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Regularization of Piecewise Smooth Two-Folds

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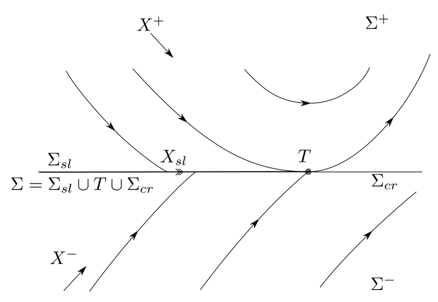
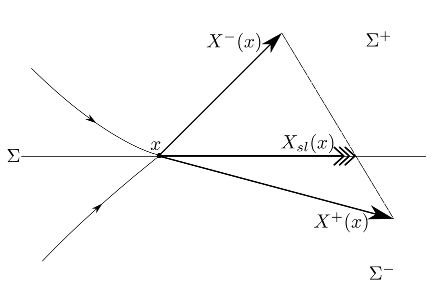
1 Piecewise Smooth Systems

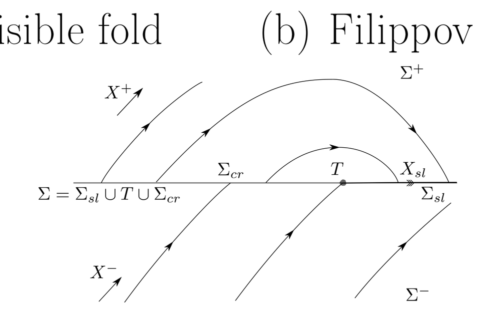
$X = (X^+, X^-)$ with vector-fields $(X^\pm, \Sigma^\pm)$ is a piecewise smooth (PWS) system. $\Sigma = \overline{\Sigma^+} \cap \overline{\Sigma^-} : f(x, y, z) = 0$ is the switching manifold. Locally we take $f(x, y, z) = y$. Σ is divided into sliding: Σ_{sl}, crossing: Σ_{cr} and tangencies: T, see Fig. (a). On Σ_{sl} we adopt the Filippov convention [2] of sliding (see Fig. (b)) to obtain a vector-field (X_{sl}, Σ_{sl}).



(a) Visible fold



(b) Filippov convention



(c) Invisible fold

2 Singularities

$p \in \Sigma$ is a tangency of $X^\pm$ with Σ if $X^\pm f(p) = 0$. A tangency is a fold if $X^\pm f(X^\pm f)(p) \neq 0$, being visible when > 0 (see Fig. (a)), invisible when < 0 (see Fig. (c)). Here $X^\pm f = X^\pm \cdot \nabla f = X_2^\pm$ is the Lie-derivative. A two-fold $p \in \Sigma$ is a fold from above and below: $X^+ f(p) = X^- f(p) = 0$.

3 Two-Folds in $\mathbb{R}^3$

Proposition. [3] Generically, a two-fold p in $\mathbb{R}^3$ is the transverse intersection of two lines $l^- : x = y = 0, z \in [-c^{-1}, c^{-1}]$, $l^+ : y = z = 0, x \in [-c^{-1}, c^{-1}]$ consisting of fold points of $X^\mp$, respectively. □

The lines $l^\pm$ divide $\Sigma : y = 0$ into four separate regions:

- Stable sliding $\Sigma_{sl}^- : x \leq 0, z \leq 0$.
- Unstable sliding $\Sigma_{sl}^+ : x \geq 0, z \geq 0$.
- Crossing downwards $\Sigma_{cr}^- : x \geq 0, z \leq 0$.
- Crossing upwards $\Sigma_{cr}^+ : x \leq 0, z \geq 0$.

See Fig. (d)-(f). A two-fold is:

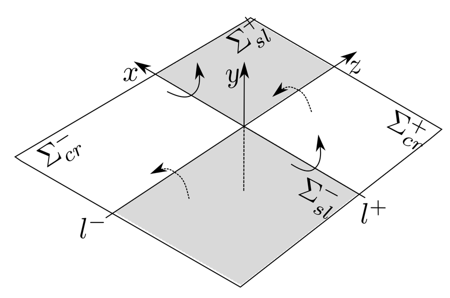
- Visible if $l^\pm$ are both visible (Fig. (d)).
- Visible-invisible if l^+ visible, l^- invisible (Fig. (e)).
- Invisible if $l^\pm$ are both invisible (Fig. (f)).

Definition. A singular canard of a PWS is a trajectory of $(\Sigma_{sl}^\pm, X_{sl}^\pm)$ having a continuation through the two-fold singularity p. □

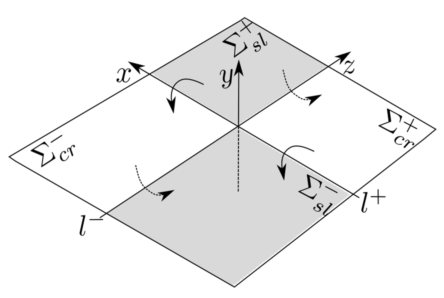
The two-fold p is an equilibrium of the vector-field $F_\mp X_{sl}^\mp$, defined in $\Sigma_{sl}^- \cup \{p\} \cup \Sigma_{sl}^-$, and with $F_\mp = H(x, z)$ a scalar smooth function which is positive (negative) for $x, z < 0$ ($x, z > 0$). Then:

Proposition. [3] Non-degenerate singular canards exists if and only if p corresponds to a node or a saddle of $F\Sigma_{sl}^-$, and an eigenspace is contained within $\Sigma_{sl} \cup \{p\}$. □

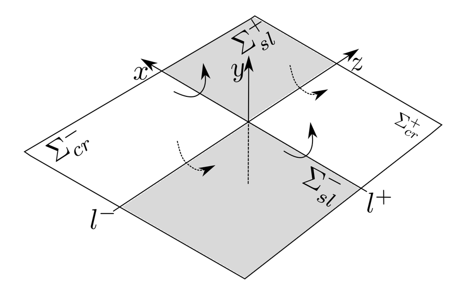
See Fig. (g).



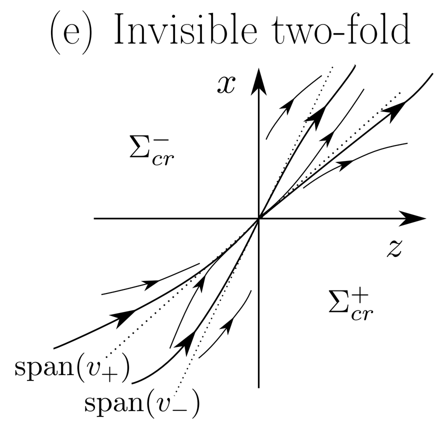
(d) Visible two-fold



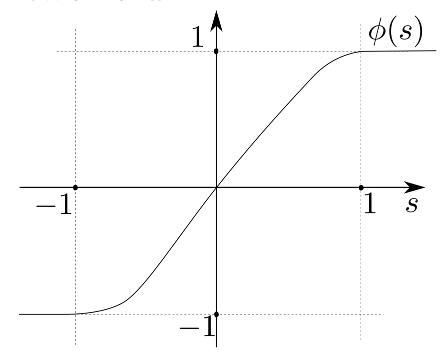
(e) Invisible two-fold



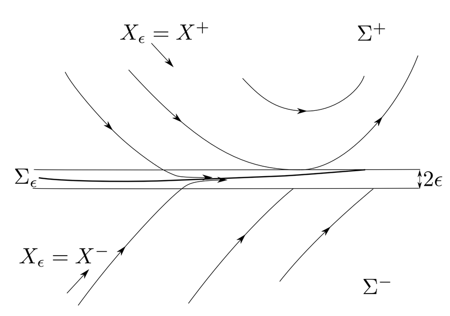
(f) Visible-invisible two-fold



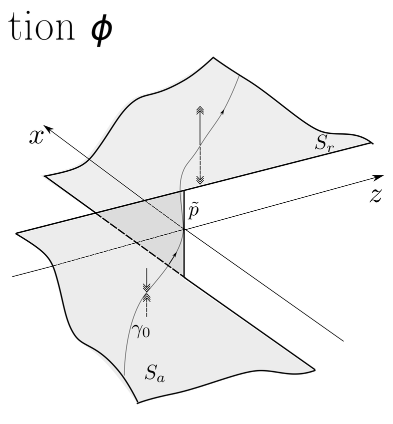
(g) Singular canards for visible-invisible case



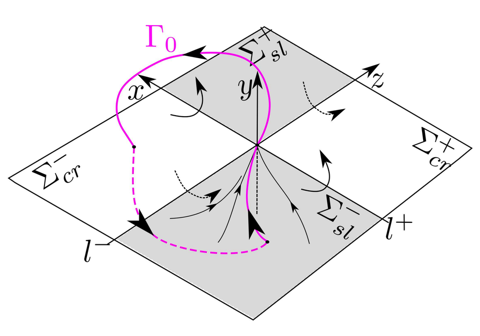
(h) Regularization function ϕ



(i) Regularized system X_ϵ



(j) Critical manifolds $S_{a,r}$ and $\tilde{p}$



(k) Singular cycle Γ_0

4 Regularization

- What happens to the two-fold/singular canards when we regularize the PWS system?
- Can we learn something about the PWS system by regularizing?

We consider the Sotomayor-Teixeira regularization [5]:

$$X_\epsilon = \frac{1}{2}X^+(1 + \phi(\epsilon^{-1}y)) + \frac{1}{2}X^-(1 - \phi(\epsilon^{-1}y)),$$

with $\epsilon \ll 1$ (see Fig. (h) and (i)). Writing $y = \epsilon\hat{y}$ we obtain a hidden slow-fast system with (x, z) slow and $\hat{y}$ fast.

Theorem. [3] X_0 has critical manifolds: $S_a = \Sigma_{sl}^-$ (attracting), $S_r = \Sigma_{sl}^+$ (repelling) and a non-hyperbolic line $\tilde{p} : x = z = 0, \hat{y} \in (-1, 1)$ (see Fig. (j)). On $S_{a,r}$: Reduced system = Filippov sliding system. □

Note that in terms of $y = \epsilon\hat{y}$ we have $\tilde{p} = p$.

5 Blowup

To study the persistence of canards we blowup the nonhyperbolic line $\tilde{p}$: $x = r\bar{x}, z = r\bar{z}, \epsilon = r^2\bar{\epsilon}, (\bar{x}, \bar{z}, \bar{\epsilon}) \in S^2$ following the formulation of Krupa and Szmolyan [4]. We study the phase space using directional charts $\kappa_1 : \bar{x} = -1$, $\kappa_3 : \bar{x} = 1$ and a rescaling chart: $\kappa_2 : \bar{\epsilon} = 1$. We obtain:

Theorem. [3] Singular canards ⇒ (Primary, maximal) Canards as transverse intersections of continuations of Fenichel slow manifolds $S_{a,\epsilon}$ and $S_{r,\epsilon}$ provided a certain non-resonance condition holds true. These maximal canards are $\mathcal{O}(\sqrt{\epsilon})$-close to the singular canards. □

Result and approach very similar to [6, 7] for folds in slow-fast systems in $\mathbb{R}^3$. But the geometry is very different.

6 Visible-Invisible Two-Fold

The two-fold is associated with forward and backwards non-uniqueness. By regularizing we can pick the “right orbits”.

Theorem. Consider the visible-invisible case and suppose as in Fig. (k) that there exists a singular cycle Γ_0 (satisfying certain non-degeneracy conditions, see also [1]). Then for $\epsilon \ll 1$ sufficiently small X_ϵ possesses an attracting limit cycle Γ_ϵ satisfying $\Gamma_\epsilon = \Gamma_0 + \mathcal{O}(\sqrt{\epsilon})$. □

PWS orbit Γ_0 is therefore distinguished, as $\Gamma_0 = \lim_{\epsilon \to 0} \Gamma_\epsilon$, among all the orbits through p. Note that these results hold true for all monotone regularization functions.

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