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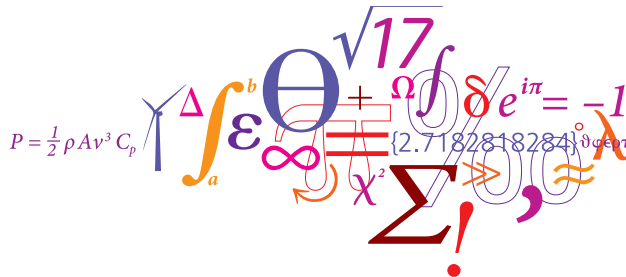
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On Solving Large-Scale Free Material Optimization Problems

Alemseged Gebrehiwot Weldeyesus, Mathias Stolpe

Technical University of Denmark (DTU)



Free Material Optimization (FMO)

- The design variable is the entire material tensor E

$$\sigma_{ij}(x) = E_{ijkl}(x)e_{kl}(x)$$

- allowed to vary freely at each point of the design domain
- the only requirement is that the material tensors should be symmetric and positive semidefinite

M. P. Bendsøe, J. M. Guedes, R. B. Haber, P. Pedersen, and J. E. Taylor. An analytical model to predict optimal material properties in the context of optimal structural design. *Journal of Applied Mechanics*, 61:930–937, 1994.

U. T. Ringertz. On finding the optimal distribution of material properties. *Structural Optimization*, 5:265–267, 1993.

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J. Haslinger, M. Kočvara, G. Leugering, and M. Stingl. Multidisciplinary free material optimization. *SIAM Journal on Applied Mathematics*, 70(7):2709–2728, 2010.

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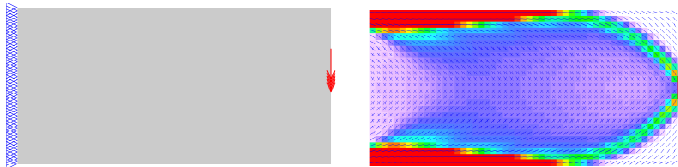
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- obtains conceptual optimal structures regarded as ultimately best designs.
- can be used to generate benchmark solutions for other models and besides to propose novel ideas for new design situations.

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Motivation

FMO problems

- demand fine finite element discretization, hence large problems
- involve matrix inequities/variables
- are modeled as nonlinear (mostly non convex) [semidefinite programming](#)
- are computationally expensive and more difficult than standard optimization problems.

Most today's methods are based on first-order method leading to many iterations.

Objective

- To develop special purpose second order interior point method that can robustly and efficiently solve large-scale FMO problems.

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- To develop special purpose second order interior point method that can robustly and efficiently solve large-scale FMO problems.
- To investigate numerically the performance of
 - some equivalent FMO problem formulations
 - the most common directions (AHO, HRVW/KSH/M, and NT directions) for solving FMO problems.

Primal problems (SAND)

Minimum compliance

$$\begin{aligned} & \text{minimize} && \sum_{\ell \in L} w_{\ell} \mathbf{f}_{\ell}^T \mathbf{u}_{\ell} \\ & \mathbf{u}_{\ell} \in \mathbb{R}^n, \mathbf{E} \in \mathbb{E} \\ & \text{subject to} && \mathbf{A}(\mathbf{E}) \mathbf{u}_{\ell} = \mathbf{f}_{\ell}, \ell \in L, \\ & && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V. \end{aligned}$$

where the discrete set of admissible materials

$$\mathbb{E} := \left\{ \mathbf{E} \in (\mathbb{R}^{dm \times d}) \mid \mathbf{E}_i = \mathbf{E}_i^T \succeq 0, \underline{\rho} \leq \text{Tr}(\mathbf{E}_i) \leq \bar{\rho}, i = 1, \dots, m \right\}$$

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 & && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V.
 \end{aligned}$$

Minimum weight

$$\begin{aligned}
 & \text{minimize} && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\
 & \mathbf{u}_{\ell} \in \mathbb{R}^n, \mathbf{E} \in \mathbb{E} \\
 & \text{subject to} && \mathbf{A}(\mathbf{E}) \mathbf{u}_{\ell} = \mathbf{f}_{\ell}, \ell \in \mathcal{L}, \\
 & && \sum_{\ell \in L} w_{\ell} \mathbf{f}_{\ell}^T \mathbf{u}_{\ell} \leq \gamma,
 \end{aligned}$$

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Primal problems (Nested)

Minimum compliance

$$\begin{aligned} & \underset{\mathbf{E} \in \mathbb{E}}{\text{minimize}} && \sum_{\ell \in L} w_{\ell} \mathbf{f}_{\ell}^T \mathbf{A}^{-1}(\mathbf{E}) \mathbf{f}_{\ell} \\ & \text{subject to} && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V. \end{aligned}$$

Minimum weight

$$\begin{aligned} & \underset{\mathbf{E} \in \mathbb{E}}{\text{minimize}} && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\ & \text{subject to} && \sum_{\ell \in L} w_{\ell} \mathbf{f}_{\ell}^T \mathbf{A}^{-1}(\mathbf{E}) \mathbf{f}_{\ell} \leq \gamma \end{aligned}$$

where the discrete set of admissible materials

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Primal problems (Linear)

Primal problems (Linear)

Minimum compliance

$$\begin{aligned}
 & \underset{\mathbf{E} \in \mathbb{E}, \varrho_\ell \geq 0}{\text{minimize}} && \sum_{\ell \in L} w_\ell \varrho_\ell \\
 & \text{subject to} && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V, \\
 & && \begin{pmatrix} \varrho_\ell & \mathbf{f}_\ell^T \\ \mathbf{f}_\ell & \mathbf{A}(\mathbf{E}) \end{pmatrix} \succeq 0, \forall \ell \in L.
 \end{aligned}$$

Minimum weight

$$\begin{aligned}
 & \underset{\mathbf{E} \in \mathbb{E}, \tau_\ell \geq 0}{\text{minimize}} && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\
 & \text{subject to} && \sum_{\ell \in \mathcal{L}} w_\ell \tau_\ell \leq \bar{\gamma}, \\
 & && \begin{pmatrix} \tau_\ell & \mathbf{f}_\ell^T \\ \mathbf{f}_\ell & \mathbf{A}(\mathbf{E}) \end{pmatrix} \succeq 0, \forall \ell \in \mathcal{L}.
 \end{aligned}$$

$$\mathbb{E} := \left\{ \mathbf{E} \in (\mathbb{R}^{dm \times d}) \mid \mathbf{E}_i = \mathbf{E}_i^T \succeq 0, \underline{\rho} \leq \text{Tr}(\mathbf{E}_i) \leq \bar{\rho}, i = 1, \dots, m \right\}$$

Very **large-scale matrix inequalities** which are difficult to deal within computations.

Dual problems

The minimum compliance primal problem

$$\begin{aligned}
 & \underset{\mathbf{u}_\ell \in \mathbb{R}^n, \mathbf{E}}{\text{minimize}} && \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell \\
 & \text{subject to} && \mathbf{A}(\mathbf{E}) \mathbf{u}_\ell = \mathbf{f}_\ell, \ell \in L, \\
 & && \mathbf{E}_i = \mathbf{E}_i^T \succeq 0, i = 1, \dots, m, \\
 & && \underline{\rho} \leq \text{Tr}(\mathbf{E}_i) \leq \bar{\rho}, i = 1, \dots, m, \\
 & && \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V.
 \end{aligned}$$

has the dual problem given by

$$\begin{aligned}
 & \underset{\substack{\mathbf{u}_1, \dots, \mathbf{u}_L \in \mathbb{R}^n \\ \alpha \in \mathbb{R}, \underline{\beta} \in \mathbb{R}^m, \underline{\beta} \in \mathbb{R}^m}}{\text{maximize}} && -\alpha \bar{V} + 2 \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell + \underline{\rho} \sum_{i=1}^m \underline{\beta}_i - \bar{\rho} \sum_{i=1}^m \bar{\beta}_i \\
 & \text{subject to} && \sum_{\ell \in L} \sum_{k=1}^{n_G} w_\ell \mathbf{B}_{i,k}^T \mathbf{u}_\ell \mathbf{u}_\ell^T \mathbf{B}_{i,k} - (\alpha - \underline{\beta}_i + \bar{\beta}_i) \mathbf{I} \preceq 0, i = 1, \dots, m \\
 & && \alpha \geq 0, \underline{\beta} \geq 0, \bar{\beta} \geq 0.
 \end{aligned}$$

Primal and dual problems

Minimum complinace problems (SAND, Nested, and Dual)

$$\begin{aligned}
 & \text{minimize}_{\mathbf{u}_\ell \in \mathbb{R}^n, \mathbf{E} \in \mathbb{E}} \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell \\
 & \text{subject to} \quad \mathbf{A}(\mathbf{E}) \mathbf{u}_\ell = \mathbf{f}_\ell, \ell \in L, \\
 & \quad \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V.
 \end{aligned}$$

$$\begin{aligned}
 & \text{minimize}_{\mathbf{E} \in \mathbb{E}} \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{A}^{-1}(\mathbf{E}) \mathbf{f}_\ell \\
 & \text{subject to} \quad \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V.
 \end{aligned}$$

$$\begin{aligned}
 & \text{maximize}_{\mathbf{u}_1, \dots, \mathbf{u}_{n_L} \in \mathbb{R}^n, \alpha \in \mathbb{R}, \underline{\beta} \in \mathbb{R}^m, \bar{\beta} \in \mathbb{R}^m} -\alpha \bar{V} + 2 \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell + \underline{\rho} \sum_{i=1}^m \underline{\beta}_i - \bar{\rho} \sum_{i=1}^m \bar{\beta}_i \\
 & \text{subject to} \quad \sum_{\ell \in L} \sum_{k=1}^{n_G} w_\ell \mathbf{B}_{i,k}^T \mathbf{u}_\ell \mathbf{u}_\ell^T \mathbf{B}_{i,k} - (\alpha - \underline{\beta}_i + \bar{\beta}_i) \mathbf{I} \preceq 0, i = 1, \dots, m \\
 & \quad \alpha \geq 0, \bar{\beta} \geq 0, \underline{\beta} \geq 0.
 \end{aligned}$$

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Minimum complinace problems (SAND, Nested, and Dual)

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$$\begin{aligned} & \text{maximize}_{\mathbf{u}_1, \dots, \mathbf{u}_{n_L} \in \mathbb{R}^n} \\ & \quad \quad \quad \alpha \in \mathbb{R}, \underline{\beta} \in \mathbb{R}^m, \bar{\beta} \in \mathbb{R}^m \end{aligned}$$

$$-\alpha \bar{V} + 2 \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell + \underline{\rho} \sum_{i=1}^m \underline{\beta}_i - \bar{\rho} \sum_{i=1}^m \bar{\beta}_i$$

$$\text{subject to} \quad \sum_{\ell \in L} \sum_{k=1}^{n_G} w_\ell \mathbf{B}_{i,k}^T \mathbf{u}_\ell \mathbf{u}_\ell^T \mathbf{B}_{i,k} - (\alpha - \underline{\beta}_i + \bar{\beta}_i) \mathbf{I} \preceq 0, i = 1, \dots, m$$

Minimum weight problems (SAND and Nested)

$$\begin{aligned} & \text{minimize}_{\mathbf{u}_\ell \in \mathbb{R}^n, \mathbf{E} \in \mathbb{E}} \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\ & \text{subject to} \quad \mathbf{A}(\mathbf{E}) \mathbf{u}_\ell = \mathbf{f}_\ell, \ell \in \mathcal{L}, \\ & \quad \quad \quad \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell \leq \gamma, \end{aligned}$$

$$\begin{aligned} & \text{minimize}_{\mathbf{E} \in \mathbb{E}} \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\ & \text{subject to} \quad \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{A}^{-1}(\mathbf{E}) \mathbf{f}_\ell \leq \gamma \end{aligned}$$

Primal-dual interior point method

Consider the nonlinear semidefinite problem

$$\begin{array}{ll} \text{minimize} & f(\mathbf{X}, \mathbf{u}) \\ \text{subject to} & \mathbf{X} \in \mathcal{S}, \mathbf{u} \in \mathbb{R}^n \\ & g_j(\mathbf{X}, \mathbf{u}) \leq 0, \quad j = 1, \dots, k, \\ & \mathbf{X}_i \succeq 0, \quad i = 1, \dots, m, \end{array} \quad (\text{P})$$

where, $\mathcal{S} = \mathbb{S}^{d_1} \times \mathbb{S}^{d_2} \times \dots \times \mathbb{S}^{d_m}$ and $(d_1, d_2, \dots, d_m) \in \mathbb{N}^m$.

Primal-dual interior point method

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$$\begin{array}{ll}
 \text{minimize} & f(\mathbf{X}, \mathbf{u}) \\
 \text{subject to} & \mathbf{X} \in \mathcal{S}, \mathbf{u} \in \mathbb{R}^n \\
 & g_j(\mathbf{X}, \mathbf{u}) \leq 0, \quad j = 1, \dots, k, \\
 & \mathbf{X}_i \succeq 0, \quad i = 1, \dots, m,
 \end{array} \tag{P}$$

where, $\mathcal{S} = \mathcal{S}^{d_1} \times \mathcal{S}^{d_2} \times \dots \times \mathcal{S}^{d_m}$ and $(d_1, d_2, \dots, d_m) \in \mathbb{N}^m$,

We introduce the slack variables $\mathbf{s} \in \mathbb{R}^k$ and barrier parameter $\mu > 0$ and formulate the associated barrier problem as

$$\begin{array}{ll}
 \text{minimize} & f(\mathbf{X}, \mathbf{u}) - \mu \sum_{i=1}^m \ln(\det(\mathbf{X}_i)) - \mu \sum_{j=1}^k \ln(s_j) \\
 \text{subject to} & \mathbf{X} \in \mathcal{S}_+, \mathbf{u} \in \mathbb{R}^n, \mathbf{s} \in \mathbb{R}_+^k \\
 & g_j(\mathbf{X}, \mathbf{u}) + s_j = 0, \quad j = 1, \dots, k.
 \end{array} \tag{BP}$$

The central idea in interior point methods is that to solve the barrier problem (BP) for a sequence of barrier parameter $\mu_k \rightarrow 0$

Optimality conditions

The first-order optimality conditions of the barrier problem (BP) are

$$\begin{pmatrix} \nabla_{\mathbf{X}} f(\mathbf{X}, \mathbf{u}) - \mathbf{Z} + \nabla_{\mathbf{X}} (g(\mathbf{X}, \mathbf{u})^T \boldsymbol{\lambda}) \\ \nabla_{\mathbf{u}} f(\mathbf{X}, \mathbf{u}) + \nabla_{\mathbf{u}} g(\mathbf{X}, \mathbf{u})^T \boldsymbol{\lambda} \\ \mathbf{S} \boldsymbol{\Lambda} \mathbf{e} - \mu \mathbf{e} \\ g(\mathbf{X}, \mathbf{u}) + \mathbf{s} \\ \mathbf{X} \mathbf{Z} - \mu \mathbf{I} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Optimality conditions

The first-order optimality conditions of the barrier problem (BP) are

$$\begin{pmatrix} \nabla_{\mathbf{X}} f(\mathbf{X}, \mathbf{u}) - \mathbf{Z} + \nabla_{\mathbf{X}}(g(\mathbf{X}, \mathbf{u})^T \boldsymbol{\lambda}) \\ \nabla_{\mathbf{u}} f(\mathbf{X}, \mathbf{u}) + \nabla_{\mathbf{u}} g(\mathbf{X}, \mathbf{u})^T \boldsymbol{\lambda} \\ \mathbf{S} \boldsymbol{\Lambda} \mathbf{e} - \mu \mathbf{e} \\ g(\mathbf{X}, \mathbf{u}) + \mathbf{s} \\ H_P(\mathbf{X} \mathbf{Z}) - \mu \mathbf{I} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

where

$H_P : \mathbb{R}^{n \times n} \rightarrow \mathbb{S}^n$, defined by $H_P(\mathbf{Q}) := \frac{1}{2} \left(\mathbf{P} \mathbf{Q} \mathbf{P}^{-1} + (\mathbf{P} \mathbf{Q} \mathbf{P}^{-1})^T \right)$.

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- $\mathbf{P} = \mathbf{I}$, AHO direction
- $\mathbf{P} = \mathbf{Z}^{1/2}$, HRVW/KSH/M direction (**H.K.M.**)
- $\mathbf{P} = \mathbf{X}^{-1/2}$, dual HRVW/KSH/M direction
- $\mathbf{P} = \mathbf{W}^{-1/2}$, NT direction
 $\mathbf{W} = \mathbf{X}^{1/2} (\mathbf{X}^{1/2} \mathbf{Z} \mathbf{X}^{1/2})^{-1/2} \mathbf{X}^{1/2}$

Applying Newton's method we solve the reduced system

$$\begin{pmatrix} \mathbf{G} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{B} \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u} \\ \Delta \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{r}}_d \\ \tilde{\mathbf{r}}_p \end{pmatrix}$$

$$\mathbf{G} = \mathbf{G}(\tilde{\mathbf{H}}^{-1}), \mathbf{A} = \mathbf{A}(\tilde{\mathbf{H}}^{-1}), \mathbf{B} = \mathbf{B}(\tilde{\mathbf{H}}^{-1})$$

$$\tilde{\mathbf{r}}_d = \tilde{\mathbf{r}}_d(\tilde{\mathbf{H}}^{-1}), \tilde{\mathbf{r}}_p = \tilde{\mathbf{r}}_p(\tilde{\mathbf{H}}^{-1})$$

$$\tilde{\mathbf{H}} = \nabla_{\mathbf{X}\mathbf{X}}^2 \mathcal{L}_\mu(\mathbf{X}, \mathbf{u}, \mathbf{s}, \boldsymbol{\lambda}) + \mathcal{F}^{-1} \boldsymbol{\mathcal{E}}$$

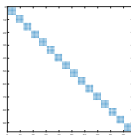
Applying Newton's method we solve the reduced system

$$\begin{pmatrix} \mathbf{G} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{B} \end{pmatrix} \begin{pmatrix} \Delta \mathbf{u} \\ \Delta \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{r}}_d \\ \tilde{\mathbf{r}}_p \end{pmatrix}$$

$$\mathbf{G} = \mathbf{G}(\tilde{\mathbf{H}}^{-1}), \mathbf{A} = \mathbf{A}(\tilde{\mathbf{H}}^{-1}), \mathbf{B} = \mathbf{B}(\tilde{\mathbf{H}}^{-1})$$

$$\tilde{\mathbf{r}}_d = \tilde{\mathbf{r}}_d(\tilde{\mathbf{H}}^{-1}), \tilde{\mathbf{r}}_p = \tilde{\mathbf{r}}_p(\tilde{\mathbf{H}}^{-1})$$

$$\tilde{\mathbf{H}} = \nabla_{\mathbf{X}\mathbf{X}}^2 \mathcal{L}_\mu(\mathbf{X}, \mathbf{u}, \mathbf{s}, \lambda) + \mathcal{F}^{-1} \mathcal{E}$$



The other search directions ($\Delta \mathbf{X}$, $\Delta \mathbf{s}$, $\Delta \mathbf{Z}$) are then determined

Problem formulations

Minimum complinace problems (SAND, Nested, and Dual)

$$\begin{aligned} & \text{minimize}_{\mathbf{u}_\ell \in \mathbb{R}^n, \mathbf{E} \in \mathbb{E}} \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell \\ & \text{subject to} \quad \mathbf{A}(\mathbf{E}) \mathbf{u}_\ell = \mathbf{f}_\ell, \ell \in L, \\ & \quad \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V. \end{aligned}$$

$$\begin{aligned} & \text{minimize}_{\mathbf{E} \in \mathbb{E}} \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{A}^{-1}(\mathbf{E}) \mathbf{f}_\ell \\ & \text{subject to} \quad \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \leq V. \end{aligned}$$

$$\begin{aligned} & \text{maximize}_{\mathbf{u}_1, \dots, \mathbf{u}_{n_L} \in \mathbb{R}^n, \alpha \in \mathbb{R}, \underline{\beta} \in \mathbb{R}^m, \bar{\beta} \in \mathbb{R}^m} -\alpha \bar{V} + 2 \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell + \underline{\rho} \sum_{i=1}^m \underline{\beta}_i - \bar{\rho} \sum_{i=1}^m \bar{\beta}_i \\ & \text{subject to} \quad \sum_{\ell \in L} \sum_{k=1}^{n_G} w_\ell \mathbf{B}_{i,k}^T \mathbf{u}_\ell \mathbf{u}_\ell^T \mathbf{B}_{i,k} - (\alpha - \underline{\beta}_i + \bar{\beta}_i) \mathbf{I} \preceq 0, i = 1, \dots, m \end{aligned}$$

Minimum weight problems (SAND and Nested)

$$\begin{aligned} & \text{minimize}_{\mathbf{u}_\ell \in \mathbb{R}^n, \mathbf{E} \in \mathbb{E}} \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\ & \text{subject to} \quad \mathbf{A}(\mathbf{E}) \mathbf{u}_\ell = \mathbf{f}_\ell, \ell \in \mathcal{L}, \\ & \quad \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{u}_\ell \leq \gamma, \end{aligned}$$

$$\begin{aligned} & \text{minimize}_{\mathbf{E} \in \mathbb{E}} \sum_{i=1}^m \text{Tr}(\mathbf{E}_i) \\ & \text{subject to} \quad \sum_{\ell \in L} w_\ell \mathbf{f}_\ell^T \mathbf{A}^{-1}(\mathbf{E}) \mathbf{f}_\ell \leq \gamma \end{aligned}$$

2D problems

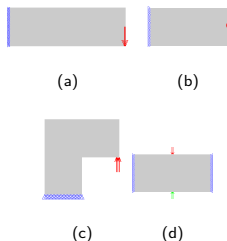


Figure: Design domains, boundary conditions, and external loads for the Cantilever benchmark problem (a), the Michell beam problem (b), the L-shape problem (c), and the two load case problem (d).

Problem instances		
Problems	No. of finite elements	No. of design variables
Cantilever I	7500	45000
Cantilever II	30000	180000
Cantilever III	120000	720000
Cantilever IV	480000	2880000
Michell I	5000	30000
Michell II	20000	120000
Michell III	80000	480000
Michell IV	320000	1920000
L-shape I	1875	11250
L-shape II	7500	45000
L-shape III	30000	180000
L-shape IV	120000	720000
Two Loads case I	5000	30000
Two Loads case II	20000	120000
Two Loads case III	80000	480000
Two Loads case IV	320000	1920000

Optimal designs and numerical results

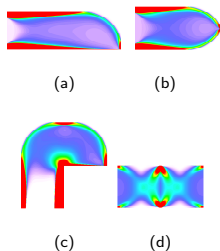


Figure: Optimal density distributions obtained by solving the minimum compliance problem for the Cantilever IV benchmark problem (a), the Michell IV beam problem (b), the L-shape IV problem (c), and the two load case IV problem (d).

Optimal designs and numerical results

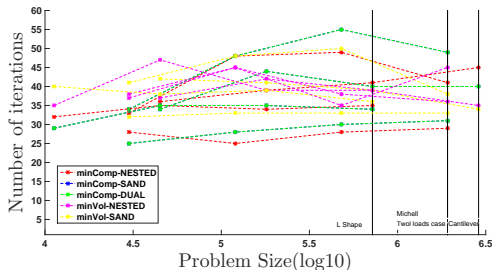
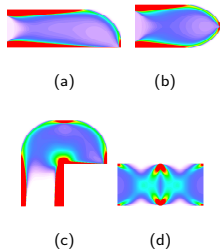


Figure: Optimal density distributions obtained by solving the minimum compliance problem for the Cantilever IV benchmark problem (a), the Michell IV beam problem (b), the L-shape IV problem (c), and the two load case IV problem (d).

Optimal designs and numerical results

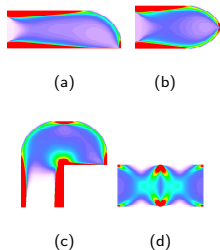
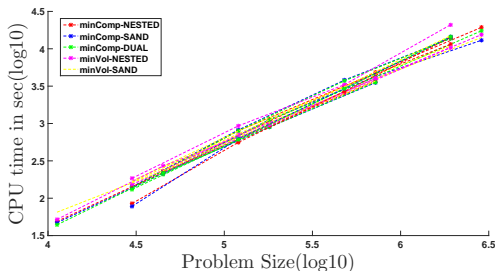
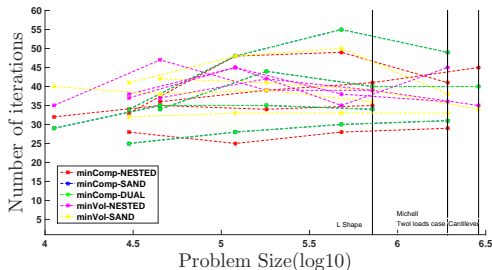
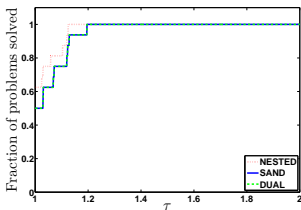


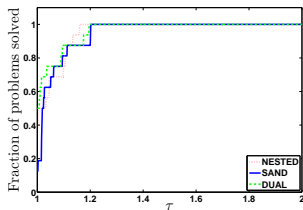
Figure: Optimal density distributions obtained by solving the minimum compliance problem for the Cantilever IV benchmark problem (a), the Michell IV beam problem (b), the L-shape IV problem (c), and the two load case IV problem (d).



Numerical results/problem formulations



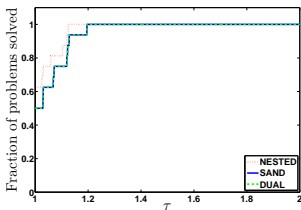
(a) Number of iterations as performance measure



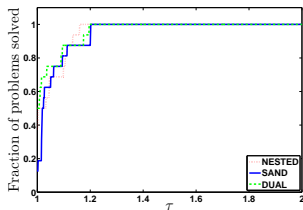
(b) CPU time as performance measure

Figure: Performance profiles for the minimum compliance problem.

Numerical results/problem formulations

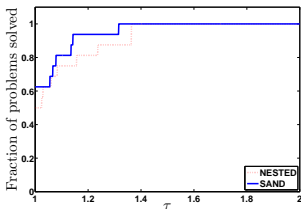


(a) Number of iterations as performance measure

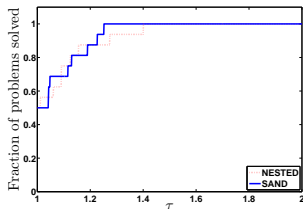


(b) CPU time as performance measure

Figure: Performance profiles for the minimum compliance problem.



(a) Number of iterations as performance measure



(b) CPU time as performance measure

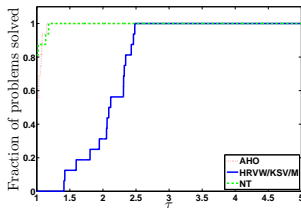
Figure: Performance profiles for the minimum weight problem.

Numerical results/search directions

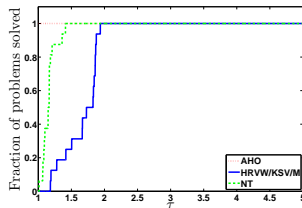
Moreover, we compare the numerical performance of the

- AHO direction
- HRVW/KSH/M direction
- NT direction

for solving the minimum compliance problem of the SAND formulation.



(a) Number of iterations as performance measure



(b) CPU time as performance measure

Figure: Performance profiles for the search directions.

Conclusions

- Efficient second order primal-dual interior point for FMO is developed.
- The method requires modest number of iterations and almost independent of problem size.
- The method has obtained solutions of good quality to the largest FMO problems to date.
- There is no clear distinction in the performance of the standard FMO problem formulations.
- We recommend the NT direction for solving FMO problems for its efficiency and robustness .

A. G. Weldeyesus and M. Stolpe. A primal-dual interior point method for large-scale free material optimization. *Computational Optimization and Applications*, 61(2):409–435, 2015.

Thank you for your attention!