16 & (-\sqrt{2}i)^2 & 4 & (\sqrt{2}i)^2 & 0 & 4]) \\
im(v^5d^3) & = (0, & [64 & (-\sqrt{2}i)^3 & 8 & (\sqrt{2}i)^3 & 0 & 8]) \\
im(x) & = (0, & [x_1 & x_2 & x_3 & x_4 & x_5 & x_6])
\end{array}$$

Thus x can be written as the combination of $[B_2]$ if

$$x = n_1(b) + n_2(a) + n_3(v^2(a+b)d) + n_4(vd) + n_5(v^3d^2) + n_6(v^5d^3),$$

where $n_i \in \mathbb{Z}_2^\wedge$ for each $i = 1, 2, 3, ..., 6$ s.t. $\sum_{i=1}^{i=6} n_i^2 \neq 0$ and n_1, n_2 are both even. This is equivalent to say that

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 16 & 64 \\ 0 & 2 & 2(-c) & -c & -2 & 2(c) \\ 0 & 0 & 0 & 2 & 4 & 8 \\ 0 & 2 & 2(c) & c & -2 & 2(-c) \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 8 & 2 & 4 & 8 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

where $c = \sqrt{2}i$. We find the solution n_i by using row-reduced matrix, i.e.,

$$\begin{pmatrix} 0 & 0 & 0 & 4 & 16 & 64 & |x_1 \\ 0 & 2 & 2(-c) & -c & -2 & 2(c) & |x_2 \\ 0 & 0 & 0 & 2 & 4 & 8 & |x_3 \\ 0 & 2 & 2(c) & c & -2 & 2(-c) & |x_4 \\ 2 & 0 & 0 & 0 & 0 & 0 & |x_5 \\ 2 & 2 & 8 & 2 & 4 & 8 & |x_6 \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 8 & 48 & |x_1 - 2x_3 = x_1' \\ 0 & 0 & 4(-c) & 2(-c) & 0 & 4(c) & |x_2 - x_4 = x_2' \\ 0 & 0 & 0 & 2 & 4 & 8 & |x_3 \\ 0 & 2 & 0 & 0 & -2 & 0 & |x_4 + (\frac{1}{2})x_2' = x_4' \\ 2 & 0 & 0 & 0 & 0 & 0 & |x_5 \\ 0 & 0 & 8 & 2 & 6 & 8 & |x_6 - x_5 - x_4' = x_6' \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 & 8 & 48 & |x_1' \\ 0 & 0 & 4(-c) & 2(-c) & 0 & 4(c) & |x_2' \\ 0 & 0 & 0 & 2 & 4 & 8 & |x_3 \\ 0 & 2 & 0 & 0 & -2 & 0 & |x_4' \\ 2 & 0 & 0 & 0 & 0 & 0 & |x_5 \\ 0 & 0 & 0 & 0 & 10 & 24 & |x_6' - c \cdot x_2' + x_3 = x_6'' \end{pmatrix}$$

$$\sim \begin{pmatrix} 0 & 0 & 0 & 0 & -12 & 0 & |x_1' - 2 \cdot x_6'' = x_1'' \\ 0 & 0 & 4(-c) & 0 & 0 & 0 & |x_2' + c \cdot x_3' - (\frac{c}{6})x_6''' = x_2'' \\ 0 & 0 & 0 & 2 & 0 & 0 & |x_3 + (\frac{1}{3})x_1'' - (\frac{1}{3})x_6''' = x_3' \\ 0 & 2 & 0 & 0 & 0 & 0 & |x_4' - (\frac{1}{6})x_1'' = x_4'' \\ 2 & 0 & 0 & 0 & 0 & 0 & |x_5 \\ 0 & 0 & 0 & 0 & 0 & 24 & |x_6'' + (\frac{5}{6})x_1'' = x_6''' \end{pmatrix}$$

Hence, $n_1 = \frac{x_5}{2}$, $n_2 = \frac{x_4''}{2}$, $n_3 = \frac{x_2''}{4(-c)}$, $n_4 = \frac{x_3'}{2}$, $n_5 = \frac{x_1''}{-12}$, $n_6 = \frac{x_6'''}{24}$.

We check only on AD^s where $s \geq 1$, and D^s where $s \geq 4$ by using table below.

n_i	AD^s, s is odd	AD^s, s is even
n_1	0	0
n_2	n_5	$\frac{1}{3}(2^s + 2 \cdot c^s)$
n_3	$\frac{1}{3}(2^{s-1} + c^{s-1})$	$\frac{1}{3}(2^{s-1} + c^{s-2})$
n_4	$\frac{-1}{9}(2^{s+2} + 8 \cdot c^{s+1})$	$\frac{1}{9}(-2^{s+2} + 4 \cdot c^s)$
n_5	$\frac{1}{3}(2^s + 2 \cdot c^{s+1})$	$\frac{1}{3}(2^s - c^s)$
n_6	$\frac{-1}{9}(2^{s-1} + c^{s+1})$	$\frac{-1}{9}(2^{s-1} + c^{s-2})$

n_i	D^s, s is odd	D^s, s is even
n_1	0	0
n_2	n_5	$n_5 + \frac{c^s}{2}$
n_3	$n_6 - \frac{n_4}{2} + \frac{c^{s-1}}{2}$	$n_6 - \frac{n_4}{2}$
n_4	$2^{s-1} - 2 \cdot n_5 - 4 \cdot n_6$	$2^{s-1} - 2 \cdot n_5 - 4 \cdot n_6$
n_5	$\frac{1}{3}(-4^{s-1} + c^{s+1}) + 2^{s-1}$	$\frac{1}{3}(-4^{s-1} + c^{s-2}) + 2^{s-1}$
n_6	$\frac{1}{9}(5 \cdot 4^{s-2} + c^{s-1}) - 2^{s-3}$	$\frac{1}{9}(5 \cdot 4^{s-2} + c^{s-4}) - 2^{s-3}$

E.g.

$$\begin{aligned} v^2ad &= \frac{-2}{3}(a) + \frac{2}{3}(v^2(a+b)d) + \frac{8}{9}(vd) + \frac{-2}{3}(v^3d^2) + \frac{1}{9}(v^5d^3), \\ v^4ad^2 &= 1 \cdot (v^2(a+b)d) + \frac{-8}{3}(vd) + 2(v^3d^2) + \frac{-1}{3}(v^5d^3), \\ v^7d^4 &= -12(a) + 3(v^2(a+b)d) + 8(vd) - 14(v^3d^2) + 7(v^5d^3), \\ v^9d^5 &= -72(a) + 18(v^2(a+b)d) + 32(vd) - 72(v^3d^2) + 32(v^5d^3). \end{aligned}$$

Also, it is immediately from the table to conclude that $n_i \in \mathbb{Z}_2^\wedge$. We now complete the proof. □

Combining all the previous results, we reach to the additive structure of $ku^*(BSD_{16})$ as:

Theorem 2.5.5. *In $ku^*(BSD_{16})$, we have $a, b \in ku^2(BSD_{16})$, $d \in ku^4(BSD_{16})$, where $a = e_{ku}(\chi_3), b = e_{ku}(\chi_2)$ and $d = e_{ku}(\sigma_1)$ s.t.*

(1) *if $k \leq 0$, then $ku^{2k}(BSD_{16}) \cong \mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^6$, which $(\mathbb{Z}_2^\wedge)^6$ is generated by $v^k\{1, vb, va, v^3(a+b)d, v^2d, v^4d^2, v^6d^3\}$,*

(2) *$ku^2(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^6$ generated by $\{b, a, v^2(a+b)d, vd, v^3d^2, v^5d^3\}$,*

(3) *$ku^4(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^6$ generated by $\{b^2, vad, v(a+b)d, d, v^2d^2, v^4d^3\}$,*

(4) *$ku^6(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^6$ generated by $\{b^3, ad, (a+b)d, vd^2, v^3d^3, v^5d^4\}$,*

(5) *if $k > 1$, then $ku^{4k+2}(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^6 \oplus (\mathbb{F}_2)^{k-1}$ with $(\mathbb{Z}_2^\wedge)^6$ generated by $\{b^{2k+1}, ad^k, (a+b)d^k, vd^{k+1}, v^3d^{k+2}, v^5d^{k+3}\}$ and $(\mathbb{F}_2)^{k-1}$ generated by $\{b^3d^{k-1} - ab^2d^{k-1}, b^5d^{k-2} - ab^4d^{k-2}, ..., b^{2k-1}d - ab^{2k-2}d\}$,*

(6) *if $k > 1$, then $ku^{4k}(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^6 \oplus (\mathbb{F}_2)^{k-1}$ with $(\mathbb{Z}_2^\wedge)^6$ generated by $\{b^{2k}, vad^k, v(a+b)d^k, d^k, v^2d^{k+1}, v^4d^{k+2}\}$ and $(\mathbb{F}_2)^{k-1}$ generated by $\{b^2d^{k-1} - ab^1d^{k-1}, b^4d^{k-2} - ab^3d^{k-2}, ..., b^{2k-2}d - ab^{2k-3}d\}$ and*

(7) *the 2 torsion is annihilated by v.*

(8) *v acts monomorphically on positive filtration and on non negative filtration if it acts on codegree which is less than or equal to 6.*

Next, to find the ring structure, representation theory and cohomology ring theory, again, play a big role. We deal with this in the next section.

§ 2.6 The multiplicative structure of $ku^*(BSD_{16})$

In this section we will find all relations of the ring $ku^*(BSD_{16})$. The very useful tool is the representation theory, cohomology ring theory and additive structure we have found. In other words, we use injectivity of homomorphism

$$ku^*(BSD_{16}) \rightarrowtail H^*(BSD_{16};\mathbb{F}_2) \oplus KU^*(BSD_{16}).$$

By theorem 2.5.5, we have that all $2, v$-torsion are in $H^*(BSD_{16};\mathbb{F}_2)$ (zero filtration), so, if we deal with elements lying on positive filtration, then it suffices to consider only on the representation theory side but if not, we need to consider on both sides.

Now, we start to find the relations. First, recall that a and b are defined to be Euler classes of one dimensional representation, χ_3 and χ_2 respectively, by lemma 1.3.4 in [14], we get immediately that

$$0 = e_{ku}(1) = e_{ku}((\chi_3)^2) = e_{ku}(\chi_3) + e_{ku}(\chi_3) - ve_{ku}(\chi_3)e_{ku}(\chi_3)$$

i.e. $va^2 = 2a$ and similarly, $vb^2 = 2b$. In fact, we can also use the relation between characteristic class of connective K-theory and representation theory (see below) to gain these relations. More precisely, since we have relation $\chi_2^2 = \chi_3^2 = 1$ and $A = va = 1 - \chi_3$, $B = vb = 1 - \chi_2$,

$$\begin{aligned} 1 = (\chi_3)^2 = (1-va)^2 &= 1 - 2va + v^2a^2 \\ 1 = \chi_2)^2 = (1-vb)^2 &= 1 - 2vb + v^2b^2 \end{aligned}$$

and hence

$$va^2 = 2a \text{ and } vb^2 = 2b, \tag{2.26}$$

because v acts monomorphically on positive filtration.

To be easier in comparing, let us recollect the representation ring we have as:

$$R(SD_{16}) = Z[\chi_2, \chi_3, \chi_4, \sigma_1, \sigma_2, \sigma_3]/R$$

$$\begin{aligned} R = (&\chi_2^2 = \chi_3^2 = \chi_4^2 = 1,\ \chi_2\chi_3 = \chi_4,\ \sigma_1^2 = \sigma_3^2 = \sigma_2 + \chi_3 + \chi_4,\ \sigma_2^2 = 1 + \chi_2 + \chi_3 + \chi_4, \\ &\sigma_1\sigma_2 = \sigma_2\sigma_3 = \sigma_1 + \sigma_3,\ \sigma_1\sigma_3 = \sigma_2 + 1 + \chi_2,\ \chi_2\sigma_1 = \sigma_1,\ \chi_2\sigma_2 = \sigma_2,\ \chi_2\sigma_3 = \sigma_3, \\ &\chi_2\sigma_2 = \chi_3\sigma_2 = \chi_4\sigma_2 = \sigma_2,\ \chi_3\sigma_1 = \chi_4\sigma_1 = \sigma_3,\ \chi_3\sigma_3 = \chi_4\sigma_3 = \sigma_1) \end{aligned}$$

and

$$\begin{aligned} \chi_2 &= 1 - vb \\ \chi_3 &= 1 - va \\ \chi_4 &= (1-vb)(1-va) \\ \sigma_1 &= 1 - v^2d + (1-vb)(1-va) \\ \sigma_2 &= 1 - v^2d_2 + 1 - vb \\ \sigma_3 &= 1 - v^2d_3 + (1-vb)(1-va). \end{aligned}$$

And the relation between connective K theory, $ku^*(BSD_{16})$, and cohomology ring theory, $H^*(SD_{16};\mathbb{F}_2)$, is given by,

$$\begin{array}{lcl} a & \mapsto & x^2, \\ b & \mapsto & y^2, \\ d, d_3 & \mapsto & P, \\ d_2 & \mapsto & 0. \end{array} \tag{2.27}$$

Now, since we have the relation $(1-\chi_2)(1-\sigma_1+\chi_4) = (1-\chi_3)(1-\chi_2)$, $v^3bd = v^2ab$ and because v is monomorphism at this point, thus

$$ab = vbd = v(a+b)d - vad. \tag{2.28}$$

Similarly, by $(1-\chi_2)(1-\sigma_2+\chi_2) = 0$, we have

$$bd_2 = 0. \tag{2.29}$$

It is not hard to check that

$$bd_3 = bd, \tag{2.30}$$

by using $(1-\chi_2)(1-\sigma_3+\chi_4) = (1-\chi_2)(1-\sigma_1+\chi_4)$ and injectivity of v. Since $\chi_3\sigma_1 = \sigma_3$, $(1-va)(1-v^2d+(1-vb)(1-va)) = 1-v^2d_3+(1-vb)(1-va)$ and hence

$$\begin{array}{rcl} d_3 & = & (1-va)d + ab \\ & = & d - 2vad + v(a+b)d. \end{array} \tag{2.31}$$

Similarly, we use $\sigma_2 = \sigma_1^2 - (\chi_3+\chi_4)$ to get

$$\begin{array}{rcl} d_2 & = & 4d - v^2d^2 - 2vad + ab \\ & = & 4d - 3vad + v(a+b)d - v^2d^2. \end{array} \tag{2.32}$$

The relation $\chi_3\sigma_1 = \chi_4\sigma_1$ gives us the relation $2ab = v^2abd$ and combining this with (2.26) and the fact that v acts monomorphically on codegree 6, theorem 2.5.5(8), we obtain

$$vabd = ab^2. \tag{2.33}$$

For regular representation, $\rho = 1+\chi_2+\chi_3+\chi_4+2(\sigma_1+\sigma_2+\sigma_3)$, we get that $\rho = 16 - 6va - 8vb + v^2ab - 12v^2d + 6v^3ad + 2v^4d^2$. Since ρ is induced up from the trivial subgroup and by Lemma 2.1.1 in [14],

$$\begin{array}{rcl} 0 = d\rho & = & 16d - 6vad - 8vbd + v^2abd - 12v^2d^2 + 6v^3ad^2 + 2v^4d^3 \\ & = & 16d - 6vad - 6vbd - 12v^2d^2 + 6v^3ad^2 + 2v^4d^3. \end{array}$$

However, we see that $\frac{d\rho}{2} \in ku^*(BSD_{16})$ and by cohomology ring, character table, $\frac{d\rho}{2} \mapsto (0,0)$, thus

$$8d - 3vad - 3vbd - 6v^2d^2 + 3v^3ad^2 + v^4d^3 = 0. \tag{2.34}$$

The relation $\chi_3\sigma_2 = \sigma_2$ give us the relation $v^2ad_2 - 2a + vab = 0$ then $v^4ad^2 = v^2a^2b - 2a + vab$ and hence

$$v^3ad^2 = 3ab - a^2 \tag{2.35}$$

The relation $\sigma_2\sigma_3 = \sigma_1 + \sigma_3$ give us the relation $v^2 d_2 d_3 + vad_2 = d_3 - d + 2d_2$ and thus

$$8d - 6v^2d^2 + v^4d^3 - v^5ad^3 + 3v^3ad^2 - 5vad + 3ab = 0.$$

Combining this relation with $\frac{d\rho}{2} = 0$, we obtain

$$v^4ad^3 = -2ad + 6bd \tag{2.36}$$

By considering on $\sigma_2^2 = 1 + \chi_2 + \chi_3 + \chi_4$, we have $4d_2 = a^2 - ab + v^2d_2^2$ and hence

$$16d - 20v^2d^2 + 8v^4d^3 - v^6d^4 - 9a^2 + 9ab = 0 \tag{2.37}$$

Furthermore, elements from $2, v$-torsion part will be necessary to fulfill relations. Here, from theorem 2.5.5, we define

$$\tau := b^2d - abd \in ku^8(BSD_{16}). \tag{2.38}$$

Hence, the torsion part of $ku^*(BSD_{16}) := TU$ is a free module over $\mathbb{F}_2[b, d]$ generated by τ, i.e.,

$$TU = \mathbb{F}_2[b, d] < \tau > . \tag{2.39}$$

The obvious relations with this element is

$$2\tau = v\tau = a\tau = 0. \tag{2.40}$$

Also, from (2.28), we have $abd = vbd^2$ and hence

$$\tau - b^2d + vbd^2 = 0. \tag{2.41}$$

Therefore, now we have a set of the relation I' in $ku^*(BSD_{16})$ as:

$$\begin{aligned} I' = \ & \{(r_1 : va^2 = 2a, r_2 : vb^2 = 2b, r_3 : ab = vbd = v(a + b)d - vad, \\ & r_4 : ab^2 = vabd, r_5 : v^3ad^2 = 3ab - a^2, r_6 : v^4ad^3 = -2ad + 6bd, \\ & r_7 : 8d - 3vad - 3vbd - 6v^2d^2 + 3v^3ad^2 + v^4d^3 = 0, \\ & r_8 : 16d - 20v^2d^2 + 8v^4d^3 - v^6d^4 - 9a^2 + 9ab = 0) \\ & r_9 : 2\tau = v\tau = a\tau = 0, r_{10} : \tau - b^2d + vbd^2 = 0\}, \end{aligned}$$

which we will see in the theorem below that I' is a complete relation set for $ku^*(BSD_{16})$.

Theorem 2.6.1. *$ku^*(BSD_{16}) = (\mathbb{Z}[v][a, b, d, \tau]/I)^\wedge_J$ where $|a| = |b| = 2, |d| = 4$ and $|\tau| = 8$, and where J is the augmentation ideal and I is the ideal*

$$\begin{aligned} I = \ & (va^2 - 2a, vb^2 - 2b, ab - vbd, \\ & ab^2 - vabd, v^3ad^2 - 3ab + a^2, v^4ad^3 + 2ad - 6bd, \\ & 8d - 3vad - 3vbd - 6v^2d^2 + 3v^3ad^2 + v^4d^3, \\ & 16d - 20v^2d^2 + 8v^4d^3 - v^6d^4 - 9a^2 + 9ab, \\ & 2\tau, v\tau, a\tau, \tau - b^2d + vbd^2). \end{aligned}$$

The natural map $ku^*(BSD_{16}) \longrightarrow H^*(BSD_{16}; \mathbb{F}_2)$ sends a to x^2, b to y^2, d_2 to 0 and d, d_3 to P.

Proof. The final statement is an evident from (2.27). Thus, now, it remains to show that I is the completed relation in $ku^*(BSD_{16})$. To show this, we need to check that multiplying any of the additive generators of theorem 2.5.5 by a, b, d, τ or v produces an element which can be written in term of that additive basis by using relation from I.

Note that multiplying by τ can be reduced obviously to the additive basis for $ku^*(BSD_{16})$. So, we consider the elements in term of v, a, b, d. Let $x \in ku^*(BSD_{16})$. It is without loss of generality to assume that $x \in R^{2m} := ku^{2m}(BSD_{16})$ for some integer m.

Case $m = k \leq 0$**:** In this case, $R^{2k} = \bigcup_{i=0}^{i=7} C_i$, where

$$\begin{array}{lll} C_0 = \{v^k\}, & C_1 = \{v^{k+i+2j}a^id^j | i,j \geq 1\}, & C_2 = \{v^{k+i}b^i | i \geq 1\}, \\ C_3 = \{v^{k+2i}d^i | i \geq 1\}, & C_4 = \{v^{k+i+j}a^ib^j | i,j \geq 1\}, & C_5 = \{v^{k+i}a^i | i \geq 1\}, \\ C_6 = \{v^{k+i+2j}b^id^j | i,j \geq 1\}, & C_7 = \{v^{k+i+j+2l}a^ib^jd^l | i,j,l \geq 1\}, & \end{array}$$

and additive basis of $ku^{2k}(BSD_{16})$ is

$$B_{2k} := \{v^k, v^{k+1}a, v^{k+1}b, v^{k+3}(a+b)d, v^{k+2}d, v^{k+4}d^2, v^{k+6}d^3\}.$$

For any element in C_5 and C_2, it is easily reduced to the combination of elements in B_{2k} via iterating of r_1 and r_2 respectively, i.e. $v^{k+i}a^i = 2^{i-1}v^{k+1}a$ and $v^{k+i}b^i = 2^{i-1}v^{k+1}b$. From r_5 and r_7, we have $ab = \frac{1}{3}(v^3ad^2) + \frac{1}{3}a^2$ and $\frac{1}{3}(v^3ad^2) = \frac{1}{3}(v(a+b)d - \frac{8}{9}d + \frac{2}{3}v^2d^2 - \frac{1}{9}v^4d^3$. Thus, combining these relations with r_1, we get that

$$v^{k+2}ab = \frac{1}{3}(v^{k+3}(a+b)d - \frac{8}{9}v^{k+2}d + \frac{2}{3}v^{k+4}d^2 - \frac{1}{9}v^{k+6}d^3 + \frac{2}{3}v^{k+1}a.$$

This follows that, for any element in C_4, $v^{k+i+j}a^ib^j = 2^{i+j-2}v^{k+2}ab$ can reduce to the combination of elements in B_{2k} via I.

By using r_3 and r_4, we have $v^3bd = v^2(vbd) = v^2ab$ and then $v^5bd^2 = v^2(v^3bd)d = v^2(v^2abd) = 2v^2ab$. So, iterating this process and using induction we can show that $v^{2j+1}bd^j = 2^{j-1}v^2ab$. Then $v^{k+i+2j}b^id^j$ can reduce to $2^{i-1}v^k(v^{2j+1}bd^j) = 2^{i+j-2}v^{k+2}ab$ and hence C_6 can be reduced to B_{2k} via I.

It is not hard now to show that the conclusion for C_7 also holds by using the previous result, i.e. $v^{2j+1}bd^j$, $v^{k+2}ad$ and r_1. For C_1, we consider the following process;

$$\begin{aligned} v^{k+3}ad &= v^{k+3}(a+b)d - v^{k+3}bd \\ v^{k+5}ad^2 &= v^{k+2}(3ab - a^2), \quad \text{because of } r_5, \\ &= 3v^{k+2}ab - 2v^{k+1}a \\ v^{k+7}ad^3 &= 3v^{k+2}(v^2abd) - 2v^{k+3}ad \\ &= 6v^{k+2}ab - 2v^{k+3}ad, \quad \text{because of } r_4, \end{aligned}$$

So, by induction, we can conclude that $v^{2j+1}ad^j$ can reduce to the combination of elements in B_{2k} via I and hence this holds for all elements in C_1. The last task for this case can be done by considering step by step as above. That is

$$\begin{aligned} v^{k+8}d^4 &= v^{k+2}(v^6d^4) \\ &= 16v^{k+2}d - 20v^{k+4}d^{k+2} + 8v^{k+6}d^3 - 18v^{k+1}a + 9v^{k+2}ab, \quad \text{because of } r_8, \\ v^{k+10}d^5 &= v^2d(v^{k+8}d^4) \\ &= 16v^{k+4}d^2 - 20v^{k+6}d^{k+3} + 8v^{k+8}d^4 - 18v^{k+3}ad + 9v^{k+4}abd. \end{aligned}$$

By repeating this process and using the previous result, we complete the proof for our fist case.

Case $m = 1$**:** In this case $R^2 = \bigcup_{i=1}^{i=7} C_i$, where

$$\begin{array}{ll} C_1 = \{v^i a^{i+1} | i \geq 0\}, & C_2 = \{v^i b^{i+1} | i \geq 0\}, \\ C_3 = \{v^{2i+1} d^{i+1} | i \geq 0\}, & C_4 = \{v^{i+j-1} a^i b^j | i, j \geq 1\}, \\ C_5 = \{v^{i+2j-1} a^i d^j | i, j \geq 1\}, & C_6 = \{v^{i+2j-1} b^i d^j | i, j \geq 1\}, \\ C_7 = \{v^{i+j+2l-1} a^i b^j d^l | i, j, l \geq 1\}, & \end{array}$$

and additive basis for $ku^2(BSD_{16})$ is

$$B_{2k} := \{a, b, v^2(a+b)d, vd, v^3d^2, v^5d^3\}.$$

The proof for this case is very similar to the first case.

Case $m = 2k, k \geq 1$: In this case $R^{4k} = \bigcup_{i=1}^{i=10} C_i$, where

$$\begin{array}{l} C_1 = \{v^i a^{2k+i} | i \geq 0\}, \\ C_2 = \{v^i b^{2k+i} | i \geq 0\}, \\ C_3 = \{v^{2i} d^{k+i} | i \geq 0\}, \\ C_4 = \{v^{j+l} a^{i+j} b^{(2k-i)+l} | 1 \leq i < 2k, j, l \geq 0\}, \\ C_5 = \{v^{j+2l} a^{2i+j} d^{(k-i)+l} | 1 \leq i < k, j, l \geq 0\}, \\ C_6 = \{v^{j+1+2l} a^{2i+j+1} d^{(k-i)+l} | 0 \leq i < k, j, l \geq 0\}, \\ C_7 = \{v^{j+2l} b^{2i+j} d^{(k-i)+l} | 1 \leq i < k, j, l \geq 0\}, \\ C_8 = \{v^{j+1+2l} b^{2i+j+1} d^{(k-i)+l} | 0 \leq i < k, j, l \geq 0\}, \\ C_9 = \{v^{l+s+2t} a^{i+l} b^{j+s} d^{k-\frac{i+j}{2}+t} | \text{ i+j is even and } 1 \leq \frac{i+j}{2} < k, l, s, t \geq 0\}, \\ C_{10} = \{v^{l+s+2t+1} a^{i+l} b^{j+s} d^{k-\frac{i+j-1}{2}+t} | \text{i+j is odd and } 1 \leq \frac{i+j-1}{2} < k, l, s, t \geq 0\}, \end{array}$$

and additive basis for torsion free part of $ku^{4k}(BSD_{16})$ is

$$B_{4k} := \{b^{2k}, vad^k, v(a+b)d^k, d^k, v^2d^{k+1}, v^4d^{k+2}\},$$

and for $2, v$-torsion part is (case $k \geq 2$)

$$TU_{4k} := \{\tau d^{k-2}, \tau b^2 d^{k-3}, ..., \tau b^{2k-4} d\}.$$

In this case note that reducing of C_2 is obvious by r_2 and we will begin with C_5 by considering r_7. By r_7, we have that $v^3ad^2 = -\frac{8}{3}d + v(a+b)d + 2v^2d^2 - \frac{1}{3}v^4d^3$ and that

$$v^3ad^i = -\frac{8}{3}d^{i-1} + v(a+b)d^{i-1} + 2v^2d^i - \frac{1}{3}v^4d^{i+1}. \tag{2.42}$$

That is v^3ad^i can be reduced to the combination of elements in $B_{4(i-1)}$ for all $i \geq 2$. Thus, by r_5 we have $a^2 = 3ab - v^3ad^2$ and hence

$$\begin{aligned} a^2d &= 3abd - v^3ad^3 \\ &= 3vbd^2 + \frac{8}{3}d^2 - v(a+b)d^2 - 2v^2d^3 + \frac{1}{3}v^4d^4 \\ &= 2v(a+b)d^2 - 3vad^2 + \frac{8}{3}d^2 - 2v^2d^3 + \frac{1}{3}v^4d^4. \end{aligned}$$

Also, it is clear that $d(B_{4i} - \{b^{2i}\}) = B_{4(i+1)} - \{b^{2(i+1)}\}$. Hence,

$$a^2d^{k-1} = 2v(a+b)d^k - 3vad^k + \frac{8}{3}d^k - 2v^2d^{k+1} + \frac{1}{3}v^4d^{k+2},$$

can be reduced to $B_{4k} - \{b^{2k}\}$ via I. Consider,

$$\begin{aligned} a^4d &= 2va^2(a+b)d^2 - 3va^3d^2 + \frac{8}{3}a^2d^2 - 2v^2a^2d^3 + \frac{1}{3}v^4a^2d^4 \\ &= \frac{2}{3}a^2d^2 + 4abd^2 - 4vad^3 + \frac{2}{3}v^3ad^4 \\ &= \frac{2}{3}a^2d^2 + 4vbd^3 - 4vad^3 + \frac{2}{3}v^3ad^4, \quad \text{because of } r_3. \end{aligned}$$

Hence, from above result a^2d^{k-1}, when $k=3$, a^4d can be reduced to $B_8 - \{b^4\}$ via I and thus the conclusion holds for a^4d^{k-2}. Again,

$$\begin{aligned} a^6d &= a^2(a^4d) \\ &= a^2[e_1va^3 + e_2v(a+b)d^3 + e_3d^3 + e_4v^2d^4 + e_5v^4d^5], e_i \in \mathbb{Z}_2^\wedge \\ &= 2e_1a^2d^3 + 2e_2a^2d^3 + 2e_2abd^3 + e_3a^2d^3 + 2e_4vad^4 + 2e_5v^3ad^5. \end{aligned}$$

Hence from the previous result, a^6d can be reduced to $B_{12} - \{b^6\}$ via I and thus the conclusion holds for a^6d^{k-3}. By analogous process as above and induction, we can conclude that $a^{2i}d^{k-i}$ can be written by using elements in B_{4k} via I.

Next, we consider

$$\begin{aligned} v^2a^2d^2 &= 2v^3(a+b)d^3 - 3v^3ad^3 + \frac{8}{3}v^2d^3 - 2v^4d^4 + \frac{1}{3}v^6d^5 \\ &= 2v^3bd^3 - v^3ad^3 + \frac{8}{3}v^2d^3 - 2v^4d^4 + \frac{1}{3}v^6d^5. \end{aligned}$$

We concentrate on each term, for the fist term, we have

$$v^3bd^3 = v^2(vbd)d^2 = 2abd = 2v\overline{bd}^2 = 2v(a+b)d^2 - 2vad^2,$$

and the last term by r_8, we have $v^6d^5 = 16d^2 - 20v^2d^3 + 8v^4d^4 - 9a^2d + 9abd$. Since a^2d and abd can reduced to $B_8 - \{b^4\}$ via I, so is $v^2a^2d^2$, i.e., $v^2a^2d^2$ can be written in the form $e_1vad^2 + e_2v(a+b)d^2 + e_3d^2 + e_4v^2d^3 + e_5v^4d^4$. For $v^4a^2d^3 = v^2d(v^2a^2d^2)$ can do analysis as above and we now can say that the result for $v^{2i}a^2d^{i+1}$ follows. By using the same process we have done, we now conclude that $v^{2j}a^{2i}d^{j+1}$ can be reduced to the combination of elements in $B_{4(i+1)} - \{b^{2(i+1)}\}$. This follows that

$$\begin{aligned} v^{j+2l}a^{2i+j}d^{(k-i)+l} &= 2^j a^{2i} v^{2l} d^{(k-i)+l} \\ &= 2^j d^{(k-i)-1}(v^{2l}a^{2i}d^{1+l}), \end{aligned}$$

i.e., C_5 can be written as the combination of elements in B_{4k} via I.

For C_6, since for $j \geq 1$, $v^{j+1+2l}a^{2i+j+1}d^{(k-i)+l} = 2v^{j+2l}a^{2i+j}d^{(k-i)+1}$ which is the case C_5, we need to consider only on $v^{2l+1}ad^{k+l}$. But this is a consequence of (2.42) together with the fact that v^6d^{k+3} can reduce to B_{4k} (by the same method as above), hence C_6 can reduce to B_{4k} via I. Also, the conclusion for C_3 is immediate because multiplying by v^2d on $B_{4k} - \{b^{2k}\})$ can reduce to itself via I (i.e., by using r_8 as C_5).

For C_1, we use r_5, i.e. $a^2 = 3ab - v^3ad^2$. Then $a^{2k} = 3a^{2k-1}b - v^3a^{2k-1}d^2$. The seconde term is reduced by C_6. The first term is easy because

$$\begin{aligned} a^{2k-1}b &= a^{2k-2}(vbd) \\ &= bd(va^2)a^{2k-4} \\ &= 2bda^{2k-3} \end{aligned}$$

and iterates this via r_1 and r_3 until we get $a^{2k-1}b = 2^{k-1}vbd^k = 2^{k-1}[v(a+b)d^k - vad^k]$. Since $v^ia^{2k+i} = 2^ia^{2k}$, we finish C_1.

For C_7 we first consider r_{10}, i.e., $b^2d = \tau + vbd^2$. It is not hard to see that $b^{2i}d^{k-i}$ can reduce to $TU_{4k} \cup (B_{4(i+1)} - \{b^{2(i+1)}\})$ via I. Precisely, by r_2, r_9 and iterating r_{10}, we have $b^2d^i = \tau d^2 + vbd^{i+1}$, $b^{2i}d = \tau b^{2(i-1)} + 2^{i-1}vbd^{i+1}$ and thus

$$b^{2i}d^{k-i} = \tau b^{2(i-1)}d^{k-i-1} + 2^{i-1}vbd^k.$$

From here, with the help of r_2, the conclusion for C_7 follows.

For C_8, this is a similar situation with C_6, i.e. we need to prove only on $v^{2l+1}bd^{k+l}$. To prove this, we use the fact that $v^{2j+1}bd^{1+j} = 2^jab$ via r_3, r_4. For C_4, C_9 and C_{10} are immediately verified by the previous results and the relation r_1, r_2 and r_3.

Case $m = 2k+1, k \geq 1$: In this case $R^{4k+2} = \bigcup_{i=1}^{i=10} C_i$, where

$C_1 = \{v^ia^{2k+1+i} | i \geq 0\}$,
$C_2 = \{v^ib^{2k+1+i} | i \geq 0\}$,
$C_3 = \{v^{2i+1}d^{k+i+1} | i \geq 0\}$,
$C_4 = \{v^{j+l}a^{i+1+j}b^{(2k-i)+l} | 0 \leq i < 2k, j, l \geq 0\}$,
$C_5 = \{v^{j+2l}a^{2i+1+j}d^{(k-i)+l} | 0 \leq i < k, j, l \geq 0\}$,
$C_6 = \{v^{j+1+2l}a^{2(i+1)+j}d^{(k-i)+l} | 0 \leq i < k, j, l \geq 0\}$,
$C_7 = \{v^{j+2l}b^{2i+1+j}d^{(k-i)+l} | 0 \leq i < k, j, l \geq 0\}$,
$C_8 = \{v^{j+1+2l}b^{2(i+1)+j}d^{(k-i)+l} | 0 \leq i < k, j, l \geq 0\}$,
$C_9 = \{v^{l+s+2t+1}a^{i+l}b^{j+s}d^{k-\frac{i+j}{2}+t+1} |$ i+j is even and $1 \leq \frac{i+j}{2} < k, l, s, t \geq 0\}$,
$C_{10} = \{v^{l+s+2t}a^{i+l}b^{j+s}d^{k-\frac{i+j-1}{2}+t} |$i+j is odd and $1 \leq \frac{i+j-1}{2} < k, l, s, t \geq 0\}$,

and the additive basis for torsion free part of $ku^{4k+2}(BSD_{16})$ is

$$B_{4k+2} := \{b^{2k+1}, ad^k, (a+b)d^k, vd^{k+1}, v^3d^{k+2}, v^5d^{k+3}\},$$

and for $2, v$-torsion part is

$$TU_{4k+2} = \{\tau bd^{k-2}, \tau b^2d^{k-3}, ..., \tau b^{2k-3}\}.$$

In this case note that C_2 is reduced obviously to B_{4k+2} by r_2 and we will begin with C_5 by considering r_7 which we have;

$$\begin{aligned} vad &= \frac{8}{3}d - vbd - 2v^2d^2 + v^3ad^2 + \frac{1}{3}v^4d^3 \\ v^2ad^2 &= \frac{8}{3}vd^2 - v^2bd^2 - 2v^3d^3 + v^4ad^3 + \frac{1}{3}v^5d^4 \end{aligned}$$

From r_6 we have $v^4ad^3 = 6bd - 2ad$ and note that

$$v^2bd^2 = vabd = ab^2 = (ab)b = (vbd)b = 2bd,$$

by r_2, r_3 and r_4. This yields that v^2ad^2 can be reduced to $B_6 - \{b^3\}$ via I. It is an experience that $d(B_{4i+2} - \{b^{2i+1}\})$ can be reduced to $B_{4(i+1)+2} - \{b^{2(i+1)+1}\}$. So, v^2ad^{i+1} and v^2bd^{i+1} can be written as the combination of elements in $B_{4i+2} - \{b^{2i+1}\}$ via I.

By r_8, we have;

$$\begin{aligned} v^6d^4 &= 16d - 20v^2d^2 + 8v^4d^3 - 9a^2 + 9ab \\ v^7d^{i+4} &= 16vd^{i+1} - 20v^3d^{i+2} + 8v^5d^{i+3} - 9va^2d^i + 9vabd^i \\ &= 16vd^{i+1} - 20v^3d^{i+2} + 8v^5d^{i+3} - 18ab^i + 18bd^i. \end{aligned}$$

Now, consider

$$\begin{aligned} v^2d(ad^i) &= v^2ad^{i+1} \\ v^2d(bd^i) &= v^2ad^2 = 2bd^i \\ v^2d(vd^{i+1}) &= v^3d^{i+1} \\ v^2d(v^3d^{i+2}) &= v^5d^{i+2} \\ v^2d(v^5d^{i+3}) &= v^7d^{i+3}. \end{aligned}$$

By using the above results, we see that $v^2d(B_{4i+2} - \{b^{2i+1}\}$ can reduce to $B_{4i+2} - \{b^{2i+1}\}$ via I and hence C_3 is proved. Next, by r_5 we have $a^2d = 3abd - v^3ad^3$. Then

$$\begin{aligned} a^3d &= 3ab^2d - v^3a^2d^3 \\ &= 6bd^2 - 2v^2ad^3. \end{aligned}$$

By the results above we can conclude that a^3d can reduce, via I, to $B_{10} - \{b^5\}$ and hence, similarly for a^3d^{k-1} and for $v^{2l}a^3d^{k-1+2l}$ to $B_{4k+2} - \{b^{2k+1}\}$. Moreover, we have

$$
\begin{aligned}
a^2[ad^i] &= a^3d^i \\
a2[bd^i] &= a^2bd^i = 2bd^i \\
a^2[vd^{i+1}] &= 2ad^{i+1} \\
a^2[v^3d^{i+2}] &= 2v^2ad^{i+2} \\
a^2[v^5d^{i+3}] &= 2(v^2d)(v^2ad^{i+2})
\end{aligned}
$$

Hence, $a^2(B_{4i+2} - \{b^{2i+1}\}$ can reduce to $B_{4(i+1)+2} - \{b^{2(i+1)+1}\}$ via I, i.e. the conclusions for $a^{2i+1}d^{k-i}$ are the same. This follows that C_5 and C_6 can be reduced as we need (by similar process as case $4k$).

For C_7 and C_8, we again use r_{10}, which yields

$$b^{2i+1}d^{k-i} = \tau b^{2i-1}d^{k-i-1} + 2^ibd^k.$$

So, by r_2, the conclusion for C_7 follows. For C_8, we use r_3 and r_4. For C_1, we use r_5, i.e. $a^2 = 3ab - v^3ad^2$, which gives

$$a^{2k+1} = 3a^{2k}b - v^3a^{2k}d^2.$$

The second term is reduced by C_6. The first term follows from case $m = 2k$ which we have $a^{2k-1}b = 2^{k-1}vbd^k$ and then $a^{2k}b = 2^kbd^k$ via r_4. For C_4, C_9 and C_{10} are immediately verified by the previous results and the relations r_1, r_2 and r_3. We now complete the proof for this theorem here. □

The complex connective K-theory of SD_{16} is quite strange when we compare with that of D_{2^n} since both of them have p-rank two but their $2, v$-torsion are different, i.e. $ku^6(BD_{2^n})$ contains $2, v$-torsion whereas $ku^6(BSD_{16})$ is torsion free. We explore the relationship on complex connective K-theory of SD_{16} and its maximal subgroups in the next section.

§ 2.7 Relations with its maximal subgroups

In this section, we will relate our results, i.e., complex connective K-theory of SD_{16}, $ku^*(BSD_{16})$, to the complex connective K-theory of its maximal subgroups $ku^*(BD_8)$, $ku^*(BQ_8)$ and $ku^*(BC_8)$ in [14]. To do this, it is simple to use the fact that all of them are embedded in their sums of cohomology ring and representation ring.

First of all, we recall and record the results from [14] as;

Proposition 2.7.1. *[14], Complex connective K-theory of D_8, Q_8 and C_8 is given by:*

1. *$ku^*(BD_8) \cong (ku^*[a,b,d]/I)^{\wedge}_J$ where $|a| = |b| = 2, |d| = 4$ s.t. $a = e_{ku}(\widehat{st}), b = e_{ku}(\widehat{s}), d = e_{ku}(\sigma_1)$ and where I is the ideal*

$$\begin{aligned} I &= (v^4d^3 - 6v^2d^2 + 8d, \\ &\quad va^2 - 2a, vb^2 - 2b, vad, 2ad, \\ &\quad ab - b^2 + vbd, \\ &\quad vbd - 4d + v^2d^2, 2bd - v^2bd^2). \end{aligned}$$

The natural map $ku^*(BD_8) \longrightarrow H^*(BD_8;\mathbb{F}_2)$ sends a to x_1^2, b to x_2^2 and d to w^2, where $H^*(BD_8;\mathbb{F}_2) = \mathbb{F}_2[x_1,x_2,w]/(x_2(x_1+x_2))$ s.t. $x_1 = w_1(\widehat{st}), x_2 = w_1(\widehat{s})$ and $w = w_2(\sigma_1)$.

2. *$ku^*(BQ_8) \cong (ku^*[a,b,q]/I)^{\wedge}_J$ where $|a| = |b| = 2, |q| = 4$ s.t. $a = e_{ku}(\psi_0 - 1), b = e_{ku}(\chi), q = e_{ku}(\psi_1)$ and where I is the ideal*

$$\begin{aligned} I &= (v^4q^3 - 6v^2q^2 + 8q, \\ &\quad va^2 - 2a, vb^2 - 2b, a^2 - vaq, b^2 - vbq, \\ &\quad ab - (vaq + vbq + v^2q^2 - 4q)). \end{aligned}$$

The natural map $ku^*(BQ_8) \longrightarrow H^*(BQ_8;\mathbb{F}_2)$ sends a to x_1^2, b to x_2^2 and q to p_1, where $H^*(BQ_8;\mathbb{F}_2) = \mathbb{F}_2[x_1,x_2,p_1]/(x_1^2 + x_1x_2 + x_2^2, x_1x_2(x_1+x_2))$.

3. *$ku^*(BC_8) \cong \mathbb{Z}[v][[y]]/(y\rho = (1-(1-vy)^3)/v)$, where $y = e_{ku}(\alpha) \in ku^2(BC_8)$. The natural map $ku^*(BC_8) \longrightarrow H^*(BC_8;\mathbb{F}_2)$ sends y to $y \in H^2(BC_8;\mathbb{F}_2)$, where $H^*(BC_8;\mathbb{F}_2) = \mathbb{F}_2[z,y]/(z^2)$ where $|z| = 1, |y| = 2$.*

The relation is given by;

Theorem 2.7.2. *With the same notations as in Theorem 2.6.1 and Proposition 2.7.1 above, we have a natural monomorphism;*

$$\begin{array}{ccccccccc} ku^*(BSD_{16}) & \longrightarrow & & ku^*(BD_8) & \oplus & ku^*(BQ_8) & \oplus & ku^*(BC_8) & \\ & & & & & & & & \\ a & \mapsto & (& 0 & , & b & , & 4y - 6vy^2 + 4v^2y^3 - v^3y^4 &) \\ b & \mapsto & (& a & , & b & , & 0 &) \\ d & \mapsto & (& d & , & q & , & 3y^2 - 3vy^3 + v^2y^4 &). \end{array}$$

Proof. Note first that, for $G = SD_{16}, D_8, Q_8$ and C_8, the natural map $ku^*(BG) \longrightarrow H^*(BG;\mathbb{F}_2) \oplus KU^*(BG) \cong R(G)^{\wedge}_J[v,v^{-1}]$ is a monomorphism. Thus it is enough to find the relations of them via cohomology ring theory and representation ring theory.

For D_8, with the notation of representations in [14] and in the proof of lemma 2.2.3, we have $\widehat{st} \leftrightarrow \varphi_4$, $\widehat{s} \leftrightarrow \varphi_3$, $\sigma_1 \leftrightarrow \sigma$ and $\widehat{s} + \widehat{t} \leftrightarrow \varphi_2 + \varphi_3$. Also, in cohomology ring $H^*(BD_8;\mathbb{F}_2)$ on both sources are related by $\overline{x} = x_2, \overline{y} = x_1, \overline{w}_2 = w$. The restriction of $R(SD_{16}) \longrightarrow R(D_8)$, recall from the proof of lemma 2.2.3, is given by $1 \mapsto 1, \chi_2 \mapsto \psi_4, \chi_3 \mapsto 1, \chi_4 \mapsto \psi_4, \sigma_1 \mapsto \sigma, \sigma_2 \mapsto \psi_2 + \psi_3$ and $\sigma_3 \mapsto \sigma$. This implies that, in periodic K-theory,

$$\begin{array}{rcl} KU^0(BSD_{16}) \ni va = 1-\chi_3 & \mapsto & 1-1=0 \in KU^0(BD_8) \\ KU^0(BSD_{16}) \ni vb = 1-\chi_2 & \mapsto & 1-\psi_4 = va \in KU^0(BD_8) \\ KU^0(BSD_{16}) \ni v^2d = 1-\sigma_1+\chi_4 & \mapsto & 1-\sigma+\psi_4 = v^2d \in KU^0(BD_8). \end{array}$$

Combining this fact with Proposition 2.2.1, we finish the proof for D_8.

For Q_8, we concentrate only on periodic K-theory since there is no $2, v$-torsion element in $ku^*(BQ_8)$. The relation of the notation of representations in [14] and in the proof of lemma 2.2.3 is $\psi_0 - 1 = \widehat{\rho} \leftrightarrow \rho_3, \chi = \widehat{j} \leftrightarrow \rho_4, \psi_1 \leftrightarrow \upsilon, \psi_2 = \widehat{j} + \widehat{\rho j} \leftrightarrow \rho_2 + \rho_4$. The restriction of $R(SD_{16}) \longrightarrow R(Q_8)$, recall from the proof of lemma 2.2.3, is given by $1 \mapsto 1, \chi_2 \mapsto \rho_4, \chi_3 \mapsto \rho_4, \chi_4 \mapsto 1, \sigma_1 \mapsto \upsilon, \sigma_2 \mapsto \rho_2 + \rho_3$ and $\sigma_3 \mapsto \upsilon$. This implies that, in periodic K-theory,

$$\begin{array}{rcl} KU^0(BSD_{16}) \ni va = 1-\chi_3 & \mapsto & 1-\rho_4 = vb \in KU^0(BQ_8) \\ KU^0(BSD_{16}) \ni vb = 1-\chi_2 & \mapsto & 1-\rho_4 = vb \in KU^0(BQ_8) \\ KU^0(BSD_{16}) \ni v^2d = 1-\sigma_1+\chi_4 & \mapsto & 2-\psi_1 = 2-\upsilon = v^2q \in KU^0(BQ_8), \end{array}$$

which completes the proof for Q_8.

Finally, for $C_8 =< s|s^8 = 1 >$, we need to check the relation only on periodic K-theory. Note that $R(C_8) = \mathbb{Z}[\alpha]/(\alpha^8 - 1)$, where $\alpha^k(s) = c^k$, $c = \frac{\sqrt{2}}{2}(1+i)$. It is not hard to see that the restriction of $R(SD_{16}) \longrightarrow R(C_8)$ is given by $1 \mapsto 1, \chi_2 \mapsto 1, \chi_3 \mapsto \alpha^4, \chi_4 \mapsto \alpha^4, \sigma_1 \mapsto \alpha + \alpha^3, \sigma_2 \mapsto \alpha^2 + \alpha^6$ and $\sigma_3 \mapsto \alpha^5 + \alpha^7$. This implies that

$$\begin{array}{rcl} va = 1-\chi_3 & \mapsto & 1-\alpha^4 = 1-(1-vy)^4 \in KU^0(BQ_8) \\ vb = 1-\chi_2 & \mapsto & 1-1=0 \in KU^0(BQ_8) \\ v^2d = 1-\sigma_1+\chi_4 & \mapsto & 1-(\alpha+\alpha^3)+\alpha^4 \\ & = & 1-((1-vy)+(1-vy)^3)+(1-vy)^4 \in KU^0(BQ_8), \end{array}$$

which completes the proof. □

Moreover, note that at E_∞-page of Adams spectral sequence for $ku^*(BSD_{16})$ and $ku^*(BSD_{2^n})$ are nearly the same. That is v acts monomorphically above the zero line at E_2-page and by similar argument as in the proof of lemma 2.4.4, v acts monomorphically above the zero line at E_∞-page for $ku^*(BSD_{2^n})$. This means that the natural map

$$ku^*(BSD_{2^n}) \rightarrowtail H^*(BSD_{2^n};\mathbb{F}_2) \oplus KU^*(BSD_{2^n}) \tag{2.43}$$

is a monomorphism. So, by using the same idea as above, we have;

Theorem 2.7.3. *In general for $n \geq 5$, we have a natural monomorphism*

$$ku^*(BSD_{2^n}) \rightarrowtail ku^*(BD_{2^{n-1}}) \oplus ku^*(BQ_{2^{n-1}}) \oplus ku^*(BC_{2^{n-1}}).$$

Proof. It is not hard to see that $SD_{2^n} = D_{2^{n-1}} \cup Q_{2^{n-1}} \cup C_{2^{n-1}}$ via the inclusion map

$$\begin{array}{llll} \Phi : D_{2^{n-1}} \longrightarrow SD_{2^n}, & \text{which sends} & s \mapsto s^2, & t \mapsto t, \\ \Psi : Q_{2^{n-1}} \longrightarrow SD_{2^n}, & \text{which sends} & \sigma \mapsto s^2, & t \mapsto st, \\ \Upsilon : C_{2^{n-1}} \longrightarrow SD_{2^n}, & \text{which sends} & s \mapsto s, & \end{array}$$

where

$$D_{2^{n-1}} = Gp < s, t \mid s^{2^{n-2}} = 1 = t^2; tst = s^{-1} >, \text{ for } n \geq 5,$$

$$Q_{2^{n-1}} = Gp < \sigma, \tau \mid \sigma^{2^{n-2}} = 1 = t^2; tst = s^{-1} >, \text{ for } n \geq 5,$$

$$C_{2^{n-1}} = Gp < s \mid s^{2^{n-2}} = 1 >, \text{ for } n \geq 5.$$

This implies that the natural map

$$R(SD_{2^n}) \rightarrowtail R(D_{2^{n-1}}) \oplus R(Q_{2^{n-1}}) \oplus R(C_{2^{n-1}}) \tag{2.44}$$

is a monomorphism. Precisely, if $\rho \in R(SD_{2^n})$ is sent to $(0, 0, 0)$, then $\rho([x]) = 0$ for every conjugacy classes of $D_{2^{n-1}}, Q_{2^{n-1}}, C_{2^{n-1}}$ which are the conjugacy classes of SD_{2^n} and thus $\rho = 0$.

On the other hand, we have (see Proposition 2.2.1 or Jon F. Carlson's homepage, [15]) the restriction map

$$H^*(BSD_{2^n}; \mathbb{F}_2) \rightarrowtail H^*(BD_{2^{n-1}}; \mathbb{F}_2) \oplus H^*(BQ_{2^{n-1}}; \mathbb{F}_2) \oplus H^*(BC_{2^{n-1}}; \mathbb{F}_2) \tag{2.45}$$

is a monomorphism. Hence, combining all informations we have so far with the fact that

$$ku^*(BG) \rightarrowtail H^*(BG; \mathbb{F}_2) \oplus KU^*(BG) \cong R(G)^{\wedge}_J[v, v^{-1}]$$

is a monomorphism for each $G = SD_{2^n}, D_{2^{n-1}}, Q_{2^{n-1}}, C_{2^{n-1}}$, we complete the proof. □

We investigate complex connective k-homology theory of SD_{16}, $ku_*(BSD_{16})$, as a module over $ku^*(BSD_{16})$ in the next chapter.

Chapter 3

Complex connective K-homology for SD_{16}

In this chapter, we will calculate $ku_*(BSD_{16})$ as a module over $ku^*(BSD_{16})$. In order to do this, we will use Greenlees spectral sequence applied to $ku^*(BSD_{16})$. That is, [14];

$$E^2_{s,t} = H_I^{-s}(ku^*(BSD_{16}))_t \Longrightarrow ku_{s+t}(BSD_{16})$$

where differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ and $I = \ker(ku^*(BSD_{16}) \to ku^*) = (a, b, d)$, ideal generated by the Euler classes a, b and d with codegree $2, 2$ and 4 respectively, (Theorem 2.6.1).

Here, the main task is the calculation of local cohomology for $R = ku^*(BSD_{16})$, $H_I^*(R)$. Before of that, let us recollect local cohomology of Greenlees spectral sequence for $ku_*(BSD_{16})$ first.

§ 3.1 Local cohomology and strategy

The definition of local cohomology which is suitable to our calculation is defined via stable Koszul complex.

Definition 3.1.1. *For a commutative ring (with unity) R and its ideal $I = (x_1, x_2, ..., x_n)$, the* stable Koszul complex *of R at I is*

$$K^\infty(x_1, x_2, ..., x_n; R) = K^\infty(x_1; R) \otimes_R K^\infty(x_2; R) \otimes_R ... \otimes_R K^\infty(x_n; R)$$

the tensor of cochain complex, where $K^\infty(x_i; R)$ is the cochain complex $(R \longrightarrow R[\frac{1}{x_i}])$, $(r \longmapsto \frac{r}{1})$, for each $i \in \{1, 2, ..., n\}$. For a module M over the ring R, local cohomology *of M* at *I is*

$$H_I^*(R; M) := H^*(K^\infty(x_1, x_2, ..., x_n; R) \otimes_R M)$$

where $H^*(C)$ is the homology of a chain complex C. In particular, we define

$$H_I^*(R) := H_I^*(R; R).$$

It is clear from the definition that $H_I^i(R; M) = 0$ *for* $i > n$.

Remark 3.1.2. *Let* R *be a ring and* (x) *be an ideal of* R. *The chain complex* $K^\infty(x) = (R \longrightarrow R[\frac{1}{x}])$ *give a natural map* $\varepsilon : K^\infty(x) \longrightarrow R$. *Precisely, there is a commutative diagram;*

$$\begin{tikzcd}
K^\infty(x) \arrow[d, "\varepsilon"] & = & (R \arrow[r] \arrow[d] & R[\frac{1}{x}]) \arrow[d] \\
R & = & (R \arrow[r] & 0).
\end{tikzcd}$$

Hence, for any ideal $I = (x_1, x_2, ..., x_m)$ *and* $J = (y_1, y_2, ..., y_n)$ *of* R, *there exists a map of chain complexes*

$$1 \otimes \varepsilon^n : K^\infty(I + J) = K^\infty(I) \otimes_R K^\infty(J) \longrightarrow K^\infty(I) = K^\infty(I) \otimes_R R.$$

After applying $\otimes_R M$, M *is a module over* R, *and taking homology, we obtain the map*

$$\eta : H^s_{I+J}(R; M) \longrightarrow H^s_I(R; M).$$

Example 3.1.3. *For* $R = \mathbb{Z}$ *and* $I = (2)$, *we have* $K^\infty(2; \mathbb{Z}) = (\mathbb{Z} \longrightarrow \mathbb{Z}[\frac{1}{2}])$. *The map in this cochain complex is clearly monomorphism and also the cokernel is easy to calculate. That is*

$$H^i_{(2)}(\mathbb{Z}) = \begin{cases} \mathbb{Z}/2^\infty, & \textit{if i=1 ;} \\ 0, & \textit{otherwise,} \end{cases}$$

where $\mathbb{Z}/2^\infty$ *is the set of rational numbers which are not integers whose denominators are a power of 2.*

Example 3.1.4. *For* $R = k[x]$, *polynomial ring over field* k *with indeterminate* x *of degree* r *and* $I = (x)$, *we have* $K^\infty(x; k[x]) = (k[x] \longrightarrow k[x][\frac{1}{x}])$. *The calculation is easier if we look at the picture below.*

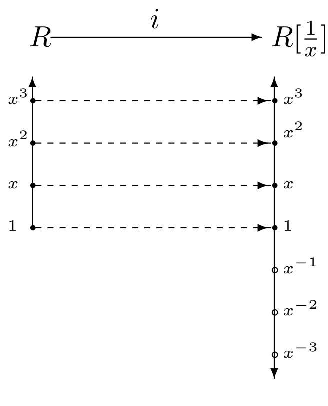


Figure 3.1: *Koszul complex of* $k[x]$ *at* (x).

This means the kernel of i *is zero and the cokenel of* i *is* $k[x, x^{-1}]/k[x]$ *which is* $\Sigma_{-r}(k[x]^\vee)$, *dual vector space of* $k[x]$ *shifted degree down by* r, *where* $k[x]^\vee := \mathrm{Hom}_k(k[x], k)$. *It follows that*

$$H^i_{(x)}(k[x]) = \begin{cases} \Sigma_{-r}(k[x]^\vee) = k[x,x^{-1}]/k[x], & \textit{if i=1 ;} \\ 0, & \textit{otherwise.} \end{cases}$$

Example 3.1.5. *For $R = k[x,y]$, polynomial ring over a field k with indeterminate x, y of degree r, s and $I = (x,y)$, we have*

$$\begin{aligned} K^\infty(I;R) &= K^\infty(x;R) \otimes_R K^\infty(y;R) \\ &= (R \longrightarrow R[\frac{1}{x}] \oplus R[\frac{1}{y}] \longrightarrow R[\frac{1}{xy}]) \end{aligned}$$

As the previous example, we illustrate the picture of Koszul complex for this ring as below.

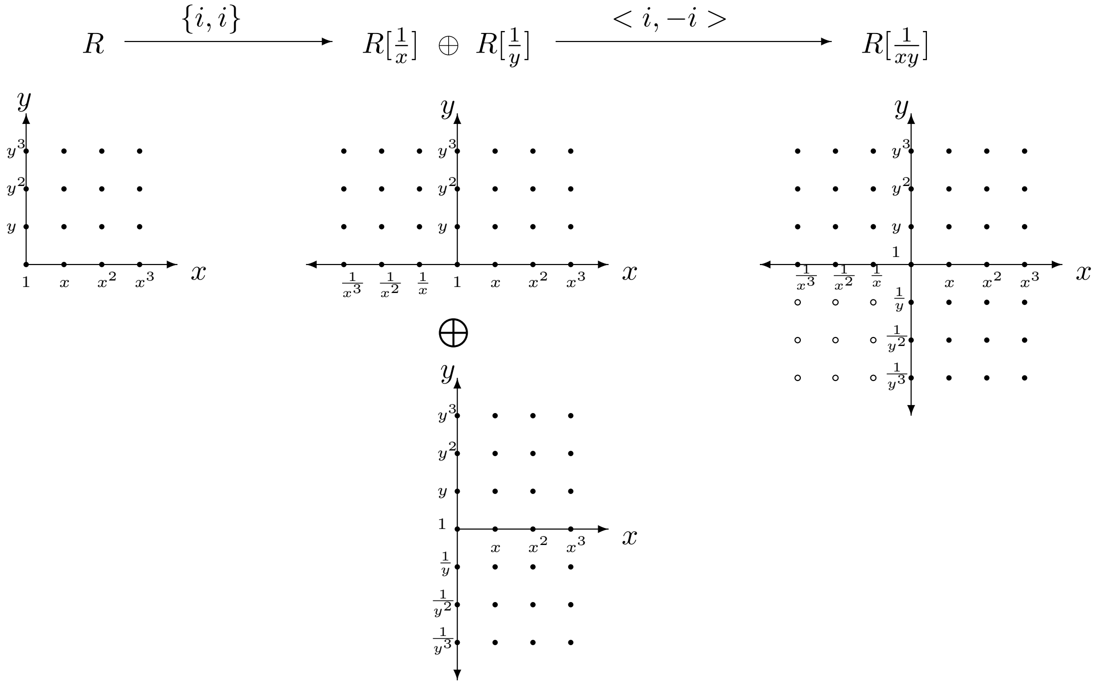


Figure 3.2: *Koszul complex of $k[x,y]$ at (x,y).*

From this figure, it is easy to see that this cochain complex is exact at the first and second term. Thus, $H^0_I(R)$ and $H^1_I(R)$ are zero. For the third term, the cokernel of $< i, -i >$ map is all the circle point in the third quadrant, which is isomorphic to $\Sigma_{-(r+s)}(R^\vee)$. Hence,

$$H^i_I(R) = \begin{cases} 0, & \textit{if i=0;} \\ 0, & \textit{if i=1;} \\ \Sigma_{-(r+s)}(R^\vee), & \textit{if i=2;} \\ 0, & \textit{otherwise.} \end{cases}$$

For a module M over a commutative ring R (with unity) with ideal I, we have other definitions of local cohomology which is defined by using functor $\Gamma_I(-)$. Here,

$$\Gamma_I(M) := \{x \in M \ : \ I^n x = 0 \text{ for some } n \in \mathbb{N} \ \}.$$

And $H_I^i()$ is defined to be the i^{th} right derived functor of Γ_I, i.e, taking injective resolution of M, applying Γ_I and taking cohomology. It is simple to show that Γ_I is left exact functor and thus

$$H_I^0(M) = \Gamma_I(M).$$

One can show that this definition and the previous definition agree for a module over Noetherian ring (see, for example, [21], page 7).

Remark 3.1.6. *For R module M, $H_I^*(M) = H_I^*(R; M)$ and $H_I^*(R) = H_I^*(R; R)$.*

Since our work involve to Noetherian ring, $ku^*(BG)$ is Noetherian ring for any finite group G, we recollect some properties relating to our calculation of local cohomology for a module over such a ring as following.

Proposition 3.1.7. *Let R be a commutaring Noetherian ring (with unity), $I \lhd R$ and M a module over R.*

1. *If L and N are R module such that $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence, then we have an induced long exact sequence*

$$0 \longrightarrow H_I^0(L) \longrightarrow H_I^0(M) \longrightarrow H_I^0(N) \longrightarrow H_I^1(L) \longrightarrow H_I^1(M) \longrightarrow H_I^1(N) \longrightarrow ...$$

2. *For J is an ideal of R, if $\sqrt{J} = \sqrt{I}$ then $H_I^i(M) = H_J^i(M)$ for all i.*
3. *For a Noetherian ring S, $\varphi : R \longrightarrow S$ a ring homomorphism and N an S module, $H_I^i(N) \cong H_{IS}^i(N)$ for each i as S module.*
4. *Let Λ be a directed set and $\{M_\lambda\}_{\lambda\in\Lambda}$ a direct system of R module. Then $\lim\limits_{\rightarrow\lambda} H_I^i(M_\lambda) \cong H_I^i(\lim\limits_{\rightarrow\lambda} M_\lambda)$.*
5. *If S is flat over R, then $H_I^i(M) \otimes_R S = H_{IS}^i(M \otimes_R S)$.*
6. *If $(R, \boldsymbol{m})$ is local, then $H_{\boldsymbol{m}}^i(M) \cong H_{\boldsymbol{m}\widehat{R}}^i(\widehat{R} \otimes_R M)$ which is isomorphic to $H_{\boldsymbol{m}\widehat{R}}^i(\widehat{M})$ if M is finitely generated.*

Proof. See, for example, [21] or [26]. □

The strategy we will use is decomposing the input of Greenlees spectral sequence, $ku^*(BSD_{16})$, as a short exact sequence

$$0 \longrightarrow TU \xrightarrow{i} ku^*(BSD_{16}) \xrightarrow{\varrho} QU \longrightarrow 0 \tag{3.1}$$

where TU is $2, v$-torsion of $ku^*(BSD_{16})$ and QU is the image of $ku^*(BSD_{16})$ in $KU^*(BSD_{16})$. This short exact sequence induces long exact sequence

$$0 \longrightarrow H_I^0(TU) \longrightarrow H_I^0(ku^*(BSD_{16})) \longrightarrow H_I^0(QU) \xrightarrow{\delta}$$

$$H_I^1(TU) \longrightarrow H_I^1(ku^*(BSD_{16})) \longrightarrow H_I^1(QU) \xrightarrow{\delta}$$

$$H_I^2(TU) \longrightarrow H_I^2(ku^*(BSD_{16})) \longrightarrow H_I^2(QU) \xrightarrow{\delta}$$

$$H_I^3(TU) \longrightarrow H_I^3(ku^*(BSD_{16})) \longrightarrow H_I^3(QU) \xrightarrow{\delta} \cdots$$

From here, instead of doing calculation of $H_I^*(ku^*(BSD_{16}))$ directly, we prefer to do calculation on $H_I^*(TU)$ and $H_I^*(QU)$. After determining differential, we obtain E_2-page, i.e., $H_I^*(ku^*(BSD_{16}))$ as original requirement.

§ 3.2 Local cohomology for v-torsion part of $ku^*(BSD_{16})$

Let $R = ku^*(BSD_{16})$. Recall from Theorem 2.5.5 that for each $k > 1$, the v- torsion of

1 $ku^{4k+2}(BSD_{16})$ is $\mathbb{F}_2^{k-1}$ generated by $\{(b^2d-abd)bd^{k-2}, (b^2d-abd)b^2d^{k-3}, ..., (b^2d-abd)b^{2k-3}\}$ and

2 $ku^{4k}(BSD_{16})$ is $\mathbb{F}_2^{k-1}$ generated by $\{(b^2d-abd)d^{k-2}, (b^2d-abd)b^2d^{k-3}, ..., (b^2d-abd)b^{2k-4}\}$.

In other words, setting $\tau = b^2d - abd$,

$$TU_{4k+2} = \{\tau bd^{k-2}, \tau b^2d^{k-3}, ..., \tau b^{2k-3}\}$$

$$TU_{4k} = \{\tau d^{k-2}, \tau b^2d^{k-3}, ..., \tau b^{2k-4}d\}.$$

The monomorphism

$$ku^*(BSD_{16}) \longrightarrow H^*(BSD_{16}; \mathbb{F}_2) \bigoplus KU^*(BSD_{16})$$

sending $a \longmapsto (x^2, A)$, $b \longmapsto (y^2, B)$ and $d \longmapsto (P, D)$ implies that $a\tau = 0$ and b, d act freely on TU. Hence, we have;

Lemma 3.2.1. *v-torsion part of $ku^*(BSD_{16})$, TU, is a free module over $PC := \mathbb{F}_2[b, d]$ generated by τ, where (codegree) $|\tau| = 8$ and $a \cdot TU = 0$.*

Note that TU can be identified to be a subring of $H^*(BSD_{16}; \mathbb{F}_2)$, i.e. $a := x^2, b := y^2$ and $d := P$. In fact, it is a $Ch^*(BSD_{16}; \mathbb{F}_2)$-module, where $Ch^*(BG; \mathbb{F}_2)$ is the Chern subring of $H^*(BG; \mathbb{F}_2)$. Moreover, we have a commutative diagram

$$\begin{tikzcd} ku^*(BSD_{16}) \arrow[d, "\eta"] \arrow[dr, "\varphi"] & \\ H^*(BSD_{16};\mathbb{Z}) \arrow[r, "\rho_2"] & H^*(BSD_{16};\mathbb{F}_2), \end{tikzcd}$$

where η is a natural map, ρ_2 is modulo by 2 map and $\varphi = \rho_2 \circ \eta$ and thus TU is an R module via φ. Then

$$\begin{aligned} H_I^*(TU) &= H^*_{(a,b,d)}(R;TU) \\ &= H^*_{\varphi(a,b,d)}(Ch^*(BSD_{16};\mathbb{F}_2);TU) \\ &= H^*_{(x^2,y^2,P)}(Ch^*(BSD_{16};\mathbb{F}_2);TU) \\ &= H^*_{(y^2,P)}(Ch^*(BSD_{16};\mathbb{F}_2);TU), [\sqrt{(y^2,P)} = (x^2,y^2,P), \because (x^2)^2 = 0] \\ &= H^*_{(b,d)}(PC,PC)\cdot\tau, [\because TU = PC\cdot\tau] \\ &= H^*_{I'}(PC)\cdot\tau, \end{aligned}$$

where $I' := (b,d)$.

By example 3.1.5, we get immediately that

Lemma 3.2.2. *Local cohomology of TU at $I = (a,b,d)$ is*

$$H^i_{I'}(PC)\cdot\tau = \Sigma_{-8}(H^i_{I'}(PC)) = \begin{cases} 0, & \text{if } i = 0; \\ 0, & \text{if } i = 1; \\ \Sigma_{-2}((\mathbb{F}_2[b,d])^\vee), & \text{if } i = 2; \\ 0, & \text{if } i \geq 3. \end{cases}$$

§ 3.3 Local cohomology for the image of $ku^*(BSD_{16})$ in $KU^*(BSD_{16})$

For $H_I^*(QU)$, we consider the additive basis of QU by recalling from Theorem 2.5.5 which asserts that, for $k \geq 0$,

$$\begin{aligned} (QU)_{2k} &= \mathbb{Z}[v] \oplus \mathbb{Z}_2^\wedge < \varrho(v^k\{vb, va, v^3(a+b)d, v^2d, v^4d^2, v^6d^3\}) >, \\ &= \mathbb{Z}[v] \oplus \mathbb{Z}_2^\wedge < B, A, (A+B)D, D, D^2, D^3 > := \overline{R}_0, \\ (QU)_{-2} &= \mathbb{Z}_2^\wedge < \varrho(\{b, a, v^2(a+b)d, vd, v^3d^2, v^5d^3\}) >, \\ &= \mathbb{Z}_2^\wedge < B, A, (A+B)D, D, D^2, D^3 >, \\ (QU)_{-4} &= \mathbb{Z}_2^\wedge < \varrho(\{b^2, vad, v(a+b)d, d, v^2d^2, v^4d^3\}) >, \\ &= \mathbb{Z}_2^\wedge < B^2, AD, (A+B)D, D, D^2, D^3 >, \\ (QU)_{-6} &= \mathbb{Z}_2^\wedge < \varrho(\{b^3, ad, (a+b)d, vd^2, v^3d^3, v^5d^4\}) >, \\ &= \mathbb{Z}_2^\wedge < B^3, AD, (A+B)D, D^2, D^3, D^4 >, \\ (QU)_{-(4k)} &= \mathbb{Z}_2^\wedge < \varrho(\{b^{2k}, vad^k, v(a+b)d^k, d^k, v^2d^{k+1}, v^4d^{k+2}\}) >, \\ &= \mathbb{Z}_2^\wedge < B^{2k}, AD^k, (A+B)D^k, D^k, D^{k+1}, D^{k+2} >, \\ (QU)_{-(4k+2)} &= \mathbb{Z}_2^\wedge < \varrho(\{b^{2k+1}, ad^k, (a+b)d^k, vd^{k+1}, v^3d^{k+2}, v^5d^{k+3}\}) >, \\ &= \mathbb{Z}_2^\wedge < B^{2k+1}, AD^k, (A+B)D^k, D^{k+1}, D^{k+2}, D^{k+3} > . \end{aligned}$$

Here, QU is a module over R via ϱ. That is

$$\begin{aligned} H_I^*(QU) &= H^*_{(a,b,d)}(R;QU) \\ &= H^*_{\varrho(a,b,d)}(QU;QU) \\ &= H^*_{I''}(QU) \end{aligned}$$

where $I'' := \varrho(a,b,d)$, the image of ϱ in $KU^*(BSD_{16})$. Since QU_i for all $i \leq -1$ are generated by Chern classes of at most 2 dimension complex representation, i.e. $\chi_2, \chi_3, \chi_4, \sigma_1, \sigma_2, \sigma_3$, we get that;

Lemma 3.3.1. *For each $i \in \mathbb{Z}$,*

$$(QU)_i = (QU)^{-i} \cong \begin{cases} 0, & \text{if } i \text{ is odd.} \\ \overline{R}_0, & \text{if } i \geq 0 \text{ and } i \text{ is even;} \\ \widehat{JU}_1, & \text{if } i = -2; \\ \widehat{JU}_2, & \text{if } i = -4; \\ \widehat{JU}_k, & \text{if } i = -2k, \end{cases}$$

where JU_1 is the augmentation ideal of QU generated by first Chern classes, JU_2 is generated by JU_1^2 and second Chern classes, and $JU_k = JU_1 JU_{k-1} + JU_2 JU_{k-2}$ for $k \geq 3$ and $\widehat{JU}_r = (JU_r)^\wedge_J$ for all r.

3.3.1 QU and $H_I^0(R)$

By lemma 3.3.1, we, instead of computing $H^*_{I''}(QU)$ directly, prefer to study from $\widehat{JU}_i$ (Modified Rees ring), which is more tidy than QU_{-i}. However, we keep the trace of such isomorphism for $i = 1, ..., 5$ which $\widehat{JU}_2, ..., \widehat{JU}_5$ will play an important role for our calculation. By definition and direct calculation, we see that

$$\begin{aligned} \widehat{JU}_1 &= \mathbb{Z}_2^\wedge < c_1^R(\chi_2), c_1^R(\chi_3), c_1^R(\chi_4), c_1^R(\sigma_1), c_1^R(\sigma_2), c_1^R(\sigma_3) > \\ &= \mathbb{Z}_2^\wedge < 1-\chi_2, 1-\chi_3, 1-\chi_4, 2-\sigma_1, 2-\sigma_2, 2-\sigma_3 > \\ &\cong \mathbb{Z}_2^\wedge < \overline{x}_1, \overline{y}_1, \overline{z}_1, \overline{t}_1, \overline{u}_1, \overline{w}_1 > \end{aligned}$$

where

$$\begin{aligned} \overline{x}_1 &= c_1^R(\chi_2) = B \\ \overline{y}_1 &= c_1^R(\chi_3) - c_1^R(\chi_4) + c_1^R(\chi_2) = \frac{2}{3}A + \frac{1}{3}(A+B)D - \frac{8}{9}D + \frac{2}{3}D^2 - \frac{1}{9}D^3 \\ \overline{z}_1 &= c_1^R(\chi_4) = B + \frac{1}{3}A - \frac{1}{3}(A+B)D + \frac{8}{9}D - \frac{2}{3}D^2 + \frac{1}{9}D^3 \\ \overline{t}_1 &= c_1^R(\sigma_2) - c_1^R(\chi_4) = \frac{5}{3}A - \frac{2}{3}(A+B)D + \frac{4}{9}D + \frac{5}{3}D^2 - \frac{4}{9}D^3 \\ \overline{u}_1 &= c_1^R(\sigma_1) + c_1^R(\sigma_3) - c_1^R(\sigma_2) - c_1^R(\chi_4) = -\frac{5}{3}A - \frac{1}{3}(A+B)D + \frac{14}{9}D - \frac{5}{3}D^2 + \frac{4}{9}D^3 \\ \overline{w}_1 &= c_1^R(\sigma_3) - c_1^R(\sigma_2) = -A - \frac{1}{3}D - D^2 + \frac{1}{3}D^2. \end{aligned}$$

On the other hand, we have $A = \overline{z}_1 - \overline{x}_1 + \overline{y}_1$, $B = \overline{x}_1$, $(A+B)D = -2\overline{w}_1 - \overline{u}_1 - \overline{t}_1 + 2\overline{y}_1$, $D = \overline{u}_1 - \overline{w}_1$, $D^2 = 2\overline{u}_1 - \overline{z}_1 + \overline{t}_1 + \overline{x}_1 - \overline{y}_1$ and $D^3 = 2\overline{w}_1 + 7\overline{u}_1 + 3\overline{t}_1 - 3\overline{y}_1$. Also, the images of $\overline{x}_1, \overline{y}_1, \overline{z}_1, \overline{t}_1, \overline{u}_1, \overline{w}_1$ in $H^*(BSD_{16};\mathbb{F}_2)$ are $y^2, 0, x^2+y^2, x^2, x^2, x^2$ respectively. Again we represent all generators of $(QU)_{-2} \cong \widehat{JU}_1$ as the image of character omitting the image of identity, because all such generators have image zero at the identity.

$$(QU)_{-2} \cong \widehat{JU}_1 = \mathbb{Z}_2^{\wedge}\left\langle \begin{array}{rl} \overline{x}_1 : & [\begin{array}{cccccc} 0 & 0 & 0 & 0 & 2 & 2 \end{array}] \\ \overline{y}_1 : & [\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 4 \end{array}] \\ \overline{z}_1 : & [\begin{array}{cccccc} 0 & 2 & 0 & 2 & 2 & 0 \end{array}] \\ \overline{t}_1 : & [\begin{array}{cccccc} 0 & 0 & 4 & 0 & 0 & 2 \end{array}] \\ \overline{u}_1 : & [\begin{array}{cccccc} 8 & 0 & 0 & 0 & 0 & 2 \end{array}] \\ \overline{w}_1 : & [\begin{array}{cccccc} 4 & c & -2 & -c & 0 & 0 \end{array}] \end{array} \right\rangle, \ c = \sqrt{2}i.$$

Next, $\widehat{JU}_2$. This is, by definition $JU_2 := JU_1^2 + c_2^R(\sigma)$, an ideal consisting of all $\alpha\beta$ for each $\alpha, \beta \in JU_1$, together with $c_2^R(\sigma_1)$, $c_2^R(\sigma_2)$ and $c_2^R(\sigma_3)$. Again by direct calculation, we get

$$\widehat{JU}_2 = \mathbb{Z}_2^{\wedge}\left\langle \begin{array}{rl} \overline{x}_2 : & [\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 4 \end{array}] \\ \overline{y}_2 : & [\begin{array}{cccccc} 0 & 0 & 0 & 0 & 4 & 0 \end{array}] \\ \overline{z}_2 : & [\begin{array}{cccccc} 0 & 0 & 8 & 0 & 0 & 0 \end{array}] \\ \overline{t}_2 : & [\begin{array}{cccccc} 0 & 2 & 4 & 2 & 0 & 0 \end{array}] \\ \overline{u}_2 : & [\begin{array}{cccccc} 0 & 2c & 0 & -2c & 0 & 0 \end{array}] \\ \overline{w}_2 : & [\begin{array}{cccccc} 4 & c & 2 & -c & 0 & 2 \end{array}] \end{array} \right\rangle,$$

where $\overline{x}_2 = \overline{x}_1\overline{t}_1 = \overline{x}_1\overline{u}_1 = \overline{t}_1\overline{u}_1$, $\overline{y}_2 = \overline{x}_1\overline{z}_1$, $\overline{z}_2 = -\overline{t}_1\overline{w}_1$, $\overline{t}_2 = c_2^R(\sigma_2)$, $\overline{u}_2 = c_2^R(\sigma_3) - c_2^R(\sigma_1)$, and $\overline{w}_2 = c_2^R(\sigma_3)$. It is not hard to see that

$$\begin{array}{rcl} \overline{x}_2 & = & (A+B)D - AD \\ \overline{y}_2 & = & B^2 - (A+B)D + AD \\ \overline{z}_2 & = & (A+B)D - 4AD + \frac{16}{3}D - \frac{1}{3}D^3 \\ \overline{t}_2 & = & (A+B)D - 3AD + 4D - D^2 \\ \overline{u}_2 & = & (A+B)D - 2AD \\ \overline{w}_2 & = & D + (A+B)D - 2AD \end{array}, \quad \begin{array}{rcl} B^2 & = & \overline{x}_2 + \overline{y}_2 \\ AD & = & \overline{x}_2 - \overline{u}_2 \\ (A+B)D & = & 2\overline{x}_2 - \overline{u}_2 \\ D & = & \overline{u}_2 - \overline{w}_2 \\ D^2 & = & 6\overline{u}_2 + 4\overline{w}_2 - \overline{x}_2 - \overline{t}_2 \\ D^3 & = & 25\overline{u}_2 + 16\overline{w}_2 - 6\overline{x}_2 - 3\overline{z}_2, \end{array}$$

and the images of $\overline{x}_2, \overline{y}_2, \overline{z}_2, \overline{t}_2, \overline{u}_2, \overline{w}_2$ in $H^*(BSD_{16};\mathbb{F}_2)$ are $0, y^4, 0, 0, 0, P$ respectively.

Next, $\widehat{JU}_3$. This is, by definition $JU_3 := JU_1 JU_2 + JU_2 JU_1 = JU_1 JU_2$, an ideal consisting of all $\alpha\beta$ for each $\alpha \in JU_1$ and $\beta \in JU_2$. Again by direct calculation, we get

$$\widehat{JU}_3 = \mathbb{Z}_2^{\wedge}\left\langle \begin{array}{rl} \overline{x}_3 : & [\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 4 \end{array}] \\ \overline{y}_3 : & [\begin{array}{cccccc} 0 & 0 & 0 & 0 & 8 & 0 \end{array}] \\ \overline{z}_3 : & [\begin{array}{cccccc} 0 & 0 & 8 & 0 & 0 & 0 \end{array}] \\ \overline{t}_3 : & [\begin{array}{cccccc} 0 & 4 & 0 & 4 & 0 & 0 \end{array}] \\ \overline{u}_3 : & [\begin{array}{cccccc} 16 & -2 & -4 & -2 & 0 & 0 \end{array}] \\ \overline{w}_3 : & [\begin{array}{cccccc} 0 & 2c & 0 & -2c & 0 & 0 \end{array}] \end{array} \right\rangle,$$

where $\overline{x}_3 = \overline{x}_1\overline{w}_2$, $\overline{y}_3 = \overline{x}_1\overline{y}_2 = \overline{z}_1\overline{y}_2$, $\overline{z}_3 = \overline{t}_1\overline{w}_2 - \overline{x}_3$, $\overline{t}_3 = -\overline{w}_1\overline{u}_2 = \overline{z}_1\overline{t}_2$, $\overline{u}_3 = \overline{w}_1\overline{w}_2$, and $\overline{w}_3 = \overline{z}_1\overline{w}_2$. It is not hard to see that

$$\begin{array}{rcl}
\overline{x}_3 & = & (A+B)D - AD \\
\overline{y}_3 & = & B^3 - 2(A+B)D + 2AD \\
\overline{z}_3 & = & (A+B)D - 4AD - \frac{4}{3}D^2 + 3D^3 - \frac{2}{3}D^4 \\
\overline{t}_3 & = & -2(A+B)D + 4AD - \frac{8}{3}D^2 + 2D^3 - \frac{1}{3}D^4 \\
\overline{u}_3 & = & D^2 - \overline{x}_3 - \overline{z}_3 \\
\overline{w}_3 & = & (A+B)D - 2AD.
\end{array}$$

On the other hand, $B^3 = \overline{y}_3 + 2\overline{x}_3$, $AD = \overline{x}_3 - \overline{w}_3$, $(A+B)D = 2\overline{x}_3 - \overline{w}_3$, $D^2 = \overline{u}_3 + \overline{x}_3$, $D^3 = \overline{w}_3 - \overline{z}_3 + 2\overline{t}_3 + 3\overline{x}_3 + 4\overline{u}_3$ and $D^4 = 16\overline{u}_3 + 10\overline{x}_3 - 6\overline{z}_3 + 9\overline{t}_3$ and the images of $\overline{x}_3, \overline{y}_3, \overline{z}_3, \overline{t}_3, \overline{u}_3, \overline{w}_3$ in $H^*(BSD_{16};\mathbb{F}_2)$ are $y^2P, y^6, (x^2+y^2)P, 0, x^2P, (x^2+y^2)P$ respectively.

Next, $\widehat{JU}_4$. By definition $JU_4 := JU_1JU_3 + JU_2JU_2 = JU_1^2JU_2 + JU_2^2$, since $JU_1^2 \subseteq JU_2$, $JU_1^2JU_2 \subseteq JU_2JU_2$ and hence $JU_4 = JU_2^2$. This is an ideal consisting of all $\alpha\beta$ for each $\alpha, \beta \in JU_2$. Again by direct calculation, we get

$$\widehat{JU}_4 = \mathbb{Z}_2^\wedge \left\langle \begin{array}{r} \overline{x}_4 : [\\ \overline{y}_4 : [\\ \overline{z}_4 : [\\ \overline{t}_4 : [\\ \overline{u}_4 : [\\ \overline{w}_4 : [\end{array} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 0 \\ 16 & -2 & 4 & -2 & 0 & 4 \\ 0 & 2c & 8 & -2c & 0 & 0 \end{array} \begin{array}{l}] \\ ] \\ ] \\ ] \\ ] \\ ] \end{array} \right\rangle,$$

where $\overline{x}_4 = \overline{x}_2\overline{w}_2$, $\overline{y}_4 = \overline{y}_2\overline{y}_2$, $\overline{z}_4 = \overline{z}_2\overline{w}_2$, $\overline{t}_4 = -\overline{u}_2\overline{w}_2$, $\overline{u}_4 = \overline{w}_2\overline{w}_2$, and $\overline{w}_4 = \overline{t}_2\overline{w}_2$. It is not hard to see that

$$\begin{array}{rcl}
\overline{x}_4 & = & (A+B)D^2 - AD^2 \\
\overline{y}_4 & = & B^4 - 2(A+B)D^2 + 2AD^2 \\
\overline{z}_4 & = & (A+B)D^2 - 4AD^2 + \frac{16}{3}D^2 - \frac{1}{3}D^4 \\
\overline{t}_4 & = & (A+B)D^2 - 2AD^2 \\
\overline{u}_4 & = & D^2 \\
\overline{w}_4 & = & \frac{4}{3}D^2 + D^3 - \frac{1}{3}D^4 - AD^2.
\end{array}$$

On the other hand, $B^4 = \overline{y}_4 + 2\overline{x}_4$, $AD^2 = \overline{x}_4 - \overline{t}_4$, $(A+B)D = 2\overline{x}_4 - \overline{t}_4$, $D^2 = \overline{u}_4$, $D^3 = 4\overline{u}_4 - \overline{z}_4 + \overline{w}_4 - \overline{x}_4 + 2\overline{t}_4$ and $D^4 = 16\overline{u}_4 - 6\overline{x}_4 - 3\overline{z}_4 + 9\overline{t}_4$ and the images of $\overline{x}_4, \overline{y}_4, \overline{z}_4, \overline{t}_4, \overline{u}_4, \overline{w}_4$ in $H^*(BSD_{16};\mathbb{F}_2)$ are $0, y^8, 0, 0, P^2, 0$ respectively.

Next, $\widehat{JU}_5$. By definition $JU_5 := JU_1JU_4 + JU_2JU_3 = JU_1JU_2^2 + JU_2^2JU_1 = JU_1JU_2^2 = JU_1JU_4$, this is an ideal consisting of all $\alpha\beta$ for each $\alpha \in JU_1$ and $\beta \in JU_4$. Again by direct calculation, we get

$$\widehat{JU}_5 = \mathbb{Z}_2^\wedge \left\langle \begin{array}{r} \overline{x}_5 : [\\ \overline{y}_5 : [\\ \overline{z}_5 : [\\ \overline{t}_5 : [\\ \overline{u}_5 : [\\ \overline{w}_5 : [\end{array} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 32 & 0 \\ 0 & 0 & 16 & 0 & 0 & 0 \\ 0 & -4 & 0 & -4 & 0 & 0 \\ 64 & -2c & -8 & 2c & 0 & 0 \\ 0 & 4c & 0 & -4c & 0 & 0 \end{array} \begin{array}{l}] \\ ] \\ ] \\ ] \\ ] \\ ] \end{array} \right\rangle,$$

where $\overline{x}_5 = \overline{x}_1\overline{u}_4$, $\overline{y}_5 = \overline{x}_1\overline{y}_4 = \overline{z}_1\overline{y}_4$, $\overline{z}_5 = \overline{t}_1\overline{u}_4 - \overline{x}_5$, $\overline{t}_5 = \overline{z}_1\overline{u}_4$, $\overline{u}_5 = \overline{w}_1\overline{u}_4$, and $\overline{w}_5 = \overline{z}_1\overline{w}_4$. It is not hard to see that

$$\begin{array}{rcl} \overline{x}_5 & = & (A+B)D^2 - AD^2 \\ \overline{y}_5 & = & B^5 - 4(A+B)D^2 + 4AD^2 \\ \overline{z}_5 & = & -5(A+B)D^2 + 8AD^2 - \frac{4}{3}D^3 + 3D^4 - \frac{2}{3}D^5 \\ \overline{t}_5 & = & -(A+B)D^2 + 2AD^2 \\ \overline{u}_5 & = & 2(A+B)D^2 - 3AD^2 - \frac{1}{3}D^3 - D^4 + \frac{1}{3}D^5 \\ \overline{w}_5 & = & 2(A+B)D^2 - 4AD^2 + \frac{8}{3}D^3 - 2D^4 + \frac{1}{3}D^5. \end{array}$$

On the other hand, $B^5 = \overline{y}_5 + 4\overline{x}_5$, $AD^2 = \overline{t}_5 + \overline{x}_5$, $(A+B)D = 2\overline{x}_5 + \overline{t}_5$, $D^3 = \overline{u}_5 + \overline{w}_5 + \overline{z}_5$, $D^4 = 4\overline{u}_5 + 3\overline{z}_5 + 2\overline{w}_5 - \overline{t}_5 - 2\overline{x}_5$ and $D^5 = 16\overline{u}_5 - 12\overline{x}_5 + 10\overline{z}_5 + 7\overline{w}_5$ and the images of $\overline{x}_5, \overline{y}_5, \overline{z}_5, \overline{t}_5, \overline{u}_5, \overline{w}_5$ in $H^*(BSD_{16};\mathbb{F}_2)$ are $y^2P^2, y^{10}, (x^2+y^2)P^2, (x^2+y^2)P^2, (x^2+y^2)P^2, 0$ respectively.

Similarly, we record $\widehat{JU}_6, \widehat{JU}_7, \widehat{JU}_8$ and $\widehat{JU}_9$ as;

$\widehat{JU}_6 = \widehat{JU}_2^3 = \widehat{JU}_2\widehat{JU}_4$ generated by

$$\begin{array}{r} \overline{x}_6 : [\\ \overline{y}_6 : [\\ \overline{z}_6 : [\\ \overline{t}_6 : [\\ \overline{u}_6 : [\\ \overline{w}_6 : [\end{array} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 64 & 0 \\ 0 & 0 & 32 & 0 & 0 & 0 \\ 0 & -4 & 16 & -4 & 0 & 0 \\ 0 & -4c & 0 & 4c & 0 & 0 \\ 64 & -2c & 8 & 2c & 0 & 8 \end{array} \begin{array}{l}] \\ ] \\ ] \\ ] \\ ] \\ ] \end{array},$$

$\widehat{JU}_7 = \widehat{JU}_1\widehat{JU}_6$ generated by

$$\begin{array}{r} \overline{x}_7 : [\\ \overline{y}_7 : [\\ \overline{z}_7 : [\\ \overline{t}_7 : [\\ \overline{u}_7 : [\\ \overline{w}_7 : [\end{array} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 0 & 128 & 0 \\ 0 & 0 & 32 & 0 & 0 & 0 \\ 0 & -8 & 0 & -8 & 0 & 0 \\ 256 & 4 & -16 & 4 & 0 & 0 \\ 0 & -4c & 0 & 4c & 0 & 0 \end{array} \begin{array}{l}] \\ ] \\ ] \\ ] \\ ] \\ ] \end{array}$$

$\widehat{JU}_8 = \widehat{JU}_2^4 = \widehat{JU}_2\widehat{JU}_6$ generated by

$$\begin{array}{r} \overline{x}_8 : [\\ \overline{y}_8 : [\\ \overline{z}_8 : [\\ \overline{t}_8 : [\\ \overline{u}_8 : [\\ \overline{w}_8 : [\end{array} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 32 \\ 0 & 0 & 0 & 0 & 256 & 0 \\ 0 & 0 & 64 & 0 & 0 & 0 \\ 0 & -8 & 0 & -8 & 0 & 0 \\ 256 & 4 & 16 & 4 & 0 & 16 \\ 0 & -4c & 32 & 4c & 0 & 0 \end{array} \begin{array}{l}] \\ ] \\ ] \\ ] \\ ] \\ ] \end{array},$$

$\widehat{JU}_9 = \widehat{JU}_1\widehat{JU}_8$ generated by

$$\begin{array}{r} \overline{x}_9 : [\\ \overline{y}_9 : [\\ \overline{z}_9 : [\\ \overline{t}_9 : [\\ \overline{u}_9 : [\\ \overline{w}_9 : [\end{array} \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 32 \\ 0 & 0 & 0 & 0 & 2^9 & 0 \\ 0 & 0 & 64 & 0 & 0 & 0 \\ 0 & 8 & 0 & 8 & 0 & 0 \\ 4^5 & 4c & -32 & -4c & 0 & 0 \\ 0 & -8c & 0 & 8c & 0 & 0 \end{array} \begin{array}{l}] \\ ] \\ ] \\ ] \\ ] \\ ] \end{array}$$

By induction, we see that $\widehat{JU}_{2k} = \widehat{JU}_2^k = \widehat{JU}_2\widehat{JU}_{2k-2}$ and $\widehat{JU}_{2k+1} = \widehat{JU}_1\widehat{JU}_{2k}$ and for $k \geq 2$ we have $\widehat{JU}_{4k+2}$, $\widehat{JU}_{4k+3}$, $\widehat{JU}_{4k+4}$ respectively as ;

$$\mathbb{Z}_2^\wedge\left\langle\begin{array}{r}\overline{x}_{4k+2}:\\ \overline{y}_{4k+2}:\\ \overline{z}_{4k+2}:\\ \overline{t}_{4k+2}:\\ \overline{u}_{4k+2}:\\ \overline{w}_{4k+2}:\end{array}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 4\cdot 4^k\\ 0 & 0 & 0 & 0 & 4\cdot 16^k & 0\\ 0 & 0 & 8\cdot 4^k & 0 & 0 & 0\\ 0 & 2(-2)^k & 4\cdot 4^k & 2(-2)^k & 0 & 0\\ 0 & 2c(-2)^k & 0 & -2c(-2)^k & 0 & 0\\ 4\cdot 16^k & c(-2)^k & 2\cdot 4^k & -c(-2)^k & 0 & 2\cdot 4^k\end{bmatrix}\right\rangle,$$

$$\mathbb{Z}_2^\wedge\left\langle\begin{array}{r}\overline{x}_{4k+3}:\\ \overline{y}_{4k+3}:\\ \overline{z}_{4k+3}:\\ \overline{t}_{4k+3}:\\ \overline{u}_{4k+3}:\\ \overline{w}_{4k+3}:\end{array}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 4\cdot 4^k\\ 0 & 0 & 0 & 0 & 8\cdot 16^k & 0\\ 0 & 0 & 8\cdot 4^k & 0 & 0 & 0\\ 0 & 4(-2)^k & 0 & 4(-2)^k & 0 & 0\\ 16\cdot 16^k & -2(-2)^k & -4\cdot 4^k & -2(-2)^k & 0 & 0\\ 0 & 2c(-2)^k & 0 & -2c(-2)^k & 0 & 0\end{bmatrix}\right\rangle,$$

$$\mathbb{Z}_2^\wedge\left\langle\begin{array}{r}\overline{x}_{4k+4}:\\ \overline{y}_{4k+4}:\\ \overline{z}_{4k+4}:\\ \overline{t}_{4k+4}:\\ \overline{u}_{4k+4}:\\ \overline{w}_{4k+4}:\end{array}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 8\cdot 4^k\\ 0 & 0 & 0 & 0 & 16\cdot 16^k & 0\\ 0 & 0 & 16\cdot 4^k & 0 & 0 & 0\\ 0 & 4(-2)^k & 0 & 4(-2)^k & 0 & 0\\ 16\cdot 16^k & -2(-2)^k & 4\cdot 4^k & -2(-2)^k & 0 & 4\cdot 4^k\\ 0 & 2c(-2)^k & 8\cdot 4^k & -2c(-2)^k & 0 & 0\end{bmatrix}\right\rangle,$$

and $\widehat{JU}_{4k+5}$ as

$$\mathbb{Z}_2^\wedge\left\langle\begin{array}{r}\overline{x}_{4k+5}:\\ \overline{y}_{4k+5}:\\ \overline{z}_{4k+5}:\\ \overline{t}_{4k+5}:\\ \overline{u}_{4k+5}:\\ \overline{w}_{4k+5}:\end{array}\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 8\cdot 4^k\\ 0 & 0 & 0 & 0 & 32\cdot 16^k & 0\\ 0 & 0 & 16\cdot 4^k & 0 & 0 & 0\\ 0 & -4(-2)^k & 0 & -4(-2)^k & 0 & 0\\ 64\cdot 16^k & -2c(-2)^k & -8\cdot 4^k & 2c(-2)^k & 0 & 0\\ 0 & 4c(-2)^k & 0 & -4c(-2)^k & 0 & 0\end{bmatrix}\right\rangle.$$

So far, we have seen the pattern of relations among $\widehat{JU}_i$'s as

$$\widehat{JU}_2 \xrightarrow[p]{\cong} \widehat{JU}_6 \xrightarrow[p]{\cong} \cdots \widehat{JU}_{4k+2} \xrightarrow[p]{\cong} \cdots$$

$$\widehat{JU}_3 \xrightarrow[p]{\cong} \widehat{JU}_7 \xrightarrow[p]{\cong} \cdots \widehat{JU}_{4k+3} \xrightarrow[p]{\cong} \cdots$$

$$\widehat{JU}_4 \xrightarrow[p]{\cong} \widehat{JU}_8 \xrightarrow[p]{\cong} \cdots \widehat{JU}_{4k+4} \xrightarrow[p]{\cong} \cdots$$

$$\widehat{JU}_5 \xrightarrow[p]{\cong} \widehat{JU}_9 \xrightarrow[p]{\cong} \cdots \widehat{JU}_{4k+5} \xrightarrow[p]{\cong} \cdots$$

where p is defined as follows.

Definition 3.3.2. *Define $p \in (QU)^8 = \widehat{JU}_4$ to be*

$$\begin{aligned} p &= \varrho(d^2 + b^4 - (2v(a+b)d^2 - 2vad^2)) \\ &= D^2 + B^4 - (2(A+B)D^2 - 2AD^2) \\ &= \begin{bmatrix} p([1]) & p([s^4]) & p([s]) & p([s^2]) & p([s^5]) & p([t]) & p([ts]) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 16 & -2 & 4 & -2 & 16 & 4 \end{bmatrix} \end{aligned}$$

where $[1]$, $[s^4]$, $[s]$, $[s^5]$, $[t]$ and $[ts]$ are conjugacy classes of SD_{16}.

Now, it is simple to see that $\varrho(a^4) = \varrho(v^5ad^2)\cdot p - \varrho(v^2a^2)\cdot p \in (p)$, principal ideal generated by p, $\varrho(b^6) = \varrho(v^3ad^2)\cdot p + \varrho(a^2)\cdot p + \rho(b^2)\cdot p \in (p)$ and $\varrho(d^3) = \varrho(d)\cdot p \in (p)$. This means the radical of (p) is I'', $\sqrt{(p)} = I''$ and hence

$$H^*_{I''}(QU) \cong H^*_{(p)}(QU). \tag{3.2}$$

Therefore $H^i_{I''}(QU) = 0$ for $i \geq 2$, $H^i_I(R) = 0$ for $i \geq 3$ and the long exact sequence from (3.1) splits as;

$$\begin{gathered} 0 \longrightarrow H^0_I(R) \longrightarrow H^0_{(p)}(QU) \longrightarrow 0 \\ 0 \longrightarrow H^1_I(R) \longrightarrow H^1_{(p)}(QU) \xrightarrow{\delta} H^2_{I'}(PC)\cdot\tau \longrightarrow H^2_I(R) \longrightarrow 0. \end{gathered} \tag{3.3}$$

The immediate result from this sequences is;

Lemma 3.3.3. *Let ρ be a regular representation, $\rho = 16\cdot 1 - 6va - 8vb + v^2ab - 12v^2d + 6v^3ad + 2v^4d^2$.*

$$(H^0_I(R))_i = \begin{cases} \mathbb{Z}\cdot\rho, & \text{if } i \text{ is non-negative even integer;} \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, $H^1_I(R) = \ker\delta$, $H^2_I(R) = \operatorname{coker}\delta$ and $H^i_I(R) = 0$ for $i \geq 3$.

Proof. The last statement is obvious from the second exact sequence in (3.3). Here we have $\varrho(\rho) = \rho = \begin{bmatrix} 16 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Since $H^0_{(p)}(QU) = \Gamma_{(p)}(QU)$, $H^0_{(p)}(QU) = \{x \in QU : (p)^n x = 0, \exists n \in \mathbb{N}\}$. Thus, the only possible element in QU satisfying $(p)^n x = 0$ for some $n \in \mathbb{N}$ is $\rho \in \overline{R}_0$ and hence all $n\cdot\rho$ for all $n \in \mathbb{Z}$. By lemma 3.3.1 and the first short exact sequence in (3.3), $H^0_I(R) \cong H^0_{(p)}(QU)$, the result follows. □

3.3.2 $E^{1\frac{1}{2}}$- PAGE

The main task in this section is to find $H^1_{(p)}(QU)$ from the short exact sequence;

$$0 \longrightarrow H^0_I(QU) \longrightarrow QU \longrightarrow QU[\tfrac{1}{p}] \longrightarrow H^1_{(p)}(QU) \longrightarrow 0.$$

Since QU is graded, so is $QU[\frac{1}{p}]$ and $H^1_{(p)}(QU)$. We calculate $H^1_{(p)}(QU)_t$ as $\text{coker}(QU_t \longrightarrow (QU[\frac{1}{p}])_t)$. Note that

$$\varinjlim(QU_t \xrightarrow{p} QU_{t-8} \xrightarrow{p} QU_{t-2(8)} \xrightarrow{p} \cdots) \cong_{(f)} (QU[\frac{1}{p}])_t$$

(where $\cong_{(f)}$ means isomorphism given by the natural map, say f), and this direct system is eventually constant at $QU_{-4} = \widehat{JU}_2$ or $QU_{-6} = \widehat{JU}_3$ or $QU_{-8} = \widehat{JU}_4$ or $QU_{-10} = \widehat{JU}_5$. By proposition A.0.4, $\widehat{JU}_\epsilon \cong_{(g)} (QU[\frac{1}{p}])_t$ for which $\epsilon \in \{2,3,4,5\}$ and hence $H^1_{(p)}(QU)_t \cong \text{coker}(QU_t \xrightarrow{p^i} \widehat{JU}_\epsilon)$ for $i = \frac{t+2\epsilon}{8} \in \mathbb{Z}$. Now, it is obvious that $H^1_{(p)}(QU)_t = 0$ for $t \leq -4$ or t is odd. For $t \geq -2$, we have

$$\begin{array}{lll}
H^1_{(p)}(QU)_{-2} \cong \widehat{JU}_5/p \cdot JU_1 & , & H^1_{(p)}(QU)_{2(4k-5)} \cong \widehat{JU}_5/p^k \cdot \overline{R}_0 \\
H^1_{(p)}(QU)_{0} \cong \widehat{JU}_4/p \cdot \overline{R}_0 & , & H^1_{(p)}(QU)_{2(4k-4)} \cong \widehat{JU}_4/p^k \cdot \overline{R}_0 \\
H^1_{(p)}(QU)_{2} \cong \widehat{JU}_3/p \cdot \overline{R}_0 & , & H^1_{(p)}(QU)_{2(4k-3)} \cong \widehat{JU}_3/p^k \cdot \overline{R}_0 \\
H^1_{(p)}(QU)_{4} \cong \widehat{JU}_2/p \cdot \overline{R}_0 & , & H^1_{(p)}(QU)_{2(4k-2)} \cong \widehat{JU}_2/p^k \cdot \overline{R}_0
\end{array}$$

for each $k \geq 2$. The isomorphism f sending $(q_1, q_2, q_3, ..., q_m, 0, 0, 0, ...)$ to $q_1 + \frac{q_2}{p} + \frac{q_3}{p^2} + ... + \frac{q_m}{p^{m-1}}$ and the isomorphism in the proposition A.0.4, yield explicitly the isomorphism g as $g(\overline{\alpha}_\epsilon) = \frac{\overline{\alpha}_\epsilon}{p^i}$ for $i = \frac{t+2\epsilon}{8} \in \mathbb{Z}$. This implies that $\overline{\alpha}_\epsilon + p^i \cdot QU_t$ in $\widehat{JU}_\epsilon / p^i \cdot QU_t$ can be identified by $\frac{\overline{\alpha}_\epsilon}{p^i} + QU_t$ in $H^1_{(p)}(QU)_t$.

Lemma 3.3.4. *For each $\alpha \in \{x, y, z, t, u, w\}$ and for $\epsilon =2,3,4$ let $\widetilde{\alpha}_{-1} = \frac{\overline{\alpha}_5}{p} + \widehat{JU}_1 \cong \overline{\alpha}_5 + p \cdot \widehat{JU}_1$, $\widetilde{\alpha}_{4i-\epsilon} = \frac{\overline{\alpha}_\epsilon}{p^i} + \overline{R}_0 \cong \overline{\alpha}_\epsilon + p^i \cdot \overline{R}_0$ for all $i \geq 1$, and let $\widetilde{\alpha}_{4j-5} = \frac{\overline{\alpha}_5}{p^j} + \overline{R}_0 \cong \overline{\alpha}_5 + p^j \cdot \overline{R}_0$, for all $j \geq 2$, we have*

- $H^1_{(p)}(QU)_{-2} = \mathbb{Z}/2 < \widetilde{x}_{-1} >$ *with* $\widetilde{x}_{-1} = \widetilde{y}_{-1} = \widetilde{z}_{-1}$, $\widetilde{t}_{-1} = \widetilde{x}_{-1} - \widetilde{y}_{-1}$ *and* $\widetilde{u}_{-1} = \widetilde{w}_{-1} = 0$.
- $H^1_{(p)}(QU)_0 = \mathbb{Z}/2 < \widetilde{y}_0 >$ *with* $\widetilde{u}_0 = \widetilde{y}_0$ *and* $\widetilde{x}_0 = \widetilde{z}_0 = \widetilde{w}_0 = \widetilde{t}_0 = 0$.
- $H^1_{(p)}(QU)_2 = \mathbb{Z}/2 < \widetilde{x}_1 > \oplus \mathbb{Z}/2 < \widetilde{z}_1 > \oplus \mathbb{Z}/4 < \widetilde{y}_1 >$ *with* $\widetilde{w}_1 = \widetilde{z}_1$, $\widetilde{u}_1 + \widetilde{x}_1 + \widetilde{z}_1 + 2\widetilde{y}_1 = 0 = \widetilde{t}_1$.
- $H^1_{(p)}(QU)_4 = \mathbb{Z}/2 < \widetilde{x}_2 > \oplus \mathbb{Z}/8 < \widetilde{y}_2 > \oplus \mathbb{Z}/16 < \widetilde{w}_2 >$ *with* $\widetilde{z}_2 = 8\widetilde{w}_2$, $\widetilde{t}_2 + \widetilde{x}_2 = 4(\widetilde{w}_2 + \widetilde{y}_2)$ *and* $\widetilde{u}_2 + 8\widetilde{w}_2 = 0$.
- $H^1_{(p)}(QU)_6 = \mathbb{Z}/2 < \widetilde{t}_3 > \oplus \mathbb{Z}/4 < \widetilde{x}_3 > \oplus \mathbb{Z}/16 < \widetilde{y}_3 > \oplus \mathbb{Z}/16 < \widetilde{u}_3 >$ *with* $\widetilde{w}_3 - 8\widetilde{u}_3 = 0$ *and* $\widetilde{t}_3 + \widetilde{z}_3 = 2\widetilde{x}_3 + 8\widetilde{y}_3 + 4\widetilde{u}_3$.

- $H^1_{(p)}(QU)_8 = \mathbb{Z}/4 < \widetilde{x}_4 > \oplus \mathbb{Z}/4 < \widetilde{w}_4 - 8\widetilde{u}_4 > \oplus \mathbb{Z}/32 < \widetilde{y}_4 > \oplus \mathbb{Z}/64 < \widetilde{u}_4 >$ *with* $\widetilde{z}_4 + 2\widetilde{w}_4 = 0$ *and* $\widetilde{t}_4 + 2\widetilde{x}_4 - 2\widetilde{w}_4 + 16\widetilde{y}_4 + 16\widetilde{u}_4 = 0$.
- $H^1_{(p)}(QU)_{10} = \mathbb{Z}/2 < \widetilde{w}_5 + 16\widetilde{u}_5 > \oplus \mathbb{Z}/4 < \widetilde{z}_5 + 8\widetilde{u}_5 > \oplus \mathbb{Z}/8 < \widetilde{x}_5 > \oplus \mathbb{Z}/64 < \widetilde{y}_5 > \oplus \mathbb{Z}/64 < \widetilde{u}_5 >$ *with* $\widetilde{t}_5 + 4\widetilde{x}_5 + 32\widetilde{y}_5 + 2\widetilde{z}_5 + 16\widetilde{u}_5 = 0$.
- $H^1_{(p)}(QU)_{12} = \mathbb{Z}/2 < \widetilde{u}_6 + 64\widetilde{w}_6 > \oplus \mathbb{Z}/8 < \widetilde{x}_6 > \oplus \mathbb{Z}/8 < \widetilde{t}_6 + 16\widetilde{w}_6 > \oplus \mathbb{Z}/128 < \widetilde{y}_6 > \oplus \mathbb{Z}/256 < \widetilde{w}_6 >$ *with* $\widetilde{z}_6 + 4\widetilde{x}_6 + 64\widetilde{y}_6 + 2\widetilde{t}_6 + 64\widetilde{w}_6 = 0$.

In general for $n \geq 3$,

- $H^1_{(p)}(QU)_{2(4n-5)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2 \cdot 4^{n-2} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/16^{n-1} \oplus \mathbb{Z}/16^{n-1}$ *generated by* $\widetilde{t}_{4n-5} + (-2)^{n-1}\widetilde{x}_{4n-5} - (-2)^{n-2}\widetilde{z}_{4n-5} + (-8)^{n-1}\widetilde{y}_{4n-5} + 4(-8)^{n-2}\widetilde{u}_{4n-5}$, $\widetilde{w}_{4n-5} + (-2)^{n-1}\widetilde{z}_{4n-5}$, $\widetilde{z}_{4n-5} + (4)^{n-1}\widetilde{u}_{4n-5}$, $\widetilde{x}_{4n-5}$, $\widetilde{y}_{4n-5}$ *and* $\widetilde{u}_{4n-5}$ *respectively.*
- $H^1_{(p)}(QU)_{2(4n-4)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/2 \cdot 16^{n-1} \oplus \mathbb{Z}/4 \cdot 16^{n-1}$ *generated by* $\widetilde{t}_{4n-4} + (-2)^{n-2}(2\widetilde{x}_{4n-4} + \widetilde{z}_{4n-4}) - 2(-8)^{n-1}(\widetilde{y}_{4n-4} + \widetilde{u}_{4n-4})$, $\widetilde{z}_{4n-4} - 2\widetilde{w}_{4n-4} - 4(-8)^{n-1}\widetilde{u}_{4n-4}$, $\widetilde{w}_{4n-4} + 2 \cdot 4^{n-1}\widetilde{u}_{4n-4}$, $\widetilde{x}_{4n-4}$, $\widetilde{y}_{4n-4}$ *and* $\widetilde{u}_{4n-4}$ *respectively.*
- $H^1_{(p)}(QU)_{2(4n-3)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-1} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/2 \cdot 4^{n-1} \oplus \mathbb{Z}/4 \cdot 16^{n-1} \oplus \mathbb{Z}/4 \cdot 16^{n-1}$ *generated by* $\widetilde{t}_{4n-3} + (-2)^{n}\widetilde{x}_{4n-3} - 4(-8)^{n-1}\widetilde{y}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3} - 2(-8)^{n-1}\widetilde{u}_{4n-3}$, $\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}$, $\widetilde{z}_{4n-3} + 2 \cdot 4^{n-1}\widetilde{u}_{4n-3}$, $\widetilde{x}_{4n-3}$, $\widetilde{y}_{4n-3}$ *and* $\widetilde{u}_{4n-3}$ *respectively.*
- $H^1_{(p)}(QU)_{2(4n-2)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-1} \oplus \mathbb{Z}/2 \cdot 4^{n-1} \oplus \mathbb{Z}/2 \cdot 4^{n-1} \oplus \mathbb{Z}/8 \cdot 16^{n-1} \oplus \mathbb{Z}/16^{n}$ *generated by* $\widetilde{z}_{4n-2} - (-2)^{n}\widetilde{x}_{4n-2} - (-8)^{n}\widetilde{y}_{4n-2} - (-8)^{n}\widetilde{w}_{4n-2} - (2 + (-2)^{n})\widetilde{t}_{4n-2}$, $\widetilde{u}_{4n-2} + (-8)^{n}\widetilde{w}_{4n-2}$, $\widetilde{t}_{4n-2} + 4^{n}\widetilde{w}_{4n-2}$, $\widetilde{x}_{4n-2}$, $\widetilde{y}_{4n-2}$ *and* $\widetilde{w}_{4n-2}$ *respectively.*

Proof. Recall from the previous subsection that $p \cdot \widehat{JU}_1 = \mathbb{Z}^{\wedge}_2 < p \cdot \overline{x}_1, p \cdot \overline{y}_1, p \cdot \overline{z}_1, p \cdot \overline{t}_1, p \cdot \overline{u}_1, p \cdot \overline{w}_1 >$ and $\widehat{JU}_5 = \mathbb{Z}^{\wedge}_2 < \overline{x}_5, \overline{y}_5, \overline{z}_5, \overline{t}_5, \overline{u}_5, \overline{w}_5 >$, we get that $p \cdot \overline{x}_1 = \overline{x}_5 + \overline{y}_5$, $p \cdot \overline{y}_1 = 2\overline{x}_5$, $p \cdot \overline{z}_1 = \overline{y}_5 + \overline{t}_5$, $p \cdot \overline{t}_1 = \overline{x}_5 + \overline{z}_5$, $p \cdot \overline{u}_1 = 2\overline{u}_5 + \overline{w}_5 + \overline{z}_5 + \overline{x}_5$ and $p \cdot \overline{w}_1 = \overline{u}_5$. Hence, we represent matrix for the computation of $\widehat{JU}_5 / p \cdot \widehat{JU}_1$ as

$$\begin{array}{r|cccccc|}
 & \overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 \\
p \cdot \overline{x}_1 : & 1 & 1 & 0 & 0 & 0 & 0 \\
p \cdot \overline{y}_1 : & 2 & 0 & 0 & 0 & 0 & 0 \\
p \cdot \overline{z}_1 : & 0 & 1 & 0 & 1 & 0 & 0 \\
p \cdot \overline{t}_1 : & 1 & 0 & 1 & 0 & 0 & 0 \\
p \cdot \overline{u}_1 : & 1 & 0 & 1 & 0 & 2 & 1 \\
p \cdot \overline{w}_1 : & 0 & 0 & 0 & 0 & 1 & 0
\end{array},$$

which we do row operations as

$$\begin{array}{c|cccccc|}
 & \overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
r_2 & 2 & 0 & 0 & 0 & 0 & 0 \\
r_3 & 0 & 1 & 0 & 1 & 0 & 0 \\
r_4 & 1 & 0 & 1 & 0 & 0 & 0 \\
r_5 & 1 & 0 & 1 & 0 & 2 & 1 \\
r_6 & 0 & 0 & 0 & 0 & 1 & 0
\end{array} \cong \begin{array}{c|cccccc|}
 & \overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_1 & 1 & 1 & 0 & 0 & 0 & 0 \\
r_2 & 2 & 0 & 0 & 0 & 0 & 0 \\
r_3 & 0 & 1 & 0 & 1 & 0 & 0 \\
r_4 & 1 & 0 & 1 & 0 & 0 & 0 \\
r_5' & 0 & 0 & 0 & 0 & 0 & 1 \\
r_6 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}, \text{ (Step } *),$$

where $r_5' = r_5 - r_4 - 2r_6$ and (Step $*$) means the last step of our row operations; after doing column operations, we get the required result. Then we do column operations and get

$$\begin{array}{r|cccccc|}
\text{new generator} \rightarrow & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
\text{column operation} \rightarrow & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_1 & 0 & 1 & 0 & 0 & 0 & 0 \\
r_2 & 2 & 0 & 0 & 0 & 0 & 0 \\
r_3 & 0 & 0 & 0 & 1 & 0 & 0 \\
r_4 & 0 & 0 & 1 & 0 & 0 & 0 \\
r_5' & 0 & 0 & 0 & 0 & 0 & 1 \\
r_6 & 0 & 0 & 0 & 0 & 1 & 0
\end{array},$$

where $c_1' = c_1 - c_2 + c_4 - c_3$, $c_2' = c_2 - c_4$ and $c_i' = c_i$ for $i = 3, 4, 5, 6$. Note that row operations do not change the basis of $\widehat{JU}_5/p \cdot \widehat{JU}_1$ but column operations will effect such a basis i.e. changing basis. More precisely, let q be an element in (Step $*$) which $q = a\overline{x}_5 + b\overline{y}_5 + c\overline{z}_5 + d\overline{t}_5 + e\overline{u}_5 + f\overline{w}_5$ where $a, b, c, d, e, f \in \mathbb{Z}_2^\wedge$. Then after column operations we get $q = (a - b - c + d)g_1 + (b - d)g_2 + cg_3 + dg_4 + eg_5 + fg_6$ and hence

$$a(\overline{x}_5 - g_1) + b(\overline{y}_5 + g_1 - g_2) + c(\overline{z}_5 + g_1 - g_3) + d(\overline{t}_5 - g_1 - g_4 + g_2) + e(\overline{u}_5 - g_5) + f(\overline{w}_5 - g_6) = 0.$$

If $q = r_6$ then $g_5 = \overline{u}_5$ and if $q = r_5'$ then $g_6 = \overline{w}_5$.
If $q = r_2$ then $g_1 = \overline{x}_5$ and if $q = r_1$ then $g_2 = \overline{x}_5 + \overline{y}_5$.
If $q = r_3$ then $g_3 = \overline{x}_5 + \overline{z}_5$ and if $q = r_4$ then $g_4 = \overline{t}_5 - \overline{x}_5 + \overline{y}_5$.

That is $\widehat{JU}_5/p \cdot \widehat{JU}_1 = \mathbb{Z}_2^\wedge/2\mathbb{Z}_2^\wedge \cong \mathbb{Z}/2$ generated by $g_1 + p \cdot \widehat{JU}_1 = \overline{x}_5 + p \cdot \widehat{JU}_1$ and $\overline{u}_5, \overline{w}_5, \overline{x}_5 + \overline{y}_5, \overline{x}_5 + \overline{z}_5, \overline{t}_5 - \overline{x}_5 + \overline{y}_5 \in p \cdot \widehat{JU}_1$ and hence $H^1_{(p)}(QU)_{-2}$ follows.

The rest of this proof relies on $p^k \cdot \overline{R}_0$ for $k > 0$ which is

$$\mathbb{Z}_2^\wedge \left\langle \begin{array}{rcccccccc}
p^k \cdot 1 : & [& 16^k & (-2)^k & 4^k & (-2)^k & 16^k & 4^k &] \\
p^k \cdot A : & [& 0 & 2(-2)^k & 0 & 2(-2)^k & 0 & 2 \cdot 4^k &] \\
p^k \cdot B : & [& 0 & 0 & 0 & 0 & 2 \cdot 16^k & 2 \cdot 4^k &] \\
p^k \cdot D : & [& 4 \cdot 16^k & -c(-2)^k & 2 \cdot 4^k & c(-2)^k & 0 & 2 \cdot 4^k &] \\
p^k \cdot (A+B)D : & [& 0 & -2c(-2)^k & 0 & 2c(-2)^k & 0 & 8 \cdot 4^k &] \\
p^k \cdot D^2 : & [& 16 \cdot 16^k & -2(-2)^k & 4 \cdot 4^k & -2(-2)^k & 0 & 4 \cdot 4^k &] \\
p^k \cdot D^3 : & [& 64 \cdot 16^k & 2c(-2)^k & 8 \cdot 4^k & -2c(-2)^k & 0 & 8 \cdot 4^k &]
\end{array} \right\rangle.$$

Here, we change this basis to

$$\mathbb{Z}_2^\wedge \left\langle \begin{array}{r} p^k\cdot 1: [\\ p^k\cdot\overline{x}_0': [\\ p^k\cdot\overline{y}_0: [\\ p^k\cdot\overline{z}_0: [\\ p^k\cdot\overline{t}_0': [\\ p^k\cdot\overline{u}_0: [\\ p^k\cdot\overline{w}_0: [\end{array} \begin{array}{cccccc} 16^k & (-2)^k & 4^k & (-2)^k & 16^k & 4^k \\ 0 & 0 & 0 & 0 & 0 & \frac{14}{3}\cdot 4^k \\ 0 & 0 & 0 & 0 & 2\cdot 16^k & 0 \\ 0 & -2(-2)^k & 0 & -2(-2)^k & 0 & 0 \\ 4\cdot 16^k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4\cdot 4^k & 0 & 0 & 0 \\ 0 & -c(-2)^k & 2\cdot 4^k & c(-2)^k & 0 & 0 \end{array} \begin{array}{l}]\\ ]\\ ]\\ ]\\ ]\\ ]\\ ] \end{array} \right\rangle,$$

where

$$\begin{aligned} p^k\cdot\overline{x}_0' &= \frac{2}{9}p^k\cdot D + p^k\cdot D^2 - \frac{2}{9}p^k\cdot D^3 - \frac{8}{3}p^k\cdot 1 + \frac{7}{3}p^k\cdot A - \frac{1}{3}p^k\cdot (A+B)D \\ &\quad + \frac{4}{3}p^k\cdot B \\ p^k\cdot\overline{y}_0 &= p^k\cdot B - \frac{7}{3}p^k\cdot\overline{x}_0' \\ p^k\cdot\overline{z}_0 &= \frac{7}{3}p^k\cdot\overline{x}_0' - p^k\cdot A \\ p^k\cdot\overline{t}_0' &= 4p^k\cdot 1 + 2p^k\cdot\overline{z}_0 - p^k\cdot\overline{u}_0 - 2p^k\cdot B + \frac{6}{7}p^k\cdot\overline{x}_0' \\ p^k\cdot\overline{u}_0 &= 8p^k\cdot 1 - 4p^k\cdot A - 2p^k\cdot D + p^k\cdot (A+B)D - \frac{6}{7}p^k\cdot\overline{x}_0' \\ p^k\cdot\overline{w}_0 &= p^k\cdot D - p^k\cdot\overline{t}_0 - \frac{7}{3}p^k\cdot\overline{x}_0'. \end{aligned}$$

We can change this basis further by

$$p^k\cdot\overline{x}_0' \to p^k\cdot\overline{x}_0 = 3p^k\cdot\overline{x}_0' - 12p^k\cdot 1 + 6p^k\cdot\overline{y}_0 - 3p^k\cdot\overline{u}_0 - 6p^k\cdot\overline{z}_0 - 3p^k\cdot\overline{t}_0'$$

and

$$p^k\cdot\overline{t}_0' \to p^k\cdot\overline{t}_0' - 4p^k\cdot 1 - 2p^k\cdot\overline{z}_0 + p^k\cdot\overline{u}_0 - 2p^k\cdot\overline{y}_0 - 2p^k\cdot\overline{x}_0 = 0.$$

Now the basis for $p^k\cdot\overline{R}_0$ for $k>0$ is reduced to

$$\mathbb{Z}_2^\wedge \left\langle \begin{array}{r} p^k\cdot 1: [\\ p^k\cdot\overline{x}_0: [\\ p^k\cdot\overline{y}_0: [\\ p^k\cdot\overline{z}_0: [\\ p^k\cdot\overline{u}_0: [\\ p^k\cdot\overline{w}_0: [\end{array} \begin{array}{cccccc} 16^k & (-2)^k & 4^k & (-2)^k & 16^k & 4^k \\ 0 & 0 & 0 & 0 & 0 & 2\cdot 4^k \\ 0 & 0 & 0 & 0 & 2\cdot 16^k & 0 \\ 0 & -2(-2)^k & 0 & -2(-2)^k & 0 & 0 \\ 0 & 0 & 4\cdot 4^k & 0 & 0 & 0 \\ 0 & -c(-2)^k & 2\cdot 4^k & c(-2)^k & 0 & 0 \end{array} \begin{array}{l}]\\ ]\\ ]\\ ]\\ ]\\ ]. \end{array} \right\rangle$$

For the calculation of $H^1_{(p)}(QU)_0 \cong \widehat{JU}_4/p\cdot\overline{R}_0$, we use the same method as above which we can represent matrix for the calculation of $\widehat{JU}_4/p\cdot\overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \overline{w}_4 & \\
p\cdot 1: & 0 & 1 & 0 & 0 & 1 & 0 & r_0 \\
p\cdot \overline{x}_0: & 1 & 0 & 0 & 0 & 0 & 0 & r_1 \\
p\cdot \overline{y}_0: & 0 & 2 & 0 & 0 & 0 & 0 & r_2 \\
p\cdot \overline{z}_0: & 0 & 0 & 0 & 1 & 0 & 0 & r_3 \\
p\cdot \overline{u}_0: & 0 & 0 & 1 & 0 & 0 & 0 & r_4 \\
p\cdot \overline{w}_0: & 0 & 0 & 0 & 0 & 0 & 1 & r_5.
\end{array}$$

After column operations, we have

$$\begin{array}{r|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & 0 & 0 & 1 & 0 \\
r_1^* & 1 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 2 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 1 & 0 & 0 \\
r_4^* & 0 & 0 & 1 & 0 & 0 & 0 \\
r_5^* & 0 & 0 & 0 & 0 & 0 & 1
\end{array},$$

where $c_2' = c_2 - c_5$ and $c_i' = c_i$ for $i \neq 2$. Thus, $\widehat{JU}_4/p\cdot\overline{R}_0 \cong \mathbb{Z}/2$ generated by $g_2 + p\cdot\overline{R}_0$ and $g_i \in p\cdot\overline{R}_0$ for all $i \neq 3$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_4$, $g_2 = \overline{y}_4$, $g_3 = \overline{z}_4$, $g_4 = \overline{t}_4$, $g_5 = \overline{u}_4 + \overline{y}_4$ and $g_6 = \overline{w}_4$ and hence the result for $H^1_{(p)}(QU)_0$ follows.

For $H^1_{(p)}(QU)_2 \cong \widehat{JU}_3/p\cdot\overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_3/p\cdot\overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_3 & \overline{y}_3 & \overline{z}_3 & \overline{t}_3 & \overline{u}_3 & \overline{w}_3 & \\
p\cdot 1: & 1 & 2 & 1 & 0 & 1 & 0 & r_0 \\
p\cdot \overline{x}_0: & 2 & 0 & 0 & 0 & 0 & 0 & r_1 \\
p\cdot \overline{y}_0: & 0 & 4 & 0 & 0 & 0 & 0 & r_2 \\
p\cdot \overline{z}_0: & 0 & 0 & 0 & 1 & 0 & 0 & r_3 \\
p\cdot \overline{u}_0: & 0 & 0 & 2 & 0 & 0 & 0 & r_4 \\
p\cdot \overline{w}_0: & 0 & 0 & 1 & 0 & 0 & 1 & r_5.
\end{array}$$

After column operations, we have

$$\begin{array}{r|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & 0 & 0 & 1 & 0 \\
r_1^* & 2 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 4 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 1 & 0 & 0 \\
r_4^* & 0 & 0 & 2 & 0 & 0 & 0 \\
r_5^* & 0 & 0 & 0 & 0 & 0 & 1
\end{array},$$

where $c_1' = c_1 - c_5$, $c_2' = c_2 - 2c_5$, $c_3' = c_3 - c_5 - c_6$ and $c_i' = c_i$ for $i = 4, 5, 6$. Thus,

$$\widehat{JU}_3/p \cdot \overline{R}_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$$

generated by $g_1 + p \cdot \overline{R}_0$, $g_3 + p \cdot \overline{R}_0$ and $g_2 + p \cdot \overline{R}_0$ respectively, and $g_i \in p \cdot \overline{R}_0$ for all $i = 4, 5, 6$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_3$, $g_2 = \overline{y}_3$, $g_3 = \overline{z}_3$, $g_4 = \overline{t}_3$, $g_5 = \overline{u}_3 + \overline{x}_3 + 2\overline{y}_3 + \overline{z}_3$ and $g_6 = \overline{w}_3 + \overline{z}_3$ and hence the result for $H^1_{(p)}(QU)_2$ follows.

For $H^1_{(p)}(QU)_4 \cong \widehat{JU}_2/p \cdot \overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_2/p \cdot \overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2 & \overline{w}_2 & \\
p \cdot 1 : & -1 & 4 & 0 & -1 & -2 & 4 & r_0 \\
p \cdot \overline{x}_0 : & 2 & 0 & 0 & 0 & 0 & 0 & r_1 \\
p \cdot \overline{y}_0 : & 0 & 8 & 0 & 0 & 0 & 0 & r_2 \\
p \cdot \overline{z}_0 : & 0 & 0 & -1 & 2 & 0 & 0 & r_3 \\
p \cdot \overline{u}_0 : & 0 & 0 & 2 & 0 & 0 & 0 & r_4 \\
p \cdot \overline{w}_0 : & 0 & 0 & 1 & 0 & 1 & 0 & r_5 .
\end{array}$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 - 2r_5^* + r_4^*]$, $r_1 = r_1^*$, $r_2 = r_2^*$
$r_3 \longrightarrow [r_3' = r_3 + 2r_0 - r_2 + r_1] \longrightarrow [r_3^* = r_3' + 4r_5^* - 2r_4^*]$,
$r_4 \longrightarrow [r_4' = r_4 + 2r_3] \longrightarrow [r_4'' = r_4' + 4r_0 - 2r_2 + 2r_1] \longrightarrow [r_4''' = r_4'' - 2r_5''] \longrightarrow [r_4^* = r_4'' + 2r_5^*]$,
$r_5 \longrightarrow [r_5' = r_5 + r_3] \longrightarrow [r_5'' = r_5' + 2r_0 - r_2 + r_1] \longrightarrow [r_5^* = r_5'' - r_4''']$,
and then we obtain (Step $*$) and the required result as

$$(\text{Step } *) = \begin{array}{r|cccccc|}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2 & \overline{w}_2 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_0^* & -1 & 4 & 0 & -1 & 0 & 4 \\
r_1^* & 2 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 8 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & -1 & 0 & 0 & 8 \\
r_4^* & 0 & 0 & 0 & 0 & 0 & 16 \\
r_5^* & 0 & 0 & 0 & 0 & 1 & 8
\end{array} \cong \begin{array}{r|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & 0 & -1 & 0 & 0 \\
r_1^* & 2 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 8 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & -1 & 0 & 0 & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 0 & 16 \\
r_5^* & 0 & 0 & 0 & 0 & 1 & 0
\end{array},$$

where $c_1' = c_1 - c_4$, $c_2' = c_2 + 4c_4$, $c_6' = c_6 + 4c_4 + 8c_3 - 8c_5$ and $c_i' = c_i$ for $i = 3, 4, 5$. Thus,

$$\widehat{JU}_2/p \cdot \overline{R}_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/16$$

generated by $g_1 + p \cdot \overline{R}_0$, $g_2 + p \cdot \overline{R}_0$ and $g_6 + p \cdot \overline{R}_0$ respectively, and $g_i \in p \cdot \overline{R}_0$ for all $i = 3, 4, 5$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_2$, $g_2 = \overline{y}_2$, $g_3 = \overline{z}_2 - 8\overline{w}_2$, $g_4 = \overline{t}_2 + \overline{x}_2 - 4(\overline{w}_2 + \overline{y}_2)$, $g_5 = \overline{u}_2 + 8\overline{w}_2$ and $g_6 = \overline{w}_2$ and hence the result for $H^1_{(p)}(QU)_4$ follows.

For $H^1_{(p)}(QU)_6 \cong \widehat{JU}_5/p^2\cdot\overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_5/p^2\cdot\overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 & \\
p^2\cdot 1: & 2 & 8 & 3 & -1 & 4 & 2 & r_0\\
p^2\cdot\overline{x}_0: & 4 & 0 & 0 & 0 & 0 & 0 & r_1\\
p^2\cdot\overline{y}_0: & 0 & 16 & 0 & 0 & 0 & 0 & r_2\\
p^2\cdot\overline{z}_0: & 0 & 0 & 0 & 2 & 0 & 0 & r_3\\
p^2\cdot\overline{u}_0: & 0 & 0 & 4 & 0 & 0 & 0 & r_4\\
p^2\cdot\overline{w}_0: & 0 & 0 & 2 & 0 & 0 & -1 & r_5.
\end{array}$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 + 2r_5 - 2r_4]$, $r_1 = r_1^*$, $r_2 = r_2^*$, $r_3 = r_3^*$
$r_4 \longrightarrow [r_4^* = r_4 + 4r_0^* + 2r_3^* - 2r_1^* - 2r_2^*]$,
$r_5 \longrightarrow [r_5^* = r_5 + 2r_0^* + r_3^* - r_1^* - r_2^*]$,
and then we obtain (Step $*$) and the required result as

$$(\text{Step } *)= \begin{array}{r|cccccc|}
 & \overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_0^* & 2 & 8 & -1 & -1 & 4 & 0 \\
r_1^* & 4 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 16 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 2 & 0 & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 16 & 0 \\
r_5^* & 0 & 0 & 0 & 0 & 8 & -1
\end{array} \cong \begin{array}{r|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & -1 & 0 & 0 & 0 \\
r_1^* & 4 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 16 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 2 & 0 & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 16 & 0 \\
r_5^* & 0 & 0 & 0 & 0 & 0 & -1
\end{array},$$

where $c_1' = c_1 + 2c_3$, $c_2' = c_2 + 8c_3$, $c_3' = c_3$, $c_4' = c_4 - c_3$, $c_5' = c_5 + 4c_3 + 8c_6$ and $c_6' = c_6$. Thus,

$$\widehat{JU}_5/p^2\cdot\overline{R}_0 \cong \mathbb{Z}/2\oplus\mathbb{Z}/4\oplus\mathbb{Z}/16\oplus\mathbb{Z}/16$$

generated by $g_4 + p^2\cdot\overline{R}_0$, $g_1 + p^2\cdot\overline{R}_0$, $g_2 + p^2\cdot\overline{R}_0$ and $g_5 + p^2\cdot\overline{R}_0$ respectively, and $g_3, g_6 \in p^2\cdot\overline{R}_0$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_5$, $g_2 = \overline{y}_5$, $g_3 = \overline{z}_5 - 2\overline{x}_5 - 8\overline{y}_5 + \overline{t}_5 - 4\overline{u}_5$, $g_4 = \overline{t}_5$, $g_5 = \overline{u}_5$ and $g_6 = \overline{w}_5 - 8\overline{u}_5$ and hence the result for $H^1_{(p)}(QU)_6$ follows.

For $H^1_{(p)}(QU)_8 \cong \widehat{JU}_4/p^2\cdot\overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_4/p^2\cdot\overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \overline{w}_4 & \\
p^2\cdot 1: & -6 & 16 & -3 & 9 & 16 & 0 & r_0\\
p^2\cdot\overline{x}_0: & 4 & 0 & 0 & 0 & 0 & 0 & r_1\\
p^2\cdot\overline{y}_0: & 0 & 32 & 0 & 0 & 0 & 0 & r_2\\
p^2\cdot\overline{z}_0: & 0 & 0 & 0 & -2 & 0 & 0 & r_3\\
p^2\cdot\overline{u}_0: & 0 & 0 & 4 & 0 & 0 & 0 & r_4\\
p^2\cdot\overline{w}_0: & 0 & 0 & 3 & 0 & 0 & -2 & r_5.
\end{array}$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 + 2r_1 + r_5 + 4r_3]$, $r_1 = r_1^*$, $r_2 = r_2^*$,
$r_3 \longrightarrow [r_3' = r_3 + 2r_0^* - r_1^* - r_2^*] \longrightarrow [r_3^* = -r_3']$
$r_4 \longrightarrow [r_4' = r_4 + 4r_5^*] \longrightarrow [r_4^* = -(r_4' - 2r_3')]$,
$r_5 \longrightarrow [r_5^* = r_5 - r_4]$,
and then we obtain (Step $*$) and the required result as

$$\begin{array}{r|cccccc|}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \overline{w}_4 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_0^* & 2 & 16 & 0 & 1 & 16 & -2 \\
r_1^* & 4 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 32 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 0 & -32 & 4 \\
r_4^* & 0 & 0 & 0 & 0 & 64 & 0 \\
r_5^* & 0 & 0 & -1 & 0 & 0 & -2
\end{array}
\cong
\begin{array}{r|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & 0 & 1 & 0 & 0 \\
r_1^* & 4 & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 32 & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 0 & 0 & 4 \\
r_4^* & 0 & 0 & 0 & 0 & 64 & 0 \\
r_5^* & 0 & 0 & -1 & 0 & 0 & 0
\end{array},$$

where $c_1' = c_1 - 2c_4$, $c_2' = c_2 - 16c_4$, $c_3' = c_3$, $c_4' = c_4$, $c_5' = c_5 + 8c_6 - 16c_3$ and $c_6' = c_6 + 2c_4 - 2c_3$. Thus,

$$\widehat{JU}_4/p^2 \cdot \overline{R}_0 \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/64$$

generated by $g_1 + p^2 \cdot \overline{R}_0$, $g_6 + p^2 \cdot \overline{R}_0$, $g_2 + p^2 \cdot \overline{R}_0$ and $g_5 + p^2 \cdot \overline{R}_0$ respectively, and $g_3, g_4 \in p^2 \cdot \overline{R}_0$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_4$, $g_2 = \overline{y}_4$, $g_3 = \overline{z}_4 + 2\overline{w}_4$, $g_4 = \overline{t}_4 + 2\overline{x}_4 + 16\overline{y}_4 - 2\overline{w}_4 + 16\overline{u}_4$, $g_5 = \overline{u}_4$ and $g_6 = \overline{w}_4 - 8\overline{u}_4$ and hence the result for $H^1_{(p)}(QU)_8$ follows.

For $H^1_{(p)}(QU)_{10} \cong \widehat{JU}_3/p^2 \cdot \overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_3/p^2 \cdot \overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_3 & \overline{y}_3 & \overline{z}_3 & \overline{t}_3 & \overline{u}_3 & \overline{w}_3 & \\
p^2 \cdot 1 : & 4 & 32 & 10 & 9 & 16 & 0 & r_0 \\
p^2 \cdot \overline{x}_0 : & 8 & 0 & 0 & 0 & 0 & 0 & r_1 \\
p^2 \cdot \overline{y}_0 : & 0 & 64 & 0 & 0 & 0 & 0 & r_2 \\
p^2 \cdot \overline{z}_0 : & 0 & 0 & 0 & -2 & 0 & 0 & r_3 \\
p^2 \cdot \overline{u}_0 : & 0 & 0 & 8 & 0 & 0 & 0 & r_4 \\
p^2 \cdot \overline{w}_0 : & 0 & 0 & 4 & 0 & 0 & -2 & r_5.
\end{array}$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 - r_4 + 4r_3]$, $r_1 = r_1^*$, $r_2 = r_2^*$,
$r_3 \longrightarrow [r_3^* = r_3 + 2r_0^* - r_1^* - r_2^* - r_5]$
$r_4 \longrightarrow [r_4^* = r_4 - 2r_5 - 2r_3^*]$,
$r_5 \longrightarrow [r_5^* = r_5 + r_3^*]$,
and then we obtain (Step $*$) and the required result as

$$\begin{array}{c|cccccc|c}
 & \overline{x}_3 & \overline{y}_3 & \overline{z}_3 & \overline{t}_3 & \overline{u}_3 & \overline{w}_3 & \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \\
r_0^* & 4 & 32 & 2 & 1 & 16 & 0 & \\
r_1^* & 8 & 0 & 0 & 0 & 0 & 0 & \\
r_2^* & 0 & 64 & 0 & 0 & 0 & 0 & \\
r_3^* & 0 & 0 & 0 & 0 & 32 & 2 & \\
r_4^* & 0 & 0 & 0 & 0 & -64 & 0 & \\
r_5^* & 0 & 0 & 4 & 0 & 32 & 0 &
\end{array}
\cong
\begin{array}{c|cccccc|c}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' & \\
r_0^* & 0 & 0 & 0 & 1 & 0 & 0 & \\
r_1^* & 8 & 0 & 0 & 0 & 0 & 0 & \\
r_2^* & 0 & 64 & 0 & 0 & 0 & 0 & \\
r_3^* & 0 & 0 & 0 & 0 & 0 & 2 & \\
r_4^* & 0 & 0 & 0 & 0 & -64 & 0 & \\
r_5^* & 0 & 0 & 4 & 0 & 0 & 0 &
\end{array},$$

where $c_1' = c_1 - 4c_4$, $c_2' = c_2 - 32c_4$, $c_3' = c_3 - 2c_4$, $c_4' = c_4$, $c_5' = c_5 - 8c_3 + 16c_6$ and $c_6' = c_6$. Thus,

$$\widehat{JU}_3/p^2 \cdot \overline{R}_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/64 \oplus \mathbb{Z}/64$$

generated by $g_6 + p^2 \cdot \overline{R}_0$, $g_3 + p^2 \cdot \overline{R}_0$, $g_1 + p^2 \cdot \overline{R}_0$, $g_2 + p^2 \cdot \overline{R}_0$ and $g_5 + p^2 \cdot \overline{R}_0$ respectively, and $g_4 \in p^2 \cdot \overline{R}_0$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_3$, $g_2 = \overline{y}_3$, $g_3 = \overline{z}_3 + 8\overline{u}_3$, $g_4 = \overline{t}_3 + 4\overline{x}_3 + 32\overline{y}_3 + 2\overline{z}_3 + 16\overline{u}_3$, $g_5 = \overline{u}_3$ and $g_6 = \overline{w}_3 + 16\overline{u}_3$ and hence the result for $H^1_{(p)}(QU)_{10}$ follows.

For $H^1_{(p)}(QU)_{12} \cong \widehat{JU}_2/p^2 \cdot \overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_2/p^2 \cdot \overline{R}_0$ as

$$\begin{array}{r|cccccc|l}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2 & \overline{w}_2 & \\
p^2 \cdot 1 : & -28 & 64 & -15 & 2 & -32 & 64 & r_0 \\
p^2 \cdot \overline{x}_0 : & 8 & 0 & 0 & 0 & 0 & 0 & r_1 \\
p^2 \cdot \overline{y}_0 : & 0 & 128 & 0 & 0 & 0 & 0 & r_2 \\
p^2 \cdot \overline{z}_0 : & 0 & 0 & 2 & -4 & 0 & 0 & r_3 \\
p^2 \cdot \overline{u}_0 : & 0 & 0 & 8 & 0 & 0 & 0 & r_4 \\
p^2 \cdot \overline{w}_0 : & 0 & 0 & 4 & 0 & -2 & 0 & r_5.
\end{array}$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 + 4r_1 + 2r_4 + 8r_4']$, $r_1 = r_1^*$, $r_2 = r_2^*$,
$r_3 \longrightarrow [r_3' = r_3 - 2r_0^* + r_1^* + r_2^*] \longrightarrow [r_3^* = -r_3']$
$r_4 \longrightarrow [r_4' = r_4 - 2r_5] \longrightarrow [r_4^* = r_4' + 2r_5']$,
$r_5 \longrightarrow [r_5' = r_5 - 4r_0^* + 2r_1^* + 2r_2^* - r_3'] \longrightarrow [r_5^* = -r_5']$,
and then we obtain (Step $*$) and the required result as

$$\begin{array}{c|cccccc|c}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2 & \overline{w}_2 & \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \\
r_0^* & 4 & 64 & 1 & 2 & 0 & 64 & \\
r_1^* & 8 & 0 & 0 & 0 & 0 & 0 & \\
r_2^* & 0 & 128 & 0 & 0 & 0 & 0 & \\
r_3^* & 0 & 0 & 0 & 8 & 0 & 128 & \\
r_4^* & 0 & 0 & 0 & 0 & 0 & 256 & \\
r_5^* & 0 & 0 & 0 & 0 & 2 & 128 &
\end{array}
\cong
\begin{array}{c|cccccc|c}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' & \\
r_0^* & 0 & 0 & 1 & 0 & 0 & 0 & \\
r_1^* & 8 & 0 & 0 & 0 & 0 & 0 & \\
r_2^* & 0 & 128 & 0 & 0 & 0 & 0 & \\
r_3^* & 0 & 0 & 0 & 8 & 0 & 0 & \\
r_4^* & 0 & 0 & 0 & 0 & 0 & 256 & \\
r_5^* & 0 & 0 & 0 & 0 & 2 & 0 &
\end{array},$$

where $c_1' = c_1 - 4c_3$, $c_2' = c_2 - 64c_3$, $c_3' = c_3$, $c_4' = c_4 - 2c_3$, $c_5' = c_5$ and $c_6' = c_6 - 16c_4 - 64c_5 - 32c_3$. Thus,

$$\widehat{JU}_2/p^2 \cdot \overline{R}_0 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/128 \oplus \mathbb{Z}/256$$

generated by $g_5 + p^2 \cdot \overline{R}_0$, $g_1 + p^2 \cdot \overline{R}_0$, $g_4 + p^2 \cdot \overline{R}_0$, $g_2 + p^2 \cdot \overline{R}_0$ and $g_6 + p^2 \cdot \overline{R}_0$ respectively, and $g_3 \in p^2 \cdot \overline{R}_0$. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_2$, $g_2 = \overline{y}_2$, $g_3 = \overline{z}_2 + 4\overline{x}_2 + 64\overline{y}_2 + 2\overline{t}_2 + 64\overline{w}_2$, $g_4 = \overline{t}_2 + 16\overline{w}_2$, $g_5 = \overline{u}_2 + 64\overline{w}_2$ and $g_6 = \overline{w}_2$ and hence the result for $H^1_{(p)}(QU)_{12}$ follows.

Now, we are going to prove the general case which separates to 4 types. Let $n \geq 3$, $r_0 = p^n \cdot 1$, $r_1 = p^n \cdot \overline{x}_0$, $r_2 = p^n \cdot \overline{y}_0$, $r_3 = p^n \cdot \overline{z}_0$, $r_4 = p^n \cdot \overline{u}_0$ and $r_5 = p^n \cdot \overline{w}_0$. Beginning with $H^1_{(p)}(QU)_{2(4n-5)} \cong \widehat{JU}_5/p^n \cdot \overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_5/p^n \cdot \overline{R}_0$ as

$$\left|\begin{array}{cccccc}
\overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 \\
2 \cdot 4^{n-2} & 8 \cdot 16^{n-2} & 2 \cdot 16^{n-2} + 4^{n-2} & -(-2)^{n-2} & 4 \cdot 16^{n-2} & 2 \cdot 16^{n-2} \\
4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 16^{n-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -(-2)^{n-1} & 0 & 0 \\
0 & 0 & 4^{n-1} & 0 & 0 & 0 \\
0 & 0 & 2 \cdot 4^{n-2} & 0 & 0 & -(-2)^{n-2}
\end{array}\right|.$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 - 2 \cdot 4^{n-3} r_4]$, $r_1 = r_1^*$, $r_2 = r_2^*$, $r_5 = r_5^*$
$r_3 \longrightarrow [r_3' = r_3 + 2r_0^* - r_1^* - r_2^*] \longrightarrow [r_3^* = r_3' + 4(-8)^{n-2} r_6 - 2(-8)^{n-2} r_4]$
$r_4 \longrightarrow [r_4^* = -(r_4 - 2r_3^*)]$,
and then we obtain (Step $*$) and the required result as

$$\begin{array}{r|cccccc|}
 & \overline{x}_5 & \overline{y}_5 & \overline{z}_5 & \overline{t}_5 & \overline{u}_5 & \overline{w}_5 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_0^* & 2 \cdot 4^{n-2} & 8 \cdot 16^{n-2} & 4^{n-2} & -(-2)^{n-2} & 4 \cdot 16^{n-2} & 2 \cdot 16^{n-2} \\
r_1^* & 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 16^{n-1} & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 2 \cdot 4^{n-2} & 0 & 8 \cdot 16^{n-2} & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 16^{n-1} & 0 \\
r_5^* & 0 & 0 & 2 \cdot 4^{n-2} & 0 & 0 & -(-2)^{n-2}
\end{array} \cong$$

$$\begin{array}{r|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & 0 & -(-2)^{n-2} & 0 & 0 \\
r_1^* & 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 16^{n-1} & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 2 \cdot 4^{n-2} & 0 & 0 & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 16^{n-1} & 0 \\
r_5^* & 0 & 0 & 0 & 0 & 0 & -(-2)^{n-2}
\end{array},$$

where

$$\begin{aligned} c_1' =& \ c_1 + 2(-2)^{n-2}c_4 \\ c_2' =& \ c_2 + 8(-8)^{n-2}c_4 \\ c_3' =& \ c_3 - (-2)^{n-1}c_6 + [(-2)^{n-2} + 4(16)^{n-2}]c_4 \\ c_4' =& \ c_4 \\ c_5' =& \ c_5 - 4^{n-1}c_3 + (-8)^{n-1}c_6 - 4^{3n-4}c_4 \\ c_6' =& \ c_6 + 2(-8)^{n-2}c_4. \end{aligned}$$

Thus,

$$\widehat{JU}_5/p^n \cdot \overline{R}_0 \cong \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2\cdot 4^{n-2} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/16^{n-1} \oplus \mathbb{Z}/16^{n-1}$$

generated by $g_4 + p^n \cdot \overline{R}_0$, $g_6 + p^n \cdot \overline{R}_0$, $g_3 + p^n \cdot \overline{R}_0$, $g_1 + p^n \cdot \overline{R}_0$, $g_2 + p^n \cdot \overline{R}_0$ and $g_5 + p^n \cdot \overline{R}_0$ respectively. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_5$, $g_2 = \overline{y}_5$, $g_3 = \overline{z}_5 + 4^{n-1}\overline{u}_5$, $g_5 = \overline{u}_5$, $g_6 = \overline{w}_5 + (-2)^{n-1}\overline{z}_5$ and

$$g_4 = \overline{t}_5 + (-2)^{n-1}\overline{x}_5 + (-8)^{n-1}\overline{y}_5 - (-2)^{n-2}\overline{z}_5 - 2(-8)^{n-2}\overline{w}_5 + 4(-8)^{n-2}\overline{u}_5.$$

Note that $2(-8)^{n-2}\overline{w}_5$ is zero and hence the result for $H^1_{(p)}(QU)_{2(4n-5)}$ follows.

Next, $H^1_{(p)}(QU)_{2(4n-4)} \cong \widehat{JU}_4/p^n \cdot \overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_4/p^n \cdot \overline{R}_0$ as

$$\left|\begin{array}{cccccc} \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \overline{w}_4 \\ e & 16^{n-1} & 4^{n-2} - 4\cdot 16^{n-2} & l & 16^{n-1} & 0 \\ 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 2\cdot 16^{n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-2)^{n-1} & 0 & 0 \\ 0 & 0 & 4^{n-1} & 0 & 0 & 0 \\ 0 & 0 & 2\cdot 4^{n-2} + (-2)^{n-2} & 0 & 0 & (-2)^{n-1} \end{array}\right|,$$

where $e = 2\cdot 4^{n-2} - 8\cdot 16^{n-2}$ and $l = (-2)^{n-2} + 8\cdot 16^{n-2}$. Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 + 2\cdot 4^{n-2}r_1^* + 4^{n-2}r_4 + 4(-8)^{n-2}r_3]$, $r_1 = r_1^*$, $r_2 = r_2^*$,
$r_3 \longrightarrow [r_3' = r_3 + 2r_0^* - r_1^* - r_2^*] \longrightarrow [r_3^* = r_3' + (-2)^{n-1}r_5^* + (-2)^{n-2}r_4^*]$
$r_4 \longrightarrow [r_4^* = -(r_4 - 2r_3')]$,
$r_5 \longrightarrow [r_5' = r_5 - 2r_0^* + r_1^* + r_2^* - r_3] \longrightarrow [r_5^* = r_5' + r_4^*]$,
and then we obtain (Step $*$) and the required result as

$$\begin{array}{c|cccccc|c} & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \overline{w}_4 & \\ & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \\ r_0^* & 2\cdot 4^{n-2} & 16^{n-1} & 4^{n-2} & (-2)^{n-2} & 16^{n-1} & 0 & \\ r_1^* & 4^{n-1} & 0 & 0 & 0 & 0 & 0 & \\ r_2^* & 0 & 2\cdot 16^{n-1} & 0 & 0 & 0 & 0 & \cong \\ r_3^* & 0 & 0 & 0 & 0 & 2\cdot 16^{n-1} & 4^{n-1} & \\ r_4^* & 0 & 0 & 0 & 0 & 4\cdot 16^{n-1} & 0 & \\ r_5^* & 0 & 0 & (-2)^{n-2} & 0 & 2\cdot 16^{n-1} & (-2)^{n-1} & \end{array}$$

$$\begin{array}{c|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c'_1 & c'_2 & c'_3 & c'_4 & c'_5 & c'_6 \\
r_0^* & 0 & 0 & 0 & (-2)^{n-2} & 0 & 0 \\
r_1^* & 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 2\cdot 16^{n-1} & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 0 & 0 & 4^{n-1} \\
r_4^* & 0 & 0 & 0 & 0 & 4\cdot 16^{n-1} & 0 \\
r_5^* & 0 & 0 & (-2)^{n-2} & 0 & 0 & 0
\end{array},$$

where

$$\begin{aligned}
c'_1 &= c_1 + (-2)^{n-1}c_4 \\
c'_2 &= c_2 + 2(-8)^{n-1}c_4 \\
c'_3 &= c_3 - (-2)^{n-2}c_4 \\
c'_4 &= c_4 \\
c'_5 &= c_5 + (4(-8)^{n-1} - 4^n)c_3 + 2\cdot 16^{n-1}c_4 - 2\cdot 4^{n-1}c_6 \\
c'_6 &= c_6 + 2c_3 + (-2)^{n-1}c_4.
\end{aligned}$$

Thus,

$$\widehat{JU}_4/p^n\cdot\overline{R}_0 \cong \mathbb{Z}/2^{n-2}\oplus\mathbb{Z}/2^{n-2}\oplus\mathbb{Z}/4^{n-1}\oplus\mathbb{Z}/4^{n-1}\oplus\mathbb{Z}/2\cdot 16^{n-1}\oplus\mathbb{Z}/4\cdot 16^{n-1}$$

generated by $g_3 + p^n\cdot\overline{R}_0$, $g_4 + p^n\cdot\overline{R}_0$, $g_1 + p^n\cdot\overline{R}_0$, $g_6 + p^n\cdot\overline{R}_0$, $g_2 + p^n\cdot\overline{R}_0$ and $g_5 + p^n\cdot\overline{R}_0$ respectively. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_4$, $g_2 = \overline{y}_4$, $g_3 = \overline{z}_4 - 2\overline{w}_4 - 4(-8)^{n-1}\overline{u}_4$, $g_5 = \overline{u}_4$, $g_6 = \overline{w}_4 + 2\cdot 4^{n-1}\overline{u}_4$,

$$g_4 = \overline{t}_4 + (-2)^{n-2}(2\overline{x}_4 + \overline{z}_4) - 2(-8)^{n-1}(\overline{y}_4 + \overline{u}_4),$$

and hence the result for $H^1_{(p)}(QU)_{2(4n-4)}$ follows.

Next, $H^1_{(p)}(QU)_{2(4n-3)} \cong \widehat{JU}_3/p^n\cdot\overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_3/p^n\cdot\overline{R}_0$ as

$$\left|\begin{array}{cccccc}
\overline{x}_3 & \overline{y}_3 & \overline{z}_3 & \overline{t}_3 & \overline{u}_3 & \overline{w}_3 \\
4^{n-1} & 2\cdot 16^{n-1} & 2\cdot 4^{n-2} + 8\cdot 16^{n-2} & 2^{4n-5} + (-2)^{n-2} & 16^{n-1} & 0 \\
2\cdot 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 4\cdot 16^{n-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (-2)^{n-1} & 0 & 0 \\
0 & 0 & 2\cdot 4^{n-1} & 0 & 0 & 0 \\
0 & 0 & 4^{n-1} & 0 & 0 & (-2)^{n-1}
\end{array}\right|.$$

Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 - 4^{n-2}r_4 + 4(-8)^{n-2}r_3]$, $r_1 = r_1^*$, $r_2 = r_2^*$,
$r_3 \longrightarrow [r_3^* = r_3 + 2r_0^* - r_1^* - r_2^*]$
$r_4 \longrightarrow [r'_4 = r_4 - 2r_3^*] \longrightarrow [r_4^* = -r'_4]$,
$r_5 \longrightarrow [r_5^* = r_5 - 2r_3^* + r_4^*]$
and then we obtain (Step $*$) and the required result as

$$\begin{array}{c|cccccc|c}
 & \overline{x}_3 & \overline{y}_3 & \overline{z}_3 & \overline{t}_3 & \overline{u}_3 & \overline{w}_3 & \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & \\
r_0^* & 4^{n-1} & 2\cdot 16^{n-1} & 2\cdot 4^{n-2} & (-2)^{n-2} & 16^{n-1} & 0 & \\
r_1^* & 2\cdot 4^{n-1} & 0 & 0 & 0 & 0 & 0 & \\
r_2^* & 0 & 4\cdot 16^{n-1} & 0 & 0 & 0 & 0 & \cong \\
r_3^* & 0 & 0 & 4^{n-1} & 0 & 2\cdot 16^{n-1} & 0 & \\
r_4^* & 0 & 0 & 0 & 0 & 4\cdot 16^{n-1} & 0 & \\
r_5^* & 0 & 0 & -4^{n-1} & 0 & 0 & (-2)^{n-1} &
\end{array}$$

$$\begin{array}{c|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & 0 & (-2)^{n-2} & 0 & 0 \\
r_1^* & 2\cdot 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 4\cdot 16^{n-1} & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 4^{n-1} & 0 & 0 & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 4\cdot 16^{n-1} & 0 \\
r_5^* & 0 & 0 & 0 & 0 & 0 & (-2)^{n-1}
\end{array},$$

where

$$\begin{aligned}
c_1' &= c_1 - (-2)^n c_4 \\
c_2' &= c_2 + 4(-8)^{n-1} c_4 \\
c_3' &= c_3 + (-2)^{n-1} c_4 + (-2)^{n-1} c_6 \\
c_4' &= c_4 \\
c_5' &= c_5 - 8\cdot 4^{n-2} c_3 - 2(-8)^{n-1} c_6 \\
c_6' &= c_6.
\end{aligned}$$

Thus,

$$\widehat{JU}_3/p^n\cdot\overline{R}_0 \cong \mathbb{Z}/2^{n-2}\oplus\mathbb{Z}/2^{n-1}\oplus\mathbb{Z}/4^{n-1}\oplus\mathbb{Z}/2\cdot 4^{n-1}\oplus\mathbb{Z}/4\cdot 16^{n-1}\oplus\mathbb{Z}/4\cdot 16^{n-1}$$

generated by $g_4+p^n\cdot\overline{R}_0,\ g_6+p^n\cdot\overline{R}_0,\ g_3+p^n\cdot\overline{R}_0,\ g_1+p^n\cdot\overline{R}_0,\ g_2+p^n\cdot\overline{R}_0$ and $g_5+p^n\cdot\overline{R}_0$ respectively. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1=\overline{x}_3,\ g_2=\overline{y}_3,\ g_3=\overline{z}_3+2\cdot 4^{n-1}\overline{u}_3,\ g_5=\overline{u}_3,\ g_6=\overline{w}_3-(-2)^{n-1}\overline{z}_3,$

$$g_4=\overline{t}_3+(-2)^n\overline{x}_3-4(-8)^{n-1}\overline{y}_3-(-2)^{n-1}\overline{z}_3-2(-8)^{n-1}\overline{u}_3,$$

and hence the result for $H^1_{(p)}(QU)_{2(4n-3)}$ follows.

Finally, $H^1_{(p)}(QU)_{2(4n-2)}\cong\widehat{JU}_2/p^n\cdot\overline{R}_0$, we can represent matrix for the calculation of $\widehat{JU}_2/p^n\cdot\overline{R}_0$ as

$$\left|\begin{array}{cccccc}
\overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2 & \overline{w}_2 \\
4^{n-1}-2\cdot 16^{n-1} & 4\cdot 16^{n-1} & h & -(-2)^{n-1} & -2\cdot 16^{n-1} & 4\cdot 16^{n-1} \\
2\cdot 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 8\cdot 16^{n-1} & 0 & 0 & 0 & 0 \\
0 & 0 & -(-2)^{n-1} & 2(-2)^{n-1} & 0 & 0 \\
0 & 0 & 2\cdot 4^{n-1} & 0 & 0 & 0 \\
0 & 0 & 4^{n-1} & 0 & (-2)^{n-1} & 0
\end{array}\right|,$$

where $h = 2 \cdot 4^{n-2} - 16^{n-1} - (-2)^{n-2}$. Now, we do row operations by
$r_0 \longrightarrow [r_0^* = r_0 + 4^{n-1}r_1^* + 2 \cdot 4^{n-2}r_4 - (-2)^{n-2}r_3 - (-8)^{n-1}r_4'],\ r_1 = r_1^*,\ r_2 = r_2^*,$
$r_3 \longrightarrow [r_3^* = r_3 + 2r_0^* - r_1^* - r_2^*]$
$r_4 \longrightarrow [r_4' = r_4 - 2r_5] \longrightarrow [r_4^* = r_4' + 2r_5^*],$
$r_5 \longrightarrow [r_5^* = r_5 + 4(-2)^{n-2}r_0^* + (-2)^{n-1}(r_1^* + r_2^*) - (1 - (-2)^{n-1})r_3^*],$
and then we obtain (Step $*$) and the required result as;

$$
\begin{array}{c|cccccc|}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2 & \overline{w}_2 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
r_0^* & 4^{n-1} & 4 \cdot 16^{n-1} & -(-2)^{n-2} & 4^{n-1} - (-2)^{n-1} & 0 & 4 \cdot 16^{n-1} \\
r_1^* & 2 \cdot 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 8 \cdot 16^{n-1} & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 2 \cdot 4^{n-1} & 0 & 8 \cdot 16^{n-1} \\
r_4^* & 0 & 0 & 0 & 0 & 0 & -16^n \\
r_5^* & 0 & 0 & 0 & 0 & (-2)^{n-1} & -8 \cdot 16^{n-1}
\end{array} \cong
$$

$$
\begin{array}{c|cccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 & g_6 \\
 & c_1' & c_2' & c_3' & c_4' & c_5' & c_6' \\
r_0^* & 0 & 0 & -(-2)^{n-2} & 0 & 0 & 0 \\
r_1^* & 2 \cdot 4^{n-1} & 0 & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 8 \cdot 16^{n-1} & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 2 \cdot 4^{n-1} & 0 & 0 \\
r_4^* & 0 & 0 & 0 & 0 & 0 & -16^n \\
r_5^* & 0 & 0 & 0 & 0 & (-2)^{n-1} & 0
\end{array},
$$

where

$$
\begin{aligned}
c_1' &= c_1 + (-2)^n c_3 \\
c_2' &= c_2 + (-8)^n c_3 \\
c_3' &= c_3 \\
c_4' &= c_4 + (2 + (-2)^n) c_3 \\
c_5' &= c_5 \\
c_6' &= c_6 - 2 \cdot 4^n c_3 - (-8)^n c_5 - 4^n c_4.
\end{aligned}
$$

Thus,

$$\widehat{JU}_2/p^n \cdot \overline{R}_0 \cong \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-1} \oplus \mathbb{Z}/2 \cdot 4^{n-1} \oplus \mathbb{Z}/2 \cdot 4^{n-1} \oplus \mathbb{Z}/8 \cdot 16^{n-1} \oplus \mathbb{Z}/16^n$$

generated by $g_3 + p^n \cdot \overline{R}_0,\ g_5 + p^n \cdot \overline{R}_0,\ g_1 + p^n \cdot \overline{R}_0,\ g_4 + p^n \cdot \overline{R}_0,\ g_2 + p^n \cdot \overline{R}_0$ and $g_6 + p^n \cdot \overline{R}_0$ respectively. By the same process as the case $H^1_{(p)}(QU)_{-2}$, we get that $g_1 = \overline{x}_2,\ g_2 = \overline{y}_2,\ g_4 = \overline{t}_2 + 4^n \overline{w}_2,\ g_5 = \overline{u}_2 + (-8)^n \overline{w}_2,\ g_6 = \overline{w}_2,$

$$g_3 = \overline{z}_2 - (-2)^n \overline{x}_2 - (-8)^n \overline{y}_2 - (-8)^n \overline{w}_2 - (2 + (-2)^n)\overline{t}_2,$$

and hence the result for $H^1_{(p)}(QU)_{2(4n-2)}$ follows. □

Accordingly, we reach to the $E^{1\frac{1}{2}}$-page of Greenlees spectral sequence as;

					degree(t)
$\vdots$	$\vdots$		$\vdots$	$\vdots$	19
0	2^6	$\xleftarrow{}\ \delta_9$	$[2]\oplus[4]\oplus[16]\oplus[32]\oplus[1024]\oplus[1024]$	$\mathbb{Z}\cdot\rho$	18
0	0		0	0	17
0	2^5	$\xleftarrow{}\ \delta_8$	$[2]\oplus[2]\oplus[16]\oplus[16]\oplus[512]\oplus[1024]$	$\mathbb{Z}\cdot\rho$	16
0	0		0	0	15
0	2^5	$\xleftarrow{}\ \delta_7$	$[2]\oplus[2]\oplus[8]\oplus[16]\oplus[256]\oplus[256]$	$\mathbb{Z}\cdot\rho$	14
0	0		0	0	13
0	2^4	$\xleftarrow{}\ \delta_6$	$[2]\oplus[8]\oplus[8]\oplus[128]\oplus[256]$	$\mathbb{Z}\cdot\rho$	12
0	0		0	0	11
0	2^4	$\xleftarrow{}\ \delta_5$	$[2]\oplus[4]\oplus[8]\oplus[64]\oplus[64]$	$\mathbb{Z}\cdot\rho$	10
0	0		0	0	9
0	2^3	$\xleftarrow{}\ \delta_4$	$[4]\oplus[4]\oplus[32]\oplus[64]$	$\mathbb{Z}\cdot\rho$	8
0	0		0	0	7
0	2^3	$\xleftarrow{}\ \delta_3$	$[2]\oplus[4]\oplus[16]\oplus[16]$	$\mathbb{Z}\cdot\rho$	6
0	0		0	0	5
0	2^2	$\xleftarrow{}\ \delta_2$	$[2]\oplus[8]\oplus[16]$	$\mathbb{Z}\cdot\rho$	4
0	0		0	0	3
0	2^2	$\xleftarrow{}\ \delta_1$	$[2]\oplus[2]\oplus[4]$	$\mathbb{Z}\cdot\rho$	2
0	0		0	0	1
0	2	$\xleftarrow{}\ \delta_0$	$[2]$	$\mathbb{Z}\cdot\rho$	0
0	0		0	0	-1
0	2	$\xleftarrow{}\ \delta_{-1}$	$[2]$	0	-2
0	0		0	0	-3
0	0		0	0	-4
$H_I^{\epsilon\geq 3}(R)$	$H^2_{I'}(PC)\cdot\tau$	$\xleftarrow{\ \delta\ }$	$H^1_{(p)}(QU)$	$H^0_I(R)$	

where $[n]:=$ cyclic group of order n, $2^r:=$ elementary abelian group of rank r.

Figure 3.3: The $E^{1\frac{1}{2}}$-page of Greenlees spectral sequence for $ku_*(BSD_{16})$

§ 3.4 E^2-page

The purpose here is to find $H^1_I(R)$ and $H^2_I(R)$ which is equivalent to find $\ker(\delta)$ and $\mathrm{coker}(\delta)$ respectively. Recall from lemma 3.2.2 that $H^2_{I'}(PC)\cdot\tau = (\mathbb{F}_2[b,d])^\vee\cdot\tau$ on which b,d act freely and a acts as zero, where the degree of a,b,d are $-2,-2,-4$ respectively. For $v\in M$, we write it image in $M^\vee$ by $\frac{1}{v}$. Since $H^2_{I'}(PC)\cdot\tau$ is annihilated by $2,a$ and δ is a module homomorphism over R, the restriction of δ to $2(H^1_{(p)}(QU))$ is a zero map. Note that the structure of $(H^2_{I'}(PC)\cdot\tau)^\vee$ is simpler than the undual one, so instead of calculating R-module homomorphism $\delta: H^1_{(p)}(QU)\longrightarrow H^2_{I'}(PC)\cdot\tau$ directly, we will do on the opposite side first, i.e. $\mathbb{F}_2[b,d]$-module homomorphism

$$(\overline{\delta})^{\vee} : (H^2_{I'}(PC)\cdot\tau)^{\vee} \longrightarrow [H^1_{(p)}(QU)/2(H^1_{(p)}(QU))]^{\vee}\,.$$

Moreover, we note that ku_* is a connective spectrum and then $ku_t(BSD_{16}) = 0$ for all $t < 0$. It follows, by $E^{1\frac{1}{2}}$-page, that δ_{-1} and δ_0 are isomorphism and δ_1 is surjective. Thus

$$(\overline{\delta}_{-1})^{\vee}(\tfrac{bd}{\tau}) = \tfrac{1}{\widetilde{x}_{-1}} \text{ and } (\overline{\delta}_0)^{\vee}(\tfrac{b^2d}{\tau}) = \tfrac{1}{\widetilde{x}_0}\,.$$

So as to identify all $(\overline{\delta}_i)^{\vee}$, it suffices to find the structure of $[H^1_{(p)}(QU)/2(H^1_{(p)}(QU))]^{\vee}$ as a module over $\mathbb{F}_2[b,d]$, i.e. how do b and d act? To do this, we simply use the character table of $\widehat{JU}_\epsilon$ for $\epsilon \in \{2,3,4,5\}$ and lemma 3.3.4 in the last section. Here we also record the action of v and a because this will be used in the calculation of R action on $ku_*(BSD_{16})$.

Generator	Image of a	Image of b	Image of d	where $(k>0)$
$\widetilde{x}_i$	$\begin{cases}\widetilde{x}_{i-1},\\ 2\widetilde{x}_{i-1},\end{cases}$	$\begin{cases}\widetilde{x}_{i-1},\\ 2\widetilde{x}_{i-1},\end{cases}$	$\begin{cases}\widetilde{x}_{i-2},\\ \widetilde{x}_{i-2},\end{cases}$	$\begin{cases} i \text{ is odd;}\\ i \text{ is even.}\end{cases}$
$\widetilde{y}_i$	0	$\widetilde{y}_{i-1}$	0	for all i.
$\widetilde{z}_i$	0	0	$\widetilde{z}_{i-2}$	for all i.
$\widetilde{t}_i$	$\begin{cases}2\widetilde{t}_{i-1}-\widetilde{z}_{i-1},\\ -2\widetilde{t}_{i-1},\\ 2\widetilde{t}_{i-1},\\ \widetilde{t}_{i-1},\end{cases}$	$\begin{cases}0,\\0,\\0,\\0,\end{cases}$	$\begin{cases}-\widetilde{w}_{i-2},\\ \widetilde{u}_{i-2},\\ -\widetilde{w}_{i-2},\\ \widetilde{z}_{i-2}-\widetilde{w}_{i-2},\end{cases}$	$\begin{cases} i=4k-5;\\ i=4k-4;\\ i=4k-3;\\ i=4k-2.\end{cases}$
$\widetilde{u}_i$	$\begin{cases}\widetilde{u}_{i-1},\\ \widetilde{x}_{i-1}+\widetilde{t}_{i-1},\\ -\widetilde{t}_{i-1},\\ 2\widetilde{w}_{i-1},\end{cases}$	$\begin{cases}0,\\ \widetilde{x}_{i-1},\\0,\\0,\end{cases}$	$\begin{cases}\widetilde{u}_{i-2}+\widetilde{t}_{i-2},\\ \widetilde{w}_{i-2}-\widetilde{u}_{i-2},\\ \widetilde{u}_{i-2}+\widetilde{w}_{i-2},\\ \widetilde{t}_{i-2},\end{cases}$	$\begin{cases} i=4k-5;\\ i=4k-4;\\ i=4k-3;\\ i=4k-2.\end{cases}$
$\widetilde{w}_i$	$\begin{cases}-2\widetilde{u}_{i-1},\\ \widetilde{w}_{i-1},\\ 2\widetilde{w}_{i-1}-\widetilde{z}_{i-1},\\ \widetilde{x}_{i-1}+\widetilde{w}_{i-1},\end{cases}$	$\begin{cases}0,\\0,\\0,\\ \widetilde{x}_{i-1},\end{cases}$	$\begin{cases}-\widetilde{t}_{i-2},\\ \widetilde{z}_{i-2}-\widetilde{t}_{i-2},\\ -\widetilde{t}_{i-2},\\ \widetilde{u}_{i-2}+\widetilde{t}_{i-2},\end{cases}$	$\begin{cases} i=4k-5;\\ i=4k-4;\\ i=4k-3;\\ i=4k-2.\end{cases}$

Table 3.4 : The action of a, b and d on $H^1_{(p)}(QU)$.

Generator	Image of v , $k \geq 0$
$\widetilde{x}_i$	$\begin{cases} \widetilde{x}_{i+1}, & \text{if } i \text{ is odd;} \\ 2\widetilde{x}_{i+1}, & \text{if } i \text{ is even.} \end{cases}$
$\widetilde{y}_i$	$2\widetilde{y}_{i+1}$ for all $i \geq -1$.
$\widetilde{z}_i$	$\begin{cases} \widetilde{z}_{i+1}, & \text{if } i \text{ is odd;} \\ 2\widetilde{z}_{i+1}, & \text{if } i \text{ is even.} \end{cases}$
$\widetilde{t}_i$	$\begin{cases} -\widetilde{t}_{i+1}, & \text{if } i = 4k-5; \\ \widetilde{t}_{i+1}, & \text{if } i = 4k-4; \\ 2\widetilde{t}_{i+1} - \widetilde{z}_{i+1}, & \text{if } i = 4k-3; \\ \widetilde{t}_{i+1} + \widetilde{z}_{i+1}, & \text{if } i = 4k-2. \end{cases}$
$\widetilde{u}_i$	$\begin{cases} 4\widetilde{u}_{i+1} + 2\widetilde{t}_{i+1} - \widetilde{w}_{i+1} - \widetilde{z}_{i+1} - 2\widetilde{x}_{i+1}, & \text{if } i = 4k-5; \\ \widetilde{u}_{i+1} - \widetilde{z}_{i+1} - \widetilde{x}_{i+1}, & \text{if } i = 4k-4; \\ 4\widetilde{w}_{i+1} - 2\widetilde{u}_{i+1} - \widetilde{z}_{i+1} - \widetilde{t}_{i+1} - 2\widetilde{x}_{i+1}, & \text{if } i = 4k-3; \\ -\widetilde{w}_{i+1}, & \text{if } i = 4k-2. \end{cases}$
$\widetilde{w}_i$	$\begin{cases} 2\widetilde{w}_{i+1} - \widetilde{z}_{i+1}, & \text{if } i = 4k-5; \\ \widetilde{w}_{i+1} + \widetilde{z}_{i+1}, & \text{if } i = 4k-4; \\ \widetilde{u}_{i+1}, & \text{if } i = 4k-3; \\ \widetilde{u}_{i+1} + \widetilde{z}_{i+1} + \widetilde{x}_{i+1}, & \text{if } i = 4k-2. \end{cases}$

Table 3.5 : The action of v on $H^1_{(p)}(QU)$.

Note that we can view $H^1_{(p)}(QU)_t/2H^1_{(p)}(QU)_t$ for each t, as a vector space over $\mathbb{F}_2$ and b, d as a linear transformation among them. Hereby, the module structure of $[H^1_{(p)}(QU)/2H^1_{(p)}(QU))]^\vee := M^\vee$ over $\mathbb{F}_2[b,d]$ given by

$$b \cdot [\tfrac{1}{\widetilde{\alpha}_i}] = [b^\vee(\tfrac{1}{\widetilde{\alpha}_i})] \text{ and } d \cdot [\tfrac{1}{\widetilde{\alpha}_i}] = [d^\vee(\tfrac{1}{\widetilde{\alpha}_i})]$$

for each $[\frac{1}{\widetilde{\alpha}_i}] \in M^\vee$, can be done by Table 3.4 and the fact that;

Lemma 3.4.1. *Let V and W be an finite vector space over a field k and T be a linear transformation, $T : V \longrightarrow W$. If T is represented by a matrix A, then $T^\vee : W^\vee \longrightarrow V^\vee$ is represented by transpose of matrix A.*

Proof. Suppose $\{v_1, ..., v_n\}$ and $\{w_1, ..., w_m\}$ are an basis for V and W respectively. The natural basis for $V^\vee$ and $W^\vee$ are $\{v_1^*, ..., v_n^*\}$ and $\{w_1^*, ..., w_n^*\}$, where $v_i^*(v_j) = \delta_{ij}$ and $w_l^*(w_j) = \delta_{lj}$, Kronecker delta function. Let A be a matrix representing for T, i.e. $T(v_i) = \sum A_{ij} w_j$, and B for $T^\vee$, i.e.$T^\vee(w_i^*) = \sum B_{ij} v_j^*$. Hence, $B_{ij} = (B_{i1}v_1^* + ... + B_{in}v_n^*)(v_j) = T^\vee(w_i^*)(v_j) = w_i^* \circ T(v_j) = w_i^*(A_{j1}w_1 + ... + A_{jm}w_m) = A_{ji}$. $\square$

Now, by setting $\alpha_i^* = [\frac{1}{\widetilde{\alpha}_i}]$ for each $\alpha = x, y, z, t, u, w$, we get that, as an abelian group;

-$(M^v)_2$ is generated by x_{-1}^* with $x_{-1}^* = y_{-1}^* = z_{-1}^*$, $t_{-1}^* = x_{-1}^* - z_{-1}^*$ and $u_{-1}^* = w_{-1}^* = 0$

- $(M^v)_0$ is generated by y_0^* with $u_0^* = y_0^*$ and $x_0^* = z_0^* = t_0^* = w_0^* = 0$
- $(M^v)_{-2}$ is generated by x_1^*, y_1^*, z_1^* with $w_1^* = z_1^*$, $t_1^* = 0$ and $u_1^* + x_1^* + z_1^* = 0$
- $(M^v)_{-4}$ is generated by x_2^*, y_2^*, w_2^* with $t_2^* = x_2^*$ and $z_2^* = u_2^* = 0$
- $(M^v)_{-6}$ is generated by $x_3^*, y_3^*, t_3^*, u_3^*$ with $w_3^* = 0$ and $t_3^* = z_3^*$
- $(M^v)_{-8}$ is generated by $x_4^*, y_4^*, u_4^*, w_4^*$ with $z_4^* = t_4^* = 0$
- $(M^v)_{-10}$ is generated by $x_5^*, y_5^*, z_5^*, u_5^*, w_5^*$ with $t_5^* = 0$
- $(M^v)_{-12}$ is generated by $x_6^*, y_6^*, t_6^*, u_6^*, w_6^*$ with $z_6^* = 0$ and
- $(M^v)_{-2n}$ is generated by $x_n^*, y_n^*, z_n^*, t_n^*, u_n^*, w_n^*$ for $n > 6$.

Hence, by Table 3.4 and lemma 3.4.1, i.e. $b \cdot \alpha^* = b^\vee(\alpha^*)$ and $d \cdot \alpha^* = d^\vee(\alpha^*)$, we obtain the structure of $M^\vee$ over $\mathbb{F}_2[b,d]$ as;

Generator	Image of b, $(i > 0)$	Image of d
x_k^*	$\begin{cases} y_0^*, & \text{if } k=-1; \\ w_{2i+1}^*, & \text{if } k=2i; \\ u_{4i}^*, & \text{if } k=4i-1; \\ w_{4i-2}^*, & \text{if } k=4i-3. \end{cases}$	$\begin{cases} x_1^*+z_1^*, & \text{if } k=-1; \\ x_3^*+u_3^*, & \text{if } k=1; \\ x_4^*+w_4^*, & \text{if } k=2; \\ x_{i+2}^*, & \text{if } k=i>2. \end{cases}$
y_k^*	$y_{k+1}^*, \forall k$	$0, \forall k$
z_k^*	$0, \forall k$	$\begin{cases} u_3^*+t_3^*, & \text{if } k=1; \\ z_5^*+w_5^*, & \text{if } k=3; \\ z_{2i+1}^*, & \text{if } k=2i-1, i>2; \\ z_{4i-2}^*+t_{4i-2}^*, & \text{if } k=4i-4, i>2; \\ z_{4i}^*+w_{4i}^*, & \text{if } k=4i-2, i>2. \end{cases}$
t_k^*	$0, \forall k$	$\begin{cases} u_3^*+t_3^*, & \text{if } k=1; \\ w_{4i}^*, & \text{if } k=4i-2, i>1; \\ w_{4i+1}^*, & \text{if } k=4i-1, i>1; \\ u_{4i+2}^*+w_{4i+2}^*, & \text{if } k=4i, i>1; \\ u_{4i+3}^*+w_{4i+3}^*, & \text{if } k=4i+1, i>1. \end{cases}$
u_k^*	$0, \forall k$	$\begin{cases} u_{2i+3}^*, & \text{if } k=2i+1, i>0; \\ w_{4i+2}^*, & \text{if } k=4i, i>0; \\ t_{4i+4}^*+u_{4i+4}^*, & \text{if } k=4i+2, i>0. \end{cases}$
w_k^*	$0, \forall k$	$\begin{cases} u_{4i}^*, & \text{if } k=4i-2, i>0; \\ t_{4i+2}^*, & \text{if } k=4i, i>0; \\ t_{4i+3}^*, & \text{if } k=4i+1, i>0; \\ t_{4i+5}^*+u_{4i+5}^*, & \text{if } k=4i+3, i>0. \end{cases}$

Table 3.6: The module structure of $[H^1_{(p)}(QU)/2H^1_{(p)}(QU))]^\vee$ over $\mathbb{F}_2[b,d]$.

Consequently, we have:

Lemma 3.4.2. *The module map over* $\mathbb{F}_2[b,d]$, $\overline{\delta}^\vee : [H_I^2(TU)]^\vee \longrightarrow [H_I^1(QU)/2H_I^1(QU)]^\vee$ *is given by* $\overline{\delta}^\vee(q) = \widetilde{x}_{-1}^*$, $\overline{\delta}^\vee(dq) = \widetilde{x}_1^* + \widetilde{z}_1^*$, $\overline{\delta}^\vee(d^2q) = \widetilde{x}_3^* + \widetilde{t}_3^*$ *and*

$$\begin{array}{lcll}
\overline{\delta}^{\vee}(b^i q) & = & \widetilde{y}^*_{i-1} & \textit{for all } i \geq 1 \\
\overline{\delta}^{\vee}(d^{2i+1} q) & = & \widetilde{x}^*_{4i+1} + \widetilde{w}^*_{4i+1} + \widetilde{z}^*_{4i+1} & \textit{for all } i \geq 1 \\
\overline{\delta}^{\vee}(d^{2i} q) & = & \widetilde{x}^*_{4i-1} + \widetilde{t}^*_{4i-1} + \widetilde{z}^*_{4i-1} & \textit{for all } i \geq 2 \\
\overline{\delta}^{\vee}(bd^{2i-1} q) & = & \widetilde{w}^*_{4i-2} \textit{and } \overline{\delta}^{\vee}(bd^{2i} q) = \widetilde{u}^*_{4i} & \textit{for all } i \geq 1
\end{array}$$

and 0 *if otherwise, where* $q = \frac{bd}{\tau}$.

Accordingly, by using lemma 3.4.1 applying on $\overline{\delta}^{\vee}$, we can find $\ker(\delta) = H^1_I(R)$ and $\operatorname{coker}(\delta) = H^2_I(R)$ as:

Lemma 3.4.3. *As an abelian group,* $H^1_I(R)_i = 0$ *for* $i < 2$ *or* i *is odd, and*

- $H^1_I(R)_2 = \mathbb{Z}/2 < \widetilde{x}_1 + \widetilde{z}_1 > \oplus \mathbb{Z}/2 < 2\widetilde{y}_1 >$ *with* $\widetilde{w}_1 + \widetilde{z}_1 = 0 = \widetilde{t}_1$, *and* $\widetilde{u}_1 + (\widetilde{x}_1 + \widetilde{z}_1) + 2\widetilde{y}_1 = 0$.
- $H^1_I(R)_4 = \mathbb{Z}/2 < \widetilde{x}_2 > \oplus \mathbb{Z}/4 < 2\widetilde{y}_2 > \oplus \mathbb{Z}/8 < 2\widetilde{w}_2 >$ *with* $\widetilde{z}_2 = 8\widetilde{w}_2$, $\widetilde{t}_2 + \widetilde{x}_2 = 4(\widetilde{w}_2 + \widetilde{y}_2)$ *and* $\widetilde{u}_2 + 8\widetilde{w}_2 = 0$.
- $H^1_I(R)_6 = \mathbb{Z}/4 < \widetilde{x}_3 + \widetilde{t}_3 > \oplus \mathbb{Z}/8 < 2\widetilde{y}_3 > \oplus \mathbb{Z}/16 < \widetilde{u}_3 >$ *with* $\widetilde{w}_3 - 8\widetilde{u}_3 = 0$ *and* $\widetilde{t}_3 + \widetilde{z}_3 = 2\widetilde{x}_3 + 8\widetilde{y}_3 + 4\widetilde{u}_3$.
- $H^1_I(R)_8 = \mathbb{Z}/4 < \widetilde{x}_4 > \oplus \mathbb{Z}/4 < \widetilde{w}_4 - 8\widetilde{u}_4 > \oplus \mathbb{Z}/16 < 2\widetilde{y}_4 > \oplus \mathbb{Z}/32 < 2\widetilde{u}_4 >$ *with* $\widetilde{z}_4 + 2\widetilde{w}_4 = 0$ *and* $\widetilde{t}_4 + 2\widetilde{x}_4 - 2\widetilde{w}_4 + 16\widetilde{y}_4 + 16\widetilde{u}_4 = 0$.
- $H^1_I(R)_{10} = \mathbb{Z}/4 < \widetilde{z}_5 + \widetilde{w}_5 + 24\widetilde{u}_5 > \oplus \mathbb{Z}/8 < \widetilde{x}_5 + \widetilde{z}_5 + 8\widetilde{u}_5 > \oplus \mathbb{Z}/32 < 2\widetilde{y}_5 > \oplus \mathbb{Z}/64 < \widetilde{u}_5 >$ *with* $\widetilde{t}_5 + 4\widetilde{x}_5 + 32\widetilde{y}_5 + 2\widetilde{z}_5 + 16\widetilde{u}_5 = 0$.
- $H^1_I(R)_{12} = \mathbb{Z}/2 < \widetilde{u}_6 + 64\widetilde{w}_6 > \oplus \mathbb{Z}/8 < \widetilde{x}_6 > \oplus \mathbb{Z}/8 < \widetilde{t}_6 + 16\widetilde{w}_6 > \oplus \mathbb{Z}/64 < 2\widetilde{y}_6 > \oplus \mathbb{Z}/128 < 2\widetilde{w}_6 >$ *with* $\widetilde{z}_6 + 4\widetilde{x}_6 + 64\widetilde{y}_6 + 2\widetilde{t}_6 + 64\widetilde{w}_6 = 0$.
- $H^1_I(R)_{14} = \mathbb{Z}/2 < \widetilde{w}_7 + 4\widetilde{z}_7 > \oplus \mathbb{Z}/8 < \widetilde{\eta}_7 + \widetilde{z}_7 + 16\widetilde{u}_7 > \oplus \mathbb{Z}/16 < \widetilde{x}_7 + \widetilde{z}_7 + 16\widetilde{u}_7 > \oplus \mathbb{Z}/128 < 2\widetilde{y}_7 > \oplus \mathbb{Z}/256 < \widetilde{u}_7 >$ *with* $2\widetilde{\eta}_7 = 0$ *where* $\widetilde{\eta}_7 = \widetilde{t}_7 + 4\widetilde{x}_7 + 2\widetilde{z}_7 + 32\widetilde{y}_7 - 32\widetilde{u}_7$.

In general,

- *for* $n \geq 4$, $H^1_I(R)_{2(4n-5)} = \mathbb{Z}/2^{n-3} \oplus \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2 \cdot 4^{n-2} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/8 \cdot 16^{n-2} \oplus \mathbb{Z}/16^{n-1}$ *generated by* $2[\widetilde{t}_{4n-5} + (-2)^{n-1}\widetilde{x}_{4n-5} - (-2)^{n-2}\widetilde{z}_{4n-5} + (-8)^{n-1}\widetilde{y}_{4n-5} - 4(-8)^{n-2}\widetilde{u}_{4n-5}]$, $\widetilde{w}_{4n-5} + (-2)^{n-1}\widetilde{z}_{4n-5}$, $[\widetilde{z}_{4n-5} + (4)^{n-1}\widetilde{u}_{4n-5}] + [\widetilde{t}_{4n-5} + (-2)^{n-1}\widetilde{x}_{4n-5} - (-2)^{n-2}\widetilde{z}_{4n-5} + (-8)^{n-1}\widetilde{y}_{4n-5} + 4(-8)^{n-2}\widetilde{u}_{4n-5}]$, $\widetilde{x}_{4n-5} + [\widetilde{z}_{4n-5} + (4)^{n-1}\widetilde{u}_{4n-5}]$, $2\widetilde{y}_{4n-5}$ *and* $\widetilde{u}_{4n-5}$ *respectively.*
- *for* $n \geq 3$, $H^1_I(R)_{2(4n-4)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/16^{n-1} \oplus \mathbb{Z}/2 \cdot 16^{n-1}$ *generated by* $\widetilde{t}_{4n-4} + (-2)^{n-2}(2\widetilde{x}_{4n-4} + \widetilde{z}_{4n-4}) - 2(-8)^{n-1}(\widetilde{y}_{4n-4} + \widetilde{u}_{4n-4})$, $\widetilde{z}_{4n-4} - 2\widetilde{w}_{4n-4} - 4(-8)^{n-1}\widetilde{u}_{4n-4}$, $\widetilde{w}_{4n-4} + 2 \cdot 4^{n-1}\widetilde{u}_{4n-4}$, $\widetilde{x}_{4n-4}$, $2\widetilde{y}_{4n-4}$ *and* $2\widetilde{u}_{4n-4}$ *respectively.*

- *for* $n \geq 3$, $H^1_I(R)_{2(4n-3)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/4^{n-1} \oplus \mathbb{Z}/2\cdot 4^{n-1} \oplus \mathbb{Z}/2\cdot 16^{n-1} \oplus \mathbb{Z}/4\cdot 16^{n-1}$ *generated by* $\widetilde{t}_{4n-3} + (-2)^n\widetilde{x}_{4n-3} - 4(-8)^{n-1}\widetilde{y}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3} - 2(-8)^{n-1}\widetilde{u}_{4n-3}$, $2[\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}]$, $[\widetilde{z}_{4n-3} + 2\cdot 4^{n-1}\widetilde{u}_{4n-3}] + [\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}]$, $\widetilde{x}_{4n-3} + [\widetilde{z}_{4n-3} + 2\cdot 4^{n-1}\widetilde{u}_{4n-3}]$, $2\widetilde{y}_{4n-3}$ *and* $\widetilde{u}_{4n-3}$ *respectively.*

- *for* $n \geq 3$, $H^1_I(R)_{2(4n-2)} = \mathbb{Z}/2^{n-2} \oplus \mathbb{Z}/2^{n-1} \oplus \mathbb{Z}/2\cdot 4^{n-1} \oplus \mathbb{Z}/2\cdot 4^{n-1} \oplus \mathbb{Z}/4\cdot 16^{n-1} \oplus \mathbb{Z}/8\cdot 16^{n-1}$ *generated by* $\widetilde{z}_{4n-2} - (-2)^n\widetilde{x}_{4n-2} - (-8)^n\widetilde{y}_{4n-2} - (-8)^n\widetilde{w}_{4n-2} - (2 + (-2)^n)\widetilde{t}_{4n-2}$, $\widetilde{u}_{4n-2} + (-8)^n\widetilde{w}_{4n-2}$, $\widetilde{t}_{4n-2} + 4^n\widetilde{w}_{4n-2}$, $\widetilde{x}_{4n-2}$, $2\widetilde{y}_{4n-2}$ *and* $2\widetilde{w}_{4n-2}$ *respectively.*

$H^2_I(R) = (\mathbb{F}_2[b,d])^\vee(\nu)$, *where* $\nu = \frac{\tau}{b^3d^2}$.

So far, we reach to the E^2-page shown as below.

$H_I^{\epsilon\geq 3}(R)$	$H^2_I(R)$	$H^1_I(R)$	$H^0_I(R)$	degree(t)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	19
0	2^4	$[2]\oplus[2]\oplus[16]\oplus[32]\oplus[512]\oplus[1024]$	$\mathbb{Z}\cdot\rho$	18
0	0	0	0	17
0	2^3	$[2]\oplus[2]\oplus[16]\oplus[16]\oplus[256]\oplus[512]$	$\mathbb{Z}\cdot\rho$	16
0	0	0	0	15
0	2^3	$[2]\oplus[8]\oplus[16]\oplus[128]\oplus[256]$	$\mathbb{Z}\cdot\rho$	14
0	0	0	0	13
0	2^2	$[2]\oplus[8]\oplus[8]\oplus[64]\oplus[128]$	$\mathbb{Z}\cdot\rho$	12
0	0	0	0	11
0	2^2	$[4]\oplus[8]\oplus[32]\oplus[64]$	$\mathbb{Z}\cdot\rho$	10
0	0	0	0	9
0	2	$[4]\oplus[4]\oplus[16]\oplus[32]$	$\mathbb{Z}\cdot\rho$	8
0	0	0	0	7
0	2	$[4]\oplus[8]\oplus[16]$	$\mathbb{Z}\cdot\rho$	6
0	0	0	0	5
0	0	$[2]\oplus[4]\oplus[8]$	$\mathbb{Z}\cdot\rho$	4
0	0	0	0	3
0	0	$[2]\oplus[2]$	$\mathbb{Z}\cdot\rho$	2
0	0	0	0	1
0	0	0	$\mathbb{Z}\cdot\rho$	0
0	0	0	0	-1
0	0	0	0	-2
0	0	0	0	-3
0	0	0	0	-4

where $[n] :=$ cyclic group of order n, $2^r :=$ elementary abelian group of rank r.

Figure 3.7: The E^2-page of Greenlees spectral sequence for $ku_*(BSD_{16})$.

§ 3.5 $ku_*(BSD_{16})$

Note that $ku_*(BG) = ku_*(pt) \oplus \widetilde{ku}_*(BG)$. In our case, $ku_*(pt) = \mathbb{Z}[v] = H^0_I(R)$. This means that there is no non-zero differential detecting on E^2-page and hence $E^2 = E^\infty$. Since this spectral sequence converges to $ku_*(BSD_{16})$, there is a filtration

$$ku_*(BSD_{16}) = F^0_* \supseteq F^1_* \supseteq F^2_* \supseteq 0,$$

which $F^0_n/F^1_n \cong E^\infty_{0,n}$, $F^1_n/F^2_n \cong E^\infty_{-1,n+1}$ and $F^2_n \cong E^\infty_{-2,n+2}$. It is clear that there is no additive extension problem and thus $ku_*(BSD_{16})$ can be read from $E^\infty = E^2$-page and the main required result follows as;

Theorem 3.5.1. *As a module over $ku^*(BSD_{16})$, $ku_*(BSD_{16}) = ku_{even}(BSD_{16}) \oplus ku_{odd}(BSD_{16})$ such that*

(1) $ku_{even}(BSD_{16}) = \mathbb{Z}[v] < \rho > \oplus \Sigma_2 H^2_I(R) = \mathbb{Z}[v] < \rho > \oplus \Sigma_2 \mathbb{F}_2[b,d])^\vee(\nu)$, *where* $\rho = 16 \cdot 1 - 6va - 8vb + v^2ab - 12v^2d + 6v^3ad + 2v^4d^2$ *and* $\nu = \frac{\tau}{b^3d^2}$
Additively, $\widetilde{ku}_{2i}(BSD_{16}) = \begin{cases} 0, & \text{if } i = 0,1 \ ; \\ (\mathbb{Z}/2)^{L(\frac{i-1}{2})}, & \text{if } i > 1 \ . \end{cases}$,
where $L(r)$:= least integer which is greater than or equal to r.

(2) $ku_{odd}(BSD_{16}) = \bigoplus_{i\geq 1} ku_{2i-1}(BSD_{16}) = \bigoplus_{i \geq 1} H^1_I(R)_{2i}$

More precisely,

- $ku_1(BSD_{16}) \cong [2] \oplus [2]$ *generated by* x_1, y_1, *with* $z_1 = t_1 = w_1 = u_1 + x_1 + y_1 = 0$.

- $ku_3(BSD_{16}) \cong [2] \oplus [4] \oplus [8]$ *generated by* x_2, y_2, w_2 *resp., with* $z_2 = 4w_2$, $t_2 = 2(w_2 + y_2)$ *and* $u_2 + 4w_2 = 0$.

- $ku_5(BSD_{16}) \cong [4] \oplus [8] \oplus [16]$ *generated by* x_3, y_3, u_3 *resp., with* $w_3 = 8u_3$, $z_3 = 2x_3 + 4y_3 + 4u_3$ *and* $t_3 = 0$.

- $ku_7(BSD_{16}) \cong [4] \oplus [4] \oplus [16] \oplus [32]$ *generated by* w_4, x_4, y_4, u_4 *resp., with* $z_4 + 2w_4 + 8u_4 = 0$ *and* $t_4 + 2x_4 + z_4 + 8y_4 + 8u_4 = 0$.

- $ku_9(BSD_{16}) \cong [4] \oplus [8] \oplus [32] \oplus [64]$ *generated by* z_5, x_5, y_5, u_5 *resp., with* $w_5 = 0$ *and* $t_5 + 4x_5 - 2z_5 + 16y_5 - 24u_5 = 0$.

- $ku_{11}(BSD_{16}) \cong [2] \oplus [8] \oplus [8] \oplus [64] \oplus [128]$ *generated by* u_6, t_6, x_6, y_6, w_6 *resp., with* $z_6 + 4x_6 + 32y_6 + 2t_6 + 16w_6 = 0$.

- $ku_{13}(BSD_{16}) \cong [2] \oplus [8] \oplus [16] \oplus [128] \oplus [256]$ *generated by* w_7, z_7, x_7, y_7, u_7, *with* $t_7 = 0$.

In general,

$$\begin{array}{rcl}
ku_{8n-11}(BSD_{16}) & \cong & [2^{n-3}] \oplus [2^{n-2}] \oplus [2 \cdot 4^{n-2}] \oplus [4^{n-1}] \oplus [8 \cdot 16^{n-2}] \oplus [16^{n-1}], \\
\textit{generated by} & & t_{4n-5}, w_{4n-5}, z_{4n-5}, x_{4n-5}, y_{4n-5}, u_{4n-5} \ \textit{resp.}, n > 3. \\
ku_{8n-9}(BSD_{16}) & \cong & [2^{n-2}] \oplus [2^{n-2}] \oplus [4^{n-1}] \oplus [4^{n-1}] \oplus [16^{n-1}] \oplus [2 \cdot 16^{n-1}], \\
\textit{generated by} & & t_{4n-4}, z_{4n-4}, w_{4n-4}, x_{4n-4}, y_{4n-4}, u_{4n-4} \ \textit{resp.}, n > 2. \\
ku_{8n-7}(BSD_{16}) & \cong & [2^{n-2}] \oplus [2^{n-2}] \oplus [4^{n-1}] \oplus [2 \cdot 4^{n-1}] \oplus [2 \cdot 16^{n-1}] \oplus [4 \cdot 16^{n-1}], \\
\textit{generated by} & & t_{4n-3}, w_{4n-3}, z_{4n-3}, x_{4n-3}, y_{4n-3}, u_{4n-3} \ \textit{resp.}, n > 2. \\
ku_{8n-5}(BSD_{16}) & \cong & [2^{n-2}] \oplus [2^{n-1}] \oplus [2 \cdot 4^{n-1}] \oplus [2 \cdot 4^{n-1}] \oplus [4 \cdot 16^{n-1}] \oplus [8 \cdot 16^{n-1}], \\
\textit{generated by} & & z_{4n-2}, u_{4n-2}, t_{4n-2}, x_{4n-2}, y_{4n-2}, w_{4n-2} \ \textit{resp.}, n > 2,
\end{array}$$

where

$$\begin{array}{rclrclrclrcl}
x_1 & = & \widetilde{x}_1 + \widetilde{z}_1, & x_2 & = & \widetilde{x}_2, & x_3 & = & \widetilde{x}_3 + \widetilde{t}_3, & x_4 & = & \widetilde{x}_4, \\
y_1 & = & 2\widetilde{y}_1, & y_2 & = & 2\widetilde{y}_2, & y_3 & = & 2\widetilde{y}_3, & y_4 & = & 2\widetilde{y}_4, \\
z_1 & = & 2\widetilde{z}_1, & z_2 & = & \widetilde{z}_2, & z_3 & = & \widetilde{z}_3 + \widetilde{t}_3, & z_4 & = & \widetilde{z}_4, \\
t_1 & = & \widetilde{t}_1, & t_2 & = & \widetilde{t}_2, & t_3 & = & 2\widetilde{t}_3, & t_4 & = & \widetilde{t}_4, \\
u_1 & = & \widetilde{u}_1, & u_2 & = & \widetilde{u}_2, & u_3 & = & \widetilde{u}_3, & u_4 & = & 2\widetilde{u}_4, \\
w_1 & = & \widetilde{w}_1 + \widetilde{z}_1, & w_2 & = & 2\widetilde{w}_2, & w_3 & = & \widetilde{w}_3, & w_4 & = & \widetilde{w}_4 - 8\widetilde{u}_4,
\end{array}$$

$$\begin{array}{rclrclrcl}
x_5 & = & \widetilde{x}_5 + \widetilde{z}_5 + 8\widetilde{u}_5, & x_6 & = & \widetilde{x}_6, & x_7 & = & \widetilde{x}_7 + \widetilde{z}_7 + 16\widetilde{u}_7, \\
y_5 & = & 2\widetilde{y}_5, & y_6 & = & 2\widetilde{y}_6, & y_7 & = & 2\widetilde{y}_7, \\
z_5 & = & \widetilde{z}_5 + \widetilde{w}_5 + 24\widetilde{u}_5, & z_6 & = & \widetilde{z}_6, & z_7 & = & \widetilde{z}_7 + 16\widetilde{u}_7 + \widetilde{\eta}_7, \\
t_5 & = & \widetilde{t}_5, & t_6 & = & \widetilde{t}_6 + 16\widetilde{w}_6, & t_7 & = & 2\widetilde{\eta}_7, \\
u_5 & = & \widetilde{u}_5, & u_6 & = & \widetilde{u}_6 + 64\widetilde{w}_6, & u_7 & = & \widetilde{u}_7, \\
w_5 & = & 2(\widetilde{w}_5 + 16\widetilde{u}_5), & w_6 & = & 2\widetilde{w}_6, & w_7 & = & \widetilde{w}_7 + 4\widetilde{z}_7,
\end{array}$$

such that $\widetilde{\eta}_7 = \widetilde{t}_7 + 4\widetilde{x}_7 + 2\widetilde{z}_7 + 64\widetilde{y}_7 - 32\widetilde{u}_7$ *and in general,*

$$\begin{array}{rcl}
x_{4n-5} & = & \widetilde{x}_{4n-5} + [\widetilde{z}_{4n-5} + 4^{n-1}\widetilde{u}_{4n-5}], \\
y_{4n-5} & = & 2\widetilde{y}_{4n-5}, \\
z_{4n-5} & = & [\widetilde{z}_{4n-5} + 4^{n-1}\widetilde{u}_{4n-5}] + \frac{1}{2}t_{4n-5}, \\
t_{4n-5} & = & 2[\widetilde{t}_{4n-5} + (-2)^{n-1}\widetilde{x}_{4n-5} - (-2)^{n-2}\widetilde{z}_{4n-5} + (-8)^{n-1}\widetilde{y}_{4n-5} + 4(-8)^{n-2}\widetilde{u}_{4n-5}], \\
u_{4n-5} & = & \widetilde{u}_{4n-5}, \\
w_{4n-5} & = & \widetilde{w}_{4n-5} + (-2)^{n-1}\widetilde{z}_{4n-5},
\end{array}$$

$$\begin{array}{rcl}
x_{4n-4} & = & \widetilde{x}_{4n-4}, \\
y_{4n-4} & = & 2\widetilde{y}_{4n-4}, \\
z_{4n-4} & = & \widetilde{z}_{4n-4} - 2\widetilde{w}_{4n-4} - 4(-8)^{n-1}\widetilde{u}_{4n-4}, \\
t_{4n-4} & = & \widetilde{t}_{4n-4} + (-2)^{n-2}(2\widetilde{x}_{4n-4} + \widetilde{z}_{4n-4}) - 2(-8)^{n-1}(\widetilde{y}_{4n-4} + \widetilde{u}_{4n-4}), \\
u_{4n-4} & = & 2\widetilde{u}_{4n-4}, \\
w_{4n-4} & = & \widetilde{w}_{4n-4} + 2 \cdot 4^{n-1}\widetilde{u}_{4n-4},
\end{array}$$

$$\begin{aligned}
x_{4n-3} &= \widetilde{x}_{4n-3} + [\widetilde{z}_{4n-3} + 2\cdot 4^{n-1}\widetilde{u}_{4n-3}],\\
y_{4n-3} &= 2\widetilde{y}_{4n-3},\\
z_{4n-3} &= [\widetilde{z}_{4n-3} + 2\cdot 4^{n-1}\widetilde{u}_{4n-3}] + [\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}],\\
t_{4n-3} &= \widetilde{t}_{4n-3} + (-2)^n\widetilde{x}_{4n-3} - 4(-8)^{n-1}\widetilde{y}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3} - 2(-8)^{n-1}\widetilde{u}_{4n-3},\\
u_{4n-3} &= \widetilde{u}_{4n-3},\\
w_{4n-3} &= 2[\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}],
\end{aligned}$$

$$\begin{aligned}
x_{4n-2} &= \widetilde{x}_{4n-2}\\
y_{4n-2} &= 2\widetilde{y}_{4n-2}\\
z_{4n-2} &= \widetilde{z}_{4n-2} - (-2)^n\widetilde{x}_{4n-2} - (-8)^n\widetilde{y}_{4n-2} - (-8)^n\widetilde{w}_{4n-2} - (2+(-2)^n)\widetilde{t}_{4n-2}\\
t_{4n-2} &= \widetilde{t}_{4n-2} + 4^n\widetilde{w}_{4n-2}\\
u_{4n-2} &= \widetilde{u}_{4n-2} + (-8)^n\widetilde{w}_{4n-2}\\
w_{4n-2} &= 2\widetilde{w}_{4n-2},
\end{aligned}$$

For the action, b, d act freely on $\widetilde{ku}_{2i}(BSD_{16})$ whereas a acts as zero and the $ku^*(BSD_{16})$ action on $ku_{odd}(BSD_{16})$ can be read from Table 2.4 and Table 2.5 with the definition above.

Proof. This is the immediate result from E^∞-page and lemma 3.4.3. □

Next we investigate some relations of $ku_*(BSD_{16})$ and $ku_*(BG)$ for $G = D_8, Q_8$ and C_8 in the section below.

§ 3.6 Relations with its maximal subgroups

In this section we aim to explicit the natural corestriction maps from $ku_*(BSD_{16})$ to $ku_*(BG)$ for $G = D_8, Q_8$ and C_8.

3.6.1 $ku_*(BSD_{16})$ AND $ku_*(BD_8)$

We first recollect the results of $ku_*(BD_8)$ from theorem 3.5.1 in [14] as;

Proposition 3.6.1. *([14]) As an $R' := ku^*(BD_8)$ module, $ku_*(BD_8) = ku_{odd}(BD_8) \oplus ku_{even}(BD_8)$ where $ku_{even}(BD_8) = \mathbb{Z}[v] < \rho' > \oplus \widetilde{ku}_{even}(BD_8)$ with $\rho' = 8 - 4va - 2v^2d - v^3bd$ and*

1. *$\widetilde{ku}_{even}(BD_8) = \Sigma^{-2}H_I^2(R') = \Sigma^2 P^\vee$, where $P = \mathbb{F}_2[a, b, d]/(ab + b^2)$. Additively, $\widetilde{ku}_{even}(BD_8) = (\mathbb{Z}/2)^i$.*
2. *$ku_{odd}(BD_8) = \Sigma^{-1}H_I^1(R')$, with additive generators a_i, b_i, c_i and d_i in $ku_{2i-1}(BD_8)$ for $i > 0$.*
   - *2.1 $ku_1(BD_8) = (\mathbb{Z}/2)^2 =< a_1 > \oplus < b_1 >$, with $d_1 = a_1$ and $c_1 = 0$.*
   - *2.2 $ku_3(BD_8) = (\mathbb{Z}/4)^3 =< a_2 > \oplus < b_2 > \oplus < d_2 >$ and $c_2 = 2a_2 + 2d_2$.*

2.3 $ku_5(BD_8) = (\mathbb{Z}/8)^3 =< a_3 > \oplus < b_3 > \oplus < d_3 >$ *and* $c_3 = 4a_3 + 4d_3$.

2.4 For $i \geq 4$, $ku_{2i-1}(BD_8) = (\mathbb{Z}/2^i)^3 \oplus A_{2i-1} =< a_i > \oplus < b_i > \oplus < d_i > \oplus A_{2i-1}$, *where*

- $A_{4n-1} = \mathbb{Z}/2^{n-1} =< c_{2n} + 2^n(a_{2n} + d_{2n}) >$,
- $A_{4n+1} = \mathbb{Z}/2^{n-1} =< c_{2n+1} + 2^{n+1}(a_{2n+1} + d_{2n+1}) >$.

The R'*-module structure is given by*

	v	a	b	d
a_i	$2a_{i+1}$	a_{i-1}	$b_{i-1} - 2c_{i-1}$	0
b_i	$2(b_{i+1} - \epsilon_{i+1}c_{i+1})$	$b_{i-1} - 2c_{i-1}$	$b_{i-1} + 2\epsilon_i c_{i-1}$	$2c_{i-2}$
c_i	$2c_{i+1}$	0	$(1+\epsilon_i)c_{i-1}$	c_{i-2}
d_i	$2d_{i+1} - \epsilon_i c_{i+1}$	0	c_{i-1}	d_{i-2}

where $\epsilon_{2i} = 0$ *and* $\epsilon_{2i+1} = 1$.

We can also explicit the generator name for $\widetilde{ku}_{even}(BD_8) = \Sigma^{-2}H^2_I(R') = \Sigma^2 P^\vee$ by doing calculation on local cohomology of $TU = \mathbb{F}_2[a,b,d]/(ab+b^2) < ad >$ at $I = (a,b,d) = \sqrt{(a,d)}$. Indeed, $H^i_I(TU) = 0$ if $i \neq 2$ and

$$H^2_I(TU) = \mathbb{F}_2[\frac{1}{a}, \frac{1}{d}] < \frac{\tau' b}{ad}, \frac{\tau'}{ad} >,$$

where $\tau' = ad$. After determining differentials $\delta : [H^1_I(QU)]_* \longrightarrow [H^2_I(TU)]_*$, we get that

$$H^2_I(ku^*(BD_8)) = \mathbb{F}_2[\frac{1}{a}, \frac{1}{d}] < \frac{\tau' b}{a^2d^2}, \frac{\tau'}{a^2d^2} > . \tag{3.4}$$

Also, the explicit name for $\widetilde{ku}_{even}(BSD_{16}) \cong \Sigma_2 H^2_I(ku^*(BSD_{16}))$ is given by

$$H^2_I(ku^*(BSD_{16})) = \mathbb{F}_2[\frac{1}{b}, \frac{1}{d}] < \frac{\tau}{b^3d^2} >, \tag{3.5}$$

where $\tau = b^2d - abd$.

The rest of this subsection is devoted to prove the following property.

Proposition 3.6.2. *The natural corestriction map from* $ku_*(BSD_{16})$ *to* $ku_*(BD_8)$ *is as follows;*

1 $ku_{even}(BSD_{16})$ *is embedded in* $ku_{even}(BD_8)$, *explicitly,*

$$\rho \mapsto 2\rho' \text{ and } \tfrac{\tau}{b^3d^2} \mapsto \tfrac{\tau'}{a^2d^2}.$$

2 $ku_{odd}(BSD_{16}) \longrightarrow ku_{odd}(BD_8)$ *is given by*

$$\begin{array}{llllll}
x_1 \mapsto c_1, & y_1 \mapsto a_1, & z_1 \mapsto 2c_1, & t_1 \mapsto 0, & u_1 \mapsto d_1 - c_1, & w_1 \mapsto c_1, \\
x_2 \mapsto 0, & y_2 \mapsto a_2, & z_2 \mapsto 2c_2, & t_2 \mapsto c_2, & u_2 \mapsto 0, & w_2 \mapsto d_2, \\
x_3 \mapsto 0, & y_3 \mapsto a_3, & z_3 \mapsto c_3, & t_3 \mapsto 0, & u_3 \mapsto d_3 - c_3, & w_3 \mapsto 0, \\
x_4 \mapsto 0, & y_4 \mapsto a_4, & z_4 \mapsto 2c_4, & t_4 \mapsto 0, & u_4 \mapsto d_4, & w_4 \mapsto c_4 - 4d_4,
\end{array}$$

$x_5 \mapsto 8d_5 - 7c_5,$	$x_6 \mapsto 0,$	$x_7 \mapsto 16d_7 - 15c_7,$
$y_5 \mapsto a_5,$	$y_6 \mapsto a_6,$	$y_7 \mapsto a_7,$
$z_5 \mapsto 24d_5 - 23c_5,$	$z_6 \mapsto 2c_6,$	$z_7 \mapsto 19c_7 - 16d_7 + 32a_7,$
$t_5 \mapsto 0,$	$t_6 \mapsto c_6 + 8d_6,$	$t_7 \mapsto 68c_7 + 64a_7 - 64d_7,$
$u_5 \mapsto d_5 - c_5,$	$u_6 \mapsto 32d_6,$	$u_7 \mapsto d_7 - c_7,$
$w_5 \mapsto 32d_5 - 32c_5,$	$w_6 \mapsto d_6,$	$w_7 \mapsto 4c_7.$

In general,

for $n \geq 4$,

$$\begin{array}{lcl}
x_{4n-5} & \mapsto & (1-4^{n-1})c_{4n-5} + 4^{n-1}d_{4n-5}, \\
y_{4n-5} & \mapsto & a_{4n-5}, \\
z_{4n-5} & \mapsto & [1-(-2)^{n-2} - 4^{n-1} - 4(-8)^{n-2}]c_{4n-5} + [4^{n-1} + 4(-8)^{n-2}]d_{4n-5} \\
 & & -4(-8)^{n-2}a_{4n-5}, \\
t_{4n-5} & \mapsto & [(-2)^{n-1} + (-8)^{n-1}]c_{4n-5} - (-8)^{n-1}d_{4n-5} + (-8)^{n-1}a_{4n-5}, \\
u_{4n-5} & \mapsto & d_{4n-5} - c_{4n-5}, \\
w_{4n-5} & \mapsto & (-2)^{n-1}c_{4n-5},
\end{array}$$

for $n \geq 3$,

$$\begin{array}{lcl}
x_{4n-4} & \mapsto & 0, \\
y_{4n-4} & \mapsto & a_{4n-4}, \\
z_{4n-4} & \mapsto & -2(-8)^{n-1}d_{4n-4}, \\
t_{4n-4} & \mapsto & -(-2)^{n-1}c_{4n-4} - (-8)^{n-1}d_{4n-4} - (-8)^{n-1}a_{4n-4}, \\
u_{4n-4} & \mapsto & d_{4n-4}, \\
w_{4n-5} & \mapsto & c_{4n-4} + 4^{n-1}d_{4n-4},
\end{array}$$

for $n \geq 3$,

$$\begin{array}{lcl}
x_{4n-3} & \mapsto & (1-2\cdot 4^{n-1})c_{4n-3} + 2\cdot 4^{n-1}d_{4n-3}, \\
y_{4n-3} & \mapsto & a_{4n-3}, \\
z_{4n-3} & \mapsto & [1-(-2)^{n-1} - 2\cdot 4^{n-1}]c_{4n-3} + 2\cdot 4^{n-1}d_{4n-3}, \\
t_{4n-3} & \mapsto & -2(-8)^{n-1}a_{4n-3} - 2(-8)^{n-1}d_{4n-3} + [2(-8)^{n-1} - (-2)^{n-1}]c_{4n-3}, \\
u_{4n-3} & \mapsto & d_{4n-3} - c_{4n-3}, \\
w_{4n-3} & \mapsto & (-2)^{n}c_{4n-3},
\end{array}$$

for $n \geq 3$,

$$\begin{array}{lcl}
x_{4n-2} & \mapsto & 0, \\
y_{4n-2} & \mapsto & a_{4n-2}, \\
z_{4n-2} & \mapsto & -(-2)^{n}c_{4n-2} + 4(-8)^{n-1}a_{4n-2} + 4(-8)^{n-1}d_{4n-2}, \\
t_{4n-2} & \mapsto & c_{4n-2} + 2\cdot 4^{n-1}d_{4n-2}, \\
u_{4n-2} & \mapsto & -4(-8)^{n-1}d_{4n-2}, \\
w_{4n-2} & \mapsto & d_{4n-2}.
\end{array}$$

Proof. Since both $ku_*(BSD_{16})$ and $ku_*(BD_8)$ are isomorphic to their E^∞-page and we choose their elements to be the elements in E^∞-page, it is enough to consider their relations at E^∞-stage, i.e. consider $H^i_I(ku^*(BSD_{16}))$ and $H^i_I(ku^*(BD_{18}))$ for $i = 0, 1, 2$. Thus this proposition is an immediate result from Theorem 2.7.2 in the last chapter. Precisely, to prove $\rho \mapsto 2\rho'$, we use the fact that $a \mapsto 0, b \mapsto a, d \mapsto d$ and the relation $v^2d^2 = 4d - vbd$ in $ku^*(BD_8)$, i.e.

$$\begin{array}{rcl} \rho & = & 16\cdot 1 - 6va - 8vb + v^2ab - 12v^2d + 6v^3ad + 2v^4d^2 \\ & \mapsto & 16 - 8va - 12v^2d + 2v^4d^2 \\ & = & 16 - 8va - 12v^2d + 2(4v^2d - v^3bd) \\ & = & 2[8 - 4va - 2v^2d - v^3bd] = 2\rho'. \end{array}$$

Also,

$$\frac{\tau}{b^3d^2} \quad \mapsto \quad \frac{a^2d}{a^3d^2} = \frac{ad}{a^2d^2} = \frac{\tau'}{a^2d^2}.$$

To prove the corestrictions in odd degree, we use the set of relations in QU_{D_8} (lemma 3.5.2, [14]) which is $\{ad = 0, va^2 = 2a, vb^2 = 2b, vbd = b^2 - ab, v^2d^2 = 4d - vbd\}$, and definition of element's in $ku_{odd}(BD_8)$ ([14], page 73-74) which is;

$$\begin{array}{llll} a_{2n} = \frac{2a^2}{y^{n+1}} & b_{2n} = \frac{2b^2}{y^{n+1}} & c_{2n} = \frac{vbd}{y^{n+1}} & d_{2n} = \frac{2d}{y^{n+1}} \\ a_{2n-1} = \frac{2a^3}{y^{n+1}} & b_{2n-1} = \frac{2b^3}{y^{n+1}} & c_{2n-1} = \frac{2bd}{y^{n+1}} & d_{2n-1} = \frac{vd^2}{y^{n+1}}, \end{array} \tag{3.6}$$

where $p = d^2 + b^4 - 2vbd^2 \mapsto d^2 + a^4 - 2vad^2 = (d + a^2)^2 = y^2$ (because $2vad^2 = 0$).

Indeed, we have (see section 2.3) the restriction of JU_2, JU_3, JU_4 and JU_5 of $QU_{SD_{16}}$ to QU_{D_8} as;

$$\begin{array}{lcl} \overline{x}_2 & \mapsto & vad \equiv 0 \\ \overline{y}_2 & \mapsto & a^2 - vad \equiv 0 \\ \overline{z}_2 & \mapsto & vad + \frac{16}{3}d - \frac{1}{3}v^4d^3 \equiv 2vbd \\ \overline{t}_2 & \mapsto & vad + 4d - v^2d^2 \equiv vbd \\ \overline{u}_2 & \mapsto & vad \equiv 0 \\ \overline{w}_2 & \mapsto & vad + d \equiv d \end{array} , \begin{array}{lcl} \overline{x}_3 & \mapsto & ad \equiv 0 \\ \overline{y}_3 & \mapsto & a^3 - 2ad \equiv a^3 \\ \overline{z}_3 & \mapsto & ad - \frac{4}{3}vd^2 + 3v^3d^3 - \frac{2}{3}v^5d^4 \equiv 2bd \\ \overline{t}_3 & \mapsto & -2ad - \frac{8}{3}vd^2 + 2v^3d^3 - \frac{1}{3}v^5d^4 \equiv 0 \\ \overline{u}_3 & \mapsto & vd^2 - im(\overline{x}_3) - im(\overline{z}_3) \equiv vd^2 - 2bd \\ \overline{w}_3 & \mapsto & ad \equiv 0 \end{array}$$

$$\begin{array}{lcl} \overline{x}_4 & \mapsto & vad^2 \equiv 0 \\ \overline{y}_4 & \mapsto & a^4 - 2vad^2 \equiv a^4 \\ \overline{z}_4 & \mapsto & vad^2 + \frac{16}{3}d^2 - \frac{1}{3}v^4d^4 \equiv 2vbd^2 \\ \overline{t}_4 & \mapsto & vad^2 \equiv 0 \\ \overline{u}_4 & \mapsto & d^2 \\ \overline{w}_4 & \mapsto & \frac{4}{3}d^2 + v^2d^3 - \frac{1}{3}v^4d^4 \equiv vbd^2 \end{array} , \begin{array}{lcl} \overline{x}_5 & \mapsto & ad^2 \equiv 0 \\ \overline{y}_5 & \mapsto & a^5 - 4ad^2 \equiv a^5 \\ \overline{z}_5 & \mapsto & -5ad^2 - \frac{4}{3}vd^3 + 3v^3d^4 - \frac{2}{3}v^5d^5 \\ & & \equiv 2bd^2 \\ \overline{t}_5 & \mapsto & -2ad^2 \equiv 0 \\ \overline{u}_5 & \mapsto & 2ad^2 - \frac{1}{3}vd^3 - v^3d^4 + \frac{1}{3}v^5d^5 \\ & & \equiv vd^3 - 2bd^2 \\ \overline{w}_5 & \mapsto & 2ad^2 + \frac{8}{3}vd^3 - 2v^3d^4 + \frac{1}{3}v^5d^5 \equiv 0 \end{array}$$

Combining this facts with $\widetilde{\alpha}_{4i-\varepsilon} = \frac{\overline{\alpha}_\varepsilon}{p^i}$ for $i \geq 1$ and $\widetilde{\alpha}_{4i-5} = \frac{\overline{\alpha}_5}{p^i}$ for $i \geq 2$, where $\varepsilon = 2, 3, 4$ and $\alpha \in \{x, y, z, t, u, w\}$, we get that;

$$\begin{array}{lcl} \widetilde{x}_{4i-2} & \mapsto & 0 \\ \widetilde{y}_{4i-2} & \mapsto & \frac{a^2}{y^{2i}} \\ \widetilde{z}_{4i-2} & \mapsto & \frac{2vbd}{y^{2i}} \\ \widetilde{t}_{4i-2} & \mapsto & \frac{vbd}{y^{2i}} \\ \widetilde{u}_{4i-2} & \mapsto & 0 \\ \widetilde{w}_{4i-2} & \mapsto & \frac{d}{y^{2i}} \end{array} , \begin{array}{lcl} \widetilde{x}_{4i-3} & \mapsto & 0 \\ \widetilde{y}_{4i-3} & \mapsto & \frac{a^3}{y^{2i}} \\ \widetilde{z}_{4i-3} & \mapsto & \frac{2bd}{y^{2i}} \\ \widetilde{t}_{4i-3} & \mapsto & 0 \\ \widetilde{u}_{4i-3} & \mapsto & \frac{vd^2}{y^{2i}} - \frac{2bd}{y^{2i}} \\ \widetilde{w}_{4i-3} & \mapsto & 0 \end{array}$$

$$\begin{array}{lll} \widetilde{x}_{4i-4} & \mapsto & 0 \\ \widetilde{y}_{4i-4} & \mapsto & \frac{a^4}{y^{2i}} \equiv \frac{a^2}{y^{2i-1}} \\ \widetilde{z}_{4i-4} & \mapsto & \frac{vbd^2}{y^{2i}} \equiv \frac{vbd}{y^{2i-1}} \\ \widetilde{t}_{4i-4} & \mapsto & 0 \\ \widetilde{u}_{4i-4} & \mapsto & \frac{d^2}{y^{2i}} \equiv \frac{d}{y^{2i-1}} \\ \widetilde{w}_{4i-4} & \mapsto & \frac{vbd^2}{y^{2i}} \equiv \frac{vbd}{y^{2i-1}} \end{array}, \quad \begin{array}{lll} \widetilde{x}_{4i-5} & \mapsto & 0 \\ \widetilde{y}_{4i-5} & \mapsto & \frac{a^5}{y^{2i}} \equiv \frac{a^3}{y^{2i-1}} \\ \widetilde{z}_{4i-5} & \mapsto & \frac{2bd^2}{y^{2i}} \equiv \frac{2bd}{y^{2i-1}} \\ \widetilde{t}_{4i-5} & \mapsto & 0 \\ \widetilde{u}_{4i-5} & \mapsto & \frac{vd^3}{y^{2i}} - \frac{2bd^2}{y^{2i}} \equiv \frac{vd^2}{y^{2i-1}} - \frac{2bd}{y^{2i-1}} \\ \widetilde{w}_{4i-5} & \mapsto & 0. \end{array}$$

Hence by (3.6) and definition in Theorem 3.5.1, we complete the proof. □

3.6.2 $ku_*(BSD_{16})$ AND $ku_*(BQ_8)$

We proceed this subsection as in the last subsection by firstly recalling $ku_*(BQ_8)$ from [14], page 69-70 as;

Proposition 3.6.3. *(cf.[14]) As a module over $R'' := ku^*(BQ_8)$,*

$$ku_*(BQ_8) = \mathbb{Z}[v] \cdot \rho'' \oplus ku_{odd}(BQ_8),$$

where $\rho'' = v^4q^2 - 6v^2q + 8$ and $ku_{odd}(BQ_8) = \Sigma^{-1}H^1_I(R'')$, $H^1_I(R'') = cok(R'' \longrightarrow R''[\frac{1}{q}])$, which can be shown explicitly as;

- *$ku_1(BQ_8) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 =< e_1 > \oplus < f_1 >$, with $g_1 = h_1 = 0$.*
- *$ku_3(BQ_8) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8 =< e_2 > \oplus < f_2 > \oplus < g_2 >$, with $h_2 = 0$.*
- *$ku_5(BQ_8) = \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8 =< e_3 > \oplus < f_3 > \oplus < g_3 >$, with $h_3 = 0$.*

In general for $k \geq 2$,

- *$ku_{4k-1}(BQ_8) = \mathbb{Z}/2^k \oplus \mathbb{Z}/2^k \oplus \mathbb{Z}/2^{2k+1} \oplus \mathbb{Z}/2^{k-1} =< e_{2k} > \oplus < f_{2k} > \oplus < g_{2k} > \oplus < h_{2k} + (2^{k+1} - 4)g_{2k} >$,*
- *$ku_{4k+1}(BQ_8) = \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2^{k+1} \oplus \mathbb{Z}/2^{2k+1} \oplus \mathbb{Z}/2^{k-1} =< e_{2k+1} > \oplus < f_{2k+1} > \oplus < g_{2k+1} > \oplus < h_{2k+1} + (2^{k+1} - 4)g_{2k+1} >$,*

where

$$\begin{array}{llll} e_{2k} = \frac{a^2}{q^{k+1}}, & f_{2k} = \frac{b^2}{q^{k+1}}, & g_{2k} = \frac{q}{q^{k+1}}, & h_{2k} = \frac{v^2q^2}{q^{k+1}}, \\ e_{2k+1} = \frac{a}{q^{k+1}}, & f_{2k+1} = \frac{b}{q^{k+1}}, & g_{2k+1} = \frac{vq}{q^{k+1}}, & h_{2k+1} = \frac{v^3q^2}{q^{k+1}}. \end{array}$$

The $ku^*(BQ_8)$ action is given by

	v	a	b	q
e_{2i}	$2e_{2i+1}$	$2e_{i-1}$	κ_{2i}	e_{2i-2}
e_{2i+1}	e_{2i+2}	e_{2i}	κ_{2i+1}	e_{2i-1}
f_{2i}	$2f_{2i+1}$	κ_{2i}	$2f_{2i-1}$	f_{2i-2}
f_{2i+1}	f_{2i+2}	κ_{2i+1}	f_{2i}	f_{2i-1}
g_i	g_{i+1}	e_{i-1}	f_{i-1}	g_{i-2}
h_{2i}	h_{2i+1}	$2e_{2i-1}$	$2f_{2i-1}$	h_{2i-2}
g_{2i+1}	$6h_{2i+2} - 8g_{2i+2}$	$2e_{2i}$	$2f_{2i}$	h_{2i-1}

where $\kappa_{2i} = 2e_{2i-1} + 2f_{2i-1} - 4g_{2i-1} + h_{2i-1}$ *and* $\kappa_{2i+1} = e_{2i} + f_{2i} - 4g_{2i} + h_{2i}$.

Note in this proposition that we have used relation

$$v^{2(n+1)}q^{n+2} = (2 \cdot 4^n - 2^n)v^2q^2 + (2^{n+2} - 4^{n+1})q$$

obtained by relation $v^4q^3 = 6v^2q^2 - 8q$ and recurrence relation method.

Some relations between $ku_*(BSD_{16})$ and $ku_*(BQ_8)$ are;

Proposition 3.6.4. *The natural corestriction map from* $ku_*(BSD_{16})$ *to* $ku_*(BQ_8)$ *is as follows;*

1 $\rho \mapsto 2\rho''$.

2 $ku_{odd}(BSD_{16}) \longrightarrow ku_{odd}(BQ_8)$ *is given by*

$$\begin{array}{llllll} x_1 \mapsto -f_1, & y_1 \mapsto 0, & z_1 \mapsto 0, & t_1 \mapsto 0, & u_1 \mapsto 0, & w_1 \mapsto 0, \\ x_2 \mapsto f_2, & y_2 \mapsto 0, & z_2 \mapsto 0, & t_2 \mapsto -f_2 + 4g_2, & u_2 \mapsto 0, & w_2 \mapsto 2g_2, \end{array}$$

$$\begin{array}{lll} x_3 \mapsto f_3, & x_4 \mapsto f_4, & x_5 \mapsto -f_5 + 7h_5 - 20g_5, \\ y_3 \mapsto 0, & y_4 \mapsto 0, & y_5 \mapsto 0, \\ z_3 \mapsto -2f_3 + 4g_3, & z_4 \mapsto -2f_4 + 8g_4 - 2h_4, & z_5 \mapsto -2f_5 + 23h_5 - 68g_5, \\ t_3 \mapsto 0, & t_4 \mapsto 0, & t_5 \mapsto 0, \\ u_3 \mapsto f_3 - 3g_3, & u_4 \mapsto 2g_4, & u_5 \mapsto f_5 - 3g_5 + h_5, \\ w_3 \mapsto 0, & w_4 \mapsto -f_4 - (h_4 + 4g_4), & w_5 \mapsto 32(f_5 - 3g_5 + h_5), \end{array}$$

$$\begin{array}{ll} x_6 \mapsto f_6, & x_7 \mapsto 15f_7 - 44g_7 + 15h_7, \\ y_6 \mapsto 0, & y_7 \mapsto 0, \\ z_6 \mapsto -2f_6 + 8g_6 - 2h_6, & z_7 \mapsto -18f_7 + 60g_7 - 19h_7, \\ t_6 \mapsto -f_6 + 20g_6 - h_6, & t_7 \mapsto -64f_7 + 208g_7 - 68h_7, \\ u_6 \mapsto 64g_6, & u_7 \mapsto f_7 - 3g_7 + h_7, \\ w_6 \mapsto 2g_6, & w_7 \mapsto -8f_7 + 16g_7 - 4h_7. \end{array}$$

In general,

for $n \geq 4$,

$$\begin{array}{lll} x_{4n-5} & \mapsto & (4^{n-1} - 1)f_{4n-5} + (4 - 3 \cdot 4^{n-1})g_{4n-5} + (4^{n-1} - 1)h_{4n-5}, \\ y_{4n-5} & \mapsto & 0, \\ z_{4n-5} & \mapsto & [4(-8)^{n-2} + 4^{n-1} - (-2)^{n-1} - 2]f_{4n-5} + [-12(-8)^{n-2} - 3 \cdot 4^{n-1} \\ & & -(-2)^n + 4]g_{4n-5} + [4(-8)^{n-2} + 4^{n-1} + (-2)^{n-2} - 1]h_{4n-5}, \\ t_{4n-5} & \mapsto & -(-8)^{n-1}f_{4n-5} + [3(-8)^{n-1} - (-2)^{n+1}]g_{4n-5} \\ & & -[(-2)^{n-1} + (-8)^{n-1}]h_{4n-5}, \\ u_{4n-5} & \mapsto & f_{4n-5} - 3g_{4n-5} + h_{4n-5}, \\ w_{4n-5} & \mapsto & (-2)^n f_{4n-5} + (-2)^{n+1}g_{4n-5} - (-2)^{n-1}h_{4n-5}, \end{array}$$

for $n \geq 3$,

$$\begin{array}{lcl}
x_{4n-4} & \mapsto & f_{4n-4}, \\
y_{4n-4} & \mapsto & 0, \\
z_{4n-4} & \mapsto & -4(-8)^{n-1}g_{4n-4}, \\
t_{4n-4} & \mapsto & [8(-2)^{n-2} - 2(-8)^{n-1}]g_{4n-4} + (-2)^{n-1}h_{4n-4}, \\
u_{4n-4} & \mapsto & 2g_{4n-4}, \\
w_{4n-5} & \mapsto & -f_{4n-4} + (2 \cdot 4^{n-1} + 4)g_{4n-4} - h_{4n-4},
\end{array}$$

for $n \geq 3$,

$$\begin{array}{lcl}
x_{4n-3} & \mapsto & (2 \cdot 4^{n-1} - 1)f_{4n-3} + (4 - 6 \cdot 4^{n-1})g_{4n-3} + (2 \cdot 4^{n-1} - 1)h_{4n-3}, \\
y_{4n-3} & \mapsto & 0, \\
z_{4n-3} & \mapsto & (2 \cdot 4^{n-1} - (-2)^n - 2)f_{4n-3} + [-6 \cdot 4^{n-1} - (-2)^{n+1} + 4]g_{4n-3} \\
 & & +(2 \cdot 4^{n-1} + (-2)^{n-1} - 1)h_{4n-3}, \\
t_{4n-3} & \mapsto & -2(-8)^{n-1}f_{4n-3} + [6(-8)^{n-1} - (-2)^{n+1}]g_{4n-3} \\
 & & +[-2(-8)^{n-1} + (-2)^{n-1}]h_{4n-3}, \\
u_{4n-3} & \mapsto & f_{4n-3} - 3g_{4n-3} + h_{4n-3}, \\
w_{4n-3} & \mapsto & (-2)^{n+1}f_{4n-3} + (-2)^{n+2}g_{4n-3} - (-2)^n h_{4n-3},
\end{array}$$

for $n \geq 3$,

$$\begin{array}{lcl}
x_{4n-2} & \mapsto & f_{4n-2}, \\
y_{4n-2} & \mapsto & 0, \\
z_{4n-2} & \mapsto & [-4(-8)^n - (-2)^{n+2}]g_{4n-2} + (-2)^n h_{4n-2}, \\
t_{4n-2} & \mapsto & -f_{4n-2} + (4 + 4^n)g_{4n-2} - h_{4n-2}, \\
u_{4n-2} & \mapsto & (-8)^n g_{4n-2}, \\
w_{4n-2} & \mapsto & 2g_{4n-2}.
\end{array}$$

Proof. As in the proof of Proposition 3.6.2, we consider relations on local cohomology. For $\rho \mapsto 2\rho''$, we use the fact that $a \mapsto b, b \mapsto b, d \mapsto q$ and the set of relations in $ku^*(BQ_8)$ (Theorem 2.4.6, [14]) which is $\{v^4q^3 = 6v^2q^2 - 8q, va^2 = 2a, vb^2 = 2b, vaq = a^2, vbq = b^2, ab = vaq + vbq + v^2q^2 - 4q\}$, i.e.

$$\begin{array}{rcl}
\rho & = & 16 \cdot 1 - 6va - 8vb + v^2ab - 12v^2d + 6v^3ad + 2v^4d^2 \\
 & \mapsto & 16 - 6vb - 8vb + v^2b^2 - 12v^2q + 6v^3bq + 2v^4q^2 \\
 & = & 16 - 14vb + (2vb) - 12v^2q + 6(2vb) + 2v^4q^2 \\
 & = & 2[8 - 6v^2q - v^4q^2] = 2\rho''.
\end{array}$$

To prove the corestrictions in odd degree, we again use the set of relations in $ku^*(BQ_8)$ and definition of element's in $ku_{odd}(BQ_8)$. Indeed, we have (see section 2.3) the restriction of JU_2, JU_3, JU_4 and JU_5 of $QU_{SD_{16}}$ to $ku^*(BQ_8)$ as;

$$\begin{array}{lcl}
\overline{x}_2 & \mapsto & vbq \equiv b^2 \\
\overline{y}_2 & \mapsto & b^2 - vbq \equiv 0 \\
\overline{z}_2 & \mapsto & -2vbq + \frac{16}{3}q - \frac{1}{3}v^4q^3 \\
 & & \equiv -2b^2 + 8q - 2v^2q^2 \\
\overline{t}_2 & \mapsto & -vbq + 4q - v^2q^2 \\
 & & \equiv -b^2 + 4q - v^2q^2 \\
\overline{u}_2 & \mapsto & 0 \\
\overline{w}_2 & \mapsto & q
\end{array}
,
\begin{array}{lcl}
\overline{x}_3 & \mapsto & bq \\
\overline{y}_3 & \mapsto & b^3 - 2bq \equiv 0 \\
\overline{z}_3 & \mapsto & -2bq - \frac{4}{3}vq^2 + 3v^3q^3 - \frac{2}{3}v^5q^4 \\
 & & \equiv -2bq + 4vq^2 - v^3q^3 \\
\overline{t}_3 & \mapsto & -\frac{8}{3}vq^2 + 2v^3q^3 - \frac{1}{3}v^5q^4 \equiv 0 \\
\overline{u}_3 & \mapsto & v^3q^3 - 3vq^2 + bq \\
\overline{w}_3 & \mapsto & 0
\end{array}$$

$$\begin{array}{lll} \overline{x}_4 & \mapsto & vbq^2 \equiv b^2q \\ \overline{y}_4 & \mapsto & b^4 - 2vbq^2 \equiv 0 \\ \overline{z}_4 & \mapsto & -2vbq^2 + \frac{16}{3}q^2 - \frac{1}{3}v^4q^4 \\ & & \equiv -2vbq^2 + 8q^2 - 2v^2q^3 \\ \bar{t}_4 & \mapsto & 0 \\ \overline{u}_4 & \mapsto & q^2 \\ \overline{w}_4 & \mapsto & \frac{4}{3}q^2 + v^2q^3 - \frac{1}{3}v^4q^4 - vbq^2 \\ & & \equiv 4q^2 - v^2q^3 - b^2q \end{array} , \quad \begin{array}{lll} \overline{x}_5 & \mapsto & bq^2 \\ \overline{y}_5 & \mapsto & b^5 - 4bq^2 \equiv 0 \\ \overline{z}_5 & \mapsto & -2bq^2 - \frac{4}{3}vq^3 + 3v^3q^4 - \frac{2}{3}v^5q^5 \\ & & \equiv -2bq^2 + 4vq^3 - v^3q^4 \\ \bar{t}_5 & \mapsto & 0 \\ \overline{u}_5 & \mapsto & bq^2 - \frac{1}{3}vq^3 - v^3q^4 + \frac{1}{3}v^5q^5 \\ & & \equiv bq^2 - 3vq^3 + v^3q^4 \\ \overline{w}_5 & \mapsto & \frac{8}{3}vq^3 - 2v^3q^4 + \frac{1}{3}v^5q^5 \equiv 0. \end{array}$$

Also, we have $p = d^2 + b^4 - 2vbd^2 \mapsto q^2 + b^4 - 2vbq^2 \equiv q^2$. Combining these facts with $\widetilde{\alpha}_{4i-\varepsilon} = \frac{\overline{\alpha}_\varepsilon}{p^i}$ for $i \geq 1$ and $\widetilde{\alpha}_{4i-5} = \frac{\overline{\alpha}_5}{p^i}$ for $i \geq 2$, where $\varepsilon = 2, 3, 4$ and $\alpha \in \{x, y, z, t, u, w\}$, we get that;

$$\begin{array}{lll} \widetilde{x}_{4i-2} & \mapsto & \frac{b^2}{q^{2i}} \\ \widetilde{y}_{4i-2} & \mapsto & 0 \\ \widetilde{z}_{4i-2} & \mapsto & -2\frac{b^2}{q^{2i}} + 8\frac{q}{q^{2i}} - 2\frac{v^2q^2}{q^{2i}} \\ \widetilde{t}_{4i-2} & \mapsto & -\frac{b^2}{q^{2i}} + 4\frac{q}{q^{2i}} - \frac{v^2q^2}{q^{2i}} \\ \widetilde{u}_{4i-2} & \mapsto & 0 \\ \widetilde{w}_{4i-2} & \mapsto & \frac{q}{q^{2i}} \end{array} , \quad \begin{array}{lll} \widetilde{x}_{4i-3} & \mapsto & \frac{b}{q^{2i-1}} \\ \widetilde{y}_{4i-3} & \mapsto & 0 \\ \widetilde{z}_{4i-3} & \mapsto & -2\frac{b}{q^{2i-1}} + 4\frac{vq}{q^{2i-1}} - \frac{v^3q^2}{q^{2i-1}} \\ \widetilde{t}_{4i-3} & \mapsto & 0 \\ \widetilde{u}_{4i-3} & \mapsto & \frac{b}{q^{2i-1}} - 3\frac{vq}{q^{2i-1}} + \frac{v^3q^2}{q^{2i-1}} \\ \widetilde{w}_{4i-3} & \mapsto & 0 \end{array}$$

$$\begin{array}{lll} \widetilde{x}_{4i-4} & \mapsto & \frac{b^2}{q^{2i-1}} \\ \widetilde{y}_{4i-4} & \mapsto & 0 \\ \widetilde{z}_{4i-4} & \mapsto & -2\frac{b^2}{q^{2i-1}} + 8\frac{q}{q^{2i-1}} - 2\frac{v^2q^2}{q^{2i-1}} \\ \widetilde{t}_{4i-4} & \mapsto & 0 \\ \widetilde{u}_{4i-4} & \mapsto & \frac{q}{q^{2i-1}} \\ \widetilde{w}_{4i-4} & \mapsto & -\frac{b^2}{q^{2i-1}} + 4\frac{q}{q^{2i-1}} - \frac{v^2q^2}{q^{2i-1}} \end{array} , \quad \begin{array}{lll} \widetilde{x}_{4i-5} & \mapsto & \frac{b}{q^{2i-2}} \\ \widetilde{y}_{4i-5} & \mapsto & 0 \\ \widetilde{z}_{4i-5} & \mapsto & -2\frac{b}{q^{2i-2}} + 4\frac{vq}{q^{2i-2}} - \frac{v^3q^2}{q^{2i-2}} \\ \widetilde{t}_{4i-5} & \mapsto & 0 \\ \widetilde{u}_{4i-5} & \mapsto & \frac{b}{q^{2i-2}} - 3\frac{vq}{q^{2i-2}} + \frac{v^3q^2}{q^{2i-2}} \\ \widetilde{w}_{4i-5} & \mapsto & 0. \end{array}$$

Hence by definition in Proposition 3.6.3 and definition in Theorem 3.5.1, we complete the proof. □

3.6.3 $ku_*(BSD_{16})$ and $ku_*(BC_8)$

To see the relation between $ku_*(BSD_{16})$ and $ku_*(BC_8)$ as in the previous subsection, we need to explicit $ku_*(BC_8)$. To do that, it is simple to use the same method as $ku_*(BSD_{16})$, i.e. by using Greenlees spectral sequence. It is enough to calculate kernel and cokernel of the map $R''' := ku^*(BC_8) \longrightarrow R'''[\frac{1}{y}]$, where $y = \frac{1-\alpha}{v} := c_1$ and $\alpha^8 = 1$. However, to be simple in comparing with $ku_*(BSD_{16})$, we intend to use $y' := c_1 + c_3 - c_4$, where

$$c_i := \frac{1 - \alpha^i}{v}$$

for $i \geq 1$ with $c_{8k+\epsilon} = c_\epsilon$ (since $c_8 = 0$). This is possible (i.e. multiplying by y' gives an isomorphism $R'''_{-2k} \xrightarrow{\cong} R'''_{-2k-2}$ for all $k \geq 1$) by Lemma 3.4.1 in [14] and changing basis.

The kernel of the map is $\mathbb{Z}[v] < \rho''' >$ which is contributed to $ku_{even}(BC_8)$, where $\rho''' = 1 + \alpha + \alpha^2 + ... + \alpha^7$. For $ku_{odd}(BC_8)$, we need to calculate the quotient groups

$$[H^1_{(y')}(R''')]_{2n} = R'''_{-2}/(y')^{n+1}R'''_{2n}$$

for all $n \geq 0$, where $R'''_{-2} = \mathbb{Z}_2^\wedge < c_1, c_2, ..., c_7 >$, $R'''_0 = \mathbb{Z}_2^\wedge < \alpha, \alpha^2, ..., \alpha^7 > \oplus \mathbb{Z}$ and $R'''_{2n} = v^n R'''_0$. It is not hard to see that the basis of $(y')^{n+1}R'''_{2n}$ is reduced to

$$\{(y')^{n+1}v^n\alpha, (y')^{n+1}v^n\alpha^2, ..., (y')^{n+1}v^n\alpha^7\},$$

and thus we do need to explicit $(y')^k v^{k-1}\alpha^j$ in term of c_i's in $ku^2(BC_8)$.

By direct calculation, we have some useful properties for our calculation which are;

$$\begin{array}{rcl} c_i\alpha^j & = & c_{i+j} - c_j \\ vc_ic_j & = & c_i + c_j - c_{i+j} \\ v[c_3c_5 - c_4c_5] & = & c_1 + c_3 - c_4 = y' \\ y'\alpha^j & = & -c_j + c_{j+1} + c_{j+3} - c_{j+4} \\ vy'c_k & = & c_1 + c_3 - c_4 + c_k - c_{k+1} - c_{k+3} + c_{k+4}. \end{array} \tag{3.7}$$

By using equation 4 and 5 in (3.7), we also have;

$$\begin{array}{rcl} (y')^2v\alpha^j & = & -2c_j + 2c_{j+1} - c_{j+2} + 2c_{j+3} - 4c_{j+4} + 2c_{j+5} - c_{j+6} + 2c_{j+7} \\ (y')^3v^3\alpha^j & = & -10c_j + 7c_{j+1} - 6c_{j+2} + 7c_{j+3} - 10c_{j+4} + 9c_{j+5} - 6c_{j+6} + 9c_{j+7}. \end{array}$$

Repeating this process, observing the pattern and using inductive proof, we get that;

$$\begin{array}{rclcl} (y')^{2k}v^{2k-1}\alpha^j & = & -(2\cdot 16^{k-1} + 4^{k-1} - (-2)^{k-1})c_j & + & 2\cdot 16^{k-1}c_{j+1} \\ & & -(2\cdot 16^{k-1} - 4^{k-1})c_{j+2} & + & 2\cdot 16^{k-1}c_{j+3} \\ & & -(2\cdot 16^{k-1} + 4^{k-1} + (-2)^{k-1})c_{j+4} & + & 2\cdot 16^{k-1}c_{j+5} \\ & & -(2\cdot 16^{k-1} - 4^{k-1})c_{j+6} & + & 2\cdot 16^{k-1}c_{j+7} \end{array} \tag{3.8}$$

$$\begin{array}{rclcl} (y')^{2k+1}v^{2k}\alpha^j & = & -(8\cdot 16^{k-1} + 2\cdot 4^{k-1})c_j & + & (8\cdot 16^{k-1} - (-2)^{k-1})c_{j+1} \\ & & -(8\cdot 16^{k-1} - 2\cdot 4^{k-1})c_{j+2} & + & (8\cdot 16^{k-1} - (-2)^{k-1})c_{j+3} \\ & & -(8\cdot 16^{k-1} + 2\cdot 4^{k-1})c_{j+4} & + & (8\cdot 16^{k-1} + (-2)^{k-1})c_{j+5} \\ & & -(8\cdot 16^{k-1} - 2\cdot 4^{k-1})c_{j+6} & + & (8\cdot 16^{k-1} + (-2)^{k-1})c_{j+7}. \end{array} \tag{3.9}$$

The results of $ku_*(BC_8)$ follow as;

Proposition 3.6.5. *With the same symbols as above,* $ku_*(BC_8) = ku_{even}(BC_8) \oplus ku_{odd}(BC_8)$, *where* $ku_{even}(BC_8) = \mathbb{Z}[v] < \rho''' >$ *and, for* $k \geq 1$, $ku_{odd}(BC_8) = ku_{4k-3}(BC_8) \oplus ku_{4k-1}(BC_8)$ *such that*

◇ $ku_{4k-3}(BC_8) = \mathbb{Z}/2^{k-1} \oplus \mathbb{Z}/2^{k-1} \oplus \mathbb{Z}/2^{k-1} \oplus \mathbb{Z}/2^{k-1} \oplus \mathbb{Z}/2^k \oplus \mathbb{Z}/4^k \oplus \mathbb{Z}/2\cdot 4^k$,

generated by

1. $c_{1,2k-1} + ((-2)^k + 1)c_{3,2k-1} - c_{7,2k-1} + (4^k - 1)c_{5,2k-1}$,
2. $c_{2,2k-1} + (2\cdot 4^{k-1} - (-2)^k)c_{1,2k-1} - ((-2)^k + 1)c_{6,2k-1} + 3(-2)^{k-1}c_{3,2k-1}$ $-3(-2)^{k-1}c_{7,2k-1} - (4^k + (-2)^k)c_{5,2k-1}$,
3. $c_{4,2k-1} + c_{6,2k-1} - c_{2,2k-1} - (2 + 5(-2)^{k-1})c_{3,2k-1} + 5(-2)^k c_{7,2k-1}$ $+(4^k + (-2)^k)c_{5,2k-1}$,
4. $c_{3,2k-1} - c_{7,2k-1} + (-2)^k c_{5,2k-1}$,

5. $c_{7,2k-1}-(-2)^k c_{5,2k-1}$,
6. $c_{5,2k-1}+c_{6,2k-1}$
7. $c_{5,2k-1}$,

respectively and

◇ $ku_{4k-1}(BC_8)=\mathbb{Z}/2^{k-1}\oplus\mathbb{Z}/2^{k-1}\oplus\mathbb{Z}/2^k\oplus\mathbb{Z}/2^k\oplus\mathbb{Z}/4^k\oplus\mathbb{Z}/2\cdot 4^k\oplus\mathbb{Z}/2\cdot 4^k$,

generated by $c_{1,2k}-c_{3,2k}-c_{5,2k}+(-2)^{k+1}c_{6,2k}+c_{7,2k}$, $c_{2,2k}-c_{4,2k}-c_{6,2k}+(-2)^{k+1}c_{7,2k}$, $c_{3,2k}$, $c_{4,2k}$, $c_{5,2k}$, $c_{6,2k}$ *and* $c_{7,2k}$ *respectively, where*

$$c_{i,2k-1}=\tfrac{c_i}{(y')^{2k}} \text{ and } c_{i,2k}=\tfrac{c_i}{(y')^{2k+1}}$$

for $i=1,2,...,7$.

Proof. It remains to do row and column operations on matrix obtained by (3.9) and (3.8) of both odd case and even case. It is not hard to see that $[H^1_{(y')}(R''')]_0=0$.

For $[H^1_{(y')}(R''')]_{4k}=R'''_{-2}/(y')^{2k+1}R'''_{4k}$, we set $r_j:=(y')^{2k+1}\alpha^j$, for $j=1,2,...,7$, and we do row operations by:

1. $r_1\longrightarrow[r'_1=r_1-r_7]\longrightarrow[r''_1=r'_1-r'_5+r'_3]\longrightarrow[r'''_1=r''_1+2r'''_3]\longrightarrow[r^4_1=r'''_1+2r^6_5]\longrightarrow[r^*_1=\frac{1}{1+5\cdot 4^k}r^4_1]$,

2. $r_2\longrightarrow[r'_2=r_2-r_6]\longrightarrow[r''_2=r'_2+2r''_6+r''_5]\longrightarrow[r'''_2=r''_2+(2(-2)^k-1)r'''_3]\longrightarrow[r^4_2=r'''_2+(-2+2(-2)^k)r^*_1]\longrightarrow[r^*_2=r^4_2+(2(-2)^k-1)r^*_5]$,

3. $r_3\longrightarrow[r'_3=r_3-r_7]\longrightarrow[r''_3=r'_3-r''_1]\longrightarrow[r'''_3=r''_3-r''_5]\longrightarrow[r^4_3=r'''_3+r^*_1]\longrightarrow[r^*_3=r^4_3+r^*_5]$,

4. $r_4\longrightarrow[r'_4=r_4-r_6]\longrightarrow[r''_4=r'_4-(-2)^k r''_3]\longrightarrow[r'''_4=r''_4-r'_2]\longrightarrow[r^4_4=r'''_4+2(-2)^k r''_5]\longrightarrow[r^5_4=r^4_4-2(-2)^k r^*_1]\longrightarrow[r^6_4=r^5_4-2(-2)^k r^*_5]\longrightarrow[r^*_4=r^6_4+r^*_2]$,

5. $r_5\longrightarrow[r'_5=r_5-r_7]\longrightarrow[r''_5=r'_5-(-2)^k r'_2]\longrightarrow[r'''_5=r''_5+2r''_7]\longrightarrow[r^4_5=r'''_5-2(-8)^k r^4_4-r'''_3]\longrightarrow[r^5_5=r^4_5+(4^k-2(-8)^k)r'''_2]\longrightarrow[r^6_5=r^5_5+(2\cdot 4^{k-1}-2(-8)^k)r'''_1]\longrightarrow[r^*_5=r^6_5-(2+10\cdot 4^{k-1})r^*_1]$,

6. $r_6\longrightarrow[r'_6=r_6+r_7]\longrightarrow[r''_6=r'_6-(-2)^{k-1}r''_2+(-2)^{k-1}r''_3]\longrightarrow[r'''_6=r''_6-r''_7]\longrightarrow[r^4_6=r'''_6+4^k r^*_1]\longrightarrow[r^5_6=r^4_6-2\cdot 4^{k-1}r^*_2]\longrightarrow[r^*_6=r^5_6+(-2)^k r^*_3+(-8)^k r^*_4]$,

7. $r_7\longrightarrow[r'_7=r_7]\longrightarrow[r''_7=r'_7+4(-8)^{k-1}r'_2-4(-8)^{k-1}r''_3-(-2)^{k-1}r''_2]\longrightarrow[r'''_7=r''_7-4^k r^*_1]\longrightarrow[r^4_7=r'''_7+2\cdot 4^{k-1}r^*_2]\longrightarrow[r^*_7=r^4_7-(-8)^k r^*_4]$.

We now get;

$$\begin{array}{cccccccc|}
 & c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\
r_1^*| & 0 & 0 & 0 & 0 & -4^k & 0 & 0 \\
r_2^*| & 0 & 0 & 0 & 0 & 0 & -2\cdot 4^k & 0 \\
r_3^*| & 0 & 0 & 0 & -(-2)^k & 0 & 0 & 0 \\
r_4^*| & 0 & 0 & -(-2)^k & 0 & 0 & 0 & 0 \\
r_5^*| & 0 & 0 & 0 & 0 & 0 & 0 & 2\cdot 4^k \\
r_6^*| & -(-2)^{k-1} & 0 & (-2)^{k-1} & 0 & (-2)^{k-1} & -4^k & -(-2)^{k-1} \\
r_7^*| & 0 & -(-2)^{k-1} & 0 & (-2)^{k-1} & 0 & (-2)^{k-1} & -4^k
\end{array}.$$

After doing some column operations, the results for $ku_{4k-1}(BC_8)$ follow.

For $[H^1_{(y')}(R''')]_{4k-2} = R'''_{-2}/(y')^{2k}R'''_{4k-2}$, we set $r_j := (y')^{2k}\alpha^j$, for $j = 1, 2, ..., 7$, and we do row operations by:

1. $r_1 \longrightarrow [r_1' = r_1 - r_5] \longrightarrow [r_1'' = r_1' + 2r_5''] \longrightarrow [r_1''' = r_1'' + (2^k - 1)r_3'] \longrightarrow [r_1^4 = r_1''' + 2(-2)^k r_3'''] \longrightarrow [r_1^5 = r_1^4 + 2r_2^6] \longrightarrow [r_1^6 = r_1^5 - 2r_2^7] \longrightarrow [r_1^* = \frac{-1}{1+3(-8)^k} r_1^6]$,

2. $r_2 \longrightarrow [r_2' = r_2 - r_6] \longrightarrow [r_2'' = r_2' + 2r_6''] \longrightarrow [r_2''' = r_2'' + (2(-2)^{k-1} - 4^k)r_5''] \longrightarrow [r_2^4 = r_2''' + (1 + (-2)^k - 2\cdot 4^{k-1} + 4(-8)^{k-1})r_3'] \longrightarrow [r_2^5 = r_2^4 + (1 + (-2)^k + 2\cdot 4^{k-1})r_1''''] \longrightarrow [r_2^6 = r_2^5 - (-2)^k r_3''''] \longrightarrow [r_2^7 = r_2^6 + ((-2)^{k-1} + 4(-8)^{k-1})r_1^5] \longrightarrow [r_2^* = r_2^7 + 12(-8)^{k-1}r_1^*]$,

3. $r_3 \longrightarrow [r_3' = r_3 - r_7] \longrightarrow [r_3'' = r_3' + 2r_7^4] \longrightarrow [r_3''' = r_3'' - (-2)^{k-1}r_1''' - r_2^5] \longrightarrow [r_3^4 = r_3''' + (4^k + 1)r_2^*] \longrightarrow [r_3^* = r_3^4 + 4^k r_1^*]$,

4. $r_4 \longrightarrow [r_4' = r_4 - r_6] \longrightarrow [r_4'' = r_4' + 2r_6'] \longrightarrow [r_4^* = r_4'' - r_3^*]$,

5. $r_5 \longrightarrow [r_5' = r_5 - r_7] \longrightarrow [r_5'' = r_5' + (-2)^{k-1}r_1'] \longrightarrow [r_5^* = r_5'' + r_3^*]$,

6. $r_6 \longrightarrow [r_6' = r_6 + r_7] \longrightarrow [r_6'' = r_6' - (-2)^{k-1}r_4''] \longrightarrow [r_6''' = r_6'' - (-2)^{k-1}r_5^*] \longrightarrow [r_6^4 = r_6''' - (-2)^k r_3^*] \longrightarrow [r_6^* = r_6^4 - r_7^*]$,

7. $r_7 \longrightarrow [r_7' = r_7 - 2\cdot 4^{k-1}r_6'] \longrightarrow [r_7'' = r_7' - 2(-8)^{k-1}r_2' + 2\cdot 4^{k-1}r_6''] \longrightarrow [r_7''' = r_7'' - 4^{k-1}r_2'' + 2(-8)^{k-1}r_1' + (-2)^{k-1}r_5''] \longrightarrow [r_7^4 = r_7''' - (2\cdot 4^{k-1} + 2(-8)^{k-1})r_3'] \longrightarrow [r_7^5 = r_7^4 + 4^{k-1}r_1^* + (-2)^{k-1}r_3^*] \longrightarrow [r_7^6 = r_7^5 + 2\cdot 4^{k-1}r_2^8 + 4^{k-1}r_1^*] \longrightarrow [r_7^* = r_7^6 - 2\cdot 4^{k-1}r_3^* + 4^{k-1}r_1^*]$.

We now get;

$$
\begin{array}{c|cccc}
 & c_1 & c_2 & c_3 & c_4 \\
r_1^* & 0 & 0 & 0 & 0 \\
r_2^* & 0 & 0 & 0 & 0 \\
r_3^* & 0 & 0 & 0 & 0 \\
r_4^* & 2\cdot 4^{k-1} & -(-2)^{k-1} & (-2)^k - 2\cdot 4^{k-1} & (-2)^{k-1} \\
r_5^* & -(-2)^{k-1} & 0 & (-2)^{k-1} + 2\cdot 4^{k-1} & 0 \\
r_6^* & 2\cdot 4^{k-1} - 2(-8)^{k-1} & -(-2)^{k-1} & 0 & 0 \\
r_7^* & 0 & 0 & -(-2)^{k-1} & 0
\end{array}
$$

$$
\begin{array}{ccc|}
c_5 & c_6 & c_7 \\
2\cdot 4^k & 0 & 0 \\
-4^k & -4^k & 0 \\
4^k & 0 & -(-2)^k \\
-2\cdot 4^{k-1} & (-2)^{k-1} & 2\cdot 4^{k-1} \\
(-2)^{k-1} & 0 & (-2)^{k-1} + 2\cdot 4^{k-1} \\
6(-8)^{k-1} - 2\cdot 4^{k-1} & (-2)^{k-1} - 2\cdot 4^{k-1} & 0 \\
2\cdot 4^{k-1} & 0 & (-2)^{k-1}
\end{array}.
$$

After doing column operations by;

$$
\begin{aligned}
c_1' &= c_1 + (2(-2)^{k-1} - 2\cdot 4^{k-1})c_2 - 2\cdot 4^{k-1}c_4, \\
c_2' &= c_2 + c_4, \\
c_3' &= (1 + 2(-2)^{k-1})c_1 + [2(-2)^{k-1} + 2\cdot 4^{k-1} - 4(-8)^{k-1}]c_2 \\
&\quad + c_3 + [2 + 2(-2)^{k-1} - 2\cdot 4^{k-1} - 4(-8)^{k-1}]c_4, \\
c_4' &= c_4, \\
c_5' &= c_5 - c_6 + (-2)^k c_7 + (8\cdot 16^{k-1} - 4(-8)^{k-1} + 2(-2)^{k-1} - 1)c_2 \\
&\quad + (1 + (-2)^k - 4^k)c_1 + (2\cdot 4^k - (-2)^k + 4(-8)^{k-1} + 8\cdot 16^{k-1})c_4, \\
c_6' &= c_6 + (1 + (-2)^k)c_2 + (-2)^k c_4, \\
c_7' &= c_7 + c_3 + (2 - (-2)^k)c_1 + (2 - 4^k - 4(-8)^{k-1})c_4 \\
&\quad + (2 - 2^k)(2(-2)^{k-1} - 2\cdot 4^{k-1})c_2,
\end{aligned}
$$

the results for $ku_{4k-3}(BC_8)$ follow. □

The corestriction of $ku_*(BSD_{16})$ to $ku_*(BC_8)$ is not hard since we have that

$$
\begin{aligned}
a &\mapsto c_4 \\
b &\mapsto 0 \\
d &\mapsto \frac{y'}{v} \\
\frac{1}{p^i} &\mapsto \frac{v^{2i}}{(y')^{2i}} \equiv \frac{v^{2i}(y')^{2i}}{(y')^{4i}},
\end{aligned}
\tag{3.10}
$$

and also by (3.7), we have that

$$
\begin{aligned}
v(y')^2 &= 2c_1 - c_2 + 2c_3 - 4c_4 + 2c_5 - c_2 + 2c_7 \\
v^2(y')^3 &= 7c_1 - 6c_2 + 7c_3 - 10c_4 + 9c_5 - 6c_2 + 9c_7 \\
v^3(y')^4 &= 32c_1 - 28c_2 + 32c_3 - 34c_4 + 32c_5 - 28c_2 + 32c_7 \\
v^4(y')^5 &= 130c_1 - 120c_2 + 130c_3 - 136c_4 + 126c_5 - 120c_2 + 126c_7.
\end{aligned}
\tag{3.11}
$$

By, again, direct calculation, observing the pattern and inductive proof, we have;

$$\begin{array}{rcllll} v^{2k}(y')^{2k} & = & [2\cdot 16^{k-1}+4^{k-1}-(-2)^{k-1}]1 & - & 2\cdot 16^{k-1}\alpha \\ & & +[2\cdot 16^{k-1}-4^{k-1}]\alpha^2 & - & 2\cdot 16^{k-1}\alpha^3 \\ & & +[2\cdot 16^{k-1}+4^{k-1}+(-2)^{k-1}]\alpha^4 & - & 2\cdot 16^{k-1}\alpha^5 \\ & & +[2\cdot 16^{k-1}-4^{k-1}]\alpha^6 & - & 2\cdot 16^{k-1}\alpha^7 \end{array} \tag{3.12}$$

and

$$\begin{array}{rcllll} v^{2k+1}(y')^{2k+1} & = & [8\cdot 16^{k-1}+2\cdot 4^{k-1}]1 & - & [8\cdot 16^{k-1}-(-2)^{k-1}]\alpha \\ & & +[8\cdot 16^{k-1}-2\cdot 4^{k-1}]\alpha^2 & - & [8\cdot 16^{k-1}-(-2)^{k-1}]\alpha^3 \\ & & +[8\cdot 16^{k-1}+2\cdot 4^{k-1}]\alpha^4 & - & [8\cdot 16^{k-1}+(-2)^{k-1}]\alpha^5 \\ & & +[8\cdot 16^{k-1}-2\cdot 4^{k-1}]\alpha^6 & - & [8\cdot 16^{k-1}+(-2)^{k-1}]\alpha^7 \end{array} \tag{3.13}$$

Thus, if we set $\alpha^* := v^n(y')^n = \sum_{i=0}^{i=7} a_{i+1}\alpha^i$ for any $n \geq 0$, then the action of α^* on c_j's is given by

$$c_j\alpha^* = \sum_{i=0}^{7} a_{i+1}c_{j+i} - (\sum_{i=1}^{7} a_{i+1}c_i).$$

The corestriction in even degree is given by $\rho \mapsto 2\rho'''$. Indeed,

$$\begin{array}{rcl} \rho & = & 16\cdot 1 - 6va - 8vb + v^2ab - 12v^2d + 6v^3ad + 2v^4d^2 \\ & \mapsto & 16 - 6vc_4 - 12v^2(\dfrac{y'}{v}) + 6v^3c_4(\dfrac{y'}{v}) + 2v^4(\dfrac{y'}{v})^2 \\ & = & 16 - 2v[c_1+c_2+c_3+c_4+c_5+c_6+c_7], \quad \text{(by (3.11))} \\ & = & 2[1+\alpha+\alpha^2+...+\alpha^7] = 2\rho'''. \end{array}$$

And in odd degree the corestriction can be read by definition in Theorem 3.5.1 and the restrictions below (which the proof is simple by using the explicit generators of JU_2, JU_3, JU_4, JU_5 and (3.10), (3.11), (3.12), (3.13) and α^* action.)

Degree $8i-5$, $i \geq 1$.

$$\begin{array}{rcl} \widetilde{x}_{4i-2} & \mapsto & 0 \\ \widetilde{y}_{4i-2} & \mapsto & 0 \\ \widetilde{z}_{4i-2} & \mapsto & [-3vc_4(\frac{y'}{v}) + \frac{16}{3}(\frac{y'}{v}) - \frac{1}{3}v^4(\frac{y'}{v})^3](\frac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\ & \equiv & [-3vc_4y' + \frac{16}{3}y' - \frac{1}{3}v^2(y')^3](\frac{v^{2i-1}(y')^{2i-1}}{(y')^{4i-1}}) \\ & = & [2(c_2-c_4+c_6)]\frac{\alpha^*}{(y')^{4i-1}} \\ & = & 4^i[c_{2,4i-2} - c_{4,4i-2} + c_{6,4i-2}] \end{array}$$

$$\begin{array}{rcl} \widetilde{t}_{4i-2} & \mapsto & [-2vc_4(\frac{y'}{v}) + 4(\frac{y'}{v}) - v^2(\frac{y'}{v})^2](\frac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\ & \equiv & [-2vc_4y' + 4y' - v(y')^2](\frac{v^{2i-1}(y')^{2i-1}}{(y')^{4i-1}}) \\ & = & [-3c_1 + c_2 - 3c_3 + 3c_4 + c_6)]\frac{\alpha^*}{(y')^{4i-1}} \\ & = & (-6\cdot 16^{i-1} + (-2)^{i-1})c_{1,4i-2} + (6\cdot 16^{i-1} - 4^{i-1})c_{2,4i-2} \\ & & +(-6\cdot 16^{i-1} + (-2)^{i-1})c_{3,4i-2} + (6\cdot 16^{i-1} + 4^{i-1} + 3(-2)^{i-1})c_{4,4i-2} \\ & & +(-6\cdot 16^{i-1} - (-2)^{i-1})c_{5,4i-2} + (6\cdot 16^{i-1} - 4^{i-1})c_{6,4i-2} \\ & & +(-6\cdot 16^{i-1} - (-2)^{i-1})c_{7,4i-2} \end{array}$$

$$\begin{aligned}
\widetilde{u}_{4i-2} &\mapsto [-vc_4(\tfrac{y'}{v})](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
&\equiv [-vc_4y'](\tfrac{v^{2i-1}(y')^{2i-1}}{(y')^{4i-1}}) \\
&= [-(c_1+c_3-c_5-c_7)]\tfrac{\alpha^*}{(y')^{4i-1}} \\
&= -(-2)^i c_{4,4i-2}
\end{aligned}$$

$$\begin{aligned}
\widetilde{w}_{4i-2} &\mapsto [(\tfrac{y'}{v}) - vc_4(\tfrac{y'}{v})](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
&\equiv [y' - vc_4y'](\tfrac{v^{2i-1}(y')^{2i-1}}{(y')^{4i-1}}) \\
&= [c_5+c_7-c_4]\tfrac{\alpha^*}{(y')^{4i-1}} \\
&= 4^{i-1}c_{2,4i-2} - (3(-2)^{i-1}+4^{i-1})c_{4,4i-2} + 4^{i-1}c_{6,4i-2}
\end{aligned}$$

Degree $8i-7$, $i \geq 1$.

$$\begin{aligned}
\widetilde{x}_{4i-3} &\mapsto 0 \\
\widetilde{y}_{4i-3} &\mapsto 0 \\
\widetilde{z}_{4i-3} &\mapsto [-3c_4(\tfrac{y'}{v}) - \tfrac{4}{3}v(\tfrac{y'}{v})^2 + 3v^3(\tfrac{y'}{v})^3 - \tfrac{2}{3}v^5(\tfrac{y'}{v})^4](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
&\equiv [-3vc_4y' - \tfrac{4}{3}v(y')^2 + 3v^2(y')^3 - \tfrac{2}{3}v^3(y')^4](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-2}}) \\
&= [-6c_1+2c_2-6c_3-2c_4+6c_5+2c_6+6c_7]\tfrac{\alpha^*}{(y')^{4i-2}} \\
&= -6\cdot(-2)^{i-1}[c_{1,4i-3}+c_{3,4i-3}-c_{5,4i-3}] + 2\cdot 4^{i-1}[c_{2,4i-3}-c_{4,4i-3}+c_{6,4i-3}]
\end{aligned}$$

$$\begin{aligned}
\widetilde{t}_{4i-3} &\mapsto [2c_4(\tfrac{y'}{v}) - \tfrac{8}{3}v(\tfrac{y'}{v})^2 + 2v^3(\tfrac{y'}{v})^3 - \tfrac{1}{3}v^5(\tfrac{y'}{v})^4](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
&\equiv [2vc_4y' - \tfrac{8}{3}v(y')^2 + 2v^2(y')^3 - \tfrac{1}{3}v^3(y')^4](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-2}}) \\
&= [2c_4]\tfrac{\alpha^*}{(y')^{4i-2}} \\
&= -(-2)^i c_{4,4i-3}
\end{aligned}$$

$$\begin{aligned}
\widetilde{u}_{4i-3} &\mapsto [v(\tfrac{y'}{v})^2](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}}) - im(\widetilde{x}_{4i-3}) - im(\widetilde{z}_{4i-3}) \\
&\equiv [v(y')^2](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-2}}) - im(\widetilde{z}_{4i-3}) \\
&= [2c_1-c_2+2c_3-4c_4+2c_5-c_6+2c_7]\tfrac{\alpha^*}{(y')^{4i-2}} - im(\widetilde{z}_{4i-3})
\end{aligned}$$

$$= \begin{cases} 8c_{1,4i-3} - 3c_{2,4i-3} + 8c_{3,4i-3} - 2c_{4,4i-3} - 4c_{5,4i-3} - 3c_{6,4i-3} - 4c_{7,4i-3}, \\ \text{if } i = 1; \\ (28\cdot 16^{i-2} - 4^{i-2} - 11(-2)^{i-2})c_{1,4i-3} + (-28\cdot 16^{i-2} - 4^{i-1} - (-2)^{i-2})c_{2,4i-3} \\ +(28\cdot 16^{i-2} + 4^{i-2} + 6(-2)^{i-1})c_{3,4i-3} + (-28\cdot 16^{i-2} + 3\cdot 4^{i-2} - (-2)^{i-1})c_{4,4i-3} \\ +(28\cdot 16^{i-2} - 4^{i-2} + 11(-2)^{i-2})c_{5,4i-3} + (-28\cdot 16^{i-2} - 3\cdot 4^{i-2} + (-2)^{i-2})c_{6,4i-3} \\ +(28\cdot 16^{i-2} + 4^{i-2} - 6(-2)^{i-1})c_{7,4i-3}, \quad \text{if } i \geq 2. \end{cases}$$

$$\begin{aligned}
\widetilde{w}_{4i-3} &\mapsto [-c_4(\tfrac{y'}{v})](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
&\equiv [-vc_4y'](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-2}}) \\
&= [-(c_1+c_3-c_5-c_7)]\tfrac{\alpha^*}{(y')^{4i-2}} \\
&= -(-2)^{i-1}[c_{1,4i-3}+c_{3,4i-3}-c_{5,4i-3}-c_{7,4i-3}]
\end{aligned}$$

Degree $8i-9$, $i \geq 2$.

$$
\begin{aligned}
\widetilde{x}_{4i-4} &\mapsto 0\\
\widetilde{y}_{4i-4} &\mapsto 0\\
\widetilde{z}_{4i-4} &\mapsto [-3vc_4(\tfrac{y'}{v})^2 + \tfrac{16}{3}(\tfrac{y'}{v})^2 - \tfrac{1}{3}v^4(\tfrac{y'}{v})^4](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}})\\
&\equiv [-3vc_4y' + \tfrac{16}{3}y' - \tfrac{1}{3}v^2(y')^3](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-3}})\\
&= [2c_2 - 2c_4 + 2c_6]\tfrac{\alpha^*}{(y')^{4i-3}}\\
&= 2\cdot 4^{i-1}[c_{2,4i-4} - c_{4,4i-4} + c_{6,4i-4}]
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{t}_{4i-4} &\mapsto [-vc_4(\tfrac{y'}{v})^2](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}})\\
&\equiv [-vc_4y'](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-3}})\\
&= [-(c_1 + c_3 - c_5 - c_7)]\tfrac{\alpha^*}{(y')^{4i-3}}\\
&= -(-2)^{i-1}[c_{1,4i-4} + c_{3,4i-4} - c_{5,4i-4} - c_{7,4i-4}]
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{u}_{4i-4} &\mapsto [(\tfrac{y'}{v})^2](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}})\\
&\equiv [y'](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-3}})\\
&= [c_1 + c_3 - c_4]\tfrac{\alpha^*}{(y')^{4i-3}}\\
&= \begin{cases} c_{1,4i-4} + c_{3,4i-4} - c_{4,4i-4}, & \text{if } i = 1;\\ (8\cdot 16^{i-2} - (-2)^{i-2})c_{1,4i-4} + (-8\cdot 16^{i-2} + 2\cdot 4^{i-2})c_{2,4i-4} \\ (8\cdot 16^{i-2} - (-2)^{i-2})c_{3,4i-4} + (-8\cdot 16^{i-2} - 2\cdot 4^{i-2})c_{4,4i-4} \\ (8\cdot 16^{i-2} + (-2)^{i-2})c_{5,4i-4} + (-8\cdot 16^{i-2} + 2\cdot 4^{i-2})c_{6,4i-4} \\ (8\cdot 16^{i-2} + (-2)^{i-2})c_{7,4i-4}, & \text{if } i \geq 2. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{w}_{4i-4} &\mapsto [-vc_4(\tfrac{y'}{v})^2 + \tfrac{4}{3}(\tfrac{y'}{v})^2 + v^2(\tfrac{y'}{v})^3 - \tfrac{1}{3}v^4(\tfrac{y'}{v})^4](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}})\\
&\equiv [-vc_4y' + \tfrac{4}{3}y' + v(y')^2 - \tfrac{1}{3}v^2(y')^3](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-3}})\\
&= [c_2 - 2c_4 + c_6]\tfrac{\alpha^*}{(y')^{4i-3}}\\
&= 4^{i-1}c_{2,4i-4} - (4^{i-1} + (-2)^{i-1})c_{4,4i-4} + 4^{i-1}c_{6,4i-4}]
\end{aligned}
$$

Degree $8i-11$, $i \geq 2$.

$$
\begin{aligned}
\widetilde{x}_{4i-5} &\mapsto 0\\
\widetilde{y}_{4i-5} &\mapsto 0\\
\widetilde{z}_{4i-5} &\mapsto [-3c_4(\tfrac{y'}{v})^2 - \tfrac{4}{3}v(\tfrac{y'}{v})^3 + 3v^3(\tfrac{y'}{v})^4 - \tfrac{2}{3}v^5(\tfrac{y'}{v})^5](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}})\\
&\equiv [-3c_4 - \tfrac{4}{3}y' + 3v(y')^2 - \tfrac{2}{3}v^2(y')^3](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-4}})\\
&= [c_2 - 7c_4 + c_6]\tfrac{\alpha^*}{(y')^{4i-4}}\\
&= 4^{i-1}c_{2,4i-5} - (4^{i-1} + 6(-2)^{i-1})c_{4,4i-5} + 4^{i-1}c_{6,4i-5}]
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{t}_{4i-5} &\mapsto [c_4(\tfrac{y'}{v})^2](\tfrac{v^{2i}(y')^{2i}}{(y')^{4i}})\\
&\equiv [c_4](\tfrac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-4}})\\
&= [c_4]\tfrac{\alpha^*}{(y')^{4i-4}}\\
&= (-2)^{i-1}c_{4,4i-5}
\end{aligned}
$$

$$
\begin{array}{rcl}
\widetilde{u}_{4i-5} & \mapsto & [-c_4(\frac{y'}{v})^2 - \frac{1}{3}v(\frac{y'}{v})^3 - v^3(\frac{y'}{v})^4 + \frac{1}{3}v^5(\frac{y'}{v})^5](\frac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
 & \equiv & [-c_4 - \frac{1}{3}y' - v(y')^2 + \frac{1}{3}v^2(y')^3](\frac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-4}}) \\
 & = & [-c_2 + c_5 - c_6 + c_7]\frac{\alpha^*}{(y')^{4i-4}} \\
 & = & \left\{ \begin{array}{l} (4\cdot 16^{i-2} - 4^{i-2} - (-2)^{i-2})c_{1,4i-5} + (-4\cdot 16^{i-2} - 4^{i-2} - (-2)^{i-2})c_{2,4i-5} \\ (4\cdot 16^{i-2} + 4^{i-2} + (-2)^{i-2})c_{3,4i-5} + (-4\cdot 16^{i-2} - 4^{i-2})c_{4,4i-5} \\ (4\cdot 16^{i-2} - 4^{i-2} + (-2)^{i-1})c_{5,4i-5} + (-4\cdot 16^{i-2} - 4^{i-2} + (-2)^{i-2})c_{6,4i-5} \\ (4\cdot 16^{i-2} + 4^{i-2} - (-2)^{i-2})c_{7,4i-5} \end{array} \right.
\end{array}
$$

$$
\begin{array}{rcl}
\widetilde{w}_{4i-5} & \mapsto & [-2c_4(\frac{y'}{v})^2 + \frac{8}{3}v(\frac{y'}{v})^3 - 2v^3(\frac{y'}{v})^4 + \frac{1}{3}v^5(\frac{y'}{v})^5](\frac{v^{2i}(y')^{2i}}{(y')^{4i}}) \\
 & \equiv & [-2c_4 + \frac{8}{3}y' - 2v(y')^2 + \frac{1}{3}v^2(y')^3](\frac{v^{2i-2}(y')^{2i-2}}{(y')^{4i-4}}) \\
 & = & [c_1 + c_3 - c_5 - c_7]\frac{\alpha^*}{(y')^{4i-4}} \\
 & = & (-2)^{i-1}[c_{1,4i-5} + c_{3,4i-5} - c_{5,4i-5} - c_{7,4i-5}]
\end{array}
$$

We finish this chapter here and next we will investigate real connective cohomology of semidihedral group of order 16, $ko^*(BSD_{16})$, by using $\eta-$Bockstein spectral sequence.

Chapter 4

Real connective K-cohomology

In this chapter, we will calculate $ko^*(BSD_{16})$ as a ring by using η-Bockstein spectral sequence (η-BSS, for short) with input $ku^*(BSD_{16})$ and output $ko^*(BSD_{16})$, i.e.

$$E_1^{*,*} = ku^*(BG)[\widetilde{\eta}] \Rightarrow ko^*(BG),$$

where $\widetilde{\eta}$ has bidegree $(1,1)$ and differential $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-1}$, [13]. Since $\eta^3 = 0$ in $ko^*(pt)$, this spectral sequence collapses at E_4-page. Thus the main task is the calculation of E_2-page, differential d_2 and d_3 which can be done by the fact that all entries above the 2-line are all zero at E_4- page together with, again, the help of representation theory (Atiyah-Segal Theorem for the real case).

§ 4.1 Bockstein spectral sequence for $ko^*(BG)$ and strategy

In theory (for the tools), the Bockstein spectral sequence that we use here originally comes from the cofibre sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \xrightarrow{r} \Sigma^2 ko\ ,$$

where c is complexification and vr is realification, see more details about how to construct this tools in [13]. In practices (for the using tools), roughly speaking, to calculate $ko^*(BG)$ for finite group G by η-BSS, we proceed by using the facts and methods from [13];

1 For E_1-page, we can fill $ku^*(BG)$ in the bottom row (i.e. the 0-line) degree by degree and then copy them along the diagonal line via $\widetilde{\eta}$.

2 For E_2-page, we need to determine differential d_1 (see details below) and put its kernel on the 0-line and its homology on the 1-line and then copy the latter along the diagonal line via $\widetilde{\eta}$.

3 $\widetilde{\eta}$ is an infinite cycle and corresponds to $\eta \in ko^{-1}(BG)$.

4 Since $\eta^3 = 0$ in $ko^*(pt)$, $d_k = 0$ for $k \geq 4$ and hence $E_4 = E_\infty$.

5 For d_2 and d_3, we use the fact that all entries above the 2-line are zero and representation theory.

6 At $E_\infty-$page, all entries in column i contribute to $ko^{-i}(BG)$.

For example, in the simple case,

$$E_1^{*,*} = ku^*(pt)[\widetilde{\eta}] = \mathbb{Z}[v, \widetilde{\eta}] \Rightarrow ko^*(pt) = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta),$$

where β is the degree 8 Bott element, α is of degree 4, and η is the image of the Hopf map in degree 1. Note that $\eta \xrightarrow{c} 0$, $\alpha \xrightarrow{c} 2v^2$ and $\beta \xrightarrow{c} v^4$ in $ku^*(pt) = \mathbb{Z}[v]$ (where c is complexification, a ring map), see details in lemma 2.2.11 in [13].

The strategy for E_2-page is firstly decomposing $R := ku^*(BG)$ as v-torsion part (TU) and torsion free part (QU), i.e. considering the short exact sequence

$$0 \longrightarrow TU \longrightarrow ku^*(BG) \longrightarrow QU \longrightarrow 0$$

which we can view as a short exact sequence of chain complex and thus, there is an induced long exact sequence

$$\begin{aligned} 0 \longrightarrow ZTU \longrightarrow ZR \longrightarrow ZQU \longrightarrow^{\delta} \\ HTU \longrightarrow HR \longrightarrow HQU \xrightarrow{\delta} HTU \longrightarrow ... \end{aligned} \tag{4.1}$$

So, instead of calculating the kernel and homology of d_1 on $ku^*(BG)$ directly, we will do that on TU and QU part and after determining differentials δ's we will get ZR and HR as we need. This is because we have a very useful tool from [13] which deal with differentials in TU part and QU part as following.

Lemma 4.1.1. *(cf.[13]) Denote: $\tau =$ complex conjugation, we have*

1 $d_1 = \begin{cases} 1+\tau, & \text{if } d_1 \text{ departs from } QU_{4k+2}; \\ 1-\tau, & \text{if } d_1 \text{ departs from } QU_{4k}. \end{cases}$

2 $d_1 = Sq^2$ *on* TU.

3 $\ker(1+\tau : RU \longrightarrow RU) = (1-\tau)RU$, $RU = \mathbb{Z}\{\rho_i\}$ *where* $\rho_i :=$ *simple representation.*

Proof. To clarify the d_1 differential of Bockstein spectral sequence on both parts, $QU \rightarrowtail KU$ and $TU \rightarrowtail H\mathbb{F}_2$, we go back to the origin of the cofibre sequence by starting with (exact couple);

$$(*) \qquad \begin{tikzcd} S^1 \arrow[r, "\eta"] & S^0 \arrow[r] & C(\eta) \arrow[r] \arrow[drr, "d_1'"] & S^2 \arrow[d, equal] & \\ & & S^3 \arrow[r, "\eta"] & S^2 \arrow[r] & \Sigma^2 C(\eta) \end{tikzcd}$$

where $C(\eta)$ is a cone of Hopf map $\eta : S^3 \simeq S(\mathbb{C}^2) \longrightarrow \mathbb{C}P^1 \simeq S^2$ with cofibre $\mathbb{C}P^2$ given by $(w, z) \longrightarrow [w, z]$. From this diagram, we smash $(*)$ with KO, ko and $H\mathbb{F}_2$

and using the fact from [2] (Wood's theorem, page 206) that $KO \wedge C(\eta) \simeq KU$ and $ko \wedge C(\eta) \simeq ku$, [13] i.e.,
smash with KO;

$$(*) \wedge_{\mathbb{S}^0} KO: \quad \begin{tikzcd} \Sigma KO \arrow[r, "\eta"] & KO \arrow[r] & KU \arrow[r] \arrow[ddrr, "1\wedge d_1'"] & \Sigma^2 KO \arrow[d, equal] & \\ & & \Sigma^3 KO \arrow[r, "\eta"] & \Sigma^2 KO \arrow[r] & \Sigma^2 KU, \end{tikzcd}$$

smash with ko;

$$(*) \wedge_{\mathbb{S}^0} ko: \quad \begin{tikzcd} \Sigma ko \arrow[r, "\eta"] & ko \arrow[r] & ku \arrow[r] \arrow[ddrr, "1\wedge d_1'"] & \Sigma^2 ko \arrow[d, equal] & \\ & & \Sigma^3 ko \arrow[r, "\eta"] & \Sigma^2 ko \arrow[r] & \Sigma^2 ku \end{tikzcd}$$

and smash with $H\mathbb{F}_2$;

$$(*) \wedge_{\mathbb{S}^0} H\mathbb{F}_2: \quad \begin{tikzcd} \Sigma H\mathbb{F}_2 \arrow[r, "\eta"] & H\mathbb{F}_2 \arrow[r] & H\mathbb{F}_2 \wedge C(\eta) \arrow[r] \arrow[ddrr, "1\wedge d_1'"] & \Sigma^2 H\mathbb{F}_2 ko \arrow[d, equal] & \\ & & \Sigma^3 H\mathbb{F}_2 \arrow[r, "\eta"] & \Sigma^2 H\mathbb{F}_2 \arrow[r] & H\mathbb{F}_2 \wedge \Sigma^2 C(\eta). \end{tikzcd}$$

Moreover, by the associativity of the smash product, note that

$$(*) \wedge_{\mathbb{S}^0} H\mathbb{F}_2 \simeq (*) \wedge_{\mathbb{S}^0} ko \wedge_{ko} H\mathbb{F}_2.$$

That is smashing with $H\mathbb{F}_2$ factors through ko. This guarantees that differentials on TU agree with differentials on $H\mathbb{F}_2$. Now, we are ready to prove this lemma.

Proof of 1:
To do this, it is useful to consider diagram (from diagram smashed with KO);

$$\begin{tikzcd} & \Sigma^2 KU \arrow[dl, "v"'] \arrow[d, "r"] & \\ KU \arrow[r, "R"] \arrow[drr, "d_1 = 1\wedge d_1'"] & \Sigma^2 KO \arrow[d, equal] & \\ & \Sigma^2 KO \arrow[r, "c"'] & \Sigma^2 KU \end{tikzcd}$$

From this diagram, we see that $d_1 = c \circ R$ and hence $d_1 : RU \cong KU \longrightarrow \Sigma^2 KU \cong v^{-1}RU$ (whenever we are dealing with classifying space for a group G, BG) is given by

$$\begin{aligned} d_1(x) &= c \circ R(x) \\ &= c(rv^{-1}x) \\ &= (1+\tau)(v^{-1}x) \\ &= v^{-1}(1-\tau)(x) \end{aligned}$$

since $cr = 1+\tau$ and $\tau(v^{-1}) = -v^{-1}$, by lemma 2.1.5 and lemma 2.2.10 in [13]. Similarly, we can show that $d_1 : v^k RU \longrightarrow v^{k-1}RU$ is given by

$$d_1(v^k x) = v^{k-1}(1+(-1)^{k-1}\tau)(x)$$

for all $x \in RU$. Combining this facts with the isomorphism $v^k RU \cong RU$ for each integer k, the results follow. (*Note:* degree of v is 2 i.e., $QU \subseteq RU$ concentrate in even degree.)

Proof of 2:

This follows because η is detected by Sq^2 in $H\mathbb{F}_2$. Precisely, by diagram smashed with $H\mathbb{F}_2$ above, we get

$$\begin{tikzcd} H\mathbb{F}_2 \wedge C(\eta) \arrow[r, "1 \wedge d_1'"] \arrow[d, "\simeq"', "SW"] & H\mathbb{F}_2 \wedge \Sigma^2 C(\eta) \arrow[d, "\simeq"', "SW"] \\ \Sigma^2 F(C(\eta), H\mathbb{F}_2) \arrow[r, "F(d_1', 1)"] & \Sigma^4 F(C(\eta), H\mathbb{F}_2) \end{tikzcd}$$

where SW is *Spanier-Whitehead* duality. After apply $\pi_2 = [S^2, -]$ to this diagram, we get

$$\begin{tikzcd} H_2(C(\eta), \mathbb{F}_2) \arrow[r, "Sq^2"] \arrow[d, "\simeq"', "SW"] & H_0(C(\eta), \mathbb{F}_2) \arrow[d, "\simeq"', "SW"] \\ H^0(C(\eta), \mathbb{F}_2) \arrow[r, "Sq^2"] & H^2(C(\eta), \mathbb{F}_2). \end{tikzcd}$$

Here $Sq^2 \neq 0$ because $C(\eta) \simeq \mathbb{C}P^2$ and $H^*(\mathbb{C}P^2_+; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^3)$ s.t. codegree of x is 2 and hence $Sq^2(x) = x^2$. We complete the proof of 2.

Remark 4.1.2. *For* $S^{n+k} \xrightarrow{f'} S^k \longrightarrow C(f')$, *if there exist cohomology operation* $\alpha : H^k \longrightarrow H^{n+k+1}$ *s.t.* $\alpha \neq 0$ *on* $H^*(C(f'))$, *we say that* α detects f'.

Proof of 3:

We can show in general for any space X which is finite Q-set where Q is a group of order 2 that

$$\mathbb{Z}X \xrightarrow{1-\tau} \mathbb{Z}X \xrightarrow{1+\tau} \mathbb{Z}X$$

is exact. This is true because X can be viewed as the disjoint union of Q fixed point and non-fixed point one, i.e,

$$X = X^Q \sqcup Q \times Y = \{x_1, x_2, ..., x_m, y_1, \tau y_1, y_2, \tau y_2, ..., y_n, \tau y_n\}.$$

So, $\ker(1+\tau)$ is free over $\mathbb{Z}$ on generator $y_1 - \tau y_1, y_2 - \tau y_2, ..., y_n - \tau y_n$ which is $(1-\tau)\mathbb{Z}X$ as required. In particular, $X = \{\text{simple complex representations}\}$ then $\mathbb{Z}X = RU$ and $Q = Gp < \tau >$ and hence $\ker(1+\tau : RU \longrightarrow RU) = (1-\tau)RU$. $\square$

§ 4.2 E_2-page for $ko^*(BSD_{16})$

The E_1-page can be filled in easily by copying both v-torsion part and torsion free part along the diagonal by $\widetilde{\eta}$. To get E_2-page of $ku^*(BSD_{16})$, we need to calculate the differential d_1 on both parts first and then calculate connecting homomorphism δ in (4.1).

4.2.1 Bockstein spectral sequence for v-torsion part of $ku^*(BSD_{16})$

Recall that for v-torsion part in $ku^*(BSD_{16})$,

$$TU = PC \cdot \tau \text{, where } PC = \mathbb{F}_2[b,d] \text{ and } \tau = b^2d - abd.$$

Moreover, we note from theorem 2.6.1 that TU is embedded in $H^*(BSD_{16};\mathbb{F}_2)$ as

$$a \longrightarrow x^2,\ b \longrightarrow y^2,\ d \longrightarrow P \text{ and } \tau \longrightarrow y^4P - x^2y^2P = y^4P.$$

Furthermore, $d_1 = Sq^2$. The action on each element of TU is given by $Sq^2(a) = Sq^2(x^2) = x^4 = 0$, $Sq^2(b) = Sq^2(y^2) = y^4 = b^2$ and $Sq^2(d) = Sq^2(P) = u^2 = (x^2+y^2)P = ad + bd$ and thus $Sq^2(\tau) = b^3d = b\tau$. Moreover, $Sq^1(d) = 0$ and $Sq^1(b) = 0$ then $Sq^1(d^j) = 0$ and $Sq^1(b^i) = 0$ for all i,j. By using this information and Cartan formula, we have

- $Sq^2(d^j\tau) = \begin{cases} bd^j\tau, & \text{if } j \text{ is even;} \\ 0, & \text{if } j \text{ is odd.} \end{cases}$
- $Sq^2(b^i\tau) = \begin{cases} b^{i+1}\tau, & \text{if } i \text{ is even;} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$
- $Sq^2(b^id^j\tau) = \begin{cases} b^{i+1}d^j\tau, & \text{if } i+j \text{ is even;} \\ 0, & \text{if } i+j \text{ is odd.} \end{cases}$

To find the kernel and image of $d_1 = Sq^2$, we will show its action on TU in the diagram below for small degree and then observe the pattern:

$$
\begin{array}{lllllllll}
\textbf{Codegree} & & & & \textbf{Generator of } TU & & & & \\
8 & \tau & & & & & & & \\
 & \downarrow & & & & & & & \\
10 & b\tau & & & & & & & \\
 & & & & & & & & \\
12 & b^2\tau & d\tau & & & & & & \\
 & \downarrow & & & & & & & \\
14 & b^3\tau & bd\tau & & & & & & \\
 & & \downarrow & & & & & & \\
16 & b^4\tau & b^2d\tau & d^2\tau & & & & & \\
 & \downarrow & & \downarrow & & & & & \\
18 & b^5\tau & b^3d\tau & bd^2\tau & & & & & \\
 & & \downarrow & & & & & & \\
20 & b^6\tau & b^4d\tau & b^2d^2\tau & d^3\tau & & & & \\
 & \downarrow & & \downarrow & & & & & \\
22 & b^7\tau & b^5d\tau & b^3d^2\tau & bd^3\tau & & & & \\
 & & \downarrow & & \downarrow & & & & \\
24 & b^8\tau & b^6d\tau & b^4d^2\tau & b^2d^3\tau & d^4\tau & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
26 & b^9\tau & b^7d\tau & b^5d^2\tau & b^3d^3\tau & bd^4\tau & & & \\
 & & \downarrow & & \downarrow & & & & \\
28 & b^{10}\tau & b^8d\tau & b^6d^2\tau & b^4d^3\tau & b^2d^4\tau & d^5\tau & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
30 & b^{11}\tau & b^9d\tau & b^7d^2\tau & b^5d^3\tau & b^3d^4\tau & bd^5\tau & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
32 & b^{12}\tau & b^{10}d\tau & b^8d^2\tau & b^6d^3\tau & b^4d^4\tau & b^2d^5\tau & d^6\tau & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
34 & b^{13}\tau & b^{11}d\tau & b^9d^2\tau & b^7d^3\tau & b^5d^4\tau & b^3d^5\tau & bd^6\tau & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
36 & b^{14}\tau & b^{12}d\tau & b^{10}d^2\tau & b^8d^3\tau & b^6d^4\tau & b^4d^5\tau & b^2d^6\tau & bd^7\tau \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow &
\end{array}
$$

Diagram 4.1: The $d_1 = Sq^2 (:=\downarrow)$ action on TU.

This diagram suggests that the kernel and homology of $d_1 = Sq^2$ are $\mathbb{F}_2[b^2, d^2] < b\tau, d\tau >$ and $\mathbb{F}_2[d^2] < d\tau >$ respectively. This can be proved easily from the information above and induction. We record these results as:

Lemma 4.2.1. *With the same notation as above,*

1) *The Sq^2-homology of TU is concentrated in degrees $-12, -20, -28, -36, ...$ being represented by $\mathbb{F}_2[d^2] < d\tau >$.*

2) *The module of Sq^2-cycles of TU is $\mathbb{F}_2[b^2, d^2] < b\tau, d\tau >$.*

4.2.2 Bockstein spectral sequence for torsion free part of $ku^*(BSD_{16})$

Now we consider the QU part starting by filling in the zero-line of $E_{1\frac{1}{2}}$-page, i.e. the kernel of differential d_1's first and then follows by filling in the positive line, i.e., homology of d_1's.

Filling in the zero-line of QU-part:
For $\ker(1-\tau : QU_{4k} \longrightarrow QU_{4k-2}) := ZQU_{4k}$, we calculate them directly but for $\ker(1+\tau : QU_{4k+2} \longrightarrow QU_{4k}) := ZQU_{4k+2}$ we will use lemma 4.1.1(3), i.e., $ZQU_{4k+2} = (1-\tau)QU_{4k+4}$.

By lemma 3.3.1 and the character table, it is not hard to see that

$$ZQU_{4k+2} = \mathbb{Z}_2^\wedge < \theta >, \tag{4.2}$$

for all $k \geq 0$, where

$$\begin{aligned} \theta = D - \tau(D) &= -\frac{4}{3}A + \frac{1}{3}(A+B)D + \frac{16}{9}D - \frac{4}{3}D^2 + \frac{2}{9}D^3 \\ &= [0, -2c, 0, 2c, 0, 0]. \end{aligned}$$

For $ZQU_{-(4k+2)}$, we calculate on $JU_k = QU_{-(2k)}$ that is $ZQU_{-(4k+2)} = (1-\tau)JU_{2(k+1)}$ and we obtain the results as

$$ZQU_{-(4k+2)} = \mathbb{Z}_2^\wedge < \theta_k >, \tag{4.3}$$

where $\theta_k = 2^{L(\frac{k}{2})}\theta$ and $L(r) :=$ greatest integer which is less than or equal to r.

Again by lemma 3.3.1 and character table, for $k \geq 0$,

$$ZQU_{4k} = \mathbb{Z} < v^{2k} > \oplus (\mathbb{Z}_2^\wedge)^5 < A, B, C, D^2, \widetilde{D}^3 >, \tag{4.4}$$

where $C = (A+B)D - 2D$ and $\widetilde{D}^3 = D^3 + 2D$. For negative degree,

$$\begin{aligned} ZQU_{-4} &= (\mathbb{Z}_2^\wedge)^5 < \overline{x}_2, \overline{y}_2, \overline{z}_2, \overline{t}_2, \overline{u}_2 - 2\overline{w}_2 >, \\ ZQU_{-8} &= (\mathbb{Z}_2^\wedge)^5 < \overline{x}_4, \overline{y}_4, \overline{z}_4, \overline{t}_4, \overline{u}_4 > . \end{aligned}$$

In general,

$$\begin{aligned} ZQU_{-8k-4} &= (\mathbb{Z}_2^\wedge)^5 < \overline{x}_{4k+2}, \overline{y}_{4k+2}, \overline{z}_{4k+2}, \overline{t}_{4k+2}, \overline{u}_{4k+2} - 2\overline{w}_{4k+2} >, \\ ZQU_{-8k} &= (\mathbb{Z}_2^\wedge)^5 < \overline{x}_{4k}, \overline{y}_{4k}, \overline{z}_{4k}, \overline{t}_{4k}, \overline{u}_{4k} > . \end{aligned} \tag{4.5}$$

Filling in the positive line of QU-part:
We calculate $H(QU)$ only one case which is $H(QU)_{4k} = ZQU_{4k}/(1+\tau)QU_{4k+2}$. For $H(QU)_{4k+2}$, this is zero for all k, since $H(QU)_{4k+2} = ZQU_{4k+2}/(1-\tau)QU_{4k+4}$ and $ZQU_{4k+2} = (1-\tau)QU_{4k+4}$ by lemma 4.1.1(3).

In non-negative degree, as in chapter 3, we represent ZQU_{4k} and $(1+\tau)QU_{4k+2}$ as character table below; $ZQU_{4k} = \mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^5$ generated by

$$\left\langle \begin{array}{r} 1: \\ A: \\ B: \\ C: \\ D^2: \\ \widetilde{D}^3: \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & -8 & 0 & -4 & 0 & 0 & 4 \\ 0 & 16 & -2 & 4 & -2 & 0 & 4 \\ 0 & 72 & 0 & 12 & 0 & 0 & 12 \end{bmatrix} \right\rangle$$

and $(1+\tau)QU_{4k+2} = \mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^5$ generated by

$$\left\langle \begin{array}{r} (1+\tau)1: \\ (1+\tau)A: \\ (1+\tau)B: \\ (1+\tau)D: \\ (1+\tau)D^2: \\ (1+\tau)D^3: \end{array} \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 4 & 4 \\ 0 & 8 & 0 & 4 & 0 & 0 & 4 \\ 0 & 32 & -4 & 8 & -4 & 0 & 8 \\ 0 & 128 & 0 & 16 & 0 & 0 & 16 \end{bmatrix} \right\rangle$$

for all $k \geq 0$. Note that

$$(1+\tau)(A+B)D = [0,0,0,0,0,0,16] = 4(1+\tau)A - \frac{16}{3}(1+\tau)D + 4(1+\tau)D^2 - \frac{2}{3}(1+\tau)D^3.$$

Now, the representing matrix for the calculation of $H(QU)_{4k}$ can be found as;

$$\begin{array}{r|cccccc|l} & 1 & A & B & C & D^2 & \widetilde{D}^3 & \\ (1+\tau)1: & 2 & 0 & 0 & 0 & 0 & 0 & r_1 \\ (1+\tau)A: & 0 & 2 & 0 & 0 & 0 & 0 & r_2 \\ (1+\tau)B: & 0 & 0 & 2 & 0 & 0 & 0 & r_3 \\ (1+\tau)D: & 0 & \frac{4}{3} & 0 & \frac{-1}{3} & \frac{4}{3} & \frac{-2}{9} & r_4 \\ (1+\tau)D^2: & 0 & 0 & 0 & 0 & 2 & 0 & r_5 \\ (1+\tau)D^3: & 0 & \frac{-8}{3} & 0 & \frac{2}{3} & \frac{-8}{3} & \frac{22}{9} & r_6 \end{array}$$

After several row operations, we obtain

$$\begin{array}{|cccccc|l} 1 & A & B & C & D^2 & \widetilde{D}^3 & \\ 2 & 0 & 0 & 0 & 0 & 0 & r_1 \\ 0 & 2 & 0 & 0 & 0 & 0 & r_2 \\ 0 & 0 & 2 & 0 & 0 & 0 & r_3 \\ 0 & 0 & 0 & 1 & 0 & 0 & r_4' \\ 0 & 0 & 0 & 0 & 2 & 0 & r_5 \\ 0 & 0 & 0 & 0 & 0 & 2 & r_6' = r_6 + 2r_4 \end{array}$$

where $r_4' = (-3)[r_4 - \frac{2}{3}r_2 - \frac{2}{3}r_5 + \frac{1}{9}r_6']$. Hence, for $k \geq 0$,

$$H(QU)_{4k} = (\mathbb{Z}/2)^5 \text{ generated by } [1], [A], [B], [D^2], [\widetilde{D}^3], \tag{4.6}$$

or more precisely by $v^{2k}, v^{2k+1}a, v^{2k+1}b, v^{2k+4}d^2, v^{2k+6}d^3 + 2v^{2k+2}d$.

In negative degree, for $k \geq 1$, $H(QU)_{-4k} = \ker(1-\tau : JU_{2k} \longrightarrow JU_{2k+1})/(1+\tau)JU_{2k-1}$ and this can be calculated by using the character table of JU_i in Chapter 3. The representing matrix for $H(QU)_{-4}$ can be found as;

$$\begin{array}{rrrrrrl}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2^* & \\
(1+\tau)\overline{x}_1 : | & 1 & 1 & 0 & 0 & 0 & |r_1 \\
(1+\tau)\overline{y}_1 : | & 2 & 0 & 0 & 0 & 0 & |r_2 \\
(1+\tau)\overline{z}_1 : | & 0 & 1 & -1 & 2 & 0 & |r_3 \\
(1+\tau)\overline{t}_1 : | & 1 & 0 & 1 & 0 & 0 & |r_4 \\
(1+\tau)\overline{u}_1 : | & -1 & 0 & -1 & 0 & -2 & |r_5 \\
(1+\tau)\overline{w}_1 : | & -1 & 0 & -1 & 0 & -1 & |r_6
\end{array},$$

where $\overline{u}_2^* = \overline{u}_2 - 2\overline{w}_2 = [-8, 0, -4, 0, 0, -4]$.

Now, using row operations, we obtain

$$\begin{array}{rrrrrrl}
 & \overline{x}_2 & \overline{y}_2 & \overline{z}_2 & \overline{t}_2 & \overline{u}_2^* & \\
| & 1 & 1 & 0 & 0 & 0 & |r_1 \\
| & 2 & 0 & 0 & 0 & 0 & |r_2 \\
| & 0 & 0 & 0 & 2 & 0 & |r_3 + r_4 - r_1 \\
| & 1 & 0 & 1 & 0 & 0 & |r_4 \\
| & 0 & 0 & 0 & 0 & 0 & |r_5 + r_4 + 2r_6' \\
| & 0 & 0 & 0 & 0 & 1 & |r_6' = r_6 - r_5
\end{array},$$

By doing column operation, i.e., changing c_1 to $c_1 - c_2 - c_3$ and using the same method in Chapter 3, it is not hard to see that

$$H(QU)_{-4} = (\mathbb{Z}/2)^2 \text{ generated by } [\overline{x}_2] \text{ and } [\overline{t}_2], \tag{4.7}$$

with $[\overline{x}_2] = [\overline{y}_2] = [\overline{z}_2]$ and $[u_2^*] = 0$, where $[\overline{\alpha}_2] = \overline{\alpha}_2 + (1+\tau)JU_1$ for $\alpha \in \{x, y, z, t, u^*\}$.

For $H(QU)_{-8} = \ker(\tau : JU_4 \longrightarrow JU_5)/(1+\tau)JU_3$, the representing matrix for $H(QU)_{-8}$ can be found as;

$$\begin{array}{rrrrrrl}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \\
(1+\tau)\overline{x}_3 : | & 1 & 0 & 0 & 0 & 0 & |r_1 \\
(1+\tau)\overline{y}_3 : | & 0 & 1 & 0 & 0 & 0 & |r_2 \\
(1+\tau)\overline{z}_3 : | & 0 & 0 & 1 & 0 & 0 & |r_3 \\
(1+\tau)\overline{t}_3 : | & 0 & 0 & 0 & 2 & 0 & |r_4 \\
(1+\tau)\overline{u}_3 : | & -1 & 0 & -1 & 0 & 2 & |r_5 \\
(1+\tau)\overline{w}_3 : | & 0 & 0 & 0 & 0 & 0 & |r_6
\end{array},$$

Again, row operations give

$$\begin{array}{c|ccccc|l} & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \\ & 1 & 0 & 0 & 0 & 0 & r_1 \\ & 0 & 1 & 0 & 0 & 0 & r_2 \\ & 0 & 0 & 1 & 0 & 0 & r_3 \\ & 0 & 0 & 0 & 2 & 0 & r_4 \\ & 0 & 0 & 0 & 0 & 2 & r_5 + r_1 + r_3 \\ & 0 & 0 & 0 & 0 & 0 & r_6 \end{array}.$$

Hence,

$$H(QU)_{-8} = (\mathbb{Z}/2)^2 \text{ generated by } [\overline{t}_4] \text{ and } [\overline{u}_4], \tag{4.8}$$

with $[\overline{x}_4] = [\overline{y}_4] = [\overline{z}_4] = 0$ where $[\overline{\alpha}_4] = \overline{\alpha}_4 + (1+\tau)JU_3$ for $\alpha \in \{x, y, z, t, u\}$.

In general, for $k \geq 1$, $H(QU)_{-8k-4} = \ker(1-\tau : JU_{4k+2} \longrightarrow JU_{4k+3})/(1+\tau)JU_{4k+1}$, the representing matrix for $H(QU)_{-8k-4}$ can be found as;

$$\begin{array}{r|ccccc|l} & \overline{x}_{4k+2} & \overline{y}_{4k+2} & \overline{z}_{4k+2} & \overline{t}_{4k+2} & \overline{u}^*_{4k+2} & \\ (1+\tau)\overline{x}_{4k+1} : & 1 & 0 & 0 & 0 & 0 & r_1 \\ (1+\tau)\overline{y}_{4k+1} : & 0 & 1 & 0 & 0 & 0 & r_2 \\ (1+\tau)\overline{z}_{4k+1} : & 0 & 0 & 1 & 0 & 0 & r_3 \\ (1+\tau)\overline{t}_{4k+1} : & 0 & 0 & -1 & 2 & 0 & r_4 \\ (1+\tau)\overline{u}_{4k+1} : & -1 & 0 & -1 & 0 & -1 & r_5 \\ (1+\tau)\overline{w}_{4k+1} : & 0 & 0 & 0 & 0 & 0 & r_6 \end{array},$$

Hence, for $k \geq 1$,

$$H(QU)_{-8k-4} = \mathbb{Z}/2 \text{ generated by } [\overline{t}_{4k+2}], \tag{4.9}$$

with $[\overline{x}_{4k+2}] = [\overline{y}_{4k+2}] = [\overline{z}_{4k+2}] = [\overline{u}^*_{4k+2}] = 0$ where $\overline{u}^*_{4k+2} = [-8 \cdot 16^k, 0, -4 \cdot 4^k, 0, 0, -4 \cdot 4^k] = \overline{u}_{4k+2} - 2\overline{w}_{4k+2}$ and $[\overline{\alpha}_{4k+2}] = \overline{\alpha}_{4k+2} + (1+\tau)JU_{4k+1}$ for $\alpha \in \{x, y, z, t, u^*\}$.

For $k \geq 2$, $H(QU)_{-8k} = \ker(1-\tau : JU_{4k} \longrightarrow JU_{4k+1})/(1+\tau)JU_{4k-1}$, the representing matrix for $H(QU)_{-8k}$ can be found as;

$$\begin{array}{r|ccccc|l} & \overline{x}_{4k} & \overline{y}_{4k} & \overline{z}_{4k} & \overline{t}_{4k} & \overline{u}_{4k} & \\ (1+\tau)\overline{x}_{4k-1} : & 1 & 0 & 0 & 0 & 0 & r_1 \\ (1+\tau)\overline{y}_{4k-1} : & 0 & 1 & 0 & 0 & 0 & r_2 \\ (1+\tau)\overline{z}_{4k-1} : & 0 & 0 & 1 & 0 & 0 & r_3 \\ (1+\tau)\overline{t}_{4k-1} : & 0 & 0 & 0 & 2 & 0 & r_4 \\ (1+\tau)\overline{u}_{4k-1} : & -1 & 0 & -1 & 0 & 2 & r_5 \\ (1+\tau)\overline{w}_{4k-1} : & 0 & 0 & 0 & 0 & 0 & r_6 \end{array},$$

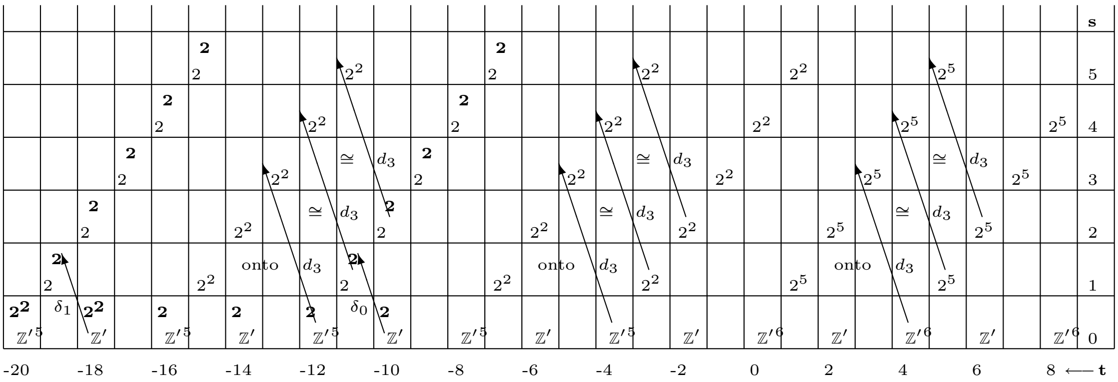
Hence, for $k \geq 2$,

$$H(QU)_{-8k} = (\mathbb{Z}/2)^2 \text{ generated by } [\overline{t}_{4k}] \text{ and } [\overline{u}_{4k}], \tag{4.10}$$

with $[\overline{x}_{4k}] = [\overline{y}_{4k}] = [\overline{z}_{4k}] = 0$ where $[\overline{\alpha}_{4k}] = \overline{\alpha}_{4k} + (1+\tau)JU_{4k-1}$ for $\alpha \in \{x, y, z, t, u\}$.

Summarizing from (4.2) to (4.10), we may display the $E_{1\frac{1}{2}}$-page of the Bockstein spectral sequence for $ko^*(BSD_{16})$ as;

Figure 4.2: $E_2(TU)\bigoplus E_2(QU) := E_{1\frac{1}{2}}(ku^*(BSD_{16}))$**-page**



Notation: $\mathbb{Z}' = \mathbb{Z}_2^\wedge$, $\mathbb{Z}'^5 = (\mathbb{Z}_2^\wedge)^5$, $\mathbb{Z}'^6 = \mathbb{Z}\oplus(\mathbb{Z}_2^\wedge)^5$, $2 = \mathbb{Z}/2$
and **2**'s come from TU part otherwise come from QU part.

To obtain the E_2-page we need to determine the connecting homomorphisms, δ's, in (4.1). In order to do this, the representation theory will be very useful.

4.2.3 Representation theory and differentials

The aim of this subsection is to prove lemma 4.2.2 below.

Lemma 4.2.2. *Connecting homomorphisms, δ_k's, are all zero in Bockstein spectral sequence for $ko^*(BSD_{16})$ and thus $E_{1\frac{1}{2}}(ku^*(BSD_{16}))$-page is $E_2(ku^*(BSD_{16}))$-page. Furthermore the d_3's leaving the $8k+4$ column are as illustrated in Figure 4.2.*

To prove this, the representation theory plays an important role. The details of representation theory involved in the computation of real connective K-cohomlogy theory can be found in [13] chapter 2. By lemma 1.2.2 and the character table of SD_{16}, we see that $1, \chi_2, \chi_3, \chi_4$ and σ_2 are real representation whereas σ_1 and σ_3 are not self-conjugate, i.e., complex representation. In fact, $\tau\sigma_1 = \sigma_3$. Thus, we have;

$$\begin{aligned} RO(SD_{16}) &= \mathbb{Z}\{1, \chi_2, \chi_3, \chi_4, \sigma_2, r\sigma_1\}, \\ RU(SD_{16}) &= \mathbb{Z}\{c1, c\chi_2, c\chi_3, c\chi_4, \sigma_1, \sigma_3\}, \\ RSp(SD_{16}) &= \mathbb{Z}\{qc1, qc\chi_2, qc\chi_3, qc\chi_4, q\sigma_1\}. \end{aligned} \tag{4.11}$$

By Atiyah-Segal theorem for the real case, we have;

Lemma 4.2.3. *Periodic real K-theory $KO^*(BSD_{16})$ is given by*

- $KO^{8k}(BSD_{16}) \cong RO(SD_{16})^\wedge_J = (\mathbb{Z}_2^\wedge)^6$,
- $KO^{8k-1}(BSD_{16}) \cong RO(SD_{16})^\wedge_J / rRU(SD_{16})^\wedge_J = (\mathbb{Z}/2)^5$,
- $KO^{8k-2}(BSD_{16}) \cong RU(SD_{16})^\wedge_J / \tilde{c}RSp(SD_{16})^\wedge_J = (\mathbb{Z}/2)^5 \oplus \mathbb{Z}_2^\wedge$,
- $KO^{8k-3}(BSD_{16}) = 0$,

- $KO^{8k-4}(BSD_{16}) \cong RSp(SD_{16})^{\wedge}_{J} = (\mathbb{Z}^{\wedge}_{2})^{6}$,
- $KO^{8k-5}(BSD_{16}) \cong RSp(SD_{16})^{\wedge}_{J}/qRU(SD_{16})^{\wedge}_{J} = 0$,
- $KO^{8k-6}(BSD_{16}) \cong RU(SD_{16})^{\wedge}_{J}/cRO(SD_{16})^{\wedge}_{J} = \mathbb{Z}^{\wedge}_{2}$,
- $KO^{8k-7}(BSD_{16}) = 0$,

where $k \in \mathbb{Z}$.

Proof. This is an immediate result from (4.11) and lemma 2.1.5 in [13]. □

Now, we are ready to prove lemma 4.2.2.

Proof. To get $E_2(ku^*(BSD_{16}))$-page, it suffices to calculate

$$\delta_k : (ZQU)_{-10-8k} \longrightarrow (HTU)_{-12-8k}$$

for each $k \geq 0$. Since all entries in $E_{1\frac{1}{2}}$-page are contained in even degree, $d_2 = 0$ and then E_2-page is equal to E_3-page. Moreover, by using the fact that this spectral must collapse at E_4-page together with lemma 4.2.3, d_3 must be surjective and isomorphism above the zero-line shown as in picture above. In particular, at degree $-12-8k$, $d_3 : (\mathbb{Z}/2)^2 \longrightarrow (\mathbb{Z}/2)^2$ must be an isomorphism. If δ_k is not zero, then rank of domain of d_3 must less than 2 and this implies that d_3 can not be isomorphism. Hence, δ_k must be zero for all k. □

There is one family of differentials still to be determined and we will deal with this in the next section.

§ 4.3 E_∞-page and additive extension problems for $ko^*(BSD_{16})$

4.3.1 E_∞-page

So as to get E_∞-page, we need to determine differential departing from the zero-line at degree $8k+4$ for all k, i.e.,

$$d_3 : Z(TU)_{8k+4} \oplus Z(QU)_{8k+4} \longrightarrow H(TU)_{8k} \oplus H(QU)_{8k}.$$

Here, $\ker d_3$ can be calculated by lemma below.

Lemma 4.3.1. *Kernel of* $d_3 : Z(TU)_{8k+4} \oplus Z(QU)_{8k+4} \longrightarrow H(TU)_{8k} \oplus H(QU)_{8k})$ *illustrated in Figure 4.2 is*

$$Sq^2TU_{8k+6} \oplus (1+\tau)QU_{8k+6}$$

for all k.

Proof. This follows from the commutative diagram below;

$$\begin{tikzcd}
H(TU)_{8k} \oplus H(QU)_{8k} \arrow[r, "\cong"] & \widetilde{\eta}H(TU)_{8k} \oplus \widetilde{\eta}H(QU)_{8k} \\
Z(TU)_{8k+4} \oplus Z(QU)_{8k+4} \arrow[u, "d_3"', "\text{onto}"] \arrow[r, "\widetilde{\eta}^*"] & H(TU)_{8k+4} \oplus H(QU)_{8k+4} \arrow[u, "d_3'"', "\cong"]
\end{tikzcd}$$

where $\widetilde{\eta}^*$ is a projective map modulo by $Sq^2TU_{8k+6} \oplus (1+\tau)QU_{8k+6}$. $\square$

Now, it is easy to see that, by this lemma and character table in Chapter 3, for $k \geq 0$,

$$E_\infty^{0,8k+4} = \ker(d_3) = \mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^5 \text{ generated by } 2v^{2k} \text{ and } 2A, 2B, \overline{D}, 2D^2, \overline{D^3}, \tag{4.12}$$

which are the image of 1 and A, B, D, D^2, D^3 under $1+\tau$, respectively. Here, v^{2k}, A, B, D^2 have the same character table as before and the character table for $\overline{D}$ and $\overline{D^3}$ is represented by $[8,0,4,0,0,4]$ and $[128,0,16,0,0,16]$ respectively.

For negative k, $E_\infty^{0,8k+4} = Sq^2TU_{8k+6} \oplus (1+\tau)JU_{-(4k+3)}$ and again these can be read from Sq^2 action in diagram of subsection 4.2.1 and JU_i in Chapter 3 easily. Explicitly,

- $E_\infty^{0,-4} = (1+\tau)JU_1 = (\mathbb{Z}_2^\wedge)^5$ generated by $\{\overline{x}_2 + \overline{y}_2, 2\overline{x}_2, \overline{y}_2 + \overline{z}_2 + 2\overline{t}_2, \overline{x}_2 + \overline{z}_2, -\overline{x}_2 - \overline{z}_2 - \overline{u^*}_2\}$.
- $E_\infty^{0,-12} = (1+\tau)JU_5 = (\mathbb{Z}_2^\wedge)^5$ generated by $\{\overline{x}_6, \overline{y}_6, \overline{z}_6, 2\overline{t}_6 - \overline{z}_6, -\overline{x}_6 - \overline{z}_6 - \overline{u^*}_6\}$.
- For $k \leq -3$, $E_\infty^{0,8k+4} = Sq^2TU_{8k+6} \oplus (1+\tau)JU_{4(k-1)+1} = (\mathbb{Z}/2)^{|k|-2} \oplus (\mathbb{Z}_2^\wedge)^5$ generated by $\{b^{4(|k|-2-i)}d^{2i+1}\tau \mid i = 0,1,2,...,|k|-3\}$ and $\{\overline{x}_{4(k-1)+2}, \overline{y}_{4(k-1)+2}, \overline{z}_{4(k-1)+2}, 2\overline{t}_{4(k-1)+2} - \overline{z}_{4(k-1)+2}, -\overline{x}_{4(k-1)+2} - \overline{z}_{4(k-1)+2} - \overline{u^*}_{4(k-1)+2}\}$.

Now, we reach the E_∞-page.

Figure 4.3: $E_\infty(ku^*(BSD_{16}))$-page

-20	-19	-18	-17	-16	-15	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8 ⟵ t
																													s
																													3
						2^2								2^2								2^5							2
					2^2								2^2								2^5								1
$\mathbf{2}$ $\mathbb{Z}'^5$		$\mathbf{2}^{\mathbf{2}}$ $\mathbb{Z}'$		$\mathbf{2}$ $\mathbb{Z}'^5$		$\mathbf{2}$ $\mathbb{Z}'$		$\mathbb{Z}'^5$		$\mathbf{2}$ $\mathbb{Z}'$		$\mathbb{Z}'^5$		$\mathbb{Z}'$		$\mathbb{Z}'^5$		$\mathbb{Z}'$		$\mathbb{Z}'^6$		$\mathbb{Z}'$		$\mathbb{Z}'^6$		$\mathbb{Z}'$		$\mathbb{Z}'^6$	0

Notation: $\mathbb{Z}' = \mathbb{Z}_2^\wedge$, $\mathbb{Z}'^5 = (\mathbb{Z}_2^\wedge)^5$, $\mathbb{Z}'^6 = \mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^5$, $2 = \mathbb{Z}/2$ and **2**'s come from TU part otherwise come from QU part.

4.3.2 ADDITIVE EXTENSION PROBLEMS

The next task is the extension problems. Here, each column of the $E_\infty^{*,-t}$-page, contributes to each $ko^t(BSD_{16})$. Precisely, $ko^t(BSD_{16})$ has filtration

$$ko^t(BSD_{16}) = F_0^t \supseteq F_1^t \supseteq F_2^t \supseteq F_3^t = 0$$

with $F_0^t/F_1^t = E_\infty^{0,-t}$, $F_1^t/F_2^t = E_\infty^{1,-t}$ and $F_2^t = E_\infty^{2,-t}$. In this case, the only extension problems come from codegree $8k-2$ for $k \geq 2$. For codegree $8k+6$, for $k \leq 0$, these short exact sequences split because they end with $\mathbb{Z}_2^\wedge$.

We claim that all this short exact sequences split. We consider firstly on codegree 14 which has two candidates, i.e, split one and non-split one. Precisely, additively, $ko^{14}(BSD_{16})$ can be $\mathbb{Z}_2^\wedge \oplus (\mathbb{Z}/2)^3$ or $\mathbb{Z}_2^\wedge \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$ but can not be $\mathbb{Z}_2^\wedge \oplus \mathbb{Z}/8$, because it reduces to two $\widetilde{\eta}$-multiple generators at E_∞ . If we can show that this codegree split, then all exact sequences will also split. This is because;

Lemma 4.3.2. *If the short exact sequence for $ko^{14}(BSD_{16})$ splits, then the short exact sequence for $ko^{8k-2}(BSD_{16})$, for each $k \geq 3$, split.*

Proof. It is clear that η-multiple elements have order 2. So, it remains to identify elements in $ko^{8k-2}(BSD_{16})$ which reduce to elements in TU at E_∞. By assumption, $b^3\tau$ in $ko^{14}(BSD_{16})$ reducing to $b^3\tau$ in E_∞ has order 2. So, $2r \cdot b^3\tau = 0$ for all $r \in ko^*(BSD_{16})$. Recall that, for $k \geq 1$, $ko^{8k}(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^5$ generated by $\{\overline{x}_{4k}, \overline{y}_{4k}, \overline{z}_{4k}, \overline{t}_{4k}, \overline{u}_{4k}\}$ which $\overline{y}_{4k}$ and $\overline{u}_{4k}$ send to $(b^{4k}, B^{4k} - 2(A+B)D^{2k} + 2AD^{2k})$ and (d^{2k}, D^{2k}) in $H^*(BSD_{16}; \mathbb{Z}/2) \oplus R(SD_{16})_J^\wedge$ respectively. Furthermore, note that $\overline{y}_4^k = \overline{y}_{4k}$ and $\overline{d}_4^k = \overline{d}_{4k}$.

Let $\{b^{4k-5-4i}d^{2i}\tau \mid i = 0,1,2,...,k-3\} \subseteq ko^{8k-2}(BSD_{16})$ reduce to (same notation!) $\{b^{4k-5-4i}d^{2i}\tau \mid i = 0,1,2,...,k-3\} \subseteq E_\infty^{0,-(8k-2)}$. It is clear that

$$b^{4k-5-4i}d^{2i}\tau = (b^4)^{k-2-i}(d^2)^i(b^3\tau)$$

for all k, which are the multiples of element $b^3\tau$ in $ko^{14}(BSD_{16})$ and $(b^4)^{k-2-i}(d^2)^i = r \in ko^{8(k-1)}(BSD_{16})$. Therefore, they are annihilated by 2 and then we complete the proof. $\square$

4.3.3 η- BOCKSTEIN SPECTRAL SEQUENCE FOR MOD 2 COEFFICIENT

In order to solve the extension problems, we will investigate η- Bockstein spectral sequence for mod 2 coefficient. That is,

$$E_1^{*,*} = ku^*(BG; \mathbb{Z}/2)[\widetilde{\eta}] \Rightarrow ko^*(BG; \mathbb{Z}/2)$$

which comes from the cofibre sequence

$$\Sigma ko/2 \xrightarrow{\eta} ko/2 \xrightarrow{c} ku/2 \xrightarrow{r} \Sigma^2 ko/2 \,.$$

For calculation of mod 2 coefficient connective K-theory, we use universal coefficient theorem for spectrum E with finite generated coefficient G (Proposition 6.6(ii) page 201, [2]), i.e.,

$$0 \longrightarrow E^n(X) \otimes G \longrightarrow (EG)^n(X) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(E^{n+1}(X), G) \longrightarrow 0.$$

In this case $G = \mathbb{Z}/2$ and $E = ku$ and thus this short exact sequence splits. That is $E_1^{*,*}$-page can be calculated by

$$ku^n(BG; \mathbb{Z}/2) \cong ku^n(BG) \otimes \mathbb{Z}/2 \oplus \mathrm{Tor}_1^{\mathbb{Z}}(ku^{n+1}(BG), \mathbb{Z}/2).$$

Note that the identity $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}^a \oplus (\mathbb{Z}/2)^b, \mathbb{Z}/2) = (\mathbb{Z}/2)^b$ is often used in our case. Also, the part $\mathrm{Tor}_1^{\mathbb{Z}}(ku^{n+1}(BG), \mathbb{Z}/2)$ for $G = SD_{16}$ comes from v-torsion part TU. Thus, all differentials from integral η-Bockstein spectral sequence are applied for mod 2 coefficient by dividing by 2. Precisely, in torsion free parts,

$$d_1 = \begin{cases} (1-\tau) mod 2, & \text{if it departs from } QU_{4n} \otimes \mathbb{Z}/2; \\ (1+\tau) mod 2, & \text{if it departs from } QU_{4n+2} \otimes \mathbb{Z}/2, \end{cases}$$

and in v torsion parts are the same, i.e., $d_1 = Sq^2$. Note that, however, in lemma 4.1.1 (3) can not applied for mod 2 coefficient case. That is $\ker(1+\tau : RU \otimes \mathbb{Z}/2 \longrightarrow RU \otimes \mathbb{Z}/2)$ can not compute from $(1-\tau)RU \otimes \mathbb{Z}/2$.

So, to find E_2-page is similar to the integral case but we need to consider kernel and image of differential carefully. Here is the E_1-page of η-Bockstein spectral sequence mod 2 coefficient:

Figure 4.4: The E_1-page of η-Bockstein spectral sequence for mod 2 coefficient

																													s
					$\mathbf{2}^4$ 2^6	$\mathbf{2}^4$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}$ 2^6	$\mathbf{2}$	$\mathbf{2}$ 2^6	$\mathbf{2}$	2^6		2^6		2^6		2^7		2^7		5
				$\mathbf{2}^4$ 2^6	$\mathbf{2}^4$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}$ 2^6	$\mathbf{2}$	$\mathbf{2}$ 2^6	$\mathbf{2}$	2^6		2^6		2^6		2^7		2^7		2^7	4
			$\mathbf{2}^4$ 2^6	$\mathbf{2}^4$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}$ 2^6	$\mathbf{2}$	$\mathbf{2}$ 2^6	$\mathbf{2}$	2^6		2^6		2^6		2^7		2^7		2^7		3
		$\mathbf{2}^4$ 2^6	$\mathbf{2}^4$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}$ 2^6	$\mathbf{2}$	$\mathbf{2}$ 2^6	$\mathbf{2}$	2^6		2^6		2^6		2^7		2^7		2^7		2^7	2
	$\mathbf{2}^4$ 2^6	$\mathbf{2}^4$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}$ 2^6	$\mathbf{2}$	$\mathbf{2}$ 2^6	$\mathbf{2}$	2^6		2^6		2^6		2^7		2^7		2^7		2^7		1
$\mathbf{2}^4$ 2^6	$\mathbf{2}^4$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^3$ 2^6	$\mathbf{2}^3$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}^2$ 2^6	$\mathbf{2}^2$	$\mathbf{2}$ 2^6	$\mathbf{2}$	$\mathbf{2}$ 2^6	$\mathbf{2}$	2^6		2^6		2^6		2^7		2^7		2^7		2^7		2^7	0
-20		-18		-16		-14		-12		-10		-8		-6		-4		-2		0		2		4		6		8 $\longleftarrow$ **t**	

Notation: 2's come from TU otherwise come from $QU \otimes \mathbb{Z}/2$.

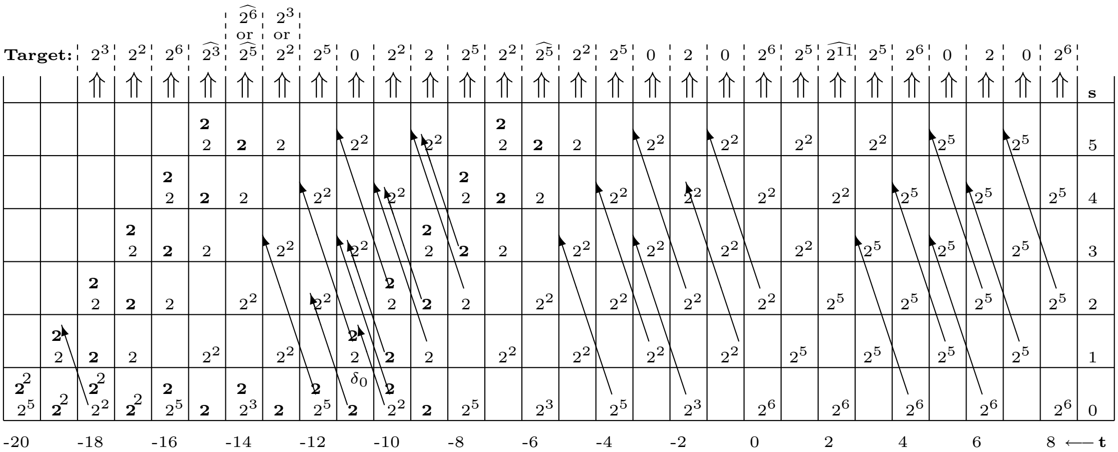
To find the E_2-page around codegree 14, we calculate $Z(QU_n \otimes \mathbb{Z}/2)$, $H(QU_n \otimes \mathbb{Z}/2)$, $Z(TU_n \otimes \mathbb{Z}/2)$ and $H(TU_n \otimes \mathbb{Z}/2)$ for $n \geq -20$. This is, again, a direct calculation from character table of QU_i in chapter 3 and Sq^2 action of TU_i in lemma 4.2.1 which we record as; (with the same notation in the integral case)

- For $n \geq 0$, $Z(QU_{4n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^6$ generated by $\{v^{2n}, A, B, C, D^2, \widetilde{D}^3\}$.
- For $n \geq 0$, $Z(QU_{4n+2} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^6$ generated by $\{v^{2n}, A, B, (A+B)D, D^2, D^3\}$.
- For $n \geq 0$, $H(QU_{4n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^5$ generated by $\{[v^{2n+1}], [A], [B], [D^2], [\widetilde{D}^3]\}$.
- For $n \geq 0$, $H(QU_{4n+2} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^5$ generated by $\{[v^{2n+1}], [A], [B], [D^2], [D^3]\}$.
- $Z(QU_{-2} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^3$ generated by $\{\overline{y}_1, \overline{t}_1 + \overline{x}_1 + \overline{z}_1, \overline{u}_1 + \overline{t}_1\}$.
- $H(QU_{-2} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ generated by $\{[\overline{y}_1], [\overline{t}_1 + \overline{x}_1 + \overline{z}_1]\}$.
- For $n \geq 1$, $Z(QU_{-4n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^5$ generated by $\{\overline{x}_{2n}, \overline{y}_{2n}, \overline{z}_{2n}, \overline{t}_{2n}, \overline{u}_{2n}\}$.
- For $n = 1, 2, 4$, $H(QU_{-4n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ generated by $\{[\overline{t}_{2n}], [\overline{u}_{2n}]\}$.
- For $n \geq 1$, $H(QU_{-4-8n} \otimes \mathbb{Z}/2) = \mathbb{Z}/2$ generated by $\{[\overline{t}_{4n+2}]\}$.
- For $n \geq 0$, $Z(QU_{-6-8n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^3$ generated by $\{\overline{t}_{4n+3}, \overline{w}_{4n+3}, \overline{u}_{4n+3} + \overline{x}_{4n+3} + \overline{z}_{4n+3}\}$.
- For $n \geq 0$, $H(QU_{-6-8n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ generated by $\{[\overline{u}_{4n+3}], [\overline{u}_{4n+3} + \overline{x}_{4n+3} + \overline{z}_{4n+3}]\}$.
- For $n \geq 1$, $Z(QU_{-2-8n} \otimes \mathbb{Z}/2) = (\mathbb{Z}/2)^2$ generated by $\{\overline{t}_{4n+1} + \overline{z}_{4n+1}, \overline{w}_{4n+1}\}$.
- For $n \geq 1$, $H(QU_{-2-8n} \otimes \mathbb{Z}/2) = \mathbb{Z}/2$ generated by $\{[\overline{t}_{4n+1} + \overline{z}_{4n+1}]\}$.

For $Z(TU_i)$ and $H(TU_i)$ part, this can be read easily from diagram 4.1.

So as to identify differentials, knowing some results, i.e., $ko^n(BSD_{16}; \mathbb{Z}/2)$ for some n, is fruitful. This is possible because all codegree n with $n \neq 8k+6$ $(k > 0)$, E_∞ of integral η-Bockstein spectral sequence split (see Figure 4.3). Thus we can identify $ko^n(BSD_{16})$ for all codegree n with $n \neq 8k+6$ $(k > 0)$. Also, since most of non-zero generators in $ko^*(BSD_{16})$ are in even degree, most of the additive structure of $ko^*(BSD_{16}; \mathbb{Z}/2)$ can be identified explicitly. After direct calculation, we record all information to find the E_∞-page of η-Bockstein spectral sequence for mod 2 coefficient as;

Figure 4.5: The $E_{1\frac{1}{2}}$-page of η-Bockstein spectral sequence for mod 2 coefficient



Notation: **2**'s come from TU otherwise come from $QU \otimes \mathbb{Z}/2$. On the Target ($ko^*(BSD_{16};\mathbb{Z}/2)$) line, $\widehat{2^k}$ means merely the order.

Description of Figure 4.5

It is obvious that there is no connecting homomorphism and d_2 departing from degree which is greater than -8. Since $ko^{-2}(BSD_{16};\mathbb{Z}/2) = (\mathbb{Z}/2)^{11}$ and $ko^{-3}(BSD_{16};\mathbb{Z}/2) = (\mathbb{Z}/2)^5$ and d_k's commute with η multiple elements, there is no non-zero differentials departing from diagonal starting from degree 0. Then d_3 departing from degree 4 and 6 in the zero-line are rank 5 and above the zero-line are isomorphisms. From a similar description, there is no non-zero differentials departing from degree -8,-6 and hence d_3 departing from degree -4 and -2 in the zero-line are rank 2 and above the zero-line are isomorphisms.

Now, we are reaching to the main point in codegree 14. Again, $ko^{15}(BSD_{16};\mathbb{Z}/2) = (\mathbb{Z}/2)^3$ and d_k's commute with η multiple elements, no-non zero differentials departing from diagonal starting from codegree 16. Then d_3 departing from diagonal line in codegree 12 must have rank 2. Consequently, connecting homomorphisms δ_0 must be 0. The dimension of $ko^{11}(BSD_{16};\mathbb{Z}/2)$ being 0 give us the rank 1 differential d_2 from codegree 11. Also there is no non-zero differtial depart from degree 14 since there must be at least rank one differential depart from diagonal line in degree 18. Hence all differentials illustrated in Figure 4.5 are determined. At this point, we have proved;

Lemma 4.3.3. *The dimension of $ko^{13}(BSD_{16};\mathbb{Z}/2)$ is 3 and the order of $ko^{14}(BSD_{16};\mathbb{Z}/2)$ is 2^6.*

Now, we are going to determine the extension problem for $ko^{14}(BSD_{16})$. To do this, we use mod 2 Bockstein spectral sequence with input $ko^*(BG;\mathbb{Z}/2)$ and output $ko^*(BG)$. We deal with this in the next subsection.

4.3.4 mod 2 - Bockstein spectral sequence for real connective k theory

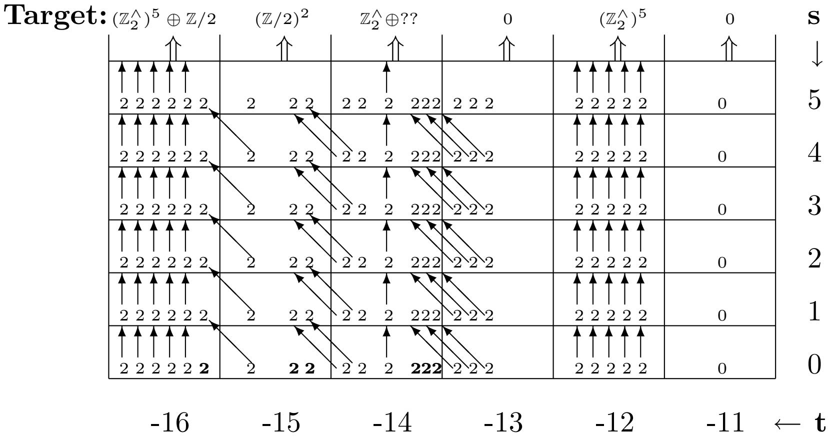
The cofibre sequence $ko \xrightarrow{2} ko \longrightarrow ko/2$ give us the mod 2 - Bockstein spectral sequence:

$$E_1^t = ko^t(BG;\mathbb{Z}/2)[\widetilde{2}] \Rightarrow ko^t(BG).$$

Since degree of $\widetilde{2}$ is $(1,0)$, E_1-page can be done similarly as the case of η but all entries in the above zero line are obtained by copying along the column (not diagonal).

In this case, we need to investigate in codegree 14 but the input in codegree 11 is zero and all differentials get back by one degree, so it is enough to consider on codegree 11 to codegree 16. By lemma 4.3.3, we have the E_1-page of mod2 - Bockstein spectral sequence as;

Figure 4.6: The E_1-page of mod2-Bockstein spectral sequence for $ko^*(BSD_{16})$



Notation: $\nwarrow$'s are differential d_1 and **2**'s are kernel of d_1 and
$2 \to 2 \to 2 \to 2 \to \ldots := \mathbb{Z}_2^\wedge$.

The precise structure of $ko^{14}(BSD_{16};\mathbb{Z}/2)$, $ko^{15}(BSD_{16};\mathbb{Z}/2)$ and all differentials illustrated in Figure 4.6 above are obtained clearly from the target $ko^*(BSD_{16})$. Thus, $ko^{14}(BSD_{16}) = \mathbb{Z}_2^\wedge \oplus (\mathbb{Z}/2)^3$.

Corollary 4.3.4. *Additive extension problems of η-Bockstein spectral sequence for $ko^*(BSD_{16})$ are trivial.*

Proof. This follows by the result above and lemma 4.3.2. □

§ 4.4 Results for $ko^*(BSD_{16})$

Since there is no additive extensions in the Bockstein spectral sequence for $ko^*(BSD_{16})$, the results (additively) can be read from E_∞-page directly, i.e., $ko^t(BSD_{16}) \cong E_\infty^{*,-t}$.

Theorem 4.4.1. *Additively,*

$ko^n(BSD_{16})$	$n \leq 0,\ (k \geq 0)$
0	$-8k-7 \leq -7$
$\mathbb{Z}_2^\wedge$	$-8k-6 \leq -6$
0	$-8k-5 \leq -5$
$\mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^5$	$-8k-4 \leq -4$
0	$-8k-3 \leq -3$
$\mathbb{Z}_2^\wedge \oplus (\mathbb{Z}/2)^5$	$-8k-2 \leq -2$
$(\mathbb{Z}/2)^5$	$-8k-1 \leq -1$
$\mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^5$	$-8k \leq 0$

,

$ko^n(BSD_{16})$	$n \geq 0$
0	$8k-7 \geq 1$
$\mathbb{Z}_2^\wedge \oplus (\mathbb{Z}/2)^{k-1}$	$8k-6 \geq 2$
0	$8k-5 \geq 3$
$(\mathbb{Z}_2^\wedge)^5$	4
$(\mathbb{Z}_2^\wedge)^5 \oplus (\mathbb{Z}/2)^{k-2}$	$8k-4 \geq 12$
0	$8k-3 \geq 5$
$\mathbb{Z}_2^\wedge \oplus (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/2)^{k-1}$	$8k-2 \geq 6$
$(\mathbb{Z}/2)^2$	$8k-1 \geq 7$
$(\mathbb{Z}_2^\wedge)^5 \oplus (\mathbb{Z}/2)^{k-1}$	$8k \geq 8$

.

Generator description:
Let $A = c_1^{RU}(\chi_3), B = c_1^{RU}(\chi_2)$ *and* $D = c_2^{RU}(\sigma_1)$ *which has character table as* $[0,2,0,2,0,2], [0,0,0,0,2,2]$ *and* $[4,-c,2,c,0,2]$ *respectively, where* $c = \sqrt{2}i$. *More over, let* $C = (A+B)D - 2D$ *and* $\widetilde{D}^3 = D^3 + 2D$. *Then the generator of* $ko^n(BSD_{16})$ *for each codegree* n *are;*

Non - positive codegree:, *for* $k \geq 0$;

- $ko^{-8k-6}(BSD_{16}) \cong \mathbb{Z}_2^\wedge < \beta^k v^3 \theta >$, *where*

$$\begin{aligned} \theta = D - \tau(D) &= -\frac{4}{3}A + \frac{1}{3}(A+B)D + \frac{16}{9}D - \frac{4}{3}D^2 + \frac{2}{9}D^3 \\ &= [0, -2c, 0, 2c, 0, 0] \end{aligned}$$

- $ko^{-8k-4}(BSD_{16}) \cong \mathbb{Z} < \beta^k 2v^2 > \oplus (\mathbb{Z}_2^\wedge)^5 < 2A, 2B, \overline{D}, 2D^2, \overline{D^3} > v^2\beta^k$, *where*

$$\begin{aligned} \overline{D} &= \frac{4}{3}A - \frac{1}{3}C + \frac{4}{3}D^2 - \frac{2}{9}\widetilde{D}^3 \\ &= [8, 0, 4, 0, 0, 4] \\ \overline{D^3} &= -\frac{8}{3}A + \frac{2}{3}C - \frac{8}{3}D^2 + \frac{22}{9}\widetilde{D}^3 \\ &= [128, 0, 16, 0, 0, 16] \end{aligned}$$

- $ko^{-8k-2}(BSD_{16}) \cong \mathbb{Z}_2^\wedge < \beta^k v\theta > \oplus (\mathbb{Z}/2)^5 < \widetilde{\eta}^2[1], \widetilde{\eta}^2[A], \widetilde{\eta}^2[B], \widetilde{\eta}^2[D^2], \widetilde{\eta}^2[\widetilde{D}^3] > \beta^k$.

- $ko^{-8k-1}(BSD_{16}) \cong (\mathbb{Z}/2)^5 < \widetilde{\eta}[1], \widetilde{\eta}[A], \widetilde{\eta}[B], \widetilde{\eta}[D^2], \widetilde{\eta}[\widetilde{D}^3] > \beta^k$.
- $ko^{-8k}(BSD_{16}) \cong \mathbb{Z} < \beta^k > \oplus (\mathbb{Z}_2^\wedge)^5 < A, B, C, D^2, \widetilde{D}^3 > \beta^k$.

Notation: $[x] = x + (1+\tau)(QU_{8k+2})$ *s.t.* $x \in \ker(1 - \tau : QU_{8k} \longrightarrow QU_{8k-2})$ *and* $\widetilde{\eta}[1] \twoheadleftarrow \eta \in ko^{-1}(pt)$, $2 \cdot v^2 \twoheadleftarrow \alpha \in ko^{-4}(pt)$ *and* $\beta \in ko^{-8}(pt)$.

Positive codegree:

- $ko^2(BSD_{16}) \cong \mathbb{Z}_2^\wedge < \theta_1 >$, *where* $\theta_1 = v^{-1}\theta = \overline{u}_1 - \overline{t}_1 - 2\overline{w}_1$.
- $ko^{8k-6}(BSD_{16}) \cong \mathbb{Z}_2^\wedge < \theta_k > \oplus (\mathbb{Z}/2)^{k-1} < b^{4(k-1-i)-3}d^{2i}\tau | i = 0, 1, ..., k-2 >$, *where* $k \geq 2$, $\tau = b^2d - abd$ *and*

$$\theta_k = \pm(1-\tau)(\overline{w}_{4(k-1)}) = \pm 2^{k-1}v^{-k}\theta_1.$$

Here, $\overline{w}_{4(k-1)} \in JU_{4(k-1)}$ *and other notations below which does not state are elements of* $JU_i's$ *in Chapter 3.*

- $ko^4(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^5 < \overline{x}_2 + \overline{y}_2, 2\overline{x}_2, 2\overline{t}_2 - \overline{z}_2 + \overline{y}_2, \overline{z}_2 + \overline{x}_2, -\overline{u^*}_2 - \overline{x}_2 - \overline{z}_2 >$ *s.t.*

$$\begin{aligned}
\overline{x}_2 + \overline{y}_2 &= (1+\tau)(\overline{x}_1) = [0,0,0,0,4,4] \\
2\overline{x}_2 &= (1+\tau)(\overline{y}_1) = [0,0,0,0,0,8] \\
2\overline{t}_2 - \overline{z}_2 + \overline{y}_2 &= (1+\tau)(\overline{z}_1) = [0,4,0,4,4,0] \\
\overline{z}_2 + \overline{x}_2 &= (1+\tau)(\overline{t}_1) = [0,0,8,0,0,4] \\
-\overline{u^*}_2 - \overline{x}_2 - \overline{z}_2 &= (1+\tau)(\overline{w}_1) = [8,0,-4,0,0,0],
\end{aligned}$$

where $\overline{u^*}_2 = \overline{u}_2 - 2\overline{w}_2 = [-8,0,-4,0,0,-4]$.

- $ko^{8k-4}(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^5 < \overline{x}_{4k-2}, \overline{y}_{4k-2}, \overline{z}_{4k-2}, 2\overline{t}_{4k-2} - \overline{z}_{4k-2}, -\overline{u^*}_{4k-2} - \overline{x}_{4k-2} - \overline{z}_{4k-2} > \oplus (\mathbb{Z}/2)^{k-2} < b^{4(k-2-i)}d^{2i+1}\tau | i = 0,1,...,k-3 >$ *s.t.*

$$\begin{aligned}
\overline{x}_{4k-2} &= (1+\tau)(\overline{x}_{4k-3}) = [0,0,0,0,0,4\cdot 4^{k-1}] \\
\overline{y}_{4k-2} &= (1+\tau)(\overline{y}_{4k-3}) = [0,0,0,0,4\cdot 16^{k-1},0] \\
\overline{z}_{4k-2} &= (1+\tau)(\overline{z}_{4k-3}) = [0,0,8\cdot 4^{k-1},0,0,0] \\
2\overline{t}_{4k-2} - \overline{z}_{4k-2} &= (1+\tau)(\overline{t}_{4k-3}) = [0,4(-2)^{k-1},0,4(-2)^{k-1},0,0] \\
-\overline{u^*}_{4k-2} - \overline{x}_{4k-2} - \overline{z}_{4k-2} &= (1+\tau)(\overline{u}_{4k-3}) = [8\cdot 16^{k-1},0,-4\cdot 4^{k-1},0,0,0],
\end{aligned}$$

where $k \geq 2$, $b^0 d\tau = 0$ *and* $\overline{u^*}_{4k-2} = \overline{u}_{4k-2} - 2\overline{w}_{4k-2} = [-8\cdot 16^{k-1}, 0, -4\cdot 4^{k-1}, 0, -4\cdot 4^{k-1}]$.

- $ko^{8k-2}(BSD_{16}) \cong \mathbb{Z}_2^\wedge < \theta'_k > \oplus (\mathbb{Z}/2)^2 < \widetilde{\eta}^2[\overline{t}_{4k}], \widetilde{\eta}^2[\overline{u}_{4k}] > \oplus (\mathbb{Z}/2)^{k-1} < b^{4(k-1-i)-1}d^{2i}\tau | i = 0,1,...,k-2 >$, *where* $k \geq 1$,

$$\begin{aligned}
\theta'_1 &= (1-\tau)(\overline{w}_2) = \overline{w}_3 \\
\theta'_k &= \pm(1-\tau)(\overline{w}_{4k-2}) = \pm 2^{k-1}\alpha'_1.
\end{aligned}$$

and $[\overline{\omega}_{4k}] = \overline{\omega}_{4k} + im(1+\tau : JU_{4k-1} \longrightarrow JU_{4k})$ *for* $\omega = t, u$.

- $ko^{8k-1}(BSD_{16}) \cong (\mathbb{Z}/2)^2 < \widetilde{\eta}[\overline{t}_{4k}], \widetilde{\eta}[\overline{u}_{4k}] >$, *where* $k \geq 1$.

- $ko^{8k}(BSD_{16}) \cong (\mathbb{Z}_2^\wedge)^5 < \overline{x}_{4k}, \overline{y}_{4k}, \overline{z}_{4k}, \overline{t}_{4k}, \overline{u}_{4k} > \oplus (\mathbb{Z}/2)^{k-1} < b^{4(k-1-i)-2}d^{2i+1}\tau | i = 0,1,...,k-2 >$, *where* $k \geq 1$.

Proof. This is an immediate results from E_∞-page, corollary 4.3.4, and character table. □

Corollary 4.4.2. *The natural homomorphism* $\beta^* : ko^*(BSD_{16}) \longrightarrow H^*(BSD_{16}; \mathbb{F}_2) \oplus KO^*(BSD_{16})$ *is a monomorphism.*

Proof. Since $ku^*(BSD_{16}) \rightarrowtail H^*(BSD_{16}; \mathbb{F}_2) \oplus KU^*(BSD_{16})$ and E_∞-page has been calculated with the initial input $ku^*(BSD_{16})$ s.t. there is no η-multiples coming from TU part and also $ko^*(BSD_{16})$ is additively isomorphic to E_∞-page, the result follows. □

We investigate the restriction map $ko^*(BSD_{16}) \longrightarrow ko^*(BG)$ for each maximal subgroup G of SD_{16} in the next section.

§ 4.5 Relations with its maximal subgroups

As the previous chapter, we will make explicit only the map at E_∞-stage because all of $ko^*(BG)$, for $G = SD_{16}, D_8, Q_8, C_8$, are isomorphic to their E_∞-page. So, the job is giving the generator names for $ko^*(BG)$'s and relating them by using Theorem 2.7.2.

4.5.1 $ko^*(BSD_{16})$ AND $ko^*(BD_8)$

By the results of $ko^*(BD_8)$ in [13], we can explicit the generator name for $ko^*(BD_8)$ by using the same symbols as in Theorem 2.5.5 in [14] (and in Proposition 2.7.1) as follows.

Proposition 4.5.1. *(cf. [13]) The additive structure of real connective K-cohomology of D_8 is isomorphically given by;*

n	$ko^n(BD_8)$	***Generators***
$-8k-7$	0	0
$-8k-6$	0	0
$-8k-5$	0	0
$-8k-4$	$\mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^4$	$\beta^k 2v^2, \beta^k 2v^2\{va, vb, v^2d, v^2d_2\}$
$-8k-3$	0	0
$-8k-2$	$[2]^5$	$\beta^k\widetilde{\eta}^2\{[1], [va], [vb], [v^2d], [v^2d_2]\}$
$-8k-1$	$[2]^5$	$\beta^k\widetilde{\eta}\{[1], [va], [vb], [v^2d], [v^2d_2]\}$
$-8k$	$\mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^4$	$\beta^k, \beta^k\{va, vb, v^2d, v^2d_2\}$
1	0	0
2	0	0
3	0	0
4	$(\mathbb{Z}_2^\wedge)^4$	$a^2, b^2, 2d, 2d_2$
5	0	0
6	$[2] \oplus 2$	$\widetilde{\eta}^2[d^2], ad$
7	$[2]$	$\widetilde{\eta}[d^2]$
8	$(\mathbb{Z}_2^\wedge)^4$	a^4, b^4, d^2, dd_2
$8k-7 \geq 9$	0	0
$8k-6 \geq 10$	2^{2k-2}	$a^{4(k-i-1)-1}d^{2i+1}, a^{4(k-i-1)-2}bd^{2i+1}$, $i = 0, ..., k-2$
$8k-5 \geq 11$	0	0
$8k-4 \geq 12$	$(\mathbb{Z}_2^\wedge)^4 \oplus 2^{2k-3}$	$a^{4k-2}, b^{4k-2}, 2d^{2k-1}, d^{2k-2}d_2$, $a^{4(k-i-1)-2}d^{2i+2}$, $i = 0, ..., k-2$ $a^{4(k-i-2)-3}bd^{2i+4}$, $i = 0, ..., k-3$
$8k-3 \geq 13$	0	0
$8k-2 \geq 14$	$[2] \oplus 2^{2k-1}$	$\widetilde{\eta}^2[d^{2k}]$, $a^{4(k-i-1)+1}d^{2i+1}$, $i = 0, ..., k-1$ $a^{4(k-i-1)}bd^{2i+1}$, $i = 0, ..., k-2$
$8k-1 \geq 15$	$[2]$	$\widetilde{\eta}[d^{2k}]$
$8k \geq 16$	$(\mathbb{Z}_2^\wedge)^4 \oplus 2^{2k-2}$	$a^{4k}, b^{4k}, d^{2k}, d^{2k-1}d_2$, $a^{4(k-i-1)}d^{2i+2}$, $a^{4(k-i-1)-1}bd^{2i+2}$, $i = 0, ..., k-2$

where $[n]$ *and* 2^m *are referred to be a cyclic group of order* n *and elementary abelian* 2*-group of rank* m *as usual and* $d_2 = 4d - v^2d^2$.

Now, the map is quickly calculated, by using Theorem 2.7.2, Proposition 4.5.1 and relations in $ku^*(BD_8)$, which we record as;

Proposition 4.5.2. *The canonical map from* $ko^*(BSD_{16})$ *to* $ko^*(BD_8)$ *is explicitly given by*

1 *For* $k \geq 0$, $ko^{-8k-6}(BSD_{16}) \mapsto 0$.

2 *For* $k \geq 0$, $ko^{-8k-4}(BSD_{16}) \longrightarrow ko^{-8k-4}(BD_8)$ *is given by;*

$$\begin{array}{ll} \beta^k 2v^2 \mapsto \beta^k 2v^2, & \beta^k v^2 2D^2 \mapsto 4(\beta^k 2v^4 d) - (\beta^k 2v^4 d_2), \\ \beta^k v^2 2A \mapsto 0, & \beta^k v^2 \overline{D} \mapsto \beta^k 2v^4 d, \\ \beta^k v^2 2B \mapsto \beta^k 2v^3 a, & \beta^k v^2 \overline{D^3} \mapsto 16(\beta^k 2v^4 d) - 6(\beta^k 2v^4 d_2). \end{array}$$

3 *For* $k \geq 0$, $ko^{-8k-2}(BSD_{16}) \longrightarrow ko^{-8k-2}(BD_8)$ *is given by;*

$$\begin{array}{lll} \beta^k v\theta \mapsto 0, & \beta^k \widetilde{\eta}^2[1] \mapsto \beta^k \widetilde{\eta}^2[1], & \beta^k \widetilde{\eta}^2[A] \mapsto 0, \\ \beta^k \widetilde{\eta}^2[B] \mapsto \beta^k \widetilde{\eta}^2[va], & \beta^k \widetilde{\eta}^2[D^2] \mapsto \beta^k \widetilde{\eta}^2[v^2 d_2], & \beta^k \widetilde{\eta}^2[\widetilde{D}^3] \mapsto 0. \end{array}$$

4 *For* $k \geq 0$, $ko^{-8k-1}(BSD_{16}) \longrightarrow ko^{-8k-1}(BD_8)$ *is given by;*

$$\begin{array}{lll} \beta^k \widetilde{\eta}[1] \mapsto \beta^k \widetilde{\eta}[1], & \beta^k \widetilde{\eta}[A] \mapsto 0, & \\ \beta^k \widetilde{\eta}[B] \mapsto \beta^k \widetilde{\eta}[va], & \beta^k \widetilde{\eta}[D^2] \mapsto \beta^k \widetilde{\eta}[v^2 d_2], & \beta^k \widetilde{\eta}[\widetilde{D}^3] \mapsto 0. \end{array}$$

5 *For* $k \geq 0$, $ko^{-8k}(BSD_{16}) \longrightarrow ko^{-8k}(BD_8)$ *is given by;*

$$\begin{array}{lll} \beta^k 1 \mapsto \beta^k 1, & \beta^k A \mapsto 0, & \beta^k B \mapsto \beta^k va, \\ \beta^k D^2 \mapsto \beta^k (4v^2 d - v^2 d_2), & \beta^k C \mapsto -2\beta^k v^2 d, & \beta^k \widetilde{D}^3 \mapsto 0. \end{array}$$

6 *For* $k \geq 1$, $ko^{8k-6}(BSD_{16}) \longrightarrow ko^{8k-6}(BD_8)$ *is given by;*

- $k = 1$, $ko^2(BSD_{16}) \mapsto 0$,
- $k \geq 2$, $\theta_k \mapsto 0$, $\quad b^{4(k-i-1)-3} d^{2i} \tau \mapsto a^{4(k-i-1)} d^{2i+1}$ *for* $i = 0, ..., k-2$.

7 *For* $k \geq 1$, $ko^{8k-4}(BSD_{16}) \longrightarrow ko^{8k-4}(BD_8)$ *is given by;*

- $k = 1$, $\begin{array}{lll} \overline{x}_2 + \overline{y}_2 \mapsto a^2, & 2\overline{x}_2 \mapsto 0, & 2\overline{t}_2 - \overline{z}_2 + \overline{y}_2 \mapsto a^2, \\ \overline{z}_2 + \overline{x}_2 \mapsto 2d_2, & -\overline{u}_2^* - \overline{x}_2 - \overline{z}_2 \mapsto 2d - 2d_2, & \end{array}$
- $k \geq 2$, $\begin{array}{ll} \overline{x}_{4k-2} \mapsto 0, & 2\overline{t}_{4k-2} - \overline{z}_{4k-2} \mapsto 0, \\ \overline{y}_{4k-2} \mapsto a^{4k-2}, & -\overline{u}_{4k-2}^* - \overline{x}_{4k-2} - \overline{z}_{4k-2} \mapsto 2d^{2k-1} - 2d^{2k-2} d_2, \\ \overline{z}_{4k-2} \mapsto 2d^{2k-2} d_2, & \end{array}$

  and for $k \geq 3$, $b^{4(k-i-2)} d^{2i+1} \tau \mapsto a^{4(k-i-1)-2} d^{2i+2}$ *for* $i = 0, ..., k-3$.

8 *For* $k \geq 1$, $ko^{8k-2}(BSD_{16}) \longrightarrow ko^{8k-2}(BD_8)$ *is given by;*
$\theta'_k \mapsto 0$, $\quad \widetilde{\eta}^2[\overline{t}_{4k}] \mapsto 0$, $\quad \widetilde{\eta}^2[\overline{u}_{4k}] \mapsto \widetilde{\eta}^2[d^{2k}]$, $b^{4(k-i-1)-1}d^{2i}\tau \mapsto a^{4(k-i-1)+1}d^{2i+1}$ *for* $i = 0, ..., k-2$.

9 *For* $k \geq 1$, $ko^{8k-1}(BSD_{16}) \longrightarrow ko^{8k-1}(BD_8)$ *is given by;*

$$\widetilde{\eta}[\overline{t}_{4k}] \mapsto 0, \qquad \widetilde{\eta}[\overline{u}_{4k}] \mapsto \widetilde{\eta}[d^{2k}].$$

10 *For* $k \geq 1$, $ko^{8k}(BSD_{16}) \longrightarrow ko^{8k}(BD_8)$ *is given by;*

$$\overline{x}_{4k} \mapsto 0, \quad \overline{y}_{4k} \mapsto a^{4k}, \quad \overline{z}_{4k} \mapsto 2d^{2k-1}d_2, \quad \overline{t}_{4k} \mapsto 0, \quad \overline{u}_{4k} \mapsto d^{2k},$$

and for $k \geq 2$, $b^{4(k-i-1)-2}d^{2i}\tau \mapsto a^{4(k-i-1)}d^{2i+2}$ *for* $i = 0, ..., k-2$.

4.5.2 $ko^*(BSD_{16})$ and $ko^*(BQ_8)$

By the results of $ko^*(BQ_8)$ in Theorem 6.4.2 in [13], we can make explicit the generator name for $ko^*(BQ_8)$ by working on representation theory only. To do this, it is simply to work out on character table of $RU(Q_8)$ which is shown in the proof of lemma 2.2.3 (i.e., we have that $1, \rho_2, \rho_3, \rho_4$ are real representations and υ is a quoternionic representation).

Precisely, with the same symbols in Theorem 2.4.6 in [14], we have character table of $va = 1-\rho_3, vb = 1-\rho_4, v^2q = 2-\upsilon$ (note that $1-\rho_2 = 4v^2q - v^4q^2 - va - vb$) as;

	$[1]$	$[s^4]$	$[s^2]$	$[ts^3]$	$[ts]$
va	0	0	2	0	2
vb	0	0	0	2	2
v^2q	0	4	2	2	2

and character table of $\widetilde{c}RSp(Q_8)$ as;

$\widetilde{c}RSp(Q_8)$	$[1]$	$[s^4]$	$[s^2]$	$[ts^3]$	$[ts]$
$\widetilde{c}qc1$	2	2	2	2	2
$\widetilde{c}qc\rho_2$	2	2	-2	-2	2
$\widetilde{c}qc\rho_3$	2	2	-2	2	-2
$\widetilde{c}qc\rho_4$	2	2	2	-2	-2
$\widetilde{c}\upsilon$	2	-2	0	0	0

because $\widetilde{c}q = 1+\tau$, where $\widetilde{c}$ is complexification from RSp to RU. From here, we get character table of $JSp^i(Q_8)$ as;

$JSp^i(Q_8)$	$[1]$	$[s^4]$	$[s^2]$	$[ts^3]$	$[ts]$
j_{1i}	0	0	2^{i+1}	0	2^{i+1}
j_{2i}	0	0	2^{i+1}	2^{i+1}	0
j_{3i}	0	0	0	2^{i+1}	2^{i+1}
j_{4i}	0	4^i	2^i	2^i	2^i

After relating these character tables to the symbols v, a, b, q, we have;

Proposition 4.5.3. *(cf. [13]) The additive structure of real connective K-cohomology of Q_8 is isomorphically given by;*

n	$ko^n(BQ_8)$	***Generators***
$-8k-7$	0	0
$-8k-6$	$[2]$	$\beta^k\widetilde{\eta}^2[v^4q]$
$-8k-5$	$[2]$	$\beta^k\widetilde{\eta}[v^4q]$
$-8k-4$	$\mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^4$	$\beta^k 2v^2, \beta^k v^2\{2va, 2vb, v^2q, 2v^4q^2\}$
$-8k-3$	0	0
$-8k-2$	$[2]^4$	$\beta^k\widetilde{\eta}^2\{[1], [va], [vb], [v^4q^2]\}$
$-8k-1$	$[2]^4$	$\beta^k\widetilde{\eta}\{[1], [va], [vb], [v^4q^2]\}$
$-8k$	$\mathbb{Z} \oplus (\mathbb{Z}_2^\wedge)^4$	$\beta^k, \beta^k\{va, vb, v^2 2q, v^4q^2\}$
$8k-7 \geq 1$	0	0
$8k-6 \geq 2$	$[2]$	$\widetilde{\eta}^2[q^{2k-1}]$
$8k-5 \geq 3$	0	$\widetilde{\eta}[q^{2k-1}]$
$8k-4 \geq 4$	$(\mathbb{Z}_2^\wedge)^4$	$vaq^{2k-1}, vbq^{2k-1}, q^{2k-1}, 2v^2q^{2k}$
$8k-3 \geq 5$	0	0
$8k-2 \geq 6$	$[2]$	$\widetilde{\eta}^2[q^{2k}]$
$8k-1 \geq 7$	$[2]$	$\widetilde{\eta}[q^{2k}]$
$8k \geq 8$	$(\mathbb{Z}_2^\wedge)^4$	$vaq^{2k}, vbq^{2k}, q^{2k}, 2v^2q^{2k+1}$

Now, the map is quickly calculated, by using Theorem 2.7.2, Proposition 4.5.3 and relations in $ku^*(BQ_8)$, which we record as;

Proposition 4.5.4. *The canonical map from $ko^*(BSD_{16})$ to $ko^*(BQ_8)$ is explicitly given by*

1 For $k \geq 0$, $ko^{-8k-6}(BSD_{16}) \mapsto 0$.

2 For $k \geq 0$, $ko^{-8k-4}(BSD_{16}) \longrightarrow ko^{-8k-4}(BQ_8)$ is given by;

$$\begin{aligned} &\beta^k 2v^2 \mapsto \beta^k 2v^2, && \beta^k v^2 2A \mapsto \beta^k 2v^3 b, && \beta^k v^2 2B \mapsto \beta^k 2v^3 b, \\ &\beta^k v^2 2D^2 \mapsto \beta^k 2v^6 q^2, && \beta^k v^2 \overline{D} \mapsto 2\beta^k v^4 q, && \beta^k v^2 \overline{D^3} \mapsto 6(\beta^k 2v^6 q^2) - 16(\beta^k v^4 q). \end{aligned}$$

3 For $k \geq 0$, $ko^{-8k-2}(BSD_{16}) \longrightarrow ko^{-8k-2}(BQ_8)$ is given by;

$$\begin{aligned} &\beta^k v\theta \mapsto 0, && \beta^k\widetilde{\eta}^2[1] \mapsto \beta^k\widetilde{\eta}^2[1], && \beta^k\widetilde{\eta}^2[A] \mapsto \beta^k\widetilde{\eta}^2[vb], \\ &\beta^k\widetilde{\eta}^2[B] \mapsto \beta^k\widetilde{\eta}^2[vb], && \beta^k\widetilde{\eta}^2[D^2] \mapsto \beta^k\widetilde{\eta}^2[v^4q^2], && \beta^k\widetilde{\eta}^2[\widetilde{D}^3] \mapsto 0. \end{aligned}$$

4 For $k \geq 0$, $ko^{-8k-1}(BSD_{16}) \longrightarrow ko^{-8k-1}(BQ_8)$ is given by;

$$\begin{aligned} &\beta^k\widetilde{\eta}[1] \mapsto \beta^k\widetilde{\eta}[1], && \beta^k\widetilde{\eta}[A] \mapsto \beta^k\widetilde{\eta}[vb], \\ &\beta^k\widetilde{\eta}[B] \mapsto \beta^k\widetilde{\eta}[vb], && \beta^k\widetilde{\eta}[D^2] \mapsto \beta^k\widetilde{\eta}[v^4q^2], && \beta^k\widetilde{\eta}[\widetilde{D}^3] \mapsto 0. \end{aligned}$$

5 *For* $k \geq 0$, $ko^{-8k}(BSD_{16}) \longrightarrow ko^{-8k}(BQ_8)$ *is given by;*

$$\begin{array}{lll} \beta^k 1 \mapsto \beta^k 1, & \beta^k A \mapsto \beta^k vb, & \beta^k B \mapsto \beta^k vb, \\ \beta^k D^2 \mapsto \beta^k v^4 q^2, & \beta^k C \mapsto 4vb - 2\beta^k v^2 q, & \beta^k \widetilde{D}^3 \mapsto 6\beta^k v^4 q^2 - 6\beta^k v^2 q. \end{array}$$

6 *For* $k \geq 1$, $ko^{8k-6}(BSD_{16}) \longrightarrow 0$.

7 *For* $k \geq 1$, $ko^{8k-4}(BSD_{16}) \longrightarrow ko^{8k-4}(BQ_8)$ *is given by;*

- $k = 1$, $\begin{array}{l} \overline{x}_2 + \overline{y}_2 \mapsto vbq, \quad 2\overline{x}_2 \mapsto 2vbq, \quad 2\overline{t}_2 - \overline{z}_2 + \overline{y}_2 \mapsto 0, \\ \overline{z}_2 + \overline{x}_2 \mapsto -vbq + 8q - 2v^2q^2, \quad -\overline{u}_2^* - \overline{x}_2 - \overline{z}_2 \mapsto vbq - 6q + 2v^2q^2, \end{array}$
- $k \geq 2$,
  $\overline{x}_{4k-2} \mapsto vbq^{2k-1}, \quad 2\overline{t}_{4k-2} - \overline{z}_{4k-2} \mapsto 0,$
  $\overline{y}_{4k-2} \mapsto 0, \quad -\overline{u}_{4k-2}^* - \overline{x}_{4k-2} - \overline{z}_{4k-2} \mapsto vbq^{2k-1} - 6q^{2k-1} + 2v^2q^{2k},$
  $\overline{z}_{4k-2} \mapsto -2vbq^{2k-1} + 8q^{2k-1} - 2v^2q^{2k},$
  and for $k \geq 3$, $b^{4(k-i-2)}d^{2i+1}\tau \mapsto 0$ *for* $i = 0, ..., k-3$.

8 *For* $k \geq 1$, $ko^{8k-2}(BSD_{16}) \longrightarrow ko^{8k-2}(BQ_8)$ *is given by;*
$\theta'_k \mapsto 0$, $\widetilde{\eta}^2[\overline{t}_{4k}] \mapsto 0$, $\widetilde{\eta}^2[\overline{u}_{4k}] \mapsto \widetilde{\eta}^2[q^{2k}]$, $b^{4(k-i-1)-1}d^{2i}\tau \mapsto 0$, *for* $i = 0, ..., k-2$.

9 *For* $k \geq 1$, $ko^{8k-1}(BSD_{16}) \longrightarrow ko^{8k-1}(BQ_8)$ *is given by;*

$$\widetilde{\eta}[\overline{t}_{4k}] \mapsto 0, \qquad \widetilde{\eta}[\overline{u}_{4k}] \mapsto \widetilde{\eta}[q^{2k}].$$

10 *For* $k \geq 1$, $ko^{8k}(BSD_{16}) \longrightarrow ko^{8k}(BQ_8)$ *is given by;*

$$\overline{x}_{4k} \mapsto vbq^{2k}, \quad \overline{y}_{4k} \mapsto 0, \quad \overline{z}_{4k} \mapsto 8q^{2k} - 2(vbq^{2k} + v^2q^{2k+1}), \quad \overline{t}_{4k} \mapsto 0, \quad \overline{u}_{4k} \mapsto q^{2k},$$

and for $k \geq 2$, $b^{4(k-i-1)-2}d^{2i}\tau \mapsto 0$ *for* $i = 0, ..., k-2$.

4.5.3 $ko^*(BSD_{16})$ and $ko^*(BC_8)$

Recall that real connective K- cohomology on non-positive codegree is obtained directly by representation theory. For $ko^*(BC_8)$, we will write out the generator of $ko^i(BC_8)$ for $i \leq 0$ by using such theory, and for $i \geq 0$ we will use the results from Theorem 6.3.1 in [13]. For the restriction map $ko^*(BSD_{16}) \longrightarrow ko^*(BC_8)$, we explicit such a map on non-negative codegree and for positive codegree, we merely investigate the kernel.

Recall that, for $i \geq 0$, $ku^{-i}(BC_8) \cong RU(C_8)_2^{\wedge} < v^i >$ and $RU(C_8) = \mathbb{Z}[\alpha]/(\alpha^8 - 1)$, where $\alpha^i(s^j) = c^{ij}$ s.t. $c = e^{\frac{i\pi}{4}} = \frac{\sqrt{2}}{2}(1+i)$. It is not hard to see that $1, \alpha^4$ are real representations and the remaining are complex representations. So, by representation theory and Theorem 6.3.1 in [13] we have;

Proposition 4.5.5. *(cf. [13]) The additive structure of real connective K-cohomology of* C_8 *on non-positive codegree is isomorphically given by;*

n	$ko^n(BC_8)$
$-8k-7$	0
$-8k-6$	$\mathbb{Z}_2^\wedge[\alpha,\alpha^2,\alpha^3,\alpha^5,\alpha^6,\alpha^7]/(\alpha+\alpha^7,\alpha^2+\alpha^6,\alpha^3+\alpha^5)$
$-8k-5$	0
$-8k-4$	$\mathbb{Z}<2>\oplus(\mathbb{Z}_2^\wedge)^4<2\alpha^4,\alpha+\alpha^7,\alpha^2+\alpha^6,\alpha^3+\alpha^5>$
$-8k-3$	0
$-8k-2$	$[2]^2<\widetilde{\eta}^2[1],\widetilde{\eta}^2[\alpha^4]>\oplus\mathbb{Z}_2^\wedge[\alpha,\alpha^2,\alpha^3,\alpha^5,\alpha^6,\alpha^7]/(\alpha+\alpha^7,\alpha^2+\alpha^6,\alpha^3+\alpha^5)$
$-8k-1$	$[2]^2<\widetilde{\eta}[1],\widetilde{\eta}[\alpha^4]>$
$-8k$	$RO(C_8)_2^\wedge=\mathbb{Z}<1>\oplus(\mathbb{Z}_2^\wedge)^4<\alpha^4,\alpha+\alpha^7,\alpha^2+\alpha^6,\alpha^3+\alpha^5>$

The additive structure of real connective K-cohomology of C_8 on positive codegree is isomorphically given by ([13]);

- $ko^{4i}(BC_8)\cong(p_1)^iRO(C_8)_2^\wedge$
- $ko^{4i+2}(BC_8)\cong(p_1)^iRU_{asc}(C_8)_2^\wedge$,

for all $i\geq 1$, where $p_1=p_1(\alpha)=c_1(\alpha)c_1(\tau\alpha)=(1-\alpha)(1-\alpha^7)$ and $RU_{asc}(C_8)_2^\wedge=(\mathbb{Z}_2^\wedge)^3<\alpha-\alpha^7,\alpha^2-\alpha^6,\alpha^3-\alpha^5>$.

Now, it is not hard to see that;

Proposition 4.5.6. *The canonical map from $ko^*(BSD_{16})$ to $ko^*(BC_8)$ on non-negative degree is explicitly given by*

1 *For $k\geq 0$, $ko^{-8k-6}(BSD_{16})\longrightarrow ko^{-8k-6}(BC_8)$ is given by;*

$$\beta^k v^3\theta\mapsto-(\alpha-\alpha^7)-(\alpha^3-\alpha^5).$$

2 *For $k\geq 0$, $ko^{-8k-4}(BSD_{16})\longrightarrow ko^{-8k-4}(BC_8)$ is given by;*

$$\begin{gathered}\beta^k2v^2\mapsto 2,\quad \beta^kv^22A\mapsto 2-2\alpha^4,\quad \beta^kv^22B\mapsto 0,\\ \beta^kv^22D^2\mapsto 4-4(\alpha+\alpha^7)+2(\alpha^2+\alpha^6)-4(\alpha^3+\alpha^5)+8\alpha^4,\\ \beta^kv^2\overline{D}\mapsto 2-(\alpha+\alpha^7)-(\alpha^3+\alpha^5)+2\alpha^4,\\ \beta^kv^2\overline{D^3}\mapsto 20-16(\alpha+\alpha^7)+12(\alpha^2+\alpha^6)-16(\alpha^3+\alpha^5)+20\alpha^4.\end{gathered}$$

3 *For $k\geq 0$, $ko^{-8k-2}(BSD_{16})\longrightarrow ko^{-8k-2}(BC_8)$ is given by;*

$$\begin{array}{lll}\beta^kv\theta\mapsto-(\alpha-\alpha^7)-(\alpha^3-\alpha^5), & \beta^k\widetilde{\eta}^2[1]\mapsto\widetilde{\eta}^2[1], & \beta^k\widetilde{\eta}^2[A]\mapsto\widetilde{\eta}^2[1]-\widetilde{\eta}^2[\alpha^4],\\ \beta^k\widetilde{\eta}^2[B]\mapsto 0, & \beta^k\widetilde{\eta}^2[D^2]\mapsto 0, & \beta^k\widetilde{\eta}^2[\widetilde{D}^3]\mapsto 0.\end{array}$$

4 *For $k\geq 0$, $ko^{-8k-1}(BSD_{16})\longrightarrow ko^{-8k-1}(BC_8)$ is given by;*

$$\begin{array}{lll}\beta^k\widetilde{\eta}[1]\mapsto\widetilde{\eta}[1], & \beta^k\widetilde{\eta}[A]\mapsto\widetilde{\eta}[1]-\widetilde{\eta}[\alpha^4], & \\ \beta^k\widetilde{\eta}[B]\mapsto 0, & \beta^k\widetilde{\eta}[D^2]\mapsto 0, & \beta^k\widetilde{\eta}[\widetilde{D}^3]\mapsto 0.\end{array}$$

5 For $k \geq 0$ *,* $ko^{-8k}(BSD_{16}) \longrightarrow ko^{-8k}(BC_8)$ *is given by;*

$$\begin{gathered}\beta^k 1 \mapsto 1, \quad \beta^k A \mapsto 1-\alpha^4, \quad \beta^k B \mapsto 0, \\ \beta^k D^2 \mapsto 2-2(\alpha+\alpha^7)+(\alpha^2+\alpha^6)-2(\alpha^3+\alpha^5)+4\alpha^4, \\ \beta^k C \mapsto -2+(\alpha+\alpha^7)+(\alpha^3+\alpha^5)-2\alpha^4, \\ \beta^k \widetilde{D}^3 \mapsto 12-9(\alpha+\alpha^7)+6(\alpha^2+\alpha^6)-9(\alpha^3+\alpha^5)+12\alpha^4.\end{gathered}$$

For positive codegree, we found that (with the help of (3.12));

1 For $k \geq 1$*,* $ko^{8k-6}(BSD_{16}) \rightarrowtail ko^{8k-6}(BC_8)$*, is monomorphism.*

2 For $k \geq 1$*,* $ko^{8k-4}(BSD_{16}) \longrightarrow ko^{8k-4}(BQ_8)$ *has kernel generated by* $\overline{x}_{4i-2}$*,* $\overline{y}_{4i-2}$ *and* $b^{4(k-i-2)}d^{2i+1}\tau$ *for* $i = 0, ..., k-3$*.*

3 For $k \geq 1$*,* $ko^{8k-2}(BSD_{16}) \longrightarrow ko^{8k-2}(BC_8)$ *has kernel generated by* $\widetilde{\eta}^2[\overline{t}_{4k}], \widetilde{\eta}^2[\overline{u}_{4k}]$ *and* $b^{4(k-i-1)-1}d^{2i}\tau$ *for* $i = 0, ..., k-2$*.*

4 For $k \geq 1$*,* $ko^{8k-1}(BSD_{16}) \longrightarrow 0$*.*

5 For $k \geq 1$*,* $ko^{8k}(BSD_{16}) \longrightarrow ko^{8k}(BQ_8)$ *has kernel generated by* $\overline{x}_{4k}, \overline{y}_{4k}$ *and for* $k \geq 2$*,* $b^{4(k-i-1)-2}d^{2i}\tau$ *for* $i = 0, ..., k-2$*.*

Accordingly, we observe from proposition 4.5.2, 4.5.4 and 4.5.6 that the canonical map from $ko^*(BSD_{16})$ to $ko^*(BD_8) \oplus ko^*(BQ_8) \oplus ko^*(BC_8)$ is not a monomorphism, e.g., $\widetilde{\eta}[t_4] \mapsto (0,0,0)$.

We investigate the real connective K homology for SD_{16} in the next chapter.

Chapter 5

Real connective K-homology

In this chapter, we will calculate $ko_*(BSD_{16})$ as a module over $ko^*(BSD_{16})$ by using the Greenlees spectral sequence with input $ko^*(BSD_{16})$ and output $ko_*(BSD_{16})$. That is by using

$$E_2^{s,t} = H_I^{-s}(ko^*(BG)_t \Rightarrow ko_{(s+t)}(BG),$$

where I is the augmentation ideal of $ko^*(BG)$.

§ 5.1 Strategy of input for $ko_*(BSD_{16})$ of Greenlees spectral sequence

5.1.1 General strategy

Strategy we have used in Chapter 3 still plays a big role in this chapter but we need more work to do. Here, for input, we consider two short exact sequences. That is

$$0 \longrightarrow ST \longrightarrow ko^*(BG) \xrightarrow{\pi^o} QO \longrightarrow 0 \tag{5.1}$$

and

$$0 \longrightarrow T \longrightarrow ko^*(BG) \xrightarrow{\pi^u} \overline{QO} \longrightarrow 0 \tag{5.2}$$

where QO is the image of $ko^*(BG)$ in $KO^*(BG)$ and $\overline{QO}$ is the image of $ko^*(BG)$ in $KU^*(BG)$ s.t. ST is the β-torsion part of $ko^*(BG)$ and T is the $\ker \pi^u$. Here, QO and $\overline{QO}$ are module over $R := ko^*(BG)$ via π^o and π^u respectively.

Moreover, let $Q\tau$ be the $\ker(i : QO \longrightarrow \overline{QO})$. By snake lemma, $Q\tau \cong \operatorname{coker}(i' : ST \longrightarrow T)$ or in other words, we have a short exact sequence

$$0 \longrightarrow ST \longrightarrow T \longrightarrow Q\tau \longrightarrow 0. \tag{5.3}$$

Generally, T will be 2- torsion part.

Thus, to find $H_I^*(ko^*(BG))$, it is convenient to calculate from the long exact sequence induced by (5.2) together with the long exact sequence induced by (5.3). Normally, $Q\tau \subseteq \tau := \eta$-multiple, since only $\eta \in KO^* = \mathbb{Z}[\alpha, \eta, \beta, (\beta)^{-1}]/(\eta^2, 2\eta, \eta\alpha, \alpha^2 - 4\beta)$ is sent to $0 \in KU^*$. Hence, if there is no η-multiple obtained from TU, v-torsion part, then $Q\tau = \tau$.

5.1.2 Strategy for $G = SD_{16}$

In this case, by E_∞-page, ST contains only elements that lie on the zero-line, i.e. $ST = TO$, where TO, by definition in [13], consists of Bockstein ∞-cycles in ZTU. Furthermore those elements come from the v-torsion part, i.e., $TO \subseteq H^*(BSD_{16}; \mathbb{F}_2)$. On the other hand, τ is η-multiples coming from the torsion free part which $TO \cap \tau = 0$ and $TO + \tau = T$. Therefore, as a 2-torsion part,

$$T \cong \tau \oplus TO$$

and hence there is also a short exact sequence

$$0 \longrightarrow \tau \longrightarrow T \longrightarrow TO \longrightarrow 0. \tag{5.4}$$

Note that τ is a module over $R := ko^*(BSD_{16})$ via π^o and TO is a module over R via φ of commutative diagram below;

$$\begin{tikzcd} ko^*(BSD_{16}) \arrow[d, "\phi^*"] \arrow[dr, "\varphi"] & \\ H^*(BSD_{16}; \mathbb{Z}) \arrow[r, "2"] & H^*(BSD_{16}; \mathbb{F}_2), \end{tikzcd} \tag{5.5}$$

where ϕ^* is induced by $\phi : ko \longrightarrow H\mathbb{Z}$. Then, we can viewed T being a module over R as a direct sum of module τ and TO over R.

Hence, in order to calculate $H_I^*(ko^*(BSD_{16}))$, we will use two short exact sequences (5.2) and (5.4). The reason to choose (5.4) instead of (5.3) will become clear later. We will start with the calculation of $H_I^*(R; T)$ in the next section.

§ 5.2 Local cohomology of 2-torsion T

To calculate $H_I^*(R; T)$, it is enough to calculate $H^*_{\varphi(I)}(\varphi(R); TO)$, $H^*_{\pi^o(I)}(\pi^o(R); \tau)$ and connecting homomorphism induced by short exact sequence (5.4), where throughout this section, R denotes $ko^*(BSD_{16})$.

5.2.1 Local cohomology of TO

In case of $G = SD_{16}$, note that, by theorem 4.4.1, $TO = \bigoplus_i TO^i$, where

$$TO^i = \begin{cases} (\mathbb{Z}/2)^{k-1}, & i = 8k-6 \geq 2; \\ (\mathbb{Z}/2)^{k-2}, & i = 8k-4 \geq 12; \\ (\mathbb{Z}/2)^{k-1}, & i = 8k-2 \geq 6; \\ (\mathbb{Z}/2)^{k-1}, & i = 8k \geq 8; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, we have a short exact sequence

$$0 \longrightarrow TO \longrightarrow \ker(Sq^2) \xrightarrow{d_3} H^*(TU; Sq^2) \longrightarrow 0 \tag{5.6}$$

where $\ker(Sq^2) = \mathbb{F}_2[b^2, d^2]\{b\tau, d\tau\}$ and $H^*(TU; Sq^2) = \mathbb{F}_2[d^2]\{d\tau\}$ and all are module over R via φ. Note in this subsection that $\tau = b^2d - abd \in TU$.

Before doing further calculation, we need to identify the ideal $I \lhd R$ of each module explicitly. To deal with this we use the fact that $H^*_I(R; M) \cong H^*_{I'}(R/ann_R(M); M)$. Let $M_1 = \ker(Sq^2)$ and $M_2 = H^*(TU; Sq^2)$. For M_ϵ, $\epsilon = 1, 2$, we calculate

$$H^*_{I'_\epsilon}(\varphi(R)/ann_{\varphi(R)}(M_\epsilon); M_\epsilon) \tag{5.7}$$

by working explicitly on $\varphi(R)/ann_{\varphi(R)}(M_\epsilon) = \varphi(R)/\{r \in \varphi(R) | rm = 0, \forall m \in M_\epsilon\}$ first.

Lemma 5.2.1. *We have,*

$$R_1 := \varphi(R)/ann_{\varphi(R)}(M_1) = \mathbb{F}_2[b^2, d^2]\{1, b\tau, d\tau\}/\mathbb{F}_2[d^2]\{d\tau\}$$

and

$$R_2 := \varphi(R)/ann_{\varphi(R)}(M_2) = \mathbb{F}_2[d^2]\{1, d\tau\}.$$

Proof. It is clear that the image of R^k for each codegree $k \leq 0$, $2R$ and η-multiples under φ are all zero, since $\varphi(v), \varphi(2)$ and $\varphi(\eta)$ are zero in $H^*(BSD_{16}; \mathbb{F}_2)$. Then $\varphi(R^k)$, for $k \leq 0$, $\varphi(2R)$ and $\varphi(\eta$-multiples) are all subset of $ann_{\varphi(R)}(M_\epsilon)$. Furthermore, by using

- explicit generators of R^k in Theorem 4.4.1,
- explicit relations of generators in $JU_i, i = 1, 2, 3, 4, 5$ in terms of Chern classes in Chapter 3,
- the fact that $JU_{4k+\varepsilon} \cong p^{k-1}JU_\varepsilon$ where $\varepsilon = 2, 3, 4, 5$ in which $\varphi(p) = P^2 + y^8 := d^2 + b^4$,
- $a := x^2$ annihilates TU and
- $\varphi(TO) \cong TO \subseteq M_1$,

we can conclude that

1 $\varphi(R^{8k-6}\setminus TO^{8k-6})$ and $\varphi(R^{8k-2}\setminus TO^{8k-2})$ are subsets of $ann_{\varphi(R)}(M_\epsilon)$ for each $k\geq 1$,

2 $\varphi(\text{generators of } R^4)=0$, except $\varphi(\overline{x}_2+\overline{y}_2)=\varphi(2\overline{t}_2-\overline{z}_2+\overline{y}_2)=b^2$,

3 $\varphi(\text{generators of torsion free part of } R^{8k-4})=0$, except $\varphi(\overline{y}_{4k-2})=b^{4k-2}$ for $k\geq 2$ and

4 $\varphi(\text{generators of torsion free part of } R^{8k})=0$, except $\varphi(\overline{y}_{4k})=b^{4k}$ and $\varphi(\overline{u}_{4k})=d^{2k}$ for $k\geq 1$.

Hence, $R_1=TO\cup\mathbb{F}_2[b^2,d^2]$. Since b^2 and $b\tau$ annihilate M_2, $R_2=\mathbb{F}_2[d^2]\{1,d\tau\}$ as required. $\square$

Note that $\varphi(\overline{x}_2+\overline{y}_2)=b^2$, $\varphi(\overline{u}_4)=d^2$ and TO are in $\varphi(I)$ where $I=\ker(ko^*(BSD_{16})\longrightarrow ko^*)$. And note further that, $(b\tau)^2=b^6d^2\in(b^2,d^2)$ and $(d\tau)^2=b^4d^4\in(b^2,d^2)$. Thus, by lemma 5.2.1 above,

$$I_1'=\sqrt{(b^2,d^2)} \text{ and } I_2'=\sqrt{(d^2)}.$$

Consequently,

$$H^*_{\varphi(I)}(\varphi(R),M_1)=H^*_{(b^2,d^2)}(R_1,M_1) \tag{5.8}$$

and

$$H^*_{\varphi(I)}(\varphi(R),M_2)=H^*_{(d^2)}(R_2,M_2). \tag{5.9}$$

Recall from definition 3.1.1 that $H^*_I(R;M):=H^*(K^\infty(I)\otimes_R M)$. Here, M_ϵ is a ring and also a module over R_ϵ in which $x\in R$ and $x\in M_\epsilon$, then

$$R_\epsilon[\frac{1}{x}]\otimes_{R_\epsilon}M_\epsilon\cong M_\epsilon[\frac{1}{x}]. \tag{5.10}$$

Therefore,

$$\begin{aligned} H^i_{\varphi(I)}(R_1;M_1) &= H^i_{(b^2,d^2)}(M_1)\\ &\cong H^i_{(b^2,d^2)}(\mathbb{F}_2[b^2,d^2]\{b\tau,d\tau\})\\ &\cong H^i_{(b^2,d^2)}(\mathbb{F}_2[b^2,d^2])\cdot b\tau\oplus H^i_{(b^2,d^2)}(\mathbb{F}_2[b^2,d^2])\cdot d\tau\\ &\cong \begin{cases}\Sigma^{-2}(\mathbb{F}_2[b^2,d^2])^\vee\bigoplus(\mathbb{F}_2[b^2,d^2])^\vee, & i=2;\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

where the last equation comes from Example 3.1.5 and

$$\begin{aligned} H^i_{\varphi(I)}(R_2;M_2) &= H^i_{(d^2)}(M_2)\\ &\cong H^i_{(d^2)}(\mathbb{F}_2[d^2]\{d\tau\})\\ &\cong H^i_{(d^2)}(\mathbb{F}_2[d^2])\cdot d\tau\\ &\cong \begin{cases}\Sigma^4(\mathbb{F}_2[d^2])^\vee, & i=1;\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

where the last equation comes from Example 3.1.4. Now, we reach to the main goal of this subsection;

Lemma 5.2.2. *Local cohomology of TO part of $ko^*(BSD_{16})$ is given by*

$$H_I^i(TO) = \begin{cases} \Sigma^4(\mathbb{F}_2[d^2])^\vee \oplus \Sigma^{-2}(\mathbb{F}_2[b^2, d^2])^\vee \oplus (\mathbb{F}_2[b^2, d^2])^\vee, & i = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows by the long exact sequence induced by (5.6) and the results above which yields the short exact sequence

$$0 \longrightarrow \Sigma^4(\mathbb{F}_2[d^2])^\vee \longrightarrow H_I^2(TO) \longrightarrow \Sigma^{-2}(\mathbb{F}_2[b^2, d^2])^\vee \oplus (\mathbb{F}_2[b^2, d^2])^\vee \longrightarrow 0.$$

This short exact sequence split because $H_I^*(TO)$ is mod 2 vector space. □

Next, we need to compute $H_I^*(\tau)$ which we are going to do this in the next subsection.

5.2.2 Local cohomology of η-multiples part

Recall from Theorem 4.4.1 for η-multiples part, τ, that

$$\tau = \mathbb{Z}/2\{\widetilde{\eta}^\epsilon[\overline{u}_{4k}], \widetilde{\eta}^\epsilon[\overline{t}_{4k}], \widetilde{\eta}^\epsilon\beta^n[1], \widetilde{\eta}^\epsilon\beta^n[A], \widetilde{\eta}^\epsilon\beta^n[B], \widetilde{\eta}^\epsilon\beta^n[D^2], \widetilde{\eta}^\epsilon\beta^n[\widetilde{D}^3]\},$$

where $\epsilon = 1, 2, k \geq 1, n \geq 0$. As the previous subsection,

$$H_I^*(\tau) := H_{\pi^o(I)}^*(\pi^o(R); \tau) \cong H_{I'}^*(R'; \tau),$$

where $R' = \pi^o(R)/ann_{\pi^o(R)}(\tau)$ and $I' \lhd R'$. So, the first task is to find $\sqrt{I'}$ explicitly. To do this, we need to consider $ann_{\pi^o(R)}(\tau)$. Obviously, it contains at least

- $\pi^o(TO)$, since this is zero via π^o;
- All stuff of degree n which is not congruent to 0 or 1 modulo 8, since there is no η-multiple on degree congruent to 0,3,4,5,6,7 modulo 8;
- $2\pi^o(R)$, since τ is annihilated by 2.

Now, we only need to consider degrees congruent to 0 or 1 modulo 8. However, whether elements in degree $n \equiv 1 \bmod 8$ are in I' or not, these elements will not be generator of it radical, since they are all η-multiples which the power three of them are all zero. This means that it is enough to consider merely on degree divided by 8 which we have the results as;

Lemma 5.2.3. *In codegree divided by 8, only $\overline{u}_{4k}, \overline{t}_{4k}, \beta^n 1, \beta^n A, \beta^n B, \beta^n D^2$ and $\beta^n \widetilde{D}^3$ for each $k \geq 1$ and $n \geq 0$, are contained in $R' := \pi^o(R)/ann_{\pi^o(R)}(\tau)$.*

Proof. From the character table of JU_i in Chapter 3, we see that

$$\overline{u}_{4k}\overline{u}_4 = [16^{k+1}, (-2)^{k+1}, 4^{k+1}, (-2)^{k+1}, 0, 4^{k+1}],$$

which is not in $(1+\tau)JU_{4(k+1)-1}$ for all $k \geq 1$, since the entry in the last coordinate in character table of every element of $(1+\tau)JU_{4k+3}$ are divisible by $2 \cdot 4^{k+1}$. This means $\overline{u}_{4k}\widetilde{\eta}^{\epsilon}\overline{u}_4 \neq 0$ and hence $\overline{u}_{4k} \in R'$ for all $k \geq 1$. Similarly for $\bar{t}_{4k}$, this is in R' because they can not kill at least $\widetilde{\eta}^{\epsilon}\overline{u}_4$. To conclude that $\beta^n 1, \beta^n A, \beta^n B, \beta^n D^2$ and $\beta^n \widetilde{D}^3$ for each $n \geq 0$, are in R', we multiply them with the element $\beta^n\widetilde{\eta}[1]$, which yields that they are not lie in $(1+\tau)RU$ or $(1+\tau)JU_3$, i.e., they are not zero.

In order to conclude that there are only these elements which are in R', we need to show that $\overline{x}_{4k}, \overline{y}_{4k}, \overline{z}_{4k}$ and C are in $ann_{\pi^o(R)}(\tau)$ for every $k \geq 1$. This means we need to check that they kill exactly every element in τ. Precisely, need to show that;

- $\overline{\alpha}_{4k}[\overline{\gamma}_{4n}] \in (1+\tau)JU_{4(k+n)-1}$ for each $k, n \geq 1, \alpha \in \{x, y, z\}$ and $\gamma \in \{t, u\}$,
- $\overline{\alpha}_{4k}[\theta] \in (1+\tau)QU_{8(n-k)+2}$ for each $k \geq 1$, $n \geq 0$, $\alpha \in \{x, y, z\}$ and $\theta \in \{\beta^n 1, \beta^n A, \beta^n B, \beta^n D^2, \beta^n \widetilde{D}^3\}$,
- $C[\overline{\gamma}_{4n}] \in (1+\tau)JU_{4n-1}$ for each $n \geq 1$ and for all $\gamma \in \{t, u\}$,
- $C[\theta] \in (1+\tau)RU$ for all $\theta \in \{1, A, B, D^2, \widetilde{D}^3\}$.

This is a routine work which we can read from the character table of JU_i and RU in Chapter 3. □

By this lemma, note that, as a set, $I' \subseteq \mathbb{Z}/2 < 1, \bar{t}_{4m}, \overline{u}_{4m}, \beta^n A, \beta^n B, \beta^n D, \beta^n \widetilde{D}^3 > \cup \tau$ for $m \geq 1$. Claim that

$$\sqrt{(\overline{u}_4)} = I'. \tag{5.11}$$

To prove this claim, we need to check only that $X^k \in (\overline{u}_4) = \{r\overline{u}_4 \mid r \in R'\}$ for some $k > 0$ and for all $X \in \{\bar{t}_{4m}, \overline{u}_{4m}, \beta^n A, \beta^n B, \beta^n D, \beta^n \widetilde{D}^3\}$. Since $\overline{u}_{4m} = (\overline{u}_4)^m$, $\overline{u}_{4m} \in (\overline{u}_4)$. It is easy to see that, by character table, $\bar{t}_4^2 = (\bar{t}_4 D^2)\overline{u}_4$ and $\bar{t}_{4m} = (\bar{t}_4\overline{u}_4^{m-2})\overline{u}_4$ for $m \geq 2$. Then $\bar{t}_{4m} \in (\overline{u}_4)$ for all $m \geq 1$. By character table again, we have $(\beta^n A)^2 = 2\beta^{2n}A, (\beta^n B)^2 = 2\beta^{2n}B, \beta^n D^2 = \beta^n(\beta \cdot 1)\overline{u}_4$ and

$$(\beta^n \widetilde{D}^3)^2 = -360\beta^{2n}A - 360\beta^{2n}D^2 + 162\beta^{2n}\widetilde{D}^3 + 90\beta^{2n}C.$$

This means $(\beta^n A)^2, (\beta^n B)^2$ and $(\beta^n(\widetilde{D}^3))^2$ are zero in R', because R' is annihilated by 2 and thus we have proved (5.11). By the same argument as (5.10), we can conclude that

$$H_I^*(\tau) \cong H_{(\overline{u}_4)}^*(\tau).$$

Therefore,

$$H_I^*(\tau) = H_{(\overline{u}_4)}^0(\tau) \oplus H_{(\overline{u}_4)}^1(\tau). \tag{5.12}$$

Now, we are ready to compute $H_I^*(\tau)$ by starting with $H_{(\overline{u}_4)}^0(\tau) \cong \Gamma_{(\overline{u}_4)}(\tau) = \{x \in \tau \mid (\overline{u}_4)^k x = 0, \exists k \geq 1\}$. It is not hard to see that multiplied by $\overline{u}_4$ is an isomorphism on negative degree, i.e., $\overline{u}_4\widetilde{\eta}^\epsilon[\overline{u}_{4k}] = \widetilde{\eta}^\epsilon[\overline{u}_{4(k+1)}]$ and $\overline{u}_4\widetilde{\eta}^\epsilon[\overline{t}_{4k}] = \widetilde{\eta}^\epsilon[\overline{t}_{4(k+1)}]$. This implies that

$$[H^0_I(\tau)]_n = 0, \text{ for all negative n.}$$

In positive degrees, we start to investigate elements of

$$\mathbb{Z}/2\{\widetilde{\eta}^\epsilon\beta^n[1], \widetilde{\eta}^\epsilon\beta^n[A], \widetilde{\eta}^\epsilon\beta^n[B], \widetilde{\eta}^\epsilon\beta^n[D^2], \widetilde{\eta}^\epsilon\beta^n[\widetilde{D}^3]\},$$

for $\epsilon = 1, 2, k \geq 1, n \geq 0$, by checking only whether that $\overline{u}_4 T$, for

$$T \in \tau' := \mathbb{Z}/2\{[1], [A], [B], [D^2], [\widetilde{D}^3]\},$$

are zero or not, since multiplied by $\overline{u}_4$ is an isomorphism on negative degree as stated above. To do this, we need to judge that $\overline{u}_4 T$, for each $T \in \tau'$, lies in $(1+\tau)\widehat{JU}_3$ or not. To be more convenient, we record $(1+\tau)\widehat{JU}_3$ as;

$$(1+\tau)\widehat{JU}_3 \;=\; \mathbb{Z}_2^\wedge \left\langle \begin{array}{rcccccccl} (1+\tau)\overline{x}_3 : & [& 0 & 0 & 0 & 0 & 0 & 8 &] \\ (1+\tau)\overline{y}_3 : & [& 0 & 0 & 0 & 0 & 16 & 0 &] \\ (1+\tau)\overline{z}_3 : & [& 0 & 0 & 16 & 0 & 0 & 0 &] \\ (1+\tau)\overline{t}_3 : & [& 0 & 8 & 0 & 8 & 0 & 0 &] \\ (1+\tau)\overline{u}_3 : & [& 32 & -4 & -8 & -4 & 0 & 0 &] \\ (1+\tau)\overline{w}_3 : & [& 0 & 0 & 0 & 0 & 0 & 0 &] \end{array} \right\rangle.$$

Now it is simple to see that $\overline{u}_4[1] = [\overline{u}_4]$, $\overline{u}_4[A] = [\overline{t}_4]$, $\overline{u}_4[D^2] = [\overline{t}_4]$, (In fact, $\overline{u}_4 D^2 = 4(1+\tau)\overline{u}_3 + 2(1+\tau)\overline{t}_3 + 3(1+\tau)\overline{z}_3 + \overline{t}_4$) and $\overline{u}_4[B]$ and $\overline{u}_4[\widetilde{D}^3]$ are in $(1+\tau)\widehat{JU}_3$. Thus,

$$[H^0_I(\tau)]_\epsilon = (\mathbb{Z}/2)^3 < \widetilde{\eta}^\epsilon[B], \widetilde{\eta}^\epsilon[\widetilde{D}^3], \widetilde{\eta}^\epsilon([A] + [D^2]) >,$$

where $\epsilon = 1, 2$.

Next, we consider elements of $\tau_{8+\epsilon}$. In this case, we need to check that whether $(\overline{u}_4)^2 T$, for each $T \in \tau'$, are in $(1+\tau)\widehat{JU}_3$ or not. This can be read from the character table above easily which we have;

$$[H^0_I(\tau)]_{8+\epsilon} = (\mathbb{Z}/2)^4 < \widetilde{\eta}^\epsilon\beta[A], \widetilde{\eta}^\epsilon\beta[B], \widetilde{\eta}^\epsilon\beta[D^2], \widetilde{\eta}^\epsilon\beta[\widetilde{D}^3] >,$$

where $\epsilon = 1, 2$.

Again, for elements of $\tau_{8k+\epsilon}$ where $k \geq 2$. In this case, we need to check that whether $(\overline{u}_4)^{k+1} T$, for each $T \in \tau'$, are in $(1+\tau)\widehat{JU}_3$ or not. This can be read from character table above easily which we have;

$$[H^0_I(\tau)]_{8k+\epsilon} = (\mathbb{Z}/2)^5 < \widetilde{\eta}^\epsilon\beta^k[1], \widetilde{\eta}^\epsilon\beta^k[A], \widetilde{\eta}^\epsilon\beta^k[B], \widetilde{\eta}^\epsilon\beta^k[D^2], \widetilde{\eta}^\epsilon\beta^k[\widetilde{D}^3] >,$$

where $\epsilon = 1, 2$ and $k \geq 2$.

For $[H^1_I(\tau)]_*$, it is now easy to determine by considering the short exact sequence;

$$0 \longrightarrow H^0_{(\overline{u}_4)}(\tau) \longrightarrow \tau \xrightarrow{i} \tau[\tfrac{1}{\overline{u}_4}] \longrightarrow H^1_{(\overline{u}_4)}(\tau) \longrightarrow 0. \tag{5.13}$$

Here, note that,

$$\begin{aligned} \tau_{\epsilon+8n}[\frac{1}{\overline{u}_4}] &\cong \varinjlim(\tau_{\epsilon+8n} \xrightarrow{\overline{u}_4} \tau_{\epsilon+8(n-1)} \xrightarrow{\overline{u}_4} \tau_{\epsilon+8(n-2)} \xrightarrow{\overline{u}_4} \cdots) \\ &\cong \tau_{\epsilon-8} \\ &= (\mathbb{Z}/2)^2 < \widetilde{\eta}^\epsilon[\overline{u}_4]/(\overline{u}_4)^{n+1}, \widetilde{\eta}^\epsilon[\overline{t}_4]/(\overline{u}_4)^{n+1} > \end{aligned}$$

where $\epsilon = 1, 2$, by Proposition A.0.4.

Since for all negative n, i is an isomorphism on those degrees, $[H^1_I(\tau)]_n = 0$, for all negative n as well. Furthermore, since $[H^0_I(\tau)]_{8k+\epsilon} = (\mathbb{Z}/2)^5$ for $k \geq 2$, i is zero map on those degrees and hence

$$[H^1_I(\tau)]_{8k+\epsilon} = (\mathbb{Z}/2)^2 < \widetilde{\eta}^\epsilon[\overline{u}_4]/(\overline{u}_4)^{k+1}, \widetilde{\eta}^\epsilon[\overline{t}_4]/(\overline{u}_4)^{k+1} > .$$

Now, it remains to calculate on degree ϵ and $\epsilon + 8$ which is simple to see that $[H^1_I(\tau)]_\epsilon = 0$ and $[H^1_I(\tau)]_{8+\epsilon} = \mathbb{Z}/2 < \widetilde{\eta}^\epsilon[\overline{u}_4]/(\overline{u}_4)^2 >$.

We summarize these results in the lemma below;

Lemma 5.2.4. *Local cohomology of η-multiples in $ko^*(BSD_{16})$ at $I = (\overline{u}_4)$ consists of two parts $H^0_{(\overline{u}_4)}(\tau)$ and $H^1_{(\overline{u}_4)}(\tau)$. Explicitly,*

$$[H^0_{(\overline{u}_4)}(\tau)]_{\epsilon+n} = \begin{cases} (\mathbb{Z}/2)^3 < \widetilde{\eta}^\epsilon[B], \widetilde{\eta}^\epsilon[\widetilde{D}^3], \widetilde{\eta}^\epsilon([A] + [D^2]) >, \\ (\mathbb{Z}/2)^4 < \widetilde{\eta}^\epsilon\beta[A], \widetilde{\eta}^\epsilon\beta[B], \widetilde{\eta}^\epsilon\beta[D^2], \widetilde{\eta}^\epsilon\beta[\widetilde{D}^3] >, \\ (\mathbb{Z}/2)^5 < \widetilde{\eta}^\epsilon\beta^k[1], \widetilde{\eta}^\epsilon\beta^k[A], \widetilde{\eta}^\epsilon\beta^k[B], \widetilde{\eta}^\epsilon\beta^k[D^2], \widetilde{\eta}^\epsilon\beta^k[\widetilde{D}^3] >, \\ 0, \end{cases}$$

where $\epsilon = 1, 2$ and $n = 0, 8, 8k$, $k \geq 2$ and otherwise respectively. And,

$$[H^1_{(\overline{u}_4)}(\tau)]_{\epsilon+n} = \begin{cases} \mathbb{Z}/2 < \widetilde{\eta}^\epsilon[\overline{u}_4]/(\overline{u}_4)^2 >, \\ (\mathbb{Z}/2)^2 < \widetilde{\eta}^\epsilon[\overline{u}_4]/(\overline{u}_4)^{k+1}, \widetilde{\eta}^\epsilon[\overline{t}_4]/(\overline{u}_4)^{k+1} >, \\ 0, \end{cases}$$

where $\epsilon = 1, 2$ and $n = 8, 8k$, $k \geq 2$ and otherwise respectively.

Consequently, by lemma 5.2.2, 5.2.4 and the long exact sequence induced by (5.4), we have;

Corollary 5.2.5. *Local cohomology of the 2-torsion part T in $ko^*(BSD_{16})$ is given by*

$$H^n_I(T) = \begin{cases} H^0_I(\tau), & \text{if } n = 0; \\ H^1_I(\tau), & \text{if } n = 1; \\ H^2_I(TO), & \text{if } n = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Note here that, if we use the long exact sequence induced by (5.3), then we will have more work to determine connecting differential $\delta : H^1_I(\tau) \longrightarrow H^2_I(TO)$. Hence, this is the answer to the question that why we prefer to use (5.4) instead of (5.3).

§ 5.3 Local cohomology of torsion free part

As in the previous process, the first thing we need to do before calculating local cohomology of $\overline{QO}$ is to determine the radical of its ideal explicitly.

5.3.1 Radical ideal for torsion free part and $H^0_I(\overline{QO})$

In the case of $\overline{QO}$ of $ko^*(BSD_{16})$, we have;

Lemma 5.3.1. *$H^*_I(\overline{QO}) \cong H^*_{(q)}(\overline{QO})$, where $q = \pi^u(\overline{u}_4 + \overline{y}_4) \in \pi^u(ko^8(BSD_{16}))$.*

Proof. To find the radical of ideal $I' \lhd \pi^u(R)$, we use the same trick as in the case of TO and τ, i.e. investigating $\overline{R} := \pi^u(R)/ann_{\pi^u(R)}(\overline{QO})$. By theorem 4.4.1 and character table, it is clear that TO and η-multiples parts kill $\overline{QO}$, then we need only to consider on the torsion free part. Again, in this case it is not hard to see that multiplying by

$$q = [16, -2, 4, -2, 16, 4] = \pi^u(\overline{u}_4 + \overline{y}_4) \in \pi^u(ko^8(BSD_{16}))$$

gives an isomorphism, for each $k \geq 1$; $\overline{QO}_{-(8k+2)} \cong q^k\overline{QO}_{-2}$, $\overline{QO}_{-(8k+6)} \cong q^k\overline{QO}_{-6}$, $\overline{QO}_{-(8k+8)} \cong q^k\overline{QO}_{-8}$ and $\overline{QO}_{-(8k+12)} \cong q^k\overline{QO}_{-12}$. Note that multiplied by q on $\overline{QO}_{-4}$ is not isomorphic to $\overline{QO}_{-12}$ because $\overline{x}_6$ can not write in term of elements in $q\overline{QO}_{-4}$.

Now, it remains to show that there exist $n \in \mathbb{N}$ such that $(\overline{QO}_\varepsilon)^n$ and $(\overline{QO}_{\geq 0})^n$ are in (q), where $\varepsilon = -2, -4, -6, -8, -12$. Since $(\overline{QO}_\varepsilon)^k \subseteq \overline{QO}_{k\varepsilon}$ and by isomorphism above (i.e., multiplied by q in negative degree) for which k is big enough, the results follows in negative degree. For non-negative degree, it suffices to consider degrees $0, 2, 4$ and 6. By the character table again, we have;

$$\begin{array}{rcl}
\theta^3 & = & \theta q \\
A^4 = (2A)^2 & = & (AD^2 - A^2)q \\
B^6 = (2B)^3 & = & (AD^2 + A^2 + B^2)q \\
C^4 = (\overline{D})^4 & = & 16\overline{D}q \\
\overline{D^3} = \overline{D}q & , & (D^2)^2 = D^2 q \\
(\widetilde{D}^3)^4 = (D^3 + 2D)^4 & = & (D^{10} + 8D^8 + 24D^6 + 32D^4 + 16D^2)q.
\end{array}$$

This shows that $\sqrt{(q)}$ contains $\overline{QO}_{\geq 0}$ and hence $\sqrt{(q)} = \sqrt{I'}$ as required. □

The consequence of this lemma is that $H^*_I(\overline{QO}) \cong H^0_{(q)}(\overline{QO}) \oplus H^1_{(q)}(\overline{QO})$. Recall that all elements of $\overline{QO}$ can be represented by character table. The character table of q is not zero except in the first coordinate, i.e., conjugacy class of identity. This suggests that each entry in character table of elements in $H^0_{(q)}(\overline{QO}) \cong \Gamma_{(q)}(\overline{QO})$ must be zero, except for the first coordinate. Such elements exist possibly in non-negative degree which is divisible by 4, i.e., on degree containing $\beta^k[1]$ and $\beta^k[2v^2]$. Precisely, we have;

Lemma 5.3.2. *For $k \geq 0$ and $\overline{QO} = Im(\pi^u : ko^*(BSD_{16}) \longrightarrow KU^*(BSD_{16}))$,*

$$[H^0_I(\overline{QO})]_i = \begin{cases} \mathbb{Z} < v^{4k}[\overline{\rho}] >, & \text{if } i = 8k; \\ \mathbb{Z} < v^{4k}[\overline{\alpha\rho}] >, & \text{if } i = 8k+4; \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \overline{\rho} &= [16,0,0,0,0,0,0] \\ &= 16 \cdot 1 - [\frac{28}{3}A + 8B - \frac{13}{3}C + \frac{4}{3}D^2 - \frac{5}{9}\widetilde{D}^3] \end{aligned}$$

and

$$\begin{aligned} \overline{\alpha\rho} &= [32,0,0,0,0,0,0] \\ &= 16 \cdot \alpha - [-8(2A) + 8(2B) - 16(2D^2) + \frac{92}{3}\overline{D} + \frac{7}{3}\overline{D^3}], \end{aligned}$$

such that $\{1, A, B, C, D^2, \widetilde{D}^3\}$ and $\{\alpha = 2v^2, 2A, 2B, \overline{D}, 2D^2, \overline{D^3}\}$ are the set of generator for $\overline{QO}_0$ and $\overline{QO}_4$ respectively (see theorem 4.4.1).

Proof. This results follows by inspection of the character table. □

5.3.2 $H^1_I(\overline{QO})$

Next, we calculate $H^1_{(q)}(\overline{QO})$ as the coker$(\overline{QO} \longrightarrow \overline{QO}[\frac{1}{q}])$ by using the fact that

$$(\overline{QO}[\frac{1}{q}])_t = \varinjlim(\overline{QO}_t \xrightarrow{q} \overline{QO}_{t-8} \xrightarrow{q} \overline{QO}_{t-2(8)} \xrightarrow{q} \cdots).$$

Note from the proof of lemma 5.3.1 that multiplying by q is eventually constant at degree $-2, -6, -8$ and -12. This implies that

$$[H^1_{(q)}(\overline{QO})]_i = 0 \text{ , for all } i < -4 \text{ and}$$

$$\overline{H}_{-4} \cong \overline{QO}_{-12}/q\overline{QO}_{-4} \quad \text{and} \quad \text{for } k \geq 1$$

$$\begin{aligned} &\overline{H}_0 \cong \overline{QO}_{-8}/q\overline{QO}_0 &&, \quad \overline{H}_{8k} \cong \overline{QO}_{-8}/q^{k+1}\overline{QO}_{8k} \\ &\overline{H}_2 \cong \overline{QO}_{-6}/q\overline{QO}_2 &&, \quad \overline{H}_{8k+2} \cong \overline{QO}_{-6}/q^{k+1}\overline{QO}_{8k+2} \\ &\overline{H}_4 \cong \overline{QO}_{-12}/q^2\overline{QO}_{-4} &&, \quad \overline{H}_{8k+4} \cong \overline{QO}_{-12}/q^{k+2}\overline{QO}_{8k+4} \\ &\overline{H}_6 \cong \overline{QO}_{-2}/q\overline{QO}_6 &&, \quad \overline{H}_{8k+6} \cong \overline{QO}_{-2}/q^{k+1}\overline{QO}_{8k+6}, \end{aligned}$$

where $\overline{H}$ denotes $H^1_{(q)}(\overline{QO})$. The rest of this subsection will devote to the proof of;

Lemma 5.3.3. *As abelian groups;*

- $\overline{H}_{-4} = \mathbb{Z}/2 < \widetilde{x}_{-2} >$ *with* $\widetilde{x}_{-2} = \widetilde{y}_{-2} = \widetilde{z}_{-2} = \widetilde{t'}_{-2}$ *and* $\widetilde{u'}_{-2} = 0$, *where* $\widetilde{\alpha}_{-2} = \frac{\overline{\alpha}_6}{q} + \overline{QO}_{-12}$ *for each* $\alpha \in \{x, y, z, t', u'\}$ *and* $\overline{t'}_6 = 2\overline{t}_6 - \overline{z}_6$, $\overline{u'}_6 = -\overline{u*}_6 - \overline{x}_6 - \overline{z}_6$,

- $\overline{H}_0 = \mathbb{Z}/2 < \widetilde{y}_0 >$ *with* $\widetilde{x}_0 = \widetilde{z}_0 = \widetilde{t}_0 = 0$ *and* $\widetilde{y}_0 = \widetilde{u}_0$, *where* $\widetilde{\alpha}_0 = \frac{\overline{\alpha}_4}{q} + \overline{QO}_{-8}$ *for each* $\alpha \in \{x, y, z, t, u\}$,

- $\overline{H}_4 = \mathbb{Z}/2 < \widetilde{z}_2 > \oplus \mathbb{Z}/4 < \widetilde{x}_2 > \oplus \mathbb{Z}/8 < \widetilde{u'}_2 > \oplus \mathbb{Z}/16 < \widetilde{y}_2 >$ *with* $\widetilde{t'}_2 - 2\widetilde{x}_2 - 8\widetilde{y}_2 - \widetilde{z}_2 - 4\widetilde{u'}_2 = 0$, *where* $\widetilde{\alpha}_2 = \frac{\overline{\alpha}_6}{q^2} + \overline{QO}_{-12}$ *for each* $\alpha \in \{x, y, z, t', u'\}$,

- *For* $k \geq 0$, $\overline{H}_{8k+2} = \mathbb{Z}/2^{k+1} < \widetilde{w}_{4k+1} >$, *where* $\widetilde{w}_{4k+1} = \frac{\overline{w}_3}{q^{k+1}} + \overline{QO}_{-6}$,

- *For* $k \geq 0$, $\overline{H}_{8k+6} = \mathbb{Z}/2^{k+1} < \widetilde{w'}_{4k+3} >$, *where* $\widetilde{w'}_{4k+3} = \frac{\overline{w'}_1}{q^{k+1}} + \overline{QO}_{-2}$ *such that* $\overline{w'}_1 = \theta_1 = \overline{u}_1 - \overline{t}_1 - 2\overline{w}_1$,

- *For* $k \geq 1$, $\overline{H}_{8k+4} = \mathbb{Z}/2^k < \widetilde{t}_{4k+2} > \oplus \mathbb{Z}/2 \cdot 4^k < \widetilde{z}_{4k+2} > \oplus \mathbb{Z}/4^{k+1} < \widetilde{x}_{4k+2} > \oplus \mathbb{Z}/8 \cdot 16^k < \widetilde{u'}_{4k+2} > \oplus \mathbb{Z}/16^{k+1} < \widetilde{y}_{4k+2} >$ *with* $\widetilde{t}_{4k+2} = \widetilde{t'}_{4k+2} + (-2)^{k+1}\widetilde{x}_{4k+2} + (-8)^{k+1}\widetilde{y}_{4k+2} - (-2)^k\widetilde{z}_{4k+2} - 4(-8)^k\widetilde{u'}_{4k+2}$, *where* $\widetilde{\alpha}_{4k+2} = \frac{\overline{\alpha}_6}{q^{k+2}} + \overline{QO}_{-12}$ *for each* $\alpha \in \{x, y, z, t', u'\}$,

- *For* $k \geq 1$, $\overline{H}_{8k} = \mathbb{Z}/2^{k-1} < \widetilde{\widehat{t}}_{4k} > \oplus \mathbb{Z}/2 \cdot 4^{k-1} < \widetilde{z}_{4k} + 4^{k+1}\widetilde{u}_{4k} > \oplus \mathbb{Z}/4^k < \widetilde{x}_{4k} > \oplus \mathbb{Z}/2 \cdot 16^k < \widetilde{y}_{4k} > \oplus \mathbb{Z}/4 \cdot 16^k < \widetilde{u}_{4k} >$ *with* $\widetilde{\widehat{t}}_{4k} = \widetilde{t}_{4k} - (-2)^k\widetilde{x}_{4k} - 2(-8)^k\widetilde{y}_{4k} + (-2)^{k-1}\widetilde{z}_{4k} - 2(-8)^k\widetilde{u}_{4k}$, *where* $\widetilde{\alpha}_{4k} = \frac{\overline{\alpha}_4}{q^{k+1}} + \overline{QO}_{-8}$ *for each* $\alpha \in \{x, y, z, t, u\}$.

The $\overline{H}_i$ *'s which are not mentioned above are all zero.*

Proof. As in Chapter 3, we do calculation by using row and column operation on generators.

For $\overline{H}_{-4}$, we can represent matrix for the calculation of $\overline{QO}_{-12}/q\overline{QO}_{-4}$ as

$$\begin{array}{r|ccccc|l} & \overline{x}_6 & \overline{y}_6 & \overline{z}_6 & \overline{t'}_6 & \overline{u'}_6 & \\ q(\overline{x}_2 + \overline{y}_2): & 1 & 1 & 0 & 0 & 0 & r_1 \\ q(2\overline{x}_2): & 2 & 0 & 0 & 0 & 0 & r_2 \\ q(2\overline{t}_2 - \overline{z}_2 + \overline{y}_2): & 0 & 1 & 0 & 1 & 0 & r_3 \\ q(\overline{x}_2 + \overline{z}_2): & 1 & 0 & 1 & 0 & 0 & r_4 \\ q(-\overline{u*}_2 - \overline{x}_2 - \overline{z}_2): & 0 & 0 & 0 & 0 & 1 & r_5 \end{array}.$$

After doing column operations, we get

$$\begin{array}{r|ccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 \\
 & c_1' & c_2' & c_3 & c_4 & c_5 \\
r_1: & 0 & 1 & 0 & 0 & 0 \\
r_2: & 2 & 0 & 0 & 0 & 0 \\
r_3: & 0 & 0 & 0 & 1 & 0 \\
r_4: & 0 & 0 & 1 & 0 & 0 \\
r_5: & 0 & 0 & 0 & 0 & 1
\end{array},$$

where $c_1' = c_1 - c_2 + c_4 - c_3$ and $c_2' = c_2 - c_4$. By using the same method as in Chapter 3, we get $g_1 = \overline{x}_6, g_2 = \overline{x}_6 + \overline{y}_6, g_3 = \overline{x}_6 + \overline{z}_6, g_4 = \overline{t'}_6 + \overline{y}_6$ and $g_5 = \overline{u'}_6$ and then the results follow.

For $\overline{H}_0$, we can represent matrix for the calculation of $\overline{QO}_{-8}/q\overline{QO}_0$ as

$$\begin{array}{r|ccccc|l}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \\
q1: & 0 & 1 & 0 & 0 & 1 & r_1 \\
qA: & 1 & 0 & 0 & -1 & 0 & r_2 \\
qB: & 1 & 2 & 0 & 0 & 0 & r_3 \\
qC: & 6 & 0 & 1 & -4 & -8 & r_4 \\
qD^2: & -6 & 0 & -3 & 9 & 16 & r_5 \\
q\widetilde{D}^3: & -30 & 0 & -15 & 36 & 72 & r_6
\end{array}.$$

Now, we do row operations by changing;
$r_1 = r_1^*,\ r_2 \longrightarrow [r_2^* = -r_5^*]$,
$r_3 \longrightarrow [r_3^* = r_3 - r_2 + r_5^*]$,
$r_4 \longrightarrow [r_4' = r_4 - 6r_2] \longrightarrow [r_4^* = r_4' - r_5' + r_5^*]$,
$r_5 \longrightarrow [r_5' = r_5 + 6r_2 + 3r_4'] \longrightarrow [r_5^* = r_5' + 8r_1 - 4r_3 - r_6'']$,
$r_6 \longrightarrow [r_6' = r_6 + 30r_2 + 15r_4'] \longrightarrow [r_6'' = r_6' - 6r_5'] \longrightarrow [r_6^* = r_6'' - 6r_5^*]$.
Then, we get Step(*) [cf. Chapter3] and the required results as

$$\begin{array}{r|ccccc|}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 \\
r_1^*: & 0 & 1 & 0 & 0 & 1 \\
r_2^*: & 1 & 0 & 0 & 0 & 0 \\
r_3^*: & 0 & 2 & 0 & 0 & 0 \\
r_4^*: & 0 & 0 & 1 & 0 & 0 \\
r_5^*: & 0 & 0 & 0 & -1 & 0 \\
r_6^*: & 0 & 0 & 0 & 0 & 0
\end{array} \cong \begin{array}{r|ccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 \\
 & c_1 & c_2 - c_5 & c_3 & c_4 & c_5 \\
r_1^*: & 0 & 0 & 0 & 0 & 1 \\
r_2^*: & 1 & 0 & 0 & 0 & 0 \\
r_3^*: & 0 & 2 & 0 & 0 & 0 \\
r_4^*: & 0 & 0 & 1 & 0 & 0 \\
r_5^*: & 0 & 0 & 0 & -1 & 0 \\
r_6^*: & 0 & 0 & 0 & 0 & 0
\end{array}.$$

By using the same method as in Chapter 3, we get $g_1 = \overline{x}_4, g_2 = \overline{y}_4, g_3 = \overline{z}_4, g_4 = \overline{t}_4$ and $g_5 = \overline{u}_4 + \overline{y}_4$ and then the results follow.

For $\overline{H}_4$, we can represent matrix for the calculation of $\overline{QO}_{-12}/q^2\overline{QO}_4$ as

$$\begin{array}{r|ccccc|l}
 & \overline{x}_6 & \overline{y}_6 & \overline{z}_6 & \overline{t'}_6 & \overline{u'}_6 & \\
q^2(\alpha): & 2 & 8 & 3 & -1 & 4 & r_1 \\
q^2(2A): & 4 & 0 & 0 & -2 & 0 & r_2 \\
q^2(2B): & 4 & 16 & 0 & 0 & 0 & r_3 \\
q^2(2D^2): & 8 & 0 & 36 & 2 & 64 & r_4 \\
q^2\overline{D}: & 4 & 0 & 10 & 0 & 16 & r_5 \\
q^2\overline{D^3}: & 16 & 0 & 136 & 0 & 256 & r_6
\end{array} .$$

Now, we do row operations by changing;
$r_1 \longrightarrow [r_1^* = r_1 + r_5^*]$,
$r_2 \longrightarrow [r_2^* = r_2 - 2r_1^* - r_5^* + r_3^*]$,
$r_3 \longrightarrow [r_3^* = r_3 + r_4^*]$,
$r_4 \longrightarrow [r_4' = r_4 + r_2 - 4r_5] \longrightarrow [r_4^* = r_4' - 2r_5' - r_6']$,
$r_5 \longrightarrow [r_5' = r_5 - 4r_1 + 2r_2 + 2r_3] \longrightarrow [r_5^* = r_5' + 3r_4^*]$,
$r_6 \longrightarrow [r_6' = r_6 - 16r_5 - 6r_4'] \longrightarrow [r_6^* = r_6' - 6r_4^*]$.
Then, we get Step(*) and the required results as;

$$\begin{array}{r|ccccc|}
 & \overline{x}_6 & \overline{y}_6 & \overline{z}_6 & \overline{t'}_6 & \overline{u'}_6 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 \\
r_1^*: & 2 & 8 & 1 & -1 & 4 \\
r_2^*: & 0 & 0 & 0 & 0 & -8 \\
r_3^*: & 0 & 16 & 0 & 0 & 0 \\
r_4^*: & -4 & 0 & 0 & 0 & 0 \\
r_5^*: & 0 & 0 & -2 & 0 & 0 \\
r_6^*: & 0 & 0 & 0 & 0 & 0
\end{array} \cong \begin{array}{r|ccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 \\
 & c_1' & c_2' & c_3' & c_4 & c_5' \\
r_1^*: & 0 & 0 & 0 & -1 & 0 \\
r_2^*: & 0 & 0 & 0 & 0 & -8 \\
r_3^*: & 0 & 16 & 0 & 0 & 0 \\
r_4^*: & -4 & 0 & 0 & 0 & 0 \\
r_5^*: & 0 & 0 & -2 & 0 & 0 \\
r_6^*: & 0 & 0 & 0 & 0 & 0
\end{array} ,$$

where $c_1' = c_1 + 2c_4$, $c_2' = c_2 + 8c_4$, $c_3' = c_3 + c_4$ and $c_5' = c_5 + 4c_4$. By using the same method as in Chapter 3, we get $g_1 = \overline{x}_6, g_2 = \overline{y}_6, g_3 = \overline{z}_6, g_4 = \overline{t'}_6 - 2\overline{x}_6 - 8\overline{y}_6 - \overline{z}_6 - 4\overline{u'}_6$ and $g_5 = \overline{u'}_6$ and then the results follow.

For $\overline{H}_{8k+4}$, $k \geq 1$, we can represent matrix for the calculation of $\overline{QO}_{-12}/q^{k+2}\overline{QO}_4$ as

$$\begin{array}{r|ccccc|l}
 & \overline{x}_6 & \overline{y}_6 & \overline{z}_6 & \overline{t'}_6 & \overline{u'}_6 & \\
q^{k+2}(\alpha): & 2\cdot 4^k & 8\cdot 16^k & 4^k + 2\cdot 16^k & -(-2)^k & 4\cdot 16^k & r_1 \\
q^{k+2}(2A): & 4\cdot 4^k & 0 & 0 & -2(-2)^k & 0 & r_2 \\
q^{k+2}(2B): & 4\cdot 4^k & 16\cdot 16^k & 0 & 0 & 0 & r_3 \\
q^{k+2}(2D^2): & 8\cdot 4^k & 0 & 4\cdot 4^k + 32\cdot 16^k & 2(-2)^k & 64\cdot 16^k & r_4 \\
q^{k+2}\overline{D}: & 4\cdot 4^k & 0 & 2\cdot 4^k + 8\cdot 16^k & 0 & 16\cdot 16^k & r_5 \\
q^{k+2}\overline{D^3}: & 16\cdot 4^k & 0 & 8\cdot 4^k + 128\cdot 16^k & 0 & 256\cdot 16^k & r_6
\end{array} .$$

Now, we do row operations by changing;
$r_1 \longrightarrow [r_1^* = r_1 + 4^k r_5^*]$,
$r_2 \longrightarrow [r_2^* = r_2 - 2r_1^* - r_5^* + r_3^* + r_4^*]$,
$r_3 \longrightarrow [r_3^* = r_3 + r_4^*]$,
$r_4 \longrightarrow [r_4' = r_4 + r_2 - 4r_5] \longrightarrow [r_4^* = r_4' - 2r_5^*]$,

$r_5 \longrightarrow [r_5^* = r_5 - 4r_1 + 2r_2 + 2r_3 + 3r_4']$,
$r_6 \longrightarrow [r_6' = r_6 - 16r_5 - 12r_4'] \longrightarrow [r_6^* = r_6' + 12r_5^*]$.
Then, we get Step(*) and the required results as;

$$
\begin{array}{c|ccccc|}
 & \overline{x}_6 & \overline{y}_6 & \overline{z}_6 & \overline{t'}_6 & \overline{u'}_6 \\
 & c_1 & c_2 & c_3 & c_4 & c_5 \\
r_1^* : & 2\cdot 4^k & 8\cdot 16^k & 4^k & -(-2)^k & 4\cdot 16^k \\
r_2^* : & 0 & 0 & 0 & 0 & -8\cdot 16^k \\
r_3^* : & 0 & 16\cdot 16^k & 0 & 0 & 0 \\
r_4^* : & -4\cdot 4^k & 0 & 0 & 0 & 0 \\
r_5^* : & 0 & 0 & -2\cdot 4^k & 0 & 0 \\
r_6^* : & 0 & 0 & 0 & 0 & 0
\end{array}
\cong
$$

$$
\begin{array}{c|ccccc|}
 & g_1 & g_2 & g_3 & g_4 & g_5 \\
 & c_1' & c_2' & c_3' & c_4 & c_5' \\
r_1^* : & 0 & 0 & 0 & -(-2)^k & 0 \\
r_2^* : & 0 & 0 & 0 & 0 & -8\cdot 16^k \\
r_3^* : & 0 & 16\cdot 16^k & 0 & 0 & 0 \\
r_4^* : & -4\cdot 4^k & 0 & 0 & 0 & 0 \\
r_5^* : & 0 & 0 & -2\cdot 4^k & 0 & 0 \\
r_6^* : & 0 & 0 & 0 & 0 & 0
\end{array}
,
$$

where $c_1' = c_1 + 2(-2)^k c_4$, $c_2' = c_2 + 8(-8)^k c_4$, $c_3' = c_3 + (-2)^k c_4$ and $c_5' = c_5 + 4(-8)^k c_4$. By using the same method as in Chapter 3, we get $g_1 = \overline{x}_6, g_2 = \overline{y}_6, g_3 = \overline{z}_6, g_4 = \overline{t'}_6 + (-2)^{k+1}\overline{x}_6 + (-8)^{k+1}\overline{y}_6 - (-2)^k \overline{z}_6 - 4(-8)^k \overline{u'}_6$ and $g_5 = \overline{u'}_6$ and then the results follow.

For $\overline{H}_{8k}, k \geq 1$, we can represent matrix for the calculation of $\overline{QO}_{-8}/q\overline{QO}_{8k}$ as

$$
\begin{array}{r|ccccc|l}
 & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \\
q^{k+1}1 : & a_1 & 16^k & 4^{k-1} - 4\cdot 16^{k-1} & (-2)^{k-1} + 8\cdot 16^{k-1} & 16^k & r_1 \\
q^{k+1}A : & 4^k & 0 & 0 & -(-2)^k & 0 & r_2 \\
q^{k+1}B : & 4^k & 2\cdot 16^k & 0 & 0 & 0 & r_3 \\
q^{k+1}C : & a_2 & 0 & 2\cdot 16^k - 4^k & -4\cdot 16^k & -8\cdot 16^k & r_4 \\
q^{k+1}D^2 : & a_3 & 0 & 4^k - 4\cdot 16^k & (-2)^k + 8\cdot 16^k & 16\cdot 16^k & r_5 \\
q^{k+1}\widetilde{D}^3 : & a_4 & 0 & 3\cdot 4^k - 18\cdot 16^k & 36\cdot 16^k & 72\cdot 16^k & r_6
\end{array}
,
$$

where $a_1 = 2\cdot 4^{k-1} - 8\cdot 16^{k-1}, a_2 = 2\cdot 4^k + 4\cdot 16^k, a_3 = 2\cdot 4^k - 8\cdot 16^k$ and $a_4 = 6\cdot 4^k - 36\cdot 16^k$.
Now, we do row operations by changing;
$r_1 \longrightarrow [r_1^* = r_1 + 2\cdot 4^{k-1} r_5^* + 4^{k-1} r_4'' - 4(-8)^{k-1} r_2']$,
$r_2 \longrightarrow [r_2' = r_2 - r_5^*] \longrightarrow [r_2^* = r_2' - 2r_1' + r_3' + r_5^*]$,
$r_3 \longrightarrow [r_3^* = r_3 - r_5^*]$,
$r_4 \longrightarrow [r_4' = r_4 + 8r_1 - 4r_3 - 4r_2] \longrightarrow [r_4'' = r_4' + 2r_5^*] \longrightarrow [r_4^* = r_4'' + 2r_2^*]$,
$r_5 \longrightarrow [r_5' = r_5 + 2r_4] \longrightarrow [r_5'' = r_5' + r_2 + r_4'] \longrightarrow [r_5^* = r_5'' - 2r_6'']$,
$r_6 \longrightarrow [r_6' = r_6 + 9r_4] \longrightarrow [r_6'' = r_6' + 6r_4' - 2r_5''] \longrightarrow [r_6^* = r_6'' - 2r_5^*]$.
Then, we get Step(*) and the required results as

$$\begin{array}{r|ccccc|l} & \overline{x}_4 & \overline{y}_4 & \overline{z}_4 & \overline{t}_4 & \overline{u}_4 & \\ & c_1 & c_2 & c_3 & c_4 & c_5 & \\ r_1^*: & 2\cdot 4^{k-1} & 16^k & 4^{k-1} & (-2)^{k-1} & 16^k & \\ r_2^*: & 0 & 0 & -2\cdot 4^{k-1} & 0 & -2\cdot 16^k & \\ r_3^*: & 0 & 2\cdot 16^k & 0 & 0 & 0 & \cong \\ r_4^*: & 0 & 0 & 0 & 0 & -4\cdot 16^k & \\ r_5^*: & 4^k & 0 & 0 & 0 & 0 & \\ r_6^*: & 0 & 0 & 0 & 0 & 0 & \end{array}$$

$$\begin{array}{r|ccccc|l} & g_1 & g_2 & g_3 & g_4 & g_5 & \\ & c_1' & c_2' & c_3' & c_4 & c_5' & \\ r_1^*: & 0 & 0 & 0 & (-2)^{k-1} & 0 & \\ r_2^*: & 0 & 0 & -2\cdot 4^{k-1} & 0 & 0 & \\ r_3^*: & 0 & 2\cdot 16^k & 0 & 0 & 0 & , \\ r_4^*: & 0 & 0 & 0 & 0 & -4\cdot 16^k & \\ r_5^*: & 4^k & 0 & 0 & 0 & 0 & \\ r_6^*: & 0 & 0 & 0 & 0 & 0 & \end{array}$$

where $c_1' = c_1 + (-2)^k c_4, c_2' = c_2 + 2(-8)^k c_4, c_3' = c_3 - (-2)^{k-1} c_4$ and $c_5' = c_5 - 4^{k+1} c_3$. By using the same method as in Chapter 3, we get $g_1 = \overline{x}_4, g_2 = \overline{y}_4, g_3 = \overline{z}_4 + 4^{k+1}\overline{u}_4, g_4 = \overline{t}_4 - (-2)^k \overline{x}_4 - 2(-8)^k \overline{y}_4 + (-2)^{k-1}\overline{z}_4 - 2(-8)^k \overline{u}_4$ and $g_5 = \overline{u}_4$ and then the results follow.

Proof of $\overline{H}_{8k+2}$ and $\overline{H}_{8k+6}$ for all $k \geq 0$ are easy because $\overline{QO}_{-2}, \overline{QO}_{-6}, \overline{QO}_{8k+2}$ and $\overline{QO}_{8k+6}$ are free abelian group with one generator and furthermore, $q^{k+1}\overline{QO}_{8k+\varepsilon} \cong \pm 2^{k+1}\overline{QO}_{-8+\varepsilon}$ for $\varepsilon = 2, 6$. The final statement in the lemma 5.3.3 is obvious. □

§ 5.4 E^2-page

In this section, we will calculate the E^2-page by using the similar process as in Chapter 3. That is by considering $E^{1\frac{1}{2}}$-page as a combination of $H_I^*(T)$ and $H_I^*(\overline{QO})$ and then determine the connecting homomorphism.

5.4.1 Connecting homomorphism δ^1

To get the E^2-page, i.e., $H_I^*(R)$, where $R = ko^*(BSD_{16})$, we need to determine the connecting homomorphism in the long exact sequence induced by (5.2);

$$0 \longrightarrow H_I^0(T) \longrightarrow H_I^0(R) \longrightarrow H_I^0(\overline{QO}) \xrightarrow{\delta^0} H_I^1(T) \longrightarrow H_I^1(R) \longrightarrow$$
$$H_I^1(\overline{QO}) \xrightarrow{\delta^1} H_I^2(T) \longrightarrow H_I^2(R) \longrightarrow 0,$$

since $H_I^i(T)$ and $H_I^{i-1}(\overline{QO})$ are zero for $i \geq 3$.

Determining δ^0 in our case is not hard. It is actually a zero map. This is because $H_I^1(T) \cong H_I^1(\tau)$ which concentrate in degree $n \equiv 1, 2 \text{mod } 8$, by corollary 5.2.5 and lemma 5.2.4, whereas $H_I^0(\overline{QO})$ concentrate in degree $n \equiv 4 \text{mod } 8$, by lemma 5.3.2. This gives two exact sequences;

$$0 \longrightarrow H_I^0(T) \longrightarrow H_I^0(R) \longrightarrow H_I^0(\overline{QO}) \longrightarrow 0 \tag{5.14}$$

and

$$0 \longrightarrow H_I^1(T) \longrightarrow H_I^1(R) \longrightarrow H_I^1(\overline{QO}) \xrightarrow{\delta^1} H_I^2(T) \longrightarrow H_I^2(R) \longrightarrow 0. \tag{5.15}$$

The short exact sequence (5.14) splits because $H_I^0(\overline{QO})$ is free abelian group over $\mathbb{Z}$, by lemma 5.3.2, and hence

$$H_I^0(R) \cong H_I^0(T) \oplus H_I^0(\overline{QO}). \tag{5.16}$$

It is clear that $H_I^i(R) = 0$ for all $i \geq 3$. To determine $H_I^1(R)$ and $H_I^2(R)$, we need to investigate connecting homomorphism δ^1. If δ^1 is determined then we will get;

$$0 \longrightarrow H_I^1(T) \longrightarrow H_I^1(R) \longrightarrow \ker(\delta^1) \longrightarrow 0 \tag{5.17}$$

and

$$H_I^2(R) \cong \text{coker}(\delta^1). \tag{5.18}$$

Now, the main task is the calculation of $\ker(\delta^1)$ and $\text{coker}(\delta^1)$. Before doing further calculation, it is useful to collect the results together, $E^{1\frac{1}{2}}$-page.

5.4.2 $E^{1\frac{1}{2}}$-PAGE

$E^{1\frac{1}{2}}$-page here is the immediate results of the previous two sections. The purpose of displaying this diagram is to provide all information involved in the calculation of the E^2-page. Precisely, need to show the motivation of using some facts of $ko_*(BSD_{16})$ to determine connecting homomorphism δ^1.

$H_I^{\epsilon\geq 3}(R)$	$H_I^2(T)$	$\xleftarrow{\delta^1}$	$H_I^1(T) \oplus H_I^1(\overline{QO})$	$H_I^0(R)$	degree(t)
$\vdots$	$\vdots$		$\vdots$	$\vdots$	27
0	2^4	$\longleftarrow \delta^1_{13}$	$2^2 \oplus [16]$	2^5	26
0	0		2^2	2^5	25
0	2^4	$\longleftarrow \delta^1_{12}$	$[4] \oplus [32] \oplus [64] \oplus [2 \cdot 16^3] \oplus [4 \cdot 16^3]$	$\mathbb{Z}$	24
0	0		0	0	23
0	2^3	$\longleftarrow \delta^1_{11}$	$[8]$	0	22
0	0		0	0	21
0	2^4	$\longleftarrow \delta^1_{10}$	$[4] \oplus [32] \oplus [64] \oplus [2048] \oplus [4096]$	$\mathbb{Z}$	20
0	0		0	0	19
0	2^3	$\longleftarrow \delta^1_{9}$	$2^2 \oplus [8]$	2^5	18
0	0		2^2	2^5	17
0	2^3	$\longleftarrow \delta^1_{8}$	$[2] \oplus [8] \oplus [16] \oplus [512] \oplus [1024]$	$\mathbb{Z}$	16
0	0		0	0	15
0	2^2	$\longleftarrow \delta^1_{7}$	$[4]$	0	14
0	0		0	0	13
0	2^3	$\longleftarrow \delta^1_{6}$	$[2] \oplus [8] \oplus [16] \oplus [128] \oplus [256]$	$\mathbb{Z}$	12
0	0		0	0	11
0	2^2	$\longleftarrow \delta^1_{5}$	$2 \oplus [4]$	2^4	10
0	0		2	2^4	9
0	2^2	$\longleftarrow \delta^1_{4}$	$[2] \oplus [4] \oplus [32] \oplus [64]$	$\mathbb{Z}$	8
0	0			0	7
0	2	$\longleftarrow \delta^1_{3}$	$[2]$	0	6
0	0		0	0	5
0	2^2	$\longleftarrow \delta^1_{2}$	$[2] \oplus [4] \oplus [8] \oplus [16]$	$\mathbb{Z}$	4
0	0		0	0	3
0	2	$\longleftarrow \delta^1_{1}$	$[2]$	2^3	2
0	0		0	2^3	1
0	2	$\longleftarrow \delta^1_{0}$	$[2]$	$\mathbb{Z}$	0
0	0		0	0	-1
0	0		0	0	-2
0	0		0	0	-3
0	2	$\longleftarrow \delta^1_{-2}$	$[2]$	0	-4

where $[n] :=$ cyclic group of order n, $2^r :=$ elementary abelian group of rank r.

Figure 5.1: The $E^{1\frac{1}{2}}$-page of Greenlees spectral sequence for $ko_*(BSD_{16})$.

5.4.3 Some structures of $H_I^2(T)$ and $H_I^1(\overline{QO})$

So as to investigate the connecting homomorphism $\delta^1 : H_I^1(\overline{QO}) \longrightarrow H_I^2(T)$, it is useful to understand the module structure of $H_I^1(\overline{QO})$ and $H_I^2(T)$ over $R := ko^*(BSD_{16})$. Recall, however, from lemma 5.2.5 and lemma 5.2.2 that,

$$H_I^2(T) = H_I^2(TO) = \Sigma^4(\mathbb{F}_2[d^2])^\vee \oplus \Sigma^{-2}(\mathbb{F}_2[b^2, d^2])^\vee \oplus (\mathbb{F}_2[b^2, d^2])^\vee .$$

Here, $H_I^2(T)$ is a module over $\mathbb{F}_2[b^2, d^2]$. Then $2[H_I^1(\overline{QO})] \subseteq \ker(\delta^1)$. Thus, it is enough (in order to investigate δ^1) to consider $H_I^1(\overline{QO})/2[H_I^1(\overline{QO})] := M$ as a module over $\mathbb{F}_2[b^2, d^2]$ as well. To do this, we need to see how b^2, d^2 act on both $H_I^2(T)$ and M.

In $H_I^2(T)$, we can write its elements explicitly (see, discussion before lemma 5.2.2) as;

Diagram 5.2: Explicit elements of $H_I^2(T)$

	-4	-2	0	2	4	6	8	10	12	14	16	18	20	22	$\cdots \leftarrow$ degree
$[H_I^2(T)]_n$:	τ^d		τ^{bd}	τ^b	$\frac{\tau^d}{d^2}$	$\frac{\tau^b}{b^2}$	$\frac{\tau^{bd}}{d^2}$	$\frac{\tau^b}{d^2}$	$\frac{\tau^d}{d^4}$	$\frac{\tau^b}{b^6}$	$\frac{\tau^{bd}}{d^4}$	$\frac{\tau^b}{d^4}$	$\frac{\tau^d}{d^6}$	$\frac{\tau^b}{b^{10}}$	
					$\frac{\tau^{bd}}{b^2}$		$\frac{\tau^{bd}}{b^4}$	$\frac{\tau^b}{b^4}$	$\frac{\tau^{bd}}{b^2d^2}$	$\frac{\tau^b}{b^2d^2}$	$\frac{\tau^{bd}}{b^4d^2}$	$\frac{\tau^b}{b^4d^2}$	$\frac{\tau^{bd}}{b^2d^2}$	$\frac{\tau^b}{b^2d^4}$	
									$\frac{\tau^{bd}}{b^6}$		$\frac{\tau^{bd}}{b^8}$	$\frac{\tau^b}{b^8}$	$\frac{\tau^{bd}}{b^6d^2}$	$\frac{\tau^b}{b^6d^2}$	
													$\frac{\tau^{bd}}{b^{10}}$		

Notation: $\tau := b^2d - abd$, $\tau^d := \frac{\tau d}{d^2}$, $\tau^{bd} := \frac{\tau d}{b^2d^2}$ and $\tau^b := \frac{\tau b}{b^2d^2}$.

The b^2, d^2 action on $H_I^2(T)$ is given by;

$$b^2 \cdot \frac{\tau^d}{d^{2j}} = 0, \; b^2 \cdot \frac{\tau^{bd}}{b^{2i}d^{2j}} = \frac{\tau^{bd}}{b^{2i-2}d^{2j}}, \; b^2 \cdot \frac{\tau^b}{b^{2i}d^{2j}} = \frac{\tau^b}{b^{2i-2}d^{2j}} \tag{5.19}$$

and

$$d^2 \cdot \frac{\tau^d}{d^{2j}} = \frac{\tau^d}{d^{2j-2}}, \; d^2 \cdot \frac{\tau^{bd}}{b^{2i}d^{2j}} = \frac{\tau^{bd}}{b^{2i}d^{2j-2}}, \; d^2 \cdot \frac{\tau^b}{b^{2i}d^{2j}} = \frac{\tau^b}{b^{2i}d^{2j-2}}, \tag{5.20}$$

where $i, j \geq 0$. This action is clear from the additive structure of $H_I^2(T)$ in the diagram above.

In $M := H_I^1(\overline{QO})/2[H_I^1(\overline{QO})]$, the action of $b^2 (= [0, 0, 0, 0, 4, 4])$ and $d^2 (= [16, -2, 4, -2, 0, 4])$ are obtained easily by calculation on character table of lemma 5.3.3, which we have;

- $b^2 M_{-4} = 0$ and $d^2 M_{-4} = 0$,
- $b^2 \widetilde{y}_0 = \widetilde{x}_{-2}$ and $d^2 M_0 = 0$

- $b^2\widetilde{y}_2 = \widetilde{y}_0, b^2\widetilde{x}_2 = 2\widetilde{x}_0 = 0 = b^2\widetilde{z}_2 = b^2\widetilde{t'}_2 = b^2\widetilde{u'}_2$ and $d^2\widetilde{\alpha}_2 = \widetilde{\alpha}_{-2} = \widetilde{x}_{-2}$ for each $\alpha \in \{x, y, z, t'\}$ whereas $d^2\widetilde{u'}_2 = \widetilde{u'}_{-2} = 0$,
- $b^2 M_6 = 0$ and $d^2 M_6 = 0$ since $M_{-2} = 0$,
- For $k \geq 1$, $b^2\widetilde{x}_{4k} = 2\widetilde{x}_{4k-2} = 0$, $b^2\widetilde{y}_{4k} = \widetilde{y}_{4k-2}$, $b^2\widetilde{z}_{4k} = b^2\widetilde{t}_{4k} = 0, b^2\widetilde{u}_{4k} = \widetilde{x}_{4k-2}$ and $d^2\widetilde{\alpha}_{4k} = \widetilde{\alpha}_{4(k-1)}$ for each $\alpha \in \{x, y, z, t, u\}$,
- For $k \geq 1$, $b^2 M_{8k+2} = 0$ and $d^2\widetilde{w}_{4k+1} = \widetilde{w}_{4(k-1)+1}$,
- For $k \geq 1$, $b^2\widetilde{x}_{4k+2} = 2\widetilde{x}_{4k} = 0$, $b^2\widetilde{y}_{4k+2} = \widetilde{y}_{4k}$, $b^2\widetilde{z}_{4k+2} = b^2\widetilde{t'}_{4k+2} = b^2\widetilde{u'}_{4k+2} = 0$ and $d^2\widetilde{\alpha}_{4k+2} = \widetilde{\alpha}_{4k-2}$ for each $\alpha \in \{x, y, z\}$, $d^2\widetilde{\alpha'}_{4k+2} = \widetilde{\alpha'}_{4k-2}$ for each $\alpha \in \{t, u\}$,
- For $k \geq 1$, $b^2 M_{8k+6} = 0$ and $d^2\widetilde{w'}_{4k+3} = \widetilde{w'}_{4(k-1)+3}$.

As we have done in Chapter 3, instead of finding δ^1 directly, we prefer to consider

$$(\delta^1)^\vee : [H_I^2(T)]^\vee \longrightarrow [H_I^1(\overline{QO})]^\vee.$$

Here, $[H_I^2(T)]^\vee$ and $M^\vee$ are also module over $\mathbb{F}_2[b^2, d^2]$. For the b^2, d^2 action on them can be obtain easily from the structure of $H_I^2(T)]$ and M together with the help of lemma 3.4.1. However, we need to determine $\delta^1_{-2} : M_{-4} \longrightarrow [H_I^2(T)]_{-4}$, $\delta^1_0 : M_0 \longrightarrow [H_I^2(T)]_0$ and $\delta^1_1 : M_2 \longrightarrow [H_I^2(T)]_2$, see $E^{1\frac{1}{2}}$-page, first. We are going to deal with this in the next subsection.

5.4.4 $ko_*(BG)$ and differential

Recall from $E^{1\frac{1}{2}}$-page that δ^1_{-2} and δ^1_0 are homomorphisms from $\mathbb{Z}/2$ to itself. These maps are actually isomorphisms because of the connectivity of $ko_*(BG)$, precisely, $ko_n(BG) = 0$ for all $n < 0$. For δ^1_1, we need more work to do. First, note that $ko_0(BG) \cong H_0(BG; \mathbb{Z})$ since BG is a connected space. Indeed, this isomorphism comes from the long exact sequence induced by the cofibre sequence (killing homotopy group)

$$ko < 1 > \longrightarrow ko \longrightarrow ko^0_\downarrow = ko(-\infty, 0] = H\mathbb{Z},$$

smashing with BG and applying π_* together with the fact that $\pi_0(ko < 1 > \wedge BG) = 0 = \pi_{-1}(ko < 1 > \wedge BG)$ and $\pi_0(ko \wedge BG) := ko_0(BG)$ and $\pi_0(H\mathbb{Z} \wedge BG) := H_0(BG; \mathbb{Z})$, where $ko < 1 >$ is 1-connected cover of spectrum ko. This implies that

$$ko_0(BSD_{16}) \cong \mathbb{Z}$$

and hence δ^1_1 can be possibly only a zero map or an isomorphism map. It depends on whether $d_2 : [H_I^0(R)]_1 \longrightarrow [H_I^2(T)]_2$ is a zero map or not.

By using the fact that δ^1_{-2} and δ^1_0 are isomorphisms, (i.e., $\delta^1_{-2}(\widetilde{x}_{-2}) = \tau^d$ and $\delta^1_0(\widetilde{y}_0) = \tau^{bd}$), and $\mathbb{F}_2[b^2, d^2]$-module structure of $H_I^2(T)$ and $H_I^1(\overline{QO})$, we can conclude that δ^1_2 is surjective. This implies that

$$ko_2(BSD_{16}) \cong (\mathbb{Z}/2)^3 < \widetilde{\eta}^2[B], \widetilde{\eta}^2[\widetilde{D}^3], \widetilde{\eta}^2([A] + [D^2]) >, \tag{5.21}$$

which contains eight elements of order two. Fortunately, we have;

Proposition 5.4.1. *For $G = SD_{16}$, the natural map $\eta_* : ko_1(BG) \longrightarrow ko_2(BG)$ is an epimorphism.*

Proof. The cofibre sequence $\Sigma ko \xrightarrow{\eta} ko \longrightarrow ku$ induces the long exact sequence;

$$\cdots \longrightarrow ko_{n-1}(BG) \longrightarrow ko_n(BG) \longrightarrow ku_n(BG) \longrightarrow ko_{n-2}(BG) \longrightarrow ko_{n-1}(BG) \longrightarrow \cdots .$$

In particular, for $G = SD_{16}$ and $n = 2$, we have

$$\cdots \longrightarrow ko_1(BG) \xrightarrow{\eta_*} ko_2(BG) \xrightarrow{r_*} ku_2(BG) \xrightarrow{\delta_*} ko_0(BG) \longrightarrow ko_1(BG) \longrightarrow \cdots .$$

Since $ku_2(BSD_{16}) \cong \mathbb{Z}$ (by Theorem 3.5.1), $ko_2(BSD_{16})$ is finite generate (by (5.21)) and r_* is homomorphism, r_* is a zero map. Therefore η_* is an epimorphism. $\square$

The consequence of this lemma is that;

Corollary 5.4.2. *δ_1^1 is an isomorphism.*

Proof. Suppose δ_1^1 is a zero map. Then d_2 is an isomorphism and hence $E_\infty^{0,1} = (\mathbb{Z}/2)^2$. Consider the commutative diagram of the natural map $\eta_* : ko_1(BSD_{16}) \longrightarrow ko_2(BSD_{16})$ below;

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^{-1,2} = \mathbb{Z}/2 \arrow[r] \arrow[d, "\eta'_*=0"] & ko_1(BSD_{16}) \arrow[r] \arrow[d, "\eta_*"] & E_\infty^{0,1} = (\mathbb{Z}/2)^2 \arrow[r] \arrow[d, "\eta''_*"] & 0 \\
0 \arrow[r] & E_\infty^{-1,3} = 0 \arrow[r] & ko_2(BSD_{16}) \arrow[r] & E_\infty^{0,2} = (\mathbb{Z}/2)^3 \arrow[r] & 0
\end{tikzcd}$$

This diagram treats that η_* can not be epimorphism which contradicts to the Proposition 5.4.1. Hence, δ_1^1 is an isomorphism as required. $\square$

Now, we already have the explicit maps

$$(\delta_{-2}^1)^\vee(\tau^{-d}) = (\widetilde{x}_{-2})^{-1},\ (\delta_0^1)^\vee(\tau^{-bd}) = (\widetilde{y}_0)^{-1} \text{ and } (\delta_1^1)^\vee(\tau^{-b}) = (\widetilde{w}_1)^{-1}. \quad (5.22)$$

Here τ^{-d}, τ^{-bd} and τ^{-b} denote the dual of τ^d, τ^{bd} and τ^b respectively and similarly for $(\widetilde{\alpha}_i)^{-1}$ denote the dual of $\widetilde{\alpha}_i$. We use the same process as in Chapter 3 to determine the module structure of $[H_I^2(T)]^\vee$ and $[H_I^1(\overline{QO})]^\vee$ over $\mathbb{F}_2[b^2, d^2]$. Then 5.22 and module structure yield all $(\delta_i^1)^\vee$ which we record as;

$$\begin{aligned}
(\delta^1)^\vee(d^2\tau^{-d}) &= (\widetilde{x}_2)^{-1} + (\widetilde{y}_2)^{-1} + (\widetilde{z}_2)^{-1} && (5.23)\\
(\delta^1)^\vee(b^2\tau^{-bd}) &= (\widetilde{y}_2)^{-1} && (5.24)\\
(\delta^1)^\vee(b^{2i}d^{2j}\tau^{-b}) &= 0, \forall i \geq 1, j \geq 0 && (5.25)\\
(\delta^1)^\vee(d^{2j}\tau^{-bd}) &= (\widetilde{y}_{4j})^{-1} + (\widetilde{u}_{4j})^{-1}, \forall j \geq 1 && (5.26)\\
(\delta^1)^\vee(d^{2j}\tau^{-b}) &= (\widetilde{w}_{4j+1})^{-1}, \forall j \geq 1 && (5.27)\\
(\delta^1)^\vee(b^{4(k-i)}d^{2i}\tau^{-bd}) &= (\widetilde{y}_{4k})^{-1}, \forall k \geq 2, k > i \geq 0 && (5.28)\\
(\delta^1)^\vee(b^{4(k-i)-2}d^{2i}\tau^{-bd}) &= (\widetilde{y}_{4k-2})^{-1}, \forall k \geq 2, k > i \geq 0 && (5.29)\\
(\delta^1)^\vee(d^{2j}\tau^{-d}) &= (\widetilde{t'}_{4j-2})^{-1} + (\widetilde{x}_{4j-2})^{-1} + (\widetilde{y}_{4j-2})^{-1} + (\widetilde{z}_{4j-2})^{-1} && (5.30)
\end{aligned}$$

where the last equation applies for $j \geq 2$.

Finally, after applying lemma 3.4.1 on $(\delta^1)^\vee$, we can find $\ker(\delta^1)$ and $\operatorname{coker}(\delta^1) = H_I^2(R)$ as:

Lemma 5.4.3. *As an abelian group, $[\ker(\delta^1)]_i = 0$ for $i < 4$ or i is odd, and*

- $[\ker(\delta^1)]_4 = \mathbb{Z}/4 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/8$ *generated by* $\widetilde{x}_2 + \widetilde{z}_2$, $\widetilde{u'}_2$ *and* $2\widetilde{y}_2$ *respectively.*
- $[\ker(\delta^1)]_6 = \mathbb{Z}/2$ *generated by* $\widetilde{w'}_3$.
- $[\ker(\delta^1)]_8 = \mathbb{Z}/2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/32$ *generated by* $\widetilde{z}_4 + 4\widetilde{u}_4$, $\widetilde{x}_4$, $2\widetilde{y}_4$ *and* $2\widetilde{u}_4$ *respectively.*
- $[\ker(\delta^1)]_{10} = \mathbb{Z}/2$ *generated by* $2\widetilde{w}_5$.
- $[\ker(\delta^1)]_{12} = \mathbb{Z}/8 \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/128 \oplus \mathbb{Z}/128$ *generated by* $\widetilde{z}_6 + \widetilde{t}_6$, $\widetilde{z}_6 + \widetilde{x}_6$, $\widetilde{u'}_6$ *and* $2\widetilde{y}_6$ *respectively.*
- *For $k \geq 2$,* $[\ker(\delta^1)]_{8k} = \mathbb{Z}/2^{k-1} \oplus \mathbb{Z}/2 \cdot 4^{k-1} \oplus \mathbb{Z}/4^k \oplus \mathbb{Z}/16^k \oplus \mathbb{Z}/2 \cdot 16^k$ *generated by* $\widetilde{\widehat{t}}_{4k}, \widetilde{z}_{4k} + 4^{k+1}\widetilde{u}_{4k}$, $\widetilde{x}_{4k}$, $2\widetilde{y}_{4k}$ *and* $2\widetilde{u}_{4k}$ *respectively.*
- *For $k \geq 2$,* $[\ker(\delta^1)]_{8k+2} = \mathbb{Z}/2^k$ *generated by* $2\widetilde{w}_{4k+1}$.
- *For $k \geq 2$,* $[\ker(\delta^1)]_{8k+4} = \mathbb{Z}/2^{k-1} \oplus \mathbb{Z}/2 \cdot 4^k \oplus \mathbb{Z}/4^{k+1} \oplus \mathbb{Z}/8 \cdot 16^k \oplus \mathbb{Z}/8 \cdot 16^k$ *generated by* $2\widetilde{t}_{4k+2}, \widetilde{t}_{4k+2} + \widetilde{z}_{4k+2}$, $\widetilde{x}_{4k+2} + \widetilde{z}_{4k+2}$, $\widetilde{u'}_{4k+2}$ *and* $2\widetilde{y}_{4k+2}$ *respectively.*
- *For $k \geq 1$,* $[\ker(\delta^1)]_{8k+6} = \mathbb{Z}/2^{k+1}$ *generated by* $\widetilde{w'}_{4k+3}$.

Furthermore, as an abelian group, $[H_I^2(R)]_i = 0$ for $i < 6$ or i is odd, and

- $[H_I^2(R)]_6 = \mathbb{Z}/2$ *generated by* $\frac{\tau^b}{b^2}$,
- $[H_I^2(R)]_8 = 0$,
- $[H_I^2(R)]_{10} = \mathbb{Z}/2$ *generated by* $\frac{\tau^b}{b^4}$,
- $[H_I^2(R)]_{12} = \mathbb{Z}/2$ *generated by* $\frac{\tau^{bd}}{b^6} + \frac{\tau^{bd}}{b^2 d^2}$,
- $[H_I^2(R)]_{14} = (\mathbb{Z}/2)^2$ *generated by* $\frac{\tau^b}{b^6}$ *and* $\frac{\tau^b}{b^2 d^2}$,
- $[H_I^2(R)]_{16} = \mathbb{Z}/2$ *generated by* $\frac{\tau^{bd}}{b^8} + \frac{\tau^{bd}}{b^4 d^2}$,
- *For $k \geq 2$,* $[H_I^2(R)]_{8k+2} = (\mathbb{Z}/2)^k$ *generated by* $\{\frac{\tau^b}{b^{4(k-i)} d^{2i}} | 0 \leq i \leq k-1\}$,
- *For $k \geq 2$,* $[H_I^2(R)]_{8k+4} = (\mathbb{Z}/2)^k$ *generated by* $\{\frac{\tau^{bd}}{b^{4(k-i)+2} d^{2i}} + \frac{\tau^{bd}}{b^{4(k-i-1)+2} d^{2(i+1)}} | 0 \leq i \leq k-1\}$,
- *For $k \geq 2$,* $[H_I^2(R)]_{8k+6} = (\mathbb{Z}/2)^{k+1}$ *generated by* $\{\frac{\tau^b}{b^{4(k-i)+2} d^{2i}} | 0 \leq i \leq k\}$,

- *For* $k \geq 3$, $[H^2_I(R)]_{8k+8} = (\mathbb{Z}/2)^k$ *generated by* $\{\frac{\tau^{bd}}{b^{4(k-i+1)}d^{2i}} + \frac{\tau^{bd}}{b^{4(k-i)}d^{2(i+1)}} | 0 \leq i \leq k-1\}$,

Now, we reach to E^2_*-page. Here, E^2_*-page denotes the E^2-page as normal but with the assumption that the short exact sequence (5.17) split (i.e., up to the additive extension problems). The notation $\oplus^*$ refers to the direct sum of groups as normal but it comes with the assumption that the extension problems associated to the groups are trivial which we will determine them later.

$H_I^{\epsilon\geq 3}(R)$	$H_I^2(R)$	$H_I^1(T) \oplus \ker(\delta^1)$	$H_I^0(R)$	degree(t)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	27
0	2^3	$2^2 \oplus^* [8]$	2^5	26
0	0	2^2	2^5	25
0	2^2	$[4] \oplus [32] \oplus [64] \oplus [16^3] \oplus [2 \cdot 16^3]$	$\mathbb{Z}$	24
0	0	0	0	23
0	2^3	$[8]$	0	22
0	0	0	0	21
0	2^2	$[2] \oplus [32] \oplus [64] \oplus [2048] \oplus [2048]$	$\mathbb{Z}$	20
0	0	0	0	19
0	2^2	$2^2 \oplus^* [4]$	2^5	18
0	0	2^2	2^5	17
0	2	$[2] \oplus [8] \oplus [16] \oplus [256] \oplus [512]$	$\mathbb{Z}$	16
0	0	0	0	15
0	2^2	$[4]$	0	14
0	0	0	0	13
0	2	$[8] \oplus [16] \oplus [128] \oplus [128]$	$\mathbb{Z}$	12
0	0	0	0	11
0	2	$2 \oplus^* [2]$	2^4	10
0	0	2	2^4	9
0	0	$[2] \oplus [4] \oplus [16] \oplus [32]$	$\mathbb{Z}$	8
0	0	0	0	7
0	2	$[2]$	0	6
0	0	0	0	5
0	0	$[4] \oplus [8] \oplus [8]$	$\mathbb{Z}$	4
0	0	0	0	3
0	0	0	2^3	2
0	0	0	2^3	1
0	0	0	$\mathbb{Z}$	0

where $[n] :=$ cyclic group of order n, $2^r :=$ elementary abelian group of rank r.

Figure 5.3: The E^2_*-page of Greenlees spectral sequence for $ko_*(BSD_{16})$.

5.4.5 Extension problems and E^2 - page

In order to get the E^2-page, we need to solve the extension problems which occur in degree $2+8k, k \geq 1$, of the second column in the E^2_*-page; viz:

$$0 \longrightarrow [H^1_I(T)]_{2+8k} \longrightarrow [H^1_I(R)]_{2+8k} \longrightarrow [\ker(\delta^1)]_{2+8k} \longrightarrow 0. \tag{5.31}$$

The strategy to solve this problems is using the action of some elements in R, precisely the action of $\beta \in R_8$ and $\overline{u}_4 \in R_{-8}$ over $[H^1_I(T)]_{2+8k}$ and $[\ker(\delta^1)]_{2+8k}$. Recall from lemma 5.2.5, lemma 5.2.4 and lemma 5.4.3 that

$$[H^1_I(T)]_{2+8k} = \begin{cases} \mathbb{Z}/2 < \widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^2 >, & \text{if } k=1; \\ (\mathbb{Z}/2)^2 < \widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^{k+1}, \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^{k+1} >, & \text{if } k \geq 2, \end{cases}$$

and $[\ker(\delta^1)]_{2+8k} = \mathbb{Z}/2^k$ generated by $2\widetilde{w}_{4k+1} = \frac{\overline{w}_3}{q^{k+1}} + \overline{QO}_{-6}$ for all $k \geq 1$.

It is not hard to see that

$$\beta \cdot (2\widetilde{w}_{4k+1}) = v^4(2\widetilde{w}_{4k+1}) = \pm 2[2\widetilde{w}_{4(k+1)+1}] \tag{5.32}$$

and

$$\begin{aligned} \beta \cdot \widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^2 &= 16\widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^3 + 9\widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^3 - 3\widetilde{\eta}^2[\overline{z}_4]/(\overline{u}_4)^3 - 3\widetilde{\eta}^2[\overline{x}_4]/(\overline{u}_4)^3 \\ &= \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^3, \\ \beta \cdot \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^2 &= -2 \cdot \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^3 = 0, \end{aligned}$$

and also in general we have;

$$\beta \cdot \widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^{k+1} = \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^{k+2} \tag{5.33}$$

$$\beta \cdot \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^{k+1} = 0 \tag{5.34}$$

for each $k \geq 1$.

We firstly consider on $[H^1_I(R)]_{10}$ and $[H^1_I(R)]_{18}$ that whether they split or not by using the action of β. There are four candidates to investigate, i.e., all split, all are non-split, $[H^1_I(R)]_{10}$ split but $[H^1_I(R)]_{18}$ is non-split and $[H^1_I(R)]_{18}$ split but $[H^1_I(R)]_{10}$ is non-split. We claim that the two latter cases can not be possible by considering on the commutative diagrams below;

$$\begin{tikzcd} \text{degree } \mathbf{10}\text{: } 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "i_1"] \arrow[d, "\beta'"] & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \arrow[r, "p_1"] \arrow[d, "\beta"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\beta''"] & 0 : \text{split} \\ \text{degree } \mathbf{18}\text{: } 0 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[r, "i_2"] & \mathbb{Z}/2 \oplus \mathbb{Z}/8 \arrow[r, "p_2"] & \mathbb{Z}/4 \arrow[r] & 0 : \text{non-split} \end{tikzcd} \tag{5.35}$$

and

$$\begin{tikzcd} \text{degree } \mathbf{10}\text{: } 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "i_1"] \arrow[d, "\beta'"] & \mathbb{Z}/4 \arrow[r, "p_1"] \arrow[d, "\beta"] & \mathbb{Z}/2 \arrow[r] \arrow[d, "\beta''"] & 0 : \text{non-split} \\ \text{degree } \mathbf{18}\text{: } 0 \arrow[r] & (\mathbb{Z}/2)^2 \arrow[r, "i_2"] & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4 \arrow[r, "p_2"] & \mathbb{Z}/4 \arrow[r] & 0 : \text{split} \end{tikzcd} \tag{5.36}$$

In diagram (5.35), $4g \in \mathbb{Z}/8$ must be detected by i_2 and β, where g is the generator of $\mathbb{Z}/8$. This means $p_2(4g) = 0$ whereas $\beta''(p_1(\beta^{-1}(4g))) \neq 0$, since β'' is injective, by (5.32). Thus these two facts lead to the contradiction by commutative property of the diagram and hence diagram (5.35) can not happen. For diagram (5.36), the injectivity of β' and β'' yields the injectivity of β. Under this monomorphism, for generator $g' \in \mathbb{Z}/4$ of degree 10, $p_2(\beta(g'))$ has order 4 whereas $\beta''(p_1(g'))$ has order 2 which contradicts to the commutative property of the diagram. So, our claim is true.

Nonetheless, in higher degree $k \geq 2$, if $[H_I^1(R)]_{2+8k}$ is non-split, then we can not conclude about $[H_I^1(R)]_{2+8(k+1)}$ by using only the action of β because β is not monomorphism (since β' is not injective in high degree, by (5.34)). However, for $k \geq 2$,

$$\begin{array}{lcccccccc}
\mathbf{2{+}8k}\colon 0 & \longrightarrow & (\mathbb{Z}/2)^2 & \xrightarrow{i_1} & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/2^k & \xrightarrow{p_1} & (\mathbb{Z}/2)^k & \longrightarrow & 0 : \text{split} \\
 & & \big\downarrow{\scriptstyle \beta'} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \beta''} & & \\
\mathbf{2{+}8(k{+}1)}\colon 0 & \longrightarrow & (\mathbb{Z}/2)^2 & \xrightarrow{i_2} & \mathbb{Z}/2 \oplus \mathbb{Z}/2^{k+2} & \xrightarrow{p_2} & \mathbb{Z}/2^{k+1} & \longrightarrow & 0 : \text{non-split}
\end{array} \tag{5.37}$$

still can not be possible, by similarly reason as the case $k = 1$. Precisely, the generator $g \in \mathbb{Z}/2^k$ in degree $2 + 8k$ must not be zero via $\beta'' \circ p_1$ because β'' is injective. By commutative diagram, $\beta(g) \neq 0 \in \mathbb{Z}/2^{k+1}$. It follows that $\beta(g) = 4h$ where h is the generator of $\mathbb{Z}/2^{k+1}$, by homomorphism property. Thus $2^{k-1}g$ is zero via $p_2 \circ \beta$ but not zero via $\beta'' \circ p_1$ which is a contradiction.

Next, we use the action of $\overline{u}_4$. It is not hard to see that

$$\overline{u}_4 \cdot \widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^{k+2} = \widetilde{\eta}^2[\overline{u}_4]/(\overline{u}_4)^{k+1} \tag{5.38}$$
$$\overline{u}_4 \cdot \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^{k+2} = \widetilde{\eta}^2[\overline{t}_4]/(\overline{u}_4)^{k+1}. \tag{5.39}$$

In other words, $\overline{u}_4 : [H_I^1(T)]_{2+8(k+1)} \longrightarrow [H_I^1(T)]_{2+8k}$ is an isomorphism for all $k \geq 2$. This implies that, for $k \geq 2$,

$$[H_I^1(R)]_{2+8k} \text{ and } [H_I^1(R)]_{2+8(k+1)} \text{ can not be both non-split.} \tag{5.40}$$

This is because if both of them are non-split, we have the commutative diagram as;

$$\begin{array}{lcccccccc}
\mathbf{2{+}8(k{+}1)}\colon 0 & \longrightarrow & (\mathbb{Z}/2)^2 & \xrightarrow{i_1} & \mathbb{Z}/2 \oplus \mathbb{Z}/2^{k+2} & \xrightarrow{p_1} & (\mathbb{Z}/2)^{k+1} & \longrightarrow & 0 : \text{non-split} \\
 & & \big\downarrow{\scriptstyle \overline{u}_4'} & & \big\downarrow{\scriptstyle \overline{u}_4} & & \big\downarrow{\scriptstyle \overline{u}_4''} & & \\
\mathbf{2{+}8k}\colon 0 & \longrightarrow & (\mathbb{Z}/2)^2 & \xrightarrow{i_2} & \mathbb{Z}/2 \oplus \mathbb{Z}/2^{k+1} & \xrightarrow{p_2} & \mathbb{Z}/2^{k} & \longrightarrow & 0 : \text{non-split}
\end{array} \tag{5.41}$$

and then $0 = \ker(i_2 \circ \overline{u}_4') = \ker(\overline{u}_4 \circ i_1) = \ker(\overline{u}_4) \cap (\mathbb{Z}/2)^2 \neq 0$, which is a contradiction.

By using the impossibility of the diagram (5.35), (5.36), (5.37) and the fact (5.40), we can conclude that the short exact sequence for $[H_I^1(R)]_{2+8k}$ split for each $k \geq 3$. More precisely,

$$\textit{if the short exact sequence for } [H_I^1(R)]_{10} \textit{ splits, then all split.} \tag{5.42}$$

Lemma 5.4.4. *For $R = ko^*(BSD_{16})$ and I which is the augmentation ideal of R,*

$$[H^1_I(R)]_{10} = \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Proof. Consider the commutative diagram below;

$$\begin{tikzcd}
0 \arrow[r] & R_{10} \arrow[r, "i_1"] \arrow[d, "\pi^u"] & R[\frac{1}{q'}]_{10} \arrow[r, "p_1"] \arrow[d, "\pi'^u"] & {[H^1_{(q')}(R)]_{10}} \arrow[r] \arrow[d, "\pi''^u"] & 0 \\
0 \arrow[r] & \overline{QO}_{10} \arrow[r, "i_2"] & \overline{QO}[\frac{1}{q}]_{10} \arrow[r, "p_2"] & {[H^1_{(q)}(\overline{QO})]_{10}} \arrow[r] & 0,
\end{tikzcd} \tag{5.43}$$

where $q = \pi^u(\overline{u}_4 + \overline{y}_4) \in \pi^u(ko^8(BSD_{16}))$ in lemma 5.3.1 and q' is the pre-image of q under π^u. We need to show that there is some elements of order 2 in $[H^1_{(q')}(R)]_{10}$ detects the generator $[g]$ of $\ker(\delta^1 : [H^1_{(q)}(\overline{QO})]_{10} \longrightarrow H^2_I(T)) = \mathbb{Z}/2$, where $g = 2\frac{\overline{w}_3}{q^2} \in \overline{QO}[\frac{1}{q}]_{10}$. By definition, $\overline{QO} = \pi^u(R)$, then we can find $\widetilde{g} \in R[\frac{1}{q'}]$ in which

$$\pi''^u([\widetilde{g}]) = [g].$$

Note that $[2g] = 0$, then $2g \in Im(i_2)$ and in fact $2g = \frac{v^4\theta}{1}$ where $v^4\theta = [0, 2c, 0, -2c, 0, 0]$, by the explicit generator in theorem 4.4.1. So, there exist $\beta\widetilde{\theta} \in R$ which $\pi^u(\beta\widetilde{\theta}) = v^4\theta$. Now, we have $i_1(\beta\widetilde{\theta})$ and $2\widetilde{g}$ has the same image, (viz; $2g$), under π'^u, by commutative property of diagram. To conclude that $i_1(\beta\widetilde{\theta}) = 2\widetilde{g}$, we need to check that they have the same image in $H^*(BSD_{16}; \mathbb{F}_2)[\frac{1}{q'}]$. This is immediate because 2 and β are zero in $H^*(BSD_{16}; \mathbb{F}_2)[\frac{1}{q'}]$. Hence,

$$2\widetilde{g} = \frac{\beta\widetilde{\theta}}{1},$$

and thus $[\widetilde{g}]$ has order 2. Note further that $H^1_I(T) = H^1_{(\overline{u}_4)}(T) = H^1_{(q')}(T)$, since $q' = \overline{u}_4 + \overline{y}_4$ and $\overline{y}_4 \in ann_{\pi^0(R)}(\tau)$, i.e., $\sqrt{(\overline{u}_4)} = \sqrt{(q')}$. Then the long exact sequence obtained by applying $H^*_{(q')}$ to (5.2),($G = SD_{16}$), splits to give the short exact sequence

$$0 \longrightarrow [H^1_{(q')}(T)]_{10} \longrightarrow [H^1_{(q')}(R)]_{10} \longrightarrow [H^1_{(q)}(\overline{QO})]_{10} \longrightarrow 0,$$

because $[H^0_{(q)}(\overline{QO})]_{10} = 0$. Also, the natural map (see remark 3.1.2 in Chapter 3) $\eta : H^s_{I+J}(R; M) \longrightarrow H^s_I(R; M)$ yields the commutative diagram;

$$\begin{tikzcd}
0 \arrow[r] & {[H^1_I(T)]_{10}} \arrow[r, "i_1"] \arrow[d, "\cong"] & {[H^1_I(R)]_{10}} \arrow[r, "p_1"] \arrow[d, "\eta"] & {[H^1_I(\overline{QO})]_{10}} \arrow[r, "\delta^1"] \arrow[d, "\cong"] & {[H^2_I(T)]_{10}} \\
0 \arrow[r] & {[H^1_{(q')}(T)]_{10}} \arrow[r, "i_2"] & {[H^1_{(q')}(R)]_{10}} \arrow[r, "p_2"] & {[H^1_{(q)}(\overline{QO})]_{10}} \arrow[r] & 0,
\end{tikzcd} \tag{5.44}$$

which treats η to be a monomorphism. Recall that $[H^1_{(q')}(T)]_{10} = \mathbb{Z}/2$ and $[H^1_{(q)}(\overline{QO})]_{10} = \mathbb{Z}/4$. If $[H^1_{(q')}(R)]_{10} = \mathbb{Z}/8$, then it contains one element of order 2, namely $[\widetilde{g}]$, and it must be detected by the image of the generator of $[H^1_{(q')}(T)]_{10}$ under i_2. This contradicts the fact that $[\widetilde{g}]$ has order 2 sending to $[g] \in [H^1_{(q)}(\overline{QO})]_{10}$ and hence

$$[H^1_{(q')}(R)]_{10} = \mathbb{Z}/2 \oplus \mathbb{Z}/4.$$

Let $(g', 0)$ and $(0, a)$ be the generator of $[H^1_{(q')}(R)]_{10}$ in which the order of $(g', 0)$ and $(0, a)$ are 2 and 4 resp. Now, we see that $[\widetilde{g}]$ must be either $(0, 2a)$ or $(g', 2a)$. Suppose, $[H^1_I(R)]_{10} = \mathbb{Z}/4$ generated by h, then $2h$ must be zero via p_1. Since η is an monomorphism, $\eta(h)$ must be either $(0, a)$ or (g', a) which the images of them in $[H^1_{(q)}(\overline{QO})]_{10}$ are of order 4. The contradiction happens with commutative property of diagram 5.44 again, because $2h$ will be not zero via this way and hence $[H^1_I(R)]_{10} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ as required. □

The immediate results from lemma 5.4.4 and the assertion (5.42) is;

Corollary 5.4.5. *The E^2-page of Greenlees spectral sequence for $ko_*(BSD_{16})$ is the E^2_*-page in Figure 5.3.*

§ 5.5 The results

5.5.1 $E^\infty-$ PAGE

It is left only to determine the differential d_2 so as to get the E^∞-page. The possible non-zero differential starting from $[H^0_I(R)]_n$ to $[H^2_I(R)]_{n+1}$, occurs in only degree n being congruent to 1 modulo 8, for $n \geq 9$.

We start to justify d_2 in degree 9 first by using the action of R over $[H^0_I(R)]_*$ and $[H^2_I(R)]_*$. Recall that $[H^0_I(R)]_*$ is a module over R via π^u and $[H^2_I(R)]_*$ is a module over R via φ in diagram (5.5). Here, we make use the action of $\overline{x}_2 + \overline{y}_2$ which its image under φ is b^2. By lemma 5.4.3, $[H^2_I(R)]_6 = \mathbb{Z}/2 < \frac{\tau^b}{b^2} >$ and $[H^2_I(R)]_{10} = \mathbb{Z}/2 < \frac{\tau^b}{b^4} >$ which is not hard to see that $b^2 : [H^2_I(R)]_{10} \xrightarrow{\cong} [H^2_I(R)]_6$ is an isomorphism. Now, consider the commutative diagram below;

$$\begin{tikzcd}
(\mathbb{Z}/2)^4 = [H^0_I(R)]_9 \arrow[r, "d_2"] \arrow[d, "b^2"] & {[H^2_I(R)]_{10}} \arrow[d, "b^2"] \\
0 = [H^0_I(R)]_5 \arrow[r, "d_2"] & {[H^2_I(R)]_6.}
\end{tikzcd}$$

The commutative property of this diagram treats d_2 to be zero in degree 9.

Next, we investigate degree 17 by using the same method as above. So, to conclude that $d_2 : [H^0_I(R)]_{17} \longrightarrow [H^2_I(R)]_{18}$ is a zero map, it is left only to check whether that $b^2 : [H^2_I(R)]_{18} \longrightarrow [H^2_I(R)]_{14}$ is an isomorphism, because $[H^0_I(R)]_{13} = 0$. It is does isomorphism since $[H^2_I(R)]_{14} = (\mathbb{Z}/2)^2 < \frac{\tau^b}{b^6}, \frac{\tau^b}{b^2d^2} >$ and $[H^2_I(R)]_{18} = (\mathbb{Z}/2)^2 < \frac{\tau^b}{b^8}, \frac{\tau^b}{b^4d^2} >$, by lemma 5.4.3.

For higher degrees, $n = 17 + 8k$ for $k \geq 1$, we use the action of β. It is simple to see that multiplying by β gives an isomorphism from $[H^0_I(R)]_{17+8(k-1)}$ to $[H^0_I(R)]_{17+8k}$. Since d_2 departing from degree 17, i.e. $E^2_{17,0}$, is a zero map, d_2 departing from degree $n = 17+8k$ for $k \geq 1$, are also zero maps. Therefore E^∞-page is E^2-page.

$H^{\epsilon\geq 3}_I(R)$	$H^2_I(R)$	$H^1_I(R)$	$H^0_I(R)$	degree(t)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	27
0	2^3	$2^2 \oplus [8]$	2^5	26
0	0	2^2	2^5	25
0	2^2	$[4] \oplus [32] \oplus [64] \oplus [16^3] \oplus [2 \cdot 16^3]$	$\mathbb{Z}$	24
0	0	0	0	23
0	2^3	$[8]$	0	22
0	0	0	0	21
0	2^2	$[2] \oplus [32] \oplus [64] \oplus [2048] \oplus [2048]$	$\mathbb{Z}$	20
0	0	0	0	19
0	2^2	$2^2 \oplus [4]$	2^5	18
0	0	2^2	2^5	17
0	2	$[2] \oplus [8] \oplus [16] \oplus [256] \oplus [512]$	$\mathbb{Z}$	16
0	0	0	0	15
0	2^2	$[4]$	0	14
0	0	0	0	13
0	2	$[8] \oplus [16] \oplus [128] \oplus [128]$	$\mathbb{Z}$	12
0	0	0	0	11
0	2	$2 \oplus [2]$	2^4	10
0	0	2	2^4	9
0	0	$[2] \oplus [4] \oplus [16] \oplus [32]$	$\mathbb{Z}$	8
0	0	0	0	7
0	2	$[2]$	0	6
0	0	0	0	5
0	0	$[4] \oplus [8] \oplus [8]$	$\mathbb{Z}$	4
0	0	0	0	3
0	0	0	2^3	2
0	0	0	2^3	1
0	0	0	$\mathbb{Z}$	0

where $[n] :=$ cyclic group of order n, $2^r :=$ elementary abelian group of rank r.

Figure 5.4: The E^∞-page of Greenlees spectral sequence for $ko_*(BSD_{16})$.

5.5.2 Results and Extension problems

The results can be read from the E_∞-page directly with the filtration given by

$$ko_n(BSD_{16}) = F_0^n \supseteq F_1^n \supseteq F_2^n \supseteq F_3^n = 0,$$

with $F_0^n/F_1^n \cong E_\infty^{0,n}, F_1^n/F_2^n \cong E_\infty^{-1,n+1}$ and $F_2^n \cong E_\infty^{-2,n+2}$. Precisely, we use two short exact sequences to determine $ko_n(BSD_{16})$, viz;

$$0 \longrightarrow F_1^n \longrightarrow ko_n(BSD_{16}) \longrightarrow E_\infty^{0,n} \longrightarrow 0,$$

and

$$0 \longrightarrow E_\infty^{-2,n+2} \longrightarrow F_1^n \longrightarrow E_\infty^{-1,n+1} \longrightarrow 0.$$

From this fact, we see that there are extension problems in degree $n \geq 8$ being congruent to $0, 1, 2$ modulo 8. In this chapter, we will solve such problems in degree $8k+2$ for all $k \geq 1$. For resolving the extension problems in degree $8k$ and $8k+1$, we wait to the next chapter.

Solving extension problems in degree $8k+2$ for all $k \geq 1$.

In these degrees, we consider the commutative diagram below;

$$\begin{tikzcd}
0 \arrow[r] & E_\infty^{-1,8k+2} \arrow[r] \arrow[d] & ko_{8k+1}(BSD_{16}) \arrow[r, "p_1"] \arrow[d, "\eta"] & E_\infty^{0,8k+1} \arrow[r] \arrow[d, "\widetilde{\eta}"] & 0 \\
0 \arrow[r] & E_\infty^{-2,8k+4} \arrow[r] & ko_{8k+2}(BSD_{16}) \arrow[r, "p_2"] & E_\infty^{0,8k+2} \arrow[r] & 0,
\end{tikzcd}$$

where the homomorphism η and $\widetilde{\eta}$ mean multiplying by η and $\widetilde{\eta}$ respectively. By isomorphism $E_\infty^{-1,8k+\epsilon} \cong [H^1_{(\overline{u}_4)}(\tau)]_{8k+\epsilon}$ for each $\epsilon = 1, 2$ (since $E_{1\frac{1}{2}}^{0,8k+\epsilon} = E_\infty^{0,8k+\epsilon}$) and by lemma 5.2.4, we get that $\widetilde{\eta} : E_\infty^{-1,8k+1} \longrightarrow E_\infty^{-1,8k+2}$ is an isomorphism. The consequence is that we can define the homomorphism $s : E_\infty^{0,8k+2} \longrightarrow ko_{8k+2}(BSD_{16})$ by setting

$$s(\widetilde{\eta}(p_1(g))) := \eta(g),$$

for all $g \in ko_{8k+1}(BSD_{16})$ which is easy to check that $p_2 \circ s = id_{E_\infty^{0,8k+2}}$. Thus, by splitting lemma, the second short exact sequence in the above diagram is additively split and therefore

$$ko_{8k+2}(BSD_{16}) \cong E_\infty^{0,8k+2} \oplus E_\infty^{-2,8k+4}. \tag{5.45}$$

We collect all results we have so far as;

Theorem 5.5.1. *Additively, $ko_*(BSD_{16})$ can tabulate as follows;*

$ko_n(BSD_{16})$	n
$[2^k] \oplus [2 \cdot 4^{k+1}] \oplus [4^{k+2}] \oplus [8 \cdot 16^{k+1}] \oplus [8 \cdot 16^{k+1}]$	$8k + 11 \geq 19$
$[2]^{\oplus 5} \oplus 2^{k+1}$	$8k + 10 \geq 18$
$< 2^7 \oplus [2^{k+1}] >$	$8k + 9 \geq 17$
$\mathbb{Z} \cdot \beta^{k+1}\rho \oplus < [2]^{\oplus 2} \oplus 2^{k+1} >$	$8k + 8 \geq 16$
$[2^k] \oplus [2 \cdot 4^k] \oplus [4^{k+1}] \oplus [16^{k+1}] \oplus [2 \cdot 16^{k+1}]$	$8k + 7 \geq 15$
2^k	$8k + 6 \geq 14$
$[2^{k+1}]$	$8k + 5 \geq 13$
$\mathbb{Z} \cdot \beta^k \alpha \oplus 2^{k+1}$	$8k + 4 \geq 12$
$[8] \oplus [16] \oplus [128] \oplus [128]$	11
$[2]^{\oplus 4} \oplus 2$	10
$< 2^5 \oplus [2] >$	9
$\mathbb{Z} \cdot \beta\rho \oplus < [2] \oplus 2 >$	8
$[2] \oplus [4] \oplus [16] \oplus [32]$	7
0	6
$[2]$	5
$\mathbb{Z} \cdot \alpha \oplus 2$	4
$[4] \oplus [8] \oplus [8]$	3
$[2]^{\oplus 3}$	2
$[2]^{\oplus 3}$	1
$\mathbb{Z} \cdot \rho$	0

,

where β is the Bott element in $KO_8(pt)$, ρ is the first Chern class of regular representation of SD_{16}, $\alpha = 2\rho$ and $[n]$ means cyclic group of order n, 2^k means elementary abelian two group of rank k and $< 2^a \oplus [2^b] >$ means abelian groups of order 2^{a+b} which is not determined yet.

In order to see more precisely structure, e.g., η-multiples, we will use Bockstein spectral sequence to calculate $ko_*(BSD_{16})$ from $ku_*(BSD_{16})$ in Chapter 3 and then compare both results. We postpone this calculation and the remaining extension problems to the next chapter.

Chapter 6

$ko_*(BSD_{16})$ by η-Bockstein spectral sequence

In this chapter, we will repeat the calculation of $ko_*(BSD_{16})$ by using the η-Bockstein spectral sequence with input $ku_*(BSD_{16})$. That is;

$$E_1^{*,*} = ku_*(BSD_{16})[\widetilde{\eta}] \Rightarrow ko_*(BSD_{16}).$$

The main purpose is to investigate η-multiple elements in $ko_*(BSD_{16})$, resolve the extension problems remaining from the last chapter and to give confidence in our calculation, i.e., both ways of the calculation (i.e., via $ko^*(BSD_{16})$ by using the Greenlees spectral sequence and via $ku_*(BSD_{16})$ by using the η-Bockstein spectral sequence) must agree. However, the extension problem in degree $8k+1$ for all $k \geq 1$ is still to be a problem by this calculation, but fortunately, this problem can be sorted out by the results of D.Bayen thesis, [7].

§ 6.1 The strategy of calculation

For E_1-page, at the zero line, it is simply to lay $ku_*(BSD_{16})$ down degree by degree. To fill elements in positive filtration, we merely copy elements in the zero line along diagonal via $\widetilde{\eta}$ (which has bidegree $(1,1)$). So as to calculate d_1 differential, we need to work precisely on elements of $ku_*(BSD_{16})$. Recall from chapter 4 that $ku_*(BSD_{16})$ is separated by two parts, i.e., even degree part and odd degree part. Evidently, we have, (in this chapter we refer R to be $ku^*(BSD_{16})$);

$$ku_{2k-1}(BSD_{16}) \cong H_I^1(R)_{2k},$$

and

$$ku_{2k}(BSD_{16}) \cong \mathbb{Z}[v^k] \oplus [H_I^2(R)]_{2k+2}.$$

Actually, in even degree part, we have a short exact sequence;

$$0 \longrightarrow [H_I^2(R)]_{*+2} \longrightarrow ku_*(BSD_{16}) \longrightarrow [H_I^0(R)]_* \longrightarrow 0. \tag{6.1}$$

Note further that $H^1_I(R)_{2k}$ and $[H^0_I(R)]_{2k}$ for all $k \geq 0$ come from QU part, the image of $ku^*(BSD_{16}))$ in $KO^*(BSD_{16})$. Patently, $[H^0_I(R)]_{2k} \subseteq QU_{2k}$ and $H^1_I(R)_{2k}$ is a quotient of $(QU[\frac{1}{p}])_{2k}$. To identify the d_1 differential on these parts, we now recall lemma 4.1.1 which asserts that

$$d_1 = \begin{cases} 1+\tau, & \text{if } d_1 \text{ departs from } QU_{4k+2}\ ; \\ 1-\tau, & \text{if } d_1 \text{ departs from } QU_{4k}. \end{cases} \tag{6.2}$$

This operation is also compatible with $QU[\frac{1}{p}]_{2k} \twoheadrightarrow H^1_I(R)_{2k}$, see [13]. Hence, we have;

$$d_1 = \begin{cases} 1+\tau, & \text{if } d_1 \text{ departs from } ku_{4k-3}(BSD_{16})\ ; \\ 1-\tau, & \text{if } d_1 \text{ departs from } ku_{4k-1}(BSD_{16}). \end{cases} \tag{6.3}$$

Clearly, d_1 departing from $[H^0_I(R)]_{2k} \subseteq QO_{2k}$ is the same map as (6.2).

For the differential on the $[H^2_I(R)]_*$ part, we note that this part is calculated from $TU \subseteq H^*(BSD_{16};\mathbb{F}_2)$ by using the Greenlees spectral sequence. Thus $[H^2_I(R)]_{*+2}$ can be identified to be the subset of $H_*(BSD_{16};\mathbb{F}_2)$. Furthermore, we have that d_1 of η-Bockstein spectral sequence on $TU \subseteq H^*(BSD_{16};\mathbb{F}_2)$ is Sq^2 operation, by lemma 4.1.1, and $H_*(BSD_{16};\mathbb{F}_2)$ is the dual of $H^*(BSD_{16};\mathbb{F}_2)$. Therefore, d_1 of η-Bockstein spectral sequence on $[H^2_I(R)]_{*+2} \hookrightarrow H_*(BSD_{16};\mathbb{F}_2)$ is the dual of Sq^2, see [13]. Namely,

$$d_1 = (Sq^2)^\vee \text{ on } [H^2_I(R)]_*, \tag{6.4}$$

where $(Sq^2)^\vee$ is the dual operation of Sq^2.

As the previous technique, we calculate d_1 on $ku_{even}(BSD_{16})$ via the long exact sequence induced by (6.1). Precisely, we calculate d_1 differential on $[H^2_I(R)]_*$, $[H^0_I(R)]_*$ and $ku_{odd}(BSD_{16})$ by using (6.4), (6.2), (6.3), respectively and record as $E_{1\frac{1}{2}}$-page. After we determine the connecting homomorphism in the long exact sequence, we obtain E_2-page. Finally, the results in Theorem 5.5.1 treat the differential d_2 and d_3 and then we obtain E_∞-page as required.

We now start with the even degree part.

§ 6.2 The d_1 differential on $ku_{even}(BSD_{16})$

Differential d_1 on $[H^0_I(R)]_*$ is simple by (6.2) which gives;

$$Z([H^0_I(R)])_{2k} := \ker(d_1 : [H^0_I(R)]_{2k} \longrightarrow [H^0_I(R)]_{2k-2}) = \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd,} \end{cases} \tag{6.5}$$

and

$$H([H^0_I(R)])_{2k} := Z([H^0_I(R)])_{2k}/d_1([H^0_I(R)]_{2k+2}) = \begin{cases} \mathbb{Z}/2, & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd.} \end{cases} \tag{6.6}$$

So, the main task in this section is to compute d_1 on $[H^2_I(R)]_*$. To do this, we need to see the embedding $i^H_* : [H^2_I(R)]_{*+2} \hookrightarrow H_*(BSD_{16};\mathbb{F}_2)$ explicitly and then use the action of $(Sq^2)^\vee$ on $H_*(BSD_{16};\mathbb{F}_2)$ to determine such d_1.

6.2.1 Explicit embedding of $[H^2_I(R)]_{*+2}$ in $H_*(BSD_{16};\mathbb{F}_2)$

To gain the main objective, we need to see the elements of $[H^2_I(R)]_{*+2}$ explicitly first. This is automatic by recalling from lemma 3.4.3 that $H^2_I(R) = (\mathbb{F}_2[b,d])^\vee(\nu)$, where $\nu = \frac{\tau}{b^3d^2}$. In other words, we have;

$$[H^2_I(R)]_* \cong \mathbb{F}_2[\frac{1}{y^2}, \frac{1}{P}] < \frac{1}{Py^2} > . \tag{6.7}$$

However, to be more general, we will make explicit such an embedding of $H^2_{(y,P)}(H^*(BSD_{2^n};\mathbb{F}_2))$ at E_∞-page of the Greenlees spectral sequence in $H_*(BSD_{2^n};\mathbb{F}_2)$. D.Benson calculated this by using the Greenlees spectral sequence in [9] (page 6-8). We record his results by adapting notation to suit our notation as;

Lemma 6.2.1. *(cf.[9]) Let $R' = H^*(BSD_{2^n};\mathbb{F}_2) = \mathbb{F}_2[x,y,u,P]/(x^2+xy, x^3, xu, (x^2+y^2)P+u^2)$ for $n \geq 4$. We have the augmentation ideal $I = \ker(\varphi : R' \longrightarrow \mathbb{F}_2) = (y,P)$ and short exact sequence;*

$$0 \longrightarrow [H^2_I(R')]_{k+2} \longrightarrow H_k(BSD_{2^n};\mathbb{F}_2) \longrightarrow [H^1_I(R')]_{k+1} \longrightarrow 0,$$

where $H^2_I(R') = \mathbb{F}_2[\frac{1}{y}, \frac{1}{P}] < \frac{u}{Py}, \frac{1}{Py} >$ and $H^1_I(R') = \mathbb{F}_2[\frac{1}{P}] < x, x^2 >$. Precisely, $H_k(BSD_{2^n};\mathbb{F}_2) \cong [H^1_I(R')]_{k+1} \oplus [H^2_I(R')]_{k+2}$, for all $k \geq 0$.

Note that $H^2_{(y,P)}(H^*(BSD_{2^n};\mathbb{F}_2))$ and $H_*(BSD_{2^n};\mathbb{F}_2)$ are module over R' via cap product. So, we can determine the embedding by using the action of R' on both sides first and then follow by using comparison on their annihilators. This is possible, because we have;

Lemma 6.2.2. *For vector space V over a field $\mathbb{F}$ of finite dimension n and X a subspace of V, we have;*

$$ann_V(ann_{V^*}(X)) = X,$$

where V^* is the dual space of V.

Proof. First, choose a basis $\{v_1, v_2, ..., v_s\}$ of X and extend to a basis for V as $\{v_i | i = 1, 2, ..., n\}$. Let $\{v_i^* | i = 1, 2, ..., n\}$ be the natural dual basis. It is simple to see that

$$\begin{aligned} ann_{V^*}(X) &= \{\theta \in V^* | \theta(v) = 0, \forall v \in X\} \\ &= Span(\{v_i^* | i \in \{s+1, s+2, s+3, ..., n\}\}), \end{aligned}$$

since $v_i^*(v_j) = \delta_{ij}$, (the Kronecker delta). Then, we get;

$$\begin{aligned} ann_V(ann_{V^*}(X)) &= \{w \in V | \theta(w) = 0, \forall \theta \in ann_{V^*}(X)\} \\ &= \{w \in V | \theta(w) = 0, \forall \theta \in Span(\{v_i^* | i \in \{s+1, s+2, s+3, ..., n\}\})\} \\ &= Span(\{v_j | j \in \{v_1, v_2, ..., v_s\}\}) \\ &= X, \end{aligned}$$

which completes the proof. $\square$

The consequence of this lemma is that;

Proposition 6.2.3. *If G is a group with $R' := H^*(BG;\mathbb{F}_2)$ finite generated in each degree, then for $g,h \in H_k(BG;\mathbb{F}_2)$, for some k, we have;*

$$ann_{R'}(g) = ann_{R'}(h) \Leftrightarrow g = h.$$

Proof. By using lemma 6.2.2 apply to $V = H_k(BG;\mathbb{F}_2)$ over the filed $\mathbb{F}_2$, we obtain

$$\begin{aligned} ann_{R'}(g) = ann_{R'}(h) &\Rightarrow [ann_{R'}(g)]_k = [ann_{R'}(h)]_k \\ &\Rightarrow ann_{R'_k}(g) = ann_{R'_k}(h) \\ &\Rightarrow ann_{(R'_k)^*}ann_{R'_k}(g) = ann_{(R'_k)^*}ann_{R'_k}(h) \\ &\Rightarrow Span_{\mathbb{F}_2}(\{g\}) = Span_{\mathbb{F}_2}(\{h\}) \\ &\Rightarrow g = h, \end{aligned}$$

as required, since $H^k(BG;\mathbb{F}_2) = (H_k(BG;\mathbb{F}_2))^*$. □

From the proof of this proposition, we see that it is enough to calculate $[ann_{R'}(g)]_k = ann_{R'_k}(g)$ and $[ann_{R'}(h)]_k = ann_{R'_k}(h)$ to conclude that $g = h$. Thus, by this facts, lemma 6.2.1 and straightforward calculation, we get;

Lemma 6.2.4. *For each $k \geq 0$, $[H_I^2(H^*(BSD_{2^n};\mathbb{F}_2))]_{k+2}$ is generated by*

$$\{\frac{u}{P^{j+1}y^{k-4j+1}} | 0 \leq j \leq L(\frac{k}{4}), k \geq 0\} \cup \{\frac{1}{P^j y^{k-4j+2}} | 1 \leq j \leq L(\frac{k+1}{4}), k \geq 3\},$$

with the explicit inclusion to $H_k(BSD_{2^n};\mathbb{F}_2)$ given by

$$\begin{aligned} \frac{u}{P^{j+1}y^{k-4j+1}} &\mapsto (P^j y^{k-4j})^\vee \\ \frac{1}{P^j y^{k-4j+2}} &\mapsto (uP^{j-1}y^{k-4j+1})^\vee, \end{aligned}$$

where $L(r) :=$ greatest integer which is less than or equal to r and $(\alpha)^\vee \in H_k(BSD_{2^n};\mathbb{F}_2)$ is the natural dual of $\alpha \in H^k(BSD_{2^n};\mathbb{F}_2)$.

In particular, by (6.7) and lemma 6.2.4 above, we have that $[H_I^2(R)]_{k+2}$ is zero unless k is even. Also, for any even integer $k \geq 4$, the explicit inclusion of $[H_I^2(R)]_{k+2} \hookrightarrow H_k(BSD_{2^n};\mathbb{F}_2)$ is given by

$$\frac{1}{P^j y^{k-4j+2}} \mapsto (uP^{j-1}y^{k-4j+1})^\vee. \tag{6.8}$$

Now, we ready to calculate d_1 on $[H_I^2(R)]_*$ which we do this in the next subsection.

6.2.2 d_1 ON $[H_I^2(R)]_{*+2}$

Now, we pay attention to the elements in $H^*(BSD_{2^n};\mathbb{F}_2)$ being of the form $(uP^{j-1}y^{k-4j+1})^\vee$ (since (6.8)) for each even integer $k \geq 4$ and regard other elements as zero. To find $(Sq^2)^\vee$ operation on these elements, we firstly calculate Sq^2 operation on $uP^{j-1}y^{k-4j+1}$. This is simple since we have: (recall from Proposition 2.2.5)

$$Sq^1(x) = x^2, Sq^1(y) = y^2, Sq^1(u) = 0, Sq^1(P) = 0$$

and

$$Sq^2(x) = 0, Sq^2(y) = 0, Sq^2(u) = Px + Py + uy^2, Sq^2(P) = Px^2 + Py^2.$$

By using Cartan's formula, we have;

$$\begin{aligned}
Sq^1(P^i) &= 0, \text{for all } i,\\
Sq^2(P^i) &= \begin{cases} P^ix^2 + P^iy^2, & \text{if } i \text{ is odd;}\\ 0, & \text{if } i \text{ is even,}\end{cases}\\
Sq^2(y^{2i}) &= \begin{cases} y^{2i+2}, & \text{if } i \text{ is odd;}\\ 0, & \text{if } i \text{ is even,}\end{cases}\\
Sq^2(uy^{2i+1}) &= \begin{cases} Pxy^{2i+1} + Py^{2i+2}, & \text{if } i \text{ is odd;}\\ Pxy^{2i+1} + Py^{2i+2} + uy^{2i+3}, & \text{if } i \text{ is even.}\end{cases}
\end{aligned}$$

Thus, setting $S := Sq^2(uP^{j-1}y^{k-4j+1})$,

$$\begin{aligned}
S &= P^{j-1}Sq^2(uy^{k-4j+1}) + Sq^1(P^{j-1})Sq^1(uy^{k-4j+1}) + uy^{k-4j+1}Sq^2(P^{j-1})\\
&= P^{j-1}Sq^2(uy^{k-4j+1}) + uy^{k-4j+1}Sq^2(P^{j-1})\\
&= \begin{cases} (P^{j-1})[Pxy^{k-4j+1} + Py^{k-4j+2}], & \text{if } k-4j \equiv 2mod4;\\ (P^{j-1})[Pxy^{k-4j+1} + Py^{k-4j+2} + uy^{k-4j+3}], & \text{if } k-4j \equiv 0mod4,\end{cases}\\
&+ \begin{cases} uy^{k-4j+1}(P^{j-1}x^2 + P^{j-1}y^2), & \text{if } j-1 \text{ is odd;}\\ 0, & \text{if } j-1 \text{ is even,}\end{cases}\\
&= \begin{cases} P^jxy^{k-4j+1} + P^jy^{k-4j+2} + P^{j-1}uy^{k-4j+3}, & \text{if } k+2j \equiv 2mod4;\\ P^jxy^{k-4j+1} + P^jy^{k-4j+2}, & \text{if } k+2j \equiv 0mod4.\end{cases}
\end{aligned}$$

Since we regard that all elements are zero except elements which are in the form $uP^{j-1}y^{k-4j+1}$, we get that

$$Sq^2(uP^{j-1}y^{k-4j+1}) = \begin{cases} uP^{j-1}y^{k-4j+3}, & \text{if } k+2j \equiv 2mod4;\\ 0, & \text{if } k+2j \equiv 0mod4.\end{cases}$$

Therefore,

$$(Sq^2)^\vee((uP^{j-1}y^{k-4j+3})^\vee) = \begin{cases} (uP^{j-1}y^{k-4j+1})^\vee, & \text{if } k+2j \equiv 2mod4;\\ 0, & \text{if } k+2j \equiv 0mod4,\end{cases}$$

or in other words, by changing k to $k-2$ (in order to suit (6.8),

$$(Sq^2)^\vee((uP^{j-1}y^{k-4j+1})^\vee) = \begin{cases} (uP^{j-1}y^{k-4j-1})^\vee, & \text{if } k+2j \equiv 0mod4;\\ 0, & \text{if } k+2j \equiv 2mod4,\end{cases}$$

and hence, by (6.8),

$$(Sq^2)^\vee\left(\frac{1}{P^j y^{k-4j+2}}\right) = \begin{cases} \frac{1}{P^j y^{k-4j}}, & \text{if } k+2j \equiv 0 mod 4 \text{ and } k \neq 4j; \\ 0, & \text{if } k+2j \equiv 2 mod 4; \\ 0, & \text{if } k = 4j, \end{cases} \tag{6.9}$$

where the third condition comes from the fact that $\frac{1}{P^j}$ for each $j \geq 1$ is not contained in $[H_I^2(R)]_*$. Here, we illustrate the diagram of $d_1 = (Sq^2)^\vee$ action on $[H_I^2(R)]_*$ in low degree as follows.

degree k	**Generator of** $[H_I^2(R)]_{k+2}$							
4	$\frac{1}{Py^2}$							
	$\uparrow$							
6	$\frac{1}{Py^4}$							
8	$\frac{1}{Py^6}$	$\frac{1}{P^2y^2}$						
	$\uparrow$							
10	$\frac{1}{Py^8}$	$\frac{1}{P^2y^4}$						
		$\uparrow$						
12	$\frac{1}{Py^{10}}$	$\frac{1}{P^2y^6}$	$\frac{1}{P^3y^2}$					
	$\uparrow$		$\uparrow$					
14	$\frac{1}{Py^{12}}$	$\frac{1}{P^2y^8}$	$\frac{1}{P^3y^4}$					
		$\uparrow$						
16	$\frac{1}{Py^{14}}$	$\frac{1}{P^2y^{10}}$	$\frac{1}{P^3y^6}$	$\frac{1}{P^4y^2}$				
	$\uparrow$		$\uparrow$					
18	$\frac{1}{Py^{16}}$	$\frac{1}{P^2y^{12}}$	$\frac{1}{P^3y^8}$	$\frac{1}{P^4y^4}$				
		$\uparrow$		$\uparrow$				
20	$\frac{1}{Py^{18}}$	$\frac{1}{P^2y^{14}}$	$\frac{1}{P^3y^{10}}$	$\frac{1}{P^4y^6}$	$\frac{1}{P^5y^2}$			
	$\uparrow$		$\uparrow$		$\uparrow$			
22	$\frac{1}{Py^{20}}$	$\frac{1}{P^2y^{16}}$	$\frac{1}{P^3y^{12}}$	$\frac{1}{P^4y^8}$	$\frac{1}{P^5y^4}$			
		$\uparrow$		$\uparrow$				
24	$\frac{1}{Py^{22}}$	$\frac{1}{P^2y^{18}}$	$\frac{1}{P^3y^{14}}$	$\frac{1}{P^4y^{10}}$	$\frac{1}{P^5y^6}$	$\frac{1}{P^6y^2}$		
	$\uparrow$		$\uparrow$		$\uparrow$			
26	$\frac{1}{Py^{24}}$	$\frac{1}{P^2y^{20}}$	$\frac{1}{P^3y^{16}}$	$\frac{1}{P^4y^{12}}$	$\frac{1}{P^5y^8}$	$\frac{1}{P^6y^4}$		
		$\uparrow$		$\uparrow$		$\uparrow$		
28	$\frac{1}{Py^{26}}$	$\frac{1}{P^2y^{22}}$	$\frac{1}{P^3y^{18}}$	$\frac{1}{P^4y^{14}}$	$\frac{1}{P^5y^{10}}$	$\frac{1}{P^6y^8}$	$\frac{1}{P^7y^2}$	
	$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$	
30	$\frac{1}{Py^{28}}$	$\frac{1}{P^2y^{24}}$	$\frac{1}{P^3y^{20}}$	$\frac{1}{P^4y^{16}}$	$\frac{1}{P^5y^{12}}$	$\frac{1}{P^6y^{10}}$	$\frac{1}{P^7y^4}$	
		$\uparrow$		$\uparrow$		$\uparrow$		
32	$\frac{1}{Py^{30}}$	$\frac{1}{P^2y^{26}}$	$\frac{1}{P^3y^{22}}$	$\frac{1}{P^4y^{18}}$	$\frac{1}{P^5y^{14}}$	$\frac{1}{P^6y^{12}}$	$\frac{1}{P^7y^6}$	$\frac{1}{P^8y^2}$
	$\uparrow$		$\uparrow$		$\uparrow$		$\uparrow$	

Diagram 6.1: The $d_1 = (Sq^2)^\vee (:=\uparrow)$ action on $[H_I^2(R)]_*$.

Now, we have;

Lemma 6.2.5. *With the notation above,*

1) *The* $(Sq^2)^\vee-$*homology of* $[H^2_I(R)]_*$ *is represented by* $\mathbb{F}_2[\frac{1}{P^2}] < \frac{1}{P^2y^2} >$.

2) *The* $(Sq^2)^\vee-$*cycles of* $[H^2_I(R)]_*$ *is represented by*

$$\mathbb{F}_2[\frac{1}{y^4}, \frac{1}{P^2}] < \frac{1}{Py^2}, \frac{1}{P^2y^4}, \frac{1}{P^2y^2} > /\mathbb{F}_2[\frac{1}{y^4}] < \frac{1}{P^2y^6} > .$$

Proof. Diagram 6.1 suggests the pattern, and the proof is the immediate result from (6.9). □

§ 6.3 d_1 differentials on $ku_{odd}(BSD_{16})$

Since we have (6.3) and Theorem 3.5.1, the calculation of kernel and homology of d_1 on $ku_{odd}(BSD_{16})$ is straightforward but needs careful work. This section is devoted to the proof of lemma 6.3.1 below.

Lemma 6.3.1. *For positive odd integers* s, *let* $M_s := ku_s(BSD_{16})$. *Let* $Z(M)_s :=$ $\ker(d_1 : M_s \longrightarrow M_{s-2})$ *and* $H(M)_s := Z(M)_s/d_1(M_{s+2})$. *Using the same symbols as in Theorem 3.5.1, we have:*

- $Z(M)_1 \cong [2] \oplus [2]$ *generated by* x_1, y_1; $H(M)_1 \cong [2] \oplus [2]$ *generated by* $[x_1], [y_1]$.
- $Z(M)_3 \cong [2] \oplus [4] \oplus [8]$ *generated by* x_2, y_2, w_2, *resp.;* $H(M)_3 = 0$.
- $Z(M)_5 \cong [2] \oplus [2] \oplus [2]$ *generated by* $2x_3, 4y_3, 8u_3$, *resp.;* $H(M)_5 = [2] \oplus [2]$ *generated by* $[2x_3]$ *and* $[4y_3]$.
- $Z(M)_7 \cong [2] \oplus [4] \oplus [16] \oplus [32]$ *generated by* $2w_4, x_4, y_4, u_4$, *resp.;* $H(M)_7 = 0$.
- $Z(M)_9 \cong [2] \oplus [2] \oplus [2] \oplus [2]$ *generated by* $2z_5, 4x_5, 16y_5, 32u_5$, *resp.;* $H(M)_9 =$ $[2] \oplus [2] \oplus [2]$ *generated by* $[2z_5], [4x_5], [16y_5]$.
- $Z(M)_{11} \cong [8] \oplus [8] \oplus [64] \oplus [128]$ *generated by* $x_6, t_6, y_6, u_6 + w_6$, *resp.;* $H(M)_{11} \cong$ $[2]$ *generated by* $[t_6]$.
- $Z(M)_{13} \cong [2] \oplus [2] \oplus [2] \oplus [4]$ *generated by* $8x_7, 64y_7, 4z_7, w'_7 = w_7 - 4z_7 + 2t_7 + 64u_7$, *resp.;* $H(M)_{13} = [2] \oplus [2] \oplus [2]$ *generated by* $[8x_7], [64y_7], [4z_7]$.
- $Z(M)_{15} \cong [2] \oplus [8] \oplus [16] \oplus [256] \oplus [512]$ *generated by* $t_8, 2w_8 + z_8, x_8, y_8, u_8$, *resp.;* $H(M)_{15} = [2]$ *generated by* $[t_8]$.
- *For* $n \geq 3$, $Z(M)_{8n-7} \cong [2] \oplus [2] \oplus [2] \oplus [2] \oplus [2^{n-1}]$ *generated by* $4^{n-1}x_{4n-3}$, $16^{n-1}y_{4n-3}, 2 \cdot 4^{n-2}z_{4n-3}, 2^{n-3}t_{4n-3}$ *and* $4(-8)^{n-1}u_{4n-3} + (-2)^n z_{4n-3} - w_{4n-3} :=$ u'_{4n-3} *resp.;* $H(M)_{8n-7} \cong [2] \oplus [2] \oplus [2] \oplus [2]$ *generated by* $[4^{n-1}x_{4n-3}], [16^{n-1}y_{4n-3}]$, $[2 \cdot 4^{n-2}z_{4n-3}], [2^{n-3}t_{4n-3}]$.

- *For* $n \geq 3$, $Z(M)_{8n-5} \cong [2^{n-2}] \oplus [2 \cdot 4^{n-1}] \oplus [2 \cdot 4^{n-1}] \oplus [4 \cdot 16^{n-1}] \oplus [8 \cdot 16^{n-1}]$ *generated by* $z_{4n-2}, t_{4n-2}, x_{4n-2}, y_{4n-2}, w_{4n-2} - u_{4n-2}$ *resp.;* $H(M)_{8n-5} = [2] \oplus [2]$ *generated by* $[t_{4n-2}]$ *and* $[z_{4n-2}]$.

- *For* $n \geq 3$, $Z(M)_{8n-3} \cong [2] \oplus [2] \oplus [2] \oplus [2] \oplus [2^n]$ *generated by* $2 \cdot 4^{n-1} x_{4n-1}$, $4 \cdot 16^{n-1} y_{4n-1}$, $4^{n-1} z_{4n-1}$, $2^{n-3} t_{4n-1} + 2^{n-2} w_{4n-1}$, *and* $w_{4n-1} + (-8)^n u_{4n-1} - (-2)^n z_{4n-1}$ *resp.;* $H(M)_{8n-3} \cong [2] \oplus [2] \oplus [2] \oplus [2]$ *generated by* $[2 \cdot 4^{n-1} x_{4n-1}]$, $[4 \cdot 16^{n-1} y_{4n-1}]$, $[4^{n-1} z_{4n-1}]$, *and* $[2^{n-3} t_{4n-1} + 2^{n-2} w_{4n-1}]$.

- *For* $n \geq 4$, $Z(M)_{8n-9} \cong [2^{n-2}] \oplus [2 \cdot 4^{n-2}] \oplus [4^{n-1}] \oplus [16^{n-1}] \oplus [2 \cdot 16^{n-1}]$ *generated by* $t_{4n-4}, 2w_{4n-4} + z_{4n-4}, x_{4n-4}, y_{4n-4}, u_{4n-4}$ *resp.;* $H(M)_{8n-9} = [2]$ *generated by* $[t_{4n-4}]$.

Proof. We use the same natation as in Theorem 3.5.1 and lemma 3.3.4, i.e., for each $\alpha \in \{x, y, z, t, u, w\}$ and for $\epsilon = 2, 3, 4$, $\widetilde{\alpha}_{4i-\epsilon} = \frac{\overline{\alpha}_\epsilon}{p^i} + \overline{R}_0 \cong \overline{\alpha}_\epsilon + p^i \cdot \overline{R}_0$ for all $i \geq 1$, and $\widetilde{\alpha}_{4j-5} = \frac{\overline{\alpha}_5}{p^j} + \overline{R}_0 \cong \overline{\alpha}_5 + p^j \cdot \overline{R}_0$, for all $j \geq 2$, with $p = [16, -2, 4, -2, 16, 4]$. So, it is possible to use the character table to calculate the kernel and homology of d_1.

Degree 1: $M_1 = ku_1(BSD_{16}) \cong [2] \oplus [2]$ generated by x_1, y_1, with $z_1 = t_1 = w_1 = u_1 + x_1 + y_1 = 0$, where

$$\begin{aligned}
x_1 = \widetilde{x}_1 + \widetilde{z}_1 &= [0, 0, 2, 0, 0, 1] + \overline{R}_0 \\
y_1 = 2\widetilde{y}_1 &= [0, 0, 0, 0, 1, 0] + \overline{R}_0 \\
z_1 = 2\widetilde{z}_1 &= [0, 0, 4, 0, 0, 0] + \overline{R}_0 \\
w_1 = \widetilde{w}_1 + \widetilde{z}_1 &= [0, -c, 2, c, 0, 0] + \overline{R}_0
\end{aligned}$$

It is clear that $d_1(M_1) = (1 + \tau)(M_1) = 0$, because $M_{-1} = 0$.

Degree 3: $M_3 = ku_3(BSD_{16}) \cong [2] \oplus [4] \oplus [8]$ generated by x_2, y_2, w_2 resp., with $z_2 = 4w_2$, $t_2 = 2(w_2 + y_2)$ and $u_2 + 4w_2 = 0$, where

$$\begin{aligned}
x_2 = \widetilde{x}_2 &= [0, 0, 0, 0, 0, 1] + \overline{R}_0 \\
y_2 = 2\widetilde{y}_2 &= [0, 0, 0, 0, \frac{1}{2}, 0] + \overline{R}_0 \\
z_2 = \widetilde{z}_2 &= [0, 0, 2, 0, 0, 0] + \overline{R}_0 \\
t_2 = \widetilde{t}_2 &= [0, -1, 1, -1, 0, 0] + \overline{R}_0 \\
u_2 = \widetilde{u}_2 &= [0, -c, 0, c, 0, 0] + \overline{R}_0 \\
w_2 = 2\widetilde{w}_2 &= [\frac{1}{2}, -c, 1, c, 0, 1] + \overline{R}_0
\end{aligned}$$

It is simple to see that $(1 - \tau)(x_2) = (1 - \tau)(y_2) = 0$ and $(1 - \tau)(w_2) = 2w_1 - z_1 \equiv 0$.

Degree 5: $M_5 = ku_5(BSD_{16}) \cong [4] \oplus [8] \oplus [16]$ generated by x_3, y_3, u_3 resp., with $w_3 = 8u_3$, $z_3 = 2x_3 + 4y_3 + 4u_3$ and $t_3 = 0$, where

$$\begin{aligned}
x_3 = \widetilde{x}_3 + \widetilde{t}_3 &= [0,-1,0,-1,0,\frac{1}{2}] + \overline{R}_0 \\
y_3 = 2\widetilde{y}_3 &= [0,0,0,0,\frac{1}{4},0] + \overline{R}_0 \\
z_3 = \widetilde{z}_3 + \widetilde{t}_3 &= [0,-1,1,-1,0,0] + \overline{R}_0 \\
t_3 = 2\widetilde{t}_3 &= [0,-2,0,-2,0,0] + \overline{R}_0 \\
u_3 = \widetilde{u}_3 &= [\frac{1}{4},\frac{-c}{2},\frac{-1}{2},\frac{c}{2},0,0] + \overline{R}_0 \\
w_3 = \widetilde{w}_3 &= [0,c,0,-c,0,0] + \overline{R}_0
\end{aligned}$$

Now, we see that $(1+\tau)(x_3) = 2t_2 - z_2 + x_2 \equiv x_2$, $(1+\tau)(y_3) = y_2$ and $(1+\tau)(w_3) = w_2 - u_2 - z_2 - x_2 \equiv w_2 - x_2$. Then $2x_3, 4y_3, 8u_3 \in Z(M)_5$ and $H(M)_3 = 0$.

Degree 7: $M_7 = ku_7(BSD_{16}) \cong [4]\oplus[4]\oplus[16]\oplus[32]$ generated by w_4, x_4, y_4, u_4 resp., with $z_4 + 2w_4 + 8u_4 = 0$ and $t_4 + 2x_4 + z_4 + 8y_4 + 8u_4 = 0$, where

$$\begin{aligned}
x_4 = \widetilde{x}_4 &= [0,0,0,0,0,\frac{1}{2}] + \overline{R}_0 \\
y_4 = 2\widetilde{y}_4 &= [0,0,0,0,\frac{1}{8},0] + \overline{R}_0 \\
z_4 = \widetilde{z}_4 &= [0,0,1,0,0,0] + \overline{R}_0 \\
t_4 = \widetilde{t}_4 &= [0,1,0,1,0,0] + \overline{R}_0 \\
u_4 = 2\widetilde{u}_4 &= [\frac{1}{8},-1,\frac{1}{2},-1,0,\frac{1}{2}] + \overline{R}_0 \\
w_4 = \widetilde{w}_4 - 8\widetilde{u}_4 &= [\frac{-1}{2},\frac{c}{2}+4,\frac{-3}{2},\frac{-c}{2}+4,0,-2] + \overline{R}_0
\end{aligned}$$

Now, we see that $(1-\tau)(x_4) = (1-\tau)(y_4) = (1-\tau)(u_4) = 0$ and $(1-\tau)(w_4) = w_3 \equiv 8u_3$. Then $2w_4, x_4, y_4, u_4 \in Z(M)_7$ and $H(M)_5 = [2]\oplus[2]$ generated by $[2x_3]$ and $[4y_3]$.

Degree 9: $M_9 = ku_9(BSD_{16}) \cong [4]\oplus[8]\oplus[32]\oplus[64]$ generated by z_5, x_5, y_5, u_5 resp., with $w_5 = 0$ and $t_5 + 4x_5 - 2z_5 + 16y_5 - 24u_5 = 0$, where

$$\begin{aligned}
x_5 = \widetilde{x}_5 + \widetilde{z}_5 + 8\widetilde{u}_5 &= [\frac{1}{2},-4,\frac{-3}{2},-4,0,\frac{1}{4}] + \overline{R}_0 \\
y_5 = 2\widetilde{y}_5 &= [0,0,0,0,\frac{1}{16},0] + \overline{R}_0 \\
z_5 = \widetilde{z}_5 + \widetilde{w}_5 + 24\widetilde{u}_5 &= [\frac{3}{2},\frac{c}{2}-12,\frac{-11}{2},\frac{-c}{2}-12,0,0] + \overline{R}_0 \\
t_5 = \widetilde{t}_5 &= [0,1,0,1,0,0] + \overline{R}_0 \\
u_5 = \widetilde{u}_5 &= [\frac{1}{16},\frac{-1}{2},\frac{-1}{4},\frac{-1}{2},0,0] + \overline{R}_0 \\
w_5 = 2(\widetilde{w}_5 + 16\widetilde{u}_5) &= [2,c-16,-8,-c-16,0,0] + \overline{R}_0
\end{aligned}$$

Now, we see that $(1+\tau)(x_5) = 8u_4 - 7x_4 - 7z_4 \equiv 8u_4 + x_4 + z_4 \equiv 2w_4 + x_4$, $(1+\tau)(y_5) = y_4$, $(1+\tau)(z_5) = 2w_4 + 32u_4 - 14z_4 \equiv 16u_4 + 2w_4$, (since $2z_4 \equiv 16u_4$) and

$(1+\tau)(u_5) = u_4 - x_4 - z_4 \equiv 9u_4 - x_4 + 2w_4$. Then $4x_5, 16y_5, 2z_5, 32u_5 \in Z(M)_9$ and $H(M)_7$ is calculated by the representing matrix below;

$$\begin{array}{rcccccl} & [2] & [4] & [16] & [32] & \\ & 2w_4 & x_4 & y_4 & u_4 & \\ (1+\tau)(x_5):| & 1 & 1 & 0 & 0 & |r_1 \\ (1+\tau)(y_5):| & 0 & 0 & 1 & 0 & |r_2 \\ (1+\tau)(z_5):| & 1 & 0 & 0 & 16 & |r_3 \\ (1+\tau)(u_5):| & 1 & -1 & 0 & 9 & |r_4 \end{array}$$

which is easy to see that $H(M)_7 = 0$.

Degree 11: $M_{11} = ku_{11}(BSD_{16}) \cong [2] \oplus [8] \oplus [8] \oplus [64] \oplus [128]$ generated by u_6, t_6, x_6, y_6, w_6 resp., with $z_6 + 4x_6 + 32y_6 + 2t_6 + 16w_6 = 0$, where

$$\begin{aligned} x_6 = \widetilde{x}_6 &= [0,0,0,0,0,\frac{1}{4}] + \overline{R}_0 \\ y_6 = 2\widetilde{y}_6 &= [0,0,0,0,\frac{1}{32},0] + \overline{R}_0 \\ z_6 = \widetilde{z}_6 &= [0,0,\frac{1}{2},0,0,0] + \overline{R}_0 \\ t_6 = \widetilde{t}_6 + 16\widetilde{w}_6 &= [\frac{1}{4}, 4c + \frac{1}{2}, \frac{9}{4}, -4c + \frac{1}{2}, 0, 2] + \overline{R}_0 \\ u_6 = \widetilde{u}_6 + 64\widetilde{w}_6 &= [1, \frac{33c}{2}, 8, \frac{-33c}{2}, 0, 8] + \overline{R}_0 \\ w_6 = 2\widetilde{w}_6 &= [\frac{1}{32}, \frac{c}{2}, \frac{1}{4}, \frac{-c}{2}, 0, \frac{1}{4}] + \overline{R}_0 \end{aligned}$$

Now, we see that $(1-\tau)(x_6) = (1-\tau)(y_6) = (1-\tau)(z_6) = 0$, $(1-\tau)(t_6) = 8(w_5 - 32u_5) \equiv 0$, $(1-\tau)(u_6) = 33(w_5 - 32u_5) \equiv 32u_5$ (since $w_5 = 0$) and $(1-\tau)(w_6) = 32u_5$. Then $x_6, t_6, y_6, u_6 + w_6 \in Z(M)_{11}$ and $H(M)_9 = [2] \oplus [2] \oplus [2]$ generated by $[4x_5], [16y_5], [2z_5]$, because $32u_5$ has been detected.

Degree 13: $M_{13} = ku_{13}(BSD_{16}) \cong [2] \oplus [8] \oplus [16] \oplus [128] \oplus [256]$ generated by w_7, z_7, x_7, y_7, u_7, with $t_7 = 0$, where $(\widetilde{\eta}_7 = \widetilde{t}_7 + 4\widetilde{x}_7 + 2\widetilde{z}_7 + 64\widetilde{y}_7 - 32\widetilde{u}_7)$

$$\begin{aligned} x_7 = \widetilde{x}_7 + \widetilde{z}_7 + 16\widetilde{u}_7 &= [\frac{1}{4}, 4c, \frac{-7}{4}, -4c, 0, \frac{1}{8}] + \overline{R}_0 \\ y_7 = 2\widetilde{y}_7 &= [0,0,0,0,\frac{1}{64},0] + \overline{R}_0 \\ z_7 = \widetilde{z}_7 + \widetilde{\eta}_7 + 16\widetilde{u}_7 &= [\frac{-1}{4}, \frac{1}{2} - 4c, \frac{11}{4}, \frac{1}{2} + 4c, \frac{1}{2}, \frac{1}{2}] + \overline{R}_0 \\ t_7 = 2\widetilde{\eta}_7 &= [-1, 1 - 16c, 9, 1 + 16c, 1, 1] + \overline{R}_0 \\ u_7 = \widetilde{u}_7 &= [\frac{1}{64}, \frac{c}{4}, \frac{-1}{8}, \frac{-c}{4}, 0, 0] + \overline{R}_0 \\ w_7 = \widetilde{w}_7 + 4\widetilde{z}_7 &= [0, \frac{-c}{2}, 1, \frac{c}{2}, 0, 0] + \overline{R}_0 \end{aligned}$$

Now observe that $w_7' = w_7 - 4z_7 + 2t_7 + 64u_7 = [0, \frac{-c}{2}, 0, \frac{c}{2}, 0, 0] + \overline{R}_0$ and $4w_7' = 0$ but $2w_7' \neq 0$. Then $(1+\tau)(w_7') = 0$ and then $w_7' \in Z(M)_{13}$ having degree 4. Moreover, we see that $(1+\tau)(x_7) = 16w_6 - 16(u_6 - 32w_6) - 15x_6 - 15z_6 \equiv 16(u_6+w_6) + x_6 + z_6 \equiv 32y_6 - 2t_6 - 3x_6$, (since $z_6 + 4x_6 + 32y_6 + 2t_6 + 16w_6 = 0$), $(1+\tau)(y_7) = y_6$, $(1+\tau)(z_7) = 2t_6 - u_6 + 17(u_6 - 32w_6) + 18z_6 - 12x_6 + 32y_6 \equiv 2t_6 - 32w_6 + 2z_6 + 4x_6 + 32y_6 \equiv 4x_6 + 32y_6 - 2t_6 - 32w_6$, $(1+\tau)(w_7) = 4z_6 \equiv 64w_6$ and $(1+\tau)(u_7) = w_6 - (u_6 - 32w_6) - x_6 - z_6 \equiv 49(u_6+w_6) + 2t_6 + 3x_6 + 32y_6$. Thus $8x_7, 64y_7, 4z_7, 128u_7 \equiv 2w_7'$ are also in $Z(M)_{13}$. Now, $H(M)_{11}$ is calculated by the representing matrix below;

$$
\begin{array}{rrrrrl}
 & [8] & [8] & [64] & [128] & \\
 & t_6 & x_6 & y_6 & u_6+w_6 & \\
(1+\tau)(x_7): | & -2 & -3 & 32 & 0 & |r_1 \\
(1+\tau)(y_7): | & 0 & 0 & 1 & 0 & |r_2 \\
(1+\tau)(z_7): | & -2 & 4 & 32 & -32 & |r_3 \\
(1+\tau)(w_7): | & 0 & 0 & 0 & 64 & |r_4 \\
(1+\tau)(u_7): | & 2 & 3 & 32 & 49 & |r_5
\end{array}
$$

which is equivalent to (by row operations)

$$
\begin{array}{rrrrrl}
 & [8] & [8] & [64] & [128] & \\
 & t_6 & x_6 & y_6 & u_6+w_6 & \\
| & 0 & 1 & 0 & 0 & |r_1' \\
| & 0 & 0 & 1 & 0 & |r_2' \\
| & -2 & 0 & 0 & 0 & |r_3' \\
| & 0 & 0 & 0 & 0 & |r_4' \\
| & 0 & 0 & 0 & 49 & |r_5'
\end{array}
$$

and hence $H(M)_{11} \cong [2]$ generated by $[t_6]$.

Degree 15: $M_{15} = ku_{15}(BSD_{16}) \cong [2]\oplus[2]\oplus[16]\oplus[16]\oplus[256]\oplus[512]$ generated by $t_8, z_8, w_8, x_8, y_8, u_8$ resp., where

$$
\begin{aligned}
x_8 = \widetilde{x}_8 &= [0,0,0,0,0,\frac{1}{8}] + \overline{R}_0 \\
y_8 = 2\widetilde{y}_8 &= [0,0,0,0,\frac{1}{128},0] + \overline{R}_0 \\
z_8 = \widetilde{z}_8 - 2\widetilde{w}_8 - 256\widetilde{u}_8 &= [-1,\frac{c}{2}-64,-16,\frac{-c}{2}-64,0,-16] + \overline{R}_0 \\
t_8 = \widetilde{t}_8 - 2(2\widetilde{x}_8 + \widetilde{z}_8) - 128(\widetilde{y}_8 + \widetilde{u}_8) &= [\frac{-1}{2},\frac{-33}{2},\frac{-17}{2},\frac{-33}{2},\frac{-1}{2},\frac{-17}{2}] + \overline{R}_0 \\
u_8 = 2\widetilde{u}_8 &= [\frac{1}{128},\frac{1}{2},\frac{1}{8},\frac{1}{2},0,\frac{1}{8}] + \overline{R}_0 \\
w_8 = \widetilde{w}_8 + 32\widetilde{u}_8 &= [\frac{1}{8},8-\frac{c}{4},\frac{17}{8},8+\frac{c}{2},0,2] + \overline{R}_0
\end{aligned}
$$

Now, we see that $(1-\tau)(x_8) = (1-\tau)(y_8) = (1-\tau)(t_8) = (1-\tau)(u_8) = 0$, $(1-\tau)(w_8) = w_7'$, $(1-\tau)(z_8) = 128u_7$ and $(1-\tau)(2w_8+z_8) = 0$ (because all elements in character table of $2w_8 + z_8$ are real) . Then $x_8, t_8, y_8, u_8, 2w_8 + x_8 \in Z(M)_{15}$. And $H(M)_{13} = [2]\oplus[2]\oplus[2]$ generated by $[8x_7], [64y_7], [4z_7]$, because w_7' and $2w_7' \equiv 128u_7$ have been detected.

Next, we start to calculate in general degree beginning with degree $8n-7$ and also, however, with degree $8n-9$.

Degree $8n-7$**,** $n \geq 3$**:** $M_{8n-7} = ku_{8n-7}(BSD_{16}) \cong [2^{n-2}] \oplus [2^{n-2}] \oplus [4^{n-1}] \oplus [2 \cdot 4^{n-1}] \oplus [2 \cdot 16^{n-1}] \oplus [4 \cdot 16^{n-1}]$ generated by $t_{4n-3}, w_{4n-3}, z_{4n-3}, x_{4n-3}, y_{4n-3}, u_{4n-3}$ resp., where

$$\begin{aligned}
x_{4n-3} &= \widetilde{x}_{4n-3} + [\widetilde{z}_{4n-3} + 2 \cdot 4^{n-1}\widetilde{u}_{4n-3}] \\
&= [\frac{2}{4^{n-1}}, -(-2)^n, -2 + \frac{2}{4^{n-1}}, -(-2)^n, 0, \frac{1}{4^{n-1}}] + \overline{R}_0 \\
y_{4n-3} &= 2\widetilde{y}_{4n-3} = [0,0,0,0,\frac{1}{16^{n-1}}, 0] + \overline{R}_0 \\
z_{4n-3} &= [\widetilde{z}_{4n-3} + 2 \cdot 4^{n-1}\widetilde{u}_{4n-3}] + [\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}] \\
&= [\frac{2}{4^{n-1}}, \frac{-c}{(-2)^{n-1}} - (-2)^n, -2 + \frac{2}{4^{n-1}} - \frac{2}{(-2)^{n-1}}, \frac{c}{(-2)^{n-1}} - (-2)^n, 0, 0] + \overline{R}_0 \\
t_{4n-3} &= \widetilde{t}_{4n-3} + (-2)^n\widetilde{x}_{4n-3} - 4(-8)^{n-1}\widetilde{y}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3} - 2(-8)^{n-1}\widetilde{u}_{4n-3} \\
&= [\frac{1}{(-2)^{n-2}}, \frac{1}{(-2)^{n-2}} - 2 \cdot 4^{n-1}, -(-2)^n + \frac{1}{(-2)^{n-2}}, \\
&\qquad \frac{1}{(-2)^{n-2}} - 2 \cdot 4^{n-1}, \frac{1}{(-2)^{n-2}}, \frac{1}{(-2)^{n-2}}] + \overline{R}_0 \\
u_{4n-3} &= \widetilde{u}_{4n-3} = [\frac{1}{16^{n-1}}, \frac{1}{(-2)^{n-1}}, \frac{-1}{4^{n-1}}, \frac{1}{(-2)^{n-1}}, 0, 0] + \overline{R}_0 \\
w_{4n-3} &= 2[\widetilde{w}_{4n-3} - (-2)^{n-1}\widetilde{z}_{4n-3}] \\
&= [0, \frac{c}{(-2)^{n-2}}, \frac{2}{(-2)^{n-2}}, \frac{-c}{(-2)^{n-2}}, 0, 0] + \overline{R}_0
\end{aligned}$$

Degree $8n-9$**,** $n \geq 3$**:** $M_{8n-9} = ku_{8n-9}(BSD_{16}) \cong [2^{n-2}] \oplus [2^{n-2}] \oplus [4^{n-1}] \oplus [4^{n-1}] \oplus [16^{n-1}] \oplus [2 \cdot 16^{n-1}]$ generated by $t_{4n-4}, z_{4n-4}, w_{4n-4}, x_{4n-4}, y_{4n-4}, u_{4n-4}$ resp., where

$$\begin{aligned}
x_{4n-4} &= \widetilde{x}_{4n-4} = [0,0,0,0,0,\frac{2}{4^{n-1}}] + \overline{R}_0 \\
y_{4n-4} &= 2\widetilde{y}_{4n-4} = [0,0,0,0,\frac{2}{16^{n-1}}, 0] + \overline{R}_0 \\
z_{4n-4} &= \widetilde{z}_{4n-4} - 2\widetilde{w}_{4n-4} - 4(-8)^{n-1}\widetilde{u}_{4n-4} \\
&= [\frac{2}{(-2)^{n-2}}, \frac{-c}{(-2)^{n-2}} - 4^n, -(-2)^{n+1}, \frac{c}{(-2)^{n-2}} - 4^n, 0, -(-2)^{n+1}] + \overline{R}_0, \\
t_{4n-4} &= \widetilde{t}_{4n-4} + (-2)^{n-2}(2\widetilde{x}_{4n-4} + \widetilde{z}_{4n-4}) - 2(-8)^{n-1}(\widetilde{y}_{4n-4} + \widetilde{u}_{4n-4}) \\
&= [\frac{1}{(-2)^{n-2}}, \frac{1}{(-2)^{n-2}} - 2 \cdot 4^{n-1}, (-2)^n + \frac{1}{(-2)^{n-2}}, \\
&\qquad \frac{1}{(-2)^{n-2}} - 2 \cdot 4^{n-1}, \frac{1}{(-2)^{n-2}}, (-2)^n + \frac{1}{(-2)^{n-2}}] + \overline{R}_0,
\end{aligned}$$

$$\begin{aligned}
u_{4n-4} &= 2\widetilde{u}_{4n-4} = [\frac{2}{16^{n-1}}, \frac{2}{(-2)^{n-1}}, \frac{2}{4^{n-1}}, \frac{2}{(-2)^{n-1}}, 0, \frac{2}{4^{n-1}}] + \overline{R}_0 \\
w_{4n-4} &= \widetilde{w}_{4n-4} + 2\cdot 4^{n-1}\widetilde{u}_{4n-4} \\
&= [\frac{2}{4^{n-1}}, 2(-2)^{n-1} - \frac{c}{(-2)^{n-1}}, 2 + \frac{2}{4^{n-1}}, 2(-2)^{n-1} + \frac{c}{(-2)^{n-1}}, 0, 2] + \overline{R}_0
\end{aligned}$$

We observe that $s := (2(-8)^{n-1}u_{4n-4} + z_{4n-4}) + (2w_{4n-4} - 2\cdot 4^{n-1}u_{4n-4}) = [0, 0, \frac{1}{4^{n-2}}, 0, 0, 0]$. Now, we have;

$$\begin{aligned}
(1+\tau)(x_{4n-3}) &= 2\cdot 4^{n-1}u_{4n-4} - 2\cdot 4^{n-1}s + s - 2\cdot 4^{n-1}x_{4n-4} + x_{4n-4}, \\
&\equiv 2\cdot 4^{n-1}u_{4n-4} + s + x_{4n-4}, \text{ since } 2\cdot 4^{n-1}s \equiv 0, \\
&\equiv 2(-8)^{n-1}u_{4n-4} + (2w_{4n-4} + z_{4n-4}) + x_{4n-4}, \text{ by replacing } s. \\
(1+\tau)(y_{4n-3}) &= y_{4n-4}. \\
(1+\tau)(z_{4n-3}) &= 2\cdot 4^{n-1}u_{4n-4} - 2\cdot 4^{n-1}s + s - (-2)^{n-1}s - 2\cdot 4^{n-1}x_{4n-4}, \\
&\equiv 4(-8)^{n-1}u_{4n-4} + (1 - (-2)^{n-1})(2w_{4n-4} + z_{4n-4}). \\
(1+\tau)(t_{4n-3}) &= 2t_{4n-4} + 2(-8)^{n-1}s + 2(-8)^{n-1}x_{4n-4} \equiv 2t_{4n-4}. \\
(1+\tau)(u_{4n-3}) &= u_{4n-4} - s - x_{4n-4}, \\
&\equiv (1 + 2(-8)^{n-1} - 2\cdot 4^{n-1})u_{4n-4} + (2w_{4n-4} + z_{4n-4}) - x_{4n-4}. \\
(1+\tau)(w_{4n-3}) &= (-2)^n s \equiv (-2)^n(2w_{4n-4} + z_{4n-4}) + 4(-8)^{n-1}u_{4n-4}.
\end{aligned}$$

Thus $4^{n-1}x_{4n-3}, 16^{n-1}y_{4n-3}, 2\cdot 4^{n-2}z_{4n-3}, 2^{n-3}t_{4n-3}, 2^{n-2}w_{4n-3}$ and $2\cdot 16^{n-1}u_{4n-3}$ are in $Z(M)_{4n-3}$. However, we also observe that

$$\widetilde{u}''_{4n-3} := 4(-8)^{n-1}u_{4n-3} + (-2)^n z_{4n-3} + ((-2)^{n-1} - 1)w_{4n-3} = [0, \frac{-c}{(-2)^{n-2}}, 0, \frac{c}{(-2)^{n-2}}, 0, 0],$$

and then

$$u'_{4n-3} := 4(-8)^{n-1}u_{4n-3} + (-2)^n z_{4n-3} - w_{4n-3} \equiv \widetilde{u}''_{4n-3}$$

having order 2^{n-1} is also contained in $Z(M)_{4n-3}$. Moreover, $2^{n-2}u'_{4n-3} \equiv 2\cdot 16^{n-1}u_{4n-3}$ and hence the result for $Z(M)_{4n-3}$ follows.

Degree $8n-5$, $n \geq 3$: $M_{8n-5} = ku_{8n-5}(BSD_{16}) \cong [2^{n-2}] \oplus [2^{n-1}] \oplus [2\cdot 4^{n-1}] \oplus [2\cdot 4^{n-1}] \oplus [4\cdot 16^{n-1}] \oplus [8\cdot 16^{n-1}]$ generated by $z_{4n-2}, u_{4n-2}, t_{4n-2}, x_{4n-2}, y_{4n-2}, w_{4n-2}$ resp., where

$$\begin{aligned}
x_{4n-2} &= \widetilde{x}_{4n-2} = [0, 0, 0, 0, 0, \frac{1}{4^{n-1}}] + \overline{R}_0 \\
y_{4n-2} &= 2\widetilde{y}_{4n-2} = [0, 0, 0, 0, \frac{8}{16^n}, 0] + \overline{R}_0 \\
z_{4n-2} &= \widetilde{z}_{4n-2} - (-2)^n\widetilde{x}_{4n-2} - (-8)^n\widetilde{y}_{4n-2} - (-8)^n\widetilde{w}_{4n-2} - (2 + (-2)^n)\widetilde{t}_{4n-2} \\
&= [\frac{-4}{(-2)^n}, -(2 + \frac{1}{(-2)^{n-2}}) - 4^n c, (-2)^{n+1} - \frac{1}{(-2)^{n-2}}, \\
&\quad -(2 + \frac{1}{(-2)^{n-2}}) + 4^n c, \frac{-4}{(-2)^n}, (-2)^{n+1} - \frac{1}{(-2)^{n-2}}] + \overline{R}_0
\end{aligned}$$

$$\begin{aligned}
t_{4n-2} &= \widetilde{t}_{4n-2} + 4^n \widetilde{w}_{4n-2} \\
&= [\frac{1}{4^{n-1}}, (-2)^n c - \frac{1}{(-2)^{n-1}}, 2 + \frac{1}{4^{n-1}}, -(-2)^n c - \frac{1}{(-2)^{n-1}}, 0, 2] + \overline{R}_0 \\
u_{4n-2} &= \widetilde{u}_{4n-2} + (-8)^n \widetilde{w}_{4n-2} \\
&= [\frac{4}{(-2)^n}, 4^n c - \frac{c}{(-2)^{n-1}}, 2(-2)^n, -4^n c + \frac{c}{(-2)^{n-1}}, 0, 2(-2)^n] + \overline{R}_0 \\
w_{4n-2} &= 2\widetilde{w}_{4n-2} = [\frac{8}{16^n}, \frac{-c}{(-2)^{n-1}}, \frac{1}{4^{n-1}}, \frac{c}{(-2)^{n-1}}, 0, \frac{1}{4^{n-1}}] + \overline{R}_0
\end{aligned}$$

Now, we get that $(1-\tau)(x_{4n-2}) = (1-\tau)(y_{4n-2}) = 0$ and, because u'_{4n-3} has order 2^{n-1}, we also get $(1-\tau)(z_{4n-2}) = (1-\tau)(t_{4n-2}) \equiv 0$. Furthermore, $(1-\tau)(u_{4n-2}) \equiv (1-\tau)(w_{4n-2}) \equiv u'_{4n-3}$, so $(1-\tau)(w_{4n-2} - u_{4n-2}) \equiv 0$ and hence the result for $Z(M)_{8n-5}$ follows. Moreover, $H(M)_{8n-7}$ is immediate since only u'_{4n-3} has been detected.

Degree $8n-11$**,** $n \geq 4$**:** $M_{8n-11} = ku_{8n-11}(BSD_{16}) \cong [2^{n-3}] \oplus [2^{n-2}] \oplus [2 \cdot 4^{n-2}] \oplus [4^{n-1}] \oplus [8 \cdot 16^{n-2}] \oplus [16^{n-1}]$ generated by $t_{4n-5}, w_{4n-5}, z_{4n-5}, x_{4n-5}, y_{4n-5}, u_{4n-5}$ resp., where

$$\begin{aligned}
x_{4n-5} &= \widetilde{x}_{4n-5} + [\widetilde{z}_{4n-5} + 4^{n-1}\widetilde{u}_{4n-5}] \\
&= [\frac{1}{4^{n-2}}, (-2)^{n-1}c, -2 + \frac{1}{4^{n-2}}, -(-2)^{n-1}c, 0, \frac{2}{4^{n-1}}] + \overline{R}_0 \\
y_{4n-5} &= 2\widetilde{y}_{4n-5} = [0,0,0,0, \frac{4}{16^{n-1}}, 0] + \overline{R}_0 \\
z_{4n-5} &= [\widetilde{z}_{4n-5} + 4^{n-1}\widetilde{u}_{4n-5}] + \frac{1}{2} t_{4n-5} \\
&= [\frac{1}{(-2)^{n-2}} + \frac{1}{4^{n-2}}, ((-2)^{n-1} - 2 \cdot 4^{n-1})c - \frac{1}{(-2)^{n-2}}, (-2)^{n-1} - 2 + \frac{1}{4^{n-2}} \\
&\quad - \frac{1}{(-2)^{n-2}}, -((-2)^{n-1} - 2 \cdot 4^{n-1})c - \frac{1}{(-2)^{n-2}}, \frac{-1}{(-2)^{n-2}}, \frac{-1}{(-2)^{n-2}}] + \overline{R}_0 \\
t_{4n-5} &= 2[\widetilde{t}_{4n-5} + (-2)^{n-1}\widetilde{x}_{4n-5} - (-2)^{n-2}\widetilde{z}_{4n-5} + (-8)^{n-1}\widetilde{y}_{4n-5} + 4(-8)^{n-2}\widetilde{u}_{4n-5}] \\
&= [\frac{2}{(-2)^{n-2}}, \frac{-2}{(-2)^{n-2}} - 4^{n-1}c, -(-2)^n - \frac{2}{(-2)^{n-2}}, \\
&\quad \frac{-2}{(-2)^{n-2}} + 4^{n-1}c, \frac{1}{(-2)^{n-3}}, \frac{1}{(-2)^{n-3}}] + \overline{R}_0 \\
u_{4n-5} &= \widetilde{u}_{4n-5} = [\frac{4}{16^{n-1}}, \frac{c}{(-2)^{n-1}}, \frac{-2}{4^{n-1}}, \frac{-c}{(-2)^{n-1}}, 0, 0] + \overline{R}_0 \\
w_{4n-5} &= \widetilde{w}_{4n-5} + (-2)^{n-1}\widetilde{z}_{4n-5} = [0, \frac{c}{(-2)^{n-2}}, \frac{1}{(-2)^{n-3}}, \frac{-c}{(-2)^{n-2}}, 0, 0] + \overline{R}_0
\end{aligned}$$

Here, we need to find $Z(M)_{8n-11} = \ker(1+\tau : ku_{8n-11}(BSD_{16}) \longrightarrow ku_{8n-13}(BSD_{16}))$ for $n \geq 4$, which is equivalent to find $Z(M)_{8n-3} = \ker(1+\tau : ku_{8(n+1)-11}(BSD_{16}) \longrightarrow$

$ku_{8n-5}(BSD_{16}))$ for $n \geq 3$. Now, we see that:

$$\begin{aligned} s' &:= z_{4n-2} + (2+(-2)^n)t_{4n-2} + (-2)^n x_{4n-2} - 4(-8)^{n-1}y_{4n-2} - 4^n w_{4n-2} \\ &= [0, 0, \frac{2}{4^{n-1}}, 0, 0, 0] \end{aligned}$$

and that:

$$\begin{aligned}
(1+\tau)x_{4n-1} &= [\frac{2}{4^{n-1}}, 0, -4+\frac{2}{4^{n-1}}, 0, 0, \frac{1}{4^{n-1}}] + \overline{R}_0 \\
&= 4^n w_{4n-2} - 4^n(u_{4n-2} + 4(-8)^{n-1}w_{4n-2}) + (1-4^n)s' + (1-4^n)x_{4n-2} \\
&\equiv 4^n w_{4n-2} + s' + x_{4n-2} \\
&= z_{4n-2} + (2+(-2)^n)t_{4n-2} + ((-2)^n+1)x_{4n-2} - 4(-8)^{n-1}y_{4n-2} \\
(1+\tau)y_{4n-1} &= y_{4n-2} \\
(1+\tau)z_{4n-1} &= [\frac{2}{(-2)^{n-1}} + \frac{2}{4^{n-1}}, \frac{-2}{(-2)^{n-1}}, 2(-2)^n - 4 + \frac{2}{4^{n-1}} - \frac{2}{(-2)^{n-1}}, \\
&\quad \frac{-2}{(-2)^{n-1}}, \frac{-2}{(-2)^{n-1}}, \frac{-2}{(-2)^{n-1}}] + \overline{R}_0 \\
&= (4(-8)^{n-1} + 4^n)w_{4n-2} - (4(-8)^{n-1} + 4^n)(u_{4n-2} + 4(-8)^{n-1}w_{4n-2}) \\
&\quad -(4(-8)^{n-1} + 4^n)x_{4n-2} - (z_{4n-2} + (-2)^n t_{4n-2}) + \\
&\quad (1 - (-2)^{n-1} - 4^n - 4(-8)^{n-1})s' \\
&\equiv 8(-8)^{n-1}w_{4n-2} + (2+(-2)^n)t_{4n-2} + (-2)^n x_{4n-2} - 4(-8)^{n-1}y_{4n-2} \\
(1+\tau)t_{4n-1} &= [\frac{4}{(-2)^{n-1}}, \frac{-4}{(-2)^{n-1}}, -2(-2)^{n+1} - \frac{4}{(-2)^{n-1}}, \\
&\quad \frac{-4}{(-2)^{n-1}}, \frac{2}{(-2)^{n-2}}, \frac{2}{(-2)^{n-2}}] + \overline{R}_0 \\
&= 8(-8)^{n-1}[w_{4n-2} - x_{4n-2} - (u_{4n-2} + 4(-8)^{n-1}w_{4n-2})] \\
&\quad -4(-8)^{n-1}s' - 2(z_{4n-2} - 2(-2)^{n-1}t_{4n-2}) + ((-2)^n - 4(-8)^{n-1})s' \\
&\equiv -2(-8)^n w_{4n-2} - 2z_{4n-2} \\
(1+\tau)u_{4n-1} &= [\frac{8}{16^n}, 0, \frac{-4}{4^n}, 0, 0, 0] + \overline{R}_0 \\
&= w_{4n-2} - (u_{4n-2} + 4(-8)^{n-1}w_{4n-2}) - x_{4n-2} - s' \\
&\equiv (w_{4n-2} - u_{4n-2}) + (4^n - 4(-8)^{n-1})w_{4n-2} - z_{4n-2} \\
&\quad -(1+(-2)^n)x_{4n-2} + 4(-8)^{n-1}y_{4n-2} - (2+(-2)^n)t_{4n-2} \\
(1+\tau)w_{4n-1} &= [0, 0, \frac{2}{(-2)^{n-2}}, 0, 0, 0] + \overline{R}_0 \\
&= (-2)^n s' \equiv 2(-2)^n)t_{4n-2} - (-8)^n w_{4n-2}
\end{aligned}$$

Therefore, it is simple now to see that $2 \cdot 4^{n-1}x_{4n-1}$, $4 \cdot 16^{n-1}y_{4n-1}$, $4^{n-1}z_{4n-1}$, $2^{n-3}t_{4n-1} + 2^{n-2}w_{4n-1}$, $8 \cdot 16^{n-1}u_{4n-1}$ are contained in $Z(M)_{8n-3}$ which their order are 2. However, we also observe that;

$$\begin{aligned} w_{4n-1} - (-2)^{n-1}[2 \cdot 4^n u_{4n-1} + t_{4n-1} - 2z_{4n-1}] &= [0, \frac{c}{(-2)^{n-1}}, 0, \frac{-c}{(-2)^{n-1}}, 0, 0] + \overline{R}_0 \\ &\equiv w_{4n-1} + (-8)^n u_{4n-1} - (-2)^n z_{4n-1} \\ &:= w'_{4n-1} \end{aligned}$$

having order 2^n with $2^{n-1}(w'_{4n-1}) \equiv 8 \cdot 16^{n-1} u_{4n-1}$. Thus, the result for $Z(M)_{8n-3}$ follows. Furthermore, $H(M)_{8n-5}$ is calculated by the representing matrix below;

$$\begin{array}{r|cccccl} & [2^{n-2}] & [2\cdot 4^{n-1}] & [2\cdot 4^{n-1}] & [4\cdot 16^{n-1}] & 8\cdot 16^{n-1} & \\ & z_{4n-2} & t_{4n-2} & x_{4n-2} & y_{4n-2} & w'_{4n-2} & \\ (1+\tau)(x_{4n-1}): & 1 & 2+(-2)^n & 1+(-2)^n & -4(-8)^{n-1} & 0 & |r_0 \\ (1+\tau)(y_{4n-1}): & 0 & 0 & 0 & 1 & 0 & |r_1 \\ (1+\tau)(z_{4n-1}): & 0 & 2+(-2)^n & (-2)^n & -4(-8)^{n-1} & 8(-8)^{n-1} & |r_2 \\ (1+\tau)(t_{4n-1}): & -2 & 0 & 0 & 0 & -2(-8)^n & |r_3 \\ (1+\tau)(u_{4n-1}): & -1 & -2-(-2)^n & -1-(-2)^n & 4(-8)^{n-1} & g & |r_4 \\ (1+\tau)(w_{4n-1}): & 0 & 2(-2)^n & 0 & 0 & -(-8)^n & |r_5, \end{array}$$

where $w'_{4n-2} = w_{4n-2} - u_{4n-2}$ and $g = 1 + 4^n - 4(-8)^{n-1}$ which is equivalent to

$$\begin{array}{|cccccl} [2^{n-2}] & [2\cdot 4^{n-1}] & [2\cdot 4^{n-1}] & [4\cdot 16^{n-1}] & 8\cdot 16^{n-1} & \\ z_{4n-2} & t_{4n-2} & x_{4n-2} & y_{4n-2} & w'_{4n-2} & \\ 1 & 0 & 1 & 0 & 0 & |r_0^* \\ 0 & 0 & 0 & 1 & 0 & |r_1^* \\ 0 & 2 & 0 & 0 & 0 & |r_2^* \\ -2 & 0 & 0 & 0 & 0 & |r_3^* \\ 0 & 0 & 0 & 0 & g & |r_4^* \\ 0 & 0 & 0 & 0 & 0 & |r_5^*, \end{array}$$

by row operations;
$r_0 \longrightarrow [r'_0 = r_0 + 4(-8)^{n-1} r_1] \longrightarrow [r_0^* = r'_0 - r'_2]$,
$r_1 = r_1^*$,
$r_2 \longrightarrow [r'_2 = r_2 + 4(-8)^{n-1} r_1 - 8(-8)^{n-1} r_4^*] \longrightarrow [r''_2 = r'_2 - (-2)^n r_0^*]$
$\longrightarrow [r_2^* = r''_2 + (-2)^{n-2} r''_5]$.
$r_3 \longrightarrow [r_3^* = r_3 + 2(-8)^n r_4^*]$
$r_4 \longrightarrow [r_4^* = r_4 + r_0]$,
$r_5 \longrightarrow [r'_5 = r_5 + (-8)^n r_4^*] \longrightarrow [r''_5 = r'_5 - 2r''_2] \longrightarrow [r_5^* = r''_5 + 2r_2^*]$.
After doing a bit column operations, the result for $H(M)_{8n-5}$ follows.

Next, for $Z(M)_{8n-9} = \ker(1-\tau : ku_{8n-9}(BSD_{16}) \longrightarrow ku_{8n-11}(BSD_{16}))$, we see that $x_{4n-4}, y_{4n-4}, t_{4n-4}$ and u_{4n-4} are in $Z(M)_{8n-9}$, because their entry in character table are all real. Moreover, $(1-\tau)z_{4n-4} = 2w'_{4(n-1)-1} = 2w'_{4n-5}$ and $(1-\tau)w_{4n-4} = w'_{4(n-1)-1} = w'_{4n-5}$ and thus $2w_{4n-4} + z_{4n-4} \in Z(M)_{8n-9}$ which yields the results for $Z(M)_{8n-9}$ and the result for $H(M)_{8n-3}$ is immediate (only the generator w'_{4n-1} has been detected).

Finally, $H(M)_{8n-9}$ can be calculated by the representing matrix (by **Degree**

$8n-7$ and **Degree** $8n-9$ above) below;

$$
\begin{array}{r|cccccl}
 & [2^{n-2}] & [2\cdot 4^{n-2}] & [4^{n-1}] & [16^{n-1}] & 2\cdot 16^{n-1} & \\
 & t_{4n-4} & 2w_{4n-4}+z_{4n-4} & x_{4n-4} & y_{4n-4} & u_{4n-4} & \\
(1+\tau)(x_{4n-3}): & 0 & 1 & 1 & 0 & 2(-8)^{n-1} & |r_0 \\
(1+\tau)(y_{4n-3}): & 0 & 0 & 0 & 1 & 0 & |r_1 \\
(1+\tau)(z_{4n-3}): & 0 & 1-(-2)^{n-1} & 0 & 0 & 4(-8)^{n-1} & |r_2 \\
(1+\tau)(t_{4n-3}): & 2 & 0 & 0 & 0 & 0 & |r_3 \\
(1+\tau)(u_{4n-3}): & 0 & 1 & -1 & 0 & h & |r_4 \\
(1+\tau)(w_{4n-3}): & 0 & (-2)^n & 0 & 0 & 4(-8)^{n-1} & |r_5,
\end{array}
$$

where $h = 1+2(-8)^{n-1}-2\cdot 4^{n-1}$, which is equivalent to

$$
\begin{array}{|cccccl}
 [2^{n-2}] & [2\cdot 4^{n-2}] & [4^{n-1}] & [16^{n-1}] & 2\cdot 16^{n-1} & \\
 t_{4n-4} & 2w_{4n-4}+z_{4n-4} & x_{4n-4} & y_{4n-4} & u_{4n-4} & \\
 0 & 0 & 1 & 0 & 0 & |r_0^* \\
 0 & 0 & 0 & 1 & 0 & |r_1^* \\
 0 & 1 & 0 & 0 & 0 & |r_2^* \\
 2 & 0 & 0 & 0 & 0 & |r_3^* \\
 0 & 0 & 0 & 0 & 1-2\cdot 4^{n-1} & |r_4^* \\
 0 & 0 & 0 & 0 & 0 & |r_5^*,
\end{array}
$$

by simple row operations and hence we complete the proof.

□

§ 6.4 E_∞-page

From the last section, lemma 6.3.1, we obtain $E_{1\frac{1}{2}}$-page. To get E_2-page, we need to determine connecting homomorphism in even degree part induced by (6.1), i.e. $\delta : Z(H_I^0(R)_*) \longrightarrow H(H_I^2(R)_{*+2})$. By (6.5) and lemma 6.2.5, we see that $Z(H_I^0(R)_*)$ contain only in degree divided by 4 whereas $H(H_I^2(R)_{*+2})$ contain only in degree divided by 8. Thus $\delta = 0$, because δ shift down degree by 2, and hence $E_{1\frac{1}{2}}-$page $\cong E_2-$page.

We can now display the E_2-page, differentials and the E_∞-page in Figure 6.2 and Figure 6.3 below, where $[n]$ means cyclic group of order n derived from $ku_{odd}(BSD_{16})$ -part and 2^k denotes elementary abelian 2 group of rank k derived from $ku_{even}(BSD_{16})$-part. The target symbols are the same meaning as in Theorem 5.5.1. All differentials are treated by Theorem 5.5.1 and the fact that $d_r = 0, \forall r \geq 4$ (because $\eta^3 = 0$).

Filtration(s) ⇓ / Degree(t) ⇒	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Target ⇒	$\mathbb{Z}\cdot\rho$	2^3	2^3	$[4]\oplus[8]\oplus[8]$	$\mathbb{Z}\cdot\alpha\oplus 2$	$[2]$	0	$[2]\oplus[4]\oplus[16]\oplus[32]$	$2\oplus\mathbb{Z}\cdot\rho\oplus 2$	$[2]\oplus 2\oplus[2]^4$	$[2]^4\oplus 2$	$[8]\oplus[16]\oplus[128]\oplus[128]$	$\mathbb{Z}\cdot\alpha\oplus 2^2$	$[4]$	2	$[2]\oplus[8]\oplus[16]\oplus[256]\oplus[512]$	$2^2\oplus 2^2\ \mathbb{Z}\cdot\rho$	$[4]\oplus[2]^5\oplus 2^2$	$[2]^5\oplus 2^2$	$[2]\oplus[32]\oplus[64]\oplus[2048]\oplus[2048]$	$2^3\oplus\mathbb{Z}\cdot\alpha$
	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑	⇑
6							$[2]$	$[2]\oplus[2]$	0	0	$[2]$	$[2]\oplus[2]$	0	0	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]$	0
5						$[2]$	$[2]\oplus[2]$	0	0	$[2]$	$[2]\oplus[2]$	0	0	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]$	0	$[2]$
4					$[2]$	$[2]\oplus[2]$	0	0	$[2]$	$[2]\oplus[2]$	0	0	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]\oplus 2$
3				$[2]$	$[2]\oplus[2]$	0	0	$[2]$	$[2]\oplus[2]$	0	0	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]\oplus[2]$
2			$[2]$	$[2]\oplus[2]$	0	0	$[2]$	$[2]\oplus[2]$	0	0	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]\oplus[2]$	0
1		$[2]$	$[2]\oplus[2]$	0	0	$[2]$	$[2]\oplus[2]$	0	0	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]$	0	$[2]$	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]\oplus[2]$	0	$[2]\oplus[2]$
0	$\mathbb{Z}\cdot\rho$	$[2]\oplus[2]$	0	$[2]\oplus[4]\oplus[8]$	$\mathbb{Z}\cdot\rho\oplus 2$	$[2]\oplus[2]\oplus[2]$	0	$[2]\oplus[4]\oplus[16]\oplus[32]$	$\mathbb{Z}\cdot\rho\oplus 2^2$	$[2]\oplus[2]\oplus[2]\oplus[2]$	2	$[8]\oplus[8]\oplus[64]\oplus[128]$	$\mathbb{Z}\cdot\rho\oplus 2^2$	$[2]\oplus[2]\oplus[2]\oplus[4]$	2	$[2]\oplus[8]\oplus[16]\oplus[256]\oplus[512]$	$\mathbb{Z}\cdot\rho\oplus 2^3$	$[2]\oplus[2]\oplus[2]\oplus[2]\oplus[4]$	2^2	$[2]\oplus[32]\oplus[32]\oplus[1024]\oplus[2048]$	$\mathbb{Z}\cdot\rho\oplus 2^3$

Figure 6.2: The E_2-page of Bockstein spectral sequence for $ko_*(BSD_{16})$ and differentials.

Target $\Rightarrow$	$\mathbb{Z}\cdot\rho$	2^3	2^3	$[4]\oplus[8]\oplus[8]$	$\mathbb{Z}\cdot\alpha\oplus 2$	$[2]$	0	$[2]\oplus[4]\oplus[16]\oplus[32]$	$2\oplus\mathbb{Z}\cdot\rho\oplus 2$	$[2]\oplus 2\oplus[2]^4$	$[2]^4\oplus 2$	$[8]\oplus[16]\oplus[128]\oplus[128]$	$\mathbb{Z}\cdot\alpha\oplus 2^2$	$[4]$	2	$[2]\oplus[8]\oplus[16]\oplus[256]\oplus[512]$	$2^2\oplus 2^2$ $\mathbb{Z}\cdot\rho$	$[4]\oplus[2]^5\oplus 2^2$	$[2]^5\oplus 2^2$	$[2]\oplus[32]\oplus[64]\oplus[2048]\oplus[2048]$	$2^3\oplus\mathbb{Z}\cdot\alpha$
Filtration(s) $\Downarrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Uparrow$
6																					
5																					
4																					
3																					
2			$[2]$	$[2]\oplus[2]$							$[2]$	$[2]\oplus[2]\oplus[2]$						$[2]$	$[2]$	$[2]\oplus[2]\oplus[2]\oplus[2]$	
1		$[2]$	$[2]\oplus[2]$							$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]$						$[2]$	$[2]\oplus 2$	$[2]\oplus[2]\oplus[2]\oplus[2]$		
0	$\mathbb{Z}\cdot\rho$	$[2]\oplus[2]$	0	$[2]\oplus[4]\oplus[8]$	$\mathbb{Z}\cdot\alpha\oplus 2$	$[2]$		$[2]\oplus[4]\oplus[16]\oplus[32]$	$\mathbb{Z}\cdot\rho\oplus 2^2$	$[2]\oplus[2]\oplus[2]\oplus[2]$	2	$[4]\oplus[8]\oplus[64]\oplus[128]$	$\mathbb{Z}\cdot\alpha\oplus 2^2$	$[4]$	2	$[2]\oplus[8]\oplus[16]\oplus[256]\oplus[512]$	$\mathbb{Z}\cdot\rho\oplus 2^3$	$[2]\oplus[2]\oplus[2]\oplus[2]\oplus[4]$	2^2	$[16]\oplus[32]\oplus[1024]\oplus[2048]$	$\mathbb{Z}\cdot\alpha\oplus 2^3$
$\Leftarrow$ **Degree(t)**	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20

Figure 6.3: The E_∞-page of Bockstein spectral sequence for $ko_*(BSD_{16})$

§ 6.5 Extension problems and results

The E_∞-page in Figure 6.3 gives η-multiple structure. Precisely, η and η^2 multiple elements of $ko_*(BSD_{16})$ are detected by the generater lying on the second (s=1) and the third row (s=2) of E_∞-page. For example, $ko_1(BSD_{16})$ contains one η-multiple generator, $ko_2(BSD_{16})$ contains one η^2-multiple and two η-multiple generator, $ko_3(BSD_{16})$ contains two η^2-multiple generator, et cetera.

More surprisingly, the extension problems of degree 8 is obviously trivial and this fact leads to:

Lemma 6.5.1. *The extension problem occurring in Greenlees spectral sequence for $ko_*(BSD_{16})$ in Theorem 5.5.1 of degree $8k$ for all $k \geq 1$ are trivial.*

Proof. Consider the commutative diagram, from Greenlees spectral sequence;

$$\begin{tikzcd} 0 \arrow[r] & 2^2 \cong E_\infty^{-2,18} \arrow[r] \arrow[d, "t\cap"] & \widetilde{ko}_{16}(BSD_{16}) \arrow[r] \arrow[d, "t\cap"] & E_\infty^{-1,17} \cong [2]^2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 2 \cong E_\infty^{-2,10} \arrow[r] & \widetilde{ko}_8(BSD_{16}) \arrow[r] & E_\infty^{-1,9} \cong [2] \arrow[r] & 0. \end{tikzcd}$$

Suppose that the first row of this diagram is non-split. So, there is $\widetilde{x} \in \widetilde{ko}_{16}(BSD_{16})$ such that $2\widetilde{x} \neq 0$ and $2\widetilde{x}$ must lie in $E_\infty^{-2,18} \cong 2^2 < \frac{\tau^b}{b^8}, \frac{\tau^b}{b^4 d^2} >$. By calculation we check that there exit $t \in ko^8(BSD_{16})$, namely $\overline{y}_4$ or $\overline{u}_4$, s.t. $t \cap (2\widetilde{x}) \neq 0$. This implies that $2(t \cap \widetilde{x}) = t \cap (2\widetilde{x})$ in non-zero in $\widetilde{ko}_8(BSD_{16})$ which is a contradiction because $\widetilde{ko}_8(BSD_{16})$ is split and contains $t \cap \widetilde{x}$. Hence, $\widetilde{ko}_{16}(BSD_{16})$ is split.

To generalise the conclusion to all degree $8k, k \geq 2$, we again consider the commutative diagram below;

$$\begin{tikzcd} 0 \arrow[r] & 2^k \cong E_\infty^{-2,8k+2} \arrow[r] \arrow[d, "\beta"] & \widetilde{ko}_{8k}(BSD_{16}) \arrow[r, "\pi_k"] \arrow[d, "\beta"] & E_\infty^{-1,8k+1} \cong [2]^2 \arrow[r] \arrow[d, "\overline{\beta}"] & 0 \\ 0 \arrow[r] & 2^{k+1} \cong E_\infty^{-2,8k+10} \arrow[r] & \widetilde{ko}_{8(k+1)}(BSD_{16}) \arrow[r, "\pi_{k+1}"] & E_\infty^{-1,8k+9} \cong [2]^2 \arrow[r] & 0. \end{tikzcd}$$

Since $\overline{\beta}$ is an isomorphism for $k \geq 2$, lemma 5.2.4, if $\widetilde{ko}_{8k}(BSD_{16})$ is split (i.e. there exist $s : E_\infty^{-1,8k+1} \longrightarrow \widetilde{ko}_{8k}(BSD_{16})$ s.t. $s \circ \pi_k = id_{E_\infty^{-1,8k+1}}$), then we can define $s' : E_\infty^{-1,8k+9} \longrightarrow \widetilde{ko}_{8(k+1)}(BSD_{16})$ by

$$s'(\overline{\beta}(x)) = \beta(s(x))$$

which is easy to see that $s' \circ \pi_{k+1} = id_{E_\infty^{-1,8k+9}}$, i.e. $\widetilde{ko}_{8(k+1)}(BSD_{16})$ is split. The assumption is possible by degree 16 and hence completes the proof. □

Next, we need to solve the extension problems of degree $8k+1$ for all $k \geq 0$. If we can show that the extension problems in degree 9 and 17 are trivial, then by the similar reason as in the second part of the proof of lemma 6.5.1 above ($E_\infty^{0,8k+1} \cong E_\infty^{0,8k+9}$ by $\overline{\beta}$), $ko_{8k+1}(BSD_{16})$ for all $k \geq 0$ are split. To solve the extension problems in degree 9 and 17, we use many techniques as we have used in chapter 4 and chapter 5, but they still remain.

Fortunately, D.Bayen, [7], calculated $ko_*(BSD_{16})$ by using Adams spectral sequence on each part of the stable splitting of BSD_{16};

$$BSD_{2^n} \simeq BSL_3(q) \vee L(2) \vee \mathbb{R}P^\infty \vee \Sigma^{-1}BS^3/BN,$$

(see details of this description in [7]). In that, he calculated and recorded separately for each part of

$$\widetilde{ko}_*(BSD_{16}) \cong \widetilde{ko}_*(BSL_3(3)) \oplus \widetilde{ko}_*(L(2)) \oplus \widetilde{ko}_*(\mathbb{R}P^\infty) \oplus \widetilde{ko}_*(\Sigma^{-1}BS^3/BN).$$

By his results, the extension problems are trivial (even through in degree $8k$ and $8k+2$).

Now, by recollecting all our results we have so far (in this chapter and in the last chapter), we get $ko_*(BSD_{16})$ as a module over $ko^*(BSD_{16})$ as;

Theorem 6.5.2. *The additive structure of $ko_*(BSD_{16})$ is given by;*

$ko_n(BSD_{16})$	n
$[2^k] \oplus [2 \cdot 4^{k+1}] \oplus [4^{k+2}] \oplus [8 \cdot 16^{k+1}] \oplus [8 \cdot 16^{k+1}]$	$8k+11 \geq 19$
$[2]^{\oplus 5} \oplus 2^{k+1}$	$8k+10 \geq 18$
$2^7 \oplus [2^{k+1}]$	$8k+9 \geq 17$
$\mathbb{Z} \cdot \beta^{k+1}\rho \oplus [2]^{\oplus 2} \oplus 2^{k+1}$	$8k+8 \geq 16$
$[2^k] \oplus [2 \cdot 4^k] \oplus [4^{k+1}] \oplus [16^{k+1}] \oplus [2 \cdot 16^{k+1}]$	$8k+7 \geq 15$
2^k	$8k+6 \geq 14$
$[2^{k+1}]$	$8k+5 \geq 13$
$\mathbb{Z} \cdot \beta^k\alpha \oplus 2^{k+1}$	$8k+4 \geq 12$
$[8] \oplus [16] \oplus [128] \oplus [128]$	11
$[2]^{\oplus 4} \oplus 2$	10
$2^5 \oplus [2]$	9
$\mathbb{Z} \cdot \beta\rho \oplus [2] \oplus 2$	8
$[2] \oplus [4] \oplus [16] \oplus [32]$	7
0	6
$[2]$	5
$\mathbb{Z} \cdot \alpha \oplus 2$	4
$[4] \oplus [8] \oplus [8]$	3
$[2]^{\oplus 3}$	2
$[2]^{\oplus 3}$	1
$\mathbb{Z} \cdot \rho$	0

,

where β is the Bott element in $KO_8(pt)$, ρ is the first Chern class of regular representation of SD_{16}, $\alpha = 2\rho$ and $[n]$ means cyclic group of order n, 2^k means elementary abelian two group of rank k.

The generator description for $ko_*(BSD_{16})$ by the Greenlees spectral sequence is as follows.

- $ko_0(BSD_{16}) \cong \mathbb{Z} < \rho >$.
- $ko_1(BSD_{16}) \cong [2]\oplus[2]\oplus[2]$ *generated by* $\widetilde{\eta}[B], \widetilde{\eta}[\widetilde{D}^3], \widetilde{\eta}([A]+[D^2])$ *such that only* $\widetilde{\eta}[\widetilde{D}^3]$ *is an* η*-multiple generator.*
- $ko_2(BSD_{16}) \cong [2]\oplus[2]\oplus[2]$ *generated by* $\widetilde{\eta}^2[B], \widetilde{\eta}^2[\widetilde{D}^3], \widetilde{\eta}^2([A]+[D^2])$ *such that* $\widetilde{\eta}^2[\widetilde{D}^3]$ *is an* η^2*-multiple generator and* $\widetilde{\eta}^2[B], \widetilde{\eta}^2([A]+[D^2])$ *are* η*-multiple generator.*
- $ko_3(BSD_{16}) \cong [4]\oplus[8]\oplus[8]$ *generated by* $\widetilde{x}_2+\widetilde{z}_2$, $\widetilde{u'}_2$ *and* $2\widetilde{y}_2$ *respectively, with* $\widetilde{t'}_2 - 2\widetilde{x}_2 - 8\widetilde{y}_2 - \widetilde{z}_2 - 4\widetilde{u'}_2 = 0$, *where* $\widetilde{\alpha}_2 = \frac{\overline{\alpha}_6}{q^2} + \overline{QO}_{-12}$ *for each* $\alpha \in \{x, y, z, t', u'\}$. *Moreover, there are two* η^2*-multiple generators from the top of* $\widetilde{x}_2+\widetilde{z}_2$ *and,* $\widetilde{u'}_2$ *or* $2\widetilde{y}_2$.
- $ko_4(BSD_{16}) \cong \mathbb{Z} < \alpha > \oplus 2 < \frac{\tau^b}{b^4} >$ *such that* $\frac{\tau^b}{b^2}$ *is embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as* $(uy)^\vee$.
- $ko_5(BSD_{16}) \cong [2] < \widetilde{w}'_3 >$, *where* $\widetilde{w'}_3 = \frac{\overline{w'}_1}{q^1} + \overline{QO}_{-2}$ *such that* $\overline{w'}_1 = \theta_1 = \overline{u}_1 - \overline{t}_1 - 2\overline{w}_1$.
- $ko_6(BSD_{16}) = 0$.
- $ko_7(BSD_{16}) \cong [2]\oplus[4]\oplus[16]\oplus[32]$ *generated by* $\widetilde{z}_4 + 4\widetilde{u}_4$, $\widetilde{x}_4$, $2\widetilde{y}_4$ *and* $2\widetilde{u}_4$ *respectively, where* $\widetilde{\alpha}_4 = \frac{\overline{\alpha}_4}{q^2} + \overline{QO}_{-8}$ *for each* $\alpha \in \{x, y, z, t, u\}$.
- $ko_8(BSD_{16}) \cong \mathbb{Z} < \beta\rho > \oplus[2] < \frac{\widetilde{\eta}[\overline{u}_4]}{(\overline{u}_4)^2} > \oplus 2 < \frac{\tau^b}{b^4} >$ *such that* $\frac{\widetilde{\eta}[\overline{u}_4]}{(\overline{u}_4)^2}$ *and* $\frac{\tau^b}{b^4}$ *are embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as* $(uPy)^\vee$ *and* $(uy^5)^\vee$ *respectively.*
- $ko_9(BSD_{16}) \cong [2]^4 < \widetilde{\eta}[A], \widetilde{\eta}[B], \widetilde{\eta}[D^2], \widetilde{\eta}[\widetilde{D}^3] > \oplus[2] < \frac{\widetilde{\eta}^2[\overline{u}_4]}{(\overline{u}_4)^2} > \oplus[2] < 2\widetilde{w}_5 >$, *where* $\widetilde{w}_5 = \frac{\overline{w}_3}{q^2} + \overline{QO}_{-6}$, *such that* $\widetilde{\eta}[\widetilde{D}^3]$ *and* $\frac{\widetilde{\eta}^2[\overline{u}_4]}{(\overline{u}_4)^2}$ *are only two* η*-multiple generators.*
- $ko_{10}(BSD_{16}) \cong [2]^4 < \widetilde{\eta}^2[A], \widetilde{\eta}^2[B], \widetilde{\eta}^2[D^2], \widetilde{\eta}^2[\widetilde{D}^3] > \oplus 2 < \frac{\tau^{bd}}{b^6} + \frac{\tau^{bd}}{b^2d^2} >$, *such that there are three* η*-multiple generators which are* $\widetilde{\eta}^2[A], \widetilde{\eta}^2[B], \widetilde{\eta}^2[D^2]$ *and only one* η^2*-multiple generator which is* $\widetilde{\eta}^2[\widetilde{D}^3]$. *Moreover* $\frac{\tau^{bd}}{b^6} + \frac{\tau^{bd}}{b^2d^2}$ *is embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as* $(uPy^3)^\vee$.
- $ko_{11}(BSD_{16}) \cong [8]\oplus[16]\oplus[128]\oplus[128]$ *generated by* $\widetilde{z}_6+\widetilde{t}_6, \widetilde{x}_6+\widetilde{z}_6$, $\widetilde{u'}_6$ *and* $2\widetilde{y}_6$ *respectively, with* $\widetilde{t}_6 = \widetilde{t'}_6 + 4\widetilde{x}_6 + 64\widetilde{y}_6 + 2\widetilde{z}_2 + 64\widetilde{u'}_6$, *where* $\widetilde{\alpha}_6 = \frac{\overline{\alpha}_6}{q^3} + \overline{QO}_{-12}$ *for each* $\alpha \in \{x, y, z, t', u'\}$. *Moreover, there are three* η^2*-multiple generators from the top of* $\widetilde{z}_6+\widetilde{t}_6$, $\widetilde{x}_6+\widetilde{z}_6$ *and,* $\widetilde{u'}_6$ *or* $2\widetilde{y}_6$.

- *For* $k \geq 1$, $ko_{8k+4}(BSD_{16}) \cong \mathbb{Z} < \beta^k\alpha > \oplus 2^{k+1} < \frac{\tau^b}{b^{4(k-i)+2}d^{2i}} | 0 \leq i \leq k >$ *such that for each* i, $\frac{\tau^b}{b^{4(k-i)+2}d^{2i}}$ *is embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as* $(uP^{2i}y^{8(k-i)+1})^\vee$.

- *For* $k \geq 1$, $ko_{8k+5}(BSD_{16}) \cong [2^{k+1}] < \widetilde{w}'_{4k+3} >$, *where* $\widetilde{w'}_{4k+3} = \frac{\overline{w'}_1}{q^{k+1}} + \overline{QO}_{-2}$ *such that* $\overline{w'}_1 = \theta_1 = \overline{u}_1 - \overline{t}_1 - 2\overline{w}_1$.

- *For* $k \geq 1$, $ko_{8k+6}(BSD_{16}) \cong 2^k < \frac{\tau^{bd}}{b^{4(k-i+1)}d^{2i}} + \frac{\tau^{bd}}{b^{4(k-i)}d^{2(i+1)}} | 0 \leq i \leq k-1 >$ *such that for each* i, $\frac{\tau^{bd}}{b^{4(k-i+1)}d^{2i}} + \frac{\tau^{bd}}{b^{4(k-i)}d^{2(i+1)}}$ *is embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as*

$$(uP^{2i-1}y^{8(k-i+1)-1})^\vee + (uP^{2i+1}y^{8(k-i)-1})^\vee,$$

(if $i = 0$, *the first term disappears).*

- *For* $k \geq 1$, $ko_{8k+7}(BSD_{16}) \cong [2^k] \oplus [2 \cdot 4^k] \oplus [4^{k+1}] \oplus [16^{k+1}] \oplus [2 \cdot 16^{k+1}]$ *generated by* $\widetilde{\widetilde{t}}_{4k+4}, \widetilde{z}_{4k+4} + 4^{k+2}\widetilde{u}_{4k+4}$, $\widetilde{x}_{4k+4}$, $2\widetilde{y}_{4k+4}$ *and* $2\widetilde{u}_{4k+4}$, *respectively, with* $\widetilde{\widetilde{t}}_{4k+4} = \widetilde{t}_{4k+4} - (-2)^{k+1}\widetilde{x}_{4k+4} - 2(-8)^{k+1}\widetilde{y}_{4k+4} + (-2)^k\widetilde{z}_{4k+4} - 2(-8)^{k+1}\widetilde{u}_{4k+4}$, *where* $\widetilde{\alpha}_{4k+4} = \frac{\overline{\alpha}_4}{q^{k+2}} + \overline{QO}_{-8}$ *for each* $\alpha \in \{x, y, z, t, u\}$.

- *For* $k \geq 1$, $ko_{8k+8}(BSD_{16}) \cong \mathbb{Z} < \beta^{k+1}\rho > \oplus [2]^2 < \frac{\widetilde{\eta}[\overline{u}_4]}{(\overline{u}_4)^{k+2}}, \frac{\widetilde{\eta}[\overline{t}_4]}{(\overline{u}_4)^{k+2}} > \oplus 2^{k+1} < \frac{\tau^b}{b^{4(k-i+1)}d^{2i}} | 0 \leq i \leq k >$ *such that* $\frac{\widetilde{\eta}[\overline{u}_4]}{(\overline{u}_4)^{k+2}}$ *and* $\frac{\tau^b}{b^{4(k-i+1)}d^{2i}}$ *for each* i, *are embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as*

$$(uyP^{2k+1})^\vee \text{ and } (uy^{8(k-i)+5}P^{2i})^\vee$$

respectively, whereas $\frac{\widetilde{\eta}[\overline{t}_4]}{(\overline{u}_4)^{k+2}}$ *is an only one* η*-multiple generator (which is actually* $\eta \cdot \widetilde{\widetilde{t}}_{4k+4}$*).*

- *For* $k \geq 1$, $ko_{8k+9}(BSD_{16}) \cong [2]^5 < \beta^{k+1}\widetilde{\eta}[1], \beta^{k+1}\widetilde{\eta}[A], \beta^{k+1}\widetilde{\eta}[B], \beta^{k+1}\widetilde{\eta}[D^2], \beta^{k+1}\widetilde{\eta}[\widetilde{D}^3] > \oplus [2]^2 < \frac{\widetilde{\eta}^2[\overline{u}_4]}{(\overline{u}_4)^{k+2}}, \frac{\widetilde{\eta}^2[\overline{t}_4]}{(\overline{u}_4)^{k+2}} > \oplus [2^{k+1}] < 2\widetilde{w}_{4k+5} >$, *where* $\widetilde{w}_{4k+5} = \frac{\overline{w}_3}{q^{k+2}} + \overline{QO}_{-6}$, *such that* $\beta^{k+1}\widetilde{\eta}[\widetilde{D}^3]$ *and* $\frac{\widetilde{\eta}^2[\overline{u}_4]}{(\overline{u}_4)^{k+2}}$ *are only two* η*-multiple generators whereas* $\frac{\widetilde{\eta}^2[\overline{t}_4]}{(\overline{u}_4)^{k+2}}$ *is only one* η^2*-multiple generator.*

- *For* $k \geq 1$, $ko_{8k+10}(BSD_{16}) \cong [2]^5 < \beta^{k+1}\widetilde{\eta}^2[1], \beta^{k+1}\widetilde{\eta}^2[A], \beta^{k+1}\widetilde{\eta}^2[B], \beta^{k+1}\widetilde{\eta}^2[D^2], \beta^{k+1}\widetilde{\eta}^2[\widetilde{D}^3] > \oplus 2^{k+1} < \frac{\tau^{bd}}{b^{4(k+1-i)+2}d^{2i}} + \frac{\tau^{bd}}{b^{4(k-i)+2}d^{2i+2}} >$, *such that there are four* η*-multiple generators which are* $\beta^{k+1}\widetilde{\eta}^2[1], \beta^{k+1}\widetilde{\eta}^2[A], \beta^{k+1}\widetilde{\eta}^2[B], \beta^{k+1}\widetilde{\eta}^2[D^2]$ *and only one* η^2*-multiple generator which is* $\beta^{k+1}\widetilde{\eta}^2[\widetilde{D}^3]$. *Moreover* $\frac{\tau^{bd}}{b^{4(k+1-i)+2}d^{2i}} + \frac{\tau^{bd}}{b^{4(k-i)+2}d^{2i+2}}$ *is embedded in* $H_*(BSD_{16};\mathbb{F}_2)$ *as*

$$(uy^{8(k+1-i)+1}P^{2i-1})^\vee + (uy^{8(k-i)+1}P^{2i+1})^\vee$$

(if $i = 0$, *the first term disappears).*

- *For* $k \geq 1$, $ko_{8k+11}(BSD_{16}) \cong [2^k] \oplus [2 \cdot 4^{k+1}] \oplus [4^{k+2}] \oplus [8 \cdot 16^{k+1}] \oplus [8 \cdot 16^{k+1}]$ *generated by* $2\widetilde{t}_{4k+6}, \widetilde{t}_{4k+6} + \widetilde{z}_{4k+6}$, $\widetilde{x}_{4k+6} + \widetilde{z}_{4k+6}$, $\widetilde{u'}_{4k+6}$ *and* $2\widetilde{y}_{4k+6}$ *respectively, with* $\widetilde{t}_{4k+6} = \widetilde{t'}_{4k+6} + (-2)^{k+2}\widetilde{x}_{4k+6} + (-8)^{k+2}\widetilde{y}_{4k+6} - (-2)^{k+1}\widetilde{z}_{4k+6} -$

$4(-8)^{k+1}\widetilde{u'}_{4k+6}$, *where* $\widetilde{\alpha}_{4k+6} = \frac{\overline{\alpha}_6}{q^{k+3}} + \overline{QO}_{-12}$ *for each* $\alpha \in \{x, y, z, t', u'\}$. *Moreover, there are four* $\underset{\sim}{\eta}^2$*-multiple generators from the top of* $2\widetilde{t}_{4k+6}$, $\widetilde{z}_{4k+6} + \widetilde{t}_{4k+6}$, $\widetilde{x}_{4k+6} + \widetilde{z}_{4k+6}$ *and,* $\widetilde{u'}_{4k+6}$ *or* $2\widetilde{y}_{4k+6}$.

Proof. This is an immediate result from theorem 5.5.1, lemma 5.4.3, lemma 6.2.4, E_∞-page of both Greenlees spectral sequence and η-BSS. Precisely, to conclude that $\widetilde{\eta}[\widetilde{D}^3]$ is an η-multiple generator we use;

$$\overline{\rho} = 16 \cdot 1 - [\frac{28}{3}A + 8B - \frac{13}{3}C + \frac{4}{3}D^2 - \frac{5}{9}\widetilde{D}^3],$$

in lemma 5.3.2 and the same conclusion for $\beta^k\widetilde{\eta}^\epsilon[\widetilde{D}^3]$ is an immediate from the case $k = 0$. To conclude η^2-multiple generators in degree $8k + 3$ for all $k \geq 0$, we use the comparison from the both spectral sequences. For the explicit embedding elements in $H_*(BSD_{16})$ are obtained by lemma 5.4.3, η-BSS and lemma 6.2.4. □

From our calculation, we see that the calculation of real connective K-homology by using Greenlees spectral sequence applying on $ko^*(BSD_{16})$ gives us the enriched structure which is suitable for GLR-conjecture but that method will not cover the η-multiples. A tool to reveal them is η-Bockstein spectral sequence applying on $ku_*(BSD_{16})$. Also, the results from D.Bayen which was obtained by using Adams spectral sequence are needed to solve the extension problems. In other words, mixing of these tools is more powerful.

All in all, from our calculations, we can conclude that even if the methods that we have used to calculate connective K-theory are different, all of them still require representation theory to determine their differentials, and surprisingly, they give the same answer.

Appendix A

Basic commutative algebra

In the calculation of connective K-homology of both real and complex theory by using Greenlees spectral sequences, we need to calculate local cohomology and in that direct limits (involving to Kozsul complex) are the main thing to deal with. Also, connective K-theory is related to the completion mod p (for our calculation, $p = 2$) of representation rings, thus p-adic integers play a role in the calculation.

A.0.1 Direct limits

Here we investigate the definition and some properties of direct limit of modules and rings by doing exercise on page 32-34 of M.F. Atiyah and I.G. Macdonald's book [6].

Definition A.0.3. *Let I be a direct set, R be a ring and let $(M_i)_{i\in I}$ be a family of R module indexed by I. A* direct system *(M_i, μ_{ij}) over the direct set I consists of an R-homomorphism $\mu_{ij} : M_i \longrightarrow M_j$ for each $i \leq j \in I$ such that*

- *μ_{ij} is the identity mapping of M_i, for all $i \in I$;*
- *$\mu_{ik} = \mu_{jk} \circ \mu_{ij}$ whenever $i \leq j \leq k$.*

Let C be the direct sum of M_i and identify each module M_i with its canonical image in C. Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. The direct limit *of the direct system (M_i, μ_{ij}) is defined to be $\varinjlim M_i := C/D$. Let $\mu : C \longrightarrow \varinjlim M_i$ be the projection and let μ_i be the restriction of μ to M_i, then $\mu_i = \mu_j \circ \mu_{ij}$ whenever $i \leq j$.*

We can use this definition directly to prove:

Proposition A.0.4. *In direct system (M_i, μ_{ij}) of an R-module over the direct set $\mathbb{N}$, if there is $N \in \mathbb{N}$ such that $\mu_{(N+k)(N+k+1)}$ are an R isomorphism for all $k \geq 0$ then $M_N \cong \varinjlim M_i$.*

Remark A.0.5. *Note that:*

- *This proposition will be true for any direct set I which can be proved by very similar way to the direct set $\mathbb{N}$.*
- *The condition that $\mu_{(N+k)(N+k+1)}$ are an R isomorphism for all $k \geq 0$ is necessary and the condition that $\mu_{(N+k)(N+k+1)}$ are an R injection for all $k \geq 0$ is not sufficient because m'_t may be not lies in the domain of $\mu^{-1}_{(t-1)t}$, e.g. in direct system of (M_i, μ_{ij}) such that μ_{ij} are all inclusion, we have*

$$\varinjlim M_i \cong \bigcup_i M_i. \tag{A.1}$$

For direct system (M_i, μ_{ij}) over a ring R where $M_i = M, \forall i \in \mathbb{N}$ and $\mu_{ij} = p^{j-i}$ for all $i \leq j \in \mathbb{N}$ and $p \in R$, we define $\tau : \bigoplus_{i\in\mathbb{N}} M_i \longrightarrow M[\frac{1}{p}]$ by

$$\tau((x_1, x_2, ..., x_k, 0, 0, 0, ...)) = x_1 + \tfrac{x_2}{p} + \tfrac{x_3}{p^2} + ... + \tfrac{x_k}{p^{k-1}}.$$

It is not hard to see that τ is surjective and $\ker \tau$ is D and hence

$$\varinjlim(M \xrightarrow{p} M \xrightarrow{p} M \xrightarrow{p} \cdots) \cong M[\frac{1}{p}] \tag{A.2}$$

By very similar process as the proof of proposition above, we get a useful lemma.

Lemma A.0.6. *In the situation of definition A.0.3, we have*

1.) *every element of $\varinjlim M_i$ can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$,*

2.) *if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ and*

3.) *if $\mu_i(x_i) = \mu_j(x'_j)$ for some $j \geq i$ then $\exists, k \geq j$ such that $\mu_{ik}(x_i) = \mu_{jk}(x'_j)$*

Proof. 1.) and 2.) are obtained directly form definition, and 3.) follows from the definition and 2.). More precisely, since $\mu_i(x_i) = \mu_j(x'_j)$, by definition, $\mu_j(\mu_{ij}(x_i)) = \mu_j(x'_j)$ and hence $\mu_j(\mu_{ij}(x_i) - x'_j) = 0$. By 2.), there exist $k \geq j$ such that $\mu_{jk}(\mu_{ij}(x_i) - x'_j) = 0$ and thus 3.) is proved. □

As the result of this lemma, we obtain universal property of direct limit:

Corollary A.0.7. *(Universal property) Let N be an R- module and for each $i \in I$ let $\alpha_i : M_i \longrightarrow N$ be an R-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : \varinjlim M_i \longrightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.*

Proof. For each $m \in \varinjlim M_i$, by lemma A.0.6, $m = \mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$ and define $\alpha(m) = \alpha_i(x_i)$. This is well defined by 3.) and the detail of checking that α is unique is a routine work. □

Definition A.0.8. *Let* (M_i, μ_{ij}) *and* (N_i, ν_{ij}) *be direct systems of* R*-module over the same directed set. Let* $\mu_i : M_i \longrightarrow \varinjlim M_i$, $\nu_i : N_i \longrightarrow \varinjlim N_i$ *be the associated homomorphisms. A* homomorphism $\Phi : (M_i, \mu_{ij}) \longrightarrow (N_i, \nu_{ij})$ *is by definition a family of* R*-module homomorphisms* $\phi_i : M_i \longrightarrow N_i$ *such that* $\phi_j \circ \mu_{ij} = \nu_{ij} \circ \phi_i$ *whenever* $i \leq j$.

This homomorphism induces a unique homomorphism $\phi : \varinjlim M_i \longrightarrow \varinjlim N_i$ given by $\phi(m) = \nu_i(\phi_i(x_i))$ for each $m = \mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$, such that $\phi \circ \mu_i = \nu_i \circ \phi_i$ for all $i \in I$.

We say that a sequence of direct systems and homomorphisms

$$(M_i, \mu_{ij}) \longrightarrow (N_i, \nu_{ij}) \longrightarrow (P_i, \eta_{ij})$$

is *exact* if the corresponding sequence of modules and module homomorphisms is exact for each $i \in I$.

Proposition A.0.9. *In direct system* (M_i, μ_{ij}) *of an* R*-module over the direct set* I, *we have*

1.) *if* $(M_i, \mu_{ij}) \longrightarrow (N_i, \nu_{ij}) \longrightarrow (P_i, \eta_{ij})$ *is an exact sequence then* $\varinjlim M_i \longrightarrow \varinjlim N_i \longrightarrow \varinjlim P_i$ *is also exact and*

2.) *any* R*-module* N, $\varinjlim(M_i \otimes N) \cong (\varinjlim M_i) \otimes N$ *where* $\varinjlim(M_i \otimes N)$ *is a direct limit of a direct system* $(M_i \otimes N, \mu_{ij} \otimes 1)$.

Proof. The first statement is obtained by lemma A.0.6, definition A.0.8 and chasing diagram. The second follows by using universal property of direct limit and universal property of tensor product of module. More clear, a homomorphism $\mu_i \otimes 1 : M_i \otimes N \longrightarrow (\varinjlim M_i) \otimes N$ for each $i \in I$ induces a homomorphism $\psi : \varinjlim(M_i \otimes N) \longrightarrow (\varinjlim M_i) \otimes N$, by corollary A.0.1. On the other hand, for each $i \in I$, let $g_i : M_i \times N \longrightarrow M_i \otimes N$ be the canonical bilinear mapping. Note that $\varinjlim(M_i \times N) = (\varinjlim M_i) \otimes N$. By passing to the limit we obtain a mapping $g : (\varinjlim M_i) \times N \longrightarrow \varinjlim(M_i \otimes N)$. It is not hard to see that g is R-bilinear and hence by universal property of tensor product we get a unique homomorphism $\phi : (\varinjlim M_i) \otimes N \longrightarrow \varinjlim(M_i \otimes N)$. Now it is simple to show that $\phi \circ \psi$ and $\psi \circ \phi$ are identity mappings which completes the proof. $\square$

In our calculations, we will need to calculate $H^1_{(q)}(R) = coker(R \longrightarrow R[\frac{1}{q}])$ for some ring R and some $q \in R$ (e.g. in Chapter 3, $R = QU$, a subring of $KU^*(BSD_{16})$). So, it is reasonable to understand $R[\frac{1}{q}]$ as a direct limit. We start with;

Definition A.0.10. *Let* $(A_i)_{i \in I}$ *be a family of rings indexed by a direct set* I, *and for each pair* $i \leq j$ *in* I *let* $\alpha_{ij} : A_i \longrightarrow A_j$ *be a ring homomorphism satisfying conditions in the definition A.0.3. Regarding each* A_i *as a* $\mathbb{Z}$*-module, we can then form the direct* *limit* $\varinjlim(A_i)$. *As in [8], define multiplicative structure of* $\varinjlim A_i$ *as follows. For the coset* $a = \oplus_{i\in I} r_i$ *and* $b = \oplus_{j\in I} r'_j$ *and take* $k \geq i, j$ *for all* $i \in \{i \in I : r_i \neq 0\} = supp(a)$, $j \in \{j \in I : r'_j \neq 0\} = supp(b)$. *Define*

$$(a+J)(b+J) := \sum_{i\in supp(a), j\in supp(b)} \alpha_{ik}(r_i)\alpha_{jk}(r'_j) + J$$

where J *is the subgroup of* $\bigoplus_{i\in I} A_i$ *generated by* $r_i - \alpha_{ij}(r_i)$ *for all* $i \leq j \in I$. *This is well defined and* $\alpha_i : A_i \longrightarrow \varinjlim(A_i)$ *are ring homomorphism, see [8], page 4.*

Now, the ring structure of $\bigoplus_{i\in I} A_i$ can be given as follows. For each $i \in I$ we identify A_i with its image in $\bigoplus_{i\in I} A_i$ and for $r_i \in R_i$, $r'_j \in R_j$. Define

$$r_i \cdot r'_j = \alpha_{i(ij)}(r_i)\alpha_{j(ij)}(r'_j) \quad ;$$

for general product, we extend this product via distributive law, [8](page8). So, we can see that, for a direct system $((A_i)_{i\in\mathbb{N}}, \alpha_{ij})$ such that $A_i = A, \forall i \in \mathbb{N}$ and $\alpha_{ij} = p^{j-i}$ for all $j \geq i$, where $p \in A$,

$$\varinjlim(A \xrightarrow{p} A \xrightarrow{p} A \xrightarrow{p} \cdots) \cong A[\frac{1}{p}] \tag{A.3}$$

A.0.2 Inverse limits, completion and p-adic integers

We review the definition and some properties of inverse limit, completion and p-adic integers from Chapter 10 in [6]. In that by definition, an *inverse system* consists of a sequence of groups $\{A_n\}$ and homomorphism $\theta_{n+1} : A_{n+1} \longrightarrow A_n$. The groups of all coherent sequence (a_n) (i.e. $a_n \in A_n$ and $\theta_{n+1}a_{n+1} = a_n$) is called the *inverse limit* of this system and usually it is written by $\varprojlim A_n$. In other words, by setting $A = \Pi_{n=1}^{\infty} A_n$, one defines $d^A : A \longrightarrow A$ by,

$$d^A(a_n) = a_n - \theta_{n+1}(a_{n+1}),$$

then $\varprojlim A_n \cong \ker(d^A)$ and the $coker(d^A)$ is normally denoted by $\varprojlim{}^1 A_n$.

The exactness properties of inverse limit is different to direct limit, i.e. it is merely left exact functor. Precisely, by proposition 10.2 in [6], one has that if $0 \longrightarrow \{A_n\} \longrightarrow \{B_n\} \longrightarrow \{C_n\} \longrightarrow 0$, is an exact sequence of inverse systems, then

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n$$

is always exact. If, moreover, $\{A_n\}$ is a surjective system (i.e. $\theta_{n+1} : A_{n+1} \longrightarrow A_n$ is surjective for all n), then

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \longrightarrow \varprojlim C_n \longrightarrow 0$$

is exact.

Closely related to the inverse limit is completion. For the topological group G topologized by the neighborhood of 0 in G, the *completion of* G, denoted by $\widehat{G}$, is the set of all equivalence classes of Cauchy sequence. It is an abelian group under the additive operation, i.e., if (x_i) and (y_j) are two represented Cauchy sequence classes in $\widehat{G}$, so is $(x_i + y_j)$. Moreover, for each $x \in G$, we define $\phi(x)$ to be the constant sequence (x) in $\widehat{G}$ for $\phi : G \longrightarrow \widehat{G}$. This is a homomorphism of abelian groups which is not in general injective. In fact, we have $\ker(\phi) = \cap U$, where U runs through all neighborhoods of 0 in G, and furthermore ϕ is injective if and only if G is Hausdorff, [6].

The relevance to the inverse limit comes from the alternative purely algebraic definitions of completion, namely, by using topologies given by the sequences of subgroups $G = G_0 \supseteq G_1 \supseteq \cdots G_n \supseteq \cdots$, i.e., $U \subseteq G$ is a neighborhood of 0 if and only if it contains some G_n. Note that the subgroups G_n of G are both open and closed in that topology. By this topology, one can form the inverse system $\{G/G_n\}$ with $\theta_{n+1} : G/G_{n+1} \longrightarrow G/G_n$ and one can show that

$$\widehat{G} \cong \varprojlim G/G_n, \tag{A.4}$$

see details of this isomorphism in page 103 of [6].

Since $\{G'/G'_n\}$ is surjective system, for any short exact sequence of groups, $0 \longrightarrow G' \longrightarrow G \xrightarrow{p} G'' \longrightarrow 0$ yields the short exact sequence,

$$0 \longrightarrow \widehat{G'} \longrightarrow \widehat{G} \longrightarrow \widehat{G''} \longrightarrow 0,$$

where G has topology defined by a sequence $\{G_n\}$ of subgroups and gives G', G'' the induces topologies, i.e. by the sequence $\{G'_n \cap G_n\}$, $\{pG_n\}$, (Corollary 10.3, [6]). The consequence is that $\widehat{G}_n$ is a subgroup of $\widehat{G}$ and

$$\widehat{G}/\widehat{G}_n \cong G/G_n \text{ and } \widehat{\widehat{G}} \cong \widehat{G}.$$

The other relevances of inverse limit is the completion of topological ring (the ring operations are continuous) given by the sequences of its ideal. For topological ring $G = A$ and its ideal $\mathfrak{a}$, the sequence of the ideal $\mathfrak{a}$ is $A = \mathfrak{a}^0 \supseteq \mathfrak{a}^1 \supseteq \mathfrak{a}^2 \supseteq \cdots$ and the $\mathfrak{a}$-*adic topology* is given by

$$A^\wedge_{\mathfrak{a}} = \varprojlim A/\mathfrak{a}^n.$$

This completion is again a topological ring and $\phi : A \longrightarrow A^\wedge_{\mathfrak{a}}$ is a continuous ring homomorphism whose kernel is $\cap\mathfrak{a}^n$. The topology is Hausdorff if and only if $\cap\mathfrak{a}^n = 0$ if and only if A is a completion ring.

Likewise for an A-module M; take $G = M$ and $G_n = \mathfrak{a}^n M$ and the completion of M is call $\mathfrak{a}$-*adic topology* on M which is

$$\widehat{M} = \varprojlim M/\mathfrak{a}^n M.$$

This is a topological $\widehat{A}$-module. If $M \longrightarrow N$ is any $A-$module homomorphism then $f(\mathfrak{a}^n M) = \mathfrak{a}^n f(M) \subseteq \mathfrak{a}^n N$, and thus f is continuous (with respect to the $\mathfrak{a}$-adic topology on M and N) and so defines $\widehat{f} : \widehat{M} \longrightarrow \widehat{N}$, page 105, [6].

The example of our interest is p-adic integers, where p is prime, i.e. by taking $A = \mathbb{Z}$ and $\mathfrak{a} = (p)$. It elements are infinite series $\sum_{n=0}^{\infty} a_n p^n$, $0 \leq a_n \leq p-1$ which $p^n \longrightarrow 0$ as $n \longrightarrow 0$, [6]. In other words,

$$\mathbb{Z}_p^{\wedge} = \varprojlim \mathbb{Z}/p^k,$$

which contains all sequence $(a_1, a_2, a_3, ...)$ such that $a_k \in \mathbb{Z}/p^k$ and $a_{k+1} \equiv a_k \, mod \, p^k$.

Here, for example, we see that $\frac{1}{3} \in \varprojlim \mathbb{Z}/2^k \cong \mathbb{Z}_2^{\wedge}$ because we can view $\frac{1}{3} = 1 + (-2) + (-2)^2 + (-2)^3 + ...$ as the coherent sequence $(a_1, a_2, a_3, ...)$, [i.e. a_i is the image of $1 + (-2) + (-2)^2 + (-2)^3 + ...$ in $\mathbb{Z}/2^i$], where

$$\begin{array}{lcl} a_1 & = & 1 \equiv 1 \quad mod 2, \\ a_2 & = & 1 + (-2) \equiv -1 \quad mod 4, \\ a_3 & = & 1 + (-2) + (-2)^2 \equiv 3 \quad mod 8, \\ a_4 & = & 1 + (-2) + (-2)^2 + (-2)^3 \equiv -5 \quad mod 16 \\ \vdots & \vdots & \vdots \end{array}$$

which is easy to see that $a_{k+1} \equiv a_k \, mod \, 2^k$. In general, we claim that $\frac{p}{q} \in \mathbb{Z}_2^{\wedge}$ for all $p, q \in \mathbb{Z}$ s.t. $(p, q) = 1$ and q is odd. To prove this, we will use another description of $\mathbb{Z}_2^{\wedge}$, i.e. as the completion of the matric space $(\mathbb{Z}, d_2)$, where

$$d_2(a, b) = \begin{cases} \frac{1}{\nu_2(a-b)}, & \text{if } a \neq b; \\ 0, & \text{if } a = b, \end{cases}$$

and $\nu_2(n) = s$ if $n = 2^s \cdot odd$.

Proposition A.0.11. *For $p, q \in \mathbb{Z}$, $(p, q) = 1$ and $q \neq 0$, we have that*

$$\tfrac{p}{q} \in \mathbb{Z}_2^{\wedge} \iff q \textit{ is odd.}$$

Proof. $(\Longleftarrow)$: It is clear that $p \in \mathbb{Z}_2^{\wedge}$ (as an eventually constant sequence), so by the closed property of ring multiplication of $\mathbb{Z}_2^{\wedge}$, we need only to check that $\frac{1}{q} \in \mathbb{Z}_2^{\wedge}$ for any odd number q. As before, we view $\frac{1}{q}$ as

$$\frac{1}{q} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ...,$$

where x is an even integer. From this form, we get coherent sequence $(q_n) = (a_1, a_2, a_3, ...)$, where $a_i = 1 + x + x^2 + ... + x^i$, the image of $\frac{1}{q}$ in $\mathbb{Z}/2^i$. It is not hard to check that (q_n) is a Cauchy sequence in the matric space $(\mathbb{Z}, d_2)$. Precisely, for $r \geq s$, $a_r - a_s = x^s + x^{s+1} + ... + x^r = x^s(1 + x + x^2 + ... + x^r)$, then $\nu_2(a_r - a_s) \geq s$ and thus

$d_2(a_r, a_s) \leq \frac{1}{s}$ which yields (q_n) to be a Cauchy sequence, i.e. $\frac{1}{q} \in \mathbb{Z}_2^\wedge$.

$(\Longrightarrow)$: Suppose q is even (p is odd) and $\frac{p}{q} \in \mathbb{Z}_2^\wedge$, then $(\frac{1}{p})(\frac{p}{q}) = \frac{1}{q} \in \mathbb{Z}_2^\wedge$, since $\frac{1}{p} \in \mathbb{Z}_2^\wedge$. This implies that q is a unit in the ring $\mathbb{Z}_2^\wedge$ and then $q \quad mod 2$ must be a unit in the ring $\mathbb{Z}/2$ via any ring homomorphism $\mathbb{Z}_2^\wedge \longrightarrow \mathbb{Z}/2$. Hence q must be odd which contradicts to the assumption. $\square$

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