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Optimization-based guidelines to retirement planning and pension product design

Ph.D. thesis

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Preface

This thesis has been prepared in fulfillment of the requirement for the Ph.D. degree at the Technical University of Denmark.

The project has been carried out from November 30, 2011 to November 30, 2014, in the Division of Management Science, Department of Management Engineering, Technical University of Denmark, under the supervision of professors David Pisinger and Alex Weissensteiner. This period includes seven months at the Department of Operations Research and Financial Engineering, Princeton University, as a visiting student research collaborator with John M. Mulvey. The project was fully funded by the Technical University of Denmark.

The dissertation consists of six academic papers on different but related topics within retirement planning. The thesis starts with an overall introduction to the project, including the background and the motivation for the research, the contribution, the summary of the papers and the discussion of the main results. Afterwards, each chapter consists of one paper, each of which can be read independently. All the papers have either been published or submitted to scientific journals within the area of operations research, financial mathematics, or actuarial science.

Acknowledgements

I started with a financial and actuarial background, but maybe by accident, or maybe with a little bit of luck, I ended up working with one of the strongest operations research groups in the world. As soon as I started my new job, I realized that I had joined a strong and interesting group of people. People with great minds and great research aspirations. In the course of time, I was introduced to operations research methods and became particularly interested in their applications to personal finance, pensions, and life insurance. I felt like I discovered a whole new world, and that I could define and solve so many interesting problems that could be applied in practice.

Being a PhD student is a unique experience. Those who have been through the PhD process, often say that it was the best time of their career, no matter which career they chose afterwards. I believe them. I got three years to follow my heart and my mind, while gaining a thorough knowledge within a few research areas and developing new skills related to research, programming, and written and spoken communication. I had a chance to travel to places I have never been to, attend and present at conferences, visit other universities, and meet inspiring people. I had a great time and I am so grateful that I had an opportunity to do that.

Nevertheless, my time as a PhD student would not have been that amazing if I did not meet some very special people, who influenced my work and my everyday life. Without any doubts the biggest thank you goes to my supervisor, professor **David Pisinger**. David, thank you for giving me so much freedom in terms of following my research interests, the content of the papers, and the choice of conferences. You always say that your PhD students are the best ever, and you are always so supportive no matter what we do. You believed in me and my research skills from the very beginning, and already in the first year you gave me so much confidence that I flew with open wings through the entire PhD. Our research areas do not necessarily overlap, and our backgrounds are so different, and despite that, you always came up with invaluable comments and questions that improved our papers, and that allowed me to see my research from a different perspective. You have always been so supportive, both professionally and personally. Thank you so much for that.

I would also like to thank my co-supervisor, professor **Alex Weissensteiner**. We started working together at the beginning of my third year, and it was the first time, when I actually had someone to discuss my research ideas in as much detail as possible and on a daily basis. Alex, you are the most research-oriented, hardworking, and honest person that I have ever worked with. Even though your research areas do not really lie within pensions and retirement products, you were always so keen to discuss whichever research topic occupied my mind—sometimes in person, other times through Skype. We often had different opinions about certain topics, but that is what makes the research discussions more interesting and fruitful, right? I have learnt a lot from you, and thanks to your supervision, implementing the models and writing the papers was easier and faster. Thank you for being so supportive during my PhD. It was a pleasure to work with you.

A special thank you goes to professor **John M. Mulvey**, who I collaborated with during my 7-month-long stay at the Department of Operations Research and Financial Engineering, at Princeton University. Thank you, John, for being so welcoming, for finding the time for research discussions, for helping to get our papers published, and for being so supportive regarding my career in academia (and my dancing interests). I had a great and inspiring time at Princeton, and it was an honor to experience the spirit of Princeton University. When mentioning Princeton, I will also take the opportunity to thank some visiting PhD students who later became my "Princeton gang": **Emilie, Flavia**, and **Tana**. You totally made my time there and I miss you so much. I love watching our pictures from Princeton—especially the recording of us going nuts on the bouncy castle.

Thanks to all the co-authors of my papers. It is incredible how many different people I had an opportunity to work with, how different their backgrounds are, how different parts of the world they come from, and how different their working habits and publishing styles are. Thank you, **Montserrat Guillén, Ana M. Peréz-Marín, John M. Mulvey, Jens Perch Nielsen, David Pisinger, Kourosh M. Rasmussen, Mogens Steffensen**, and **Alex Weissensteiner**. Each paper that you co-authored became better thanks to your contribution, and reminds me of the time we spent discussing the papers.

Thanks to my colleagues from the **Operations Research group**. I enjoyed your company every single day. Thanks for your contribution to our seminars, for the knowledge you shared with me, for our lunch discussions, Friday cakes, movie nights, Friday beers, cycling competitions, and other fun stuff that we have done together. Thank you for making every day of my work more fun and for being inspirational in my research.

I will finish these acknowledgements by saying thanks to my husband **Andrew**—for listening, for always being positive, and for taking care of me by not letting me work too much.

Cheers,
Aga

Summary

Pension systems differ across countries and are subject to country specific regulations. In addition, various countries offer different types of pension plans to their citizens. In some pension plans individuals have a lot of flexibility regarding managing their savings; in others, pension funds or life insurers manage the members' savings without taking into account the individuals' preferences. Nevertheless, in most of the pension plans individuals face some challenges. They have to decide on the investment and consumption of their retirement savings either by choosing a pension provider and a specific pension product, or by choosing a particular annuity, and, possibly, a life insurance policy. In some countries they have to make decisions as soon as they are employed; in other countries, not until they approach the time of retirement.

To help individuals make the right decisions regarding their retirement savings, this thesis presents some optimization techniques that could be applied by pension providers and financial advisers to provide individuals with such guidelines. For a given objective function and a number of constraints, we search for the optimal solution, which indicates, for example, how to invest the savings, which annuity to purchase, and which level of death benefit to choose. In most of the papers we follow a classical approach and maximize the expected CRRA utility of either the final wealth, or the consumption and bequest amount.

Scholars refer to the aforementioned decisions as the consumption-investment problems. A classical approach to solving these problems is stochastic optimal control (SOC), which aims to find a closed-form solution to a given problem. However, as the explicit solution exists only for relatively simple models, this approach cannot be applied to the problems considered in this thesis. Therefore, in this thesis we use a multi-stage stochastic programming (MSP) approach that is known for its practical applications and thus broadly used in operations research. In a few chapters of the thesis we apply a mixed approach, i.e., a combination of the MSP and SOC approaches, and in one of the chapters we apply Monte Carlo simulations.

Each chapter of the thesis focuses on different challenges that individuals face, given a different set of constraints, determined either by national regulations or by individuals' personal preferences. The first two chapters deal with defined contribution pension plans, where an individual makes the consumption, investment, and life insurance decisions, both before and after retirement. Our results indicate that retirement savings management differs for each individual, and that it should not only depend on the individual's degree of risk aversion or time left to retirement, but also characteristics such as current wealth, expected income before and after retirement, expected pension contributions,

impatience factor, lifetime expectancy, and preferences regarding portfolio composition and the level of death benefit. Consequently, pension providers should offer variable life annuities that are tailored to the individuals' needs in terms of the underlying asset allocation, the payout profile, and the level of death benefit.

The next three chapters discuss the purchase of the right annuity. We start with investigating whether individuals should wait until retirement to purchase a life annuity providing fixed payments, whether they should invest in this annuity already some years before retirement by purchasing a deferred annuity with the same fixed payments, or whether they should invest their savings in stocks and bonds. Despite simple model assumptions, our findings indicate that individuals should invest part of their savings in deferred annuities. The proportion in these products increases with the degree of risk aversion and the expected lifetime, and decreases with the bequest motive.

Afterwards, we investigate the optimal annuity choice under inflation risk, which is often ignored both by practitioners advising on the retirement planning and by scholars investigating the consumption-investment problems. We search for an optimal level of retirement income in real terms, given investment opportunities in inflation-linked, nominal, and variable annuities, as well as in stocks and bonds. Our findings show that real annuities are a crucial asset in every portfolio, and that trying to hedge inflation without investing in inflation-linked products leads to a lower and more volatile retirement income.

In the last chapter discussing the annuity purchase, we differentiate between a wide variety of annuities. In addition to stocks and bonds with different maturities, we search for the optimal investment in annuities contingent on a single and joint lifetime, with fixed and variable payments, with immediate and deferred payments, and with temporary and life long payments. We conclude that the optimal portfolio for a single and a two-person household not only requires frequent rebalancing, but also consists of too many different assets. Accordingly, we argue that despite so many annuities available in the market, the products that individuals need the most are not available.

Finally, in the last chapter of this thesis we apply Monte Carlo simulations to investigate the value of the interest rate guarantee incorporated in the Danish with-profit products. In these products the pension provider offers its members a guaranteed interest rate as well as some bonus rate that depends on the pension provider's realized investment returns. We argue that with-profit products with such a bonus mechanism often provide lower returns than pure unit-linked products without any guarantee because, to meet the solvency requirements, the pension provider has to invest individuals' assets more conservatively.

Resumé (Summary in Danish)

Pensionssystemer afviger fra land til land og er underlagt nationale direktiver. Desuden tilbyder forskellige lande divergerende pensionsordninger til deres borgere. I nogle pensionsordninger har enkeltpersoner masser af fleksibilitet med hensyn til at styre deres opsparing; i andre, administrerer pensionskasser eller livsforsikringsselskaber medlemmernes opsparing uden at tage hensyn til enkeltpersonernes præferencer. I de fleste pensionsordninger står enkeltpersoner alligevel over for nogle udfordringer. De skal beslutte investering og forbrug af deres pensionsopsparing enten ved at vælge en pensionsudbyder og et specifikt pensionsprodukt eller ved at købe en bestemt livrente og eventuelt en livsforsikring. I nogle lande skal de træffe beslutninger, så snart de bliver ansat i en virksomhed; i andre lande, kan det gøres når de nærmer sig pensionsalderen.

For at hjælpe enkeltpersoner til at træffe de rigtige beslutninger vedrørende deres pensionsopsparing, præsenterer denne afhandling nogle optimeringsteknikker, som kan anvendes af pensionsudbydere og finansielle rådgivere til at give enkeltpersoner denne type anbefalinger. Til en given objektfunktion og en række betingelser, søger vi efter den optimale løsning, som angiver, for eksempel, hvordan man bør investere sin pensionsopsparing, hvilken livrente man bør købe, eller hvilken dødsfaldssum man bør vælge. I de fleste artikler følger vi standarden og maksimerer den forventede CRRA nyttefunktion af enten den endelige formue, eller forbrug og arvebeløb.

Forskere refererer til de nævnte beslutninger for forbrugs- og investeringsproblemer. En klassisk metode til at løse disse problemer er at anvende stokastisk optimal kontrol (SOC), der sigter mod at finde en eksplicit løsning på et givent problem. Imidlertid, da eksplicitte løsninger kun eksisterer for relativt enkle modeller, kan denne fremgangsmåde ikke anvendes på de problemer, der behandles i denne afhandling. Derfor bruger vi stokastisk programmering (MSP), der er kendt for sine praktiske anvendelser og dermed bredt anvendt i operationsanalyse. I enkelte kapitler af denne afhandling anvender vi en blandet metode, dvs. en kombination af MSP og SOC fremgangsmåderne, og i et af kapitlerne anvender vi Monte Carlo simuleringer.

Kapitlerne i afhandlingen fokuserer på hver sin række af udfordringer, som enkeltpersoner står over for ved pensionering, givet nationale direktiver eller personlige præferencer. De to første kapitler omhandler bidragsbaserede pensionsordninger, hvor en person tager beslutninger om forbrug, investeringer, og livsforsikring, både før og efter pensioneringen. Vores resultater viser, at styring af pensionsopsparing bør tilpasses hver enkeltperson, og at den ikke blot bør afhænge af den enkelte persons risikoaversion eller tid til pension, men også af parametre såsom nuværende formue, forventet indkomst før og efter pensionen, forventede pensionsbidrag, utålmodighedsfaktor, forventet

restlevetid, og præferencer ved porteføljesammensætning og størrelse af dødsfaldssum. Følgeligt burde pensionsselskaber tilbyde variable livrenter, der er skræddersyede til enkeltpersoners behov med hensyn til den underliggende portefølje, udbetalingsprofil, og størrelse af dødsfaldssum.

De næste tre kapitler diskuterer køb af livsvarig livrente. Vi starter med at undersøge, om enkeltpersoner burde vente til pensionen med at købe en livsvarig livrente med faste ydelser, om de burde investere i denne livrente allerede nogle år før pensionen ved at købe en opsat livsvarig livrente med de samme faste ydelser, eller om de burde investere deres opsparing i aktier og obligationer. Trods enkle modelantagelser viser vores resultater, at personerne bør investere en del af deres opsparing i opsatte livsvarige livrenter. Investeringsandelen i disse produkter stiger med risikoaversion og med forventet restlevetid, og falder med arvemotiv.

Derefter undersøger vi det optimale valg af livrente givet inflationsrisiko. Denne risiko ignoreres ofte af både praktikere, som rådgiver om pensionsplanlægning, og af forskere, som undersøger forbrugs- og investeringsproblemer. Vi søger efter en optimal pensionsindkomst i nutidskroner, givne investeringsmuligheder i inflationsindekserede, nominelle, og variable livsvarige livrenter, samt i aktier og obligationer. Vores resultater viser, at reelle livsvarige livrenter er et afgørende aktiv i enhver portefølje, og at et forsøg på at afdække inflationen uden at investere i inflationsindekserede produkter fører til et lavere og mere ustabilt pensionsindkomst.

I det tredje kapitel omhandlende køb af livrenter, skelner vi mellem en række livrenter. Ud over aktier og obligationer med forskellige løbetider, søger vi efter den optimale investering i livrenter betingede af en enkeltperson liv og et fælles liv, med faste og variable ydelser, med øjeblikkelige og opsatte ydelser, og med ophørende og livsvarige ydelser. Vi konkluderer, at den optimale portefølje for en- og to-personers husstande ikke blot kræver en hyppig rebalancering af portefølje, men også består af alt for mange forskellige aktiver. Derfor konkluderer vi, at på trods af de mange livrenter, der er tilgængelige på markedet, findes de produkter, som enkeltpersoner har brug for, ikke.

I det sidste kapitel af denne afhandling, anvender vi Monte Carlo simuleringer for at undersøge værdien af en garanti inkluderet i de danske gennemsnitsrenteprodukter. I disse produkter tilbyder pensionsselskaberne deres medlemmer en garanteret rente samt en bonusrente, der afhænger af pensionsudbyderens realiserede investeringsafkast. Vi finder, at gennemsnitsrenteprodukter med en sådan bonusmekanisme ofte giver lavere afkast end rene unit-linked produkter uden garanti, fordi pensionsudbydere er nødt til at følge en konservativ investeringsstrategi for at opfylde solvenskrav.

List of papers

- Konicz, A. K., Pisinger, D., Rasmussen, K. M., and Steffensen, M. A combined stochastic programming and optimal control approach to personal finance and pensions.
  OR Spectrum, DOI: 10.1007/s00291-014-0375-6, Konicz et al. (2014a)
- Konicz, A. K. and Mulvey, J. M. Optimal savings management for individuals with defined contribution pension plans.
  European Journal of Operational Research, DOI: 10.1016/j.ejor.2014.11.016, Konicz and Mulvey (2014b)
- Konicz, A. K. and Mulvey, J. M. Applying a stochastic financial planning system for an individual: immediate or deferred life annuities?
  The Journal of Retirement, 1(2):46-60, 2013, Konicz and Mulvey (2013)
- Konicz, A. K., Pisinger, D., and Weissensteiner, A. Optimal annuity portfolio under inflation risk.
  Submitted, Konicz et al. (2014b)
- Konicz, A. K., Pisinger, D., and Weissensteiner, A. Optimal retirement planning with a focus on single and multilife annuities.
  Submitted, Konicz et al. (2014c)
- Guillén, M., Konicz, A. K., Nielsen, J. P., and Pérez-Marín, A. M. Do not pay for a Danish interest guarantee. The law of the triple blow.
  Annals of Actuarial Science, 7(2):192-209, 2013, Guillén et al. (2013a)

Chapter 1

Introduction

1.1 Background

Pension systems vary across the world, nevertheless their main purpose is to provide retirement income for all citizens in a given country. Pension systems are often based on three pillars, each having a different objective regarding the level of the pension income and the group of recipients. The first pillar corresponds to pensions from the public sector and is financed from general tax revenues. The aim of the public pension is to provide basic coverage preventing poverty in old age. The second pillar is often referred to as labor market pension or occupational pension, and is designated to maintain an equivalent standard of living compared to the period of employment. This pillar is funded by members' and employers' contributions. The third pillar provides an individual supplement and typically relates to privately funded voluntary accounts such as individual savings plans.

The amount of savings invested in the second and third pillar is significant. The study based on 13 major pension markets (U.S., U.K., Japan, Australia, Canada, Netherlands, Switzerland, Germany, Brazil, South Africa, France, Ireland and Hong Kong) reports that the global pension assets in 2013 have reached USD 31,980 bn, see TowersWatson (2014). The largest pension market is in the U.S. (59% of the global pension assets), followed by the U.K. (10.2%), and Japan (10.1%), see Fig. 1.1.

Specification of the benefits further divides the second and third pillar of a pension system into two groups. A *defined benefit* (DB) pension plan determines a level of benefit in relation to salary near retirement or through employment. The employer and, possibly, the employee, contribute throughout the employment to reach the target level of benefit. Depending on the investment return and demographic experience (the expected lifetime of the plan members), the contributions are adjusted to meet the target level of benefit, see Dickson et al. (2009). In a *defined contribution* (DC) pension plan the employer, the employee, or both, on a regular basis contribute a fixed percentage of the salary into the employee's individual account. Then, the pension provider invests the savings in the employee's account and accredits the capital gains (or losses) to the account. Depending on the country, upon retirement the employee can either convert the value of the savings

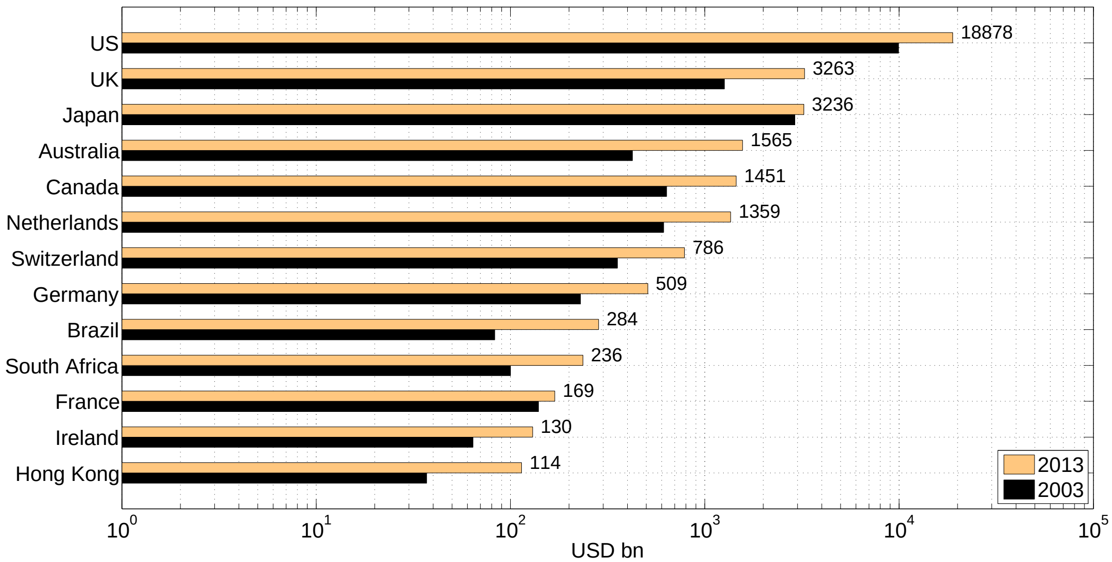


Figure 1.1: Global pension assets in 2003 and 2013 in billion USD. The figure is reproduced from the data provided in TowersWatson (2014).

to an annuity that would provide a stream of cash-flows during retirement (e.g. in the U.K.), or may simply draw funds (e.g., in the U.S. and Canada).

In recent years, defined contribution plans have spread all over the world. These plans are dominant in the private sector in many countries; even the countries with a long history of DB plans have been showing signs of a shift to DC. In contrast to defined benefit plans, defined contribution plans are easier and cheaper to administer, are more transparent, and are more flexible in terms of capturing the individuals' needs. At the end of 2013 the assets invested in DC plans in the seven largest pension markets represented 47% of total pension assets, and were primarily invested in equities (52%), bonds (28%), cash (1%), and other assets including property and other alternatives, see TowersWatson (2014). The allocation to bonds and cash has been decreasing in recent years in favor of allocations to equities.

1.2 Research motivation

The purpose of this thesis is to provide individuals with DC pension plans some guidelines regarding management of their retirement savings. Even though the degree of the members' involvement in managing their retirement sav-

"If I could see the future, I'd feel so much better about my pension plan."

ings varies across the countries, all individuals covered by DC pension plans face some challenges. Sometimes individuals are responsible for making all decisions regarding their savings; at other times they have freedom to make certain choices.

For example, the most important decision for plan members in the U.K. takes place upon their retirement when they have to convert part of the savings accumulated on their retirement account into an annuity. An *annuity* is a sum of money payable yearly or at other regular intervals, see Dellinger (2006). We can further categorize annuities by the conditions under which this sum is paid. For example, annuities can be contingent on a *single lifetime* (pay out conditionally on a person's survival), contingent on a *joint lifetime* with a spouse (pay out conditionally on the survival of either of the spouses), or simply *non-life contingent* (pay out independently of a person's survival). Annuities can provide fixed constant, increasing or decreasing payments (*fixed income annuities*), or payments that vary with a return of some underlying portfolio (*variable annuities*). Annuity payments may be temporary for, e.g., 10–25 years (*term/temporary annuities*) or life long (*whole life annuities*), and they may start upon the purchase of the annuity (*immediate annuity*), or at a certain time in the future (*deferred annuity*). In addition, life contingent annuities may include a guarantee related to the individual's time of death such as a guarantee ensuring that in the event of the individual's death, the dependants receive either a lump sum payment or a sum of the cash-flows for a certain period. Figure 1.2 shows an example of a deferred whole life annuity contingent on a single life.

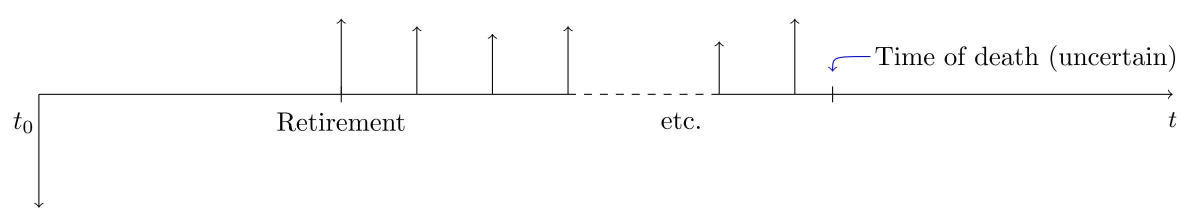


Figure 1.2: Cash-flows (indicated by the arrows pointing upwards) from a whole life deferred annuity. The individual purchases this annuity upon time t_0 (indicated by the arrow pointing down), but does not receive the payments until retirement. From retirement and onwards, the individual will receive the payments (whose level differs depending whether it is a fixed or variable annuity) as long as she is alive.

Having so many annuity products available in the market, it is not surprising that plan members find it difficult when it comes to deciding which annuity to buy. Because purchasing an annuity concerns a significant amount of money that individuals have been accumulating during their entire employment period, this choice is one of the most important financial decisions. The fact that annuities have either prohibitive surrender charges or are irreversible (once purchased they can never be sold), does not make the choice easier, and implies that buying the wrong annuity may have long-term consequences. Accordingly, individuals are reluctant to buy any annuity, and this behaviour is often referred by scholars and practitioners as the *annuity puzzle*.

Americans face similar challenges to British with respect to choosing the right annuity. For Americans making decisions is even more difficult because they do not have to convert their savings to an annuity but can simply draw from the savings account. If they decide not to purchase an annuity, how can they know how much of the savings to draw without a danger of outliving their resources? Moreover, members of American defined contribution plans are also responsible for the decisions regarding investment of the contributions. For example, in the American Individual Retirement Accounts (IRAs) and 401(k) plans, members decide on the investment strategy by selecting a certain mutual fund, individual stocks or other financial products. The savings account is directly linked to market returns, therefore a wrong investment may seriously deplete the savings, leaving individuals without sufficient income for old age. Consequently, an average American citizen must seek costly professional financial advice to make sound investment decisions.

Danish citizens covered by the DC plans face different challenges than British or Americans. In particular, they often do not have much freedom with respect to managing their retirement savings. The labor market and occupational pension are based on agreements between Danish employers and pension funds or life insurers. Within such an agreement, the pension provider manages the contributions of all the employees in a given company by offering all the employees particular pension products. These products typically determine not only the investment of savings during the employment period, but also the distribution of the benefits and the investment of residual savings after retirement. In addition, Danish pension products include by default a life insurance and a disability insurance policy.

The Danish pension industry offers two types of products: traditional with-profit products and unit-linked products.[1] *With-profit products* aim at distributing investment and insurance risks between the pension provider and its members. These products guarantee some minimum interest rate, which may further be topped off with a bonus rate, whose level depends on the realized return on the investment strategy managed by the pension provider. *Unit-linked products* are more flexible, and to some degree give individuals freedom to decide on the investment of their savings. The member's account is linked directly to market returns, thus the value of the account fluctuates according to the return from the chosen investment strategy. Similar to the U.S., the employee can choose her investment strategy from a list of specific funds, stocks or other financial assets, and rebalance the portfolio as frequently as needed. Alternatively, the pension provider offers a range of specific investment strategies (e.g., conservative, medium, aggressive, etc., see Fig. 1.3) corresponding to different risk profiles. In these strategies, the pension provider regularly adjusts the allocations to particular assets in line with a classical life cycle asset allocation model—the closer to retirement individuals are, the more conservative the portfolio they should hold. Independent of the investment strategy, the members can optionally purchase a guarantee ensuring some minimum return on the savings.

While an average Danish citizen does not have to make any decisions regarding her retirement savings at all and still can be certain to receive sufficient retirement income and protection in the event of disability or death, pension products offered in the Danish market are often generic and not

[1]When using the term *unit-linked* products (which is more familiar to academics and practitioners), we refer to the entire class of the Danish *market interest rate products* (*markedsrenteprodukter* in Danish), which include both the *life cycle* products or the *unit-linked* products.

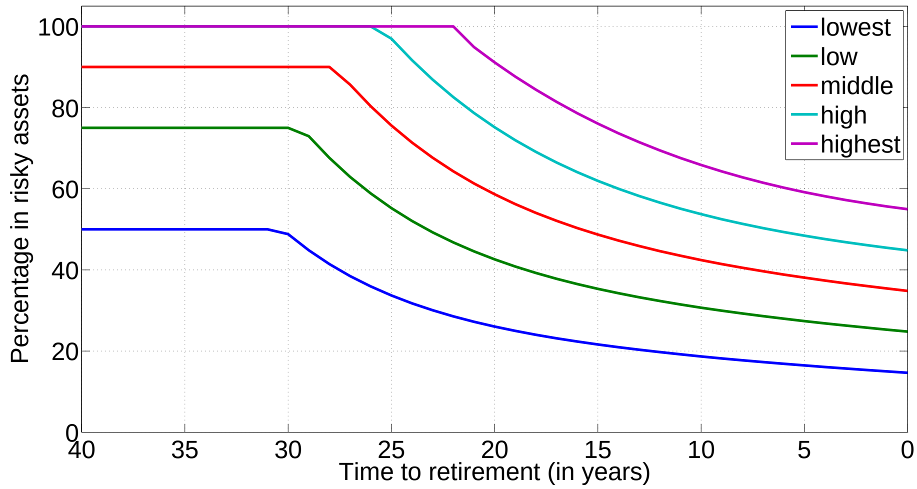


Figure 1.3: An example of a Danish pension product (Danica Balance) with a life cycle investment strategy. Source: data from Danica's homepage `http://www.danicapension.dk/da-dk/Medarbejdere-og-private/Opsparing/Danica-Balance/Pages/balancekurver.aspx`

customized to the individuals' needs. A natural question arises regarding possible improvements of these products. How to design pension products that capture not only the individuals' risk aversion, but also preferences regarding the distribution of the benefits? Should the pension products differ for employees with different current economic situations, expected future income, and expected public pension? Finally, how to choose the right level of the death benefit?

1.3 Methods

Scholars categorize the aforementioned questions to a general class of consumption-investment problems. In this type of problem they search for the optimal consumption and investment decisions that maximize the utility of either the consumption or the final wealth of the individual. A *utility function* is a concept in economics used to measure the individual's satisfaction from having a certain good, or in other words, to measure the amount of risk the individual is willing to undertake in the hope of attaining a greater consumption. A utility function increases with wealth and consumption, implying that the individual prefers more wealth than less wealth, and that she is never satiated (never meets a level at which she would not prefer to have more wealth). Scholars typically assume in their research that individuals are risk-averse—i.e., given a choice of two investments with a similar expected return, they will prefer the one with the lower risk—thus the utility function is concave. A more risk-averse investor would rather invest in assets having low risk (and providing low returns); a less risk-averse investor would invest in assets with higher risk (and higher expected returns).

A classical approach to solving consumption-investment problems is a *stochastic optimal control* (SOC) approach, which is a subfield of control theory. SOC assumes that random noise with a known probability affects the development of the state variables. Given an objective function dependent on a certain stochastic process, SOC seeks for the stochastic (path-dependent) controls. Despite the presence of random noise, these controls aim to manage the stochastic process to behave in a certain way, and thus to optimize the objective function. To solve the problem, scholars usually try to guess a solution by making an ansatz for the optimal value function. By defining the Hamilton-Jacobi-Bellman equation and applying the verification theorem (see, e.g., Björk, 2004), they can identify the assumed function as the optimal value function and the assumed controls as the optimal controls. Accordingly, the solution (the set of optimal controls) to such a problem is derived in a closed-form.

The field of stochastic control with its applications to finance has developed significantly since the 1970s, when Merton (1969, 1971) used this approach to study optimal portfolios with a risk-free asset and risky assets. His work, and that of Yaari (1965), Samuelson (1969), Merton (1969, 1971), and Richard (1975), inspired a vast number of researchers to study various investment-consumption problems by means of stochastic optimal control. The main advantage of a SOC approach is the analytical form of the optimal solution, which is easy to interpret and implement. However, the explicit solution in many cases does not exist. Practical assumptions such as including bounds on certain variables, adding constraints defining taxes or other regulations in a given country, and using more complex models to describe the uncertainty of asset returns, makes the optimization problem so complicated that an explicit solution simply does not exist.

The ability to handle more practical constraints is necessary when searching for the optimal decisions regarding retirement planning. Therefore, in this thesis we have applied numerical methods characteristic for operations research and financial engineering. *Operations research* (OR) is a discipline that deals with the application of advanced analytical methods to help make better decisions, see Informs (2014). It originated during World War II, when a number of researchers were asked to apply a scientific approach to deal with various strategic and tactical problems, see Hillier et al. (2010). As their research played a major role in winning a number of battles, the success of OR spread to other disciplines, and is now successfully applied in business and industry. The numerical methods applied in operations research include among others, mathematical optimization, simulation, econometric methods, decision analysis, etc. Modern OR has a number of sub-disciplines, one of which is *financial engineering*, broadly defined as a field involving financial theory, the methods of engineering, the tools of mathematics and the practice of programming.[2]

In particular, to find the optimal solution to consumption-investment problems faced by individuals with defined contribution pension plans, we use a *multi-stage stochastic programming* (MSP) approach. MSP is a general purpose framework for modeling optimization problems and consists of a scenario tree representing the range of possible outcomes and of an optimization module. A *scenario tree* represents the uncertainty—typically related to asset returns, but possibly also about other unknowns such as salary progression or time of death. As shown on Fig. 1.4, a scenario tree consists of nodes uniquely assigned to certain periods, and representing possible outcomes for the uncertainties. A root node n_0 at the initial stage t_0 is the ancestor for all the nodes at the subsequent

[2] Wikipedia, `http://en.wikipedia.org/wiki/Financial_engineering`. Accessed September 23, 2014.

stage. These nodes are further the ancestors for their children nodes, etc., until the final stage. As the nodes at the final stage have no children, they are called the leaves. A scenario is a single branch from the root node to a given leaf, thus each scenario consists of a leaf node and all its predecessors including the root node. Consequently, the number of scenarios in the tree equals the number of leaves. Each node has assigned a probability related to the occurrence of the particular outcome, thus the probabilities of all the nodes belonging to the same period sums to one. The probability of each scenario is equal to the product of the probabilities of all the nodes in the scenario.

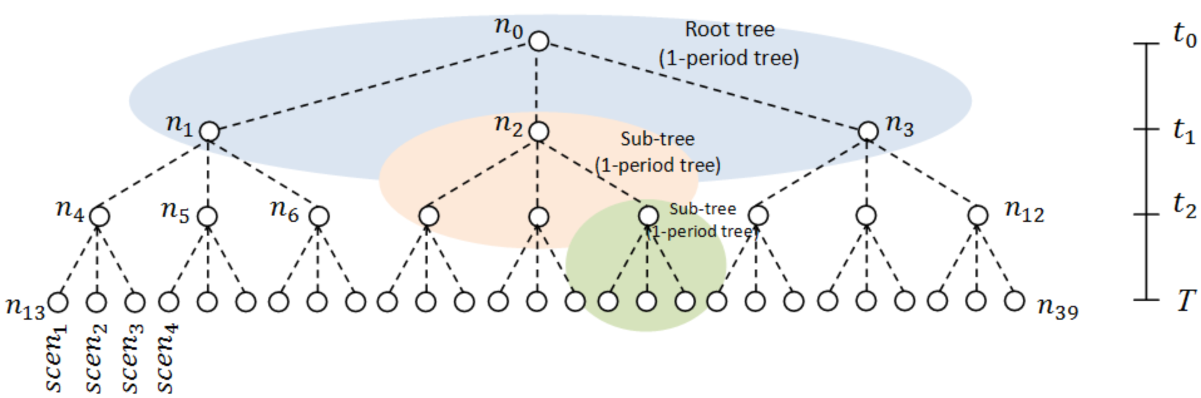


Figure 1.4: An example of a scenario tree with three periods defined by time points t_0, t_1, t_2 and T, a branching factor of 3, and $3^3 = 27$ scenarios defined as a single path from the root node to the leaf.

An *optimization module* consists of the objective function and the set of constraints defined for each scenario. The optimal decisions along the tree are computed numerically at each node of the tree, given the information available at that point. Decisions do not depend on the future observations but anticipate possible future realizations of the random vector. After the outcomes have been observed, the decisions for the next period are made depending on both the realizations of the random vector and the decisions made in the previous stage. This combination of the anticipative and adaptive models in one mathematical framework makes this approach particularly appealing in financial applications; the investor can specify the composition of a portfolio by taking into account possible future movements of asset returns (anticipation), and rebalance the portfolio (take recourse decisions) as prices change. Appendix 3.A in Chapter 3 provides more details about the formulation of a multi-stage stochastic program, whereas the studies such as Ziemba and Mulvey (1998), Carino et al. (1998), Carino and Ziemba (1998), and Mulvey et al. (2003, 2008), present some applications of MSP to finance.

While in many cases the MSP approach allows for finding the optimal solution to problems for which a closed-form solution (derived by the means of stochastic optimal control) either does not exist or is not trivial to find, MSP has also some drawbacks. For example, the optimal solution found by the optimization software is numerical, therefore scholars often find it challenging to understand the solution. Yet, the main disadvantage of MSP is the limited ability to handle many time periods under enough uncertainty related to, e.g., asset returns, inflation index, time of death, or salary progression. The scenario tree grows exponentially with the number of time periods, and solving the problem soon becomes computationally intractable. To exemplify, choosing a five-stage model with a constant branching factor of 10 leads to $10^5 = 100,000$ scenarios. Thus, depending on the size of the branching factor, scholars typically can solve the problems with up to 7 stages in a reasonable computational time. Consequently, to deal with the curse of dimensionality inherent in MSP, researchers choose either long and increasing intervals between the decisions, see, e.g., Carino et al. (1998), Dempster et al. (2003), and Consigli et al. (2012), add a steady state terminal value term to the objective function and approximate the infinite horizon problem using the dual equilibrium technique, see, e.g., Grinold (1977, 1983) and Carino et al. (1998), or apply different scenario reduction algorithms, see, e.g., Heitsch and Römisch (2009a,b).

To overcome the curse of dimensionality, in this thesis we often apply a mixed approach, which is a combination of multi-stage stochastic programming with stochastic optimal control. Specifically, we model the problem using the MSP approach until some horizon, and upon this horizon we insert into the objective the optimal value function calculated explicitly using SOC. The main advantage of the mixed approach is obtaining the optimal solution that accounts for the entire lifetime of an individual, practical constraints, and sufficient uncertainty, that allows for short intervals between the decisions, and that can be computed in a short time. Alternatively, we handle the consumption-investment problems with a long-time horizon by solving the problem using MSP until some fixed point in time, and by adding a terminal constraint ensuring lifetime income for an individual. In such cases, we argue that solving the problem only up to some horizon is sufficient because in practice financial advisers would hold regular meetings with their customers, and rerun the model with a new horizon and with new parameters reflecting the current state of the economy.

In this thesis we also work with Monte Carlo simulations. *Monte Carlo* means using random numbers in scientific computing, and is named after the famous gambling facilities city in Monaco; *simulation* means producing random variables with a certain distribution. Thus, for a given stochastic process driven by random variables, it is possible to estimate its probabilistic distribution by repeatedly sampling thousands of possible outcomes of the random variables. Because the resulting distribution of the stochastic process can be used to calculate quantities such as the process's expectation or a certain risk measure, Monte Carlo methods are essential in pricing of derivative securities and in risk management, see, e.g., Glasserman (2004).

1.4 Contribution

This thesis consists of six papers, each with its own contribution to the scientific literature. The papers have either been published or submitted to journals within financial, actuarial, or operations research field.

1.4.1 List of papers

- **Chapter 2**, Konicz, A. K., Pisinger, D., Rasmussen, K. M., and Steffensen, M. A combined stochastic programming and optimal control approach to personal finance and pensions. *OR Spectrum*, 2014, DOI: 10.1007/s00291-014-0375-6, Konicz et al. (2014a)
  - Presented at the conferences: the 5^{th} Nordic Optimization Symposium, Trondheim, Norway, June 2012; and the 25^{th} European Conference on Operational Research, Vilnius, Lithuania, July 2012; as well as at the seminars: DTU Management Engineering, Lyngby, Denmark, June 2012 and December 2012; and ORFE, Princeton University, Princeton, USA, March 2013.
- **Chapter 3**, Konicz, A. K. and Mulvey, J. M. Optimal savings management for individuals with defined contribution pension plans. *European Journal of Operational Research*, 2014, DOI: 10.1016/j.ejor.2014.11.016, Konicz and Mulvey (2014b)
  - Presented at the 17^{th} International Congress on Insurance: Mathematics and Economics, Copenhagen, Denmark, July 2013; XIII International Conference on Stochastic Programming, Bergamo, Italy, July 2013; the 48^{th} Actuarial Research Conference, Philadelphia, USA, August 2013; and the 30^{th} International Congress of Actuaries, Washington, DC, USA, April 2014.
  - Published in online proceedings of the 30^{th} International Congress of Actuaries under the title “On improving pension product design”, Konicz and Mulvey (2014a). The main difference between the journal version of this paper and the version in the proceedings is the choice of the scenario generation method. In the proceedings we have applied one of the traditional moment matching techniques to generate the scenarios; in the paper published in the journal we have developed an algorithm that generates scenarios with both asset returns and death events.
- **Chapter 4**, Konicz, A. K. and Mulvey, J. M. Applying a stochastic financial planning system for an individual: Immediate or deferred life annuities? *The Journal of Retirement*, 1(2):46-60, 2013, Konicz and Mulvey (2013)
- **Chapter 5**, Konicz, A. K., Pisinger, D., and Weissensteiner, A. Optimal annuity portfolio under inflation risk. Submitted, Konicz et al. (2014b)
  - Published as a technical report by DTU Management Engineering.
- **Chapter 6**, Konicz, A. K., Pisinger, D., and Weissensteiner, A. Optimal retirement planning with a focus on single and multilife annuities. Submitted, Konicz et al. (2014c)

  - Published as a technical report by DTU Management Engineering.
  - Presented at the 11^{th} International Conference in Computational Management Science, Lisbon, Portugal, May 2014; and at the seminar Recent Research Results in Operations Research, Operations Management and Financial Engineering, DTU Management Engineering, Lyngby, Denmark, May 2014.

- **Chapter 7**, Guillén, M., Konicz, A. K., Nielsen, J. P., and Pérez-Marín, A. M. Do not pay for a Danish interest guarantee. The law of the triple blow.
  Annals of Actuarial Science, 7(2):192-209, 2013, Guillén et al. (2013a)

1.4.2 A brief summary of the papers

All of the papers focus on retirement planning. Nevertheless, to accommodate a variety of challenges that individuals with different pension plans face (mentioned in Section 1.2), the papers differ in the model setup, underlying assumptions, constraints, and numerical methods. Each paper can be read independently from the rest of the thesis and in an arbitrary order. Each chapter includes one paper and ends with a bibliography and, where relevant, appendices.

The first two chapters (Chapters 2 and 3) refer to defined contribution pension plans where, for an individual with uncertain lifetime, we search for the optimal consumption (after retirement) and the optimal investment and life insurance (before and after retirement). When investigating this problem we do not focus on any particular country, but the problem is certainly relevant for pension markets similar to the Danish one. We explore how optimal decisions differ for individuals characterized by various economical and personal characteristics, and further analyze the impact of certain practical aspects such as short sale constraints, transaction costs, taxes on capital gains, as well as bounds on portfolio composition, on the level of pension benefits, and on the sum insured. Accordingly, we investigate how to design pension products that are not generic but are tailored to the individuals' needs.

The next three chapters discuss the purchase of the right annuity. Specifically, Chapter 4 investigates whether individuals should wait until retirement to purchase an annuity paying a life long fixed income, whether they should invest in this annuity already some years before retirement by purchasing a deferred annuity with the same fixed income, or whether they should simply purchase stocks and bonds. Chapter 5 focuses on real consumption by searching for optimal allocation of retirement savings to inflation-linked, nominal, and variable annuities, as well as to stocks and bonds. Chapter 6 investigates the optimal investment strategy given stocks and bonds, and a wide range of annuities providing payments under different circumstances. We consider annuities contingent on a single or a joint lifetime, with fixed and variable payments, with immediate and deferred payments, and with temporary and life long payments. In all three chapters we either add constraints on irreversibility of the annuities, or introduce high surrender charges combined with a free withdrawal amount. The findings from these three chapters are mostly relevant for individuals with defined contribution plans in countries similar to the U.S. and U.K.

Finally, in the last chapter (Chapter 7) we focus explicitly on the Danish pension market and compare with-profit products to unit-linked products without a guarantee. We simulate the final wealth of the individual having a with-profit product, and calculate the tail value at risk of the final wealth. Afterwards, we search for a unit-linked product with the equivalent financial risk, and compare the internal rate of return of both products.

Table 1.1 shows the similarities and differences between the papers under various criteria. In the majority of the papers (Chapters 2–6) we apply some optimization techniques and follow a classical approach of maximizing the expected constant relative risk aversion (CRRA) utility of either the final wealth or the consumption/pension benefits, and often of the bequest motive. To find the optimal solution we use a multi-stage stochastic programming approach (Chapters 4–6), a mixed approach, i.e., a combination of multi-stage stochastic programming and stochastic optimal control (Chapters 2 and 3), and Monte Carlo simulations (Chapter 7). Whenever we use the multi-stage stochastic programming (MSP) approach, we apply the moment-matching technique to generate the scenarios introduced by Høyland and Wallace (2001), Høyland et al. (2003), or Konicz and Mulvey (2014b). The latter method allows us to generate the scenarios both with asset returns and death events, given that these two types of uncertainty are independent of each other. In each paper we have also chosen a different investment universe from the following assets: cash, bonds (either with a flat interest rate or a return defined by a certain term structure), stocks, and a range of annuities. We further assume that the asset returns either follow the Black-Scholes model (Chapters 2, 4 and 7), are log-normally distributed (Chapter 3), or follow the vector autoregressive model VAR(1) (Chapters 5 and 6).

As summarized in Table 1.1, and in line with other scholars applying multi-stage stochastic programming, our programs have 4 to 5 periods. We further set the length of the periods to 1–5 years. The branching factor is in most of the papers constant for each period, and depends on the number of underlying assets. The higher the number of assets, the higher the branching factor we have to choose. Consequently, the numerical results for the chapters including a broader investment universe (Chapters 5 and 6) are based on a larger number of scenarios ($14^4 = 38,416$ and $10^4 = 10,000$, respectively), where the largest problem has over 2 million variables and 1.5 million constraints. The results in Chapters 2–4 are based on a smaller number of scenarios ($4^5 = 1,024$, $10 \cdot 6^3 = 2,160$, and $5^4 = 625$, respectively) and include approximately 5–12 thousand of variables and constraints. To ensure that the results for the smaller optimization problems are robust, we additionally rerun the programs for a certain number of different scenario trees (either 10 or 50). We are able to solve all our problems to optimality in a short computational time (less than 10 seconds for the smaller problems, 1–12 minutes for the larger problems) using—in most of the cases—the nonlinear solver MOSEK.[3]

[3]The optimization module consists in most of the papers of a non-linear objective function and linear constraints. Initially, we have linearized the objective by a piece-wise linear function and solved it using a linear solver. Nevertheless, we found that MOSEK (which is a nonlinear solver using an interior point algorithm) is more efficient and is best suited to solve this type of problems in terms of speed, robustness, and accuracy.

Criterion	**Chapter 2** Konicz et al. (2014a)	**Chapter 3** Konicz and Mulvey (2014b)	**Chapter 4** Konicz and Mulvey (2013)	**Chapter 5** Konicz et al. (2014b)	**Chapter 6** Konicz et al. (2014c)	**Chapter 7** Guillén et al. (2013a)
Objective	CRRA utility of consumption and of bequest	CRRA utility of pension benefits and of bequest	CRRA utility of final wealth and of bequest	CRRA utility of real consumption, quadratic disutility of deviations from a real target	CRRA utility of consumption and of bequest	—
Decisions	Consumption, investment, life insurance	Pension benefits, investment, life insurance	Investment, bequest amount	Consumption, investment	Consumption, investment, bequest amount	—
Decision period (before/after retirement)	Before and after	Before and after	Before	After	After	—
Available asset classes	Bonds, stocks	Cash, bonds, stocks	Bonds, stocks, immediate/deferred fixed rate annuities	Bonds, stocks, inflation-linked and nominal fixed rate annuities, variable annuities	Bonds, stocks, pure endowments, variable pure endowments, single and J&S whole life annuities	Bonds, stocks
Number of assets	3	3	2	5	up to 29	2
Single/Household	Single	Single	Single	Single	Household	Single
Additional constraints	Bounds on portfolio composition, bounds on sum insured, transaction costs, taxes on capital gains	Short sale constraints, bounds on the value of portfolio, on the level of benefits, and on the sum insured	Short sale constraints, irreversibility of annuities	Short sale constraints, irreversibility of annuities	Short sale constraints, transaction costs, free withdrawal amount and surrender charges	—
Modeling uncertainty	Black-Scholes model	Log-normal distribution	Black-Scholes model	Nelson/Siegel model, VAR(1)	Nelson/Siegel model, VAR(1)	Black-Scholes model

Table 1.1: Major similarities and differences between the papers included in the thesis.

Criterion	**Chapter 2** Konicz et al. (2014a)	**Chapter 3** Konicz and Mulvey (2014b)	**Chapter 4** Konicz and Mulvey (2013)	**Chapter 5** Konicz et al. (2014b)	**Chapter 6** Konicz et al. (2014c)	**Chapter 7** Guillén et al. (2013a)
Numerical method	MSP and SOC	MSP and SOC	MSP	MSP	MSP	Monte Carlo Simulations
Scenario generation method	Høyland and Wallace (2001)	Høyland and Wallace (2001) and Konicz and Mulvey (2014b)	Høyland and Wallace (2001) and Konicz and Mulvey (2014b)	Høyland et al. (2003)	Høyland et al. (2003)	—
Number of periods	5	4	4	4	4	—
Length of periods (in years)	{1,1,1,1,1}	{1,3,3,3} after retirement, {5,5,5,5} before retirement	{5,5,5,5}	{5,5,5,5}	{5,5,5,5}	—
Branching factor	4	{10,6,6,6}	5	14	10	—
Number of scenarios	1,024	2,160	625	38,416	10,000	10,000
Number of different scenario trees	50	50	10	1	1	—
Number of variables	11,260	9,546	4,175	118,203	2,058,869	
Number of constraints	9,895	12,075	5,282	97,518	1,489,986	
Solver (for the optimization module)	MOSEK	MOSEK	CONOPT	MOSEK (power utility max.), CPLEX (disutility min.)	MOSEK	—
Computational time	<10 s.	<10 s.	<3 s.	<1.5 min.	3–12 min.	~5 min.

Table 1.1: (continued)

1.4.3 Novelty

Researchers typically model financial and actuarial problems in a continuous-time setup and try to solve them by deriving a closed-form solution. In contrast to such approach, we solve these problems using a (discrete-time) multi-stage stochastic programming approach, which within financial applications has mainly been applied to asset liability management (ALM), see, e.g., Berger and Mulvey (1998), Ziemba and Mulvey (1998), Carino et al. (1998), Dempster and Consigli (1998), Gondzio and Kouwenberg (2001), and Ferstl and Weissensteiner (2010, 2011). Applying the MSP approach to individual asset liability management (iALM) is new and has only been investigated in recent years, see, e.g., Consiglio et al. (2004), Consigli (2007), Dempster and Medova (2011), Consigli et al. (2012), Kim et al. (2012), and Pedersen et al. (2013). These studies, however, are more general, and focus on personal finance for individuals or households, and not specifically on applications to defined contribution pension plans or to a pension product design. Consequently, the novelty of our research lies in defining and solving consumption-investment problems relevant to actuarial practice, in particular to pensions and life insurance, using the MSP approach.

When defining consumption-investment problems, we have either extended the existing models or proposed new ones. For example, we have extended Yaari (1965) by including among the asset universe a variety of different life annuities (Chapter 6); we have extended Richard (1975) by adding bounds on the asset allocation and on the level of death sum, and by adding transaction costs and taxes on capital gains during the first year decisions (Chapter 2); we have extended Gerrard et al. (2006) by introducing a stochastic and inflation-linked target in the objective function (Chapter 5); we have extended Geyer et al. (2009a) by adding life insurance to their model (Chapter 2); we have extended Koijen et al. (2011) by allowing for rebalancing decisions during retirement and by expanding the investment universe to include liquid assets (Chapter 5); finally, we have extended Guillén et al. (2013b) by applying their performance measurement methodology described in that paper to a new type of products, namely, with-profit products.

We have also defined some concepts and constraints that are new to the literature. For example, we have introduced a definition for a bequest amount (included in the objective function) for a case when the individual has no opportunity to buy life insurance (Chapters 4 and 6). The bequest amount equals the value of non-life contingent assets (such as stocks and bonds) and assets contingent on the life of the particular dependant. To account for the illiquidity of life contingent assets we have introduced either constraints that do not allow for selling these assets (Chapters 4 and 5), or constraints defining a free withdrawal amount (defined as percentage of the initial investment) and the surrender charges applied when the individual sells an amount of annuities higher than the free withdrawal amount (Chapter 6). In the last chapter (Chapter 7) we have developed a new model for asset liability management for the pension funds offering Danish with-profit products. Because of the guarantees incorporated in these products, the pension funds have to employ an investment strategy that does not violate the solvency capital requirements.

We can also note some contributions to the methodology. To deal with the curse of dimensionality inherent in the MSP approach for problems with a long-time horizon, we have further extended the mixed approach introduced by Geyer et al. (2009a). We find this approach appealing because it allows for a model of the entire lifetime of an individual under realistic assumptions, given short time intervals between the decisions and enough uncertainty. In addition, the mixed approach allows us to obtain the optimal solution in a short computational time. When combining stochastic programming with stochastic optimal control, we have derived (by means of stochastic optimal control) the explicit solution for a deferred variable annuity (Chapter 2) and for an immediate variable annuity under the constraint that the individual receives a public pension (Chapter 3). The derived closed-form formulae provide intuition behind the optimal pension products and their design.

Another contribution to the methodology concerns development of a moment-matching scenario generation technique that allows us to generate scenario trees with two independent sources of uncertainty, e.g., asset returns and death events (Chapter 3). Such scenario trees are relevant when modelling multi-person households' decisions, such as finding the optimal consumption and investment strategy of the household after the death of one of the spouses, or investigating whether, upon the death of one of the spouses, the dependant should receive a lump sum death benefit or a death benefit paid in instalments. Generating scenario trees with independent sources of uncertainty is also relevant when modelling the uncertainty of a future income or becoming ill. Another contribution to scenario generation methods is the application of the techniques Høyland and Wallace (2001) and Høyland et al. (2003) to generate scenarios with a joint distribution of nominal and real term structure, as well as stock returns and inflation index (Chapter 5).

The last contribution to the methodology relates to the definition of the considered optimization problems. We have defined most of the problems using a non-linear objective function, linear constraints, and continuous (not integer) variables. Thus, in contrast to other studies applying the MSP approach to the CRRA utility maximization problem (such as Geyer et al., 2009a,b), we were able to solve the problems with an efficient non-linear solver, accordingly avoiding the approximation error stemming from the linearization of the objective function.

Finally, based on the results of these particular papers we can also give some practical recommendations to individuals regarding their retirement planning. These recommendations are the topic of the next section.

1.5 Discussion of the main results

Recommendations on retirement planning differ with respect to the country, the regulations of the pension plans, and the products that individuals have access to. We base our guidelines on numerical results; however, these are often sensitive to the model assumptions and to changes in the underlying stochastic processes, parameters, and constraints. Nevertheless, trying to generalize our findings, we summarize this introduction chapter by discussing some rules of thumb, which we have explored from the particular chapters of this thesis.

1.5.1 Management of retirement savings tailored to individuals' needs

From Chapters 2 and 3, we learn that the decisions regarding investment, retirement income, and the size of death sum do not only depend on the individual's age (or time left to retirement), nor do they solely depend on the risk preferences, but depend also on: 1) economical characteristics—such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g., retirement state pension), and 2) personal characteristics—such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition and on the level of death sum. These decisions are further affected by constraints specific to a given country or pension plan, such as transaction costs and taxes, and are particularly affected by the realized asset returns.

For example, the optimal investment strategy always follows a classical life cycle strategy, i.e., the closer to retirement, the more conservative the individual's portfolio (see, e.g., Merton (1969) and Campbell and Viceira (2002)). In fact, the investment strategy in some pension products (e.g., those shown on Fig. 1.3) already follows a life cycle investment strategy. However, Richard (1975) showed that the weight of risky assets increases with the expected future income, implying that young individuals who expect to work for many years should significantly short their portfolio (borrow money) to invest in risky assets. Following such an optimal investment strategy is unrealistic, therefore including constraints prohibiting short sales or defining bounds on portfolio composition (e.g., as modelled in Chapters 2 and 3) is crucial in such an optimization problem, and has a tremendous effect on the investment strategy. For example, notice the difference in the optimal asset allocation for a 45-year old individual with and without the short selling constraints shown on Fig. 1.5. We further extended Richard (1975)'s results by showing the impact of capital gains taxes on the optimal investment strategy. For example, assuming the existence of a risk-free asset (such as in the Black-Scholes model), subtracting taxes only from the positive returns makes the risk-free asset more attractive; the capital gains on risky assets are lower, whereas the losses are as severe as they were before. Consequently, as shown in Chapter 2 the optimal allocation to risky assets under 20% capital gain taxes can even be half as much as the original allocation.

Scholars and practitioners often focus retirement savings management on the investment strategy. However, choosing the right level of retirement income and of the sum insured is as important as following the right investment strategy. Our results indicate that depending on the degree of risk aversion, the time preference (impatience) factor, and the expected lifetime (or health status), the individual should choose constant, increasing, or decreasing level of retirement income, or even a combination of those. To exemplify, Fig. 1.6 shows that the expected optimal payout curve is decreasing either for an individual with average expected lifetime but with a high time preference factor (i.e., an individual who values the benefits in early retirement years more than those in later years), or for an individual who expects to live shorter than others (e.g., an individual who is ill and thus has a higher mortality rate). The degree of risk aversion does not directly affect the sign of the slope of the optimal payout curve but it affects its magnitude; the less risk-averse the investor, the higher the difference between benefits in the early and late retirement years.

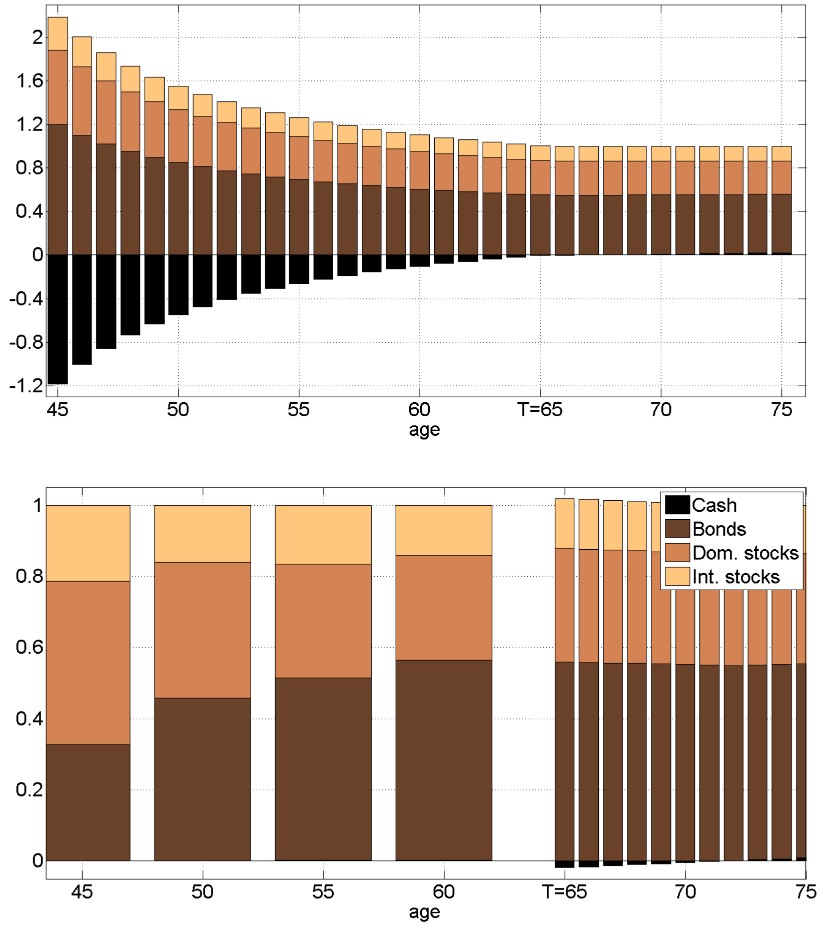


Figure 1.5: The optimal investment strategy for a 45-year old individual without (top figure) and with (bottom figure) the short selling constraints. Parameters: initial wealth $x_0 = 130,000$ (EUR), initial labor income $l_0 = 50,000$ (EUR), yearly increase in the labor income $y_l = 2\%$, pension contributions $p^{fixed} = 10\%$, expected retirement state pension $b_t^{state} = 4,000$ (EUR), risk tolerance $\gamma = -4$, impatience factor $\rho = 0.119$. Source: presentation of the paper Konicz and Mulvey (2014a) at the 30^{th} International Congress of Actuaries, Washington, DC.

Finally, the choice of the level of death sum is not trivial at all. The most common insurance policies in the market are those offering term insurance with a constant death sum. When such a policy expires and the individual is still alive, the individual can purchase a new policy with term insurance with a lower constant death sum. We find that the optimal bequest amount for an individual obtaining the CRRA utility from consumption and from the bequest amount, follows the optimal payout profile; depending on the individual's economical and personal characteristics, the bequest amount is constant, increasing or decreasing. Whenever the bequest amount is defined as the sum of the retirement savings and the death sum, the optimal level of death sum depends on the value of savings. For example, to keep a constant bequest amount before retirement (when the individual accumulates the savings), the level of death sum must be decreasing proportionally to the increase in savings. In line with Richard (1975), if the current wealth is relatively large and the expected future income is low, the individual does not need life insurance; in fact, the level of death

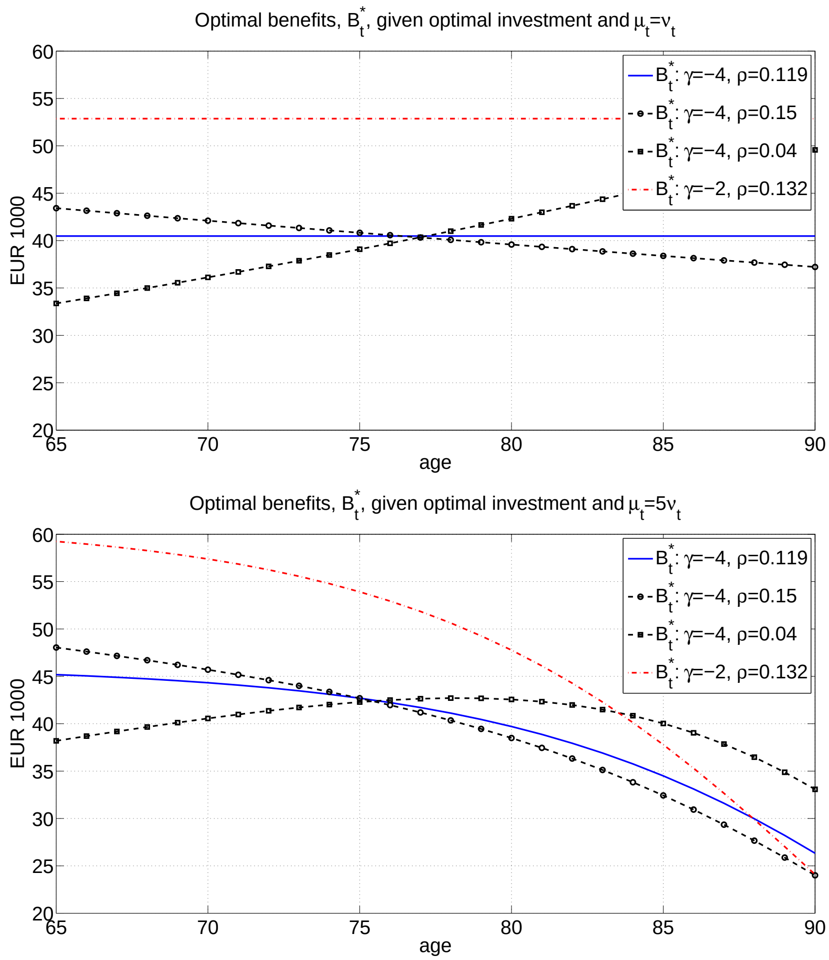


Figure 1.6: The optimal payout profile for 65-year old individuals characterized by different level of risk tolerance γ, impatience factor ρ, and subjective mortality rate μ_t. The payout profile on the top figure is for an individual with an average lifetime expectancy; on the bottom for an individual with a lifetime expectancy shorter than for an average individual, $\mu_t = 5\nu_t$. In both cases we have assumed that the individual follows the optimal investment strategy. Other parameters: initial wealth $x_0 = 650$, weight on the bequest motive $k = 3125$. The amounts are in EUR 1,000. Source: Chapter 3.

sum is negative. To obtain a more realistic solution in Chapter 2 we extend Richard (1975)'s model by adding short sale constraints and a non-negativity constraint on the death sum, and find that its optimal level is positive and converges to zero, as shown on Fig. 1.7 (the case for $\pi_t \leq 1$ and $I_t \geq 0$). Any constraints on the death benefit directly affect the optimal payout profile, which, in line with the objective function, must be proportional to the amount of bequest.

1.5.2 Investment return on life contingent products

Whole life annuities are the only products that provide a lifelong income, therefore financial advisers mainly consider them as insurance against the longevity risk. Our findings in Chapters 4–6 not only confirm this fact, but also indicate that life annuities (either whole life or temporary) are a good investment. For accepting the risk of dying early and loosing their savings, individuals receive an

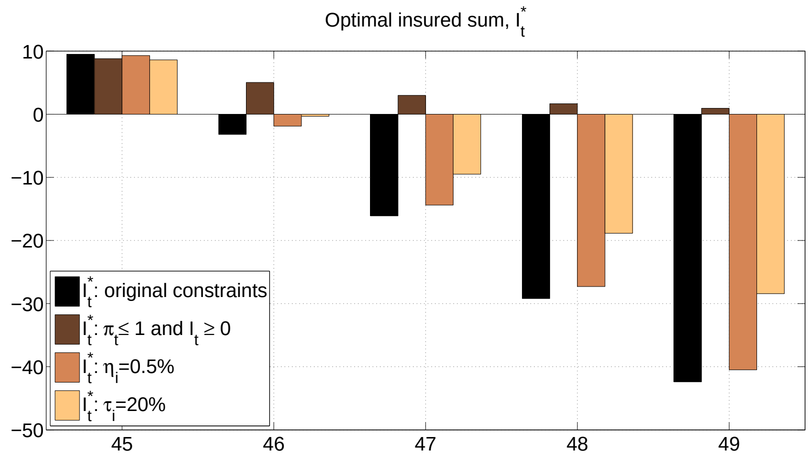


Figure 1.7: The expected optimal level of death sum under different constraints for a 45-year old individual. Parameters: initial wealth $x_0 = 60$, labor income $l_0 = 27$, risk tolerance $\gamma = -3$, impatience factor $\rho = 0.04$. The amounts are in EUR 1,000. Source: Chapter 2.

extra return (a survival credit) related to their mortality rate. Accordingly, fixed income annuities (or variable annuities) provide a higher return than long term bonds (or the corresponding equity index), while exposing the individual to similar financial risk, see Fig. 1.8.

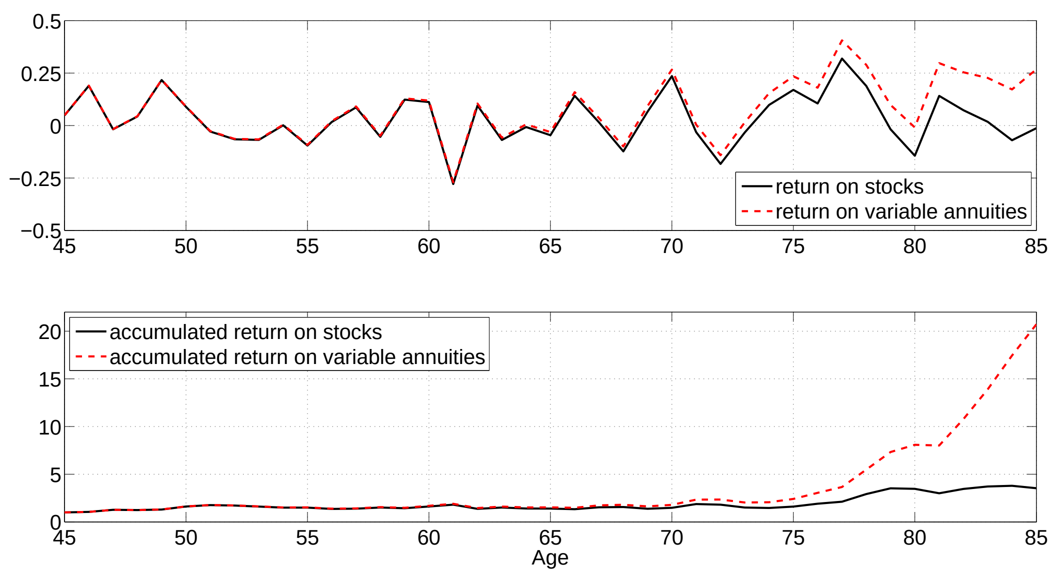


Figure 1.8: One possible realization of the yearly returns (top figure) and the accumulated returns (bottom figure) on stocks and on variable annuities (with an underlying portfolio consisting of stocks) for a 45-year old individual. The yearly survival credit is equal to the difference between the yearly returns on stocks and on variable annuities, and is calculated from the mortality tables for Danish women, Finanstilsynet (2010). We assume that stocks follow the Geometric Brownian Motion with the expected yearly return of 5% and volatility 16%.

Already Yaari (1965) showed that an investor without a bequest motive maximizing the CRRA utility of consumption should invest in fixed income annuities rather than in bonds. In Chapter 6 we further extend this result to a broader class of annuities. In particular, we argue that despite prohibitive surrender charges (defined jointly with a free withdrawal amount), the individual without a bequest motive always chooses life contingent assets (contingent on either a single lifetime or on a joint lifetime) such as fixed rate and variable annuities, or temporary and whole life annuities. Individuals simply prefer a higher interest rate to full liquidity. In the presence of a bequest motive they still invest in life contingent products, however, to ensure a bequest amount to their dependants, they invest also in inheritable assets such as stocks and bonds. The higher the bequest motive, the more the individual allocates to these assets.

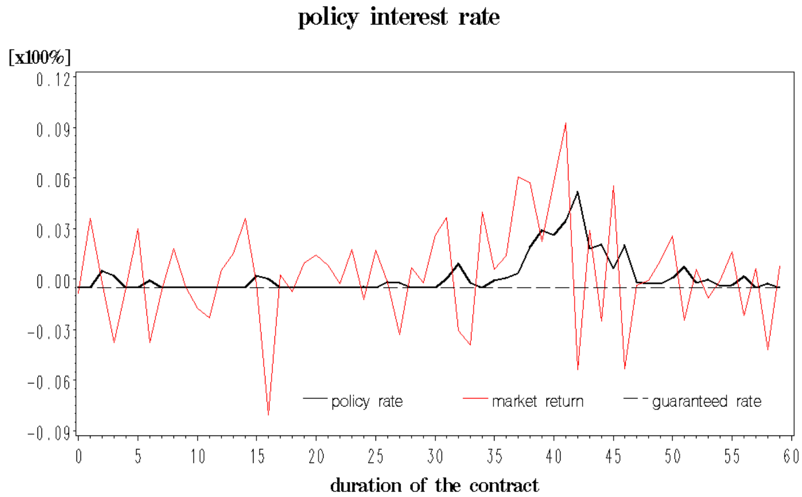
Furthermore, the results from Chapter 4 indicate that even though the survival credit on life annuities is lower for younger people, it is still worth the investment. As individuals typically cannot receive annuity payments before retirement, they should purchase deferred annuities, i.e., annuities which start paying out upon retirement. By doing so individuals have an opportunity to earn the survival credit also during the accumulation period. In this chapter we have assumed that the individual has access to fixed income annuities, which are irreversible. The optimal portfolio consists of these annuities (purchased as early as possible), as well as stocks and bonds. The allocation to these assets varies with the degree of risk aversion and with the weight on the bequest motive—the lower the bequest motive, the higher the investment in deferred annuities.

In addition, Chapter 5 (which is the only chapter where we distinguish between the real and nominal consumption) suggests that also under inflation risk, individuals without a bequest motive favour life annuities to stocks and bonds. Inflation-linked annuities provide the income adjusted to inflation, no matter how high the inflation is. We argue that ignoring inflation is myopic, and specifically individuals investing in only fixed rate annuities may experience a dramatic decrease in their purchasing power during retirement. Because inflation-linked annuities provide a rather low return (in the low inflation times that we currently have), our findings indicate that most investors should allocate their savings to both inflation-linked and variable annuities; the higher the risk aversion, the more important inflation-linked annuities are. Having access to variable annuities, individuals without a bequest motive have no desire to invest in stocks.

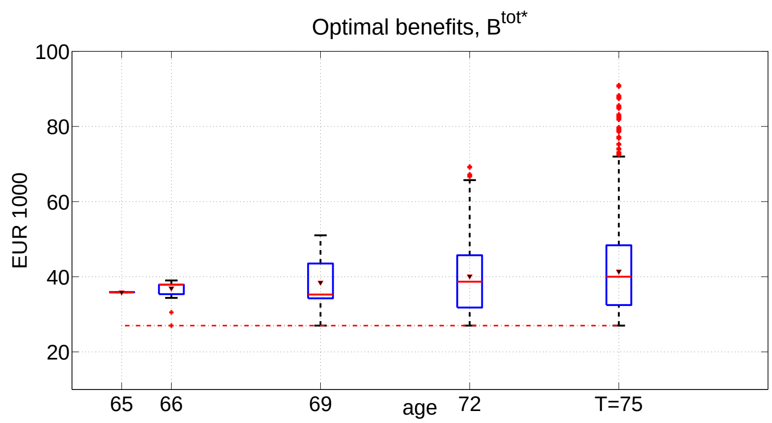
Summing up, individuals always seek an optimal balance between investments in life contingent and in non-life contingent assets (e.g., stocks and bonds). Investment in life contingent products such as fixed rate and variable annuities, immediate and deferred life annuities, and nominal and inflation-linked annuities, is not only for the protection against the longevity risk, but also due to the high reward-to-variability ratio (the Sharpe ratio). This ratio is always higher for life contingent assets than for non-life contingent assets. Thus, having access to a wide range of life annuities, the reason for purchasing stocks and bonds is to account for the bequest motive (if the individual does not have a life insurance policy), and to ensure liquidity of the savings.

1.5.3 Products with guarantees

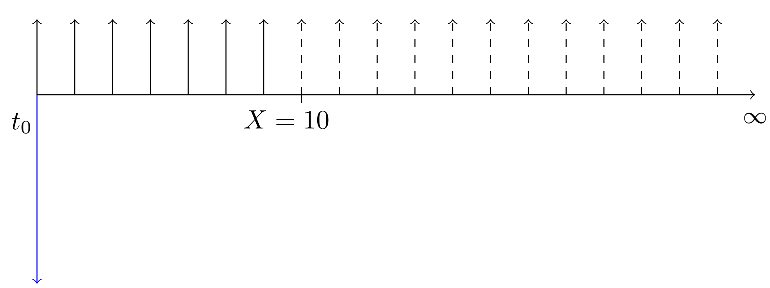
In this thesis we also consider different types of guarantees attached to pension products. Chapters 3 and 7 consider the guarantees related to market returns; and chapter 6 to the case of early death.



(a) A simulation of the policy rate in a with-profit pension product offered by *Codan* during the period 1991–2000. The policy rate is equal to the guaranteed rate (here assumed 0%) and some bonus depending on market returns. Source: Chapter 7.



(b) The probability distribution of the optimal total benefits in one scenario tree given a guarantee of the minimum level of benefits $b_t^{min} = 27$ (the red dashed line), and the minimum level o savings upon horizon, $x_T^{min} = 350$. The central mark in each box denotes the median, the triangle marker denotes the mean, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually with the red crosses. All the amounts are in EUR 1,000. Source: Chapter 3.



(c) Cash-flows from an immediate fixed income annuity with $X = 10$ years guaranteed. The blue line pointing downwards indicates the purchase of the annuity, the continuous arrows pointing upwards indicate the non-life contingent cash-flows, and the dashed arrows pointing upwards indicate the cash-flows conditional on the annuitant's survival. Source: Chapter 6.

Figure 1.9: Different types of guarantees attached to retirement products.

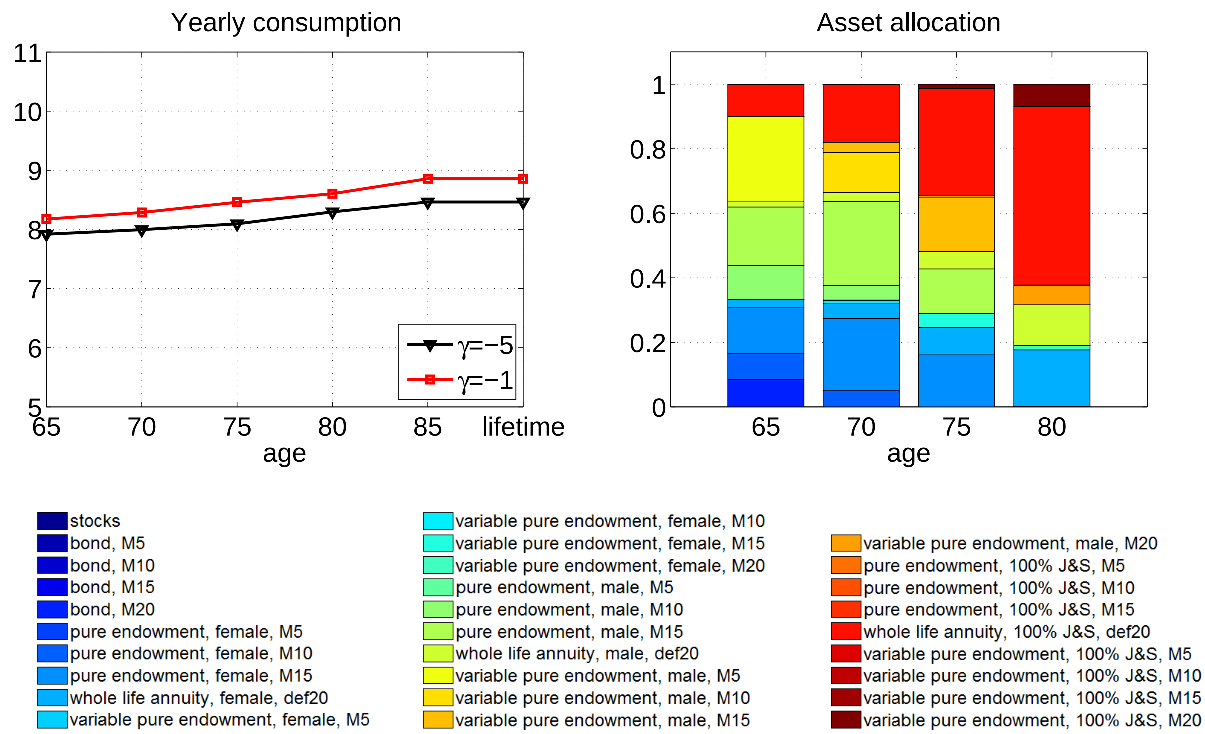
For example, with-profit products (Chapter 7) guarantee a certain interest rate, which may be topped off with a bonus rate, see Fig. 1.9a. The bonus rate depends on the realized returns on the pension provider's investment strategy; if the returns are high, the company redistributes a fraction of the surplus between the customers as a bonus rate. The company is furthermore restricted by the solvency capital requirement—they must invest such that in 95% of possible scenarios the company remains solvent. The higher the guaranteed rate is, the higher the necessary solvency capital requirement. Our results indicate that with-profit products provide, on average, a lower return than unit-linked products with no guarantee and with the same level of risk (measured as the tail value at risk). The lower return is a result of the company's bonus distribution policy and the opportunity cost corresponding to the solvency requirements.

We investigate another type of a guarantee in Chapter 3, which focuses on finding the optimal level of benefits for an individual maximizing the utility of the retirement benefits and of the bequest. As the optimal investment strategy includes allocations to risky assets, the benefits vary for each scenario depending on the actual asset returns. Therefore, the individual may be interested in adding a lower bound on the level of the benefits, as shown e.g., on Fig. 1.9b. Such a guarantee, however, implies that the individual has to follow a more conservative investment strategy, and in the years with low investment returns, withdraw more of the savings from her savings account. Consequently, the individual has, on average, lower benefits, lower death sum, and less wealth in the savings account for the future years.

Chapter 6 considers the guarantees that are particularly popular in the U.S., namely, guarantees related to the unknown time of death of the individual. These guarantees are incorporated in products such as life annuities with some guaranteed period (during this guaranteed period the life annuity pays out the cash-flows independently whether the individual is alive or not, see Fig. 1.9c), or life annuities with cash or instalment refund (upon the individual's death the dependants receive the payments—either as a lump sum or in instalments—equal to the difference between the initial investment and the value of the cash-flows that have been paid out while the individual was alive). Individuals often cannot directly see the price of these guarantees because annuity providers or financial advisers often present only the expected payouts from the particular annuities. The actuarially fair price of the cash or instalment refund is approximately equal to the survival credit, thus, as explained in Section 1.5.2, the returns on annuities with such a guarantee are no different than the returns on liquid assets with equivalent financial risk. In this case individuals would be better off investing in liquid assets, simply because in contrast to annuities these assets do not have prohibitive surrender charges. Our findings further suggest that at any time during the retirement period the optimal investment portfolio for a one- or a two-person household with a bequest motive consists of life and non-life contingent assets. If the portfolio consisted of non-life contingent assets before some fixed time point, and of life contingent assets afterwards, the annuity with a guaranteed period (Fig. 1.9c) would be the optimal product. Because the time of death is uncertain and because the individual obtains the utility from having a bequest motive during the entire lifetime, individuals must protect their dependants during the whole period (by either purchasing a life insurance policy or investing in non-life contingent assets).

1.5.4 Pension product design

The numerical results obtained in this thesis most often suggest a well-diversified portfolio. To exemplify, in Chapter 6 the optimal portfolio for a two-person household with a bequest motive for the spouse consists of fifteen different annuity products, see Fig. 1.10. In addition, to follow the optimal investment strategy the household has to rebalance the portfolio frequently. Such a solution indicates that the optimal pension product for individuals or households is not included in the considered asset universe (and possibly is not available in the market); if it was, the optimal investment strategy would suggest to spend the entire savings on that particular product.



Age	65	70	75	80	85
Consumption	7.9	8.0	8.1	8.3	8.5
Bequest for the husband	47.0	36.1	30.4	25.9	14.7
Bequest for the wife	27.1	27.3	26.8	25.6	16.3
Bequest for the children	5.3	0.0	0.0	0.1	0.0

Figure 1.10: Optimal asset allocation, consumption (in £) and bequest (in £) for a two-person household with retirement savings of £100, risk tolerance $\gamma = -5$, and a moderate bequest motive for the spouse, $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$ (average across 10,000 scenarios). Source: Chapter 6.

Following such a dynamic investment strategy to obtain the optimal level of retirement income and bequest amount sounds too complicated for most households. Therefore, we argue that instead of managing retirement savings themselves, households should be offered one pension product that is tailored to their needs; a product that provides the cash-flows equal to the household's optimal consumption, and that pays out a death benefit equal to the household's optimal bequest amount. If such a product was available, the individuals would invest their entire savings in it, without being concerned about rebalancing the portfolio. As most of the numerical results from Chapters 2—6 imply, the optimal consumption and bequest amount significantly differ from individual to individual, and from household to household, depending on the individuals' economical and personal characteristics, and on the regulations related to specific countries (transaction costs, public pension, taxes, etc.). Thus, the optimal pension product is different for each household.

But how can annuity providers design optimal pension products? In Chapters 2 and 3 we introduce the concept of an *optimal annuity* and, using a stochastic optimal control approach, derive the explicit formula for the optimal level of benefits for a CRRA utility function. Assuming a simple model (the continuous-time setup with risk-free investment and no additional income), the benefits provided by the optimal annuity are defined as a proportion of the current value of the savings X_t,

$$B_t^* = \frac{X_t}{\bar{a}_{y+t}^*},$$

where

$$\bar{a}_{y+t}^* \quad = \quad \int_t^{\widetilde{T}} e^{-\int_t^s \left(\bar{r}+\bar{\mu}_\tau\right)d\tau} ds, \quad \bar{r} = \tfrac{1}{1-\gamma}\rho - \tfrac{\gamma}{1-\gamma}r, \quad \bar{\mu}_\tau = \tfrac{1}{1-\gamma}\mu_\tau - \tfrac{\gamma}{1-\gamma}\nu_\tau,$$

and where $\bar{r}$ and $\bar{\mu}$ are the utility adjusted interest rate and mortality rate. These parameters depend on the individual's current age y, the degree of risk aversion $1-\gamma$, the impatience factor ρ, the risk-free rate r, and the subjective and pricing mortality rates μ_t and ν_t, and determine the benefits to be either constant, increasing or decreasing with time. Furthermore, the optimal level of death sum is proportional to the level of benefits by a factor of $(k\mu_t/\nu_t)^{1/(1-\gamma)}$, where k is the weight on the bequest amount relative to the level of the cash-flows. Thus, the optimal level of death sum also accounts for the individual's preferences.

A similar closed-form solution can be derived for an optimal variable annuity (under the assumption that the market is determined by the Black-Scholes model). The parameters characterizing individuals determine the expected cash-flow from this annuity as either constant, increasing or decreasing, but the realized cash-flows fluctuate with the actual returns of the underlying risky portfolio. In addition, Richard (1975) has derived the explicit solution for an optimal variable annuity and an optimal (variable) death sum, given that the individual has a deterministic labor income and a life insurance policy. We contribute to this strand of literature by deriving the explicit formula for a deferred variable annuity (in Chapter 2), and for an annuity and death sum offered by a pension provider in a defined contribution pension plan, given that the individual also expects the benefits from the first pillar, i.e., the public pension (in Chapter 3).

Deriving the closed form solution for an optimal product, that takes into account the households' economical and personal characteristics and links their level of benefits and death sum to a well-diversified portfolio (restricted by the short sales constraints and providing some inflation protection), is not trivial and cannot be achieved by means of stochastic optimal control. Nevertheless, from the MSP approach, we gain some insights about designing optimal pension products. For example, Chapter 6 suggests that the optimal product for a two-person household maximizing the CRRA utility of consumption and bequest is a special case of a joint and survivor annuity with variable and increasing payments, with a certain reduction of payments after the first death, and with a life insurance policy. In addition, in countries where pricing of annuities and life insurance is gender-specific, the parameters defining this annuity are also gender dependent.

Finally, we must remember that designing optimal pension products depends in the first place on the choice of the objective function. For example, Chapter 5 shows how different the optimal solution is for an individual maximizing the CRRA utility of real consumption compared to an individual minimizing squared deviations from some inflation-adjusted target. Thus, whenever practitioners find the optimization results counter-intuitive (such as the finding that we do not recommend products with guarantees to most individuals), they should question the choice of the objective function. Throughout the thesis we have chosen objective functions common in finance and economics, but maybe to design better pension products researchers should focus on introducing different objective functions? We leave the question open to future research, nevertheless, because the main purpose of retirement planning is protection against the longevity risk, we argue that the appropriate objective function should span the entire lifetime of an individual, implying that the optimal pension product should provide income as long as the individual is alive.

1.6 Conclusions and future work

This thesis focuses on challenges faced by individuals with defined contribution pension plans. Depending on national regulations and on a particular pension plan, individuals either are responsible for making decisions regarding their retirement savings, or have freedom to make certain choices such as the annuity purchase, the allocation of the savings, or the choice of life insurance. These decisions are definitely not easy to make, especially because they concern a significant amount of savings, and because a bad decision may result in not having sufficient income to cover standard living costs. Consequently, individuals often have to seek costly guidance from financial advisers.

We have developed a number of optimization models that could be used either by financial advisers to provide individuals with some guidelines regarding management of their retirement savings, or by pension providers to design better pension products. We have solved the problems using a multi-stage stochastic programming approach, which is known for its practical applications, and therefore frequently applied in operations research. Accordingly, we have obtained numerical results that account for the entire lifetime of an individual, while allowing for practical constraints and enough uncertainty related to asset returns and expected lifetime. The MSP framework is flexible in terms of defining the objective function, the constraints, and the underlying stochastic processes, and can further be applied to discover reliable and robust policy rules, which can afterwards be employed in practice using Monte Carlo simulations.

The numerical results indicate that retirement savings management should be customized to individuals' needs, and that products available in the market are too generic to be optimal under the considered objective functions. Independent of the individual's economical and personal characteristics, the optimal product should provide payments as long as the person is alive (such as, e.g., a whole life annuity); however, the underlying portfolio, the payout profile, and the level of death benefit should be dynamic and dependent on the individual's preferences. In addition, our results indicate that under the considered objective functions, the life contingent products are not only insurance against the longevity risk, but also a good investment opportunity. Moreover, individuals should think twice before purchasing products with guarantees because these guarantees are often costly.

This thesis raises a number of interesting questions for future research, both from a methodology point of view and in terms of defining new models. More research should be done on modelling the long-term horizon problems with an MSP approach. To tackle these problems we have either applied a mixed approach (i.e., upon a fixed short-term horizon we have approximated our problem with a simpler one that has an explicit solution), or chosen a terminal condition to ensure the lifetime income (e.g., annuitization of the final wealth). Both methods have some drawbacks—in the first case the optimal decisions before and after the horizon are made under different model assumptions; in the second, the terminal condition not only affects the optimal solution before the horizon, but also does not allow for any decisions after the horizon.

Furthermore, the models for retirement planning should capture in more detail the specific national regulations such as taxes, limits on the contributions to certain products, and restrictions regarding pension withdrawals. The inclusion of these constraints is not trivial, especially when dealing with long-term horizon and the necessity of making more frequent (e.g., yearly) decisions. We believe that these constraints may significantly affect optimal decisions.

In addition, more work should be done towards developing new methods to generate scenarios with death events. These scenarios are relevant when modeling households' decisions. For example, in line with the current literature, we have investigated the optimal investment and consumption decisions only as long as both spouses are alive, ensuring that in the case of death of one of the spouses, the survivor would receive some bequest amount. Having scenarios with death events for both spouses would allow for investigating the survivor's optimal decisions after the death of the first spouse. Methods for generating scenarios with more independent sources of uncertainty would furthermore be relevant for models taking into account, e.g., an uncertain salary progression or probability of becoming ill.

Finally, we suggest more research towards an improved pension product design. Our results sometimes indicate solutions that some individuals find counter-intuitive. For example, our findings suggest to buy inflation-linked annuities (while in reality individuals are particularly reluctant to purchase these kind of products), or to discourage purchases of products with guarantees (while in reality individuals prefer some kind of a guarantee). We suggest that researchers should dedicate more time to investigate new objective functions, which may be more suitable for designing pension products. Moreover, our results regarding pension product design are based on the individual's point of view—we have not investigated challenges that pension providers or life insurers may have when offering such products in the market.

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Chapter 2

A combined stochastic programming and optimal control approach to personal finance and pensions

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REGULAR ARTICLE

A combined stochastic programming and optimal control approach to personal finance and pensions

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Abstract We combine a dynamic programming approach (stochastic optimal control) with a multi-stage stochastic programming approach (MSP) in order to solve various problems in personal finance and pensions. Both optimization methods are integrated into one MSP formulation, making it possible to achieve a solution within a short computational time. The solution takes into account the entire lifetime of an individual, while focusing on practical constraints, such as limits on portfolio composition, limits on the sum insured, inclusion of transaction costs, and taxes on capital gains, during the first years of a contract. Two applications are considered: (A) optimal investment, consumption and sum insured for an individual maximizing the expected utility of consumption and bequest, and (B) optimal investment for a pension saver who wishes to maximize the expected utility of retirement benefits. Numerical results show that among the considered practical constraints, the presence of taxes affects the optimal controls the most. Furthermore, the individual's preferences, such as impatience level and risk aversion, have even a higher impact on the controlled processes than the taxes on capital gains.

Keywords Dynamic programming · Multi-period stochastic programming · Power utility · Personal finance · Retirement

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1 Introduction

The purpose of this paper is to formulate and solve two optimization problems relevant for personal finance and pensions in a multi-period stochastic framework. Problem (A) investigates the optimal investment, consumption and sum insured for an individual maximizing the utility of consumption and a terminal utility of leaving a positive amount of money upon death over an uncertain lifetime. Problem (B) is relevant for a pension product design in a defined contribution plan. A pension saver wishes to maximize the utility of the future retirement benefits by controlling the investment both before and after retirement as well as the level of the benefits that will be received after retirement. The solution determines a pension product that reflects the individual's risk preferences and impatience level.

A classical approach for solving investment-consumption optimization problems is to apply stochastic optimal control, also referred to continuous time and state dependent dynamic programming. Stochastic optimal control is common in financial and actuarial literature and focuses on deriving the explicit (analytical) solutions to a given model. The original problem of optimizing utility of consumption and terminal wealth over a fixed time horizon was defined by Merton (1969, 1971). This work has inspired many researchers who either expanded the original model or investigated different objective functions, by introducing, for example, the lifetime uncertainty, sum insured and the labor income, (Richard 1975), stochastic interest rate, (Munk and Sørensen 2004), salary uncertainty, (Cairns et al. 2006), multi-person household, (Bruhn and Steffensen 2011), borrowing constraints, (Byung and Yong 2011), or constant linear taxation, (Bruhn 2013). Applications within defined contribution pension scheme with a focus on the investment strategy either during the accumulation or post retirement phase (decumulation) together with the optimal time of annuitization, have been considered by Milevsky and Young (2007) and Gerrard et al. (2004, 2012).

The main advantage of a stochastic optimal control approach is the analytical form of the optimal solution, which is easy to interpret and implement. However, the explicit solution in many cases does not exist, especially, when dealing with constraints, such as limits on portfolio composition, limits on the sum insured, inclusion of transaction costs, or taxes on capital gains. To overcome these limitations, researchers refer to numerical methods, such as applied in this paper multi-stage stochastic programming (MSP) approach. MSP is a general purpose framework for modelling optimization problems, broadly applied in operations research. The optimization problem is based on a scenario tree that represents the range of possible outcomes for the uncertainties. Rather than finding a generic optimal policy, the optimal solution for the specified decision variables is computed numerically at each node in the scenario tree. MSP can handle a wide variety of objective functions and can easily address realistic considerations and constraints, as long as they have an algebraic form.

Applying stochastic programming is more common within portfolio management and asset-liability management than within the areas concerning the individual investors. See, e.g., Mulvey et al. (2006, 2007) who argue that multi-period investment models combined with Monte Carlo simulation can address important considerations for long-term investors, and Ferstl and Weissensteiner (2010) who present a stochastic linear programming model for optimal asset allocation in a situation where a financial company wishes to minimize the conditional value at risk. Nevertheless, applying stochastic programming or other numerical approaches to find the optimal decisions from the individual investors' point of view can also be found in the literature. See, e.g., Horneff et al. (2008) considering an individual characterized by Epstein-Zin preferences, Cai and Ge (2012) investigating the asset allocation for an investor with a loss aversion objective, a predetermined objective and a greedy objective, Consigli et al. (2012) comparing investment opportunities offered by traditional pension products and unit-linked contracts with variable life annuities, and Blake et al. (2013) deriving the optimal investment for a loss averse pension saver with an interim and final target.

However, the main limitation of a multi-stage stochastic programming approach is the curse of dimensionality, i.e. the problem size grows quickly as a function of the number of periods and scenarios. In particular, taking into consideration the entire lifetime of an individual, can be challenging in terms of computational tractability. Several studies investigate methods that can overcome this issue. For example, Grinold (1977, 1983) presents and compares different methods of approximating the general multi-stage optimization problem upon horizon, and concludes that a dual equilibrium technique will give improving approximations of the optimal solution as the horizon increases. Barro and Canestrelli (2005, 2006) show that a dynamic portfolio problem defined as a multistage stochastic program can be rewritten as a discrete time optimal control problem, which allows for solving large optimization problems in a low computational time, whereas Barro and Canestrelli (2011) further extend this work to a broader class of multistage stochastic programming problems.

Finally, Geyer et al. (2009a) argue that stochastic optimal control and stochastic programming can be combined and integrated into one multi-stage programming formulation. The authors model an investment-consumption problem using stochastic linear programming and combine it with the closed-form solution obtained by Richard (1975). They show that the mixed approach can accurately replicate the first-stage investment and consumption decisions derived by Richard (1975), and argue that stochastic programming and stochastic optimal control complement each other, especially in the areas where one or the other does not perform well on its own.

Inspired by the advantages of the mixed approach, we apply this optimization method to two problems relevant for personal finance and pensions. Problem (A), similarly to Geyer et al. (2009a), is based on the explicit solution derived by Richard (1975). Our paper, however, differs from the model presented in Geyer et al. (2009a) in several aspects. The most important improvement is to include the sum insured in the model. The explicit solutions for optimal consumption, investment and sum insured derived by Richard (1975) assume a life insurance policy in the model, therefore, the sum insured must be included in the MSP formulation. Furthermore, we compare the optimal controls at all stages, and not only at the first stage as in Geyer et al. (2009a). Finally, instead of approximating the objective function by a piecewise-linear interpolant, we solve the problem directly using a nonlinear solver, thus removing the approximation error from the results.

Our work differs from Richard (1975) by including realistic constrains such as limits on portfolio composition, limits on the sum insured, inclusion of transaction costs and taxes on capital gains. We show that the optimal investment and sum insured derived by Richard (1975) are for a large variety of parameters problematic from a practical point of view, and the inclusion of more realistic constraints affects the optimal solution significantly.

Another contribution of this paper is the formulation and solution of problem (B), which is relevant for pension product design in defined contribution pension plans. The uniqueness of this problem from the stochastic optimal control perspective lies in solving for the optimal consumption only during the period after retirement, while the optimal investment is solved for the entire lifetime of an individual. None of the aforementioned studies investigating the optimal investment-consumption problems consider the optimal controls over different periods. Therefore, we derive an explicit formula for the optimal value function and the optimal controls using Hamilton–Jacobi–Bellman techniques for the proposed model. We further solve problem (B) using a mixed optimization approach.

The paper is organized as follows. Section 2 describes problems (A) and (B) in more detail together with their optimal solutions obtained through dynamic programming. Section 3 explains how to bridge between dynamic programming and multi-stage stochastic programming by incorporating both optimization methods in one MSP formulation. Section 4 focuses on defining the MSP model, i.e. the objective function, constraints, and scenario tree generation. Section 5 presents the numerical results of the optimal controls obtained by two different optimization methods and the impact on the controls of the modifications considered during the first years of the contract. Finally, Sect. 6 summarizes the paper and suggests future work. The paper includes two appendices: Appendix 1 explains the application of the mutual fund theorem, and Appendix 2 presents a derivation of the optimal value function and controls for problem (B).

2 Model description

The section presents the general model assumptions relevant for problems (A) and (B).

The individual invests the savings in N assets: one risk-free (bonds) and $N-1$ risky assets (stocks). The economy is represented by a standard Brownian motion W defined on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of W, and the asset prices S_t^i are modeled by a geometric Brownian motion,

$$\begin{aligned} \mathrm{d}S_t^i &= \alpha_i S_t^i \mathrm{d}t + \sigma_i S_t^i \mathrm{d}W_t^i, \\ S_0^i &= s_0^i > 0, \end{aligned} \tag{1}$$

where α_i and σ_i are constants, and $\mathrm{d}W_t^i$ and $\mathrm{d}W_t^j$ are correlated with a coefficient ϕ_{ij}, for all $i, j = 1, \dots, N-1$. The risk-free asset is defined by $\alpha_N = r$ and $\sigma_N = 0$. Merton (1971) shows that without loss of generality we can assume that all the risky assets are included in one mutual fund, the price of which is modeled by a geometric Brownian motion,

$$\mathrm{d}S_t = \alpha S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

where α is the expected rate of return on the risky fund and σ is the volatility of the risky fund. See Appendix 1 for details. We also assume that the individual is allowed to borrow money at the risk-free rate r to buy the risky assets, and to take a short position in those assets.

Further, assume that $\mathbb{P}$ and $\mathbb{P}^*$ are equivalent probability measures on the measurable space $(\Omega, \mathcal{F})$. $\mathbb{P}$ denotes the objective measure, whereas $\mathbb{P}^*$ is used by the pension fund for pricing both market and life insurance risk. Thus, consistently with Richard (1975)'s assumptions, we consider life insurance policies as standard tradable financial contracts.[1]

The individual has an uncertain lifetime modeled by a finite state Markov chain Z, defined on a measurable space $(\Omega, \mathcal{F})$. The state process Z indicates whether the person is alive or not; it takes values in $\{0, 1\}$, and starts in 0 at time 0, i.e. the person is alive. The mortality rates, μ_t and μ_t^* (respectively, under $\mathbb{P}$ and $\mathbb{P}^*$) are defined by the jump intensities of the process Z. They are assumed to be continuous and deterministic (as defined later in Sect. 5) and satisfy $\mu_t \to \infty$ and $\mu_t^* \to \infty$, which implies that $\lim_{t\to\infty} \mathbb{P}(Z_t = 1) = \lim_{t\to\infty} \mathbb{P}^*(Z_t = 1) = 1$.

Finally, we assume that the individual is risk averse and has a CRRA utility function u characterized by a constant relative risk aversion $1-\gamma$, constant elasticity of intertemporal substitution $1/(1-\gamma)$ (EIS) and time-dependent weights w_t reflecting the importance of present consumption in contrast to future consumption characterized by the impatience factor ρ,

$$u(t, c) = \frac{1}{\gamma} w_t^{1-\gamma} c^{\gamma}, \quad \text{where} \quad w_t^{1-\gamma} = \mathrm{e}^{-\rho t}.$$

Parameter γ is defined for $(-\infty, 1)\backslash\{0\}$, whereas for $\gamma = 0$ we have the case of a logarithmic utility.

2.1 Problem (A): optimal investment, consumption and sum insured

We keep the original settings defined in Richard (1975) and recall only the most important assumptions and results that are crucial for this paper.

Savings dynamics. The cash-flows accompanying the savings account are formalized by the continuous processes: a deterministic labor income l_t and a consumption process c_t.[2] The individual has a bequest motive and leaves upon death the amount on the savings account and the sum insured, $X_t + I_t$ to her heirs. The premium for the coverage is $\mu_t^* I_t$, where μ_t^* is the natural premium intensity decided by the life insurance company, also called pricing mortality. In particular, we allow for negative

[1] A similar approach to life insurance contracts can be found in Kraft and Steffensen (2008). The assumption about tradability of life insurance is not substantially different from considering a case where policy holders are allowed to make alterations to their contracts. Apart from realistic issues with health and other types of assymetric information (which do not appear in our model), this is certainly what appears in practice.

[2] The model could be extended by adding a stochastic labor income. However, the explicit solutions to problems (A) and (B) can be derived only if the labor income is assumed to be spanned by the stock risk. Otherwise, explicit solutions to the control problems do not exist.

I_t, which means that the person sells the amount I_t to the pension fund and purchases life annuities. The wealth dynamics while the person is alive develop as follows:

$$\begin{aligned} \mathrm{d}X_t &= \big(r + \pi_t(\alpha - r)\big)X_t\mathrm{d}t + \pi_t\sigma X_t\mathrm{d}W_t + l_t\mathrm{d}t - c_t\mathrm{d}t - \mu_t^* I_t\mathrm{d}t, \\ X_0 &= x_0, \end{aligned} \tag{2A}$$

where r is the return on the risk-free asset, α is the expected rate of return on the mutual fund, σ is the volatility of the mutual fund, π is the proportion invested in the mutual fund, and $1 - \pi$ in the risk-free asset.

Optimization problem. As the individual has a bequest motive, she obtains a utility from leaving the money to her heirs upon death:

$$U(t, x) = \frac{1}{\gamma} v_t^{1-\gamma} x^{\gamma},$$

where v_t is the weight for this utility at time t. We define v_t in terms of the weights for the utility of consumption w_t, thus v_t denotes the weight put on her heir's consumption relative to her own,

$$v_t^{1-\gamma} = \lambda^{-\gamma} w_t^{1-\gamma},$$

where λ is a constant. The individual can control consumption, investment and sum insured, to maximize the utility of consumption and bequest. Given the wealth dynamics, X_t, the problem is mathematically formulated as:

$$\begin{aligned} V^A(t, x) &= \sup_{\pi, c, I \in \mathcal{Q}[t, \widetilde{T})} E_{t,x}\left[\int_t^{\widetilde{T}} \mathrm{e}^{-\int_t^s \mu_\tau \mathrm{d}\tau}\Big(u(s, c_s) + \mu_s U(s, X_s + I_s)\Big)\mathrm{d}s\right], \\ V^A(\widetilde{T}, x) &= 0, \end{aligned} \tag{3A}$$

where $E_{t,x}$ is the conditional expectation under $\mathbb{P}$, given that the person is alive at time t and holds wealth $X_t = x$, and $\mathcal{Q}[t, \widetilde{T})$ is the set of control processes for the time $[t, \widetilde{T})$ which are admissible at time t. $\widetilde{T}$ is a fixed time point at which the investor is dead with certainty. Both utilities are multiplied by the factor $\mathrm{e}^{-\int_t^s \mu_\tau \mathrm{d}\tau}$ denoting the probability that the individual survives until time $s > t$, given she has survived until time t. Furthermore, the utility of bequest is multiplied by the mortality intensity rate μ_s, which represents the probability that the person dies within a short period after time s.

The optimal value function for this problem is given by

$$V^A(t, x) = \frac{1}{\gamma} f_A(t)^{1-\gamma}\big(x + g_A(t)\big)^{\gamma}, \tag{4A}$$

where

$$f_A(t) = \int_t^{\widetilde{T}} \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \left(\mu_\tau - \gamma(\mu_\tau^* + \varphi)\right)\mathrm{d}\tau} \left[w_s + \left(\frac{\mu_s}{(\mu_s^*)^\gamma}\right)^{\frac{1}{1-\gamma}} v_s \right] \mathrm{d}s, \varphi = r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}, \tag{5A}$$

$$g_A(t) = \int_t^{\widetilde{T}} \mathrm{e}^{-\int_t^s (r+\mu_\tau^*)\mathrm{d}\tau} l_s \mathrm{d}s, \tag{6A}$$

and the optimal investment, consumption and sum insured are of the form:

$$\begin{aligned} \pi_t^* &= \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{X_t + g_A(t)}{X_t}, \qquad c_t^* = \frac{w_t}{f_A(t)}\big(X_t + g_A(t)\big), \\ I_t^* &= \left(\frac{\mu_t}{\mu_t^*}\right)^{\frac{1}{1-\gamma}} \frac{v_t}{f_A(t)} \left(X_t + g_A(t)\right) - X_t. \end{aligned} \tag{7A}$$

Equations (4A)–(7A) correspond to Equations (25) and (40)–(44) in Richard (1975).

2.2 Problem (B): optimal investment with optimal life annuities

The application of problem (B) is relevant for a pension product design in a defined contribution pension plan, both occupational and private. As seen in most of the European pension markets, premiums are defined as a fixed percentage of the salary. In the occupational pension plans, this percentage is typically decided by the employer, whereas private plans allow the customer to choose both the size and the frequency of the premiums. When entering the contract, the individual is often given a choice of different pension products characterized by various investment strategies and a type of benefits. For example, one can choose between a conservative, moderate or aggressive investment strategy, but it is the pension fund that decides how to invest the savings such that it reflects the customer's preferences. In a different kind of product, such as a unit-linked product, it is the pension saver who can decide on the portfolio allocation and can adjust it during the contract as she wishes. Both types of products typically offer an option to add a guarantee to the contract, but in this work we assume that the benefits are directly linked to the market and no guarantees are provided. In terms of choosing the type of the benefits, one can typically choose between a lump sum payment ($\widetilde{T} = T$), life annuities ($\widetilde{T} \to \infty$) or temporary life annuities (10–25 years, $\widetilde{T} < \infty$). The size of the benefits is calculated by the pension fund according to the actuarially fair principles.

The considered optimization problem allows for controlling the investment strategy both before and after retirement, as well as the size and the distribution of the benefits after retirement. The optimal solution determines a product that is customized to the individual's risk and impatience preferences. In case of death, the pension fund inherits all the individual's savings.

Savings dynamics. We split the problem into a period before retirement (accumulation phase, $t < T$) and after retirement (decumulation phase, $t \geq T$). We further denote a fixed percentage of the labor income contributed by the pension saver to the retirement savings by l_t, and the benefits that the pension saver receives after retirement by c_t. Thus, $l_t = 0$ for $t \in [T, \widetilde{T})$ and $c_t = 0$ for $t \in [t_0, T)$. Since the person has no bequest motive, she has an additional income of $\mu_t^* X_t$, which is the amount that the pension fund pays to become her only inheritor. With these assumptions, the savings dynamics are of the form,

$$\mathrm{d}X_t = \begin{cases} \left(r + \pi_t(\alpha - r) + \mu_t^*\right) X_t \mathrm{d}t + \pi_t \sigma X_t \mathrm{d}W_t + l_t \mathrm{d}t, & t \in [t_0, T), \\ \left(r + \pi_t(\alpha - r) + \mu_t^*\right) X_t \mathrm{d}t + \pi_t \sigma X_t \mathrm{d}W_t - c_t \mathrm{d}t, & t \in [T, \widetilde{T}), \end{cases} \qquad \text{(2B)}$$
$$X_t = x_0.$$

Optimization problem. The pension saver wishes to maximize the utility of her pension benefits, or in other words, the utility of consumption after retirement, while being able to control the investment strategy and the distribution of the benefits. She obtains no utility from consumption before retirement, i.e. $u(t, c) = 0$ for $t \in [t_0, T)$, thus the problem is formulated as follows:

$$V^B(t, x) = \sup_{\substack{\pi \in \mathcal{Q}[t, \widetilde{T}), \\ c \in \mathcal{Q}[\max(t,T), \widetilde{T})}} E_{t,x} \left[\int_{\max(t,T)}^{\widetilde{T}} \mathrm{e}^{-\int_t^s \mu_\tau \mathrm{d}\tau} u(s, c_s) \mathrm{d}s \right], \qquad \text{(3B)}$$
$$V^B(\widetilde{T}, x) = 0,$$

where, as before, $E_{t,x}$ is the conditional expectation under $\mathbb{P}$, given that the person is alive at time t and holds savings $X_t = x$, and $\mathcal{Q}[t, \widetilde{T})$ is the set of control processes for time $[t, \widetilde{T})$ that are admissible at time t. As in problem (A), we take into consideration the uncertain lifetime of the individual by multiplying the utility function by the factor $\mathrm{e}^{-\int_t^s \mu_\tau \mathrm{d}\tau}$. At time $\widetilde{T}$, the investor is assumed to be dead with certainty.

Even though the individual obtains no utility from consumption before retirement, the investment process π_t is controlled both before and after retirement. Thus, the novelty of this problem from the stochastic optimal control point of view lies in defining the optimal consumption and investment over different (but partially overlapping) periods. The research to date has tended to focus on deriving the explicit solution for the controls over the same period; either until retirement, see, e.g., Cairns et al. (2006) and Bruhn and Steffensen (2011), after retirement, see, e.g., Gerrard et al. (2004, 2006) and He and Liang (2013), or generally over the life cycle, see, e.g., Merton (1969, 1971), Richard (1975), Milevsky and Young (2007), and Kraft and Steffensen (2008). Because, to the best of our knowledge, neither problem (B) nor the case of deriving optimal controls over different periods has been considered by other researchers, we derive the optimal value function and the optimal controls. For details, see Appendix 2.

The optimal value function is of the form:

$$V^B(t, x) = \frac{1}{\gamma} f_B(t)^{1-\gamma} \left(x + g_B(t)\right)^{\gamma}, \qquad \text{(4B)}$$

where

$$f_B(t) = \begin{cases} \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^T (\mu_\tau - \gamma(\mu_\tau^* + \varphi))\mathrm{d}\tau} \int_T^{\widetilde{T}} \mathrm{e}^{-\frac{1}{1-\gamma}\int_T^s \left(\mu_\tau - \gamma(\mu_\tau^* + \varphi)\right)\mathrm{d}\tau} w_s \,\mathrm{d}s, & t \in [t_0, T), \\ \int_t^{\widetilde{T}} \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \left(\mu_\tau - \gamma(\mu_\tau^* + \varphi)\right)\mathrm{d}\tau} w_s \,\mathrm{d}s, & t \in [T, \widetilde{T}), \end{cases} \tag{5B}$$

$$g_B(t) = \begin{cases} \int_t^T \mathrm{e}^{-\int_t^s (r + \mu_\tau^*)\mathrm{d}\tau} l_s \mathrm{d}s, & t \in [t_0, T), \\ 0, & t \in [T, \widetilde{T}), \end{cases} \tag{6B}$$

$$\varphi = r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)},$$

and the optimal controls are given by

$$\pi_t^* = \frac{\alpha - r}{\sigma^2(1-\gamma)} \frac{X_t + g_B(t)}{X_t}, \quad t \in [t, \widetilde{T}), \qquad c_t^* = \frac{w_t}{f_B(t)} X_t, \quad t \in [T, \widetilde{T}). \tag{7B}$$

Since the utility function u is concave in c, the retirement benefits must be non-negative. Furthermore, we must have that $X_{\widetilde{T}} = 0$, which ensures that all the savings that belong to the pension saver have been paid out during the distribution phase.

In both problems, the functions $g_A(t)$ and $g_B(t)$ are of the same form. The first function has been defined by Richard (1975) as *human capital* and represents the expected present value of future earnings, or, in other words, what an individual's labor force is worth in the financial market. Function $g_B(t)$ represents the present value of the future premiums, i.e. the fraction of the human capital that is transfered to the pension savings account.

3 Combined MSP and optimal control

As shown in the previous section, we can apply stochastic optimal control to derive the optimal solution for the above presented problems (A) and (B). However, as soon as we add some more realistic constraints, such as limits on portfolio composition, limits on the sum insured, an inclusion of transaction costs and taxes on capital gains, the explicit solution does not exist. We show later in Sect. 5.1 that adding these constraints affects the optimal decisions. For example, the optimal investment obtained by Richard (1975) implies shorting the risk-free asset and borrowing money to invest in the mutual fund, which is often a limitation for a private investor. Moreover, the sum insured is for most of the individual's lifetime negative, which cannot happen in practice. Finally, while transaction costs have a minor impact on the optimal controls, the presence of taxes on capital gains affects the optimal investment strategy significantly.

To focus on the practicalities of the problem, we choose the MSP framework. While the strength of this approach lies in the ability to address realistic constraints, its weakness lies in the limited ability to handle many time periods under sufficient uncertainty. The scenario tree grows exponentially with the number of time periods, and solving the problem soon becomes computationally intractable. When dealing with the curse of dimensionality, researchers choose either long and increasing intervals between the decisions, see, e.g., Carino et al. (1998), Dempster et al. (2003) and Consigli et al. (2012), add a steady state terminal value term to the objective function and approximate the infinite horizon problem using the dual equilibrium technique, see, e.g., Grinold (1977, 1983) and Carino et al. (1998), or apply different scenario reduction algorithms, see, e.g., Heitsch and Römisch (2009a, b).

These methods, however, also have some drawbacks. For example, choosing long intervals between the stages implies that the individual does not have an opportunity to change her decisions for a considerable amount of time, and the optimal decisions may differ from those made more frequently. Furthermore, the impact of choosing increasing rebalancing intervals relative to constant intervals remains unknown. The dual equilibrium technique implies the approximation of the infinite horizon according to some assumed discount factor, and does not include the uncertainties during the steady state. Finally, despite the increasing research on scenario reduction algorithms, modelling the entire lifetime of an individual with yearly intervals still remains computationally intractable. One can simplify the tree during the later years to a single branch tree but if the branching factor at any node is lower than the number of non-redundant assets, the scenarios may allow for arbitrage opportunities, which would affect the optimal decisions, see, e.g., Kouwenberg (2001), Klaasen (2002) and Geyer et al. (2014).

In the light of the limitations of stochastic optimal control and multi-stage stochastic programming, a mixed approach sounds appealing. Applying the MSP approach during the first years of the contract and stochastic optimal control during the remaining lifetime of the individual, see Fig. 1, allows us to model the entire lifetime of the individual under realistic assumptions, with short intervals between the decisions, and in a short computational time. The decisions are made under enough uncertainty both during the period modeled by MSP, and in the later years modeled by stochastic optimal control.

Even though the mixed approach assumes the more realistic constraints only during the first few years, they affect the optimal controls (see Sect. 5). We argue that the initial decisions are the most important because the customer needs an advice about her personal finance and pension at the present moment. A financial advisor or a pension fund typically would hold regular meetings with the customer and rerun the model for a different set of parameters than those initially chosen. This is necessary not only because the expectations about the economy change, but also because the customer

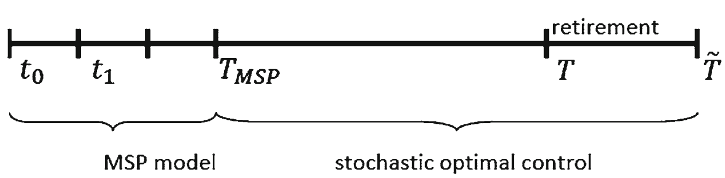

Fig. 1 The first years decisions are solved by the MSP model, whereas the decisions over the remaining lifetime of the individual are solved using HJB techniques.

might change her risk and impatience preferences. To what extent it is possible to neglect certain constraints and having a simpler model in the long run, is not trivial and left to future research.

When applying the mixed approach, we first split the lifetime of the individual into two periods: $t \in [t_0, T_{\text{MSP}})$ and $t \in [T_{\text{MSP}}, \widetilde{T})$, and rewrite the objective functions for problems (A) and (B) defined in Eqs. (3A) and (3B), respectively, using their recursive properties, i.e.

$$V^A(t,x) = \sup_{\pi,c,I \in \mathcal{Q}[t,T_{\text{MSP}})} E_{t,x}\Bigg[\int_t^{T_{\text{MSP}}} \mathrm{e}^{-\int_t^s \mu_\tau d\tau}\Big(u(s,c_s) + \mu_s U(s, X_s + I_s)\Big)ds + \mathrm{e}^{-\int_t^{T_{\text{MSP}}} \mu_\tau d\tau} V^A(T_{\text{MSP}}, X_{T_{\text{MSP}}})\Bigg], \tag{8A}$$

$$V^B(t,x) = \sup_{\substack{\pi \in \mathcal{Q}[t,T_{\text{MSP}}),\\ c \in \mathcal{Q}[\max(t,T),T_{\text{MSP}})}} E_{t,x}\Bigg[\int_{\max(t,T)}^{T_{\text{MSP}}} \mathrm{e}^{-\int_t^s \mu_\tau d\tau} u(s,c_s)ds + \mathrm{e}^{-\int_t^{T_{\text{MSP}}} \mu_\tau d\tau} V^B(T_{\text{MSP}}, X_{T_{\text{MSP}}})\Bigg]. \tag{8B}$$

Then, we calculate the optimal value functions $V^A(T_{\text{MSP}}, X_{T_{\text{MSP}}})$ and $V^B(T_{\text{MSP}}, X_{T_{\text{MSP}}})$ derived using a stochastic optimal control approach [Eqs. (4A) and (4B)], respectively), and insert them into Eqs. (8A) and (8B). These equations define the objective functions for problems (A) and (B) for the multi-period stochastic model. In this way, two optimization methods are integrated into one MSP formulation, and the optimal decisions are made during the entire lifetime of the individual.

4 MSP formulation

The main elements of multi-stage stochastic programming are a scenario tree and an optimization model.

A scenario tree consists of nodes $n \in \mathcal{N}_t$ representing the range of possible outcomes for the uncertainties. All the nodes are uniquely assigned to stages $t = t_0, \ldots, T_{\text{MSP}}$ such that at the first stage there is a root node n_0, at the subsequent stages, $n \in \mathcal{N}_t$, $t > t_0$, each node has a unique ancestor n^-, and at all the stages except for the final stage, $n \in \mathcal{N}_t$, $t < T_{\text{MSP}}$, each node has children nodes n^+. The nodes at the final stage T_{MSP} are called the leaves. A scenario $\mathcal{S}^n$ consists of a leaf n and all its predecessors $n^-, n^{--}, \ldots, n_0$, or, in other words, it is defined by a single branch from the root to the leaf. The number of scenarios in the tree equals the number of leaves. Each node has a probability $p_{t,n}$, so that $\forall_t \ \sum_{n \in \mathcal{N}_t} p_{t,n} = 1$, thus, each scenario $\mathcal{S}^n$ has a probability equal to the product of the probabilities of all the nodes in the scenario, $p_{\mathcal{S}^n} = p_{T_{\text{MSP}},n} \cdot p_{T_{\text{MSP}}-1,n^-} \cdot p_{T_{\text{MSP}}-2,n^{--}} \cdot \ldots \cdot p_{t_0,n_0}$.

The optimal decisions along the tree are computed numerically at each node of the tree, given the information available at that point. Decisions do not depend on the future observations but anticipate possible future realizations of the random vector. After the outcomes have been observed, the decisions for the next period are made and depend both on the realizations of the random vector and the decisions made in the previous stage. This combination of anticipative and adaptive models in one mathematical framework makes this approach particularly appealing in financial applications; the investor can specify the composition of a portfolio by taking into account possible future movements of asset returns (anticipation), and rebalance the portfolio (take recourse decisions) as prices change, see Zenios (2008). The applications of MSP specifically in individual asset liability management can be found in, e.g., Ziemba and Mulvey (1998), Kim et al. (2012) and Konicz and Mulvey (2013), whereas for a general introduction to stochastic programming we refer to Birge and Louveaux (1997) and Ruszczyński and Shapiro (2003).

4.1 Objective and constraints

The first years decisions for the optimization problems considered in this paper are modeled by a T_{MSP}-period model. The decision variables are defined with respect to the nodes n in the scenario tree, where $\mathcal{N}_t$ denotes the set of nodes corresponding to stage t, and $\mathcal{J}$ is the set of available financial assets. For each period $t = t_0, \ldots, T_{\text{MSP}}$, node $n \in \mathcal{N}_t$, and asset class $i \in \mathcal{J}$, we define the following decision variables and parameters:

Parameters	
x_0	initial value of the savings,
l_t	labor income/premiums paid to the savings account at time t,
μ_t	mortality rate for an individual aged $y+t$,
q_{y+t}	probability that an individual aged $y+t$ dies during the following period,
q^*_{y+t}	probability that an individual aged $y+t$ dies during the following period used for pricing life insurance/life annuities,
$p_{t,n}$	probability of node $n \in \mathcal{N}_t$, obtained via scenario generation, see Sect. 4.3,
$r_{i,t,n}$	return on asset i at node $n \in \mathcal{N}_t$, obtained via scenario generation.

Decision variables	
$\widetilde{P}_{i,t,n} \geq 0$	amount of asset i purchased in period t and node n,
$\widetilde{S}_{i,t,n} \geq 0$	amount of asset i sold in period t and node n,
$\widetilde{X}_{i,t,n}$	holdings of asset i at the beginning of period t, at node n, after rebalancing,
$\widetilde{X}^-_{t,n}$	holdings in all assets at the beginning of period t, at node n, before rebalancing, $\widetilde{X}^-_{t_0,n_0} = x_0, \quad \widetilde{X}^-_{t,n} = \sum_{i\in\mathcal{J}}(1+r_{i,t,n})\widetilde{X}_{i,t-1,n^-}, t = t_1, \ldots, T_{\text{MSP}}, \ n \in \mathcal{N}_t$,
$\widetilde{C}_{t,n}$	consumption/benefit in period t and node n,
$\widetilde{I}_{t,n}$	sum insured in period t and node n (problem (A) only).

We use capital letters with tilde to denote the variables of the MSP formulation as opposed to the lowercase letters denoting the parameters for the model. Expression $\mathbb{1}_{\{(\cdot)=t\}}$ denotes an indicator function equal to 1 if $(\cdot) = t$ and 0 otherwise.

The entire MSP formulation that replicates the assumptions made in the dynamic programming model setup, consists of the objective function and three constraints:

the budget constraint, the asset inventory constraint and the constraint on positivity of certain variables. With only these three constraints, we can replicate the continuous time models presented in Sect. 2. Later, in Sect. 4.2, we modify these equations to investigate the impact of various factors, such as limits on portfolio composition, limits on the sum insured, transaction costs, and taxes on capital gains.

4.1.1 Problem (A)

The problem of optimizing the expected utility of consumption and bequest, where the investment, consumption and sum insured are the controlled processes, can be modeled with the set of the following equations.

The objective function (9A) is obtained by rewriting Eq. (8A) to a discrete time formulation, i.e. substituting the integrals with the sums and replacing the expectation operator E with its discrete definition. The budget constraint (10A), or alternatively, the cash flow balance constraint, specifies that the amount invested in the purchase of new securities and the amount spent on consumption must be equal to the amount gained from the sale of the securities, the labor income paid to the savings account, and any initial savings x_0 that the person has at the beginning of the contract. Moreover, to the left- hand side of the equation, we add the term $q^*_{y+t}\widetilde{I}_{t,n}$ denoting the price for the life insurance that the investor pays at each period. The inclusion of the sum insured variable $\widetilde{I}_{t,n}$ in the budget constraint and in the objective function for problem (A) is an important part of the model that distinguishes our work from Geyer et al. (2009a). Constraint (11A) calculates the value of the savings, which is equal to the accumulated capital gains/losses on the assets held in the portfolio and the value of the assets purchased in the current period, minus the value of the assets sold in the current period. The continuous time versions of the problems assume the utility function u to be strictly concave in c, and U to be strictly concave in $X_t + I_t$. Furthermore, for $\gamma < 0$, we have $u(0) = U(0) = -\infty$, therefore we define the positivity constraints (12A). For $\gamma \in (0, 1)$, we have $u(0) = 0$ and $U(0) = 0$, so the positivity constraints are substituted with the non-negativity constraints.

This leads to the model:

$$\max \sum_{s=t_0}^{T_{\text{MSP}}-1} \sum_{n\in\mathcal{N}_s} p_{s,n} \cdot \mathrm{e}^{-\int_{t_0}^{s}\mu_\tau d\tau} \cdot \left[u(s, \widetilde{C}_{s,n}) + q_{y+s}\, U\left(s, \widetilde{X}^-_{s,n} + \widetilde{I}_{s,n}\right)\right]$$
$$+ \sum_{n\in\mathcal{N}_{T_{\text{MSP}}}} p_{T_{\text{MSP}},n} \cdot \mathrm{e}^{-\int_{t_0}^{T_{\text{MSP}}}\mu_\tau d\tau} \cdot V^A\left(T_{\text{MSP}}, \widetilde{X}^-_{T_{\text{MSP}},n}\right), \tag{9A}$$

$$\sum_{i\in\mathcal{J}} \widetilde{P}_{i,t,n} + \widetilde{C}_{t,n} + q^*_{y+t}\widetilde{I}_{t,n} = x_0 \mathbb{1}_{\{t=t_0\}} + \sum_{i\in\mathcal{J}} \widetilde{S}_{i,t,n} + l_t, \quad t = t_0, \ldots, T_{\text{MSP}} - 1,\ n \in \mathcal{N}_t, \tag{10A}$$

$$\widetilde{X}_{i,t,n} = \left(1 + r_{i,t,n}\right)\widetilde{X}_{i,t-1,n^{-}}\mathbb{1}_{\{t>t_0\}} + \widetilde{P}_{i,t,n} - \widetilde{S}_{i,t,n}, \quad t = t_0, \ldots, T_{\text{MSP}} - 1, n \in \mathcal{N}_t, \; i \in \mathcal{J}, \tag{11A}$$

$$\widetilde{C}_{t,n} > 0, \quad \widetilde{X}^{-}_{t,n} + \widetilde{I}_{t,n} > 0, \qquad t = t_0, \ldots, T_{\text{MSP}} - 1, \; n \in \mathcal{N}_t. \tag{12A}$$

4.1.2 Problem (B)

The problem of maximizing the expected utility of retirement benefits, where the investment process is controlled both before and after retirement, whereas the benefits are controlled only after retirement, can be modeled using an MSP formulation as follows.

The objective function (9B) is a discretized version of Eq. (8B). The budget constraint (10B) specifies that the amount invested in the purchase of new securities and the consumed amount must be equal to the amount gained from the sale of the securities, the premium paid to the savings account, and any savings x_0 that the person owns at the beginning of the contract. We also add to the right-hand side of the balance equation the term $q^*_{y+t}\widetilde{X}^{-}_{t,n}$ denoting the price that the pension fund pays the pension saver to be her only inheritor. Furthermore, we add the pension benefit to the side of the outgoing payments for $t \geq T$. Constraint (11B) calculates the value of the savings on the pension account. This constraint is identical to the asset inventory balance for problem (A). The utility function u is assumed to be strictly concave in c, and for $\gamma < 0$, we have $u(0) = U(0) = -\infty$. Therefore, we assume that the benefits are positive, (12B), or, for $\gamma \in (0, 1)$, non-negative.

This leads to the following MSP formulation:

$$\max \sum_{s=\max(t_0,T)}^{T_{\text{MSP}}-1} \sum_{n \in \mathcal{N}_s} p_{s,n} \cdot \mathrm{e}^{-\int_{t_0}^{s} \mu_\tau d\tau} u(s, \widetilde{C}_{s,n}) + \sum_{n \in \mathcal{N}_{T_{\text{MSP}}}} p_{T_{\text{MSP}},n} \cdot \mathrm{e}^{-\int_{t_0}^{T_{\text{MSP}}} \mu_\tau d\tau} \cdot V^B\left(T_{\text{MSP}}, \widetilde{X}^{-}_{T_{\text{MSP}},n}\right), \tag{9B}$$

$$\sum_{i \in \mathcal{J}} \widetilde{P}_{i,t,n} + \widetilde{C}_{t,n}\mathbb{1}_{\{t \geq T\}} = x_0\mathbb{1}_{\{t=t_0\}} + \sum_{i \in \mathcal{J}} \widetilde{S}_{i,t,n} + l_t + q^*_{y+t}\widetilde{X}^{-}_{t,n}, \quad t = t_0, \ldots, T_{\text{MSP}} - 1, \; n \in \mathcal{N}_t, \tag{10B}$$

$$\widetilde{X}_{i,t,n} = \left(1 + r_{i,t,n}\right)\widetilde{X}_{i,t-1,n^{-}}\mathbb{1}_{\{t>t_0\}} + \widetilde{P}_{i,t,n} - \widetilde{S}_{i,t,n}, \quad t = t_0, \ldots, T_{\text{MSP}} - 1, \; n \in \mathcal{N}_t, \; i \in \mathcal{J}, \tag{11B}$$

$$\widetilde{C}_{t,n} > 0, \qquad t = \max(t_0, T), \ldots, T_{\text{MSP}} - 1, \; n \in \mathcal{N}_t. \tag{12B}$$

4.2 Additional constraints

We have chosen to model the first years decisions with MSP to account for practical constraints such as limits on portfolio composition, limits on the sum insured, transaction costs, and taxes on capital gains. These constraints are commonly used in financial applications solved with MSP approach, see, e.g., Geyer et al. (2009a) and Ferstl and Weissensteiner (2011), but rare in investment-consumption problems solved with stochastic optimal control. We present below how to modify the original constraints and how to model the aforementioned constraints in the MSP formulation presented in Sect. 4.1.

Limits on portfolio composition. A feature that is interesting from the point of view of a private investor, a pension saver and a pension fund, is the limit on the portfolio composition. For example, no pension fund allows for borrowing or short selling of the assets. Alternatively, if the investor has certain preferences about the minimum and maximum percentage of her wealth invested in a certain asset, we would like to include it in the optimization model. The limits on the portfolio composition can be incorporated in the MSP formulation by adding the following constraints:

$$\widetilde{X}_{i,t,n} \geq d_i \sum_{j \in \mathcal{J}} \widetilde{X}_{j,t,n}, \qquad \widetilde{X}_{i,t,n} \leq u_i \sum_{j \in \mathcal{J}} \widetilde{X}_{j,t,n},$$
$$t = t_0, \ldots, T_{\text{MSP}} - 1,\ n \in \mathcal{N}_t,\ i \in \mathcal{J}, \tag{16}$$

where d_i and u_i are the lower and upper limits for the holdings of asset i. In particular, d_i and u_i can be extended to be time dependent, which would be suitable for a person with specific preferences only during certain periods.

Limits on the sum insured. Richard (1975)'s model does not assume any constraints on the size or the sign of the sum insured I_t. Including such constraints in the dynamic programming approach is definitely not trivial to solve, but the solution has been studied using other methods, see Nielsen and Steffensen (2008). The MSP model allows for adding the limits on the sum insured in a straightforward way:

$$\widetilde{I}_{t,n} \geq d_{ins}, \qquad \widetilde{I}_{t,n} \leq u_{ins}, \quad t = t_0, \ldots, T_{MSP} - 1,\ n \in \mathcal{N}_t. \tag{17}$$

As will be shown in Sect. 5.1, being able to control the sign of the sum insured is important from a practical point of view. A negative sum insured means that it is the individual who sells the life insurance to the pension fund for the price of $q^*_{y+t}\widetilde{I}_{t,n}$. We could interpret this situation as investing a part of the savings in a life annuity, however, receiving an annuity income at any age before retirement sounds unrealistic.

Transaction costs. Similarly, investigating how the transaction costs η_i affect the optimal controls is of practical importance. We add the costs as a percentage of the traded amount by modifying the budget equations (10A) and (10B). We subtract the costs from the amount of the assets sold in a given period, and add the costs to the amount purchased. The modified budget equations are defined as follows:

$$\sum_{i \in \mathcal{J}} \widetilde{P}_{i,t,n}(1 + \eta_i) + \widetilde{C}_{t,n} + q^*_{y+t}\widetilde{I}_{t,n} = x_0 \mathbb{1}_{\{t=t_0\}} + \sum_{i \in \mathcal{J}} \widetilde{S}_{i,t,n}(1 - \eta_i) + l_t, \tag{10A'}$$

$$\sum_{i \in \mathcal{J}} \widetilde{P}_{i,t,n}(1 + \eta_i) + \widetilde{C}_{t,n} \mathbb{1}_{\{t \geq T\}} = x_0 \mathbb{1}_{\{t=t_0\}} + \sum_{i \in \mathcal{J}} \widetilde{S}_{i,t,n}(1 - \eta_i) + l_t + q^*_{y+t}\widetilde{X}^-_{t,n}, \tag{10B'}$$

for $t = t_0, \ldots, T_{\text{MSP}} - 1$ and $n \in \mathcal{N}_t$.

Taxes on capital gains. We further consider taxes on capital gains τ_i. Generating a scenario tree for the MSP model allows us to subtract taxes only from the positive capital gains, i.e.

$$r_{i,t,n}^{net} = \begin{cases} r_{i,t,n}(1-\tau_i), & r_{i,t,n} > 0, \\ r_{i,t,n}, & r_{i,t,n} \leq 0. \end{cases} \tag{18}$$

This requires changing the asset inventory balance constraints (11A) and (11B) as follows:

$$\begin{aligned} \widetilde{X}_{i,t,n} &= \left(1 + r_{i,t,n}^{\text{net}}\right)\widetilde{X}_{i,t-1,n^-}\mathbb{1}_{\{t>t_0\}} + \widetilde{P}_{i,t,n} - \widetilde{S}_{i,t,n}, \\ & \quad t = t_0, \ldots, T_{\text{MSP}} - 1,\ n \in \mathcal{N}_t,\ i \in \mathcal{J}. \end{aligned} \tag{11A',11B'}$$

Subtracting taxes only from the positive asset returns is convenient in contrast to the stochastic optimal control approach, where the asset prices are modeled by the Black Scholes model, and one can only adjust the expected returns and volatility of the asset returns by introducing $\alpha_i^{\text{net}} = \alpha_i(1-\tau_i)$ and $\sigma_i^{\text{net}} = \sigma_i(1-\tau_i)$, see, e.g., Bruhn (2013). The taxes are then subtracted from both positive (gains) and negative (losses) returns, which can be interpreted as a possibility to deduct the taxes from the negative capital income.

Other modifications. There are plenty of economic factors whose impact on the optimal control would be interesting to investigate but are beyond the scope of this paper. Probably, the most relevant from a practical perspective is modelling the asset returns using a different model than Black Scholes. Geyer et al. (2009b) extended their previous work by applying the first-order unrestricted vector autoregressive process [VAR(1)] to model asset returns and found that time-varying investment opportunities significantly affect the asset allocation.

4.3 Scenario generation

The uncertainty associated with the market returns is modeled by an $N-1$-dimensional random process. The multivariate return process evolves in discrete time, and the underlying probability distributions are approximated by discrete distributions in terms of a scenario tree.

We have tested a number of scenario generation methods for stochastic programming, including sampling, simulation, scenario reduction techniques and the methods based on matching the statistical properties of the underlying process, see, e.g., Kaut and Wallace (2005) and Heitsch and Römisch (2009a). Since we combine stochastic programming with optimal control in one mathematical framework, our goal is to generate a scenario tree with the prices for the securities following the Black Scholes model defined in Eq. (1). Specifically, we aim for constructing a multi-period scenario tree with a discrete representation of a normal distribution $\mathcal{N}\left(\alpha_i - {\sigma_i}^2/2,\ \sigma_i\right)$, where the returns in two adjacent periods are independent and identically distributed. Therefore, the most suitable approach in our study is a moment matching method.

The moment matching approach was first described in Høyland and Wallace (2001). This method is based on solving a nonlinear optimization problem where both the asset returns and the probabilities of each node are the decision variables defining the scenario tree. Since then, further improvements to this method have been made. For example, to ensure that the required moments are matched, Høyland et al. (2003) suggest to perform a number of transformations rather than solving a nonlinear optimization problem, whereas Gülpınar et al. (2004) present a combined simulation and optimization approach. In the latter study, the asset returns are first simulated and then fixed in the optimization model, so that the only decision variables in the model are the probabilities associated with each node. This method can moreover be applied to generate the entire tree at once, and not node by node, as in the aforementioned approaches. Ji et al. (2005) was the first to propose a linear programming (LP) moment matching approach, though only for a single period tree. Similarly as in the previous paper, the outcomes of asset returns must be predetermined. Inspired by that work, Xu et al. (2012) design a new moment matching approach that combines simulation, the K-means clustering approach, and linear moment matching to generate multi-stage scenario tree. This method ensures that the statistical properties are matched well, the generated scenario tree has a moderate size, the solution time is reduced, and at least two branches are derived from each non-leaf node. Finally, Chen and Xu (2013) improve this work by removing the simulation component and applying the K-means clustering method directly onto the historical dataset combined with LP moment matching. This approach significantly reduces the computational time while preserving the required statistical properties.

Despite the advantages of the newest scenario generation methods described above, for the purpose of our study we choose one of the older algorithms, namely Høyland and Wallace (2001), which we further improve by adding a priori the non-arbitrage constraints, see Klaasen (2002). This method matches the statistical properties of a geometric Brownian motion better than other algorithms in a situation when having as low a branching factor as possible is a priority. The algorithms described in Xu et al. (2012) and Chen and Xu (2013) are definitely more efficient and can capture more complex models for asset returns, such as the vector autoregressive and multivariate generalized autoregressive conditional heteroscedasticity models, but they require a larger branching factor. For our choice of assets and the distribution parameters, a satisfactory statistical match is obtained for a branching factor of at least eight, whereas the algorithm presented in Høyland and Wallace (2001) allows to generate the required scenario tree with only four branches. Our priority is to investigate the impact of some realistic constraints that are hard to implement in a stochastic optimal control approach, therefore to study more time periods, we choose a scenario tree with fewer branches.

5 Numerical results

This section illustrates problems (A) and (B). We first present the explicit solutions obtained using stochastic optimal control and the solutions obtained by the mixed approach presented in Sect. 4.1. Afterwards, we add or modify the particular constraints as described in Sect. 4.2 and investigate their impact on the optimal controls during the first years of the contract.

Parameters. If not specified in the captions of the tables and figures, the following parameters have been chosen for testing the models:

- Market: the number of assets, $N = 3$; two risky and one risk-free asset, the expected rates of returns on the assets are, respectively, $\alpha_1 = 0.05, \alpha_2 = 0.07, r = \alpha_3 = 0.02$; the volatility of the assets, $\sigma_1 = 0.2$, $\sigma_2 = 0.25$, $\sigma_3 = 0$; the correlation between the risky assets is $\phi_{12} = \phi_{21} = 0.5$. The parameters are adjusted for inflation.
- Utility function: risk aversion, $1 - \gamma = 4$, corresponding to the optimal proportion in risky asset after retirement π_t^* equal to 25 %; the impatience factor for the utility weights, $\rho = 0.04$, chosen such that $\rho \geq r$; the weight on the utility of bequest relatively to the utility of consumption [problem (A) only], $\lambda = 5$.
- Lifetime uncertainty: the mortality intensity model, $\mu_t = \mu_t^* = 10^{\beta+\delta(y+t)-10}$, where y is the age of the person at time t_0, $\beta = 4.59364$ and $\delta = 0.05032$.[3] The model has been calibrated to the mortality rates among Danish women over age 40, Finanstilsynet (2010). We further approximate the probability that an individual aged $y + t$ dies during the next year by the mortality rate, i.e. $q_{y+t} \approx \mu_t$ and $q_{y+t}^* \approx \mu_t^*$.
- Age at the beginning of the contract: problem (A) and (B) before retirement, $y = 45$, problem (B) after retirement, $y = 70$.
- Cash-flows:
  - Problem (A): labor income $l_t = 27{,}000$ EUR, corresponding to the average Danish disposable income (after taxes) for a 45-year-old individual, Danmarks Statistik (2010); average savings of a 45-year-old individual, $x_0 = 60{,}000$ EUR.
  - Problem (B) before retirement: premiums, $l_t = 4{,}000$ EUR, corresponding to 15 % of the salary; average savings on the pension account of a 45-year-old individual, $x_0 = 75{,}000$ EUR.
  - Problem (B) after retirement: premiums, $l_t = 0$, average savings on the pension account of a 70-year-old individual, $x_0 = 225{,}000$ EUR.

Similarly as Geyer et al. (2009a), we consider small-scale optimization problems rather than large-scale problems. In the beginning of our analysis, we are interested in replicating the optimal controls obtained from the continuous time models. The small-scale models are sufficient to evaluate whether a similar solution can be achieved by running the MSP model. With only three assets: one risk-free and two risky assets, the first four moments of a normal distribution can be approximated with 4 nodes. A scenario tree with 6 periods and a constant branching factor of 4 gives in total $4^5 = 1{,}024$ scenarios. Additionally, we rerun each program 50 times for different

[3] In principle, the mortality intensity model does not assume that an individual is dead at time $\widetilde{T}$ with probability 1. However, for $\widetilde{T} = 110$, this error is negligible.

scenario trees, and present the results in terms of means and standard errors from the sampling. We use the notation X_t^*, π_t^*, c_t^* and I_t^* for the expected optimal value of savings, investment, consumption, and sum insured (average across all the scenarios) obtained both from the Richard (1975)'s model and from the combined model.

Problems (A) and (B) have been implemented using Matlab 8.2.0.701 (R2013b) for calculation of the explicit solutions, GAMS 24.1.3 with CONOPT 3 (3.15L) solver for scenario generation, and GAMS with MOSEK 7.0.0.75 solver for the MSP formulation and solution of the stochastic programming models. The running time of each model for one scenario tree takes only a few seconds on a Dell computer with an Intel Core i5-2520M 2.50 GHz processor and 4 GB RAM. Due to the linearity of the constraints, one can also approximate the objective function by a piecewise-linear interpolant, see e.g. Kontogiorgis (2000) and Rasmussen (2011), and solve the problem using a linear solver. We have tested a number of linear and nonlinear solvers, and we find that for the considered problems MOSEK, which uses an interior point algorithm, is the best suited solver in terms of speed, robustness and accuracy.

5.1 Numerical results for problem (A)

We start by analyzing the optimal controls derived explicitly in Richard (1975) and the corresponding optimal controls obtained from the MSP formulation. Figure 2 shows the development in the expected values of the savings account X_t^*, consumption c_t^*, sum insured I_t^*, the proportion in both risky assets, π_t^*, and the proportion in the first risky asset π_t^{*1} during the entire lifetime of the individual. Both Fig. 2 and Table 1 show that the optimal decisions calculated by the MSP model closely follow the explicit solutions, nevertheless, because we compare a continuous time model with a combined continuous and discrete time model, we expect some small discrepancies. The explicit solutions are calculated based on the assumption that the labor income, consumption, sum insured, and capital gains occur simultaneously. In the MSP formulation, we first account for the cash-flows, and then make the investment decisions based on the value of the savings after the cash-flows. Therefore, the investment control obtained in the MSP formulation is compared with the average continuous time investment over the two adjacent time periods. The standard deviations calculated from testing different scenario trees (shown in parenthesis in Table 1) are negligibly small, which means that increasing the number of scenarios will not necessarily improve the precision of the results.

Richard (1975)'s model implies that, given our choice of parameters, a 45-year-old individual should invest $\pi_{t_0}^* = 180\ \%$ of her savings in risky assets, with the ratios of the particular assets given by $\pi_{t_0}^{*1} = 0.33\pi_{t_0}^*$ and $\pi_{t_0}^{*2} = 0.67\pi_{t_0}^*$. The optimal solution favours the asset with a higher return but also a higher volatility, i.e. $\pi_t^{*1} < \pi_t^{*2}, \forall_t$. These proportions decrease smoothly over time as the individual increases her savings. After retirement π_t^* remains constant, which is a classical result known in the literature. The optimal consumption rate increases at a very slow pace over the lifetime. The increase is caused by the choice of the impatience factor ρ and risk aversion $1-\gamma$ in relation to the choice of market parameters α_i, σ_i, and the mortality rates, as explained in more detail later in the analysis of Fig. 4.

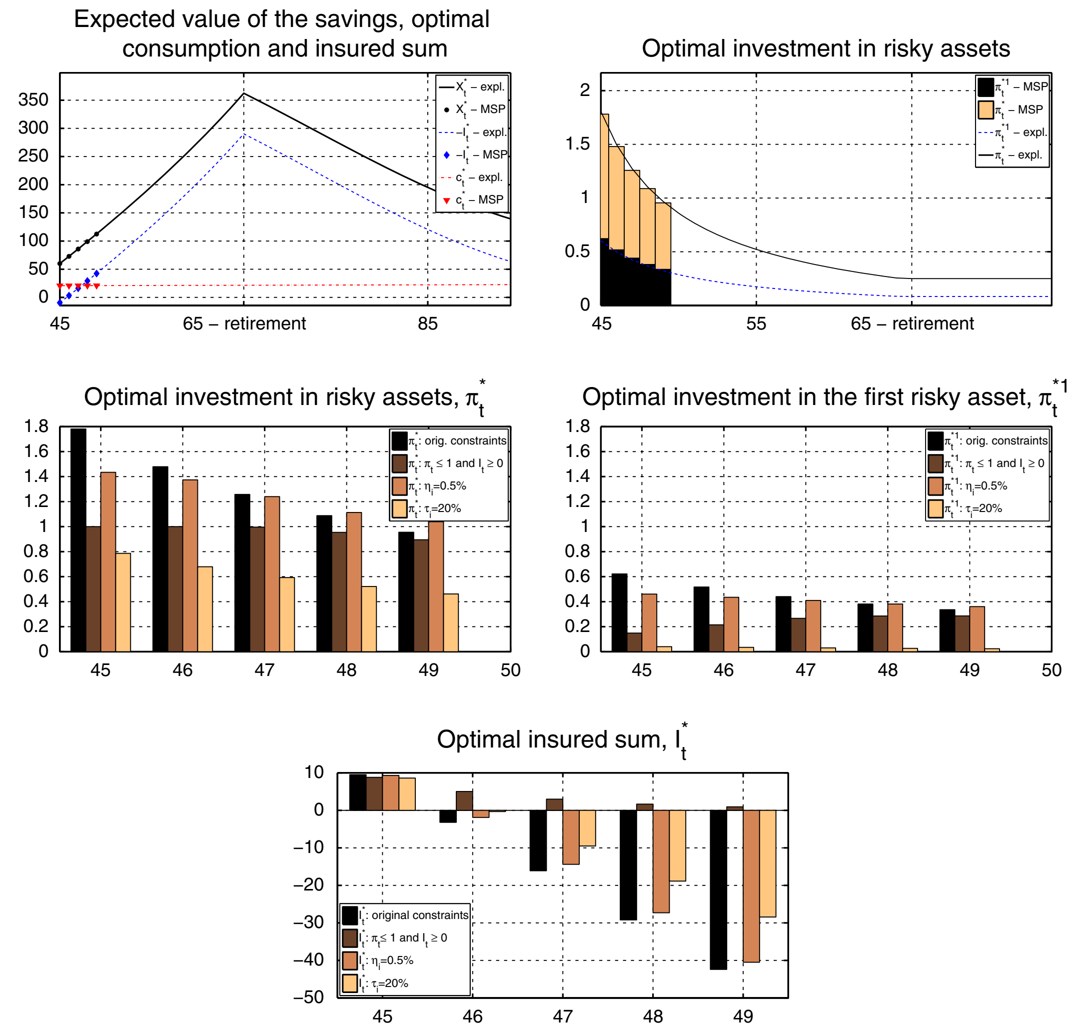

Fig. 2 Problem (A). Explicit solution and MSP solution for the expected value of the savings, consumption and sum insured (*left*, *top*, in 1,000 EUR), and investment in both risky assets and in the first risky asset (*right*, *top*, $\pi_t^* = \pi_t^{*1} + \pi_t^{*2}$). The MSP solution for the expected optimal investment in both risky assets (*left*, *middle*) and in the first risky asset (*right*, *middle*) under different constraints. The MSP solution for the expected optimal sum insured under different constraints (*bottom*, in 1,000 EUR).

Interestingly, the sum insured I_t^* is positive only during the first year of the contract. Afterwards I_t^* is negative, which is also consistent with Richard (1975)'s conclusions. Because the amount paid out to the inheritors upon the individual's death is equal to the value of the savings plus the sum insured, I_t^* can be negative as long as $X_t^* + I_t^* > 0$. The size and the sign of sum insured depend on parameter λ; by choosing $\lambda = 1$ the individual puts an equal weight on her consumption and the bequest motive, and the optimal bequest amount is equal to the optimal consumption while the person is alive, i.e. $X_t^* + I_t^* = c_t^*$, whereas choosing $\lambda = 5$ (as shown in Fig. 2; Table 1) corresponds to the optimal death sum approximately equal to three and a half years of optimal consumption, i.e. $X_t^* + I_t^* = \lambda^{-\gamma/(1-\gamma)} c_t^* \approx 3.5 c_t^*$.

The example takes as input realistic parameters, however, some of the optimal controls are problematic from a practical point of view. First, the proportion in risky assets is higher than 100 % for $t \leq t_3$. It may happen that the individual does not have an

Table 1 Problem (A). Explicit solution and MSP solution with various modifications for the expected value of the savings, consumption and sum insured (in 1,000 EUR), and investment in both risky assets and in the first risky asset ($\pi_t^* = \pi_t^{*1} + \pi_t^{*2}$)

Control	Model	t_0	t_1	t_2	t_3	$T_{MSP} - 1$
X_t^*	Expl.	60.0	72.9	86.0	99.2	112.6
	Orig. constraints	60.0	72.8	85.8	99.0	112.4
		(0.0)	(0.0)	(0.1)	(0.1)	(0.1)
	$\pi_t \leq 1$ and $I_t \geq 0$	60.0	71.0	82.6	94.9	107.7
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\eta_i = 0.5\%$	60.0	71.2	83.8	96.8	110.2
		(0.0)	(0.0)	(0.0)	(0.1)	(0.1)
	$\tau_i = 20\%$	60.0	68.6	77.5	86.6	95.9
		(0.0)	(0.2)	(0.3)	(0.5)	(0.7)
π_t^*	Expl.	1.80	1.50	1.27	1.10	0.97
	Orig. constraints	1.78	1.48	1.26	1.09	0.96
		(0.01)	(0.00)	(0.00)	(0.00)	(0.00)
	$\pi_t \leq 1$ and $I_t \geq 0$	1.00	1.00	1.00	0.95	0.89
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\eta_i = 0.5\%$	1.43	1.37	1.24	1.11	1.04
		(0.01)	(0.01)	(0.01)	(0.00)	(0.00)
	$\tau_i = 20\%$	0.80	0.69	0.60	0.53	0.47
		(0.08)	(0.07)	(0.06)	(0.05)	(0.05)
π_t^{*1}	Expl.	0.60	0.50	0.42	0.37	0.32
	Orig. constraints	0.62	0.52	0.44	0.38	0.34
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
	$\pi_t \leq 1$ and $I_t \geq 0$	0.15	0.21	0.27	0.29	0.29
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
	$\eta_i = 0.5\%$	0.46	0.44	0.41	0.38	0.36
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
	$\tau_i = 20\%$	0.04	0.04	0.03	0.03	0.03
		(0.05)	(0.05)	(0.04)	(0.04)	(0.03)
c_t^*	Expl.	20.8	20.8	20.9	20.9	20.9
	Orig. constraints	20.8	20.8	20.9	20.9	20.9
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\pi_t \leq 1$ and $I_t \geq 0$	20.6	20.6	20.6	20.7	20.7
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\eta_i = 0.5\%$	20.7	20.7	20.8	20.8	20.8
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\tau_i = 20\%$	20.5	20.4	20.3	20.2	20.1
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)

Table 1 continued

Control	Model	t_0	t_1	t_2	t_3	$T_{MSP} - 1$
I_t^*	Expl.	9.5	−3.2	−16.2	−29.3	−42.6
	Orig. constraints	9.5	−3.2	−16.1	−29.2	−42.4
		(0.0)	(0.0)	(0.1)	(0.1)	(0.1)
	$\pi_t \leq 1$ and $I_t \geq 0$	8.8	5.0	3.0	1.7	0.9
		(0.0)	(0.4)	(0.1)	(0.1)	(0.0)
	$\eta_i = 0.5\%$	9.3	−1.9	−14.4	−27.3	−40.5
		(0.0)	(0.0)	(0.0)	(0.1)	(0.1)
	$\tau_i = 20\%$	8.6	−0.4	−9.6	−18.9	−28.6
		(0.0)	(0.1)	(0.3)	(0.4)	(0.5)

The numbers are presented in terms of means and standard errors (in parenthesis)

opportunity to borrow money and to short the risk-free asset. Second, the sum insured is negative for $t > t_0$, which means that rather than purchasing a life insurance, the individual should sell a life insurance to the pension company. Specifically, she should give up the part I_t^* of her savings upon death in return for the extra premium $q_{y+t}^* I_t^*$. Theoretically, we could interpret this result as investing a part of the savings in a life annuity, however, receiving an annuity income as early as at age 45 sounds unrealistic.

The main reason for choosing a multi-stage stochastic programming approach is to include the constraints that ensure the realistic solution for a given problem. As we have shown that the closed-form solutions can be replicated by the MSP model well, we apply the presented optimization approach to analyze the optimal controls under the effects of various modifications during the first years of the contract.

Limits on portfolio composition and on the sum insured. Figure 2 (middle left, middle right, and bottom) and Table 1 show the optimal solutions that simultaneously account for two additional constraints: (i) the individual does not have a possibility to borrow money to invest in other assets (this constraint is defined in Eq. (16) for $u_i = 100\ \%$), and (ii) the individual is not allowed to sell the life insurance to the pension fund, $I_t > 0$, as defined in Eq. (17). As expected, these constraints affect the optimal controls. The optimal investment π_t^* is now equal to its upper limit 100 % and decreases faster than in the original model setup. Adding limits on portfolio composition also implies a change in the allocation between the risky assets. Before, the ratio of the first risky asset to the total risky investment was given by $\pi_t^{*1}/\pi_t^* \approx 0.35$. Under the new constraints, this ratio equals to 0.15 at t_0, and converges with time to the original optimal ratio, which further implies that π_t^{*1} increases until the original optimal ratio between the risky assets is reached. The optimal sum insured is positive and converges to its lower limit 0. Finally, the value of the savings, and as follows, the optimal consumption, are lower due to more conservative investment and positive costs for the sum insured $q_{y+t}^* I_t$.

Transaction costs. We also investigate the effect of transaction costs η_i. These have been defined as a constant percentage of a traded amount and are the same for purchases and sales and for all the assets, see Eq. (10A'). We have investigated the costs of $\eta_i = 0.5\ \%$ (shown on Fig. 2 middle left and middle right) and $\eta_i = 1\ \%$, and arrived to similar conclusions. After adding the transaction costs, the amount invested in risky assets decreases during the first few years, but after that period it stays above the original π_t^*. We conclude that introducing the transaction costs implies that the development of the risky investment over time is smoother, i.e. it decreases with time but the difference between the maximum and minimum value is smaller. A smoother investment strategy implies a lower traded amount, thus lower costs. Other optimal controls, i.e. consumption and sum insured, are not affected much by the transaction costs. Similarly to the total savings, the optimal consumption and the absolute value of the sum insured are slightly lower.

Taxes on capital gains. Finally, we investigate the impact of capital gains taxes on the optimal decisions. We have integrated a linear taxation τ_i on the positive returns on asset i, as defined in Eq. (18). In comparison with other constraints, taxes on capital gains are the most influential for the optimal decisions. Already after 5 years, a taxation of $\tau_i = 20\ \%$ causes a loss of 15 % of the expected savings relatively to the original problem. Less savings imply less money to consume and to purchase life insurance. Both the consumption and the absolute value of the sum insured decrease, but the biggest impact is noticeable on the investment controls. The optimal percentage in stocks is only $\pi_{t_0}^* = 80\ \%$ compared to the original $\pi_{t_0}^* = 178\ \%$, while the proportion in the first risky asset is $\pi_{t_0}^{*1} = 4\ \%$. Subtracting taxes only from the positive returns makes the scenario tree asymmetric and favours the risk-free asset (the positive outcomes are lower, whereas the negative outcomes are as severe as they were before). A similar conclusion has been drawn by Geyer et al. (2009a). The MSP solution is also more volatile when the taxes are included than for other considered constraints; the investment controls vary between 3 and 8 % for different scenario trees.

5.2 Numerical results for problem (B)

Problem (B) is relevant for a pension product design in a defined contribution pension plan. While the individual can choose between different types of benefits (lump sum payment, temporary benefits, and life long benefits), she cannot control how the benefits will be distributed during retirement. This decision belongs to the pension fund. However, the pension fund can take into consideration the pension saver's risk preferences γ, the impatience factor ρ, the market parameters hidden under the variable φ, and both the regular and pricing mortality intensities μ_t and μ_t^*, and design the payout profile such that the individual's preferences are included.

Optimal annuity concept. The optimal pension benefits are of the form $c_t^* = \frac{w_t}{f_B(t)} X_t$, see Eq. (7B). Inserting the definitions of functions w_t and $f_B(t)$ leads to

$$c_t^* = \frac{X_t}{\bar{a}_{y+t}}, \quad t \in [T, \widetilde{T}), \tag{19}$$

where $\bar{a}_{y+t}$ is a traditional single-life annuity that continuously pays 1 EUR per year to an individual aged $y + t$, i.e.,

$$\bar{a}_{y+t} = \int_t^{\widetilde{T}} \mathrm{e}^{-\int_t^s \left(\bar{r}+\bar{\mu}_\tau\right)\mathrm{d}\tau} \mathrm{d}s,$$

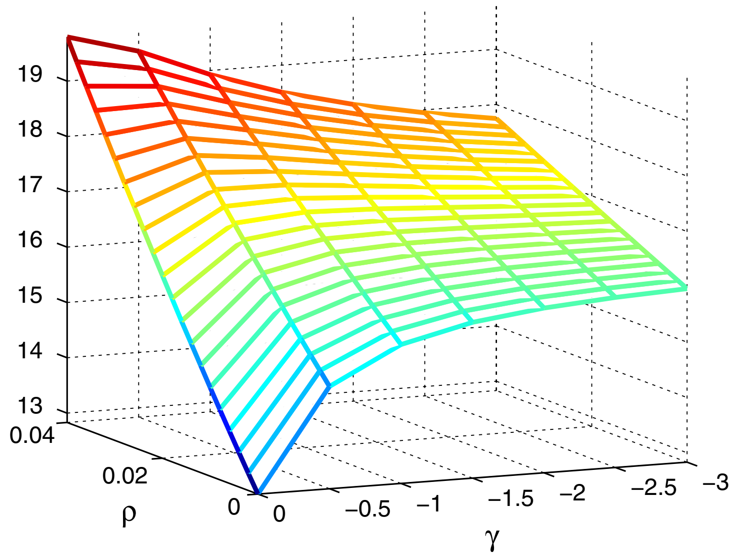

Fig. 3 Sensitivity of the optimal benefits at the beginning of retirement c_T^* to risk aversion $1-\gamma$ and impatience factor ρ (in 1,000 EUR). The individual's age upon retirement is 65, and the value of savings is $x_T = 265$. Other parameters are as specified in Sect. 5.

where

$$\bar{r} = \frac{1}{1-\gamma}\rho - \frac{\gamma}{1-\gamma}\varphi, \qquad \varphi = r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}, \qquad \bar{\mu}_\tau = \frac{1}{1-\gamma}\mu_\tau - \frac{\gamma}{1-\gamma}\mu_\tau^*.$$

Parameters $\bar{r}$ and $\bar{\mu_t}$ are the utility adjusted interest rate and mortality rate, respectively.

Figures 3 and 4 show the expected optimal benefits c_t^* for individuals with different risk aversion $1-\gamma$ and impatience factor ρ. Figure 3 shows the initial value of the benefit upon retirement, c_T^*, and its sensitivity to changes in γ and ρ. Given the value of the savings account $X_T = 265{,}000$ EUR, the size of the benefits upon T varies between $12{,}800$ EUR and $19{,}500$ EUR. The lowest initial benefit is for patient individuals who prefer more risky investment, i.e. those characterized by e.g. $\rho = 0.0$ and $\gamma = 0$, whereas the highest initial benefits will be received still by those who prefer more risky investment but are impatient, i.e. $\rho = 0.04$ and $\gamma = 0$. The tendency is similar for the risk averse retirees—the impatient ones will receive the highest annuities in the beginning of the retirement.

Figure 4 shows the optimal payout profile during retirement for different values of γ and ρ. Depending on these parameters, the annuity payments can be constant, decreasing or increasing. Specifically, it is the value of the impatience factor ρ relatively to the average return on investment adjusted for the risk aversion level $1-\gamma$ that controls the distribution of the payments. Impatient individuals (e.g. with $\rho = 0.08$) wish to receive the highest benefits in the first years of the retirement, whereas those more patient ($\rho = 0.02$) receive lower benefits in the beginning. More patient individuals have an opportunity to earn more in capital gains, leading to higher benefits during the individual's older years. Parameter γ affects mostly the size of the benefits; the more risky investment gives on average a higher return, thus higher benefits. Further-

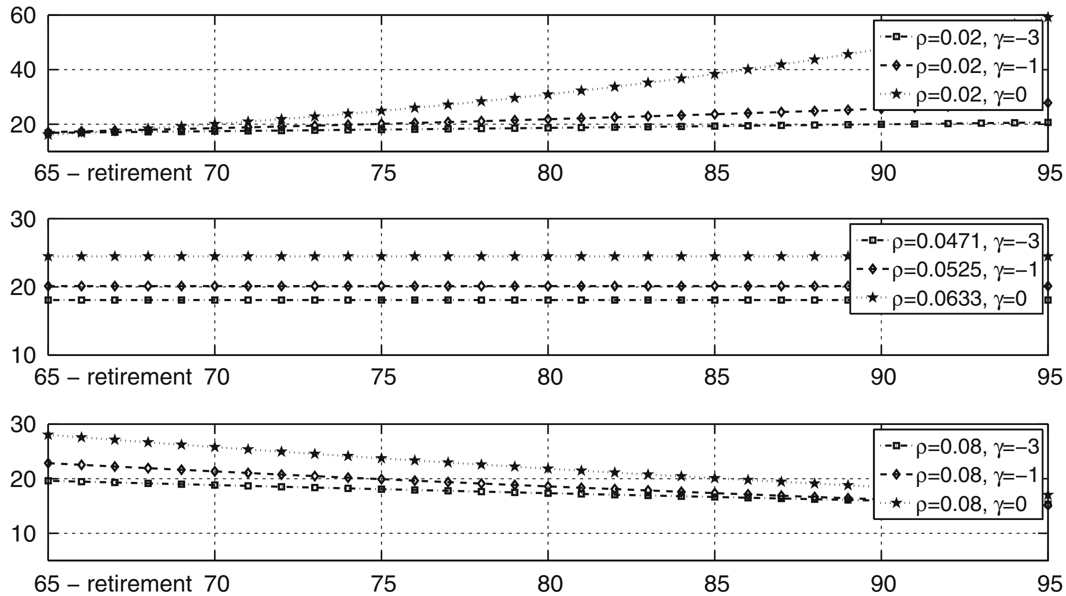

Fig. 4 Optimal benefits c_t^* for the first 30 years of retirement. Sensitivity of the benefits to various choices of the impatience factors and risk preferences (in 1,000 EUR). The individual's age upon retirement is 65, and the value of savings is $x_T = 265$. Other parameters are as specified in Sect. 5.

more, risk averse persons (e.g. $\gamma = -3$) will receive each year smoother benefits (the difference between c_{65}^* and c_{95}^* is smaller) than an individual with a low risk aversion, such as $\gamma = 0$.

Basic constraints. Figure 5 and Table 2 present the optimal solution for problem (B) before retirement (for a 45-year-old individual) and after retirement (for a 70-year-old individual). Figure 5 shows the development in the expected values of the savings account X_t^* and optimal benefits c_t^* (in 1,000 EUR) before retirement (top, left) and after retirement (middle, left). The figures on the right show the optimal proportion in the first risky asset π_t^{*1} before retirement (top, right) and after retirement (middle, right).

Given that upon age 45 the individual has $x_0 = 75{,}000$ EUR on her savings account and continues to pay the premiums of $l_t = 4{,}000$ EUR every year, which are then invested in a rather conservative way, i.e. $\gamma = -3$, she should expect that her savings upon retirement will reach 265,000 EUR, leading to the pension benefits at almost $c_t^* = 18{,}000$ EUR at age 65. The size and the development of the benefits is optimal given she puts more weight on the benefits just after retirement than on those in the future, and given she is risk averse, i.e. $\rho = 0.04$ and $\gamma = -3$. The optimal investment is more conservative than in problem (A) because it is proportional to the sum of the value of the savings and the present value of the pension contributions relatively to the value of savings, see Eq. (7B). Because the pension contributions l_t are much lower than the labor income in problem (A), π_t^* is much lower than in problem (A), starting with 45 % upon age 45 and decreasing afterwards. After retirement, the optimal proportion in risky assets remains constant at 25 %. The ratio of the investment in the first risky asset to the total risky assets remains unchanged, i.e. $\pi_t^{*1}/\pi_t^* = 33$ %, both before and after retirement.

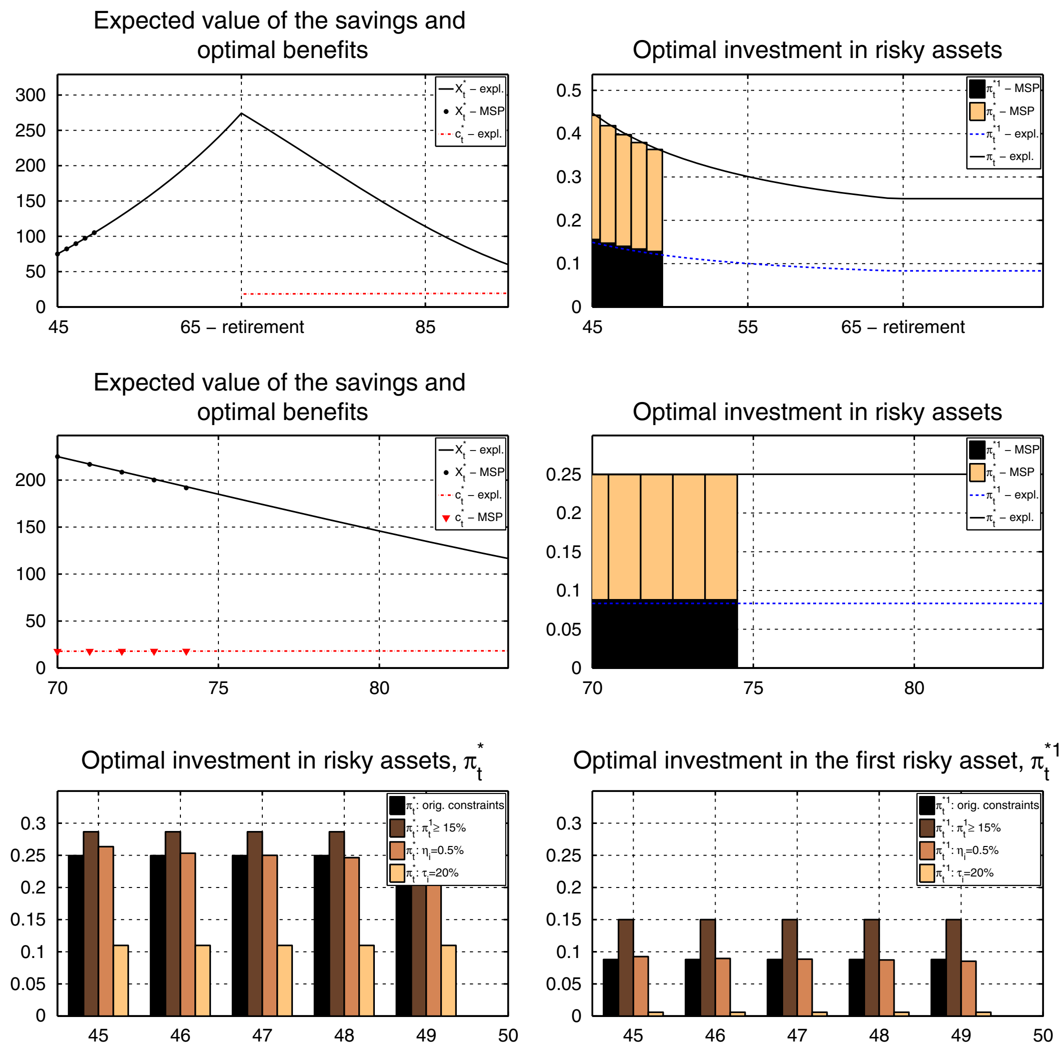

Fig. 5 Problem (B) before and after retirement. Explicit solution and MSP solution for the expected value of the savings and benefits before retirement (*left*, *top*). Explicit solution and MSP solution for the expected value of the savings and benefits after retirement (*left*, *middle*). Optimal investment in risky assets ($\pi_t^* = \pi_t^{*1} + \pi_t^{*2}$) before retirement (*right*, *top*) and after retirement (*right*, *middle*). The MSP solution for the expected optimal investment after retirement under different constraints in both risky assets (*bottom*, *left*) and in the first risky asset (*bottom*, *right*).

Similarly as in problem (A), we conclude that the MSP solution closely follows the optimal controls derived explicitly. The model provides a realistic solution that can be applied by a pension fund. Nevertheless, there is a number of constraints whose effect on the optimal control is worth investigating. Incorporating the pension saver's preferences such as the limits on the portfolio composition, or other practical constraints such as transaction costs and taxes on capital gains using stochastic optimal control is not trivial, therefore, to investigate the desired factors and to add flexibility to the model, we choose the mixed approach.

Limits on portfolio composition. Assume a 70-year-old pension saver who simply is not satisfied with the way the pension fund invests her savings. In particular, she believes that the first risky asset is more valuable and wishes to increase the allocation

Table 2 Problem (B) before and after retirement. Explicit solution and MSP solution with various modifications for the expected value of the savings and optimal benefits (in 1,000 EUR), and optimal investment in both risky assets and in the first risky asset ($\pi_t^* = \pi_t^{*1} + \pi_t^{*2}$)

Control	Model	t_0	t_1	t_2	t_3	$T_{\text{MSP}} - 1$
before retirement						
X_t^*	Expl.	75.0	82.2	89.5	97.1	105.0
	Orig. constraints	75.0	82.2	89.6	97.3	105.1
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
π_t^*	Expl.	0.45	0.42	0.40	0.38	0.37
	Orig. constraints	0.44	0.42	0.40	0.38	0.36
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
π_t^{*1}	Expl.	0.15	0.14	0.13	0.13	0.12
	Orig. constraints	0.16	0.15	0.14	0.13	0.13
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
after retirement						
X_t^*	Expl.	225.0	217.0	209.0	201.0	193.0
	Orig. constraints	225.0	216.7	208.4	200.1	191.8
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\pi_t^1 \geq 15$ %	225.0	216.9	208.7	200.5	192.3
		(0.0)	(0.0)	(0.0)	(0.0)	(0.1)
	$\eta_i = 0.5$ %	225.0	215.9	207.6	199.3	190.9
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\tau_i = 20$ %	225.0	214.5	204.0	193.7	183.5
		(0.0)	(0.1)	(0.1)	(0.2)	(0.3)
π_t^*	Expl.	0.25	0.25	0.25	0.25	0.25
	Orig. constraints	0.25	0.25	0.25	0.25	0.25
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\pi_t^1 \geq 15$ %	0.29	0.29	0.29	0.29	0.29
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\eta_i = 0.5$ %	0.26	0.25	0.25	0.25	0.24
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\tau_i = 20$ %	0.11	0.11	0.11	0.11	0.11
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)
π_t^{*1}	Expl.	0.08	0.08	0.08	0.08	0.08
	Orig. constraints	0.09	0.09	0.09	0.09	0.09
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\pi_t^1 \geq 15$ %	0.15	0.15	0.15	0.15	0.15
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\eta_i = 0.5$ %	0.09	0.09	0.09	0.09	0.09
		(0.00)	(0.00)	(0.00)	(0.00)	(0.00)
	$\tau_i = 20$ %	0.01	0.01	0.01	0.01	0.01
		(0.01)	(0.01)	(0.01)	(0.01)	(0.01)

Table 2 continued

Control	Model	t_0	t_1	t_2	t_3	$T_{MSP}-1$
c_t^*	Expl.	17.8	17.9	17.9	17.9	18.0
	Orig. constraints	17.8	17.8	17.8	17.9	17.9
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\pi_t^1 \geq 15\ \%$	17.8	17.8	17.9	17.9	18.0
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\eta_i = 0.5\ \%$	17.7	17.7	17.7	17.8	17.8
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)
	$\tau_i = 20\ \%$	17.3	17.3	17.2	17.1	17.0
		(0.0)	(0.0)	(0.0)	(0.0)	(0.0)

The numbers are presented in terms of means and standard errors (in parenthesis)

in this asset to at least 15 %. Such a constraint can easily be incorporated in the MSP model by adding Eq. (16), where $d_1 = 15\ \%$. The optimal solution under such a constraint is shown in Figure 5 (middle left and right) and Table 2. Both the total investment in risky assets and the investment in each of the risky assets remain constant as before, but the values are now higher. The pension saver's preferences are taken into account and π_t^{*1} increases from 9 to 15 %, whereas the investment in both risky assets increases from 25 to 29 %. The increase in the allocation in the second risky asset π_t^{*2} can be explained by the existence of an optimal ratio between the risky assets that the model attempts to preserve.

The MSP formulation allows for adding various limits on the portfolio. The limits can be both upper and lower bounds, and separately for each asset class. We have investigated various combinations of limits on particular assets, and without presenting the numbers, we came to the following conclusions. None of these constraints have much effect on the overall value of the savings or on the optimal benefits, but adding a limit even only to one asset class, directly affects the distribution between all three asset classes. Imposing a higher percentage on an asset class than the percentage given by the optimal control in the case without constraints, results in setting the allocation in this asset exactly to this limit. And, vice versa, limiting the amount of savings allocated to a given asset class when the original optimal investment control suggests a higher amount, results in a new optimal solution that is equal to the upper limit.

Transaction costs. When analyzing the effect of transaction costs in problem (A), we concluded that adding transaction costs defined as a percentage of a traded amount implies a smoother development in the proportion invested in risky assets. The results for problem (B) after retirement show that the development of π_t^* is no longer constant, but is slowly decreasing. The reason is that while the value of the savings decreases, trying to rebalance the portfolio each year to a constant proportion is more expensive than rebalancing to the decreasing proportion. As before, the changes in the optimal benefits are so small that it is hard to conclude how the transaction costs affect the benefits, except for the obvious fact, that the savings are slightly lower, which implies lower benefits.

Taxes on capital gains. Finally, to strengthen the conclusions that we drew from investigating the effect of adding taxes on the positive returns in problem (A), we repeat this investigation for problem (B) after retirement. The results are similar: the impact of taxes on the optimal investment is significant. Without taxes the optimal investment in both risky assets was given by $\pi_t^* = 25\ \%$ and in the first risky asset by $\pi_t^{*1} = 9\ \%$. With the presence of taxes, the optimal controls decreased to $\pi_t^* = 11\ \%$ and $\pi_t^{*1} = 1\ \%$. We explain this result by the asymmetry in the scenario tree caused by subtracting the taxes only from the positive capital gains. It is interesting, though, that taxes affect the level of the investment controls but not their development over time, i.e. the proportions remain constant as in the case of the original problem. Furthermore, the MSP solution across different scenario trees is not as volatile as for problem (A). The standard deviations (shown in parenthesis in Table 2) are low, which indicates a stability for different scenario trees. In terms of other controls, similarly as in problem (A), the presence of taxes significantly affects the value of the savings, implying lower and decreasing pension benefits.

6 Conclusions and future work

We have presented a model combining two optimization technologies: multi-period stochastic programming and stochastic optimal control, to obtain optimal decisions related to various problems within personal finance and retirement. The optimization methods complement each other well. MSP allows for building flexible and practical models that normally are difficult to solve with the stochastic optimal control approach, whereas the explicit formulas obtained by the stochastic optimal control approach cover any number of periods and are useful for the validation of the results obtained by the MSP. The considered problems have been solved using a nonlinear solver.

The numerical results show that among the considered limits on portfolio composition, limits on the sum insured, transaction costs and taxes on capital gains, the presence of taxes affects the optimal controls the most, and the changes in both the value of savings and the controlled processes are significant.

Except for the contribution regarding computational methods, the paper introduces an important aspect regarding designing pension benefits in a defined contribution pension plan. Problem (B) suggests how the pension fund could decide on the initial size of the benefits and their development during the decumulation phase so that the individual's risk and impatience preferences are captured. Interestingly, the pension saver's preferences have a higher impact on the size and development of the benefits than limits on portfolio composition, transaction costs or taxes on capital gains.

The model can be improved in various ways. The most interesting would be the extension of the MSP part to even more periods while still keeping the running time of the program within a couple of minutes. Another suggestion is to introduce the bond market instead of the risk-free asset and/or the uncertainty on the labor income/premiums. Finally, it could be of interest to consider other utility functions such as a loss aversion framework, where an individual can control both the gains and losses in savings relative to a pre-defined target, or a multi-objective function, which takes into account other preferences than risk aversion and impatience.

Application of the mutual fund theorem

The mutual fund theorem was originally defined in Merton (1971) for an individual with a known lifetime who wishes to maximize the expected utility of consumption. Richard (1975) extended this result and proved that the theorem holds also for an investor with the uncertain time of death optimizing the utility of consumption and bequest.

Theorem 1 (Merton 1971, p. 384). *Given n assets with prices S_i whose changes are log-normally distributed, then (1) there exists a unique pair of "mutual funds" constructed from linear combinations of these assets such that, independent of preferences (i.e., the form of utility function), wealth distribution, [probability distribution of lifetime or life insurance opportunities,* Richard (1975), p. 194*], or time horizon, individuals will be indifferent between choosing from a linear combination of these two funds or a linear combination of the original n assets.*

Corollary 1 (Merton 1971, p. 386). *If one of the assets is "risk-free" (say the Nth), then one mutual fund can be chosen to contain only the risk-free security and the other to contain only the risky assets in the proportions:*

$$\forall_{i=1,\dots,N-1} \quad \theta_i = \frac{\sum_{j=1}^{N-1}[\sigma_{ij}]^{-1}(\alpha_j - r)}{\sum_{k=1}^{N-1}\sum_{j=1}^{N-1}[\sigma_{kj}]^{-1}(\alpha_j - r)},$$

where $\{\alpha_i, \sigma_{ij}\}$ define the physical distribution of the returns, $\sigma_{ij} = \sigma_i\sigma_j\phi_{ij}$, and ϕ_{ij} is the correlation coefficient between assets i and j.

Thus, without loss of generality, we can work with only two assets: a risk-free asset and a mutual fund of risky assets with a log-normally distributed price S_t

$$\mathrm{d}S_t = \alpha S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t,$$

where

$$\alpha = \sum_{i=1}^{N-1}\theta_i\alpha_i, \qquad \sigma^2 = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\theta_i\theta_j\sigma_{ij}, \qquad \mathrm{d}W = \sum_{i=1}^{N-1}\theta_i\frac{\sigma_i}{\sigma}\mathrm{d}W_i.$$

In our study, we assume one risk-free asset and a mutual fund consisting of $N-1$ risky assets. The optimal proportion between the risk-free asset and the mutual fund during the period $[T_{\mathrm{MSP}}, \widetilde{T})$ is calculated explicitly using stochastic optimal control (Hamilton-Jacobi-Bellman techniques). Thereafter, we apply the mutual fund theorem to find the optimal allocation between the risky assets in the mutual fund.

Derivation of the optimal controls for problem (B)

We derive the optimal value function and the optimal controls for the problem of maximizing the utility of consumption, where the control processes are specified over different time periods: investment is controlled both before and after retirement, $\pi_t \in \mathcal{Q}[t_0, \widetilde{T})$, and consumption is controlled only after retirement, $c_t \in \mathcal{Q}[\max(t_0, T), \widetilde{T})$. The problem is solved first for the period *after retirement*, where for the optimal solution we refer to Bruhn and Steffensen (2011), and then for the period *before retirement*. In the latter, we insert the optimal value function obtained in *after retirement* period as the boundary condition at time T.

The problem of optimizing expected utility of consumption *after retirement* in the case where the person controls investment π_t^*, consumption c_t^*, and life insurance I_t^*, and has no bequest motive, has been solved in Bruhn and Steffensen (2011). The authors derive the optimal value function

$$V(t,x) = \frac{1}{\gamma} f_t^{1-\gamma} x^{\gamma}, \tag{20}$$

and the optimal controls

$$\pi_t^* X_t = \frac{\alpha - r}{\sigma^2(1-\gamma)} X_t, \qquad c_t^* = \frac{w_t}{f_t} X_t, \qquad I_t^* = 0,$$

where

$$f_t = \int_t^{\widetilde{T}} \mathrm{e}^{-\frac{1}{1-\gamma}\int_t^s \left(\mu_\tau - \gamma(\mu_\tau^* + \varphi)\right)d\tau} w_s ds, \qquad \varphi = r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)}. \tag{21}$$

To solve the problem *before retirement*, we apply Hamilton–Jacobi–Bellman techniques. We first define the HJB equation based on the savings dynamics defined in Eq. (2B) for the considered period,

$$\begin{aligned}
&\frac{\partial V(t,x)}{\partial t} - \mu_t V + \sup_{\pi} \left\{ l_t \frac{\partial V(t,x)}{\partial x} + \left(r + \pi_t(\alpha - r) + \mu_t^*\right) x \frac{\partial V(t,x)}{\partial x} \right.\\
&\left. + \frac{1}{2}\pi_t^2 \sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} \right\} = 0,\\
&\qquad V(T,x) = \frac{1}{\gamma} f_T^{1-\gamma} X_T^{\gamma},
\end{aligned}$$

where the boundary condition is equal to the optimal value function at time T given by Eq. (20). Then, we guess the solution and verify its correctness,

$$V(t,x) = \frac{1}{\gamma} f_t^{1-\gamma} \left(x + g_t\right)^{\gamma}.$$

We insert the derivatives to the HJB equation,

$$
\begin{aligned}
&\frac{1-\gamma}{\gamma} f_t^{-\gamma} \frac{\partial f_t}{\partial t} \big(x+g_t\big)^{\gamma} + f_t^{1-\gamma} \big(x+g_t\big)^{\gamma-1} \frac{\partial g_t}{\partial t} \\
&= \mu_t \frac{1}{\gamma} f_t^{1-\gamma} (x+g_t)^{\gamma} - l_t f_t^{1-\gamma} \big(x+g_t\big)^{\gamma-1} \\
&\quad - (r+\mu_t^*)(x+g_t) f_t^{1-\gamma} (x+g_t)^{\gamma-1} + (r+\mu_t^*) g_t f_t^{1-\gamma} (x+g_t)^{\gamma-1} \\
&\quad - \frac{1}{1-\gamma} \frac{(\alpha-r)^2}{2\sigma^2} f_t^{1-\gamma} (x+g_t)^{\gamma},
\end{aligned}
$$

and find the functions f_t and g_t defined for $t \in [t_0, T)$:

$$
\begin{aligned}
g_t &= \int_t^T \mathrm{e}^{-\int_t^s (r+\mu_\tau^*) d\tau} l_s ds, \\
f_t &= \mathrm{e}^{-\frac{1}{1-\gamma} \int_t^T (\mu_\tau - \gamma(\mu_\tau^* + \varphi)) d\tau} f_T \\
&= \mathrm{e}^{-\frac{1}{1-\gamma} \int_t^T (\mu_\tau - \gamma(\mu_\tau^* + \varphi)) d\tau} \int_T^{\widetilde{T}} \mathrm{e}^{-\frac{1}{1-\gamma} \int_T^s \left(\mu_\tau - \gamma(\mu_\tau^* + \varphi)\right) d\tau} w_s \, ds,
\end{aligned}
$$

where in the last equality we inserted function f_T specified in Eq. (21).

The optimal investment is then given by

$$
\frac{\partial}{\partial \pi}: \qquad \pi_t^* = -\frac{\alpha-r}{\sigma^2 x} \frac{\frac{\partial V(t,x)}{\partial x}}{\frac{\partial^2 V(t,x)}{\partial x^2}} = \frac{1}{1-\gamma} \frac{\alpha-r}{\sigma^2} \frac{x+g_t}{x}.
$$

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Chapter 3

Optimal savings management for individuals with defined contribution pension plans

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Optimal savings management for individuals with defined contribution pension plans

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Abstract

The paper provides some guidelines to individuals with defined contribution (DC) pension plans on how to manage pension savings both before and after retirement. We argue that decisions regarding investment, annuity payments, and the size of death sum should not only depend on the individual's age (or time left to retirement), nor should they solely depend on the risk preferences, but should also capture: 1) economical characteristics—such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g. retirement state pension), and 2) personal characteristics—such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition. Specifically, the decisions are optimal under the expected CRRA utility function and are subject to the constraints characterizing the individual.

The problem is solved via a model that combines two optimization approaches: stochastic optimal control and multi-stage stochastic programming. The first method is common in financial and actuarial literature, but produces theoretical results. However, the latter, which is characteristic for operations research, has practical applications. We present the operations research methods which have potential to stimulate new thinking and add to actuarial practice.

Keywords: Finance • Utility theory • Stochastic programming • Stochastic optimal control

3.1 Introduction

Recent years have seen a decided worldwide shift from defined benefits (DB) pension plans toward defined contribution (DC). The number of participants in DC plans is quickly expanding because these plans are not only easier and cheaper to administer, but also more transparent and more flexible. Furthermore they can better capture the individual's needs. However, DC plans also pose some challenges, namely, the participants often do not know how to manage their saving and investment decisions.

In some countries, such as the U.S., most DC decisions are made by the individual with little advice from the employer. In contrast, in countries such as Denmark, the sponsoring organizations, including life insurers, suggest a dynamic investment strategy suitable to the individual's age and risk preferences. Individuals in most of the countries also have to decide on how to spend the amount accumulated on their pension savings account. Should they follow a certain withdrawal rate rule, or should they purchase annuities that will provide with regular payments during retirement? This task is not easy, especially when life insurers offer a wide variety of annuity products (e.g. fixed or variable, deferred or immediate, term or whole-life). How can the individuals know, which product is best for them?

There is one more decision they have to keep in mind. Namely, what to do with the savings in case of their death? Do they want to bequeath the savings to their heirs, or maybe purchase an annuity product combined with a life insurance policy? What level of death sum should they choose?

We argue that aforementioned decisions should differ for each individual and should account for the following factors: 1) economical characteristics—such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g. retirement state pension), and 2) personal characteristics—such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition.

To help the individuals manage the savings and investment decisions we build an optimization-based financial planning model. Because such a model can be complicated and difficult to solve, we propose to combine two popular methodologies: multi-period stochastic programming (MSP) and stochastic optimal control (SOC), also referred to continuous-time and state dependent dynamic programming. The latter method is common in financial and actuarial literature, and its main advantage is the analytical form of the optimal solution, which is easy to understand and implement. See, for example, Yaari (1965), Samuelson (1969), Merton (1969, 1971), Richard (1975) and Campbell and Viceira (2002), for optimal decisions regarding investment, consumption and sum insured. However, the main drawback of this approach is that the explicit solution in many cases does not exist.

On the contrary, MSP, which is characteristic for operations research, has practical application and complements the SOC approach, especially in terms of adding realistic constraints and modeling more complicated processes. In a stochastic programming approach we model the possible outcomes for the uncertainties in a scenario tree, and numerically compute the optimal solution at each node of the tree. See, for example, Carino et al. (1998) and Carino and Ziemba (1998), who formulate a financial planning model for one of the biggest Japanese property and casualty insurer, Mulvey et al. (2003), who present a multi-period stochastic network model for integrating corporate financial and pension planning, and Mulvey et al. (2008) who expand this work by adding the borrowing decisions. The applications of MSP to individual asset-liability management can be found, for example, in Ziemba and Mulvey (1998), Kim et al. (2012) and Konicz and Mulvey (2013). However, the main drawback of this optimization method is the limited ability to handle many periods under enough uncertainty about future asset returns and human lifetime. Especially, modeling the entire lifetime of an individual is challenging in terms of computational tractability.

To benefit from both optimization approaches and to avoid the aforementioned drawbacks, we combine multi-stage stochastic programming and stochastic optimal control into one mathematical framework. We solve the problem using MSP approach up to some horizon T, and to ensure that the model accounts for the entire lifetime of an individual, we insert the *end effect* in the objective function of MSP. The end effect is determined by the optimal value function calculated explicitly via SOC technique. This function covers the period from the horizon T to the individual's death. Combining these two optimization approaches is new and has only been investigated in Geyer et al. (2009) and Konicz et al. (2014). The presented MSP framework can be posed and solved with reasonable efficiency, while providing reliable and robust insights. These policy rules can further be implemented using Monte Carlo simulations, which are simpler and more likely to be employed in practice than complicated stochastic models, see, e.g., Mulvey et al. (2008).

The paper is organized as follows. Section 3.2 describes the economical and personal characteristics that we take into account when advising on how to manage the pension savings. Section 3.3 presents the financial planning model. Section 3.4 explains the intuition behind the optimal solution obtained from MSP model. Section 3.5 includes numerical examples illustrating the application of the model for different individuals. Section 3.6 concludes. Finally, Appendix 3.A introduces multi-stage stochastic programs and Appendix 3.B presents details of the explicit solution derived via SOC approach.

3.2 Economical and personal characteristics

We argue that management of savings in DC pension plan should account for economical and personal characteristics, and it should be tailored to a customer. Our model takes into account the following factors.

3.2.1 Economical characteristics

Current value on the savings account The value of the individual's account, X_t, develops according to the initial savings x_0, contributed premiums, capital gains including dividends, insurance coverage, accredited survival credit and the benefits paid after retirement—all these elements are described below.

Premiums Until retirement the individual contributes to the savings account. The premiums P_t^{tot} consist of a fixed percentage p^{fixed} of the labor income l_t, which is in many countries mandatory and decided by the employer, and the additional voluntary contributions, $p^{vol}\, l_t$. The latter may be of interest of an individual who wishes to increase the future benefits.

$$P_t^{tot} \quad = \quad (p^{fixed} + p^{vol}) l_t, \qquad p^{fixed} \in [0,1],\ p^{vol} \in [0, 1 - p^{fixed}].$$

The labor income l_t is deterministic and increases with a salary growth rate y_l, $l_t = l_0 e^{y_l t}$, where l_0 is the level of the labor income at the current time t_0. Both the premiums and the labor income are positive only until retirement, $t < T_R$; otherwise 0.

State retirement pension After retirement the individual has no other income than state retirement pension, b_t^{state}. This income is typically financed on a pay-as-you-go basis from general tax revenues, and ensures a basic standard of living for old age. It often depends on the level of the individual's income before retirement, but not on the income from the DC plan. We assume that the state retirement pension consists of the life long, yearly adjusted payments.

3.2.2 Personal preferences

Risk aversion The individual is risk averse and obtains a utility u from the total pension benefits B_t^{tot} and from leaving money upon death to the heirs, Beq_t. The utility function is characterized by a constant relative risk aversion (CRRA) $1-\gamma$ and time dependent weights w_t:

$$u(t, B_t^{tot}) = \frac{1}{\gamma} w_t^{1-\gamma} \big(B_t^{tot}\big)^{\gamma}, \qquad u(t, Beq_t) = \frac{1}{\gamma} w_t^{1-\gamma} \big(Beq_t\big)^{\gamma},$$

where $\gamma \in (-\infty, 1) \setminus \{0\}$, whereas $\gamma = 0$ implies the logarithmic utility. Time dependent weights w_t include an impatience weighted interest factor ρ,

$$w_t = e^{-\rho\, t/(1-\gamma)},$$

which allows the individual to specify how important the benefits and the death sum are at the present moment relatively to how important these payments would be in the future. Thus, $\rho = 0$ implies that the current and future payments are equally important for the individual, and $\rho > 0$ reflects that the weight on the future payments decreases exponentially with time.

Lifetime expectancy The individual has an uncertain lifetime, which we model with two kinds of mortality rates: μ_t and ν_t. The first function denotes the subjective mortality rate and reflects the individual's expectation about her mortality rate. The lifetime expectancy is either based on the individual's lifestyle and health status or simply on the individual's opinion. For example, does she live a healthy lifestyle and therefore expect to live longer than others? Is she a regular smoker or maybe seriously ill? Does she expect to live longer than an average individual despite a smoking habit? The choice of the subjective mortality rate μ_t affects the decisions regarding the payout profile as well as the decision about purchasing life insurance.

The second function, ν_t, also referred to pricing mortality, is used by life insurers for calculating the price of their life contingent products. Especially in European countries, due to legislation, both the survival credit and the price for life insurance are calculated under unisex criteria, and the individual is not even subject to health screening, see Rocha et al. (2010). A person with a cancer disease, heart attack, a regular smoker or an overweight person has the right to the same benefits as a healthy individual. Pricing mortality rates are typically reported in the actuarial life tables.

Payout profile Upon retirement, the individual starts receiving retirement income consisting of two types of benefits: the state retirement pension b_t^{state}, and the benefits from the DC plan B_t:

$$B_t^{tot} = b_t^{state} + B_t, \quad t \geq T_R.$$

The benefits from the DC plan, also known as the labor market pension (occupational pension, 2^{nd} pillar) and the individual retirement accounts (private pension, 3^{rd} pillar), are one of the most important decision variables in the optimization problem. Individuals can choose among the following possibilities:

- *Duration of the payments.* Is the individual interested in receiving a lump sum benefit upon retirement, regular payments over a period of 10 or 25 years (term annuity), or regular payments as long as she is alive (life annuity)? We can control the duration in the model by choosing the appropriate value of $\widetilde{T}$, which denotes the time of receiving the last benefit.

- *Payout curve.* If interested in annuities, does the person prefer to receive constant, increasing or maybe decreasing payments? By inserting the right parameters in the utility function: the impatience weighted interest factor ρ and risk aversion $1 - \gamma$, we can control the payout curve. Another important factor to consider is the lifetime expectancy, which depends on the subjective mortality rate μ_t. A person with health problems might want to spend more of her savings during the first years of retirement, whereas a person who expects to live long would want to make sure she would never outlive her resources.

- *Size of the payments.* To increase the size of the benefits, the individual can either increase the premiums or choose a more aggressive investment strategy, for example, by choosing equity-linked (variable) annuities, as defined in, e.g., Blake et al. (2003). These products differ from traditional fixed rate annuities that offer a constant level of payments during retirement in a way that their size is regularly adjusted to account for capital gains and losses. This adjustment is necessary to avoid the danger of running out of the resources before death. However, equity-linked annuities are directly linked to market returns, therefore are risky.

Bequest motive Not surprisingly, the marital status and dependants play a crucial role in the choice of the pension product. A single individual would be interested in a life annuity. In this type of contract an individual agrees to give up the savings upon death, which are then inherited by the life insurer. Thus, in contrast to other products, life annuities provide an additional return arising from mortality risk sharing. This return is often called a survival credit and is proportional to the value of the individual's savings, i.e. $\nu_t X_t$.

The individual with dependants would bequeath some death sum Beq_t to her heirs. We assume that upon death, life insurer inherits the value of the savings, but pays out Beq_t to the dependants. The size of death benefit can be proportional to the value of the account through a factor ins_t,

$$Beq_t \quad = \quad ins_t X_t,$$

or can be a decision variable in the program. In the latter case, we introduce an additional parameter k, which defines the strength of the bequest motive relatively to the received benefits. The actuarially fair price for the insurance coverage is equal to $\nu_t Beq_t$. Because choosing larger death benefit leads to lower annuity payments, and vice versa, it is not easy to find the right balance between the level of annuity payments and the level of death sum.

Portfolio composition Retirement savings can be allocated to a number of financial assets. In the U.S. the individuals have lots of flexibility and can invest in leveraged financial products. In countries such as Denmark, the individuals have a limited list of assets to choose from. For example, life insurers offering unit-linked products allow for investments within their own list of mutual funds and ETFs, which replicate stock indices for different regions and industries, corporate and government bonds with different maturities, commodities, etc.

Our model allows the individual to include her preferences regarding asset allocation. For simplicity, we consider portfolios composed of positions in three asset classes: cash (corresponding to a 3-month short rate), an aggregate bond index including both government and corporate bonds with different durations, and a stock index.

3.3 Multi-stage stochastic program formulation

Stochastic programming is a general purpose framework for modeling optimization problems. We include a brief introduction to stochastic programming in Appendix 3.A, whereas more details can be found in the classical books on the subject, for example, Birge and Louveaux (1997), Zenios (2008), and Shapiro et al. (2009). Appendix 3.B includes a description of the mortality model necessary for generating the scenario trees.

3.3.1 Scenario generation

The range of possible outcomes for the uncertainties is modeled by a *scenario tree*, which consists of nodes $n \in \mathcal{N}_t$ uniquely assigned to stages $t = 1, \ldots, T$. Each node has a probability $prob_n$, so that $\forall_t \sum_{n \in \mathcal{N}_t} prob_n = 1$. At the first stage we have only one root node n_0, whereas the number of nodes at other stages corresponds to possible values of random vector ξ_t. Every node $n \in \mathcal{N}_t$, $t > t_0$, has a unique ancestor n^-, and every node $n \in \mathcal{N}_t$, $t < T$, has children nodes n^+. The nodes with no children are called the leaves. A scenario $\mathcal{S}^n$ is a set of all predecessors of a leaf $n : n^-, n^{--}, ..., n_0$, or equivalently, a single branch from the root to the leaf. The number of scenarios in the tree equals the number of leaves.

When applying stochastic programming one often speaks of the curse of dimensionality. The size of the tree grows exponentially with the number of periods, and the problem quickly becomes too large to be computationally tractable. Therefore, to solve the problems with a long-term horizon T, the scholars often choose longer intervals between the stages, see, e.g., Carino et al. (1998) and Dempster et al. (2003), gradually reduce the number of scenarios, see, e.g., Heitsch and Römisch

(2009a,b), or, as applied in this paper, approximate the problem upon horizon by solving a simplified model via stochastic optimal control approach, see, e.g., Geyer et al. (2009) and Konicz et al. (2014).

A considerable amount of literature focuses on scenario generation methods for stochastic programming. Among different approaches we can distinguish sampling, simulation, scenario reduction techniques and moment matching methods. For the purpose of our study we have chosen the technique that matches the statistical properties (the first four moments and the correlations) of the underlying processes. This approach has been introduced by Høyland and Wallace (2001) and Høyland et al. (2003), who suggest solving a nonlinear optimization problem that minimizes the distance between the properties of the generated tree and of the underlying process. Both the asset returns and the probabilities of each node are the decision variables in this formulation.

During recent years several authors have been investigating possible improvements of the moment matching approach. Ji et al. (2005) show that if one can predetermine the outcomes of the asset returns (e.g. by simulation) and choose the probabilities of the nodes to be the only variables in the model, then it is possible to match the statistical properties of the underlying process with a linear optimization problem. This method is further improved by Xu et al. (2012) who combine the simulation, the K-means clustering approach, and the linear moment matching, and by Chen and Xu (2013), who remove the simulation component and applies the K-means clustering method directly onto the historical dataset.

For the purpose of our study, we have to generate a scenario tree that includes the uncertainties of both the assets returns and the individual's lifetime. We are not familiar with any literature that explains how to combine two independent sources of uncertainty into one scenario tree, nor how to include the events of death in a scenario tree, therefore, we developed a new algorithm to generate the scenario trees. We have already applied this procedure in Konicz and Mulvey (2013); however, we did not explain it in detail.

1. Generate a tree based on a chosen structure (number of periods and branching factors).

2. Randomly choose the nodes at which death occurs. Denote them by $n \in \mathcal{N}_t^{dead}$ and remove all their children nodes. All other nodes belong to the set $\mathcal{N}_t^{alive}$ as shown on Fig. 3.1.

3. Assign the *transition probabilities* $\widehat{prob}_n$ to nodes $n \in \mathcal{N}_t^{dead}$. These probabilities define the chances of a person dying between the two stages and express the individual's lifetime expectancy. We follow the standard actuarial notation and denote the probability that a $y+t$-year old individual survives the next period by $\tilde{p}_{y+t}$ and the probability that a $y+t$-year old individual dies between stages t and $t+1$ by $\tilde{q}_{y+t}$ so that $\tilde{p}_{y+t} + \tilde{q}_{y+t} = 1$. (For the mortality rate model see Eq. (3.B.1) in Appendix 3.B.) Thus, for each $n \in \mathcal{N}_t$, the children nodes at which death occurs, $n^+ \in \mathcal{N}_{t+1}^{dead}$, must satisfy $\sum_{n^+ \in \mathcal{N}_{t+1}^{dead}} \widehat{prob}_{n^+} = \tilde{q}_{y+t}$.

4. Having fixed the transition probabilities $\widehat{prob}_n$ of nodes $n \in \mathcal{N}_t^{dead}$, $\forall t$, we can choose any of the aforementioned moment matching algorithms and add the asset returns. Let the transition probabilities of nodes $n \in \mathcal{N}_t^{alive}$, $\forall t$, be random variables in the optimization program. Then,

either let the asset returns be random variables and apply Høyland and Wallace (2001), or predetermine the asset returns and apply the algorithms presented in Xu et al. (2012) or Chen and Xu (2013). (The parameters for the asset returns are shown in Table 3.B.1 in Appendix 3.B.)

5. After specifying the transition probabilities $\widehat{prob}_n$ and the asset returns at each node, we can calculate the probability of each scenario in the tree as a product of the transition probabilities of all the nodes in a given scenario, see Fig. 3.1. Specifically, assuming that the individual is y-years old at the initial stage t_0, the probability of the individual surviving until stage $t+1$ is equal to $\tilde{p}_y \cdot \ldots \cdot \tilde{p}_{y+t-1} \cdot \tilde{p}_{y+t} = {}_t\tilde{p}_y \cdot \tilde{p}_{y+t} = {}_{t+1}\tilde{p}_y$, and the probability of the individual dying between stages t and $t+1$ is equal to $\tilde{p}_y \cdot \ldots \cdot \tilde{p}_{y+t-1} \cdot \tilde{q}_{y+t} = {}_t\tilde{p}_y \cdot \tilde{q}_{y+t}$. In what follows, $\sum_{n \in \mathcal{N}_{t+1}^{alive}} prob_n = {}_{t+1}\tilde{p}_y$, $\sum_{n \in \mathcal{N}_{t+1}^{dead}} prob_n = {}_t\tilde{p}_y \cdot \tilde{q}_{y+t}$ and $\sum_{n \in (\mathcal{N}_{t+1}^{alive} + \mathcal{N}_{t+1}^{dead})} prob_n = {}_t\tilde{p}_y$.

6. To ensure that the asset returns are independent from the events of death, repeat the algorithm for a larger number of trees.

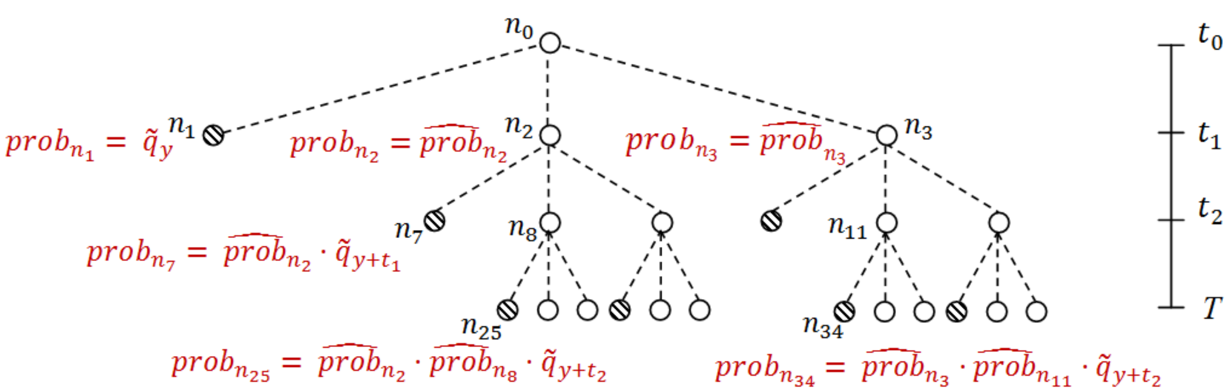


Figure 3.1: A 3-period scenario tree with a constant branching factor 3 and $3^3 = 27$ scenarios. The events of death occur at the colored nodes. The scenarios consist of a leaf node and its predecessors, thus, the scenarios in which the person died have a shorter length. The probabilities of each node are consistent with the survival and death probabilities.

The presented approach can be applied in various financial optimization problems. For example, one can follow the same procedure to generate a scenario tree with the uncertainty around the labor income (given that the labor income is independent of the asset returns).

3.3.2 Optimization module

Once generating a scenario tree, we can calculate the investment, annuity payout and bequest decisions. The optimal solution depends on the possible future realizations of the asset returns and death events, and on the decisions made in the previous stage. For example, as illustrated on Fig. 3.2, the yearly returns on bonds and stocks at node n_2 are 0.103 and 0.261, respectively, whereas

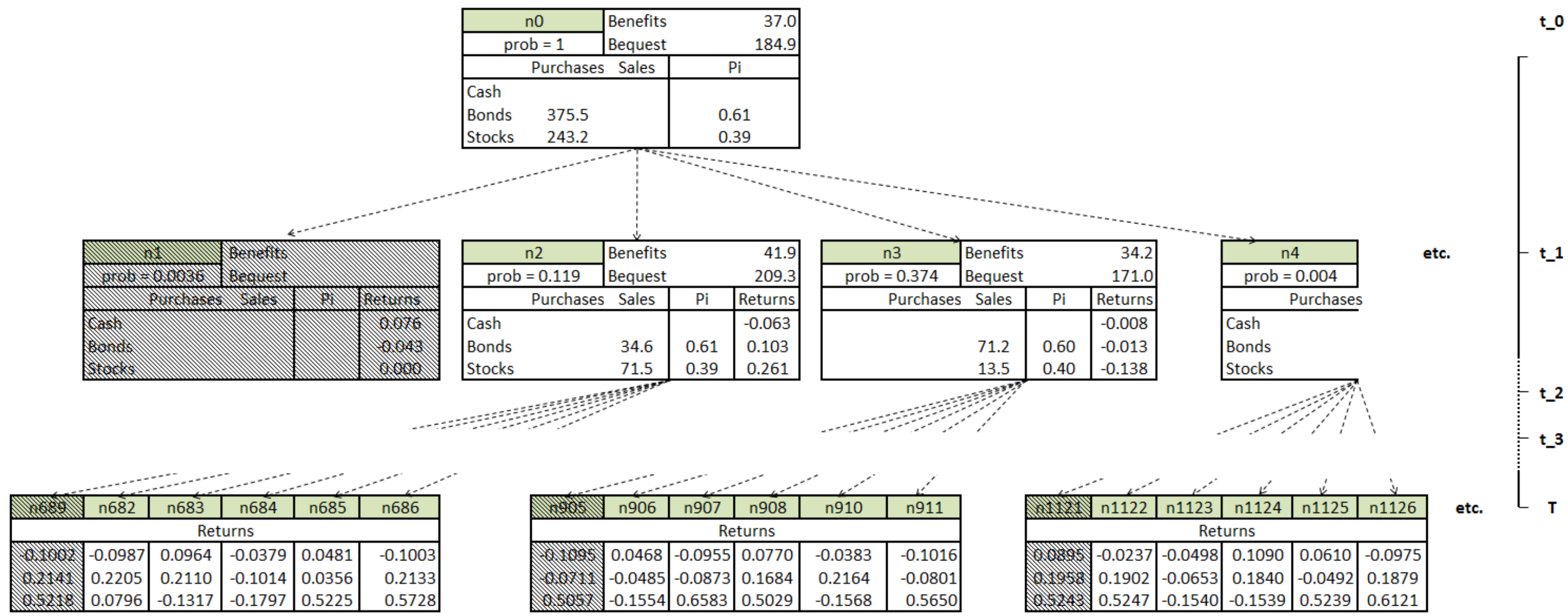


Figure 3.2: A fragment of the multi-stage problem. The total benefits, the size of death benefit and the rebalancing decisions (purchases and sales) are calculated numerically at each node. The colored columns denote the nodes with the events of death. Pi denotes the proportion in each asset class. The periods between the decisions are of $\Delta t = \{1, 3, 3, 3\}$, thus the returns at t_1 are the yearly returns, whereas the returns in the following periods are accumulated over 3 years. The amounts are in EUR 1,000. Other parameters: $age_0 = 65$, $x_0 = 650$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.04$, $k = 5$, $d_i = 0$, $u_i = 1$ and $T = 10$.

the yearly returns on bonds and stocks at node n_3 are -0.013 and -0.138, respectively. The optimal benefits at these nodes are EUR 41,900 and EUR 34,200, and the optimal bequest amount is EUR 209,300 and EUR 171,000, respectively. The optimal asset allocation is similar at both nodes (61% and 39% versus 60% and 40% in bonds and stocks, respectively); however, to rebalance the portfolio to the given allocation, at each node the individual sells different amounts of the assets.

We define the optimization model by introducing the following parameters and decision variables depending on the period $t \in [t_0, T]$, node $n \in \mathcal{N}_t$, and asset class $i \in \mathcal{I}$:

Parameters

T_R	retirement time
T	the end of decision horizon and the beginning of the remaining period modeled by the end effect
$\widetilde{T}$	expiration of the contract, i.e. the last benefit is paid out
$prob_n$	probability of being at node n
x_0	initial value of savings
x_T^{min}	minimum level of savings upon horizon T
b_t^{state}	state retirement pension received at time t
b_t^{min}	minimum level of benefits received at time t
l_t	labor income at time t
p^{fixed}	fixed percentage of the labor income defining the mandatory premiums
p^{vol}	fixed percentage of the labor income defining the maximum voluntary premiums
ins_t	proportion of the savings defining the death sum
k	weight on the bequest motive relatively to the size of the benefits
$\tilde{q}_y$	probability that an y-year-old individual dies during the following period (individual's expectation)
q_y	probability that an y-year-old individual dies during the following period (insurer's expectation)
$r_{i,t,n}$	return on asset i at node n corresponding to stage t.

Decision variables

$X_{i,t,n}^{\rightarrow}$	amount allocated to asset class i, at the beginning of period t, at node n, before rebalancing and any cash-flows
$X_{i,t,n}$	amount allocated to asset class i, at the beginning of period t, at node n, after rebalancing and any cash-flows
$X_{i,t,n}^{buy}$	amount of asset class i purchased in period t, at node n
$X_{i,t,n}^{sell}$	amount of asset class i sold in period t, at node n
$B_{t,n}$	benefits (annuity payments) from the DC pension plan received in period t, node n
$B_{t,n}^{tot}$	total benefits received in period t, node n
$P_{t,n}^{tot}$	total premiums (mandatory and voluntary) paid in period t, node n
$Beq_{t,n}$	death sum paid to the heirs upon the individual's death in period t, node n.

All the savings are initially allocated to cash (denoted by asset class $i = 1$), thus $X_{1,t_0,n_0}^{\rightarrow} = x_0$ and $X_{i,t_0,n_0}^{\rightarrow} = 0$, $\forall_{i \neq 1}$. The administration costs, transaction costs, and the taxes have been ignored for simplicity.

The objective function, Eq. (3.3.1), which we aim to maximize, consists of three terms: (i) the expected utility of total retirement benefits paid while the person is alive, (ii) the expected utility of death sum paid to the heirs upon the individual's death, and (iii) the end effect described in detail in Appendix 3.B. The budget constraint, Eq. (3.3.2), specifies the cash-flows accompanying the savings account: the incoming payments (capital gains, the amount gained from the sales of the securities, premiums, and survival credit) and the outgoing payments (the amount spent on the purchase of new securities, annuity payments, and insurance coverage). The next constraint, (3.3.3), defines the asset inventory balance. We first account for the returns earned during the previous period, Eq. (3.3.4), and then rebalance the amount by purchasing or selling a given asset. In (3.3.5) we define the total premiums paid to the savings account as the sum of the mandatory and voluntary contributions. Constraint (3.3.6) defines the total benefits as the sum of the state retirement pension and the annuity payments received from the DC plan. By including equations (3.3.7) and (3.3.8), we ensure that the benefits and the value of the savings do not fall below the certain pre-specified levels b_t^{min} and x_T^{min}, respectively. Constraint (3.3.9) defines the death benefit as a fraction of the savings, whereas (3.3.10) defines the actuarially fair survival credit that the individual receives for each period she survives. If the individual wishes to bequeath exactly the value of the savings, i.e. $ins_t = 1$, then the survival credit is equal to the price of the death benefit, and the last two terms in the budget constraint (3.3.2) cancel out. In a case when the individual is interested in the optimal death sum, constraint (3.3.9) is no longer necessary and should be removed. Equations (3.3.11)-(3.3.12) define the limits on portfolio composition. These can reflect the regulatory constraints, for example, the individual is not allowed to either hold a short position in a given asset i or to borrow money to invest in this asset ($d_i = 0$ and $u_i = 1$), or they can reflect the individual's preferences on portfolio composition. Finally, we include Eq. (3.3.13) to distinguish between the purchases and sales, and to ensure that the annuity payments and the death sum are positive.

$$\begin{aligned}\text{maximize} \quad & \sum_{s=\max(t_0,T_R)}^{T-1} \sum_{n\in\mathcal{N}_s^{alive}} u\left(s, B_{s,n}^{tot}\right)\cdot prob_n + \sum_{s=t_0}^{T-1}\sum_{n\in\mathcal{N}_s^{alive}} k\,\tilde{q}_{y+s}\, u\left(s, Beq_{s,n}\right)\cdot prob_n \\ + & \sum_{n\in\mathcal{N}_T^{alive}} V\left(T, \sum_i X_{i,T,n}^{\rightarrow}\right)\cdot prob_n, \qquad (3.3.1)\end{aligned}$$

subject to

$$X_{1,t,n} = X_{1,t,n}^{\rightarrow} + \sum_{i\neq 1} X_{i,t,n}^{sell} - \sum_{i\neq 1} X_{i,t,n}^{buy} + P_{t,n}^{tot}\mathbf{1}_{\{t<T_R\}} - B_{t,n}\mathbf{1}_{\{t\geq T_R\}} + R_{t,n}^{surv} - q_{y+t}Beq_{t,n},$$
$$t\in\{t_0,\ldots,T-1\},\ n\in\mathcal{N}_t^{alive}, \qquad (3.3.2)$$
$$X_{i,t,n} = X_{i,t,n}^{\rightarrow} + X_{i,t,n}^{buy} - X_{i,t,n}^{sell}, \qquad t\in\{t_0,\ldots,T-1\},\ n\in\mathcal{N}_t^{alive},\ i\neq 1, \qquad (3.3.3)$$
$$X_{i,t,n}^{\rightarrow} = (1+r_{i,t,n})X_{i,t-1,n^-}, \qquad t\in\{t_1,\ldots,T\},\ n\in\mathcal{N}_t^{alive},\ i\in\mathcal{I}, \qquad (3.3.4)$$
$$P_{t,n}^{tot} \leq (p^{vol}+p^{fixed})l_t, \qquad t\in\{t_0,\ldots,T-1\},\ n\in\mathcal{N}_t^{alive}, \qquad (3.3.5)$$
$$B_{t,n}^{tot} = B_{t,n} + b_t^{state}, \qquad t\in\{t_0,\ldots,T-1\},\ n\in\mathcal{N}_t^{alive}, \qquad (3.3.6)$$

$$B^{tot}_{t,n} \geq b^{min}_t, \qquad t \in \{t_0, \ldots, T-1\},\ n \in \mathcal{N}^{alive}_t, \tag{3.3.7}$$

$$\sum_i X^{\rightarrow}_{i,T,n} \geq x^{min}_T, \qquad n \in \mathcal{N}^{alive}_T, \tag{3.3.8}$$

$$Beq_{t,n} = ins_t \sum_i X^{\rightarrow}_{i,t,n}, \qquad t \in \{t_0, \ldots, T-1\},\ n \in \mathcal{N}^{alive}_t, \tag{3.3.9}$$

$$R^{surv}_{t,n} = q_{y+t} \sum_i X^{\rightarrow}_{i,t,n}, \qquad t \in \{t_0, \ldots, T-1\},\ n \in \mathcal{N}^{alive}_t, \tag{3.3.10}$$

$$X_{i,t,n} \leq u_i \sum_i X_{i,t,n}, \qquad t \in \{t_0, \ldots, T-1\},\ n \in \mathcal{N}^{alive}_t,\ i \in \mathcal{I}, \tag{3.3.11}$$

$$X_{i,t,n} \geq d_i \sum_i X_{i,t,n}, \qquad t \in \{t_0, \ldots, T-1\},\ n \in \mathcal{N}^{alive}_t,\ i \in \mathcal{I}, \tag{3.3.12}$$

$$X^{buy}_{i,t,n} \geq 0,\ X^{sell}_{i,t,n} \geq 0,\ B_{t,n} \geq 0,\ Beq_{t,n} \geq 0, \qquad t \in \{t_0, \ldots, T-1\},\ n \in \mathcal{N}^{alive}_t,\ i \in \mathcal{I}. \tag{3.3.13}$$

Expression $\mathbf{1}_{\{(\cdot)=t\}}$ denotes an indicator function equal to 1 if $(\cdot) = t$ and 0 otherwise.

3.4 Intuition behind the optimal policy

Stochastic programming is a general purpose framework for modeling optimization problems where an objective function can take a variety of forms. Rather than finding a generic optimal policy, the optimal decisions are computed numerically at each node in the scenario tree. MSP can easily address realistic considerations and constraints, as long as they have an algebraic form, see, e.g., Carino et al. (1998) and Carino and Ziemba (1998), who formulate a financial planning model for one of the biggest Japanese property and casualty insurer. However, the numerical solution may be difficult to interpret.

On the contrary, the analytical form of the optimal solution is the main advantage of a stochastic optimal control approach. SOC, however, is best applicable for simple models because deriving the explicit solution for many problems is not trivial (as, for example, for the problem presented above). Nevertheless, to gain more insights about the optimal solution for our model, we take a closer look at the explicit formulae obtained via SOC approach for a simplified model.

Specifically, to be able to derive the explicit solution we have to simplify the model by introducing the following assumptions: i) a continuous-time setting, ii) no upper or lower bounds on the variables (such as those in equations (3.3.7)-(3.3.8) and (3.3.11)-(3.3.13)), iii) a risk-free return on cash, and iv) either a deterministic or optimal death benefit. Otherwise, obtaining the analytical solution is non-trivial. Because the explicit solution for the case with two sources of retirement income (state retirement pension and benefits from the DC plan) has not been presented in the literature, we derive the optimal decisions in Appendix 3.B.

Optimal investment The optimal investment decision for the presented model is of the form obtained by Richard (1975). Equation (3.B.9) for optimal proportion in risky portfolio Π^*_t, indicates that the optimal investment decision depends on the risk aversion, the market parameters, the value of the savings, and the present value of the expected retirement state pension g_t; whereas

Eq. (3.B.10) for the proportions between the risky assets θ_i specified by the mutual fund theorem, indicates that these proportions depend on the expected returns of the risky assets, their volatilities, and the correlations between them. If the individual expects no retirement state pension, the optimal strategy suggests a fixed-mix portfolio, as shown by Merton (1969, 1971). Otherwise the individual should decrease the percentage in the risky portfolio as g_t decreases.

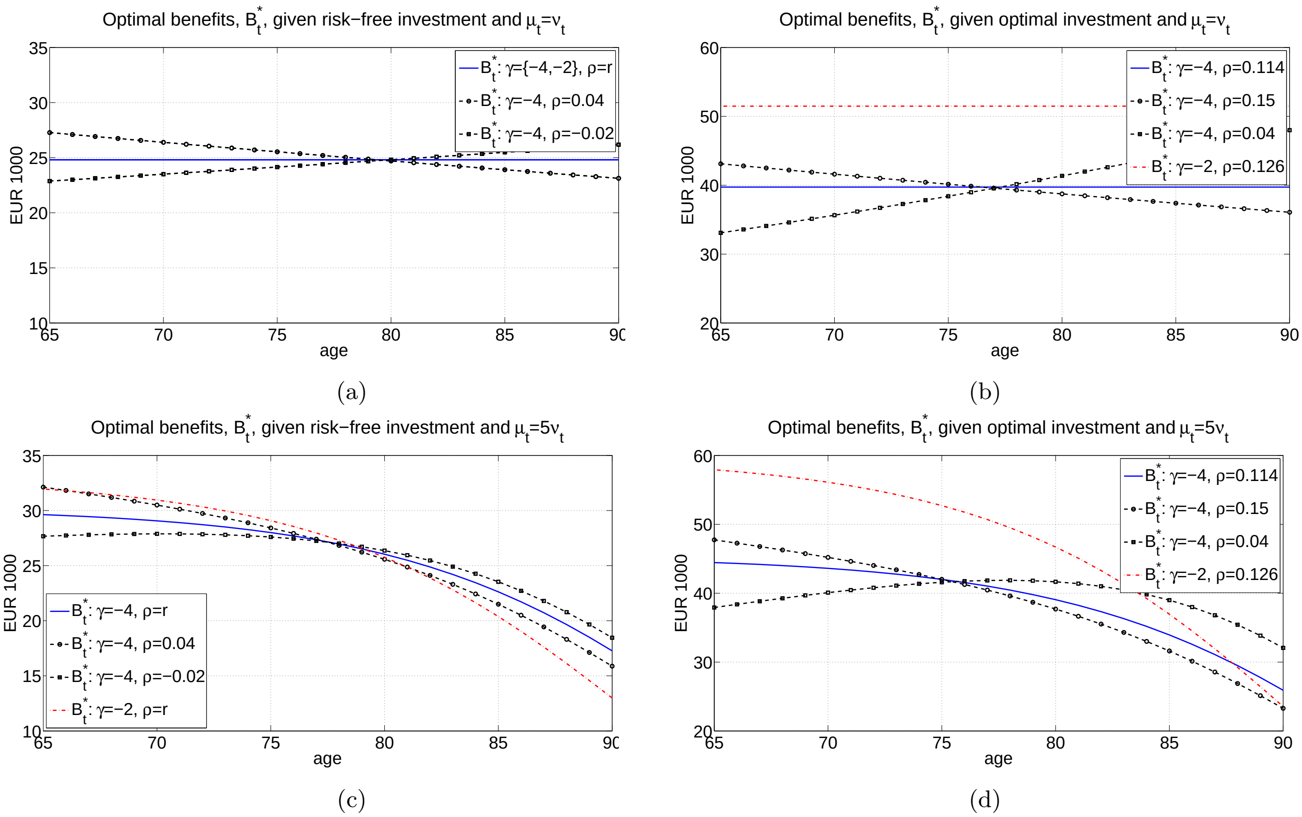
Optimal annuity payments Not less important is to determine the optimal annuity payments. In particular, we investigate whether there exists a withdrawal rate, according to which the accumulated savings should be spent, as in, e.g., Bengen (1994) and Horneff et al. (2008).

To understand formula (3.B.6) for the optimal benefits sum B_t^*, let us focus on the individual upon retirement (i.e. 65-year old), and let us assume the risk-free investment, so we can separate the annuity payments decision from the investment decision. Given that the subjective mortality rate is equal to the pricing mortality rate, $\mu_t = \nu_t$, the payout curve is: constant if the impatience weighted interest factor is equal to the risk-free rate, $\rho = r$, decreasing for impatient individuals, $\rho > r$, and increasing for patient ones, $\rho < r$. The parameter γ controls the slope of the payout curve. For the less risk averse individuals (such as $\gamma = -2$), the difference between the benefits received at the beginning and at the end of retirement is bigger than for moderately risk averse persons with $\gamma = -4$. The optimal payout profile for different choices of γ and ρ is illustrated on Fig. 3.3 a. None of these payout profiles is better than the other; they are all optimal for individuals with different preferences.

Investing in risky assets directly affects the size of annuity payments. To avoid the danger of running out of the resources before the individual's death, the benefits must be adjusted each year to account for the capital gains and losses. Nevertheless, despite these adjustments, we can still control the expected payout curve; we can choose parameters (γ, ρ, μ_t) such that the expected payout curve is constant, increasing or decreasing. Given the optimal investment strategy, Eq. (3.B.9), we obtain constant expected annuity payments for $\rho = r + (2-\gamma)\frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}$ and $\mu_t = \nu_t$. Any other choice of ρ and μ_t leads to either increasing or decreasing payout curve, as shown on Figs. 3.3 b and 3.3 d.

Following this argumentation, one can recognize that formula (3.B.6) defines equity-linked annuity payments. The more aggressive investment strategy, the higher expected benefits. A person interested in a constant payout curve should expect: EUR 51,500 given $\gamma = -2$ and $\rho = 12.6\%$, EUR 39,700 given $\gamma = -4$ and $\rho = 11.4\%$, and EUR 24,800 given the risk-free investment, $\rho = r$ and any choice of γ. Formula (3.B.6) also shows, that indeed there exists an optimal withdrawal rate $1/\bar{a}_t^*$ that depends on the constants γ and ρ characterizing the individual's risk tolerance and impatience, and on the subjective mortality rate μ_t. Interestingly, the withdrawal rate is not only a fraction of the savings at a given time, but also of the present value of the expected retirement state pension. Accordingly, the size of the benefits expected from the state retirement pension affects the optimal size of the payments from the DC plan. The optimal withdrawal rates for different values of γ, ρ and μ_t are presented in Table 3.1.

Finally, Figs. 3.3 c and 3.3 d show how the subjective lifetime expectancy affects the optimal payout curve. The choice of $\mu_t = 5\nu_t$ indicates that the individual expects to die earlier than an



(a) (b) (c) (d)

Figure 3.3: The optimal annuity payments for a given investment strategy and parameters γ, ρ and μ_t. Figs. (a) and (c) assume the risk-free investment, Figs. (b) and (d) assume the optimal investment: {cash, bonds, stocks}={11.6%, 52.5%, 35.8%} for $\gamma = -4$ and {cash, bonds, stocks}={-47.3%, 87.6%, 59.7%} for $\gamma = -2$. Furthermore, Figs. (c) and (d) assume the expected lifetime shorter than for an average individual, $\mu_t = 5\nu_t$. Parameters: $age_0 = 65$, $x_0 = 650$, $k = 3125$, and $b_t^{state} = 0$. The amounts are in EUR 1,000.

Age	65	70	75	80	85	90
Risk-free investment, moderate risk aversion ($\gamma = -4$)						
Constant benefits, $\rho = r$	3.8%	4.4%	5.3%	6.4%	7.8%	9.6%
Decreasing benefits, $\rho = 0.04$	4.2	4.8	5.6	6.7	8.1	9.9
Increasing benefits, $\rho = -0.02$	3.5	4.1	5.0	6.1	7.5	9.3
Shorter expected lifetime, $\mu_t = 5\nu_t$, $\rho = r$	4.6	5.5	6.7	8.3	10.5	13.0
Optimal investment, moderate risk aversion ($\gamma = -4$), i.e. {cash, bonds, stocks}={11.6%, 52.5%, 35.8%}						
Constant benefits, $\rho = 0.114$	6.1	6.7	7.4	8.4	9.7	11.3
Decreasing benefits, $\rho = 0.15$	6.6	7.2	7.9	8.9	10.1	11.7
Increasing benefits, $\rho = 0.04$	5.1	5.7	6.5	7.5	8.9	10.6
Shorter expected lifetime, $\mu_t = 5\nu_t$, $\rho = 0.126$	6.8	7.7	8.8	10.4	12.4	14.7
Optimal investment, lower risk aversion ($\gamma = -2$), i.e. {cash, bonds, stocks}={-47.3%, 87.6%, 59.7%}						
Constant benefits, $\rho = 0.126$	7.9	8.4	9.1	10.0	11.1	12.5

Table 3.1: Optimal withdrawal rates for a given investment strategy and parameters γ, ρ and μ_t. Parameters: $age_0 = 65$, $x_0 = 650$, $k = 3125$, and $b_t^{state} = 0$.

average individual assumed by the life insurer. Specifically, for the chosen mortality model, such a choice of μ_t corresponds to the expected lifetime of 78.7 years, with 70.2% chances of survival until age 75 and only 18.5% chances of survival until age 85 (given that the individual is alive at age 65). Independently of the choice of γ and ρ, the payout curve is no longer constant, but decreases proportionally to the probability of survival. This result indicates that a life annuity with a decreasing payout curve is preferable than, for example, a term annuity, which pays constant benefits for 10 or 25 years. A similar conclusion has been drawn by Milevsky and Huang (2011), who argue that "the optimal ... behavior in the face of personal longevity risk is to consume in proportion to survival probabilities—adjusted upward for pension income and downward for longevity risk aversion—as opposed to blindly withdrawing constant income for life".

Optimal death sum Equation (3.B.7) for the bequest Beq_t^* shows that the optimal death sum is a linear function of the optimal annuity payments. Both decisions are proportional by the factor $(k\mu_t/\nu_t)^{1/(1-\gamma)}$, which changes with the strength of the bequest motive k, risk aversion, and the relation between the subjective and pricing mortality rates. Therefore, similarly as expected annuity payments, the expected death sum can be constant, increasing or decreasing, whereas the actual size of the death sum depends on the realized portfolio returns. Formula (3.B.7) defines moreover the optimal death sum rate as a proportion $1/\bar{a}_t^* \, (k\mu_t/\nu_t)^{1/(1-\gamma)}$ of the current savings and the present value of expected state retirement pension.

3.5 Numerical results

To present the application of the model we have chosen a number of individuals with different economical and personal characteristics. Even small-scale optimization problems, such as problems based on 2,160 scenarios (four periods with branching factors {10,6,6,6}) with 12,075 constraints and 9,546 variables, are sufficient to present the applications of the model. The MSP formulation can be implemented on a personal computer and takes only a few seconds to run. We implemented the program on a Dell computer with an Intel Core i5-2520M 2.50 GHz processor and 4 GB RAM, using MATLAB 8.2.0.713 (R2013b), and GAMS 24.1.3 with non-linear solver MOSEK 7.0.0.75. The optimization module can also be solved with a linear or quadratic solver, such as CPLEX, but the objective function has to be linearly or quadratically approximated. Furthermore, to check the robustness of the results, we rerun the model for 50 different scenario trees. Thus, the results are based on $2,160 \cdot 50 = 108,000$ scenarios.

The numerical examples provide some guidelines to individuals in DC pension plans on how to manage their savings both before and after retirement. These guidelines can also be used by life insurers for designing pension products that are highly customized to the individuals' needs. We show that pension savings management is based on three important decisions: investment, annuity payments and the level of death sum, and that these decisions vary substantially for individuals with different economical and personal preferences. Furthermore, the optimal decisions depend on the realizations of the risky assets, and are regularly adjusted to account for the changes in the financial market.

Because the optimal solution incorporates the investment in risky assets while guaranteeing income as long as the person is alive, it defines *optimal equity-linked life annuities.* Depending on the time of the purchase of the product, we distinguish between *immediate* and *deferred* annuities. Both products start paying out the benefits upon retirement, but deferred annuity is purchased when the individual is still employed. Furthermore, during the deferment period the premiums are invested according to the optimal investment strategy, and the life insurance policy is effective.

3.5.1 Optimal immediate equity-linked life annuity

We start with a 65-year old female.[1] She is just retiring and is interested in purchasing an immediate equity-linked life annuity. She has saved $x_0 = 650,000$ (EUR) on her pension account, has moderate risk aversion $1 - \gamma = 5$, and has an expected lifetime as assumed by the insurer ($\mu_t = \nu_t$), i.e. on average until age 89.1. During retirement she expects the benefits from the state, $b_t^{state} = 4,000$ (EUR). When asked about the preferable payout profile, she chooses life long increasing payments. Such a payout curve can be obtained for, e.g., $\rho = 0.04$. She has a bequest motive but is not sure how much money to bequeath to her heirs, and how it will affect the level of the annuity payments. Therefore, we investigate three cases: no bequest motive, $k = 0$, the death sum equal to the level of savings, $ins_t = 1$, and the death sum equal to the sum of the benefits received over 5 years (obtained for $k = 5^{1-\gamma}\nu_t/\mu_t = 3125$). The life insurer does not allow for having a short position in any asset or for borrowing money to invest in any asset, thus $d_i = 0$ and $u_i = 1$.

Sections (a), (b) and (c) in Table 3.2 present the optimal decisions for a person with such characteristics. The first 10 years of the retirement are modeled using MSP approach with the intervals between the decisions of $\Delta t = \{1, 3, 3, 3\}$ years. Thereafter, we approximate the model with its simpler continuous-time version that can be solved explicitly using SOC approach.

Reading Table 3.2, we can observe three important facts. First, the optimal investment strategy is almost identical for the considered different weights on the bequest motive. This result can be surprising at first, but Eq. (3.B.9) for the optimal investment in risky assets, states that the only parameters that influence the investment decisions are the market parameters, risk aversion, the current level of savings, and the present value of expected retirement state pension. The majority of savings upon retirement is invested in bonds (57%) and stocks (40%), and these proportions change slowly so that the individual invests less in risky assets as the present value of expected state retirement benefits decreases.

Second, as chosen by the individual, the expected annuity payments increase. A person without a bequest motive would receive the highest payments, not only because she does not pay for the insurance coverage, but also because she receives a survival credit for each year she survives. Upon retirement our individual will obtain on average the total yearly benefits of EUR 42,700, 36,200

[1] We follow the approach common for the most European countries, where the price of annuities does not depend on the gender, i.e. even though women are expected to live longer, they are entitled to receive the same benefits as men.

Age	MSP 65	MSP 66	MSP 69	MSP 72	SOC age_T=75	SOC 80	SOC 85
(a) Without a bequest motive, $k = 0$							
Cash	3%	0%	1%	0%	2%	3%	4%
Bonds	57	62	62	63	58	58	57
Stocks	40	38	37	37	40	39	39
Total benefits, B_t^{tot*}	€42.7	€43.4	€45.3	€47.3	€49.0	€52.7	€56.8
Bequest amount, Beq_t^*	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Value of savings, X_t^*	650.0	643.2	614.6	578.4	534.3	461.7	379.0
(b) With a death sum equal to the value of savings, $ins_t = 1$							
Cash	3%	0%	1%	1%	3%	4%	6%
Bonds	57	62	62	63	58	57	56
Stocks	40	38	37	36	39	39	38
Total benefits, B_t^{tot*}	€36.2	€36.7	€38.2	€39.6	€41.0	€44.2	€47.6
Bequest amount, Beq_t^*	650.0	647.6	632.2	608.2	205.0	220.8	237.9
Value of savings, X_t^*	650.0	647.6	632.2	608.2	573.8	541.1	501.8
(c) With an optimal death sum given $k = 3125$							
Cash	3%	0%	1%	1%	3%	5%	6%
Bonds	57	62	62	63	58	57	56
Stocks	40	38	37	36	39	38	38
Total benefits, B_t^{tot*}	€37.1	€37.7	€39.4	€41.1	€42.5	€45.8	€49.4
Bequest amount, Beq_t^*	185.4	188.5	196.8	205.6	212.7	229.1	246.8
Value of savings, X_t^*	650.0	648.4	637.2	620.4	597.5	563.1	521.9
(d) With a minimum level of benefits, $b_t^{min} = 27$*, minimum level of savings upon horizon,* $x_T = 350$*, and optimal death sum given* $k = 3125$							
Cash	14%	1%	2%	1%	3%	4%	6%
Bonds	53	63	62	63	58	57	56
Stocks	33	36	36	36	39	39	38
Total benefits, B_t^{tot*}	€36.3	€37.1	€38.9	€40.7	€42.3	€45.6	€49.1
Bequest amount, Beq_t^*	181.4	185.3	194.2	203.3	211.4	227.8	245.4
Value of savings, X_t^*	650.0	645.5	633.4	616.3	593.6	559.5	518.6
(e) With shorter expected lifetime than an average individual, $\mu_t = 5\nu_t$*, and optimal death sum given* $k = 3125$							
Cash	2%	1%	2%	4%	2%	2%	3%
Bonds	57	64	61	65	59	58	58
Stocks	40	35	37	32	40	40	39
Total benefits, B_t^{tot*}	€42.1	€42.7	€43.8	€44.8	€44.8	€44.9	€42.0
Bequest amount, Beq_t^*	289.1	292.0	298.7	304.3	308.4	309.0	289.2
Value of savings, X_t^*	650.0	643.1	605.8	563.0	509.0	427.8	330.5

Table 3.2: The optimal asset allocation, total benefits and size of death sum for a 65-year old individual given: (a) no bequest motive, (b) a death sum equal to the value of savings, (c) an optimal death sum given $k = 3125$, (d) a minimum level of benefits $b_t^{min} = 27$ and a minimum level of savings upon horizon $x_T^{min} = 350$, and (e) a subjective lifetime expectancy $\mu_t = 5\nu_t$. The numbers are presented in terms of means across the nodes associated with each period and the scenario trees. Parameters: $age_0 = 65$, $x_0 = 650$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.04$, $d_i = 0$, $u_i = 1$, $\mu_t = \nu_t$, $T = 10$, and $\Delta t = \{1, 3, 3, 3\}$. The asset allocations are in percentages and other amounts are in EUR 1,000.

and 37,100, respectively for the cases without a bequest motive, with a death sum equal to the level of savings, and with an optimal death sum given $k = 3125$. Twenty years after retirement the payments are expected to increase to EUR 56,800, 47,600 and 49,400, respectively.

Third, looking closer at the case with the optimal death sum, we find that the death benefit increases with time, and that it is much lower than the value of savings. As explained in Sec. 3.4, the optimal death sum is proportional to the annuity payments by a factor $(k\mu_t/\nu_t)^{1/(1-\gamma)} = 5$. Therefore, increasing annuity payments imply increasing death benefit. Furthermore, Beq_t^* is much lower than the value of savings for most of the retirement; it is higher than the value of savings only during the very late years (e.g. later than age 95), when the individual has already spent most of her savings.

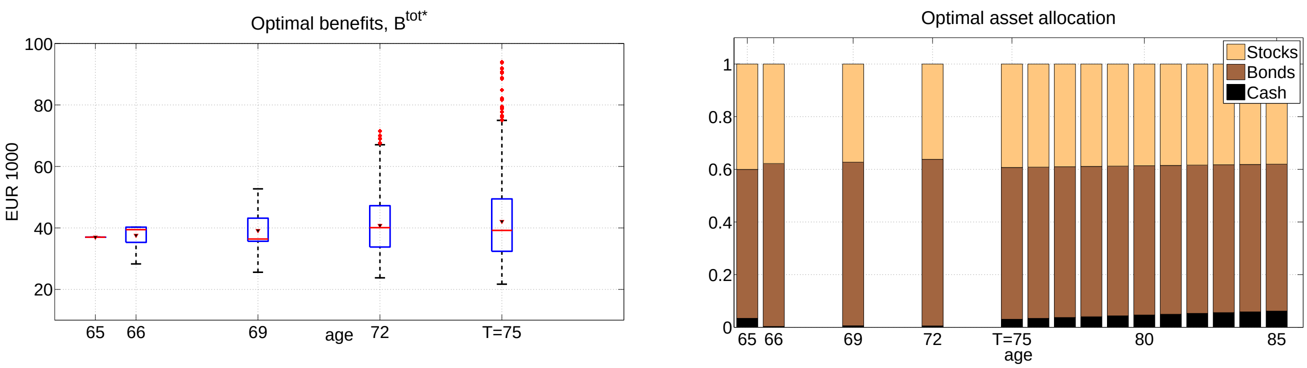
Recall that Table 3.2 presents the means across the nodes assigned to each time period, across the scenarios, and across different scenario trees. Equity-linked payments depend on the realizations of the risky assets, and their payments may vary significantly, especially after a longer period such as 10 years. Even though the expected benefits are increasing, if for 10 years in a row the risky assets bring losses, our individual may receive a significantly lower amount than expected. Fig. 3.4 a (left) emphasizes this fact and shows that in the worst case scenario the individual with an optimal death sum may end up only with EUR 23,800 upon age 75 instead of expected EUR 42,500. On the other hand, after a long period of higher asset returns than assumed, the benefits may even be twice as high as expected.

To mitigate the risk of receiving too low payments, we can add a lower limit on the benefits' size by adding constraints (3.3.7) and (3.3.8) in the MSP formulation. The results in Table 3.2, section (d), and Fig. 3.4 b, show that these constraints affect the optimal decisions. The asset allocation is more conservative during the first years of retirement, and leads on average to lower annuity payments and death sum. Other studies have shown that adding the guarantees to pension products increases the life insurer's liabilities, and thus prevents them from offering greater investment opportunities, see e.g., Guillén et al. (2013). In this example, a guarantee that the minimum payment never falls below $b_t^{min} = 27,000$ (EUR) is only added for the first 10 years after retirement. Nevertheless, such a guarantee reduces the expected yearly benefits from EUR 37,100 and 49,400 (upon ages 65 and 85) to EUR 36,300 and 49,100, respectively.

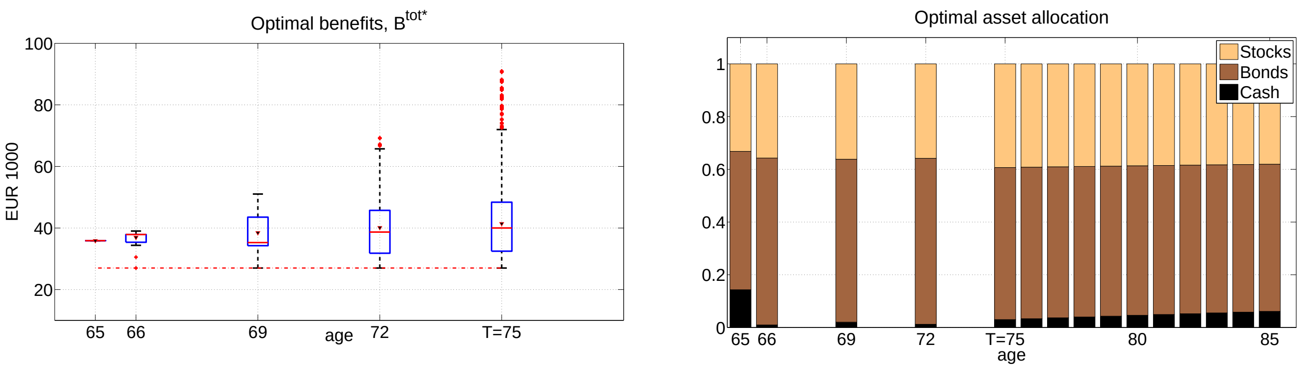
Finally, what can we recommend if our individual has a bad health condition and expects to die earlier than an average individual? To illustrate such a case, we choose $\mu_t = 5\nu_t$, that is, the expected lifetime of our individual is 78.7 years, which is approximately 10 years shorter than what the insurer assumes. Given that as in many European countries the survival credit and the price for life insurance are calculated under unisex criteria and are not subject to health screening, the individual should spend more savings during the first years of the retirement. The optimal solution (Table 3.2, section (e)) clearly suggests to change the payout curve so that the expected benefits decrease proportionally to the probability of survival, ${}_t\tilde{p}_y$, and to increase the death sum. The optimal investment strategy remains similar as in the case with the average lifetime expectancy.

3.5.2 Optimal deferred equity-linked life annuity

This section focuses on the decisions that an individual faces during the accumulation phase (i.e. before retirement). Our person is a 45-year old female with initial savings of $x_0 = 130,000$ (EUR)



(a)



(b)

Figure 3.4: The probability distribution of the optimal total benefits in one scenario tree, $B_t^{tot*} = B_t^* + b_t^{state}$, and the optimal investment in terms of means across the nodes and the scenario trees given: (a) an optimal death sum given $k = 3125$, and (b) an optimal death sum given $k = 3125$, the minimum level of benefits $b_t^{min} = 27$, and the minimum level o savings upon horizon, $x_T^{min} = 350$. The central mark in each box denotes the median, the triangle marker denotes the mean, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually with the red crosses. The red dashed line denotes the minimum level of benefits b_t^{min}. Parameters: $age_0 = 65$, $x_0 = 650$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.04$, $\mu_t = \nu_t$, $d_i = 0$, $u_i = 1$, $T = 10$, and $\Delta t = \{1, 3, 3, 3\}$. All the amounts are in EUR 1,000.

and pension contributions of 10% of her salary. The yearly salary, $l_t = 50,000$ (EUR), increases every year with $y_l = 2\%$. Having an average lifetime expectancy and anticipating $b_t^{state} = 4,000$ (EUR) from the retirement state pension, she would like to purchase an annuity that starts constant payments in 20 years (upon her retirement). She describes herself as moderate risk averse (e.g. $\gamma = -4$), therefore would like to invest some of her savings in risky assets. She has no further preferences on the portfolio composition but she faces short sales constraints on all assets.

The optimal decisions in this example are the investment strategy before and after retirement and the annuity payments after retirement. We also investigate the cases with and without a bequest motive. We divide the period of 20 years into 4 periods of 5 years each—we make the decisions every fifth year. The solution after retirement is calculated analytically using Hamilton–Jacobi–Bellman techniques for the simplified model.

Table 3.3 shows that, similarly to the previous case, the bequest motive has a minor effect on the optimal asset allocation. The overall investment strategy suggests to decrease the risk as the individual ages. The optimal portfolio consisting of 0% in cash, 41% in bonds, and 59% in stocks, (upon age 45), smoothly changes to {1%, 59%, 40%}, respectively, upon age 75. For the period $[t_0, T)$, we assume that cash has a volatility of 3.8% and that the individual is not allowed to have a short position in any asset or to borrow money to invest in any asset, whereas for the period $[T, \widetilde{T})$ the model is simplified: cash is assumed to be risk-free and no constraints on the portfolio allocation are imposed. Otherwise finding the analytical solution is non-trivial. This difference in the assumptions causes the fluctuations in cash holdings between the periods covered by two different optimization approaches.

The bequest motive affects the level of annuity payments, but not the payout curve: the expected benefits are constant. A person without a bequest motive will receive the highest benefits, $E[B_t^{tot*}] = 47,900$ (EUR) per year, a person with a death sum equal to the value of savings ($ins_t = 1$) will receive payments of EUR 41,600 per year, and a person with an optimal death sum given parameter $k = 5^{1-\gamma} = 3125$ will receive payments of EUR 41,800 per year. Notice how small the difference between the annuity payments in the last two cases is. Because the probability that a 45-year old person survives until age 65 is high, the price for the life insurance is low. Therefore, the value of savings upon retirement X_T^* is similar in both cases, and implies the annuity payments of approximately the same level. After retirement the optimal death sum is constant and proportional to the annuity payments by factor 5.

Nevertheless, the numbers in Table 3.3 are the means across the scenarios, whereas their actual values depend on the realizations of the asset returns. Figure 3.5 a (left) shows the probability distribution of the savings upon retirement for the case with the optimal death sum (for one scenario tree). After contributing to the pension account for 20 years and allocating the portfolio according to the optimal investment strategy, the individual should expect to save up EUR 596,700 upon retirement. This amount gives the expected total benefits of EUR 41,800, which is almost 60% of the individual's salary level upon retirement.

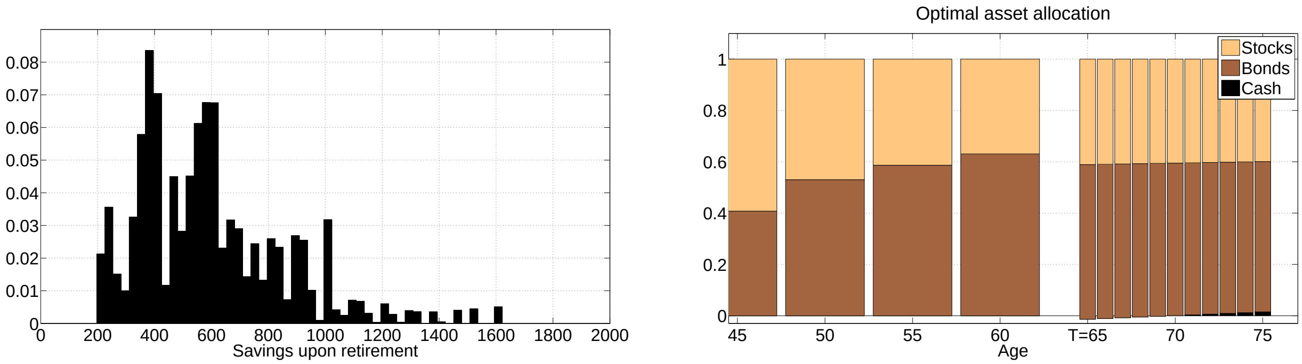
However, this amount can be much lower: in a scenario with long periods of negative returns, the person may end up with only EUR 200,000 on her savings account, which would provide the yearly retirement income of EUR 17,600. Thus, she may choose to increase the premiums by additional 5% and add a lower limit on the size of the savings upon retirement, for example, $x_T^{min} > 290,000$ (EUR). This limit corresponds to the minimum level of benefits $b_t^{min} = 23,100$ (EUR). As illustrated in Table 3.3 and Fig 3.5 b, both the probability distribution of savings and the optimal asset allocation change. The probability distribution has shifted to the right and the investment strategy implies slightly more conservative portfolio.

Is it possible to choose a higher limit x_T^{min} solely by adjusting the investment strategy? The answer depends on the available assets and their returns' distribution. To be certain that the value of savings will not fall below a pre-specified limit, we must employ a more conservative investment strategy. However, if the strategy is too conservative, it may not be possible to reach this level. For example, for the choice of the parameters as in Fig. 3.5 b, the program returns an optimal solution

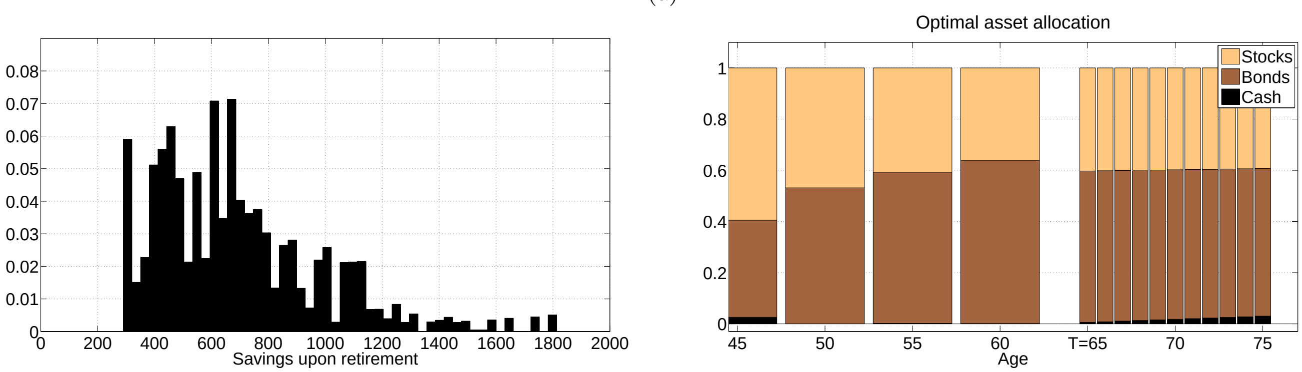
	MSP				SOC		
Age	45	50	55	60	age_T=65	70	75
(a) Without a bequest motive, $k = 0$							
Cash	0%	0%	0%	0%	-1%	0%	1%
Bonds	41	53	59	63	60	60	59
Stocks	59	47	41	37	41	40	40
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€47.9	€47.9	€47.9
Bequest amount, Beq_t^*	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Value of savings, X_t^*	130.0	209.4	309.8	438.9	604.2	534.9	456.6
(b) With a death sum equal to the value of savings, $ins_t = 1$							
Cash	0%	0%	0%	0%	-1%	0%	1%
Bonds	41	53	59	63	60	59	59
Stocks	59	47	41	37	41	41	40
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€41.6	€41.6	€41.6
Bequest amount, Beq_t^*	130.0	209.2	308.9	435.5	208.2	208.2	208.2
Value of savings, X_t^*	130.0	209.2	308.9	435.5	593.2	550.7	501.8
(c) With an optimal death sum given $k = 3125$							
Cash	0%	0%	0%	0%	-1%	0%	1%
Bonds	41	53	59	63	60	59	59
Stocks	59	47	41	37	41	41	40
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€41.8	€41.8	€41.8
Bequest amount, Beq_t^*	221.5	217.0	213.5	211.5	209.2	209.2	209.2
Value of savings, X_t^*	130.0	209.0	308.4	435.5	596.7	553.8	504.6
(d) With minimum level of savings upon retirement, $x_T^{min} = 290$, additional contributions $p^{vol} = 5\%$, and optimal death sum given $k = 3125$							
Cash	3%	0%	0%	0%	1%	2%	3%
Bonds	38	53	59	64	59	58	58
Stocks	59	47	41	36	40	40	39
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€48.4	€48.4	€48.4
Bequest amount, Beq_t^*	253.4	249.9	246.0	244.1	242.1	242.1	242.1
Value of savings, X_t^*	130.0	225.5	347.9	504.7	704.2	652.2	593.0
(e) With shorter expected lifetime than an average individual, $\mu_t = 5\nu_t$ and optimal death sum given $k = 3125$							
Cash	0%	0%	0%	0%	-2%	-1%	0%
Bonds	41	53	60	65	61	60	60
Stocks	59	47	40	35	41	41	41
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€45.7	€44.8	€43.2
Bequest amount, Beq_t^*	342.6	335.1	328.1	321.4	314.4	308.5	297.2
Value of savings, X_t^*	130.0	208.8	306.9	429.3	580.2	510.5	430.4

Table 3.3: The optimal asset allocation, total benefits and size of death sum for a 45-year old individual given: (a) no bequest motive, (b) a death sum equal to the value of savings, (c) an optimal death sum given $k = 3125$, (d) a minimum level of savings upon horizon $x_T^{min} = 290$ and additional contributions $p^{vol} = 5\%$, and (e) a subjective lifetime expectancy $\mu_t = 5\nu_t$. The numbers are presented in terms of means across the nodes associated with each period and the scenario trees. Parameters: $age_0 = 45$, $x_0 = 130$, $l_0 = 50$, $y_l = 2\%$, $p^{fixed} = 10\%$, $p^{vol} = 0$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.114$, $d_i = 0$, $u_i = 1$, $\mu_t = \nu_t$, $T = 20$, and $\Delta t = \{5, 5, 5, 5\}$. The asset allocations are in percentages and other amounts are in EUR 1,000.

for all the considered 50 different scenario trees. If we increase x_T^{min} to EUR 350,000, the problem is feasible only for 15 out of 50 different scenario trees (30%), whereas for $x_T^{min} = 380,000$ (EUR) the problem is feasible only for 2% of the scenario trees.



(a)



(b)

Figure 3.5: The probability distribution of the value of savings upon retirement X_T^* for one scenario tree, and the optimal investment in terms of means across the nodes and the scenario trees given: (a) an optimal death sum given $k = 3125$, and (b) an optimal death sum given $k = 3125$, the additional contributions $p^{vol} = 5\%$, and the minimum level o savings upon horizon, $x_T^{min} = 290$. Parameters: $age_0 = 45$, $x_0 = 130$, $l_0 = 50$, $y_l = 2\%$, $p^{fixed} = 10\%$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.119$, $\mu_t = \nu_t$, $d_i = 0$, $u_i = 1$, $T = 20$, and $\Delta t = \{5, 5, 5, 5\}$. The asset allocations are in percentages and other amounts are in EUR 1,000.

Finally, are optimal deferred life annuities still attractive if one expects to die earlier than an average person? We investigate the case for $\mu_t = 5\nu_t$, i.e. the expected lifetime of the individual is 78.7 years and the probability that she survives until age 85 is only 18%. A closer look at the optimal decisions (Table 3.3, section (e)) reveals that it is optimal for the person to invest in deferred life annuities only if the payout curve is decreasing. The optimal withdrawal rate is proportional to the probability of survival, therefore she should spend more savings in the beginning of retirement. The initial payment is EUR 4,000 higher than in the case with the average lifetime expectancy (compare with Table 3.3, section (c)), and the optimal death benefit increases significantly and stays above EUR 300,000 until age 72.

3.6 Conclusions and future work

This paper provides some guidelines to individuals with defined contribution pension plans. We argue that the decisions regarding the asset allocation, the annuity payments, and the size of death sum should be highly customized. With several numerical examples we have illustrated how the optimal decisions depend on: 1) economical characteristics—such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g. retirement state pension), and 2) personal characteristics—such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition.

To help individuals manage their pension savings, we have built a model that combines two optimization techniques: multi-stage stochastic programming and stochastic optimal control. This mixed approach has practical applications and generates results that are not only consistent with common knowledge about life-cycle asset allocation, but are also realistic. The presented model is flexible and can be applied either by financial advisers in countries where individuals have lots of flexibility in managing their pension savings, or by life insurers in countries where individuals are less involved in the savings and investment decisions. The framework can also be applied to discover reliable and robust policy rules, which can further be employed in practice using Monte Carlo simulations. Because the operations research methods are not common in the actuarial literature, we argue that the presented optimization approach has potential to stimulate new thinking and add to actuarial practise.

Even though the paper shows an application of decision making in pension plans, it includes several contributions to the current literature focusing on optimization methods. First, we have further developed the approach of combining multi-stage stochastic programming and stochastic optimal control. Only a few papers have been investigating this approach, and this area is fairly unknown. Second, we defined a procedure for generating scenario trees with two independent sources of uncertainty: time of death and asset returns. Third, by including another source of retirement income in the objective function, we expanded the classical problem of optimal investment, consumption and life insurance, and (using SOC method) derived the explicit solution.

This work could be improved in various ways. Investigation of the impact of the administration costs, transaction costs and taxes is definitely relevant from a practical point of view. Taxes are especially important, since in many countries life annuities are tax deferred investment vehicles, and therefore preferred to personal investment. Furthermore, one could incorporate in the model other sources of uncertainty such as stochastic mortality risk and uncertain salary progression.

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Appendix

3.A A short introduction to multi-stage stochastic programming

This appendix briefly introduces a multi-stage program with recourse. For a more detailed theory, see, e.g., Birge and Louveaux (1997), Zenios (2008) and Shapiro et al. (2009).

To start with, let us formulate a two-stage version of the problem with recourse. We keep the notation from the aforementioned books, and define $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space, ω is an element (outcome) of a sample space Ω, and $\xi = \xi(\omega)$ is a random vector which belongs to the probability space with support $\Xi = \{\xi \in \mathbb{R}^N \mid 0 \leq \xi < \infty\}$. We need two vectors for decision variables to distinguish between the anticipative and adaptive policy:

- $y_0 \in \mathbb{R}^{n_0}$ - a vector of first-stage decisions, which are made before the random variables are observed; the decisions do not depend on the future observations but anticipate possible future realizations of the random vector,
- $y_1(\xi) \in \mathbb{R}^{n_1}$ - a random vector of second-stage decisions which are made after the random variables have been observed. They are constrained by decisions y_0 and depend on the realizations of the random vector ξ.

Once a first-stage decision y_0 has been made, some realization of the random vector can be observed. Then, the second-stage problem seeks a decision vector $y_1(\xi)$ that optimizes the function $f_1(y_1(\xi);\xi)$ for a given value of the first-stage decision y_0 and the random parameters $\{T_{0,1}(\xi), W_1(\xi), h_1(\xi) \mid \xi \in \Xi\}$. Combining both stages, the two-stage problem with recourse is an optimization problem in the first-stage variable y_0, which maximizes the function $f_0(y_0)$ and the expected value of the function $f_1(y_1)$ of the second-stage decision:

$$
\begin{aligned}
\max_{y_0} \quad & f_0(y_0) + E\left[\max_{y_1} f_1(y_1(\xi);\xi) \,\middle|\, \mathcal{F}\right], \\
\text{s.t.} \quad & W_0 y_0 = h_0, \\
& T_{0,1}(\xi) y_0 + W_1(\xi) y_1(\xi) = h_1(\xi), \\
& y_0 \geq 0, \; y_1(\xi) \geq 0.
\end{aligned}
\tag{3.A.1}
$$

The recourse problem can be extended to a multi-period stochastic program, where observations and decisions are made at T different stages, which correspond to time instances when some information is revealed and a decision can be made. Let the random variable ξ have support $\Xi_1 \times \Xi_2 \times \cdots \Xi_T$ and the observations are captured in the information sets $\{\mathcal{F}_t\}_{t=1}^{T}$ with $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_T$. For each stage $t = 1, \ldots, T$, $y_t(\omega) \in \mathbb{R}^{n_t}$ denotes the recourse decision variable vector optimizing the random objective function $f_t(y_t(\xi_t);\xi_t)$, given the random parameters $\{T_{t-1,t}(\omega_t), W_t(\omega_t), h_t(\omega_t) \mid \xi_t \in \Xi_t\}$. Then, the following actions are taken at each stage:

$$
\begin{aligned}
&\text{decision } y_0 \rightarrow \text{observation } \xi_1 := (T_{0,1}, W_1, h_1) \rightarrow \text{decision } y_1 \rightarrow \cdots \\
&\rightarrow \text{observation } \xi_T := (T_{T-1,T}, W_T, h_T) \rightarrow \text{decision } y_T,
\end{aligned}
$$

which can be formulated as the following multi-stage program:

$$\begin{aligned}
\max_{y_0} \quad & f_0(y_0) + E\left[\max_{y_1} f_1(y_1;\xi_1) + \ldots + E\left[\max_{y_T} f_T(y_T;\xi_T) \,\middle|\, \mathcal{F}_T\right] \ldots \,\middle|\, \mathcal{F}_1\right], && (3.A.2)\\
\text{s.t.} \quad & W_0 y_0 = h_0, &&\\
& T_{t-1,t}(\xi_t) y_{t-1}(\xi_{t-1}) + W_t(\xi_t) y_t(\xi_t) = h_t(\xi_t), && t = 1,\ldots,T,\\
& y_0 \geq 0,\; y_t(\xi_t) \geq 0, && t = 1,\ldots,T.
\end{aligned}$$

By the tower property of conditional expectation we can rewrite the objective function of the above problem as:

$$\max_{y_0,y_1,\ldots,y_T} \; f_0(y_0) + \sum_{t=1}^{T} E\left[f_t(y_t;\xi_t) \,\middle|\, \mathcal{F}_1\right]. \tag{3.A.3}$$

Finally, if the random vector ξ_t has a discrete distribution with a finite number $n \in \mathcal{N}_t$ of possible realizations $\xi_{t,n}$ with the corresponding probabilities $prob_n$, the equation (3.A.3) can be rewritten as follows:

$$\max_{y_0,y_1,\ldots,y_T} \; f_0(y_0) + \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} f_t(y_{t,n};\xi_{t,n}) \cdot prob_n, \tag{3.A.4}$$

where $y_{t,n}$ is the decision variable corresponding to time t and realization n.

3.B The end effect

The main drawback of multi-stage stochastic programs is the limited ability to handle many time periods under sufficient uncertainty. The scenario tree grows exponentially with each time period, therefore solving the problem becomes soon computationally intractable. To ensure, that the optimization problem covers the decisions for the entire lifetime of the individual, we incorporate the *end effect* in the objective function of the MSP formulation, Eq. (3.3.1). The end effect is equal to the optimal value function, which can be calculated explicitly using stochastic optimal control (Hamilton-Jacobi-Bellman techniques), and covers the remaining years, i.e. the interval $[T, \widetilde{T})$. However, to be able to derive the explicit solution, we have to simplify the model by introducing the following assumptions: i) a continuous-time setting, ii) no upper or lower bounds on the variables (such as those in equations (3.3.7)-(3.3.8) and (3.3.11)-(3.3.13)), iii) a risk-free return on cash, and iv) either a deterministic or optimal death benefit; as it has been done in the classical literature on optimal consumption and investment, Merton (1969, 1971), and life insurance, Richard (1975) and Kraft and Steffensen (2008).

Assume that the economy is represented by a standard Brownian motion W defined on the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of W. The space is equipped with the equivalent probability measures: objective measure $\mathbb{P}$ and the martingale measure $\mathbb{P}^*$. The latter is used by the insurer to price the financial assets and life insurance, and to calculate the level of the benefits. The individual invests the proportion $1 - \Pi_t$ of her savings in a risk-free asset (cash) with a constant interest rate r, and the proportion Π_t in a mutual fund consisting of $N - 1$ assets, which prices are log-normally distributed. Then, the mutual fund follows the dynamics

$dS_t = \alpha S_t dt + \sigma S_t dW_t$, where

$$\alpha = \sum_{i=1}^{N-1} \theta_i \alpha_i, \qquad \sigma^2 = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \theta_i \theta_j \sigma_{ij}, \qquad dW = \sum_{i=1}^{N-1} \theta_i \frac{\sigma_i}{\sigma} dW_i,$$

θ_i is the proportion of asset i in the mutual fund, and $\{\alpha_i, \sigma_{ij}\}$ define the physical distribution of the returns. The assets are correlated with the coefficient $corr_{ij}$, thus $\sigma_{ij} = \sigma_i \sigma_j corr_{ij}$.

The individual has an uncertain lifetime modeled by a time-heterogeneous continuous-time Markov chain Z, defined on a measurable space $(\Omega, \mathcal{F})$. The process Z has two states $\{0, 1\} = \{alive, dead\}$, indicating whether the person is alive or not. For $s > t$, the transition probabilities between the states are given by

$$\begin{bmatrix} P(Z_s = 0 | Z_t = 0) & P(Z_s = 1 | Z_t = 0) \\ P(Z_s = 0 | Z_t = 1) & P(Z_s = 1 | Z_t = 1) \end{bmatrix} = \begin{cases} \begin{bmatrix} {}_s\tilde{p}_t & 1 - {}_s\tilde{p}_t \\ 0 & 1 \end{bmatrix}, & \text{under } \mathbb{P}, \\ \begin{bmatrix} {}_s p_t & 1 - {}_s p_t \\ 0 & 1 \end{bmatrix}, & \text{under } \mathbb{P}^*, \end{cases}$$

where ${}_s\tilde{p}_t = e^{-\int_t^s \mu_\tau d\tau}$, ${}_s p_t = e^{-\int_t^s \nu_\tau d\tau}$, and μ_τ and ν_τ are the jump intensities of the process Z, i.e.,

$$\lim_{dt \to 0} \frac{P(Z_{t+dt} = 0 | Z_t = 0)}{dt} = \begin{cases} 1 - \mu_t, & \text{under } \mathbb{P}, \\ 1 - \nu_t, & \text{under } \mathbb{P}^*, \end{cases}$$

$$\lim_{dt \to 0} \frac{P(Z_{t+dt} = 1 | Z_t = 0)}{dt} = \begin{cases} \mu_t, & \text{under } \mathbb{P}, \\ \nu_t, & \text{under } \mathbb{P}^*. \end{cases}$$

The mortality rates μ_τ and ν_τ are assumed to be continuous, deterministic, and satisfying $\mu_t \to \infty$ and $\nu_t \to \infty$, which further implies that $\lim_{t\to\infty} \mathbb{P}(Z_t = 1) = \lim_{t\to\infty} \mathbb{P}^*(Z_t = 1) = 1$. We have calibrated ν_t to the Danish mortality rates and obtained a satisfactory curve fit for a function

$$\nu_t = a_1 \exp\left(-\left(\frac{t - b_1}{c_1} \right)^2 \right) + a_2 \exp\left(-\left(\frac{t - b_2}{c_2} \right)^2 \right), \tag{3.B.1}$$

where constants $a_1, b_1, c_1, a_2, b_2, c_2$ are defined in Table 3.B.1. Data, which include the mortality improvements, can be downloaded from Danish Financial Supervisory Authority website, see Finanstilsynet (2012). We further assume that the subjective mortality rate μ_t is proportional to ν_t.

During retirement, $T \geq T_R$, the savings develop according to the incoming and outgoing payments. The incoming cash-flows are determined by the capital gains earned on the investment and the survival credit $\nu_t X_t$ paid by the life insurer. The outgoing payments consist of continuously paid premiums $\nu_t Beq_t$ for the death sum Beq_t, and the benefits B_t. Upon death the life insurer inherits the value of savings X_t but pays out the lump sum Beq_t to the heirs. Therefore, the dynamics of the savings account, while the person is alive, are given by

$$\begin{aligned} dX_t &= \big(r + \Pi_t(\alpha - r)\big) X_t dt + \Pi_t \sigma X_t dW_t - \nu_t Beq_t dt + \nu_t X_t dt - B_t dt, \\ X_0 &= x_T, \end{aligned} \tag{3.B.2}$$

where (X_t, B_t, Beq_t) are continuous-time variables corresponding to variables $(\sum_i X_{i,t,n}, B_{t,n}, Beq_{t,n})$

defined in the MSP formulation. Note that in the continuous-time framework we do not distinguish between the value of the savings before and after rebalancing. Moreover, rather than keeping the track of the traded amounts, we calculate the optimal asset allocation Π_t directly.

The objective is to maximize the expected utility of total benefits and bequest, given that the individual is alive at time t and has $X_t = x_t$ on her savings account:

$$\begin{aligned} V(t,x) &= \sup_{(\Pi_t, B_t, Beq_t)\in\mathcal{Q}[t,\widetilde{T})} E_{t,x}\left[\int_t^{\widetilde{T}} {}_s\tilde{p}_{y+t}\Big(u(s, b_s^{state} + B_s) + \mu_s\, k\, u(s, Beq_s)\Big)ds\right], \\ V(\widetilde{T}, x) &= 0. \end{aligned}$$

The expression $E_{t,x}$ denotes the conditional expectation under $\mathbb{P}$, whereas $\mathcal{Q}$ is the set of control processes that are admissible at time t. Both utilities are multiplied by the subjective probability that a y-year-old individual survives until time $s > t$, given she has survived until time t,

$${}_s\tilde{p}_{y+t} = e^{-\int_t^s \mu_{y+\tau} d\tau}.$$

The utility of bequest is moreover multiplied by the probability of dying shortly after surviving until time s, μ_s. Parameter k denotes the weight on the bequest motive relatively to the benefits, and $\widetilde{T}$ is a fixed time point at which the individual is dead with certainty.

This simplified problem can be solved explicitly using the Hamilton-Jacobi-Bellman techniques. In this appendix we derive the optimal value function and the optimal controls only for the period after retirement, $t \geq T_R$. The case for $t < T_R$ is slightly more complicated but can be derived in a similar way.

Based on the savings dynamics, Eq. (3.B.2), the HJB equation for the considered period is defined as follows,

$$\begin{aligned} \frac{\partial V(t,x)}{\partial t} &- \mu_t V + \sup_{(\Pi_t, B_t, Beq_t)} \left\{ \frac{1}{\gamma} w_t^{1-\gamma}(B_t + b_t^{state})^\gamma - B_t \frac{\partial V(t,x)}{\partial x} + \mu_t k \frac{1}{\gamma} w_t^{1-\gamma} Beq_t^\gamma \right. \\ &- \left. \nu_t Beq_t \frac{\partial V(t,x)}{\partial x} + (r + \Pi_t(\alpha - r) + \nu_t) x \frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\Pi_t^2 \sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} \right\} = 0, \\ V(\widetilde{T}, x) &= 0. \end{aligned}$$

We guess the solution

$$V(t,x) = \frac{1}{\gamma} f_t^{1-\gamma}\big(x + g_t\big)^\gamma, \tag{3.B.3}$$

where f_t and g_t are deterministic functions of time satisfying $f_{\widetilde{T}} = 0$ and $g_{\widetilde{T}} = 0$, and verify that our guess is correct. To derive the functions f_t and g_t we plug in the derivatives of the function

$V(t,x)$ to the HJB equation,

$$
\begin{aligned}
& \frac{1-\gamma}{\gamma} f_t^{-\gamma} \frac{\partial f_t}{\partial t} (x+g_t)^{\gamma} + f_t^{1-\gamma} (x+g_t)^{\gamma-1} \frac{\partial g_t}{\partial t} - \mu_t \frac{1}{\gamma} f_t^{1-\gamma} (x+g_t)^{\gamma} \\
+ \quad & \frac{1}{\gamma} w_t^{1-\gamma} \left(\frac{w_t}{f_t}(x+g_t) \right)^{\gamma} - \left(\frac{w_t}{f_t}(x+g_t) - b_t^{state} \right) f_t^{1-\gamma} (x+g_t)^{\gamma-1} \\
+ \quad & \mu_t \frac{1}{\gamma} k w_t^{1-\gamma} \left(k \frac{\mu_t}{\nu_t} \right)^{\gamma/(1-\gamma)} \frac{w_t^{\gamma}}{f_t^{\gamma}} (x+g_t)^{\gamma} - \nu_t \left(k \frac{\mu_t}{\nu_t} \right)^{1/(1-\gamma)} \frac{w_t}{f_t} (x+g_t) f_t^{1-\gamma} (x+g_t)^{\gamma-1} \\
+ \quad & (r+\nu_t) x f_t^{1-\gamma} (x+g_t)^{\gamma-1} + \frac{1}{1-\gamma} \frac{(\alpha-r)^2}{2\sigma^2} f_t^{1-\gamma} (x+g_t)^{\gamma} = 0.
\end{aligned}
$$

We further rewrite the term $x(x+g_t)^{\gamma-1} = (x+g_t)^{\gamma} - g_t(x+g_t)^{\gamma-1}$ and set the coefficients of the independent x terms of order $(x+g_t)^{\gamma-1}$ and $(x+g_t)^{\gamma}$ to 0, i.e.,

$(x+g_t)^{\gamma-1}$:

$$
f_t^{1-\gamma} (x+g_t)^{\gamma-1} \frac{\partial g_t}{\partial t} + b_t^{state} f_t^{1-\gamma} (x+g_t)^{\gamma-1} - (r+\nu_t) g_t f_t^{1-\gamma} (x+g_t)^{\gamma-1} = 0
$$

$(x+g_t)^{\gamma}$:

$$
\begin{aligned}
& \frac{1-\gamma}{\gamma} f_t^{-\gamma} \frac{\partial f_t}{\partial t} (x+g_t)^{\gamma} - \mu_t \frac{1}{\gamma} f_t^{1-\gamma} (x+g_t)^{\gamma} + \frac{1}{\gamma} w_t^{1-\gamma} \left(\frac{w_t}{f_t}(x+g_t) \right)^{\gamma} \\
- \quad & \frac{w_t}{f_t} f_t^{1-\gamma} (x+g_t)^{\gamma} + \mu_t \frac{1}{\gamma} k w_t^{1-\gamma} \left(k \frac{\mu_t}{\nu_t} \right)^{\gamma/(1-\gamma)} \frac{w_t^{\gamma}}{f_t^{\gamma}} (x+g_t)^{\gamma} \\
- \quad & \nu_t \left(k \frac{\mu_t}{\nu_t} \right)^{1/(1-\gamma)} \frac{w_t}{f_t} f_t^{1-\gamma} (x+g_t)^{\gamma} + (r+\nu_t) f_t^{1-\gamma} (x+g_t)^{\gamma} + \frac{1}{1-\gamma} \frac{(\alpha-r)^2}{2\sigma^2} f_t^{1-\gamma} (x+g_t)^{\gamma} = 0
\end{aligned}
$$

and obtain respectively

$$
g_t = \int_t^{\widetilde{T}} {}_s p_{y+t} \, e^{-r(s-t)} \, b_s^{state} ds, \tag{3.B.4}
$$

$$
f_t = \int_t^{\widetilde{T}} ({}_s \tilde{p}_{y+t})^{1/(1-\gamma)} ({}_s p_{y+t})^{-\gamma/(1-\gamma)} e^{\gamma/(1-\gamma)\varphi(s-t)} \left[w_s \left(1 + \left(k \frac{\mu_s}{\nu_s^{\gamma}} \right)^{1/(1-\gamma)} \right) \right] ds, \tag{3.B.5}
$$

where ${}_s p_{y+t} = e^{-\int_t^s \nu_{y+\tau} d\tau}$ and $\varphi = r + \dfrac{(\alpha-r)^2}{2\sigma^2(1-\gamma)}$. Beginning the determination of the three optimal controls, the total optimal benefits and the optimal size of the death sum are given by:

$$
\begin{aligned}
\frac{\partial}{\partial B} \quad : \quad & w_t^{1-\gamma} (B_t + b_t^{state})^{\gamma-1} - \frac{\partial V(t,x)}{\partial x} = 0 \\
\Rightarrow \quad & B_t^* + b_t^{state} = \frac{w_t}{f_t}(x+g_t) = 1/\bar{a}_t^* \, (x+g_t),
\end{aligned} \tag{3.B.6}
$$

$$
\begin{aligned}
\frac{\partial}{\partial Beq} \quad : \quad & \mu_t k w_t^{1-\gamma} Beq_t^{\gamma-1} - \nu_t \frac{\partial V(t,x)}{\partial x} = 0 \\
\Rightarrow \quad & Beq_t^* = \left(k \frac{\mu_t}{\nu_t} \right)^{1/(1-\gamma)} \frac{w_t}{f_t}(x+g_t) = \left(k \frac{\mu_t}{\nu_t} \right)^{1/(1-\gamma)} 1/\bar{a}_t^* \, (x+g_t),
\end{aligned} \tag{3.B.7}
$$

where

$$\bar{a}_t^* \quad = \quad \int_t^{\widetilde{T}} e^{-\int_t^s \left(\bar{\mu}_\tau + \bar{r}\right) d\tau} \left(1 + \left(k \frac{\mu_s}{\nu_s^\gamma}\right)^{1/(1-\gamma)}\right) ds, \tag{3.B.8}$$

and $\bar{r} = \frac{1}{1-\gamma}\rho - \frac{\gamma}{1-\gamma}\varphi$ and $\bar{\mu}_t = \frac{1}{1-\gamma}\mu_t - \frac{\gamma}{1-\gamma}\nu_t$. The optimal proportion of the savings invested in the mutual fund is given by

$$\frac{\partial}{\partial \Pi} : \ (\alpha - r)x \frac{\partial V(t,x)}{\partial x} + \Pi_t \sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} = 0 \qquad \Rightarrow \qquad \Pi_t^* = \frac{\alpha - r}{(1-\gamma)\sigma^2} \frac{x + g_t}{x}, \tag{3.B.9}$$

and the proportions between the risky assets in the mutual fund are specified by the mutual fund theorem, see Merton (1969) and Richard (1975):

$$\forall_{i=1,\ldots,N-1} \quad \theta_i = \frac{\sum_{j=1}^{N-1} [\sigma_{ij}]^{-1} (\alpha_j - r)}{\sum_{k=1}^{N-1} \sum_{j=1}^{N-1} [\sigma_{kj}]^{-1} (\alpha_j - r)}, \qquad \sum_i \theta_i = 1. \tag{3.B.10}$$

	Real values		Correlations		
Asset class	Long-term rate	Volatility	Cash	Bonds	Stocks
Cash	0.7%	3.80%	1.00	0.30	-0.05
Bonds	2.4%	7.10%		1.00	0.15
Stocks	8.2%	19.7%			1.00

	a_1	b_1	c_1	a_2	b_2	c_2
ν_t	-2.531	123.5	10.57	1.041e+15	660.7	93.88

Table 3.B.1: Statistical properties of the considered asset classes estimated as the historical real values, and constants for the mortality rate model.

Chapter 4

Applying a stochastic financial planning system for an individual: immediate or deferred life annuities?

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Applying a Stochastic Financial Planning System to an Individual: *Immediate or Deferred Life Annuities?*

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Ideally, an individual ought to make her investment decisions in an optimal fashion, with regard to risk tolerances, expected longevity, goals, return expectations, loss aversion, family situations, and other factors. In the real world, however, there are severe limitations to our ability to render optimal decisions.

What are the major barriers? First, individuals rarely have the time or expertise to conduct a proper wealth management exercise. Second, seeking sound advice can be a challenge. Most individuals cannot judge the ability of professionals to help them and rely on a wide variety of "experts", such as advisors of friends, the prestige of the advisor's organization, and so on. Third, the development of an optimal wealth management exercise can be expensive (in time and computations) and difficult. Even listing and ordering goals in a quantifying manner can be a large first step.

Another barrier is the complexity of the products available for individuals. Certainly, there are a massive number of individual securities, mutual funds, and more recent innovations, such as exchange-traded funds (ETFs), available for immediate purchase. Similarly, securities firms have constructed a wide range of annuities that can be purchased by individuals, firms, and family trusts and offices. However, in most cases, individuals do not know which annuity product is suitable for their needs.

To overcome the aforementioned barriers, in this article we propose a formal asset and liability management system (ALM-I) for an individual investor. While the focus is on the choice between different annuities, the model can be readily extended to more general financial planning problems for an individual. The proposed stochastic investment system draws upon advanced portfolio theory for large institutional investors such as global pension plans. A multiperiod, financial optimization model underlies the process.

LITERATURE

Previous research on stochastic financial planning for individual investors has been relatively limited, due to the complexity of the problem and the aforementioned barriers. Research includes the work by Berger and Mulvey [1998], Campbell and Viceira [2002], Dempster and Consigli [1998], Kim et al. [2012], and Konicz et al. [2013]. These efforts can be divided into three fundamental frameworks: 1) Monte Carlo simulation, 2) multistage stochastic programs (MSP), and 3) stochastic control (SC). Ziemba and Mulvey [1998] provide details. Because of the need for simplicity, the majority of systems in real-world operations employ Monte Carlo simulation. Both MSP and SC suffer from the curse of dimensionality—in which the size of the optimization problem is an exponential function of the number of assets and time periods, and the size of the state space. Accordingly, these approaches, while theoretically sound, have not attained many users relative to the potential size of the population in need of assistance. In this article, we show that the MSP framework can be posed and solved with reasonable efficiency, given the power of today's computers, and the recommendations can provide insights that are difficult to attain from the standpoint of a Monte Carlo simulation.

Evaluating annuities in a stochastic financial planning system can be a daunting task in the most general setting. The selection process can be complicated by individual circumstances, including the health of the investor, current interest rates, the needs of the investor's family, the level of payments, optionality of the contracts, and job security, among others. We simplify these issues in our model so that the ALM-I system can be described without too much clutter. Nevertheless, the ensuing recommendations are reasonable, and lessons can be learned from this study.

Asset allocation in the presence of annuity markets has been investigated in several studies. A common approach is to maximize the expected utility—either CRRA (constant relative risk aversion) or Epstein/Zin—over the wealth or consumption at retirement, and potentially with a bequest motive. Several conclusions have been drawn. Yaari [1965] was the first to show that individuals can gain from having an opportunity to invest in annuities in addition to bonds, and that in the case of no bequest motive, she would invest only in annuities. Charupat and Milevsky [2002] derive the optimal allocation between fixed and variable annuities during the payout phase and conclude that the solution is identical to the classical Merton [1969, 1971] solution. Koijen et al. [2011] obtain the optimal allocation between the nominal, inflation-linked, and variable annuities upon retirement; they find that the allocation to nominal annuities has a marginal impact for various risk aversion levels. We are not the first to find out that is optimal to invest at least some savings in deferred life annuities. Milevsky and Young [2007] point out that individuals should annuitize a fraction of wealth as soon as they have an opportunity to do so. In fact, even in the presence of a bequest motive, individuals should hold some annuities, and the proportion of annuities to other assets increases with the higher levels of risk aversion, greater investment volatility, and better health status. Scott et al. [2007] show that even a small fraction of wealth invested in deferred annuities can enable much larger welfare gains than are achieved with similar allocations to immediate annuities. In addition, it can also be shown that the attractiveness of deferred annuities relative to immediate annuities depends on the size of the annuity budget. Other relevant studies include Horneff et al. [2008] and Horneff et al. [2009] who argue that in a portfolio consisting of stocks, bonds, and annuities, the optimal stock fraction exhibits the typical life-cycle pattern; however, instead of shifting from stocks to bonds over time the household should shift to annuities, as these are a close substitute to bonds.

METHODOLOGY

To help the individual make decisions on whether or not to purchase deferred annuity contracts (DA) we build a stochastic optimization model. We propose a multiperiod stochastic programming (MSP). This approach is a general-purpose decision framework with an objective function that can take a variety of forms and numerous realistic constraints in an algebraic form. Stochastic programming is characteristic for operations research, has highly practical applications, and is especially helpful in terms of adding realistic constraints and modeling more complicated processes (see, for instance, Birge and Louveaux [1997], Ziemba and Mulvey [1998], and Mulvey [2004]. The underlying framework depends upon scenario trees that represent the range of possible outcomes for the future uncertainties (Dupačová et al. [2000], Heitsch and Römisch [2009] and Mulvey et al. [2000]). The MSP optimal solution is computed numerically at each node in the scenario tree, for the specified decision variables. The main drawback is that the problem size grows quickly as a function of the number of periods and scenarios, thus we often have to choose between how long the modeling period can be versus how often we make decisions. Taking the entire lifetime of an individual into consideration can be especially challenging in terms of computational feasibility.

Most applications of stochastic programming are published in the operations research journals. See for instance Mulvey et al. [2003], who present an MSP for integrating corporate financial and pension planning, and Mulvey et al. [2008], who expand this work by adding borrowing decisions. Cariño et al. [1998] and Cariño and Ziemba [1998] show the practical implications of MSP by developing a new asset-liability management model for the second-largest Japanese property and casualty insurer, Yasuda Kasai. MSP applied to individual asset liability management has been considered by Berger and Mulvey [1998], Consigli et al. [2012], Kim et al. [2012], and Konicz et al. [2013].

SCENARIO TREE

The main elements of a stochastic planning system are a stochastic scenario generator and an optimization module. The first, and to some degree, the most important element, involves the scenario generator, which drives the underlying stochastic processes. To model possible outcomes for the uncertainties, we generate a scenario tree with different outcomes at each node, as shown on Exhibit 1. Nodes along the tree are uniquely assigned to each period t and denoted by $n \in N_t$. Every node $n \in N_t$, $t > t_0$, has a unique ancestor n^- and children nodes n^+. The nodes with no children are called the leaves. A scenario S^n is a set of all predecessors of a leaf n: n^-, n^{--}, …, n_0, or equivalently, a single branch from the root to the leaf. The number of scenarios in the tree equals the number of leaves. To incorporate the uncertainty of the individual's lifetime in the scenario tree, we define a set of nodes at which the person is alive by $n \in N_t^{alive}$ and at which death occurs by $n \in N_t^{dead}$. These nodes are marked on Exhibit 1 with a white color and with a pattern, respectively. The probabilities associated with each node, pr_n, are consistent with the actuarial mortality tables. Thus, $\Sigma_{n \in N_t^{alive}} pr_n = {}_t p_x$ and $\Sigma_{n \in N_t^{dead}} pr_n = {}_{t-1} p_x q_{x+t-1}$, where ${}_t p_x$ is the probability that an x-year-old individual survives to at least age $x + t$ and ${}_t p_x q_{x+t}$ denotes the probability that an x-year-old individual dies between ages $x + t$ and $x + t + 1$.

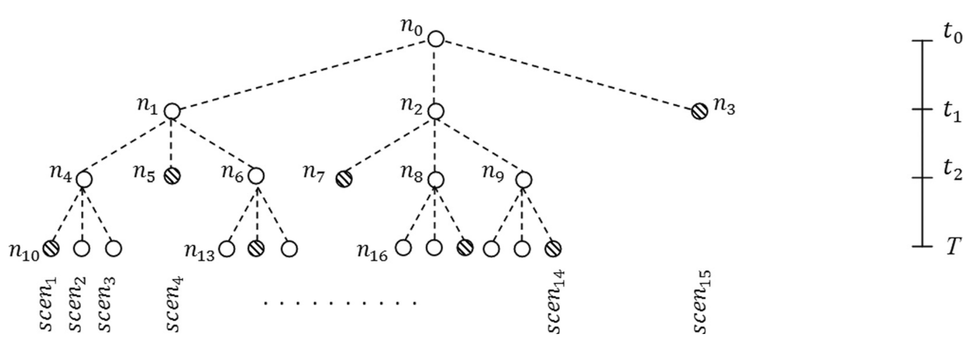
OPTIMIZATION MODULE

The optimization module consists of the objective function and the constraints defined for each node of the tree. A multiperiod stochastic program combines anticipative and adaptive models in a common mathematical framework. Decisions along the tree do not depend on the future observations but anticipate possible future realizations of the random vector. After the random variables have been observed, the decisions for the next period are made and depend both on the realizations of the random vector and the decisions made in the previous stage. For example, the investor specifies the composition of a portfolio by taking into account both possible future movements of asset returns (anticipation) and then she rebalances the portfolio as prices change (adaptation) (Zenios [2007]).

In our study we consider a 45-year-old individual concerned about her retirement income. She is planning to purchase a single life immediate annuity (IA) upon retirement (at age 65), but because she has still 20 years until retirement, she tries to make wise decisions regarding investment of her savings. The main decisions in the optimization problem are: 1) whether to purchase today a deferred life annuity (DA) with the fixed payments starting upon retirement, 2) whether to put

EXHIBIT 1
A Multi-Stage Scenario Tree

A three-period scenario tree with a constant branching factor bf = 3. The person is alive at the white nodes, dead at the colored nodes. The scenarios consist of a leaf node and its predecessors, thus, the scenarios in which the person died have a shorter length. The probabilities of each node are consistent with the mortality rates.



the money in the retirement savings account, where it would be invested in stocks and bonds according to some investment strategy, and 3) whether to partially invest in the DA contracts and the financial assets.

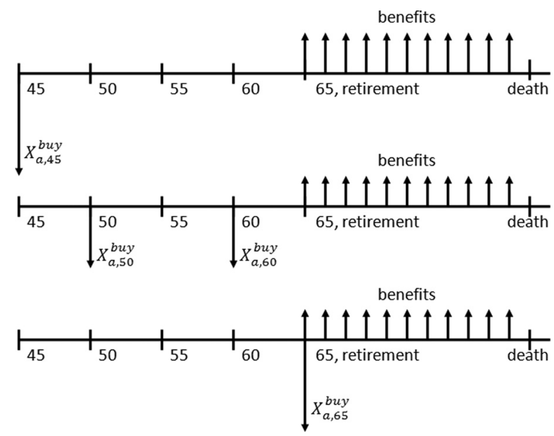
A deferred annuity contract guarantees a stream of payments starting upon retirement, and is an attractive option because it protects against the uncertain future. While the value of a retirement account fluctuates according to market performance, the income from the DA contract is fixed and not affected by future interest rates. One can also argue that purchasing this contract removes the risk of impulsive spending of the retirement account capital, since withdrawal of savings invested is typically not allowed in DA contracts. On the other hand, a single life annuity provides no bequest. Upon death, all the savings invested in this contract are kept by the insurer.

The price of annuities depends on five main parameters: the age and sex/gender of an individual, mortality rates, the interest rate, the length of the deferment, and the load factor. Given the same interest rate, the value of a DA contract with payments starting upon retirement is, at any time during the deferment period, lower than the value of an IA that will be purchased upon the expiration of the deferment period. This is the case because there is a probability that the person dies during the deferment period and the insurer keeps the principal. Therefore, the difference between the prices of these two contracts is determined by the probability of survival from the date of purchase of the DA to the start of payments. In reality, however, the interest as well as the mortality rates change. An increase in either rate would reduce the cost of both annuities. Therefore, if one expects higher rates in the future, she would find the current annuity prices too expensive and would be willing to postpone the purchase. On the contrary, a decrease in both rates would increase the annuity prices, thus the individual might want to purchase annuities as soon as possible. In our study, the interest rate and mortality rates are assumed to be deterministic, thus both annuities and bonds are considered risk-free.

We allow for rebalancing the portfolio every fifth year until retirement. Therefore, if the individual has not spent all her savings on the purchase of DA_{45} (deferred annuity purchased at age 45), she may purchase another deferred annuity five years later, namely DA_{50}, for the amount accumulated during this period on the savings account. Similarly as in the previous contract, DA_{50} will

EXHIBIT 2
Structure of Deferred Annuity (DA) Contracts

The individual makes decisions every fifth year on how much to spend for a purchase of a DA contract guaranteeing the income for life starting upon retirement, $X^{buy}_{a,age}$. The residual amount is invested in stocks and bonds. Upon retirement, the individual spends all the remaining savings on an immediate life annuity (IA).



start payments at age 65. See Exhibit 2 for the structure of the payments corresponding to different annuity contracts. Alternatively, she may revise the investment strategy driving the savings account and rebalance the portfolio optimally. The financial assets held on the account (stocks and bonds) may be traded as required, but the annuity contracts are nonreversible, i.e. once purchased must not be sold. The individual is allowed to rebalance her portfolio again at ages 55 and 60, making decisions on whether to purchase deferred annuities DA_{55} and DA_{60}. Finally, if she is alive upon retirement, she purchases an immediate single life fixed annuity IA_{65} for the amount accumulated on the savings account. If she dies before retirement, the heirs receive the value accumulated on the savings account.

OBJECTIVE

The optimal decisions are naturally driven by the objective function. From an individual's perspective, it sounds reasonable to be interested in maximizing the entire retirement income, or equivalently, the present value of the contract providing the payments. Since in our model, our individual is allowed to make decisions only during the accumulation phase (upon retirement she converts all the savings into an immediate life annuity contract given she is alive), we are interested in the present value of all annuity contracts upon retirement. Additionally, we may add a term to our objective representing a bequest motive. A person with a bequest motive seeks an optimal balance between the savings invested in the deferred annuities and the retirement savings account. Then, upon death, the heirs are protected and receive a bequest amount equal to the value of the savings account.

Finally, we adopt a utility function to understand how the various allocations between the value of all deferred annuity contracts and the value of the savings account would be assessed by people with different levels of risk aversion. We choose a CRRA utility with a constant risk aversion γ reflecting how much risk the individual is willing to undertake in order to gain some additional wealth. The larger the positive values of γ, the larger risk aversion. Risk aversion parameter equal to 0 represents risk neutrality, whereas the values above 5 reflect very risk-averse individuals.

Therefore, the objective function is of the form:

$$\sum_{n\in N_T^{alive}} \frac{1}{1-\gamma}(W_{T,n}+A_{T,n})^{1-\gamma}\cdot pr_n + \sum_{n\in N_T^{dead}} K\frac{1}{1-\gamma}(W_{T,n})^{1-\gamma}\cdot pr_n \tag{1}$$

where $W_{T,n}$ is the value of the savings account upon retirement that will be spent to purchase an IA if the individual is alive, or paid out to the heirs, otherwise, and $A_{T,n}$ is the present value of all the DA contracts purchased in the previous years. Both variables depend on a node $n \in N_T$, which represents different scenarios. Parameter K denotes the weight on the bequest motive relatively to the value of the DA contracts.

In the next section we present the numerical results of our study, while for the interested reader we describe the details on the optimization procedure in the Appendix.

NUMERICAL RESULTS

We have investigated a number of different cases. We start with analyzing 45-year-old males and females who are expected to retire at age 65, and are characterized by different degrees of risk aversion and different bequest motives. Their lifetime expectancies are consistent with Annuity 2000 Basic Mortality Table. For ease of exposition, we chose an initial investment of \$100. Later in this section we also investigate the effects of different subjective lifetime expectancies, additional contributions, and late retirement.

We present the results in terms of means across the scenarios.

Risk Aversion

First we investigate the case for a 45-year-old woman without a bequest motive ($K = 0$). Exhibit 3 shows the value of the portfolio, savings, and the value of all DA contracts upon retirement. In addition, we present the amount spent at the time of the purchase on each DA contract, and the yearly retirement income generated by the deferred and immediate annuity contracts.

The optimal solution suggests that deferred annuities are attractive and worth purchasing for most degrees of risk aversion, however the proportion of savings spent on these contracts varies depending on the degree of risk aversion. The least risk-averse investor ($\gamma = 2$) invests 18% of her initial savings in DA_{45}, which upon retirement constitutes 11% of her yearly retirement income. The biggest proportion of the retirement income comes from IA_{65} and is equal to 67% (see Exhibit 4).[1] During the accumulation the optimal investment strategy suggests a high percentage in stocks (on average 60%) and a small percentage in bonds (starts with 19% and decreases with time to 0). Such an aggressive investment strategy leads on average to the value of the portfolio equal to \$303 upon retirement, which over 20 years gives a return of 6.15% on the initial investment of \$100. The more risk-averse women invest \$48 (for $\gamma = 4$) and \$63 ($\gamma = 6$) of their initial savings in DA_{45} contracts and similarly as the first investor never invest more than 20% in bonds. Upon retirement the income from the immediate annuity consists of 38% and 25% of total retirement income for the moderate and very risk-averse person, respectively.

Bequest Motive

Exhibit 4 also compares the results for both male and female with different risk aversions and with and without bequest motives. The value of the portfolio of a moderate risk-averse male, $\gamma = 4$, with a bequest motive,

EXHIBIT 3

Value of Portfolio, Savings, and All DA Contracts at Retirement, Amount Spent on DA Contracts at Purchase, and Yearly Retirement Income Generated by All Annuity Contracts for Males and Females with Different Degrees of Risk Aversion

	Risk Aversion	Value of Portfolio	Value of Savings	Value of DA Contracts	Amount Spent on DA Contracts at the Time of the Purchase				Yearly Retirement Income
	γ	W_T + A_T	W_T	A_T	DA_45	DA_50	DA_55	DA_60	
Without a Bequest Motive									
	2	$303	$204	$99	$18	$6	$18	$28	$17
Female	4	244	95	149	48	8	18	21	13
	6	223	61	162	63	7	14	15	12
	2	310	201	109	25	14	16	14	18
Male	4	253	92	161	57	13	13	11	15
	6	234	61	174	70	10	10	8	14
With a Bequest Motive									
	2	$304	$248	$55	$14	$6	$8	$9	$17
Female	4	242	196	46	23	2	1	0	13
	6	219	185	34	18	0	0	0	12
	2	317	264	53	18	7	5	1	19
Male	4	249	202	47	23	2	0	0	15
	6	224	191	33	17	0	0	0	13

Notes: Based on a $100 investment at age 45. Numbers are rounded to $1.

is equal to $249 (based on invested $100) and consists of $202 of savings that is converted to the immediate life annuity and $47 in deferred annuity contracts. He purchases DA_{45} for $23, DA_{50} for $2 and, if he is still alive upon retirement, IA_{65} for the value of the savings, $249. His yearly retirement income is $15 and is generated primarily by DA_{45} contract (23%) and IA_{65}. $2 spent on DA_{50} generate less than 1% of his yearly income. If he dies before retirement, he does not purchase the immediate annuity but the value of savings, $249, is paid out to the heirs.

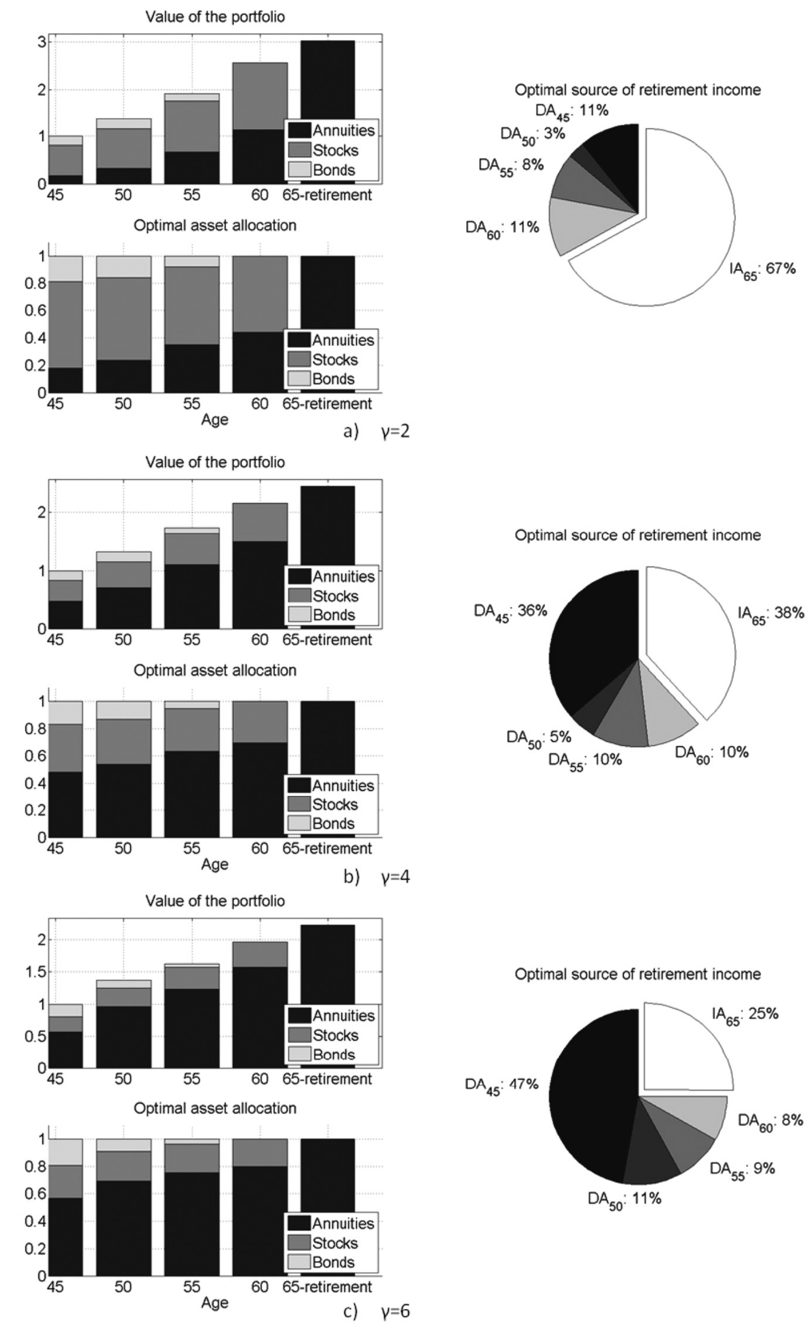
Exhibit 5 shows four of the cases presented in Exhibit 3: female without a bequest motive, female with a bequest motive, male without a bequest motive, and male with a bequest motive, all with a moderate degree of risk aversion ($\gamma = 4$).

In the absence of a bequest motive, the initial amounts that women spent on DA_{45}, stocks, and bonds are $48, $35, and $17, respectively, whereas for men we have $57, $34, and $9. We observe a few interesting patterns. First, despite the absence of a bequest motive, it is optimal for both genders to have a small proportion of bonds in their portfolios. Both bonds and annuities are characterized by a similar level of risk, but the rate of return on life contingent annuities is higher than on bonds. On the other hand, bonds, in contrast to annuities, are liquid. The motive for liquidity in the model is the ability to rebalance the portfolio optimally. For instance, if the stock market crashes in a certain period, the investor's post-crash portfolio may be too conservative for her degree of risk aversion. Then, the only way to rebalance the portfolio in favor of stocks is to sell some of the bonds. We conclude that having bonds in the portfolio gives the individual more flexibility to rebalance the portfolio in the future, but it comes with a cost of the excess return generated by annuities. Another observation is that the men's and women's optimal portfolios differ mostly with the ratio between bonds and annuities. Given the same risk aversion, annuities are slightly more preferable by men, mostly because their life expectancy is on average shorter than for women, which implies the lower prices of annuities. Women value the liquidity of bonds more at the beginning of the deferment period but as they get older the amount invested in bonds decreases to 0. We connect this fact to the probabilities of survival—the annuities become more attractive than bonds as the individuals age.

The decisions change quite a lot if our individual has a bequest motive. Since upon death the heirs receive only the value accumulated on the savings account, the proportion of the savings to the value of all annuity contracts upon retirement is much higher for all the considered cases with a bequest motive. Without a bequest motive, this ratio is around 0.35 for the most conserva-

EXHIBIT 4

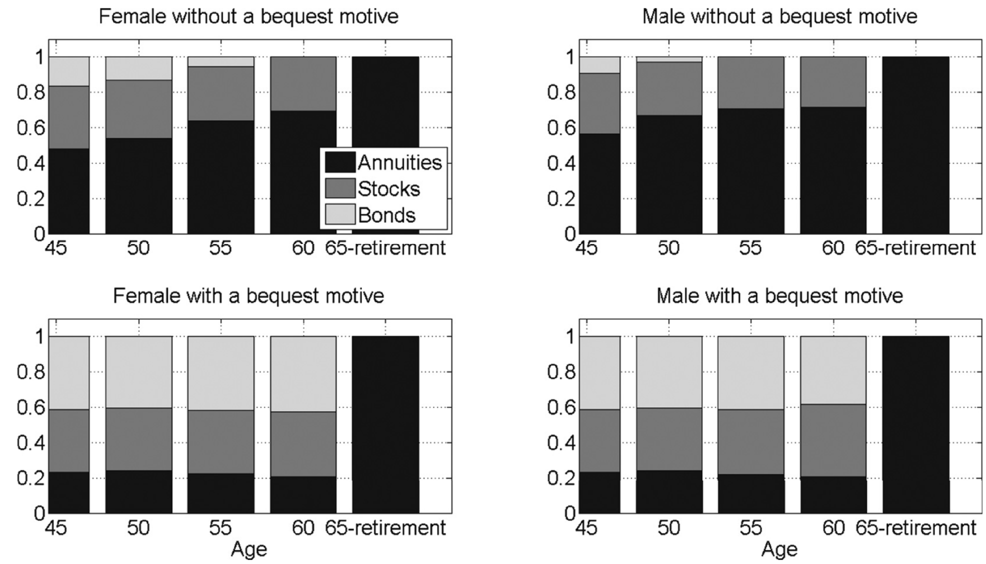
Value of Portfolio, Optimal Asset Allocation, and Optimal Source of Retirement Income According to Risk-Aversion Levels for a 45-Year-Old Woman



Notes: With least risk-averse, γ = 2 (top figure), moderately risk-averse γ = 4 (middle), and very risk-averse γ = 6 (bottom). Assumed no bequest motive, K = 0.

Exhibit 5

The Optimal Asset Allocation for a 45-Year-Old Female and Male (with and without a Bequest Motive)



Notes: Assumed moderate degree of risk-aversion $\gamma = 4$.

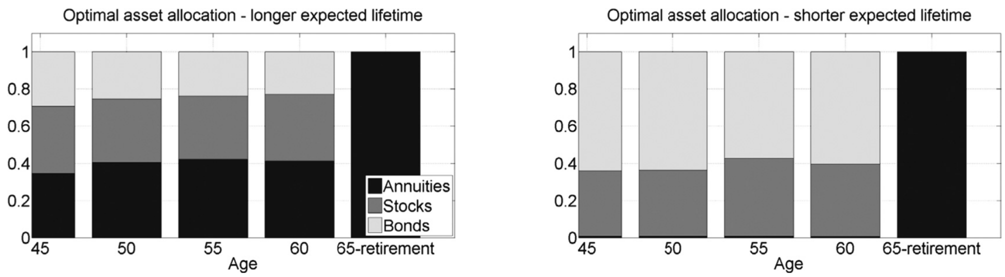
tive investors, 0.6 for moderate risk-averse, and around 2 for the least risk-averse. Choosing $K = 1$ implies the investment decisions that lead to a 4.2–5.7 ratio of savings to annuities, again depending on the risk aversion. As shown on 0, the asset allocation for moderately risk-averse males and females is similar. During the accumulation phase, the savings are invested primarily in DA_{45}, which constitutes 23% of the value of the portfolio and remains almost constant until retirement. DA_{45} is preferable to other deferred annuities because it is the cheapest. The residual amount of savings is invested in stocks and bonds with a similar fixed weight between these two.

Subjective Lifetime Expectancy

Taking account of varying subjective life expectancy will also affect the decisions on purchases of deferred annuity contracts. Exhibits 6 and 7 depict the results for a 45-year-old woman assuming subjectively perceived lifetime expectancies of 75.5 and 100.2 years, moderate risk-aversion, and the presence of a bequest motive. These numbers can be compared with our benchmark case (Exhibit 3, Row 8), where we assumed the life expectancy of 86.9 years. Clearly, the expected lifetime affects the women's decisions. A woman who expects to die almost 12 years earlier than the average will not invest in deferred annuities at all. To be more accurate, the results show \$1 invested in D_{45}, which is negligibly small compared to the total value of savings. This result is intuitively obvious. The individual with both a short lifetime expectancy and a bequest motive would protect the heirs and allocate all the savings in other financial assets. The allocation between the stocks and bonds is approximately fixed and defined by the risk aversion. Thus, for $\gamma = 4$, the female allocates almost 40% of total wealth to stocks and 60% to bonds during the accumulation phase. The optimal solution for a female who expects to live to 100 suggests a higher percentage in DA contracts than in the benchmark case with a life expectancy of 86.9 years. Upon retirement the entire portfolio consists of 40% in annuities, 38% in stocks, and 22% in bonds. Interestingly, both investment strategies, despite being so different, generate a similar yearly retirement income: \$14 and \$13 for a shorter and longer expected lifetime, respectively, which corresponds to 5.7% and 5.3% of the value of the portfolio upon retirement.

EXHIBIT 6

Optimal Asset Allocation for a 45-Year-Old Female with Different Expected Lifetimes



Notes: Based on 75.5 years(right) and 100.2 years (left), moderate degree of risk aversion $\gamma = 4$, bequest motive $K = 1$.

EXHIBIT 7

Value of Portfolio, Savings, and All DA Contracts at Retirement, Amount Spent on DA Contracts at Purchase, and Yearly Retirement Income Generated by All Annuity Contracts

	Expected Lifetime	Value of Portfolio W_T + A_T	Value of Savings W_T	Value of DA Contracts A_T	Amount Spent on DA Contracts at the Time of the Purchase DA_45	DA_50	DA_55	DA_60	Yearly Retirement Income
Risk Aversion = 4, With a Bequest Motive (K=1)									
Female	75.5	$245	$244	$2	$1	$0	$0	$0	$14
	100.2	244	154	90	35	7	7	6	13

Notes: Based on a $100 investment at age 45 for a female with moderate risk aversion $\gamma = 4$, bequest motive $K = 1$, and different expected lifetime. The numbers are rounded to $1.

Additional Contributions

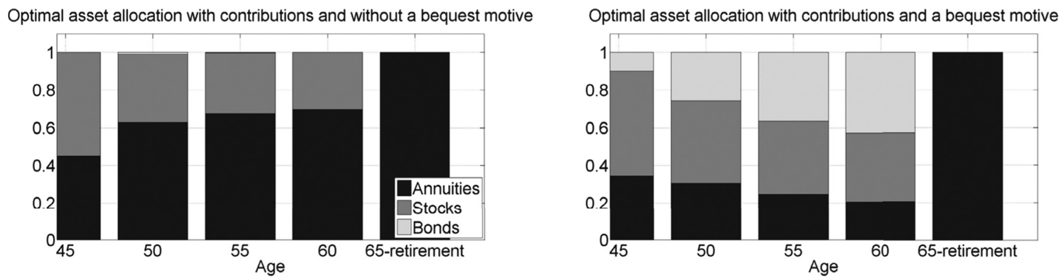
Another interesting topic to investigate is how the investment decisions would change if we allow for additional contributions during the accumulation phase. Thus, at age 45, our individual still has $100 to invest in annuities, stocks, and bonds, but every fifth year, when she rebalances the portfolio, she contributes an additional $25. The results are presented in Exhibits 8 and 9. The individual without a bequest motive spends all her contributions on the DA contracts, whereas the individual with a bequest motive is still reluctant to invest in the DA contracts other than D_{45}. Notice that upon retirement the ratio of the value of savings to the value of DA contracts is similar to the case without additional contributions—0.6 without a bequest motive and 4.3 with a bequest motive. For the individual without a bequest motive, this ratio remains similar despite the fact that she spends so much money on the DA contracts. The additional contributions allow the individual to keep a higher amount in stocks during the deferment period (she does not need to sell them to purchase deferred annuities), which results in the faster growth of the value of savings relative to the value of annuities. Also, in the absence of a bequest motive, the individual no longer invests in bonds. We conclude that $25 paid each period to the savings account is sufficient in terms of liquidity. Finally, a comparison between Exhibit 5 (top and bottom, left) and Exhibit 8 show that, independent of a bequest motive, the investment strategy is more aggressive during the first 10 years of the accumulation phase, which stems from the fact that there is no uncertainty related to the additional contributions. The yearly retirement income is naturally much higher and equal to $21 as opposed to $13 without contributions.

Late Retirement

Finally, in our last study we test how the optimal solution differs for a different initial age. Thus, we choose a 65-year-old individual who is still active and

EXHIBIT 8

Optimal Asset Allocation for a 45-Year Old Female Contributing $25 at Ages 50, 55 and 60, with a Moderate Degree of Risk Aversion



Notes: $\gamma = 4$*, without (left) and with (right) a bequest motive.*

EXHIBIT 9

Value of Portfolio, Savings, and All DA Contracts at Retirement, Amount Spent on DA Contracts at Purchase, and Yearly Retirement Income Generated by All Annuity Contracts, Based on a $100 Investment at Age 45 and Additional Contributions of $25 at Ages 50, 55 and 60, for a Female With Moderate Risk Aversion ($\gamma = 4$), with and Without a Bequest Motive

	Bequest Motive	Value of Portfolio W_T + A_T	Value of Savings W_T	Value of DA Contracts A_T	Amount Spent on DA Contracts at the Time of the Purchase DA_45	DA_50	DA_55	DA_60	Yearly Retirement Income
Contributions of $25 at Ages 50, 55, 60; Risk Aversion = 4									
Female	0	$382	$150	$232	$45	$33	$38	$40	$21
Female	1	380	308	72	34	5	2	0	21

Notes: Numbers are rounded to $1.

plans to retire at age 85. She can invest in deferred life annuities which guarantee payments starting at age 85, i.e., DA_{65}, DA_{70}, DA_{75}, DA_{80}. Upon retirement, if she is still alive, she converts the value of the savings account to purchase an immediate life annuity IA_{85}.

Exhibits 10 and 11 show that without a bequest motive, the individual invests a much higher proportion of the savings in the DA contracts (69%) than a 45-year-old (48%), and at age 80 her portfolio consists primarily of DAs. In a case with a bequest motive, the optimal allocation to annuities is similar, whereas the allocation between stocks and bonds changes. (Compare with Exhibit 5, bottom left panel). Interestingly, the female increases the proportion allocated to stocks as she approaches retirement. This fact can be explained by her exceptionally late retirement age. Investing only 21% of the initial savings at age 65 in DA_{65} which starts payout at age 85, generates almost two and a half times more retirement income than investing the same amount in DA_{45} at age 45, which starts payout at age 65; therefore the investor is willing to invest more aggressively in the stock market in order to increase the value of her savings that would be inherited by the heirs if she died before age 85.

SUMMARY AND FUTURE WORK

In this article we have built a stochastic financial planning system that provides guidelines on whether or not to invest in deferred life annuities, and if so, who would benefit the most from purchasing them.

Our study shows that most individuals would benefit from purchasing deferred annuity contracts and should purchase them as early as possible. Especially those without a bequest motive should invest a proportion of their savings in deferred annuities every time

Exhibit 10

The Optimal Asset Allocation for a 65-Year-Old Female Expected to Retire at Age 85, with a Moderate Degree of Risk Aversion (γ = 4), with and without a Bequest Motive

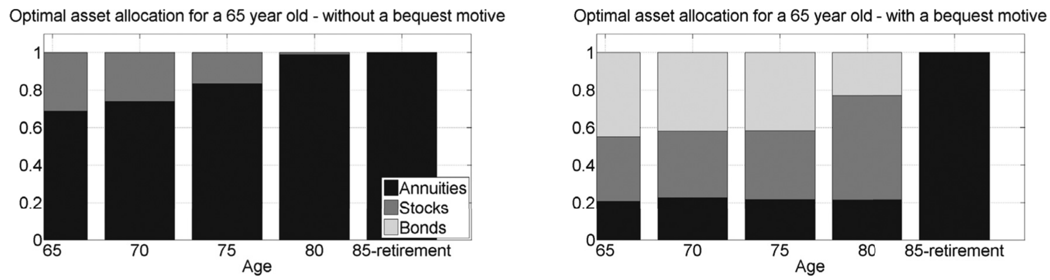


Exhibit 11

Value of Portfolio, Savings, and All DA Contracts at Retirement, Amount Spent on DA Contracts at Purchase, and Yearly Retirement Income Generated by All Annuity Contracts, Based on a $100 Investment at Age 65 for a Female Expected to Retire at Age 85 with Moderate Risk Aversion (γ = 4), with and without a Bequest Motive

	Bequest Motive	Value of Portfolio W_T + A_T	Value of Savings W_T	Value of DA Contracts A_T	Amount Spent on DA Contracts at the Time of the Purchase DA_65	DA_70	DA_75	DA_80	Yearly Retirement Income
Current age 65, Retirement age 85, Risk Aversion = 4									
Female	0	$302	$57	$246	$69	$9	$20	$40	$36
Female	1	274	223	51	21	2	1	0	32

Notes: Numbers are rounded to $1.

they rebalance the portfolio. The allocation changes based on the degree of risk aversion—the most aggressive individuals should allocate less to these contracts, such as 20%–25%, whereas the conservative individuals should allocate up to 60%–70% of their savings. These proportions are higher for those who want to postpone their retirement to a much older age.

The presence or absence of a bequest motive has a very important bearing on the recommendations of this study and shows that the individuals with a strong bequest motive would be rather reluctant to purchase deferred annuities. They prefer to accumulate savings on their retirement account so that their heirs are protected in case of death. However, even with a bequest motive, individuals would spend 15%–25% of savings on the deferred annuity at the initial stage, and for the rest of the accumulation period they would invest primarily in other financial assets.

The only case where the individuals should not invest in DAs is when they expect to die much earlier than the average individual and have a bequest motive.

Finally, we argue that DA contracts are more preferred by men than by women, simply because their life expectancies are shorter, which implies cheaper annuity prices.

Nowadays, one can find a fairly wide choice of lifetime income products in the market. In our study we have focused only on a single "life only" annuity; therefore it would be of interest to extend this paper by including other annuity options, such as "10-year certain" annuities, "joint and survivor" annuities, variable payout annuities, or annuities with different deferment periods, interest rate, and mortality assumptions.

Appendix

The following procedure explains the technicalities behind the stochastic optimization problem.

1. A scenario tree structure is created. The probability of the nodes at which death occurs is consistent with the mortality rates based on Annuity 2000 Basic Mortality

Table and include the mortality improvements given by Projection Scale G for both men and women.

Based on the scenario tree structure, the returns for stocks are generated. Bond returns are constant for each node. We use the moment matching approach described in (Høyland and Wallace [2001]), where we match the first four moments of a normal distribution given the probabilities defined in step 1. Alternatively, the returns can be generated by applying other methods, see for instance (Kaut and Wallace [2005]) for the guidelines on which scenario generation methods are suitable for a given problem. For simplicity, we assume two asset classes: bonds and stocks. Bonds are assumed to be long-term with an expected return of 3%, and stocks are assumed to be normally distributed with expected return of 8.2% and volatility 19.7%. The return on bonds is also used as a discounting factor for calculating the prices of the annuities.

2. The annuity prices are calculated according to the actuarial formula, (Dickson et al. [2009]):

$$ {}_{t|}\ddot{a}_x = v_t^t\, p_x \ddot{a}_{x+t}, \quad \ddot{a}_x = \sum_{k=0}^{\infty} v^k\, {}_k p_x \tag{A-1} $$

where in the actuarial notation ${}_{t|}\ddot{a}_x$ denotes a deferred annuity payable to an individual now aged x under which annual payments of 1 will commence at age $x+t$, and $\ddot{a}_x$ is a whole-life annuity-due, i.e., an annuity of 1 per year payable annually in advance throughout the lifetime of an individual now aged x, $v^t = \left(\frac{1}{1+r}\right)^t$ is a discounting factor, and ${}_t p_x$ is the probability that an x-year-old individual survives at least until age $x+t$. The load factor is assumed to be 0.

3. The annuity prices and the generated scenario tree with the asset returns and death events are the most important input data to our model. Based on these, we can formulate the optimization module, so that at each node of the tree, the individual makes a decision regarding the purchase of deferred annuities and the financial assets: stocks and bonds, such that the objective function defined in Equation (A-1) is maximized.

4. A full description of the parameters, decision variables, and constraints is listed below. We have used optimization software GAMS with a nonlinear solver CONOPT to formulate and solve the model, and Matlab to calculate all the necessary input parameters. The scenario tree consists of 5 periods covering in total 20 years. The number of branches at each node is constant and equal to 5, thus in each scenario tree we have $5^4 = 625$ different scenarios. Additionally, we reran the program for 10 different scenario trees to make sure there is enough uncertainty. The results presented in the tables and figures are the means across the nodes associated with a given period and across all the scenario trees.

For each period $t \in [t_0, T]$, node $n \in \left\{N_t^{alive}, N_t^{dead}\right\}$, and asset class $i \in I = \{1,2,3\}$={cash, bonds, stocks}, we define the following parameters and decision variables.

Parameters

T	retirement time
$r_{i,t,n}$	return on asset i at node n corresponding to stage t
x_o	initial savings
c_t	contributions; $c_t > 0$ for the case described in the paragraph "Additional contributions"; otherwise $c_t = 0$
K	weight on bequest motive
γ	risk aversion
pr_n	probability of being at node n
${}_{T-t\|}\ddot{a}_{x+t}$	value at time t of a deferred life annuity-due contract starting payouts upon retirement T to an individual who is $x+t$ years old at the time of purchasing.

Decisions

$X_{i,t,n}^{\rightarrow}$	amount allocated to asset class i, at the beginning of period t, at node n, before rebalancing
$X_{a,t,n}^{\rightarrow}$	amount allocated to deferred annuity at the beginning of period t, at node n, before rebalancing
$X_{i,t,n}$	amount allocated to asset class i, at the beginning of period t at node n, after rebalancing
$X_{a,t,n}$	amount allocated to deferred annuity at the beginning of period t, at node n, after rebalancing
$X_{i,t,n}^{buy}$	amount of asset class i purchased for rebalancing in period t, at node n
$X_{a,t,n}^{buy}$	amount spent for a purchase of deferred annuity for rebalancing in period t, at node n
$X_{i,t,n}^{sell}$	amount of asset class i sold for rebalancing in period t, at node n
$B_{t,n}^{def}$	benefits guaranteed from a deferred annuity contract purchased in period t, node n
$W_{T,n}$	value of savings allocated to all asset classes
$A_{T,n}$	value of all deferred annuities purchased in the past.

The contract starts with an initial capital of x_0, thus, $X_{1,t_0,n_0}^{\rightarrow} = x_0, X_{i,t_0,n_0}^{\rightarrow} = 0, \forall_{i \neq 1}$, and $X_{a,t_0,n_0}^{\rightarrow} = 0$.

The objective function (Equation A-2), which we aim to maximize, consists of the expected utility of the present value of all deferred annuity contracts purchased during the accumulation phase and providing the payments for life plus the value of the savings upon retirement, and the expected utility of bequest equal to the value of the savings upon retirement. The budget constraint (Equation A-3) specifies the cash flows accompanying the savings account. At the beginning of each period t, the value of the cash account is equal to its value from the last period plus any incoming payments, i.e., the amount gained from the sales of the securities and the additional pension contributions c_t, minus the outgoing payments, i.e. the amount spent on the purchase of new securities. We assume no interest on the cash account ($r_{1,t,n} = 0$), which implies that the investor will never keep any savings in cash, but distribute them among stocks, bonds, and annuities, i.e., $X_{1,t,n} = 0$, $\forall t > t_0$. The next constraints, Equations (A-4) and (A-5), define the asset inventory balance. We first account for the returns earned during the last period as shown in Equation (A-6) and then rebalance the amount by purchasing or selling the given asset. The annuity contracts are nonreversible, thus they must not be sold. The present value of all purchased annuities, Equation (A-7), is calculated as a sum of the present values of all deferred annuities purchased in the previous periods that belong to the same scenario S^n. The next constraint (Equation (A-8)) defines the level of yearly income generated by each deferred annuity.

The value of savings upon retirement is defined in Equation (A-9), and the value of the purchased deferred annuities in Equation (A-10). Constraints in Equations (A-11) and (A-12) define the limits on portfolio composition, i.e., gearing and short-selling of any asset is not allowed. Finally, we include Equation (A-13) to distinguish between the purchases and sales and to ensure that once purchased annuities cannot be sold.

$$\sum_{n\in N_T^{alive}} \frac{1}{1-\gamma}(W_{T,n} + A_{T,n})^{1-\gamma}\cdot pr_n + \sum_{n\in N_T^{dead}} K\frac{1}{1-\gamma}(W_{T,n})^{1-\gamma}\cdot pr_n \tag{A-2}$$

$$X_{1,t,n} = X_{1,t,n}^{\rightarrow} + \sum_{i\neq 1} X_{i,t,n}^{sell} - \sum_{i\neq 1} X_{i,t,n}^{buy} - X_{a,t,n}^{buy} + c_t, \quad t\in\{t_0, \ldots, T-1\},\ n\in N_t^{alive} \tag{A-3}$$

$$X_{i,t,n} = X_{1,t,n}^{\rightarrow} + X_{i,t,n}^{buy} - X_{i,t,n}^{sell}, \quad t\in\{t_0, \ldots, T-1\},\ n\in N_t^{alive},\ i\in I\setminus\{1\} \tag{A-4}$$

$$X_{a,t,n} = X_{a,t,n}^{\rightarrow} + X_{a,t,n}^{buy}, \quad t\in\{t_0, \ldots, T-1\}, n\in N_t^{alive} \tag{A-5}$$

$$X_{i,t,n}^{\rightarrow} = (1+r_{i,t,n})X_{i,t-,n-,} \quad t\in\{t_1, \ldots, T-1\}, n\in N_t^{alive+dead}, i\in I \tag{A-6}$$

$$X_{a,t,n}^{\rightarrow} = \sum_{n'\in S^n} B_{t',n'}^{def}\cdot {}_{T-t|}\ddot{a}_{x+t}, \quad t\in\{t_0,\ldots,T\}, n\in N_t^{alive} \tag{A-7}$$

$$B_{t,n}^{def} = \frac{X_{a,t,n}^{buy}}{{}_{T-t|}\ddot{a}_{x+t}}, \quad t\in\{t_0,\ldots,T-1\}, n\in N_t^{alive} \tag{A-8}$$

$$W_{T,n} = \sum_i X_{i,T,n}^{\rightarrow}, \quad n\in N_T^{alive+dead} \tag{A-9}$$

$$A_{T,n} = X_{a,t,n}^{\rightarrow}, \quad n\in N_T^{alive} \tag{A-10}$$

$$X_{i,t,n} \le \sum_i X_{i,t,n}, \quad t\in\{t_0, \ldots, T-1\}, n\in N_t^{alive},\ i\in I \tag{A-11}$$

$$X_{i,t,n} \ge 0, \quad t\in\{t_0, \ldots, T-1\}, n\in N_t^{alive},\ i\in I \tag{A-12}$$

$$X_{i,t,n}^{buy}\ge 0,\ X_{i,t,n}^{sell}\ge 0,\ X_{a,\tau,n}^{buy}\ge 0, \quad t\in\{t_0, \ldots, T-1\},\ n\in N_t^{alive},\ i\in I \tag{A-13}$$

ENDNOTES

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[1]Note that in all the exhibits, the optimal asset allocation upon retirement shows 100% in annuities. It is an assumption in our model that, if the individual is alive upon retirement, she annuitizes all her savings. It should not be interpreted to mean that it is optimal to annuitize all the savings upon retirement.

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Erratum

The following is a list of typos found in the published version of “Applying a stochastic financial planning system to an individual: immediate or deferred life annuities?”

Page	Wrong	Correct
104	return of 6.15%	yearly return of 5.7%
107	As shown on 0	As shown on Exhibit 3
111	defined in Equation (A-1)	defined in Equation (A-2)
112, Eq. (A-6)	$t \in \{t_1 \ldots, T-1\}$	$t \in \{t_1 \ldots, T\}$
112, Eq. (A-10)	$A_{T,n} = X^{\rightarrow}_{a,t,n}$	$A_{T,n} = X^{\rightarrow}_{a,T,n}$

Chapter 5

Optimal annuity portfolio under inflation risk

Submitted

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Optimal annuity portfolio under inflation risk

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Abstract

The paper investigates the importance of inflation-linked annuities in retirement planning for individuals with uncertain lifetime. Given the investment opportunities in nominal, inflation-linked, and variable annuities, as well as in bonds and stocks, we search for the consumption and investment decisions under two different objective functions: 1) maximization of the expected CRRA utility function of real consumption, and 2) minimization of the expected quadratic loss function penalizing any deviations from an inflation-adjusted target. When optimizing the objective, we allow for rebalancing the portfolio during retirement by buying additional annuities and by trading bonds and stocks. To find the optimal solution, we apply a multi-stage stochastic programming approach. Our findings indicate that independently of the considered objective function and risk aversion, real annuities are a crucial asset in every portfolio. In addition, without investing in real annuities, the retiree has to rebalance the portfolio more frequently, and still obtains a lower and more volatile real consumption.

Keywords: Inflation-linked annuity • Retirement planning • CRRA utility • Loss disutility • Multi-stage stochastic programming

5.1 Introduction

When planning for retirement, individuals often decide to purchase nominal annuities that guarantee lifelong fixed income. However, what they tend to forget is that these products expose them to substantial uncertainty of the real value of their income. In the event of high inflation, the purchasing power of income may not be sufficient to cover standard living costs.

To accommodate the individuals' needs, during the last decades a new type of financial products became available in the market; namely, inflation-linked (real) annuities. The income provided by these products increases each year with the Retail Price Index (RPI) or Consumer Price Index (CPI) measuring inflation, thus giving the annuitants a natural protection against inflation. As investigated in Brown et al. (2000, 2001), real annuities are available at least in the British and American market, and are offered especially by private insurers.

Many scholars have investigated the demand for the inflation-linked products, mostly in the expected utility maximization framework, see, e.g. Fischer (1975), Brown et al. (2001), Campbell and Viceira (2001), Brennan and Xia (2002), Soares and Warshawsky (2003), Koijen et al. (2011), Han and Hung (2012), Peijnenburg et al. (2013), and Kwak and Lim (2014). These studies have shown that the availability of inflation-linked assets is beneficial for individuals. Moreover, Fischer

(1975) and Kwak and Lim (2014) argue that the demand for inflation-linked bonds increases with a decline in the correlation between stocks and inflation rate; Soares and Warshawsky (2003) argue that the prices of inflation-linked annuities are much less volatile than the prices of nominal fixed and increasing annuities; whereas Campbell and Viceira (2001), Koijen et al. (2011) and Han and Hung (2012) show that inflation-linked products are most beneficial for conservative investors. In addition, Koijen et al. (2011) argue that in the presence of real annuities, independently of the risk aversion level, individuals allocate only a marginal amount to nominal annuities, whereas Kwak and Lim (2014) prove that inflation-linked bonds are not only a perfect hedge for inflation risk, but also serve as investment opportunity and portfolio diversification.

Despite these scientific findings in favour of inflation-linked products, as recently investigated in the UK, about 95% of retirees do not purchase inflation-linked annuities, see Cowie (2011) and Towler (2013). British pension experts explain that retirees' reluctance towards real annuities is related to the high price of these products. Because insurers do not know how high inflation will be in the future, they price real annuities more conservatively than nominal annuities. Therefore, it may take many years before the payments from real annuities exceed those from nominal annuities with constant payments. To exemplify, Hyde (2013) shows that the initial income from inflation-linked annuities can even be 40% lower than from annuities with constant nominal payments. Given a realized inflation of 3% per year, a retiree would have to wait until age 82 before the lower payouts from the real annuities exceeded those from the nominal annuities with constant payments. Accordingly, given the lifetime expectancy of nearly 90 years, individuals find this waiting period too long, and prefer to take the risk of experiencing a severe shrinkage in their income.

On the other hand, retirees who are concerned about inflation risk, often prefer to invest in stocks. According to a common belief, stocks are a natural hedge against inflation. However, several empirical studies have reported a negative correlation between the stock returns and the inflation rate, see, e.g., Fama (1981), Geske and Roll (1983), Lee (1992), and Brown et al. (2001). Moreover, Attié and Roache (2009) show that assets such as commodities, which are known to be an effective hedge against inflation in short-term investment strategies, may not work over longer horizons. Thus, retirees owning stocks or variable annuities (annuities whose income is linked to stock returns) should not feel certain about the real value of their income.

The purpose of this paper is threefold: 1) to analyze how important inflation-linked products are for individuals with different retirement goals, 2) to investigate which parameters and constraints affect the decisions of inflation concerned individuals the most, and 3) to investigate how the retirees can protect themselves against inflation risk without investing in inflation-linked products. To support our argumentation, we suggest an optimization model whose solution provides recommendations regarding how to ensure the retirement income in real terms.

As the main goal of retirement planning is to make sure that the life-long income is sufficient to cover standard living costs independently of the level of inflation, we model the optimization problem under two different objective functions. The first objective is a maximization of the expected constant relative risk aversion (CRRA) utility of real consumption, as investigated, e.g., in

Brown et al. (2001), Koijen et al. (2011), and Peijnenburg et al. (2013). The second objective is a minimization of the expected disutility, defined by a quadratic loss function penalizing deviations from a certain inflation-linked target. Neither the quadratic disutility nor the target-based approach is new in the literature on retirement planning, see, e.g., Cairns (2000), Gerrard et al. (2004), Blake et al. (2013), and Di Giacinto et al. (2014), however, none of the studies considered inflation risk or the presence of inflation-linked products, and only Blake et al. (2013) allowed for a path-dependent target. When searching for the optimal investment and consumption decisions under both objective functions, we allow the individual to allocate his savings to nominal annuities, inflation-linked annuities (RPI-adjusted), variable annuities and stocks (whose returns are linked to the MSCI UK stock index), and long-term bonds.

Most of the aforementioned studies focus on deriving an explicit solution to a given problem by using dynamic programming. However, solving a problem of wealth allocation to nominal, real and variable annuities is too complicated for this approach. While Koijen et al. (2011) and Peijnenburg et al. (2013) use a simulation-based approach, we apply multi-stage stochastic programming (MSP). This approach is common in financial engineering and operations research, and is especially handy when it comes to incorporating realistic constraints into the model, see, e.g., Carino et al. (1998), Mulvey et al. (2008), Ferstl and Weissensteiner (2011), and Konicz et al. (2014a). By choosing the MSP approach, we are able to find the optimal dynamic strategy consisting of the annuities, stocks and bonds, thus, in contrast to the mentioned simulation-based studies, we do not only allow for rebalancing the portfolio, but also expand the investment opportunities by adding liquid assets (bonds and stocks). We further assume that annuities are irreversible (once purchased they can never be sold), therefore rebalancing decisions concern purchases of annuities, and purchases and sales of bonds and stocks. MSP also allows us to investigate the optimal strategy under different practical constraints such as liquidity preferences or exclusion of certain products. For example, we can explore how the optimal allocation changes when the individual prefers to hold some of his savings in liquid assets, and whether any of the products providing nominal income are able to hedge against inflation.

In addition, our study contributes to the strand of literature developing scenario generation methods for multi-stage stochastic programs under inflation risk. The prices and the cash-flows from the annuities are stochastic and vary with the development of the inflation index, stock returns, and nominal and real yield curves. We model jointly these three sources of uncertainty with a vector autoregressive VAR(1) process, thus we can fully explore time-varying investment opportunities.

The remainder of the paper is organized as follows. Section 5.2 introduces two different objective functions that take into account inflation risk. Section 5.3 explains the main characteristics of the products that individuals can purchase to optimize their objective, and Section 5.4 describes how we model the uncertainty of the stock returns, the inflation index, and the real and nominal term structure necessary to price and to calculate the cash-flows from the considered products. Section 5.5 presents the multi-stage stochastic programming approach, in particular the concept of a scenario tree and the formulation of the optimization problems. Section 5.6 illustrates and analyzes the optimal solution, and Section 5.7 concludes and suggests future work.

5.2 Objective

To investigate the importance of inflation-linked annuities in retirement planning, we explore two optimization problems that differ mostly with respect to the objective function.

CRRA utility maximization When defining the first problem we follow a classical approach (as, e.g., Brown et al., 2001; Koijen et al., 2011) and consider an individual with an uncertain time of death who obtains a constant relative risk aversion (CRRA) utility from the real consumption,

$$\max \quad \mathbb{E}_{t_0,w_0}\left[\sum_{t=t_0}^{\infty} {}_tp_x\,\mathcal{U}\left(t,C_t\right)\right], \qquad \mathcal{U}\left(t,C_t\right)=\frac{1}{\gamma}e^{-\rho t}\left(\frac{C_t}{I_t}\right)^{\gamma}, \tag{5.2.1}$$

where C_t is the nominal consumption at time t, and I_t is the inflation index. Function $\mathcal{U}$ denotes a utility function with a risk aversion $1-\gamma$ and a time preference factor ρ reflecting how important the current consumption is relatively to the consumption in the future. The operator $\mathbb{E}$ denotes the expectation under the physical probability measure $\mathbb{P}$, given that at time t_0 the individual has an initial wealth w_0. In addition, we multiply the utility at each period by the probability that a retiree aged x survives until time t, ${}_tp_x$, which we calculate from mortality tables.[1]

Loss disutility minimization In the second problem we consider an individual minimizing a loss function $\mathcal{L}$ that penalizes squared deviations from a certain target $\widehat{C}$. The target reflects the level of consumption necessary to cover the standard living costs, therefore, to account for an increase in these costs caused by inflation, we multiply $\widehat{C}$ by the RPI index:

$$\min \quad \mathbb{E}_{t_0,w_0}\left[\sum_{t=t_0}^{\infty} {}_tp_x\,\mathcal{L}\left(t,C_t\right)\right], \qquad \mathcal{L}\left(t,C_t\right)=e^{-\upsilon t}\left(\widehat{C}\cdot I_t-C_t\right)^2. \tag{5.2.2}$$

Parameter υ denotes a subjective discount factor reflecting the importance of minimizing current deviations from the target relatively to minimizing deviations in the future, and ${}_tp_x$ is the survival probability until time t of a retiree aged x. In addition, to define the risk aversion, we set a lower bound κ on the target income,

$$\widehat{C} \geq \kappa. \tag{5.2.3}$$

A high lower bound implies that the individual has to follow a more aggressive strategy to reach the target, therefore high κ specifies less risk averse individuals.

[1] We use British mortality tables for males based on 2000-2006 experience from UK self-administered pension schemes. Source: `http://www.actuaries.org.uk/research-and-resources/documents/s1pml-all-pensioners-excluding-dependants-male-lives`.

While some retirees may consider penalizing deviations above the target as a drawback, doing so prevents them from exposure to unnecessary financial risk; once they achieve the target, they can follow a more risk averse strategy. A loss function defined as the sum of squared deviations from a certain target is not new and has been considered by, e.g., Gerrard et al. (2004) and Di Giacinto et al. (2014), who applied a dynamic programming approach to find the optimal solution to the problem in a closed form. However, none of these studies searches for the optimal target level, adds constraints on this level, nor includes inflation risk and inflation-linked products.

5.3 Available assets

When optimizing the considered objective functions, we search for an optimal allocation to annuities, long-term bonds, and stocks. We assume that the annuities are *whole life*, i.e. pay as long as the retiree is alive, and are payable *in arrears*, i.e. the first payment commences not upon the purchase, but in the beginning of the next period, see Fig. 5.1. We further distinguish between three types of annuities common in the market: nominal, inflation-linked, and variable annuities.

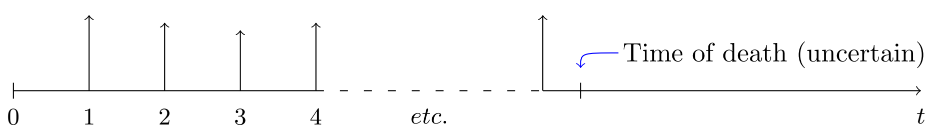


Figure 5.1: Cash-flows (indicated by arrows pointing upwards) from whole life annuities in arrears purchased upon time 0.

Nominal annuities provide fixed payments cf_t^N, which are determined at the time of the purchase,

$$cf_1^N = e^{\delta}, \qquad cf_t^N = cf_{t-1}^N \cdot e^{\delta\Delta t} = e^{\delta t}, \tag{5.3.1}$$

and which can be constant ($\delta = 0$), increasing ($\delta > 0$) or decreasing ($\delta < 0$). At any time t during the retirement period, the price of this annuity (for an individual aged x upon retirement) is given by

$$price_t^N = \sum_{s=t+1}^{\infty} {}_{s-t}p_{x+t} \cdot e^{\delta s - y(\boldsymbol{\beta}_t^N, s)(s-t)}, \tag{5.3.2}$$

where, following the actuarial notation, ${}_{s-t}p_{x+t}$ is a survival probability until time s for an individual aged $x+t$, and $y(\boldsymbol{\beta}_t^N, s)$ is a nominal interest rate p.a. over the period $[t, s)$. We explain vector $\boldsymbol{\beta}_t^N$ later in Section 5.4.

Inflation-linked (real) annuities provide income linked to realized inflation rate reported as Retail

Price Index (RPI),

$$cf_1^R = I_1, \qquad cf_t^R = cf_{t-1}^R \cdot e^{rpi(t-1,t)} = I_t, \tag{5.3.3}$$

where $rpi(t-1,t)$ is the inflation rate realized over the period $[t-1,t)$, and $I_t = \prod_{s=1}^t e^{rpi(s-1,s)}$ is the level of the inflation index (RPI).[2] Consequently, neither the retiree nor the annuity provider knows the level of the cash-flows at the purchase of the product—this value is revealed upon the payment, when the actual realized inflation is measured. To price real annuities, their providers use the real interest rates $y(\boldsymbol{\beta}_t^R, s)$, which are known at time t for all maturities $s > t$,

$$price_t^R = \sum_{s=t+1}^{\infty} {}_{s-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}_t^R,s)(s-t)} \cdot I_t. \tag{5.3.4}$$

Furthermore, to convert the price of real annuities to nominal terms, we multiply it by the current level of inflation index I_t.

Variable annuities provide nominal income that is linked to risky assets such as bonds or equities. Similarly to real annuities, the cash-flows from variable annuities are unknown upon the purchase of the product, and are first revealed when the actual stock returns are observed. The size of the payments depends on some assumed yearly interest rate (AIR), i.e. upon time t the annuitant receives the cash-flows equal to the excess return from stock returns r over the AIR $\bar{r}$,

$$cf_1^V = e^{r(0,1)-\bar{r}}, \qquad cf_t^V = cf_{t-1}^V \cdot e^{r(t-1,t)-\bar{r}\Delta t}, \tag{5.3.5}$$

where $r(t-1,t)$ is the stock return realized over the period $[t-1,t)$. The common rates for AIR are between 3% and 7% p.a., see Dellinger (2006). The lower the AIR, the higher the expected excess return $r(t-1,t) - \bar{r}\Delta t$ (implying that the annuity payments are likely to be increasing over time), and the higher the price, i.e.

$$price_t^V = \sum_{s=t+1}^{\infty} {}_{s-t}p_{x+t} \cdot e^{-\bar{r}(s-t)} \cdot cf_t^V. \tag{5.3.6}$$

When purchasing annuities, individuals take the risk of dying early, in particular, before receiving the payments whose accumulated value exceeds the value of the initial investment. Therefore, in contrast to non-life contingent assets, annuities provide a higher return. The excess return over the return provided by the assets with the same financial risk is called a *survival credit*, and its value depends on the individual's survival probability. Accordingly, the survival credit is included in the pricing formulae for the annuities in a form of the discounting factor ${}_{s-t}p_{x+t}$. The older the individual, the lower his survival probability, and therefore the lower the annuity price (or, equivalently, the higher the return).

[2] We normalize the inflation index by assuming that $I_0 = 1$.

5.4 Modeling uncertainty

Term structure of nominal and real interest rates To model the uncertainty of the interest rates, we use nominal (N) and real (R) UK yield curves with a monthly frequency. We further use the parametric Nelson and Siegel (1987) and Diebold and Li (2006) framework to condense the information in the following three-factor term structure of the nominal ($i = N$) and real ($i = R$) interest rates:

$$y(\boldsymbol{\beta}_t^i, s) = \beta_{1,t}^i + \beta_{2,t}^i \left(\frac{1 - e^{-\lambda_t^i s}}{\lambda_t^i s} \right) + \beta_{3,t}^i \left(\frac{1 - e^{-\lambda_t^i s}}{\lambda_t^i s} - e^{-\lambda_t^i s} \right), \tag{5.4.1}$$

where the parameter vector $\boldsymbol{\beta}_t^i = [\beta_{1,t}^i, \beta_{2,t}^i, \beta_{3,t}^i]^\top$ defines the level, slope and curvature of the term structure of interest rates, and $y(\boldsymbol{\beta}_t^i, s)$ defines the (continuously compounded) nominal/real spot rate for maturity s at time t.

Time-varying opportunities of term structure, inflation and stock returns The difference between nominal and real yields for different maturities is called the *break-even inflation* rate and can be interpreted as a result of expected inflation plus a premium for inflation risk minus a liquidity premium (given that nominal bonds are more liquid than the inflation-linked ones). A strand of literature tries to back out the components of the break-even inflation with different term structure models, see e.g. Joyce et al. (2010), Christensen et al. (2010), and Geyer et al. (2012), nevertheless, for the purpose of this paper, we refrain from this attempt and include directly realized log inflation rpi_t from the UK RPI.[3]

In line with Barberis (2000) and Campbell et al. (2003), we model time-varying realized stock returns, realized inflation, and nominal and real yield curves (represented by $\boldsymbol{\beta}_t^N$ and $\boldsymbol{\beta}_t^R$) with a VAR(1) model:

$$\boldsymbol{\xi}_t = \mathbf{c} + \mathbf{A}\boldsymbol{\xi}_{t-1} + \mathbf{u}_t, \tag{5.4.2}$$

with $\boldsymbol{\xi}_t = \left[r_t, rpi_t, \boldsymbol{\beta}_t^N, \boldsymbol{\beta}_t^R\right]'$, and where $\mathbf{c}$ is the (8×1) vector of intercepts, $\mathbf{A}$ is the (8×8) matrix of slope coefficients and $\mathbf{u}_t$ is the (8×1) vector of i.i.d. innovations with $\mathbf{u} \sim N(0, \boldsymbol{\Sigma})$. The covariance of the innovations $\boldsymbol{\Sigma}$ is given by $\mathbb{E}(\mathbf{u}\,\mathbf{u}^\top)$. Thus, we allow the shocks to be cross-sectionally correlated, but assume that they are homoskedastic and independently distributed over time. The stochastic process in equation (5.4.2) is stable if eigenvalues of $\mathbf{A}$ have modulus less than one, and has the unconditional expected mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Gamma}$ for the steady state at $t = \infty$ given by:

$$\begin{aligned} \boldsymbol{\mu} &:= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c}, \\ \mathrm{vec}(\boldsymbol{\Gamma}) &:= (\mathbf{I} - \mathbf{A} \otimes \mathbf{A})^{-1}\mathrm{vec}(\boldsymbol{\Sigma}), \end{aligned}$$

where $\mathbf{I}$ refers to the identity matrix, the symbol $\otimes$ is the Kronecker product and $\mathrm{vec}(\cdot)$ transforms a $(K \times K)$ matrix into a $(K^2 \times 1)$ vector by stacking the columns, see e.g. Lütkepohl (2005).

[3] `http://www.ons.gov.uk/`

When using decision steps longer than one month but calibrating the VAR process to monthly data, we follow Pedersen et al. (2013) to calculate aggregated stock returns and inflation between two decision stages. For notation brevity, we define $\boldsymbol{\zeta}_\tau$ as the vector of *cumulated* stock returns, *cumulated* inflation, and the Nelson/Siegel parameters,[4] and introduce an indicator matrix $\mathbf{J} = \mathrm{diag}(1,1,0,0,0,0,0,0)$. Then, the expectation and the covariance of $\boldsymbol{\zeta}_\tau$ for a general number of time time steps τ (i.e., months) of Eq. (5.4.2) are given by:

$$\mathbb{E}(\boldsymbol{\zeta}_\tau) = \left(\left(\sum_{i=1}^{\tau-1}(\mathbf{I}+\mathbf{J}(\tau-i))\,\mathbf{A}^{i-1}\right)+\mathbf{A}^{\tau-1}\right)\mathbf{c}+\left(\mathbf{A}^{\tau}+\sum_{i=1}^{\tau-1}\mathbf{J}\,\mathbf{A}^{i}\right)\boldsymbol{\xi}_0, \tag{5.4.3}$$

and

$$\begin{aligned}\mathbb{V}(\boldsymbol{\zeta}_\tau) &= \boldsymbol{\Sigma}\\ &+ (\mathbf{J}+\mathbf{A})\,\boldsymbol{\Sigma}\,(\mathbf{J}+\mathbf{A})^\top\\ &+ \left(\mathbf{J}+\mathbf{J}\,\mathbf{A}+\mathbf{A}^2\right)\boldsymbol{\Sigma}\left(\mathbf{J}+\mathbf{J}\,\mathbf{A}+\mathbf{A}^2\right)^\top\\ &+ \ldots\\ &+ \left(\mathbf{A}^{\tau-1}+\sum_{i=1}^{\tau-1}\mathbf{J}\,\mathbf{A}^{i-1}\right)\boldsymbol{\Sigma}\left(\mathbf{A}^{\tau-1}+\sum_{i=1}^{\tau-1}\mathbf{J}\,\mathbf{A}^{i-1}\right)^\top.\end{aligned} \tag{5.4.4}$$

5.5 MSP formulation

A multi-stage stochastic program consists of a scenario tree and an optimization module, both explained in the following subsections. The optimal decisions in the multi-stage stochastic programming are computed numerically at each node of the tree, see, e.g., Birge and Louveaux (1997) and Zenios (2008). They depend on the decisions made at the ancestor nodes, on the realizations of the random vector, and on the anticipation of the possible future outcomes. Because the multi-stage stochastic programming approach combines anticipative and adaptive models in one mathematical framework, it is particularly appealing in financial applications. For example, an investor composes his portfolio given the anticipation of possible future movements of asset prices, and rebalances the portfolio as prices change.

Throughout this section, we use capital letters to denote the variables and lower-case letters to specify the parameters.

5.5.1 Scenario tree and scenario tree generation

A *scenario tree* represents uncertainty in the MSP approach. As shown on Fig. 5.1, a scenario tree consists of nodes $n \in \mathcal{N}_t$ uniquely assigned to periods t, and representing possible outcomes for the uncertainties, $\boldsymbol{\zeta}_{t,n} = \left[r_{t,n}, rpi_{t,n}, \boldsymbol{\beta}^N_{t,n}, \boldsymbol{\beta}^R_{t,n}\right]'$. The initial stage t_0 has only one node n_0, which is the

[4] While in $\boldsymbol{\xi}$ realized inflation and stock returns are on a monthly basis, $\boldsymbol{\zeta}_\tau$ cumulates τ monthly rates. The Nelson/Siegel parameter vector is the same for $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$.

ancestor for all the nodes at the subsequent stage t_1. These nodes are further the ancestors for their children nodes, etc., until the final stage T. As the nodes at the final stage have no children, they are called the leaves. We define a scenario $\mathcal{S}^n$ as a single branch from the root node to the leaf, i.e. each scenario consists of a leaf node n and all its predecessors up to the root node n_0. Consequently, the number of scenarios in the tree equals the number of leaves. Each node has a probability pr_n, so that $\forall_t \sum_{n\in\mathcal{N}_t} pr_n = 1$, implying that the probability of each scenario $\mathcal{S}^n$ is equal to the product of the probabilities of all the nodes in the scenario.

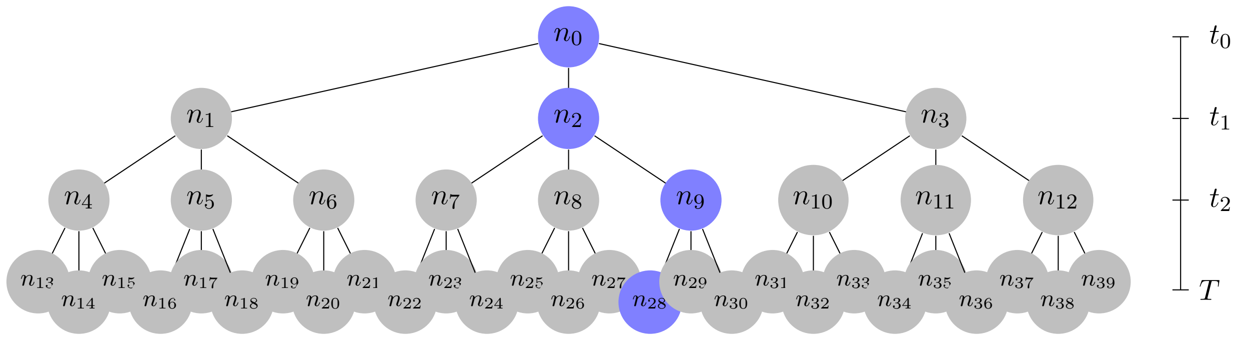


Figure 5.1: An example of a scenario tree with three periods, a branching factor of 3, and $3^3 = 27$ scenarios defined as a single path from the root node to the leaf (such as the path marked in blue).

When modelling the uncertainty for MSP, we have to consider the *curse of dimensionality* inherent in this approach—the scenario tree grows exponentially with the number of periods. Therefore, to generate scenarios, we approximate the discrete-time multivariate process in Eq. (5.4.2) with a few mass points, accordingly reducing the computational complexity. We start the tree construction from the unconditional expected values as done, e.g., by Campbell et al. (2003) and Ferstl and Weissensteiner (2011), and use the technique proposed by Høyland and Wallace (2001) and Høyland et al. (2003) to match the first four moments and the correlations. For the subsequent time periods we use the conditional expected value and covariance (Eqs. (5.4.3) and (5.4.4)).

Given the generated stock returns, inflation rate, and the parameters for the term structure of nominal and real interest rates, $\boldsymbol{\zeta}_{t,n} = \left[r_{t,n}, rpi_{t,n}, \boldsymbol{\beta}^N_{t,n}, \boldsymbol{\beta}^R_{t,n}\right]'$, we can calculate the input parameters for the MSP. In particular, we rewrite to a nodal form formulae (5.3.1)–(5.3.6) for the annuity cash-flows and prices,

$$cf^N_{t,n} = cf^R_{t,n} = cf^V_{t,n} = 0, \qquad t = t_0,\ n \in \mathcal{N}_t, \tag{5.5.1}$$

$$cf^N_{t,n} = e^{\delta\Delta t}, \qquad t = t_1,\ n \in \mathcal{N}_t, \tag{5.5.2}$$

$$cf^R_{t,n} = e^{rpi_n(t-1,t)}, \qquad t = t_1,\ n \in \mathcal{N}_t, \tag{5.5.3}$$

$$cf^V_{t,n} = e^{r_n(t-1,t)-\bar{r}\Delta t}, \qquad t = t_1,\ n \in \mathcal{N}_t, \tag{5.5.4}$$

$$cf^N_{t,n} = cf^N_{t-1,n^-} \cdot e^{\delta\Delta t}, \qquad t = t_2,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.5}$$

$$cf^R_{t,n} = cf^R_{t-1,n^-} \cdot e^{rpi_n(t-1,t)}, \qquad t = t_2,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.6}$$

$$cf^V_{t,n} = cf^V_{t-1,n^-} \cdot e^{r_n(t-1,t)-\bar{r}\Delta t}, \qquad t = t_2,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.7}$$

$$price^N_{t,n} = \sum_{s=t+1}^{\omega-x} {}_{s-t}p_{x+t} \cdot e^{\delta\Delta s - y(\boldsymbol{\beta}^N_{t,n},s)(s-t)}, \qquad t = t_0,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.8}$$

$$price^R_{t,n} = \sum_{s=t+1}^{\omega-x} {}_{s-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}^R_{t,n},s)(s-t)} \cdot I_{t,n}, \qquad t = t_0,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.9}$$

$$price^V_{t,n} = \sum_{s=t+1}^{\omega-x} {}_{s-t}p_{x+t} \cdot e^{-\bar{r}(s-t)} \cdot cf^V_{t,n}, \qquad t = t_0,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.10}$$

and define the prices of stocks and bonds with maturity s at each node of the tree,

$$price^S_{t,n} = 1, \qquad t = t_0,\ n \in \mathcal{N}_t, \tag{5.5.11}$$

$$price^S_{t,n} = price^S_{t-1,n^-} \cdot e^{r(t-1,t)}, \qquad t = t_1,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.12}$$

$$price^{B,s}_{t,n} = 1, \qquad t = t_0,\ n \in \mathcal{N}_t, \tag{5.5.13}$$

$$price^{B,s}_{t,n} = price^{B,s}_{t-1,n^-} \cdot e^{y(\boldsymbol{\beta}^N_{t-1,n^-},s)\Delta t}, \qquad t = t_1,\dots,T,\ n \in \mathcal{N}_t, \tag{5.5.14}$$

where x is the age of the individual at time t_0, ω is the maximum age upon which the individual is dead with certainty, and n^- denotes the ancestor of node n.

5.5.2 Optimization module

The optimization module consists of an objective function and a set of constraints defined for each scenario. We first present the problem in the nodal representation for the CRRA utility maximization, and afterwards for the disutility minimization.

CRRA utility maximization The curse of dimensionality characteristic for MSP does not allow us to make optimal decisions for the entire lifetime of the individual, therefore we must simplify the model. In particular, we choose some horizon T and define the scenario tree only up to this horizon. To make sure that the individual has enough savings for the rest of his life, we further maximize the utility of the final wealth upon horizon T.[5] Consequently, we calculate the optimal consumption and asset allocation only up to time $T-1$; from T and onwards the individual no longer rebalances the portfolio, but consumes the savings in liquid assets and cash-flows from the annuities that he has purchased during the period $[t_0, T-1]$, see Fig. 5.2.

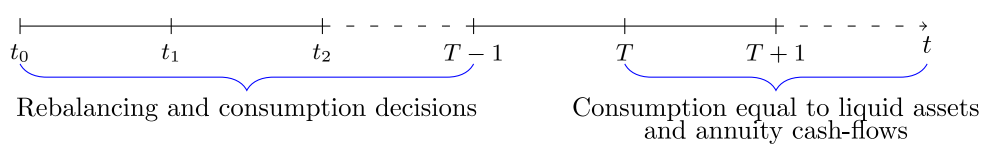


Figure 5.2: Overview of the model.

[5]We argue that solving the problem only up to some horizon T is sufficient because in practice financial advisers would hold regular meetings with their customers, and rerun the model with an updated horizon and with the new parameters reflecting the current state of the economy.

We rewrite the objective function defined in Eq. (5.5.1) to the following nodal representation:

$$\max \quad \sum_{t=t_0}^{T-1} {}_t p_x \sum_{n\in\mathcal{N}_t} f_{t,n}^{1-\gamma} \cdot \mathcal{U}\left(t, C_{t,n}\right) \cdot pr_n + {}_T p_x \sum_{n\in\mathcal{N}_T} f_{T,n}^{1-\gamma} \cdot \mathcal{U}\left(T, W_{T,n}\right) \cdot pr_n, \tag{5.5.15}$$

where

$$\mathcal{U}\left(t, C_{t,n}\right) = \frac{1}{\gamma} e^{-\rho t} \left(\frac{C_{t,n}}{I_{t,n}}\right)^{\gamma}, \quad \mathcal{U}\left(T, W_{T,n}\right) = \frac{1}{\gamma} e^{-\rho T} \left(\frac{W_{T,n}}{I_{T,n}}\right)^{\gamma}, \tag{5.5.16}$$

and where $C_{t,n}$ is the consumption during the subsequent period, $W_{T,n}$ is the value of wealth upon horizon, pr_n is the probability of being at node n, and $f_{t,n}$ is the multiplier accounting for the length of the subsequent interval. We calculate $f_{t,n}$ as the price of a nominal term annuity with the length of the payments equal to the length of the subsequent period, i.e.

$$f_{t,n} \quad = \quad \begin{cases} \sum_{s=t}^{t+\Delta t-1} {}_{s-t}p_{x+t}\, e^{-y(\boldsymbol{\beta}_{t,n}^N, s)(s-t)}, & t = t_0, \ldots, T-1,\ n \in \mathcal{N}_t, \\ \sum_{s=t}^{\omega-x-t-1} {}_{s-t}p_{x+t}\, e^{-y(\boldsymbol{\beta}_{t,n}^N, s)(s-t)}, & t = T,\ n \in \mathcal{N}_t, \end{cases} \tag{5.5.17}$$

where ω is the maximum age, at which the individual is assumed to be dead with certainty. The multiplier $f_{t,n}$ is necessary to measure the utility of the yearly consumption $\mathcal{U}(t, C_{t,n}/f_{t,n})$ obtained during Δt, i.e. $f_{t,n} \cdot \mathcal{U}(t, C_{t,n}/f_{t,n}) = f_{t,n}^{1-\gamma} \cdot \mathcal{U}(t, C_{t,n})$, and to attribute the value to the utility of the final wealth so that the retiree never outlives his income. By definition of the utility function, we further have that $C_{t,n} > 0$ and $W_{T,n} > 0$ for $\gamma \in (-\infty, 1) \setminus \{0\}$.

Let $\mathcal{J}$ denote a set of all available assets: bonds, stocks, and nominal, real, and variable annuities, let $\mathcal{K} \subset \mathcal{J}$ denote the subset including only the liquid assets, i.e. bonds and stocks, and let $\mathcal{A} \subset \mathcal{J}$ denote the subset including only annuities. For each asset $j \in \mathcal{J}$, we define the variable $Buy_{t,n}^j$ denoting the number of units of asset j purchased at time t and node n, and the variable $Hold_{t,n}^j$ denoting the number of units of asset j held at time t and node n. Because annuities are often irreversible (i.e. once purchased they can never be sold) or have prohibitive transaction costs, we do not allow for selling these products. Nevertheless, we define the variable $Sell_{t,n}^j$ for bonds and stocks. Then, the optimization problem consists of the following constraints: the budget constraint, the inventory constraint, and the non-negativity constraint.

In the *budget constraint* we equal the outgoing payments (consumption and the purchase of new assets) to the incoming payments (the initial wealth, the cash-flows from the annuities, and the capital gains earned on sales),

$$C_{t,n} + \sum_{j\in\mathcal{J}} price_{t,n}^j Buy_{t,n}^j \mathbf{1}_{\{t<T\}} = w_0 \mathbf{1}_{\{t=t_0\}} + \sum_{j\in\mathcal{A}} cf_{t,n}^j Hold_{t-1,n^-}^j + \sum_{j\in\mathcal{K}} price_{t,n}^j Sell_{t,n}^j \mathbf{1}_{\{t<T\}},$$
$$t = t_0, \ldots, T,\ n \in \mathcal{N}_t. \tag{5.5.18}$$

We further calculate the value of savings upon horizon as the sum of the market value of the assets held in the portfolio and the cash-flows provided by these annuities,[6]

$$W_{t,n} \quad = \quad \sum_{j\in\mathcal{A}} cf^{j}_{t,n} Hold^{j}_{t-1,n^-} + \sum_{j\in\mathcal{J}} price^{j}_{t,n} Hold^{j}_{t-1,n^-}, \qquad t = T,\, n \in \mathcal{N}_t. \tag{5.5.19}$$

The *inventory constraint* keeps track of the current holdings in a given asset,

$$Hold^{j}_{t,n} = Hold^{j}_{t-1,n^-}\mathbf{1}_{\{t>t_0\}} + Buy^{j}_{t,n} - Sell^{j}_{t,n}\mathbf{1}_{\{j\in\mathcal{K}\}}, \qquad t = t_1,\ldots,T-1,\ n\in\mathcal{N}_t,\ j\in\mathcal{J}. \tag{5.5.20}$$

Finally, we add the *non-negativity constraints* on the purchase, hold and sale variables,

$$Buy^{j}_{t,n} \geq 0,\ Hold^{j}_{t,n} \geq 0, \qquad t = t_0,\ldots,T-1,\ n\in\mathcal{N}_t,\ j\in\mathcal{J}, \tag{5.5.21}$$

$$Sell^{j}_{t,n} \geq 0, \qquad t = t_0,\ldots,T-1,\ n\in\mathcal{N}_t,\ j\in\mathcal{K}, \tag{5.5.22}$$

implying that we do not allow for borrowing money or for having a short position in any asset.

Loss disutility minimization To model the loss disutility minimization problem in a multi-stage stochastic programming framework, we rewrite Eq. (5.2.2) to the following nodal representation:

$$\min \quad \sum_{t=t_0}^{T} {}_tp_x \sum_{n\in\mathcal{N}_t} \mathcal{L}\left(t, C_{t,n}\right)\cdot pr_n, \qquad \mathcal{L}\left(t, C_{t,n}\right) = e^{-\upsilon t}\left(\widehat{C}\cdot I_{t,n} - C_{t,n}\right)^2. \tag{5.5.23}$$

Similarly to the CRRA utility maximization problem, to ensure computational tractability, we sum the disutility function up to a finite horizon T. Thus, the retiree makes consumption and investment decisions only up to time $T-1$, whereas from T and onwards he consumes his savings in bonds and stocks, and the income from the annuities that he has purchased during the period $[t_0, T-1]$.

The model constraints are identical to the case of the power utility maximization, i.e. the model comprises the budget constraint (5.5.18), the inventory constraint (5.5.20), and the non-negativity constraints (5.5.21)–(5.5.22). Furthermore, we add constraint (5.2.3), which defines the risk aversion of the retiree. The higher the target κ, the less risk averse the retiree is.

5.6 Numerical results

We start this section by explaining the choice of the model parameters and by calculating the annuity prices and cash-flows. Afterwards, we illustrate the optimal solution from both optimization problems.

[6]By definition of annuities in arrears, the current price does not include the current annuity cash-flows.

5.6.1 Model parameters

Parameters for the term structures, inflation index and stock returns To determine the parameters for the term structure, (5.4.1), we use nominal and real UK yield curves with a monthly frequency from October 1992 (when the inflation-targeting of the Bank of England began, see, e.g., Joyce et al., 2010) to March 2014.[7] We follow Diebold and Li (2006) and fix λ^i to be time-independent and omit therefore the subscript t. The parameter λ^i determines the maximum of the factor loading for the curvature. While a lower value ensures a better fit for long maturities, increasing this value enhances the fit for short maturities. We optimize λ^i separately for nominal and real yields by minimizing the sum of squared differences between observed yields and fitted values from our model. Therefore we solve an iterative, nonlinear optimization problem, where in each iteration $\boldsymbol{\beta}_t^i$ parameters are determined by OLS. Gilli et al. (2010) point out that this estimation through OLS might be prone to a collinearity problem for certain λ^i values, which is particularly relevant if the ultimate goal is to model the evolution of yield curves over time. Therefore, to avoid such a problem we restrict λ^i such that the correlation between the second and third factor loading is in the interval [-0.7, 0.7]. The corresponding admissible range for λ^i depends on the maturities for which we have observed yields. Nominal yields in our data set start at maturities of 1 year while real yields are only available for 2.5 years or more. For nominal yields, the restriction turns out to be non-binding, and the optimal $\lambda^N = 0.42$. For real yields, however, the optimal $\lambda^R = 0.34$ is at the upper end of its admissible range. In both cases the fit is very accurate, in most cases the coefficient of determination is above 0.99.

The estimated parameters $\mathbf{c}$ and $\mathbf{A}$ denoting the intercepts and slope coefficients for the VAR(1) model are shown in Table 5.1. All eigenvalues in $\mathbf{A}$ have modulus less than one, thus the process is stable. In the steady state, both yield curves are increasing (15y nominal yields at 3.4% p.a. and 15y real yields at 0.1% p.a.), the average break-even inflation rate (calculated as the difference between the nominal and real yields across all the maturities) is equal to $\mathbb{E}[bei] = 3.35\%$ p.a., and the expected realized RPI is equal to $\mathbb{E}[rpi_t] = 3.92\%$ p.a. The analyzed historical data from the period October 1992 to March 2014 show that the average realized inflation rate has been higher than the break-even inflation, implying that the premium for inflation risk minus the liquidity premium has been negative. While most people expect a positive inflation risk premium, empirical evidence shows that it does not need to be the case. Especially, during the periods of relative illiquidity of inflation-linked bonds, the inflation risk premium is more likely to be negative. Durham (2006) estimates the inflation risk during 2000-2006 and shows that prior to 2003 the inflation premium was mostly negative. From 2003 to 2008, as also shown by Christensen et al. (2010), the inflation risk premium fluctuated around zero within ±50 basis points. Following Christensen et al. (2010), we lower the expected realized inflation by 0.57% so that $\mathbb{E}[rpi_t] = \mathbb{E}[bei]$, see Fig. 5.1. Stocks have a drift of 7.00% p.a., a volatility of 15.22%, and are nearly uncorrelated to the inflation rates (0.0375).

[7] `http://www.bankofengland.co.uk/statistics/pages/yieldcurve/default.aspx`

	r_t	rpi_t	$\beta^N_{1,t}$	$\beta^N_{2,t}$	$\beta^N_{3,t}$	$\beta^R_{1,t}$	$\beta^R_{2,t}$	$\beta^R_{3,t}$
c	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	[0.68]	[0.82]	[1.10]	[-0.36]	[-0.55]	[0.41]	[-2.08]	[-0.88]
r_{t-1}	0.01	0.00	0.01	0.01	0.00	0.00	-0.02	-0.01
	[0.18]	[0.57]	[1.51]	[1.26]	[-0.25]	[0.56]	[-2.04]	[-0.94]
rpi_{t-1}	-0.59	-0.01	-0.06	0.11	0.35	-0.02	0.40	0.07
	[-0.86]	[-0.13]	[-1.24]	[1.54]	[2.79]	[-0.50]	[3.97]	[0.46]
$\beta^N_{1,t-1}$	-0.53	0.11	1.01	-0.03	-0.07	-0.01	0.09	0.17
	[-1.01]	[2.30]	[28.34]	[-0.52]	[-0.75]	[-0.24]	[1.18]	[1.57]
$\beta^N_{2,t-1}$	-0.13	0.10	0.01	1.01	-0.09	-0.02	0.14	0.06
	[-0.49]	[4.13]	[0.60]	[38.34]	[-1.88]	[-1.99]	[3.60]	[1.03]
$\beta^N_{3,t-1}$	-0.21	0.05	0.03	0.02	0.92	0.00	0.00	0.08
	[-1.28]	[3.09]	[2.27]	[1.33]	[30.23]	[0.00]	[0.12]	[2.19]
$\beta^R_{1,t-1}$	1.23	-0.19	-0.08	0.07	0.18	0.99	-0.02	-0.32
	[1.53]	[-2.55]	[-1.54]	[0.91]	[1.21]	[25.72]	[-0.21]	[-1.86]
$\beta^R_{2,t-1}$	0.02	-0.13	-0.02	-0.06	0.11	0.03	0.87	-0.10
	[0.08]	[-5.03]	[-1.03]	[-2.09]	[2.17]	[2.07]	[21.29]	[-1.64]
$\beta^R_{3,t-1}$	0.14	-0.04	-0.01	-0.01	-0.03	0.01	-0.02	0.86
	[0.74]	[-2.17]	[-0.91]	[-0.28]	[-0.91]	[0.88]	[-0.73]	[21.19]
R^2	0.03	0.11	0.97	0.97	0.94	0.97	0.94	0.87

Table 5.1: VAR(1) parameters and t-statistics (in squared brackets) for stock returns, inflation rate, and nominal and real yield curves, estimated from monthly data covering October 1992 to March 2014.

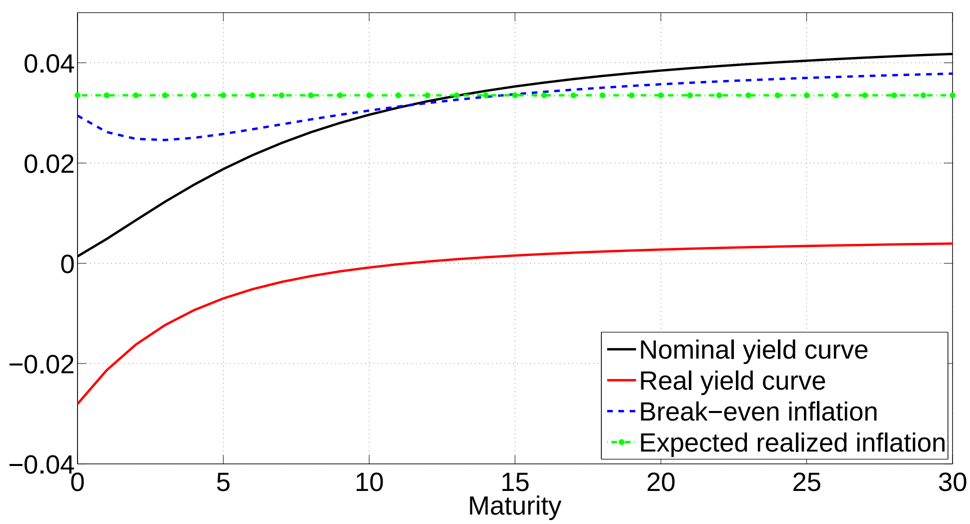


Figure 5.1: The expected nominal and real yield curve, break-even inflation and the realized inflation at the steady state, estimated from monthly data covering October 1992 to March 2014. The expected realized inflation has been lowered by 57bp to equal the average break-even inflation rate (calculated as the difference between the nominal and real yields across all the maturities).

Available assets Among the available assets we consider long-term nominal bonds with a spot rate $y(\boldsymbol{\beta}^N_t, 30)$, the MSCI UK stock index, and three types of whole life annuities in arrears: nominal annuities with constant payments ($\delta = 0$), inflation-linked (real) annuities, and variable annuities with the assumed interest rate $\bar{r} = 5\%$. We assume that all the annuities are fairly priced (equations (5.5.1)–(5.5.10)), and provide cash-flows every fifth year, starting in the period following the purchase and ending upon the individual's death. The cash-flows from nominal annuities are constant (in nominal terms), whereas the cash-flows from real and variable annuities vary with inflation and stock returns, respectively. In addition, the prices of all annuities vary across scenarios due to changes in the inflation index, stock returns, and nominal and real yield curves. Table 5.2 shows the expected prices and cash-flows in nominal terms of each annuity, and their development over time for the first 20 years after retirement. Upon retirement the cheapest product is the variable annuity; the most expensive, the real annuity.

	Prices (£), $\mathbb{E}[price_t^j]$					Cash-flows (£), $\mathbb{E}[cf_t^j]$				
Annuity	65	70	75	80	85	65	70	75	80	85
Nominal annuities	2.02	1.63	1.22	0.85	0.54	-	1.00	1.00	1.00	1.00
Real annuities	2.86	2.57	2.17	1.72	1.25	-	1.18	1.40	1.66	1.96
Variable annuities	1.65	1.54	1.37	1.13	0.85	-	1.17	1.37	1.62	1.91

Table 5.2: Expected prices and cash-flows in nominal terms from whole life annuities in arrears paying $cf_{t,n}^j$ every fifth year.

Break-even age Towler (2013) argues that 95% of British retirees do not bother to protect their savings against inflation by purchasing real annuities, and that a possible explanation for such behaviour is that retirees find inflation-linked products too expensive. Before deciding which annuity to buy, they often compare the payout from different products. The cash-flows from real annuities are always lower for some years after the purchase than the cash-flows from nominal annuities with constant payments, and it may take many years to see the benefits of real annuities. To exemplify, given our choice of parameters, for a lump sum of £100 the retiree can purchase 49.5 units of nominal annuities or 35 units of real annuities. The income upon age 70 from real annuities is £41.3 $(35 \cdot 1.18)$, which is 20% lower than the income from nominal annuities with constant payments. Given a realized inflation equal to $\mathbb{E}[rpi_t] = 3.35\%$, it will take 16 years before the smaller payouts from the inflation-linked annuity exceed the fixed rate payouts. Figure 5.2 shows that the break-even age (i.e. the time when the accumulated expected payouts from both annuities are equal) is around age 81. If the realized inflation rate is lower than expected,[8] the individual has to wait even longer for his payments to exceed those from the nominal annuity.

5.6.2 Optimization-based results

Looking at the break-even age, not surprisingly, the individuals feel reluctant to purchase real annuities. Thus, to further investigate whether these products are a good investment, we consider an optimization-based approach. We search for an optimal investment strategy with and without the inflation-linked products, and analyze the corresponding optimal level of consumption. We choose the decision intervals of $\Delta t = 5$ years, and the horizon $T = 20$ years. Accordingly, the individual makes consumption and investment decisions, starting at his retirement (assumed age 65), until

[8]The inflation target range of the Bank of England is between 1 and 3%, see `http://www.bankofengland.co.uk/monetarypolicy/Pages/framework/framework.aspx`.

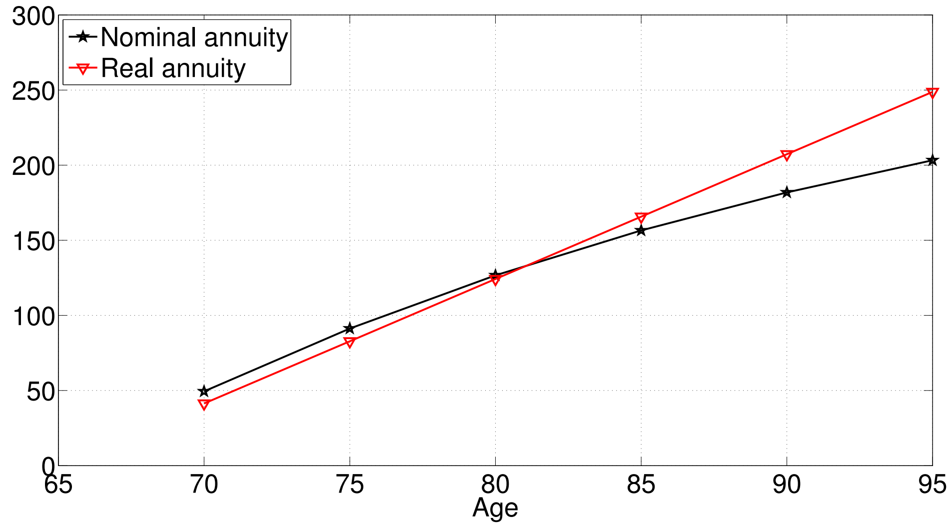


Figure 5.2: The accumulated expected payout in real terms from a nominal annuity with constant payments and from an inflation-linked annuity. The annuities are purchased upon retirement (age 65) for a lump sum of £100, and pay out in arrears every fifth year.

he reaches age 85. Upon that age and until his death, his consumption is equal to the value of liquid assets and the cash-flows from the whole life annuities purchased during the first 20 years of retirement. For the ease of presentation, we assume that upon retirement the individual has $w_0 = 100$ (£) on his savings account.

As described in the previous section, we study two different objective functions resulting in different consumption and investment decisions. We implement the multi-stage stochastic models in GAMS 24.1.3, using MOSEK 7.0.0.75 to solve the power utility maximization problem, and CPLEX 12.5.1.0 to solve the loss minimization problem. The scenario tree has four stages, each with a branching factor of 14 (which is the minimum number of branches providing the sufficient statistical match of the first four moments and the correlations between the underlying processes). Consequently, the number of scenarios in the tree is equal to $14^4 = 38,416$. The running time on a Dell computer with an Intel Core i5-2520M 2.50 GHz processor and 4 GB RAM is approximately 1.5 minutes.

To get an economic intuition for the optimal decisions (which are non-linear in the state variables), we follow Koijen et al. (2011) and approximate the strategy using linear decision rules. We run multilinear regressions to examine the optimal conditional and unconditional asset allocation, however, in contrast to the mentioned study, we investigate how the optimal decisions are affected by conditional future state variables relatively to the current state. In addition, we also examine which variables influence the consumption decision. Among the expected state variables of the successor nodes, we consider stock returns and changes in the level of inflation, long-term real, and long-term nominal interest rates:

$$
\begin{aligned}
Y^1_{t,n} &= \mathbb{E}[r_{t+1}], \qquad \mathbb{E}[r_{t+1}] = \sum_{n^+ \in \mathcal{N}_{t+1}} r_{t+1,n^+} \cdot pr_{n^+}, \\
Y^2_{t,n} &= \mathbb{E}[rpi_{t+1}] - rpi_{t,n}, \qquad \mathbb{E}[rpi_{t+1}] = \sum_{n^+ \in \mathcal{N}_{t+1}} rpi_{t+1,n^+} \cdot pr_{n^+}, \\
Y^3_{t,n} &= \mathbb{E}[y(\boldsymbol{\beta}^R_{t+1}, 30)] - y(\boldsymbol{\beta}^R_{t,n}, 30), \qquad \mathbb{E}[y(\boldsymbol{\beta}^R_{t+1}, 30)] = \sum_{n^+ \in \mathcal{N}_{t+1}} y(\boldsymbol{\beta}^R_{t+1,n^+}, 30) \cdot pr_{n^+}, \\
Y^4_{t,n} &= \mathbb{E}[y(\boldsymbol{\beta}^N_{t+1}, 30)] - y(\boldsymbol{\beta}^N_{t,n}, 30), \qquad \mathbb{E}[y(\boldsymbol{\beta}^N_{t+1}, 30)] = \sum_{n^+ \in \mathcal{N}_{t+1}} y(\boldsymbol{\beta}^N_{t+1,n^+}, 30) \cdot pr_{n^+},
\end{aligned}
$$

where n^+ denotes the successors of node n. We further normalize the state variables

$$
\widetilde{Y}^i_{t,n} = \frac{Y^i_{t,n} - \mathbb{E}(Y^i_t)}{\sigma(Y^i_t)}, \qquad i = 1, \ldots, 4,
$$

where Y^i_t is a vector of the i-th state variable at all nodes assigned to stage t, so that we can approximate the optimal decisions by

$$
\frac{C_{t,n}}{w_0} \approx \alpha_t^{C,0}, \qquad t = t_0,\ n \in \mathcal{N}_t, \tag{5.6.1}
$$

$$
\frac{C_{t,n}}{\sum_{j \in \mathcal{A}} cf^j_{t,n} Hold^j_{t-1,n^-} + \sum_{j \in \mathcal{K}} price^j_{t,n} Hold^j_{t-1,n^-}} \approx \alpha_t^{C,0} + \sum_{i=1}^{4} \alpha_t^{C,i} \widetilde{Y}^i_{t,n}, t = t_1, \ldots, T-1,\ n \in \mathcal{N}_t, \tag{5.6.2}
$$

$$
\frac{price^j_{t,n} Buy^j_{t,n}}{I_{t,n}} \approx \alpha_t^{j,0,b}, \qquad t = t_0,\ n \in \mathcal{N}_t,\ j \in \mathcal{J}, \tag{5.6.3}
$$

$$
\frac{price^j_{t,n} Buy^j_{t,n}}{I_{t,n}} \approx \alpha_t^{j,0,b} + \sum_{i=1}^{4} \alpha_t^{j,i,b} \widetilde{Y}^i_{t,n}, \qquad t = t_1, \ldots, T-1,\ n \in \mathcal{N}_t,\ j \in \mathcal{J}, \tag{5.6.4}
$$

$$
\frac{price^j_{t,n} Sell^j_{t,n}}{I_{t,n}} \approx \alpha_t^{j,0,s}, \qquad t = t_0,\ n \in \mathcal{N}_t,\ j \in \mathcal{K}, \tag{5.6.5}
$$

$$
\frac{price^j_{t,n} Sell^j_{t,n}}{I_{t,n}} \approx \alpha_t^{j,0,s} + \sum_{i=1}^{4} \alpha_t^{j,i,s} \widetilde{Y}^i_{t,n}, \qquad t = t_1, \ldots, T-1,\ n \in \mathcal{N}_t,\ j \in \mathcal{K}. \tag{5.6.6}
$$

At period t, (5.6.1) and (5.6.2) approximate the consumption relative to the liquid wealth, which consists of the cash-flows from the purchased annuities and the value in bonds and stocks, whereas (5.6.3)–(5.6.4) and (5.6.5)–(5.6.6) approximate, respectively, the purchase and the sale amount in real terms. Accordingly, the terms $\alpha_t^{C,0}$, $\alpha_t^{j,0,b}$, and $\alpha_t^{j,0,s}$ represent the unconditional relative consumption, purchase and sale amount, and the slope coefficients $\alpha_t^{C,i}$, $\alpha_t^{j,i,b}$ and $\alpha_t^{j,i,s}$ determine the change in the corresponding variables for a one standard deviation increase in the corresponding i-th state variable.

Power utility maximization Figure 5.3a shows the expected optimal consumption and retirement income for an individual with risk aversion $\gamma = -7$ and $\gamma = -2$. Consistently with Yaari (1965), in the absence of a bequest motive the individual holds his assets in life contingent annuities rather than in bonds and stocks, and consistently with Milevsky and Young (2007), he does not delay his annuitization decision, but purchases annuities as soon as he seizes a chance to do so. As also shown in Table 5.3, the asset allocation varies with the risk aversion. The more risk averse retiree ($\gamma = -7$) initially allocates 15% of the portfolio to nominal annuities, 50% to real annuities, and 33% to variable annuities, whereas the less risk averse retiree ($\gamma = -2$) allocates, respectively, 0%, 20%, and 80%. Thus, in line with Campbell and Viceira (2001), Koijen et al. (2011), and Han and Hung (2012)—the more risk averse the investor, the more he is concerned about the uncertainty of his real income.

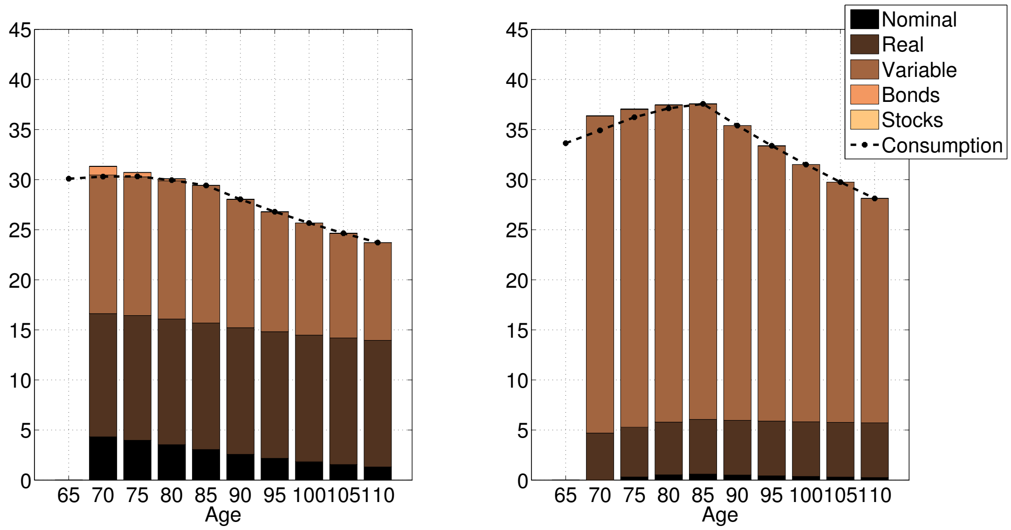
Power utility maximization		$\gamma = -7$				$\gamma = -2$			
		65	70	75	80	65	70	75	80
Asset allocation	Nominal	15	14	13	12	0	1	2	2
	Real	50	51	52	53	20	21	23	24
	Variable	33	34	35	35	80	78	75	74
	Bonds	2	1	0	0	0	0	0	0
	Stocks	0	0	0	0	0	0	0	0
Consumption	$\mathbb{E}[C_t]$	30.1	30.3	30.3	29.5	33.6	34.9	36.2	37.7
	$\sigma[C_t]$	0.0	4.1	6.9	12.0	0.0	10.1	16.2	26.2

Disutility minimization		$\kappa = 28.5$ (£)				$\kappa = 38.8$ (£)			
		65	70	75	80	65	70	75	80
Asset allocation	Nominal	0	2	2	2	0	8	9	9
	Real	91	91	90	90	57	61	60	59
	Variable	4	5	6	6	18	23	27	27
	Bonds	0	0	1	1	0	2	3	2
	Stocks	5	2	1	1	25	6	1	3
Consumption	$\mathbb{E}[C_t]$	25.6	26.4	26.9	27.6	24.3	28.2	31.0	34.2
	$\sigma[C_t]$	0.0	1.0	1.0	1.5	0.0	5.0	5.1	7.5

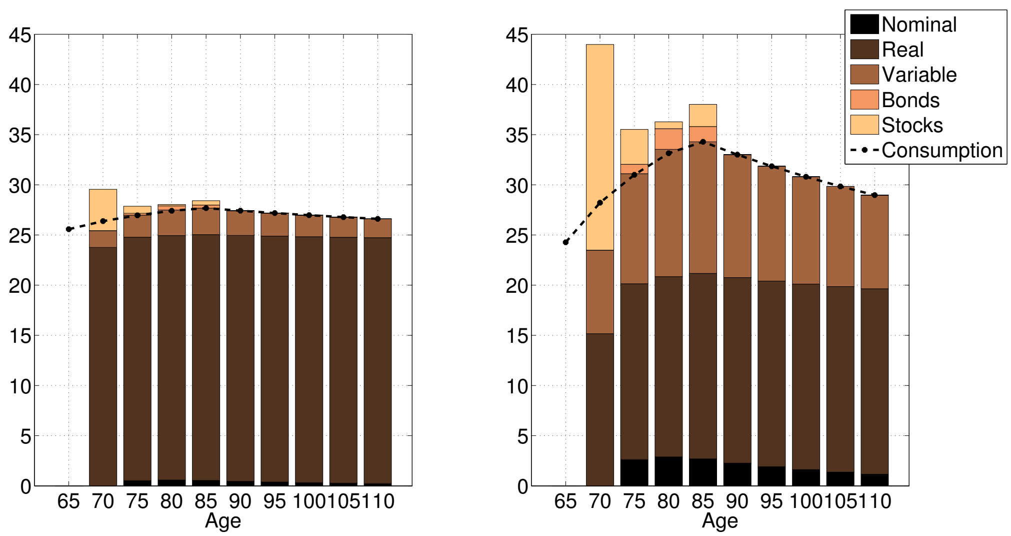
Table 5.3: The expected optimal asset allocation (rounded to the nearest percent) and consumption (in £) under the power utility maximization with $\rho = 0.04$ and the disutility minimization with $\upsilon = 0$.

The more risk averse individual expects a 5-year consumption level upon retirement of £30.1, and this amount decreases over time to £24 upon survival until age 110 (see Fig. 5.3a). The less risk averse retiree consumes initially £33.6, and increases his consumption until horizon T. Because he cannot control the consumption after the horizon, his retirement income decreases to £28 upon age 110. Looking at the volatility of consumption, we conclude that the consumption level varies significantly with each scenario. Real consumption is more than twice as volatile for the retiree with $\gamma = -2$ than for $\gamma = -7$, and its standard deviation at age 80 is as high as £26.2.

We further observe that during retirement the individual consumes almost the entire cash-flow from the annuities (the black line indicating consumption on Fig. 5.3a is nearly as high as the bars showing the retirement income). Table 5.4 illustrates these findings in detail. Upon retirement the



(a) Power utility maximization, $\rho = 0.04$, $\gamma = -7$ (left) and $\gamma = -2$ (right).



(b) Disutility minimization, $\upsilon = 0.0$, $\kappa = 28.5$ (left) and $\kappa = 38.8$ (right).

Figure 5.3: Expected optimal consumption and retirement income in real terms (in £). Retirement income consists of the cash-flows from the annuities and the amount earned from selling bonds and stocks. After T (age 85) the individual consumes the income from the annuities purchased during the first 20 years and any savings that he has in stocks and bonds.

less risk averse individual consumes $\alpha_{t_0}^{C,0} = 33.6\%$ of his savings (£33.6), and spends the rest on the purchase of real (£13.4) and variable (£52.9) annuities. At the later stages, the unconditional consumption increases by $\alpha_t^{C,1} = 3.2\%$ per one standard deviation when the retiree expects high stock returns in the next period, and by $\alpha_t^{C,4} = 1.0\%$ per one standard deviation when he expects an increase in nominal interest rates.

Moreover, independently of the risk aversion, the unconditional purchase of any asset during retirement is marginal ($\alpha_t^{j,0} < 2$ (£), $\forall_{j \in \mathcal{J}}$), which indicates that the main investment and consumption decisions are made upon retirement. Afterwards the individual makes only small re-adjustments to

	Consumption	Purchase (£)					Sale (£)	
	(%)	Nominal	Real	Variable	Bonds	Stocks	Bonds	Stocks
Power utility maximization, $\gamma = -7$								
Constant, $\alpha_{t_0}^{j,0}$	30.1	10.4	35.1	23.1	1.3	0.0	0.0	0.0
Constant, $\alpha_t^{j,0}$	98.4	1.1	1.3	0.4	0.4	0.2	0.8	0.2
Stock returns, $\alpha_t^{j,1}$	1.9	-0.3	-0.6	0.0	-0.2	0.1	0.1	-0.1
Inflation, $\alpha_t^{j,2}$	0.1	0.0	-0.1	-0.1	0.0	-0.2	0.0	0.1
Real returns $\alpha_t^{j,3}$	0.1	0.1	-0.3	0.0	0.0	0.3	0.0	0.0
Nominal returns, $\alpha_t^{j,4}$	0.8	-0.4	-0.2	0.0	-0.1	0.0	0.0	-0.1
Power utility maximization, $\gamma = -2$								
Constant, $\alpha_{t_0}^{j,0}$	33.6	0.0	13.4	52.9	0.0	0.0	0.0	0.0
Constant, $\alpha_t^{j,0}$	98.0	1.5	1.9	0.5	0.0	0.0	0.0	0.0
Stock returns, $\alpha_t^{j,1}$	3.2	-0.9	-2.0	-0.2	0.0	0.0	0.0	0.0
Inflation, $\alpha_t^{j,2}$	0.1	-0.0	-0.1	-0.2	0.0	0.0	0.0	0.0
Real returns $\alpha_t^{j,3}$	0.5	0.0	-0.8	-0.1	0.0	0.0	0.0	0.0
Nominal returns, $\alpha_t^{j,4}$	1.0	-0.4	-0.5	0.2	0.0	0.0	0.0	0.0

	Consumption	Purchase (£)					Sale (£)	
	(%)	Nominal	Real	Variable	Bonds	Stocks	Bonds	Stocks
Disutility minimization, $\kappa = 28.5$ **(£)**								
Constant, $\alpha_{t_0}^{j,0}$	25.6	0.0	67.9	2.8	0.0	3.7	0.0	0.0
Constant, $\alpha_t^{j,0}$	95.1	0.9	1.2	0.7	1.4	1.9	1.0	1.8
Stock returns, $\alpha_t^{j,1}$	-0.1	0.5	0.1	0.2	0.3	-5.3	-0.7	-0.2
Inflation, $\alpha_t^{j,2}$	0.2	-0.1	-0.3	-0.0	-0.3	3.2	0.5	-0.0
Real returns $\alpha_t^{j,3}$	-0.3	0.3	0.0	0.1	0.6	3.0	-0.5	-0.0
Nominal returns, $\alpha_t^{j,4}$	0.3	0.3	0.4	0.1	-1.1	-7.3	-0.7	-0.2
Disutility minimization, $\kappa = 38.8$ **(£)**								
Constant, $\alpha_{t_0}^{j,0}$	24.3	0.0	43.3	13.9	0.0	18.5	0.0	0.0
Constant, $\alpha_t^{j,0}$	83.4	4.7	6.0	3.6	7.0	10.0	5.1	8.9
Stock returns, $\alpha_t^{j,1}$	-1.3	2.7	0.6	0.9	1.3	-26.1	-3.5	-1.0
Inflation, $\alpha_t^{j,2}$	0.8	-0.4	-1.5	-0.0	-1.6	15.8	2.3	-0.1
Real returns $\alpha_t^{j,3}$	-1.6	1.6	0.2	0.7	2.9	14.6	-2.7	-0.1
Nominal returns, $\alpha_t^{j,4}$	-0.2	1.2	1.8	0.5	-5.3	-35.6	-3.6	-1.2

Table 5.4: Regression coefficients indicating the conditional and unconditional optimal consumption relative to the liquid wealth (in %) and conditional and unconditional optimal traded amount (in £, given that the individual trades at all). The regression coefficients are calculated jointly for $t = \{t_1, t_2\}$, i.e. upon ages 70 and 75.

the portfolio. Konicz et al. (2014b) show that when a retiree has access to immediate and deferred annuities, both with different maturities, he never consumes the entire cash-flows from annuities, but keeps a certain amount for rebalancing purposes. In addition, he invests in liquid assets (stocks and bonds) and explores time-varying investment opportunities more frequently when he has a bequest motive. In this study we consider only life long immediate annuities and no bequest motive, therefore we observe a different behaviour of a retiree.

Disutility minimization Afterwards, we also analyze the optimal solution for an individual with a different objective, i.e. the minimization of squared deviations from a target consumption $\widehat{C}$, which is adjusted to inflation. We allow the target to be a variable in the program, but to account for the risk aversion we set a lower limit on this target (Eq. (5.2.3)). The higher the limit, the more aggressive the investment strategy, and the lower the risk aversion. We calculate κ in relation to the payout from the real annuities. In particular, given that investing 100% of the savings in inflation-linked annuities provides 5-year real cash-flows of £25.9, we specify different degrees of risk aversion by increasing this level by 10% and 50%. Thus, we define a more risk averse individual by $\kappa = 28.5$ (£) and a less risk averse individual by $\kappa = 33.8$ (£).

Figure 5.3b shows the expected optimal consumption and retirement income for a disutility minimizing investor. We observe that independently of the risk aversion, upon retirement the individual invests in real and variable annuities, as well as in stocks. The ratio between these assets varies with the degree of risk aversion—91% in real annuities, 4% in variable annuities, and 5% in stocks for the more risk averse investor; 57%, 18%, and 25%, respectively, for the less risk averse investor. In addition, the less risk averse retiree also invests in nominal annuities and bonds. Table 5.3 presents further details regarding the optimal expected level of consumption and asset allocation.

Individuals minimizing squared deviations from the target rebalance their portfolio more frequently than those maximizing the CRRA utility, see Table 5.4. For example, upon retirement the less risk averse individual with $\kappa = 38.8$ (£) invests £18.5 in stocks. From the capital gains he purchases (upon ages 70 and 75) on average £4.7 of nominal annuities, £6.0 of real annuities, £3.6 of variable annuities, £7.0 of bonds and £10.0 of stocks. He buys all assets when expecting an increase in the real rates ($\alpha_t^{j,3} > 0$, $\forall_{j \in \mathcal{J}}$). Moreover, he purchases the annuities when expecting high stock returns and an increase in nominal returns ($\alpha_t^{j,1}, \alpha_t^{j,4} > 0$, $\forall_{j \in \mathcal{A}}$), bonds when expecting high stock returns ($\alpha_t^{B,1} = 1.3$ (£)), and stocks when expecting an increase in the inflation rate ($\alpha_t^{S,2} = 15.8$ (£)). Once the individual achieves the target, he penalizes deviations above the target and follows a more risk averse strategy. Therefore, when expecting high stock returns he purchases bonds rather than stocks.

The optimal solution is a trade-off between reaching the target and minimizing squared deviations from the target. On average the retiree does not reach the target but his consumption increases with inflation. Comparing the consumption and investment decisions in the disutility minimization framework and in the power utility maximization framework, we find that the latter leads on average to a higher but more volatile consumption (achieved by following a significantly more aggressive investment strategy). Upon age 80, the individual with $\gamma = -2$ consumes £37.7, whereas the individual with $\kappa = 38.8$ (£) consumes £34.2. The volatility of the consumption is £26.2 and £7.5, respectively. Given the anticipated stock returns, individuals consume their savings differently, i.e. when expecting high stock returns, individuals maximizing the CRRA utility consume more, whereas individuals minimizing their loss function consume less.

Parameter sensitivity To check the robustness of the optimal solution, we perform a sensitivity analysis. We start by investigating different levels of the inflation risk premium. Christensen et al.

(2010) show that over the past several years the implied inflation risk premium has fluctuated mainly within a band of ±50 basis points around zero, therefore we investigate the impact of this parameter on the optimal solution by running the optimization programs for different values of the inflation risk premium. Table 5.5 shows that independently of the objective function and the degree of risk aversion, the level of risk premium significantly affects the optimal asset allocation in favor of either real or nominal bonds—the lower the expected realized inflation, the higher the investment in nominal annuities (and vice versa). In particular, given that the premium for inflation risk minus the liquidity premium is equal to 50 bp, individuals with $\gamma = -7$, $\gamma = -2$, $\kappa = 28.5$ (£), and $\kappa = 38.8$ (£), invest respectively 68%, 18%, 67%, and 56% of the initial savings in nominal annuities. In addition, except for the retiree with $\kappa = 28.5$ (£), the individuals do not allocate any of their savings to real annuities.

Power utility maximization	$\gamma = -7$					$\gamma = -2$				
Criteria	Nominal	Real	Variable	Bonds	Stocks	Nominal	Real	Variable	Bonds	Stocks
Benchmark	15	50	33	2	0	0	20	80	0	0
+50 bp	0	66	31	3	0	0	28	72	0	0
–50 bp	68	0	32	0	0	18	0	82	0	0
$\rho = 0.10$	25	38	33	3	0	0	14	80	7	0
$\rho = -0.10$	0	69	31	0	0	0	22	78	0	0
$age_0 = 60$	0	58	29	12	0	0	19	73	8	0
$age_0 = 70$	20	45	35	0	0	0	17	83	0	0
$\delta = \mathbb{E}[rpi_t]$	0	62	33	4	0	0	20	80	0	0
$\delta = -\mathbb{E}[rpi_t]$	17	51	32	0	0	0	20	80	0	0
$\eta = 0.1$	17	40	33	10	0	0	13	77	10	0
$\eta = 0.3$	6	28	35	31	0	0	0	70	28	2

Disutility minimization	$\kappa = 28.5$ (£)					$\kappa = 38.8$ (£)				
Criteria	Nominal	Real	Variable	Bonds	Stocks	Nominal	Real	Variable	Bonds	Stocks
Benchmark	0	91	4	0	5	0	57	18	0	24
+50 bp	0	94	4	0	2	0	71	19	0	10
–50 bp	67	24	4	0	5	56	0	18	0	26
$\nu = 0.10$	0	84	8	4	4	0	13	42	21	24
$\nu = -0.10$	0	95	1	0	4	0	78	6	0	16
$age_0 = 60$	0	87	2	1	10	0	35	12	5	48
$age_0 = 70$	0	94	6	0	0	0	72	28	0	0
$\delta = \mathbb{E}[rpi_t]$	0	91	4	0	5	0	58	18	0	23
$\delta = -\mathbb{E}[rpi_t]$	0	91	4	0	5	0	57	19	0	25
$\eta = 0.1$	0	85	5	0	10	0	51	19	0	30
$\eta = 0.3$	0	62	8	10	21	0	29	22	0	49

Table 5.5: Parameters sensitivity of the optimal asset allocation (in %) upon the retirement age. Benchmark: $\mathbb{E}[rpi_t] = \mathbb{E}[bei]$, $\rho = 0.04$, $\nu = 0.0$, $age_0 = 65$, $\delta = 0$, $T = 20$, $\Delta t = 5$. The term ±50 bp denotes an increase/decrease in the expected realized inflation by ±0.5%.

The optimal solution is also affected by the choice of the time preference factors ρ and ν. In the CRRA utility framework parameter ρ controls the desired consumption path, see, e.g., Konicz and Mulvey (2014). The higher the time preference factor, the more important the current consumption is to the individual, thus he consumes more in the beginning of the retirement. Therefore, impatient individuals (e.g. with $\rho = 0.10$) allocate less of their savings to real annuities (whose value follows the inflation index) and purchase more nominal and variable annuities. More patient individuals (e.g. with $\rho = -0.10$) prefer higher consumption when they become older, therefore they increase the allocation to real annuities. In the disutility minimization framework a low time preference factor (such as $\nu = -0.1$) reflects the importance of minimizing squared deviations from the target in the older age rather than shortly after retirement. Consequently, the optimal portfolio for low values of ν is less volatile than for high values of ν, i.e. consists of more real annuities rather than stocks and variable annuities.

We further investigate how the optimal portfolio depends on the retirement age.[9] The survival credit (the excess return provided by annuities over the liquid assets with the same risk) is lower for younger individuals, therefore, under both objective functions, younger individuals prefer flexibility and allocate more savings to liquid assets. This shift to liquid assets is the most visible for the less risk averse retiree minimizing his disutility. He allocates 48%, 24%, and 0% of his savings to stocks, if he retires, respectively, upon age 60, 65, and 70. In addition, no matter what the risk aversion is, retirees minimizing squared deviations from the inflation-linked target increase the allocation to real and variable annuities with age. In contrast, CRRA utility maximizing individuals decrease their allocation to real annuities with age, implying that the inflation risk is more important when planning for the longer horizon.

The results also indicate that nominal annuities with increasing payments equal to the expected realized inflation ($\delta = 3.35\%$) are not desirable by any of the individuals. The price of these annuities is similar to the nominal price of real annuities, however, in real terms, nominal annuities are more risky, and therefore less preferable. Furthermore, the considered individuals are upon retirement indifferent between the nominal annuities with constant payments and the nominal annuities with payments decreasing with the expected realized inflation ($\delta = -3.35\%$).

Finally, because of the absence of the bequest motive, especially retirees who obtain a CRRA utility from their consumption are reluctant to invest in bonds and stocks. They prefer to earn the survival credit rather than to explore the time-varying investment opportunities. Therefore, we investigate the optimal asset allocation for individuals who face some liquidity constraints. In particular, we add the following constraint to the MSP formulation:

$$\sum_{j \in \mathcal{K}} price^j_{t,n} Hold^j_{t,n} \geq \eta \sum_{j \in \mathcal{J}} price^j_{t,n} Hold^j_{t,n}, \qquad t = t_0, \ldots, T-1,\ n \in \mathcal{N}_t, \tag{5.6.7}$$

where η is the minimum weight of the portfolio allocated to liquid assets (either bonds or stocks) at each stage. We read from Table 5.5 that the CRRA utility maximizing retiree allocates his

[9] When pricing annuities we assume the maximum survival age $\omega = 110$. Changing the retirement age affects the time until the maximum survival age, and accordingly the level of the annuity income and the level of the expected break-even inflation (which is calculated as an average across all the maturities). Thus, to compare the results with the benchmark case, we adjust the expected realized inflation so that $\mathbb{E}[rpi_t] = \mathbb{E}[bei]$ for the particular retirement age, i.e. $\mathbb{E}[rpi_t] = 3.42\%$ for $age_0 = 60$ and $\mathbb{E}[rpi_t] = 3.30\%$ for $age_0 = 70$. Because in the benchmark case for the disutility minimization we have specified κ as 110% and 150% of the retirement income obtained from investing the entire portfolio in real annuities, we have to adjust these values for each retirement age. Consequently, we set $\kappa = 23.86$ (£) for the more risk averse individual and $\kappa = 32.54$ (£) for the less risk averse retiree, both retiring upon $age_0 = 60$. For $age_0 = 70$, we set $\kappa = 34.8$ (£) and $\kappa = 47.5$ (£), respectively.

portfolio to bonds as exactly imposed by the lower bound, at the same time decreasing the weight in real annuities. The disutility maximizing individual facing liquidity constraints also reduces his allocation to real annuities, though, primarily in favor of stocks.

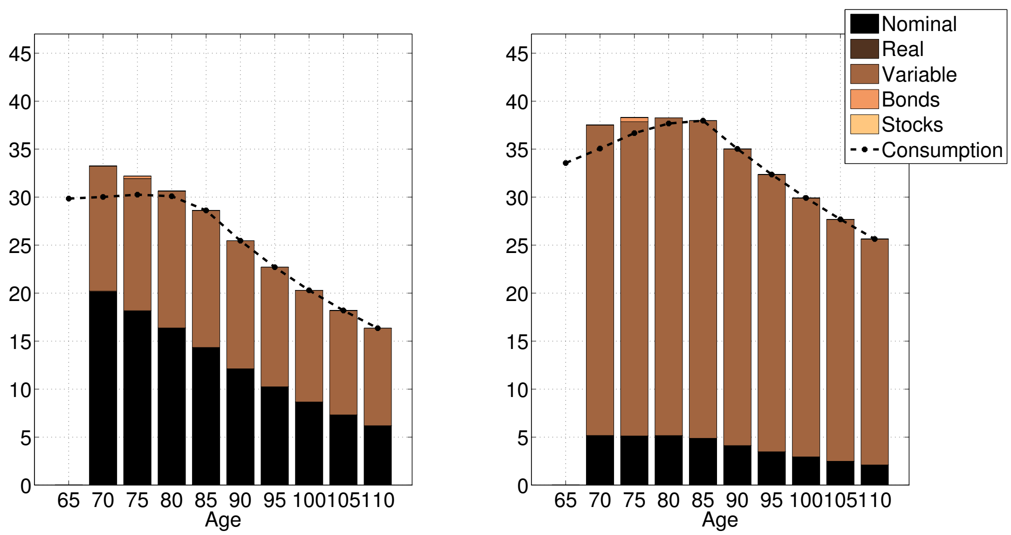
No access to inflation-linked annuities The results from the considered optimization models indicate that in most of the cases retirees should allocate significant proportions of their retirement savings to real annuities. The weight in these assets differs with the objective function and the risk aversion. Nevertheless, having in mind that 95% of British retirees are reluctant to purchase inflation-linked annuities, we explore how they can optimally allocate their savings without investing in real annuities. In particular, we solve the same two optimization problems, but with variable $Buy^R_{t,n}$ set to zero.

Power utility maximization		$\gamma = -7$				$\gamma = -2$			
		65	70	75	80	65	70	75	80
Asset allocation	Nominal	69	66	64	62	19	19	20	21
	Real	0	0	0	0	0	0	0	0
	Variable	31	34	36	38	81	80	80	79
	Bonds	0	0	0	0	0	1	0	0
	Stocks	0	0	0	0	0	0	0	0
Consumption	$\mathbb{E}[C_t]$	29.9	30.1	30.3	28.8	33.6	35.1	36.7	38.1
	$\sigma[C_t]$	0.0	3.9	6.6	12.1	0.0	10.3	16.6	27.1

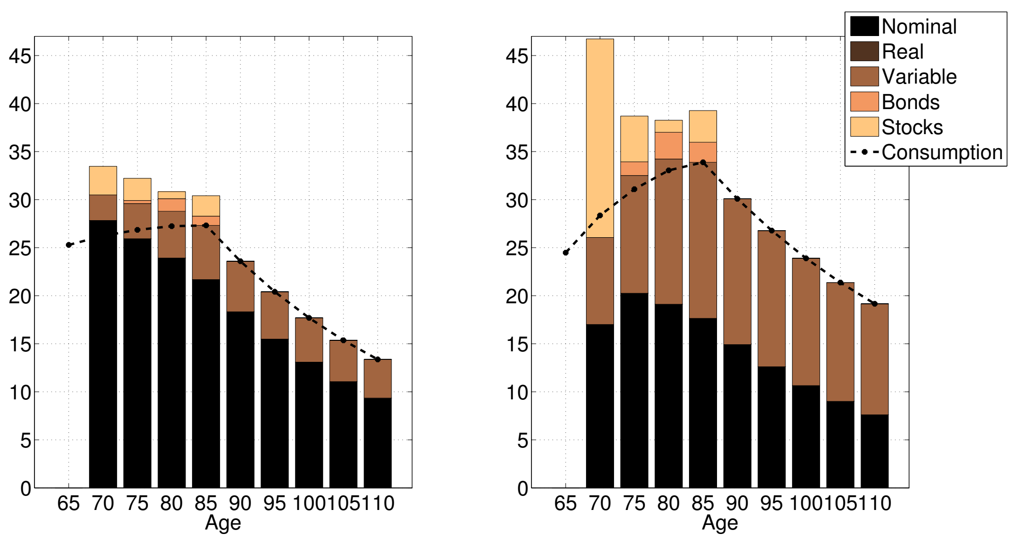
Disutility minimization		$\kappa = 28.5$ (£)				$\kappa = 38.8$ (£)			
		65	70	75	80	65	70	75	80
Asset allocation	Nominal	89	86	83	79	54	63	60	58
	Real	0	0	0	0	0	0	0	0
	Variable	6	8	11	14	20	26	33	36
	Bonds	0	2	3	2	0	4	5	2
	Stocks	5	4	3	5	26	7	2	4
Consumption	$\mathbb{E}[C_t]$	25.3	26.2	26.9	27.3	24.5	28.4	31.1	33.8
	$\sigma[C_t]$	0.0	1.2	1.4	2.8	0.0	5.1	5.3	8.5

Table 5.6: The expected optimal asset allocation (rounded to the nearest percent) and consumption (in £) under the power utility maximization with $\rho = 0.04$ and the disutility minimization with $\upsilon = 0$, and given no investment in real annuities.

Figure 5.4a shows the results for an individual who maximizes the expected CRRA utility of consumption, while 5.4b for an individual who penalizes deviations from the target—both under the assumption of no investment in inflation-linked annuities. We observe that individuals try to hedge inflation risk by allocating the wealth to variable and nominal annuities, and the disutility minimizing individual additionally invests in bonds and stocks (see also Table 5.6). In most of the considered cases, retirees who decide not to allocate their savings to real annuities face lower and more volatile retirement income. The power utility maximizing individual mostly shifts his allocation from real annuities to nominal annuities, keeping the allocation to variable annuities almost unchanged. The disutility minimizing individual follows a more aggressive investment strategy than if he had access



(a) Power utility maximization, $\rho = 0.04$, $\gamma = -7$ (left) and $\gamma = -2$ (right).



(b) Disutility minimization, $\upsilon = 0.0$, $\kappa = 28.5$ (left) and $\kappa = 38.8$ (right).

Figure 5.4: Expected optimal consumption and retirement income in real terms (in £) without investment in real annuities. Retirement income consists of the cash-flows from the annuities and the amount earned from selling bonds and stocks. After T (age 85) the individual consumes the income from the annuities purchased during the first 20 years and any savings that he has in stocks and bonds.

to real annuities. To minimize the distance from the inflation-linked target he allocates most of the portfolio to nominal annuities, while also increasing the weight in stocks and variable annuities.

The regression results on the relative consumption and on the buy and sell amount (presented in Table 5.7) show that the lack of investment in inflation-linked annuities requires more frequent rebalancing. Accordingly, to benefit from time-varying investment opportunities, the retiree consumes less. To exemplify, the disutility minimizing retiree with $\kappa = 28.5$ (£) consumes 81% of his liquid wealth, whereas the retiree with $\kappa = 33.8$ (£) consumes 76% (in contrast to 95% and 83%, respectively, if they had access to real annuities). He spends the residual amount on the purchase

	Consumption	Purchase (£)				Sale (£)	
	(%)	Nominal	Variable	Bonds	Stocks	Bonds	Stocks
Power utility maximization, $\gamma = -7$							
Constant, $\alpha_{t_0}^{j,0}$	29.8	48.4	21.8	0.0	0.0	0.0	0.0
Constant, $\alpha_t^{j,0}$	94.0	2.3	1.3	1.8	0.0	3.5	0.0
Stock returns, $\alpha_t^{j,1}$	2.1	-0.5	0.2	0.0	0.0	-0.2	0.0
Inflation, $\alpha_t^{j,2}$	0.1	-0.1	0.0	1.7	0.0	0.1	0.0
Real returns $\alpha_t^{j,3}$	-0.6	0.2	0.4	0.0	0.0	-0.1	0.0
Nominal returns, $\alpha_t^{j,4}$	0.7	-0.2	0.3	0.0	0.0	-0.3	0.0
Power utility maximization, $\gamma = -2$							
$\alpha_{t_0}^{j,0}$	33.6	12.4	54.1	0.0	0.0	0.0	0.0
Constant, $\alpha_t^{j,0}$	95.8	2.5	1.3	0.0	0.0	0.0	0.0
Stock returns, $\alpha_t^{j,1}$	3.7	-1.6	0.0	0.0	0.0	0.0	0.0
Inflation, $\alpha_t^{j,2}$	0.2	-0.1	0.0	0.0	0.0	0.0	0.0
Real returns $\alpha_t^{j,3}$	-0.1	-0.3	0.4	0.0	0.0	0.0	0.0
Nominal returns, $\alpha_t^{j,4}$	1.0	-0.5	0.4	0.0	0.0	0.0	0.0

	Consumption	Purchase (£)				Sale (£)	
	(%)	Nominal	Variable	Bonds	Stocks	Bonds	Stocks
Disutility minimization, $\kappa = 28.5$ **(£)**							
Constant, $\alpha_{t_0}^{j,0}$	25.3	66.6	4.4	0.0	3.7	0.0	0.0
Constant, $\alpha_t^{j,0}$	80.9	4.2	2.8	4.4	2.9	3.7	4.7
Stock returns, $\alpha_t^{j,1}$	-1.1	0.0	1.5	1.9	-1.4	1.0	1.2
Inflation, $\alpha_t^{j,2}$	-0.4	-0.2	-0.3	-0.4	1.4	-0.6	-0.6
Real returns $\alpha_t^{j,3}$	-1.2	0.3	0.9	0.1	-1.1	0.2	0.7
Nominal returns, $\alpha_t^{j,4}$	0.3	0.4	1.2	1.1	-1.2	1.3	0.9
Disutility minimization, $\kappa = 38.8$ **(£)**							
$\alpha_{t_0}^{j,0}$	24.5	40.7	15.1	0.0	19.7	0.0	0.0
Constant, $\alpha_t^{j,0}$	76.3	7.4	5.7	8.8	7.9	6.0	11.1
Stock returns, $\alpha_t^{j,1}$	-3.0	1.7	1.7	3.6	-9.6	-2.3	-1.2
Inflation, $\alpha_t^{j,2}$	1.0	-1.0	-0.3	-1.2	5.4	1.1	-0.2
Real returns $\alpha_t^{j,3}$	-3.1	1.3	1.5	0.9	2.7	-0.3	-0.2
Nominal returns, $\alpha_t^{j,4}$	-1.2	2.1	1.0	-0.7	-13.5	-2.4	-0.7

Table 5.7: Regression coefficients indicating the conditional and unconditional optimal consumption relative to the liquid wealth (in %) and conditional and unconditional optimal traded amount (in £, given that the individual trades at all), given no investment in inflation-linked annuities. The regression coefficients are calculated jointly for $t = \{t_1, t_2\}$, i.e. upon ages 70 and 75.

of all available assets, following a similar investment strategy as if he had access to inflation-linked annuities. Specifically, to minimize squared deviations from the inflation-linked target, he buys all available assets when expecting an increase in the real rates, buys nominal and variable annuities when anticipating high stock returns and an increase in nominal returns, buys bonds when anticipating high stock returns, and buys stocks when anticipating an increase in the inflation rate.

5.7 Conclusions and future work

This paper studies the optimal consumption and investment decisions for a retiree facing inflation risk. Having access to nominal, real, and variable annuities, as well as to bonds and stocks, the individual optimizes his decisions under two different objectives: 1) maximization of the expected CRRA utility of real consumption, and 2) minimization of the expected disutility defined in terms of squared deviations from an inflation-linked target. In the first case we extend the literature by allowing to invest in both the annuities and the liquid assets, and to rebalance the portfolio during retirement. In the second case, we contribute to the literature by defining the aforementioned loss disutility, and by illustrating the optimal solution that minimizes this function. To solve both optimization problems, we apply a multi-stage stochastic programming approach.

Our findings show that independently of the considered objective function and risk aversion, the optimal asset allocation comprises real annuities. Variable annuities are the second crucial asset in the portfolio, even for risk averse investors. Numerical results show that retirees maximizing the CRRA utility of consumption make most of their decisions upon retirement, and even though they are allowed to rebalance the portfolio, they make only small re-adjustments during the retirement period. In contrast, individuals minimizing squared deviations from the inflation-linked target explore the time-varying investment opportunities by allocating more savings to the liquid assets, and by rebalancing the portfolio more frequently during the retirement period.

The allocation to nominal annuities is in most of the cases marginal; however, the optimal solution is sensitive to the inflation risk premium—the higher the premium, the higher the allocation to nominal annuities. Retirees who choose not to invest in real annuities (or do not have access to these products), try to hedge their portfolios against inflation by purchasing primarily nominal and variable annuities. Consequently, the real consumption is more volatile than if they invested in real annuities, and in most of the cases lower. The lack of real annuities in the portfolio also requires more frequent rebalancing.

The model could be improved in multiple ways. Some straightforward extensions include adding a bequest motive, additional contributions, and/or other annuities such as joint life or nominal annuities with different payouts. Among more advanced improvements, an inclusion of other inflation-linked products, e.g., those offering a deflation floor to the initial cash-flow, would definitely be worth to investigate.

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Chapter 6

Optimal retirement planning with a focus on single and multilife annuities

Submitted

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Abstract

We optimize the asset allocation, consumption, and bequest decisions of an investor with uncertain lifetime. The asset menu consists of stocks, zero coupon bonds and pure endowments with different maturities, and whole life annuities. The pure endowments pay either fixed or variable benefits, which, similarly to the annuities, are contingent on either a single or a joint lifetime. We model the stock returns and the parameters for the term structure with a vector autoregressive model, thus allowing for time-varying investment opportunities. To find the optimal solution we use a multi-stage stochastic programming approach, which allows for including complex surrender charges on pure endowments and annuities, as well as transaction costs on stocks and bonds. Our findings indicate that despite high surrender charges, life contingent products should be the primary asset class in a portfolio. Individuals should purchase these products not necessarily to consume the specific cash-flows, but to be able to rebalance the portfolio. We argue that the optimal retirement product for a household is much more complex than any of those available in the market, therefore annuity providers should design the products that are customized to the households' needs.

Keywords: Annuity • Household • Stochastic programming

6.1 Introduction

Annuities are the only investment vehicles that provide income for life, no matter how long one lives. This unique feature attracts many individuals who try to structure their income during retirement. In some countries the purchase of annuities has even been (or still is) mandatory;[1] in others, having an annuity is highly recommended as well as advertised.

Purchasing an annuity is not an easy task for a number of reasons. First, this decision concerns a significant amount of money. Second, it is the annuity provider who upon the individual's death inherits the potential residual wealth. Third, annuities have high surrender charges if sold before maturity. Fourth, there are so many different annuity products in the market that individuals simply do not know which product to buy.[2] An annuity purchase decision is not just a question of whether or not to buy an annuity. It is a question of buying the right one. Should the individuals purchase a whole life annuity or maybe a temporary annuity that provides the income only for a designated

[1] For example, UK had the compulsory annuity purchase until April 2011.

[2] See e.g., Annuity Shopper at `http://www.immediateannuities.com/pdfs/as/annuity-shopper-2013-10.pdf`

period? Should the annuity cover a single life or maybe both the wife's and the husband's joint life? What about the bequest? Is it worth to invest in an annuity with some guaranteed period?

Not surprisingly, having carefully built up the savings, the individuals find an annuity purchase to be one of the most important financial decisions in their life, and definitely worth seeking professional advice. However, such an advice can be costly and is rarely based on optimization approaches. An alternative is to use different rules of thumb for retirement planning. For example, to diversify the portfolio such that the percentage of bonds is equal to the individual's age, or to withdraw 4 percent of the retirement savings each year during retirement.

The purpose of this paper is to propose an optimization model that helps individuals to invest their retirement savings. The investment menu consists of stocks, zero coupon bonds and pure endowments with different maturities, and whole life annuities. The pure endowments pay either fixed or variable benefits, which, similarly to the whole life annuities, are contingent on either a single or a joint lifetime. These basic products allow us to replicate more complex annuity products, such as temporary life annuities, deferred annuities, and annuities with a guaranteed period. The objective in our model is to maximize the expected utility of consumption and bequest for either a one-person or a two-person household during the retirement period, given the individual's risk aversion, personal preferences, and lifetime expectancy.

Although several studies have discussed investment strategies that include annuities, they all focus on the distribution between stocks, bonds, and either whole life fixed or variable annuity products. Very few studies investigate the preferences between single and multilife annuities, and none of the studies tries to exploit other annuity options such as the preferences between the temporary and the whole life annuities, or life annuities with a guaranteed period. In other words, scholars have paid far too little attention to advise the individuals which annuity product fits best to their needs.

Yaari (1965) was the first to show that individuals gain from investing in annuities, and in particular, if no bequest is present, they should choose annuities rather than bonds. Milevsky and Young (2007) argue that even in the presence of a bequest motive individuals should hold some fixed annuities, and the proportion in the annuities should increase with the level of risk aversion. Horneff et al. (2008), moreover, show that the optimal stock fraction follows the well-known life-cycle pattern with the shift towards fixed annuities instead of bonds, and that bonds should be chosen only in the case of a substantial bequest motive. Horneff et al. (2009) extend this work by including variable annuities and argue that the attractiveness of these products lies in their high expected return consisting of an equity premium as well as a survival credit. Another class of annuities investigated in the recent years are deferred annuities. Both Scott et al. (2007) and Horneff et al. (2010) show that households benefit substantially from holding this type of annuities in their portfolios. Finally, Hubener et al. (2014) investigate the retirement plan for households and show that retired couples should primarily invest in joint and survivor annuities.

The aforementioned studies tend to focus on finding the explicit solution for the optimal consumption and portfolio allocation problems. However, in order to keep analytical tractability, one must assume a relatively simple model. Therefore, these studies do not implement transaction costs or surrender charges, and they assume bonds to be risk free. In this paper we apply a multi-stage stochastic programming (MSP) approach, known for its practical application. See, e.g., Mulvey et al. (2003) and Mulvey et al. (2008) for the application in defined benefit pension plans, or Ferstl and Weissensteiner (2011), Konicz and Mulvey (2013) and Konicz et al. (2014) for the application in the field of individual asset liability management. This numerical method allows to obtain the optimal solution under realistic assumptions. For example, we can easily add constraints on the asset allocation and on personal preferences, as well as include transaction costs on stocks and bonds, and complex surrender charges on pure endowments and annuities. The latter are the penalties for selling the life contingent products before maturity. In addition, the MSP approach allows for considering time-varying investment opportunities, i.e. the parameters for the term structure necessary for pricing bonds and pure endowments, and the returns on stocks and variable pure endowments follow a vector-autoregressive model. We generate the scenario trees with the joint evolution of the term structure and the stock market using the approach in Høyland and Wallace (2001) and Høyland et al. (2003), by matching the first four moments and the correlations of the underlying processes.

Our findings regarding the optimal asset allocation support the results obtained by the aforementioned studies, i.e. the allocation between the liquid and illiquid assets, and between the assets with different financial risk, highly depends on the bequest motive and the level of risk aversion. However, in contrast to other studies, our findings also exploit different annuity payout options such as the number of lives covered, the designated period, and the variability of the cash-flows. Among other conclusions, we show that despite high surrender charges, individuals should diversify their portfolios by investing in a wide variety of pure endowments and annuities. They should purchase these products not necessarily to consume the specific cash-flows, but to be able to rebalance the portfolio during retirement. Finally, two-person households should hold the largest number of different products and benefit especially from investing in variable joint and survivor pure endowments as well as in variable pure endowments contingent on the husband's life. We summarize by arguing that the right annuity for a household is much more complex than any of those available in the market, and therefore annuity providers should design the products highly customized to the household's needs.

The structure of the paper is as follows. Sections 6.2 and 6.3 describe the general model setup, available assets together with the accompanying costs, and details regarding the model uncertainty. Section 6.4 presents the multi-stage stochastic model and describes the scenario generation method, the objective function and the constraints. Section 6.5 analyzes the numerical results for a one- and a two-person household, and Section 6.6 concludes and suggests directions towards retirement products design.

6.2 Model description

We seek for the optimal consumption, investment, and bequest decisions for individuals and couples upon retirement. When defining the objective function, we follow a classical approach, e.g., Richard (1975), and maximize the expected utility of consumption and bequest during retirement and until some horizon T, given that the lifetime of the individual is uncertain,

$$\max \ \mathbb{E}\left[\sum_{t=t_0}^{T} {}_tp_x u(t, C_t) + \sum_{t=t_0}^{T-1} {}_tp_x \, q_{x+t} \, k \, u(t, W_t)\right], \tag{6.2.1}$$

where u is a utility function with a constant relative risk aversion (CRRA) equal to $1-\gamma$, and with a time preference (impatience) factor ρ, i.e.

$$u(t, C_t) = \tfrac{1}{\gamma} e^{-\rho t} C_t^{\gamma}. \tag{6.2.2}$$

We assume the same utility function for the consumption C_t and for the bequest W_t, with a scalar k reflecting the strength of the bequest motive relative to consumption. Furthermore, we multiply the utility of consumption by the survival probability of an x-year-old person ${}_tp_x$, and the utility of bequest by the probability that an x-year-old person survives until time t but dies during the following period ${}_tp_x \, q_{x+t}$. When maximizing the objective, the individual can invest in stocks, zero coupon bonds and pure endowments with different maturities, and whole life annuities—all described in more detail in the following subsection. Upon horizon T the individual annuitizes her wealth, i.e. spends all the cash-flows provided by the assets held in the portfolio and all the savings invested in liquid assets on a purchase of the whole life fixed income annuities.

For a two-person household, we maximize the expected consumption of two individuals, given that they are both alive. The objective further captures different possibilities for the bequest motive, depending on whether the bequest is for the children (in case of the death of both parents) or specifically for the spouse (in case of the death of one of the spouses). In the first case the couple chooses a positive k; otherwise, a positive $k^{(x)}$, $k^{(y)}$, or both. Thus, we extend the objective function suggested in Bruhn and Steffensen (2011) by introducing the following objective:

$$\begin{aligned} \max \qquad & \mathbb{E}\Big[\sum_{t=t_0}^{T} {}_tp_{x:y} u(t, C_t) + \sum_{t=t_0}^{T-1} {}_tp_{x:y} \, q_{\overline{x+t:y+t}} \, k \, u\big(t, W_t\big) \\ + \quad & \sum_{t=t_0}^{T-1} {}_tp_{x:y} \Big(p_{x+t} \, q_{y+t} \, k^{(x)} \, u\big(t, W_t + A_t^{(x)}\big) + q_{x+t} \, p_{y+t} \, k^{(y)} \, u\big(t, W_t + A_t^{(y)}\big)\Big)\Big], \end{aligned} \tag{6.2.3}$$

where in line with the International Actuarial Notation ${}_tp_{x:y}$ denotes the joint survival probability until time t of the wife aged x and the husband aged y, and $q_{\overline{x+t:y+t}}$ denotes that both the $x+t$-year old wife and the $y+t$-year old husband die during the following period. We assume that the lives of the spouses are independent, thus ${}_tp_{x:y} = {}_tp_x \, {}_tp_y$ and $q_{\overline{x+t:y+t}} = q_{x+t} \, q_{y+t}$, where the probabilities ${}_tp_x$ and ${}_tp_y$ correspond to the survival probabilities of the wife and the husband, respectively, and

q_{x+t} and q_{y+t} correspond to death probabilities during the following period. Upon the couple's death, their children inherit amount W_t, upon the husband's death, his wife inherits $W_t + A_t^{(x)}$, and upon the wife's death, her husband inherits $W_t + A_t^{(y)}$. We define these bequest amounts later in Sect. 6.4.2. Similarly to the case of a single individual, upon horizon T the couple annuitize their wealth by purchasing whole life fixed annuities.

6.2.1 Available assets

When planning for retirement, households typically consider stocks, bonds, and different annuities. Because annuities play the most important role in retirement planning we start by defining these products and explain their payout options.

We choose to refer to annuities simply as "a sum of money payable yearly or at other regular intervals", see Dellinger (2006). Depending on the conditions under which this sum is paid, we categorize annuities into the following categories: number of lives covered, payout option, first payment date, and measure, pattern, and frequency of the benefits. Despite so many different annuities, we can replicate their payout by using a linear combination of four basic financial products: a zero coupon bond, a pure endowment, a 100% joint and surivivor pure endowment, and a variable pure endowment.

A zero coupon bond pays $cf_M^b = 1$ upon maturity M, therefore its price at any time $t \leq M$ is given by

$$price_t^b = e^{-y(\boldsymbol{\beta}_t, M)(M-t)}, \tag{6.2.4}$$

where $y(\boldsymbol{\beta}_t, M)$ is the interest rate p.a. over the period $[t, M)$ and $\boldsymbol{\beta}_t$ is the parameter defining the term structure described later in Sect. 6.3.

A pure endowment is similar to a zero coupon bond and pays $cf_M^a = 1$ upon maturity M. However, a pure endowment is life contingent, therefore pays conditionally on the individual's survival. Because the lifetime of an individual is uncertain, insurers use mortality tables to price the life contingent products. The expected lifetimes of women and men are different, therefore the prices are gender-specific[3]—products contingent on a female lifetime are more expensive than those contingent on a male lifetime because women on average live longer. The price at time t of a pure endowment paying out at time M is then given by

$$price_t^a = \begin{cases} {}_{M-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}_t, M)(M-t)}, & \text{for a female,} \\ {}_{M-t}p_{y+t} \cdot e^{-y(\boldsymbol{\beta}_t, M)(M-t)}, & \text{for a male,} \end{cases} \tag{6.2.5}$$

where ${}_{M-t}p_{x+t}$ (${}_{M-t}p_{y+t}$) is the probability that a female aged $x+t$ (male aged $y+t$) survives until

[3]In the last years the European Court of Justice ruled that gender may not be used in pricing of life contingent products, and since December 2012 the unisex pricing is in force according to the European law, see e.g `http://www.thisismoney.co.uk/money/pensions/article-1713762/How-EU-gender-rule-hits-your-pension.html` Nevertheless, in our model we take a more general approach, and price the products according to the individual's gender, as it is still the case in many countries such as the U.S., see e.g., Annuity Shopper at `http://www.immediateannuities.com/pdfs/as/annuity-shopper-2013-10.pdf`.

time M.

A pure endowment conditional on the joint lifetime such as *a 100% J&S pure endowment* pays $cf^a_M = 1$ upon maturity M on condition that at least one of the annuitants is alive. The price of this product is of the same form as the price of a pure endowment contingent on a single life, but with the suitable survival probability, i.e.

$$price^a_t = {}_{M-t}p_{\overline{x+t:y+t}} \cdot e^{-y(\boldsymbol{\beta}_t, M)(M-t)}, \tag{6.2.6}$$

where ${}_{M-t}p_{\overline{x+t:y+t}}$ is the probability that at least one of the spouses aged $x+t$ and $y+t$, respectively, survives until time M.

Finally, a *variable pure endowment* is also a life contingent product, however, in contrast to a pure endowment, the level of the payment is linked to some (risky) portfolio backing this product (e.g., a portfolio replicating the FTSE100 index). As the return on this portfolio is uncertain, neither the annuity provider nor the annuitant knows the value of the future cash-flow before the maturity. Therefore, to price this product, annuity providers choose some deterministic Assumed Interest Rate $\bar{r}$, and define the cash-flow as the excess return on the underlying portfolio over $\bar{r}$, i.e.

$$cf^a_M = e^{r(t,M) - \bar{r}(M-t)}, \tag{6.2.7}$$

and the prices as follows:

$$price^a_t = \begin{cases} {}_{M-t}p_{x+t} \cdot e^{-\bar{r}(M-t)} \cdot e^{r(t_0,t) - \bar{r}(t-t_0)}, & \text{for a female,} \\ {}_{M-t}p_{y+t} \cdot e^{-\bar{r}(M-t)} \cdot e^{r(t_0,t) - \bar{r}(t-t_0)}, & \text{for a male,} \end{cases} \tag{6.2.8}$$

where $r(t, M)$ is the realized return on the portfolio backing the annuity over the period $[t, M)$. We adjust the price by the excess return $e^{r(t_0,t)-\bar{r}(t-t_0)}$ to account for the realized stock returns, thus no matter when the individual buys the variable pure endowment, the return on this product reflects the realized stock return from the initial time t_0. Correspondingly, a variable pure endowment conditional on the survival of at least one annuitant has a price of

$$price^a_t = {}_{M-t}p_{\overline{x+t:y+t}} \cdot e^{-\bar{r}(M-t)} \cdot e^{r(t_0,t) - \bar{r}(t-t_0)}, \tag{6.2.9}$$

and provides the payment (6.2.7) as long as at least one of the annuitants is alive.

Having access to these four basic assets, i.e., a zero coupon bond, a pure endowment, a 100% J&S pure endowment, and a variable pure endowment, we can replicate the payout from a wide range of annuities. Depending on the number of lives covered, we distinguish between: *single life annuities*—payout is linked to the survival of one individual; and $Y\%$ *joint and survivor annuities*—payout is linked to the survival of two lives, i.e. after the first death of either annuitant, the annuity pays $Y\%$ of the initial benefit as long as the survivor is alive. Independently of the number of lives covered, the annuities can be further classified with respect to their different payout options such as the designated period and the guaranteed period: *whole life annuities*—pay as long as the annuitant is alive; *temporary life annuities*—pay for the shorter of M years or the life of the annuitant, typically, $M = \{10, 15, 20, 25\}$ years; and *life annuities with X years guaranteed*—pay as long as the annuitant is alive, if the annuitant dies within the first X years, payments continue to the dependants until the end of the X^{th} year. We also distinguish between fixed and variable annuities: *fixed annuities*—pay fixed benefits, typically constant or increasing; *variable annuities*—pay variable benefits that vary with the return on the portfolio that backs the annuity. Finally, we categorize the annuities by the time of the initial payment: *immediate annuities*—payments commence upon the purchase of the annuity; and *deferred annuities*— payments commence at some future date.

For example, a *whole life annuity deferred 20 years* for a female currently aged x (see Fig. 6.1, left) starts the payment upon the deferment time and pays until her death. To replicate the payout from this annuity, we can combine pure endowments with maturities s such that $20 \leq s \leq \infty$.[4] Then, the price of the a deferred whole life annuity during the deferment period is given by

$$price_t^a = \sum_{s=M}^{\omega-x-1} {}_{s-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}_t,s)(s-t)}, \tag{6.2.10}$$

where ω is the age upon which the individual is assumed to be dead with certainty.

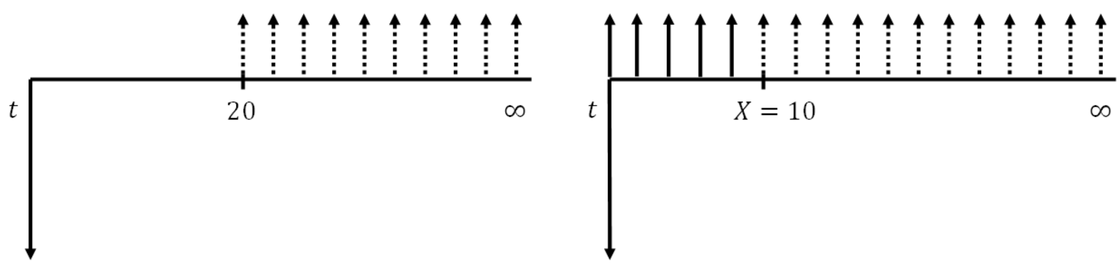


Figure 6.1: Annuity cash-flows. The arrows pointing downwards denote the purchase of the annuity; pointing upwards, the payments. The continuous line arrows indicate the certain payments; the dashed line arrows, the payments conditional on the annuitant's survival. The left figure shows that the cash-flow from a whole life annuity deferred 20 years can be replicated by a number of pure endowments with the shortest maturity equal to the time of the first payment, and the longest maturity equal to the maximum survival age. The right figure shows that the cash-flow from an immediate annuity with $X = 10$ years guaranteed can be replicated by a combination of zero coupon bonds with the maturities $M < X$, and pure endowments with $M \geq X$.

6.2.2 Costs

We assume the transaction costs for stocks and bonds, and the mortality and expense charges (M&E) and surrender charges for annuities and pure endowments. M&E charge pays for the insurance guarantee, commissions, selling, and administrative expenses of the contract, therefore its level is equal to a percentage of the investment value paid upon the purchase of the product. Surrender

[4] In practice, annuity providers assume that individuals are dead with probability one upon age 110, thus the maturity date for the last pure endowment is set to the 109th birthday of the annuitant.

charges, which are the penalties for withdrawals exceeding a certain free amount specified in the contract, are paid upon the sale. We define the variable $Withdraw_t^a$ denoting the actual amount that the annuitant receives from the sale of the pure endowment or annuity a as follows:

$$Withdraw_t^a = \begin{cases} Free_t^a + (price_t^a Sell_t^a - Free_t^a)(1 - sc_t^a), & price_t^a Sell_t^a > Free_t^a, \\ price_t^a Sell_t^a, & price_t^a Sell_t^a \leq Free_t^a, \end{cases} \tag{6.2.11}$$

where $price_t^a Sell_t^a$ is the market value of annuity a at time t, sc_t^a are the surrender charges on annuity a, and $Free_t^a$ is the free withdrawal amount determined as a percentage fw^a of the initial investment,

$$Free_t^a = \begin{cases} fw_t^a \, price_t^a Buy_t^a, & t = t_0, \\ Free_{t-1}^a + fw_t^a \, price_t^a Buy_t^a - \min(Free_{t-1}^a, price_t^a Sell_t^a), & t > t_0, \end{cases} \tag{6.2.12}$$

where $price_t^a Buy_t^a$ is the market value of annuity a upon the purchase at time t. The free withdrawal amount is adjusted every time the individual trades the annuity.

6.3 Modeling uncertainty

The optimal solution to the considered problem is stochastic and depends on the uncertainty of the term structure and the stock returns.

Term structure of interest rates To model the evolution of the yield curve, we propose the Nelson/Siegel model for two reasons. First, this parametric model can condense the entire yield curve to a few parameters, so we can ensure computational tractability by keeping the branching factor of the scenario tree low. Second, parsimonious models are better suited for predictions out-of-sample than theoretically more advanced affine term structure models, see, e.g., Diebold and Li (2006), Ang et al. (2007), and Coroneo et al. (2011). The three-factor model for the spot rates is given by:

$$y(\boldsymbol{\beta}_t, M) = \beta_{1,t} + \beta_{2,t}\left(\frac{1 - e^{-\lambda_t M}}{\lambda_t M}\right) + \beta_{3,t}\left(\frac{1 - e^{-\lambda_t M}}{\lambda_t M} - e^{-\lambda_t M}\right), \tag{6.3.1}$$

where $y(\boldsymbol{\beta}_t, M)$ denotes the yearly (continuously compounded) spot rate for maturity M at time t given the parameter vector $\boldsymbol{\beta}_t = [\beta_{1,t}, \beta_{2,t}, \beta_{3,t}]^\top$ for the level, slope, and curvature of the term structure of interest rates.

Time-varying investment opportunities of the term structure and stock returns We model the time-varying investment opportunities with a VAR(1)-process, see, e.g., Barberis (2000), Campbell et al. (2003), and Brandt et al. (2005), for an application in asset allocation decisions, and Boender et al. (2005) and Ferstl and Weissensteiner (2011) for the combined evolution of interest rates and equity returns. We use the following $(K \times 1)$ parameter vector $\boldsymbol{\xi}_t$ (with $K = 4$):

$$\boldsymbol{\xi}_t = \begin{bmatrix} r_t \\ \boldsymbol{\beta}_t \end{bmatrix}, \tag{6.3.2}$$

where r_t refers to the log equity return and $\boldsymbol{\beta}_t$ to the (3×1) vector of Nelson/Siegel parameters. The VAR(1) process can be written as:

$$\boldsymbol{\xi}_t = \mathbf{c} + \mathbf{A}\boldsymbol{\xi}_{t-1} + \mathbf{u}_t, \tag{6.3.3}$$

where $\mathbf{c}$ is the $(K \times 1)$ vector of intercepts, $\mathbf{A}$ is the $(K \times K)$ matrix of slope coefficients, and $\mathbf{u}_t$ is the $(K \times 1)$ vector of i.i.d. innovations with $\mathbf{u} \sim N(0, \boldsymbol{\Sigma})$. The covariance of the innovations $\boldsymbol{\Sigma}$ is given by $\mathbb{E}(\mathbf{u}\,\mathbf{u}^\top)$. Thus, we allow the shocks to be cross-sectionally correlated, but assume that they are homoskedastic and independently distributed over time. If all eigenvalues of $\mathbf{A}$ have modulus less than one, the stochastic process (6.3.3) is stable with unconditional expected mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Gamma}$ for the steady state at $t = \infty$ (see e.g., Lütkepohl, 2005):

$$\begin{aligned} \boldsymbol{\mu} : \quad &= \quad (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c} && (6.3.4)\\ \operatorname{vec}(\boldsymbol{\Gamma}) : \quad &= \quad (\mathbf{I} - \mathbf{A} \otimes \mathbf{A})^{-1}\operatorname{vec}(\boldsymbol{\Sigma}), && (6.3.5) \end{aligned}$$

where $\mathbf{I}$ refers to the identity matrix, the symbol $\otimes$ is the Kronecker product and $\operatorname{vec}(\cdot)$ transforms a $(K \times K)$ matrix into a $(K^2 \times 1)$ vector by stacking the columns.

When using decision steps longer than one month but calibrating the VAR process to monthly data, we follow Pedersen et al. (2013) to calculate the aggregated stock returns between two decision stages. For notation brevity, we define $\boldsymbol{\zeta}_\tau$ as the vector of *cumulated* stock returns and the Nelson/Siegel parameters,[5] and introduce an indicator matrix $\mathbf{J} = \operatorname{diag}(1, 0, 0, 0)$. Then, the expectation and the covariance of $\boldsymbol{\zeta}_\tau$ of Eq. (6.3.3) for a general number of time time steps τ (i.e., months) are given by:

$$\mathbb{E}(\boldsymbol{\zeta}_\tau) = \left(\left(\sum_{i=1}^{\tau-1} (\mathbf{I} + \mathbf{J}(\tau - i))\, \mathbf{A}^{i-1} \right) + \mathbf{A}^{\tau-1} \right) \mathbf{c} + \left(\mathbf{A}^\tau + \sum_{i=1}^{\tau-1} \mathbf{J}\, \mathbf{A}^i \right) \boldsymbol{\xi}_0, \tag{6.3.6}$$

and

$$\begin{aligned} \mathbb{V}(\boldsymbol{\zeta}_\tau) = \boldsymbol{\Sigma} \\ &+ (\mathbf{J} + \mathbf{A})\, \boldsymbol{\Sigma}\, (\mathbf{J} + \mathbf{A})^\top \\ &+ \left(\mathbf{J} + \mathbf{J}\,\mathbf{A} + \mathbf{A}^2\right) \boldsymbol{\Sigma} \left(\mathbf{J} + \mathbf{J}\,\mathbf{A} + \mathbf{A}^2\right)^\top \\ &+ \ldots \\ &+ \left(\mathbf{A}^{\tau-1} + \sum_{i=1}^{\tau-1} \mathbf{J}\, \mathbf{A}^{i-1} \right) \boldsymbol{\Sigma} \left(\mathbf{A}^{\tau-1} + \sum_{i=1}^{\tau-1} \mathbf{J}\, \mathbf{A}^{i-1} \right)^\top . \end{aligned} \tag{6.3.7}$$

[5] The Nelson/Siegel parameter vector is the same for $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$.

6.4 MSP formulation

Multi-stage stochastic programming is a general purpose framework for modelling optimization problems, broadly applied in operations research. MSP consists of a scenario tree representing the range of possible outcomes, and of an optimization module including the objective function and the set of constraints. (We explain both concepts in the following subsections.) MSP can handle a wide variety of objective functions and can easily address realistic considerations and constraints, as long as they have an algebraic form. A program solver computes the optimal decisions, which depend on the decisions made at the ancestor nodes, on the realizations of the random vector, and on the anticipation of the possible future outcomes. The multi-stage stochastic programming approach combines anticipative and adaptive models in one mathematical framework, therefore it is particularly appealing in financial applications. For example, an investor composes his portfolio given the anticipation of possible future movements of asset prices, and rebalances the portfolio as prices change. For more details about the MSP approach, see, e.g., Birge and Louveaux (1997) and Zenios (2008).

6.4.1 Scenario tree and scenario tree generation

The uncertainty in the MSP approach is represented by a *scenario tree*, which consists of nodes $n \in \mathcal{N}_t$ uniquely assigned to periods t. The initial stage t_0 of the tree has only one node n_0, which is the ancestor for all the nodes at the subsequent stage t_1. These nodes are further the ancestors for their children nodes, etc., until the final stage T. As the nodes at the final stage have no children, they are called the leaves. We define a scenario $\mathcal{S}^n$ as a single branch from the root node to the leaf, i.e. each scenario consists of a leaf node n and all its predecessors up to the root node n_0. Consequently, the number of scenarios in the tree equals the number of leaves. Each node has a probability pr_n, so that $\forall_t \ \sum_{n\in\mathcal{N}_t} pr_n = 1$, implying that the probability of each scenario $\mathcal{S}^n$ is equal to the product of the probabilities of all the nodes in the scenario.

The main limitation of a multi-stage stochastic programming approach is related to the *curse of dimensionality*, i.e. the problem size grows exponentially as a function of the number of periods and scenarios, and the problem quickly becomes too large to be computationally tractable. Therefore, to generate scenarios, we have to approximate the discrete-time multivariate process in Eq. (6.3.3) with a few mass points, accordingly reducing the computational complexity. A considerable amount of literature focuses on scenario generation methods such as sampling, simulation, scenario reduction techniques and moment matching methods. For the purpose of our study we have chosen the technique that matches the statistical properties (the first four moments and the correlations) of the underlying processes. This approach has been introduced by Høyland and Wallace (2001) and Høyland et al. (2003), who suggest solving a nonlinear optimization problem that minimizes the distance between the properties of the generated tree and of the underlying process. We start the tree construction from the unconditional expected values as done, e.g., by Campbell et al. (2003) and Ferstl and Weissensteiner (2011). For the subsequent time periods we use the conditional expected value and covariance (Eqs. (6.3.6) and (6.3.7)).

Having generated the stock returns and the parameters for the term structure interest rates, $\boldsymbol{\zeta}_{t,n} = [r_{t,n}, \boldsymbol{\beta}_{t,n}]'$, we can calculate the input parameters for the MSP. In particular, we rewrite to a nodal form formulae presented in Sect. 6.2.1 for the asset prices and cash-flows. For example, contingent on a female lifetime, the prices at node n and time t of a pure endowment with maturity M, Eq. (6.2.5), of a variable pure endowment with maturity M, Eq. (6.2.8), and of a whole life annuity deferred until time M, Eq. (6.2.10), are given by:

$$\begin{aligned}
price^a_{t,n} &= {}_{M-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}_{t,n},M)(M-t)}, && \text{where } a \text{ is a pure endowment with mat. } M,\\
price^a_{t,n} &= {}_{M-t}p_{x+t} \cdot e^{-\bar{r}(M-t)} \cdot e^{r_n(t_0,t)-\bar{r}(t-t_0)}, && \text{where } a \text{ is a variable pure endowment with mat. } M,\\
price^a_{t,n} &= \sum_{s=M}^{\omega-x-1} {}_{s-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}_{t,n},s)(s-t)}, && \text{where } a \text{ is a whole life annuity def. until time } M.
\end{aligned}$$

Afterwards, we can calculate the optimal investment, consumption, and bequest decisions, which depend on the possible future realizations of the asset prices and the cash-flows at each node of the tree, and on the decisions made in the previous stage.

6.4.2 Optimization module

The optimization module consists of an objective function and a set of constraints defined for each scenario. We start the formulation of the multi-stage stochastic program by summarizing in Table 6.1 the sets, the parameters, and the variables included in the program. Each variable starts with a capital letter and has a subscript $\{t, n\}$ implying that its value depends on the node n and stage t. Subscript $\{t^-, n^-\}$ denotes the predecessor of node $n \in \mathcal{N}_t$.

A one-person household model

The objective is to maximize the expected utility of consumption and bequest for an individual with an uncertain lifetime. In our discrete time and state framework the objective function defined in (6.2.1) is given by,

$$\max \sum_{t=t_0}^{T} \sum_{n\in\mathcal{N}_t} {}_tp_x\, \alpha_{t,n}\, u(t, C_{t,n})\, prob_n + \sum_{t=t_0}^{T-1} \sum_{n\in\mathcal{N}_t} {}_tp_x\, q_{x+t}\, k\, u(t, W_{t,n})\, prob_n, \tag{6.4.1}$$

where $u(t, C_{t,n}) = \frac{1}{\gamma} e^{-\rho t} C_{t,n}$ and

$$\alpha_{t,n} = \begin{cases} \sum_{s=t}^{t+\Delta t-1} {}_{s-t}p_{x+t} \cdot e^{-y(\boldsymbol{\beta}_{t,n},s)(s-t)}, & t < T,\\ \sum_{s=t}^{\omega-x-1} {}_{s-t}p_{x+t} \cdot e^{-y\boldsymbol{\beta}_{t,n},s)(s-t)}, & t = T. \end{cases} \tag{6.4.2}$$

Variable $C_{t,n}$ denotes the yearly consumption, thus to account for the length of the subsequent interval Δt, we multiply the utility of consumption by $\alpha_{t,n}$. The multiplier $\alpha_{t,n}$ is equal to the value of a single life annuity providing the income during the subsequent period.

Sets	
$\mathcal{N}_t$	nodes at period t,
$\mathcal{S}$	stocks,
$\mathcal{B}$	bonds,
$\mathcal{A}$	pure endowments and annuities,
$\mathcal{A}^{(\infty)}$	subset of $\mathcal{A}$ including only the whole life annuities with fixed payments,
$\mathcal{A}^{(x)}$ ($\mathcal{A}^{(y)}$)	subset of $\mathcal{A}$ including only the pure endowments and annuities contingent on the wife's (husband's) lifetime,
$\mathcal{A}^{(J\&S)}$	subset of $\mathcal{A}$ including only the 100% J&S pure endowments and annuities,
Parameters	
$cf^a_{t,n}$ ($cf^b_{t,n}$)	cash-flow generated by pure endowment/annuity a (bond b) at time t, node n,
$price^a_{t,n}$ ($price^b_{t,n}$, $price^s_{t,n}$)	price of pure endowment/annuity a (bond b, stock s) at time t, node n,
me^a	M&E, mortality and expense charge for pure endowment/annuity a,
sc^a_t	surrender charge on pure endowment/annuity a, at time t,
tc^b (tc^s)	transaction costs for bond b, stock s,
fw^a	percentage rate of a free withdrawal on pure endowment/annuity a,
w_0	initial savings,
ρ	time preference (impatience) factor,
ω	assumed maximum age at which an individual would die with certainty,
k ($k^{(x)}$, $k^{(y)}$)	weight on the bequest motive for the children (wife, husband),
${}_tp_x$ (${}_tp_y$)	probability that a female aged x survives until age $x+t$ (a male aged y survives until age $y+t$),
q_{x+t} (q_{y+t})	probability that a female aged $x+t$ (a male aged $y+t$) dies during the following period,
${}_tp_{x:y}$	probability that both a female aged x and a male aged y survive until ages $x+t$ and $y+t$,
$q_{\overline{x+t:y+t}}$	probability that both a female aged $x+t$ and a male aged $y+t$ die during the following period,
$prob_n$	probability of node n,
$\alpha_{t,n}$	multiplier for the utility of consumption at node n, stage t,
Δt	length of the subsequent period,
T	horizon (annuitization time),
Variables	
$Buy^a_{t,n}$ ($Buy^b_{t,n}$, $Buy^s_{t,n}$)	units of pure endowment/annuity a (bond b, stock s) purchased at time t, node n,
$Sell^a_{t,n}$ ($Sell^b_{t,n}$, $Sell^s_{t,n}$)	units of pure endowment/annuity a (bond b, stock s) sold at time t, node n,
$Hold^a_{t,n}$ ($Hold^b_{t,n}$, $Holds^s_{t,n}$)	units of pure endowment/annuity a (bond b, stock s) held at time t, node n, $Hold^a_{t_0^-,n_0^-} = Hold^b_{t_0^-,n_0^-} = Hold^s_{t_0^-,n_0^-} = 0$,
$W_{t,n}$	total wealth (after rebalancing) in bonds and stocks at time t, node n,
$C_{t,n}$	consumption at time t, node n,
$A^{(x)}_{t,n}$ ($A^{(y)}_{t,n}$)	wealth in pure endowments/annuities contingent on a female (male) lifetime at time t, node n,
$Free^a_{t,n}$	free withdrawal amount on pure endowment/annuity a,
$Withdraw^a_{t,n}$	actual amount received from the sales of pure endowment/annuity a at time t, node n,
$Y^{a+}_{t,n}$, $Y^{a-}_{t,n}$	auxiliary variables for pure endowment/annuity a.

Table 6.1: Parameters and variables for the multi-stage stochastic program.

The individual makes consumption, investment, and bequest decisions until horizon T. Upon T he annuitizes his wealth, i.e. rebalances the portfolio for the last time by spending all the liquid wealth on the purchase of the whole life fixed annuity, see Fig. 6.1. Thus, as long as the individual is alive, he receives the cash-flows provided by the whole life fixed annuity, and he consumes exactly these cash-flows. He has no bequest motive after T.

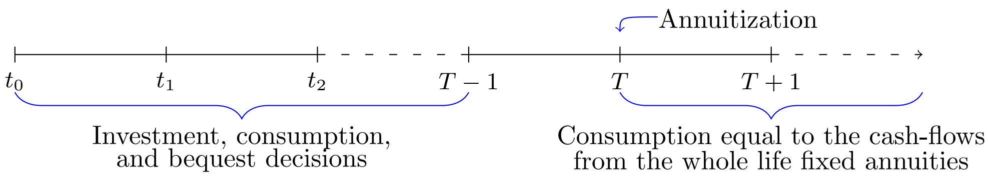


Figure 6.1: Overview of the model.

The firsts constraint in the program is the budget constraint, (6.4.3), which controls all the incoming and outgoing payments. The portfolio is self financing, i.e. there is no other exogenous income than the initial savings w_0 at time t_0. At each period, the individual spends some of the savings for consumption and finances the purchase of new assets from the cash-flows from the bonds and pure endowments purchased in the previous periods, and from selling the assets. The individual pays transaction costs tc^s and tc^b on stocks and bonds, respectively, the mortality and expense me^a charges upon the purchase of the life contingent assets, and the surrender charges sc_t^a upon the sale of the life contingent assets. Thus, we formulate the budget constraint for $t = t_0, \dots, T-1,\ n \in \mathcal{N}_t$ as follows:

$$\begin{aligned}
\alpha_{t,n}\, C_{t,n} \;=\;& w_0 1_{\{t=t_0\}} \\
&+ \sum_{a\in\mathcal{A}} Withdraw_{t,n}^a - \sum_{a\in\mathcal{A}} price_{t,n}^a Buy_{t,n}^a (1+me^a) + \sum_{a\in\mathcal{A}} cf_{t,n}^a Hold_{t,n}^a \\
&+ \sum_{b\in\mathcal{B}} price_{t,n}^b Sell_{t,n}^b (1-tc^b) - \sum_{b\in\mathcal{B}} price_{t,n}^b Buy_{t,n}^b (1+tc^b) + \sum_{b\in\mathcal{B}} cf_{t,n}^b Hold_{t,n}^b \\
&+ price_{t,n}^s Sell_{t,n}^s (1-tc^s) - price_{t,n}^s Buy_{t,n}^s (1+tc^s),
\end{aligned} \tag{6.4.3}$$

where $1_{\{t=(\cdot)\}}$ denotes the indicator function equal to 1 if $t = (\cdot)$, and 0 otherwise.

To implement the surrender charges using linear constraints, we rewrite (6.2.11)-(6.2.12) in the following way:

$$price_{t,n}^a Sell_{t,n}^a + Y_{t,n}^{a+} = Free_{t,n}^a + Y_{t,n}^{a-} \tag{6.4.4}$$

$$Free_{t,n}^a = fw^a\, price_{t,n}^a Buy_{t,n}^a + Y_{t^-,n^-}^{a+} \tag{6.4.5}$$

$$Withdraw_{t,n}^a = price_{t,n}^a Sell_{t,n}^a - Y_{t,n}^{a-}\, sc_t^a \tag{6.4.6}$$

$$Y_{t,n}^{a+} \geq 0, \quad Y_{t,n}^{a-} \geq 0, \quad Free_{t,n}^a \geq 0, \tag{6.4.7}$$

for $t = t_0, \dots, T,\ n \in \mathcal{N}_t$. In this reformulation, we have introduced two auxiliary variables $Y_{t,n}^{a+}$ and $Y_{t,n}^{a-}$, only one of which is positive at each node n. If the individual sells annuities for the value that is lower or equal the free withdrawal amount, $price^a_{t,n}Sell^a_{t,n} - Free^a_{t,n} \leq 0$, then $Y^{a+}_{t,n} \geq 0$ and $Y^{a-}_{t,n} = 0$, and the individual does not pay any charges. However, if he sells more than the free withdrawal amount, $price^a_{t,n}Sell^a_{t,n} - Free^a_{t,n} > 0$, then $Y^{a-}_{t,n} \geq 0$ and $Y^{a+}_{t,n} = 0$, and he pays surrender charges on the amount $Y^{a-}_{t,n}$. The variable $Withdraw^a_{t,n}$ is the actual amount the individual receives from selling the life contingent products.

Upon horizon T, conditional on being alive, the individual invests all the remaining wealth in the whole life annuity with fixed and constant payments, thus consumes $C_{T,n}$ (equal to the level of the whole life annuity payments) until his death. Thus, the terminal condition for all $n \in \mathcal{N}_T$ is as follows:

$$
\begin{aligned}
0 &= \sum_{a\in\mathcal{A}\setminus\mathcal{A}^{(\infty)}} Withdraw^a_{T,n} - \sum_{a\in\mathcal{A}^{(\infty)}} price^a_{T,n}Buy^a_{T,n}(1+me^a) + \sum_{a\in\mathcal{A}\setminus\mathcal{A}^{(\infty)}} cf^a_{T,n}Hold^a_{T,n} \\
&+ \sum_{b\in\mathcal{B}} price^b_{T,n}Sell^b_{T,n}(1-tc^b) + \sum_{b\in\mathcal{B}} cf^b_{T,n}Hold^b_{T,n} \\
&+ price^s_{T,n}Sell^s_{T,n}(1-tc^s) && (6.4.8) \\
C_{T,n} &= \sum_{a\in\mathcal{A}^{(\infty)}} cf^a_{T,n}Hold^a_{T,n}. && (6.4.9)
\end{aligned}
$$

The bequest amount is equal to the value of non-life contingent assets, i.e. stocks and bonds, after rebalancing,

$$
W_{t,n} = \sum_{b\in\mathcal{B}} \big(price^b_{t,n} - cf^b_{t,n}\big)Hold^b_{t,n} + price^s_{t,n}Hold^s_{t,n}, \qquad t = t_0\ldots,T-1,\ n \in \mathcal{N}_t. \tag{6.4.10}
$$

Finally, for each asset class we define the inventory constraints,

$$
Hold^i_{t,n} = Hold^i_{t^-,n^-} + Buy^i_{t,n} - Sell^i_{t,n}, \qquad t = t_0,\ldots,T,\ n \in \mathcal{N}_t,\ i \in \{\mathcal{A},\mathcal{B},\mathcal{S}\}, \tag{6.4.11}
$$

with $Buy^i_{T,n} = 0$, for $n \in \mathcal{N}_T$ and $i \in \{\mathcal{A},\mathcal{B},\mathcal{S}\} \setminus \mathcal{A}^{(\infty)}$, reflecting the annuitization, and impose non-negativity constraints on the consumption, purchase, sale, and hold variables,

$$
\begin{aligned}
&C_{t,n} \geq 0, && t = t_0,\ldots,T,\ n \in \mathcal{N}_t, && (6.4.12)\\
&Buy^i_{t,n} \geq 0,\ Sell^i_{t,n} \geq 0,\ Hold^i_{t,n} \geq 0, && t = t_0,\ldots,T,\ n \in \mathcal{N}_t,\ i \in \{\mathcal{A},\mathcal{B},\mathcal{S}\}. && (6.4.13)
\end{aligned}
$$

A two-person household model

The multi-stage stochastic programming formulation for a two-person household is similar to the case of a one-person household. The objective is to maximize the expected utility of consumption given that both the wife and the husband are alive, and of the bequest in three possible situations: (1) both the wife and the husband die, (2) the wife is alive and the husband dies, and (3) the wife dies and the husband is alive. Therefore, by choosing parameters k, $k^{(x)}$ and $k^{(y)}$, we model the strength of the bequest motive for the children, or specifically for one of the spouses,

$$\begin{aligned} \max \quad & \sum_{t=t_0}^{T} \sum_{n \in \mathcal{N}_t} {}_t p_{x:y} \, \alpha_{t,n} \, u(t, C_{t,n}) \, prob_n + \sum_{t=t_0}^{T-1} \sum_{n \in \mathcal{N}_t} {}_t p_{x:y} \, q_{\overline{x+t:y+t}} \, k \, u(t, W_{t,n}) \, prob_n \\ + \; & \sum_{t=t_0}^{T-1} \sum_{n \in \mathcal{N}_t} {}_t p_{x:y} \Big(p_{x+t} \, q_{y+t} \, k^{(x)} \, u\big(t, W_{t,n} + A^{(x)}_{t,n}\big) + q_{x+t} \, p_{y+t} \, k^{(y)} \, u\big(t, W_{t,n} + A^{(y)}_{t,n}\big) \Big) \, prob_n. \end{aligned} \tag{6.4.14}$$

The model for a two-person household consists of all the constraints for a single person household, i.e. (6.4.3)-(6.4.13), and the constraints defining the values $A^{(x)}_{t,n}$ and $A^{(y)}_{t,n}$ included in the objective function. Upon the husband's death, the wife inherits the savings invested in stocks and bonds W_t, and the value of pure endowments and annuities contingent on her lifetime,

$$A^{(x)}_{t,n} = \sum_{a \in \mathcal{A}^{(x)} \cup \mathcal{A}^{(J\&S)}} price^a_{t,n} Hold^a_{t,n}, \qquad t = t_0, \ldots, T-1, \; n \in \mathcal{N}_t, \tag{6.4.15}$$

where the subset $\mathcal{A}^{(x)}$ includes the pure endowments and whole life annuities contingent on the wife's lifetime, and the subset $\mathcal{A}^{(J\&S)}$ includes the pure endowments and whole life annuities contingent on the survival of at least one of the spouses. Similarly, upon the wife's death, the husband inherits $W_{t,n} + A^{(y)}_{t,n}$, where

$$A^{(y)}_{t,n} = \sum_{a \in \mathcal{A}^{(y)} \cup \mathcal{A}^{(J\&S)}} price^a_{t,n} Hold^a_{t,n}, \qquad t = t_0, \ldots, T-1, \; n \in \mathcal{N}_t, \tag{6.4.16}$$

and where the subset $\mathcal{A}^{(y)}$ includes all the assets contingent on the husband's lifetime.

The terminal conditions (6.4.8)-(6.4.9) ensure that if both spouses are alive upon horizon, they annuitize their wealth by purchasing whole life fixed annuities. Specifically, they can purchase either the whole life annuity contingent on the wife's life, on the husband's life, or on the joint life, or any combination of these annuities. Moreover, for ethical reasons, we add one more constraint imposing that after the horizon both spouses should receive equal benefits,

$$\sum_{a \in \mathcal{A}^{(x)}} cf^a_{T,n} Hold^a_{T,n} = \sum_{a \in \mathcal{A}^{(y)}} cf^a_{T,n} Hold^a_{T,n}, \qquad n \in \mathcal{N}_T. \tag{6.4.17}$$

6.5 Numerical results

In the beginning of this section we explain the choice of the model parameters and the available assets. Afterwards, we illustrate the optimal investment, consumption, and bequest decisions for a one- and a two-person household.

6.5.1 Parameters

MSP formulation The optimization model is based on a scenario tree with 4 periods of length 5 years. Thus, $T = 20$ and $\Delta t = 5$. The branching factor at each node is set to 10, therefore, in our model we have 10,000 scenarios. We implemented the program on a Dell computer with an Intel Core i5-2520M 2.50 GHZ processor and 4 GB RAM, using Matlab 8.2.0.713 (R2013b), and GAMS 24.1.3 with the non-linear solver MOSEK 7.0.0.75. The computational time varies between 3 and 12 minutes, depending on the choice of the parameters.

Uncertainty parameters To model the joint evolution of the term structure of interest rates and of the stock market, we use data from the UK market. In particular, we use historical zero coupon interest rates available at the Bank of England's website,[6] and adjusted prices of the FTSE100 index from Thomson Reuters Datastream. We consider monthly data from January 1993 to December 2013.

To calculate the term structure of the interest rates, (6.3.1), we follow Diebold and Li (2006) and fix λ_t so that it minimizes the mean squared error in our data set, i.e. $\lambda_t = 0.4218$. Then, the estimation of the remaining parameters $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ simplifies to an ordinary least square (OLS) regression. In addition, fixing λ leads to stability of the estimated parameters over time—a property required to capture the dynamic behaviour with a stochastic model. Table 6.1 presents the parameters $\mathbf{c}$ and $\mathbf{A}$ for the intercepts and slope coefficients for the vector autoregressive process, (6.3.3). All eigenvalues of the matrix $\mathbf{A}$ have modulus less than one, therefore the process is stable. The monthly Nelson/Siegel coefficients $\beta_{i,t-1}$ are highly persistent as well as statistically significant. Table 6.2 shows the unconditional expected mean $\boldsymbol{\mu}$ in the steady state, as well as the monthly standard deviations (on the main diagonal) and cross correlations of residuals (above the main diagonal). The term structure in the steady state is concave and increasing, see Fig. 6.1. Furthermore, we test the plausibility of the yield curves in our scenario tree. Table 6.3 presents the percentiles of spot rates with different maturities for cumulative probabilities p_z, with $p_z \in \{0.1, 0.5, 0.9\}$ at the final decision stage. Intuitively, the variability of the spot rates is higher for short-rate yields. The expected return and volatility for equities are equal to 5.7% and 14.14% p.a., respectively.

	A				**c**	R^2
	r_{t-1}	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$		
r_t	0.0168	0.1827	-0.0566	-0.0464	-0.0077	0.0099
	(0.2633)	(0.9175)	(-0.3906)	(-0.4548)	(-0.7485)	
$\beta_{1,t}$	-0.0025	0.9537	-0.0136	0.0145	0.0022	0.9689
	(-0.5894)	(72.2721)	(-1.4129)	(2.1377)	(3.1581)	
$\beta_{2,t}$	0.0181	0.0357	0.9912	0.0093	-0.0022	0.9715
	(2.8948)	(1.8239)	(69.6406)	(0.9316)	(-2.1291)	
$\beta_{3,t}$	-0.0126	0.0452	0.0196	0.9582	-0.0022	0.9405
	(-1.0655)	(1.2231)	(0.7267)	(50.5729)	(-1.1514)	

Table 6.1: VAR(1) parameters and t-statistics (in parentheses) for monthly data from Jan 1993 to Dec 2013.

[6]Bank of England, `http://www.bankofengland.co.uk/statistics/pages/yieldcurve/default.aspx`

	r	β_1	β_2	β_3
$\boldsymbol{\mu}$	0.0047	0.0526	-0.0368	-0.0143
r	4.0900	-0.0770	0.0860	-0.0066
β_1		0.2710	-0.7383	-0.2168
β_2			0.4016	0.0271
β_3				0.7594

Table 6.2: Unconditional expected values $\boldsymbol{\mu}$ for the steady state, cross correlations, and standard deviations (multiplied by 100) of residuals for monthly data from Jan 1993 to Dec 2013.

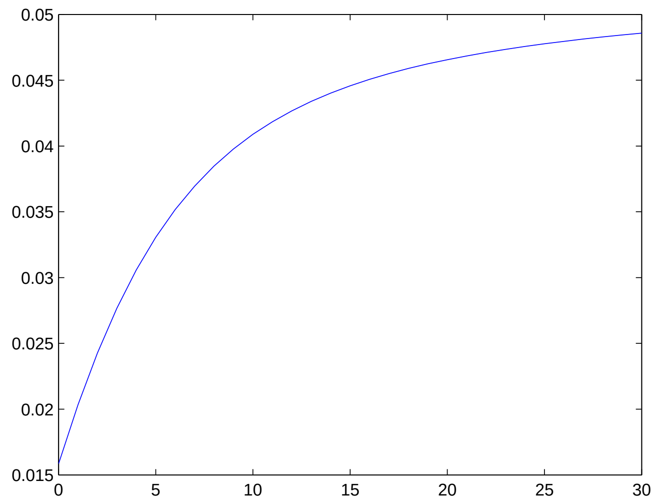


Figure 6.1: Term structure of interest rates for the steady state.

	5y	10y	15y	20y	25y	30y
$p_z = 0.1$	0.0057	0.0195	0.0261	0.0290	0.0305	0.0315
$p_z = 0.5$	0.0325	0.0403	0.0442	0.0461	0.0474	0.0482
$p_z = 0.9$	0.0620	0.0630	0.0636	0.0643	0.0655	0.0661

Table 6.3: Percentiles of the yield curve for different maturities at the final stage.

Available assets and costs The investment universe consists of non-life and life contingent assets. The non-life contingent asset menu consists of stocks with an average return 5.7% p.a., and zero coupon bonds with maturities $M \leq T$ defined with 5-year-long intervals. We define the maturities for the initial time t_0, thus they correspond to the age of the individual at the subsequent stages, see Fig. 6.2. Upon retirement individual can purchase bonds with maturities $M = \{5, 10, 15, 20\}$ years, providing the return $y(\boldsymbol{\beta}_t, M)$ p.a. and paying the cash-flows when the individual is 70, 75, 80, and 85 years old, respectively. At the next stage, the bond with a maturity of 5 years expires, and the maturity of the bonds available upon retirement decreases by 5 years. Consequently, the individual has access only to bonds with $M = \{5, 10, 15\}$, etc. Finally, upon horizon T no bonds are available. The transaction costs for both stocks and bonds are set to $tc^b = tc^s = 1\%$.

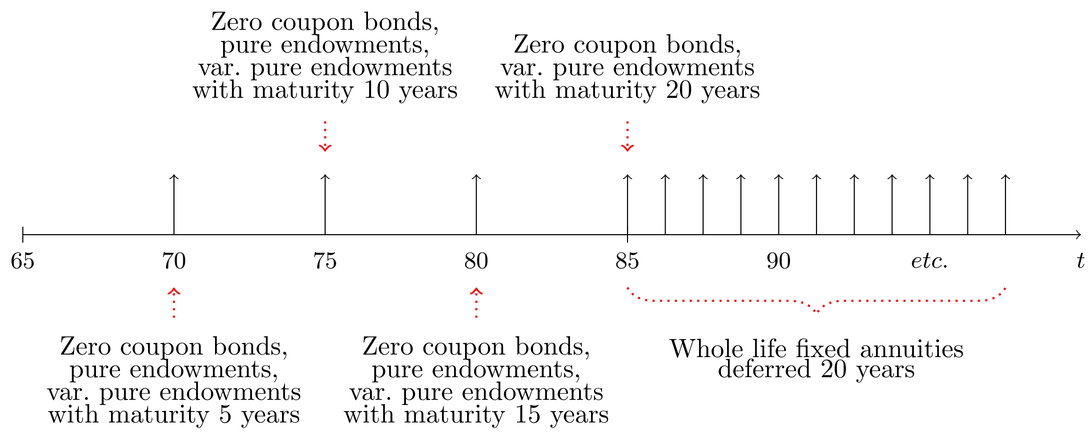


Figure 6.2: The timing of the cash-flows (indicated by the black continuous-line arrows pointing upwards) provided by the zero coupon bonds, pure endowments, variable pure endowments, and annuities available upon retirement.

The availability of pure endowments and annuities depends on the gender of the individual, and whether we consider a single or a two-person household. Therefore, we specify the available life contingent products in the following subsections corresponding to a particular optimization problem. To calculate the prices, we use the British mortality tables based on 2000-2006 experience from UK self-administered pension schemes.[7] Independently of the gender, the M&E charge is set to 1%, thus is equal to the transaction costs for stocks and bonds. Surrender charges sc_t^a are much higher but decrease with the time to maturity as shown in Table 6.4. The free withdrawal rate is set to $fw^a = 10\%$.[8]

	Time of withdrawal (years), t				
Product	0	5	10	15	20
Pure endowment, M5	5%				
Pure endowment, M10	10%	5%			
Pure endowment, M15	15%	10%	5%		
Pure endowment, M20	20%	15%	10%	5%	
Whole life annuity, def20	20%	20%	20%	20%	20%

Table 6.4: Surrender charges sc_t^a for pure endowments maturing in 5, 10, 15 and 20 years and a whole life annuity deferred 20 years.

[7] http://www.actuaries.org.uk/research-and-resources/documents/s1pml-all-pensioners-excluding-dependants-male-lives

[8] In practice, surrender charges differ between the annuity providers and can range from 1% to 20% depending on the type of annuity. The closer to the maturity, the lower the charges. See financial websites for private investors, e.g., http://money.cnn.com/retirement/guide/annuities_basics.moneymag/index9.htm and http://www.fool.com/retirement/annuities/annuities02.htm.

6.5.2 Optimal decisions for a one-person household

We start by analyzing the numerical results for a single household. Our individual is a 65-year-old woman upon retirement, with initial savings of $w_0 = 100$ (£) and time preference factor $\rho = 5\%$.[9] Upon horizon T, the individual is 85 years old. Among the assets contingent on her lifetime, she has access to pure endowments with the maturities of 5, 10, and 15 years, and a whole life annuity deferred 20 years. Thus, together with stocks and bonds, the investment universe consists of 9 basic assets, which can be combined linearly to replicate the cash-flows from more annuities. As shown on Fig. 6.2, similarly to zero coupon bonds, the maturities of the life contingent products are defined for the initial time t_0, and decrease with each period. Thus, any products maturing in 5, 10, 15, and 20 years should be understood as maturing upon ages 70, 75, 80, and 85, respectively. Accordingly, the only product that can be purchased upon the horizon is the whole life immediate annuity, which pays until the death of the individual. The individual has a possibility to trade the assets at any time before horizon T (when she becomes 85 years old).

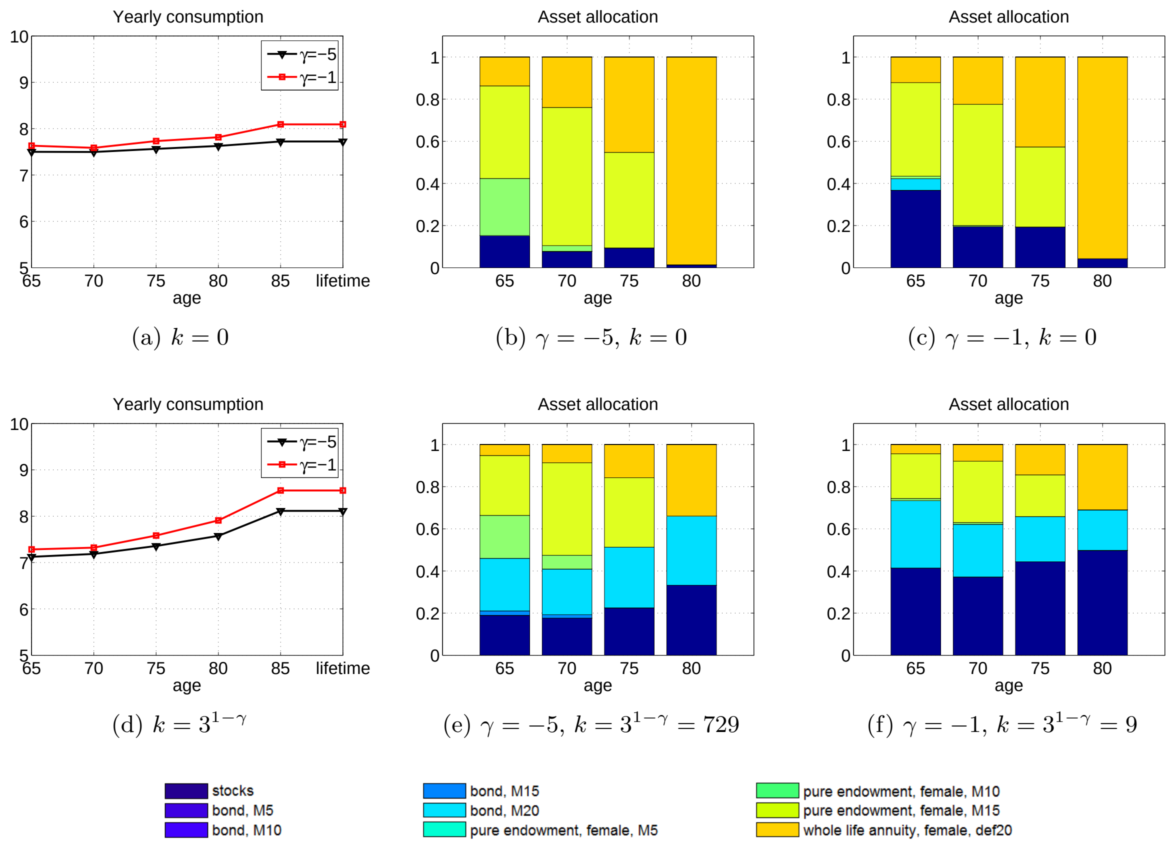
Fixed annuities Figure 6.3 shows the optimal consumption and asset allocation for different levels of risk aversion and with and without a bequest motive. Parameters $\gamma = -5$ and $\gamma = -1$ describe a conservative and a more aggressive investor, respectively, whereas $k = 0$ and $k = 3^{1-\gamma}$ describe the person without and with a moderate bequest motive, respectively.

The yearly consumption in all the cases is around 7-7.5% of the initial savings w_0 upon retirement, and increases with time, because the time preference factor ρ is lower than the expected return on the portfolio. The terminal condition upon age 85 implies that the individual annuitizes her wealth, therefore from age 85 until the death of the individual the consumption is constant. The less risk averse investor follows a more aggressive investment strategy (which on average provides a higher return), thus is able to consume more. During the first 20 years after retirement the investor gradually purchases the whole life annuity deferred until age 85. She spends the residual amount of her savings on the pure endowments with the maturities of 10 and 15 years, the zero coupon bonds with a maturity of 20 years, and stocks, nevertheless, the proportions between these assets differ with the level of risk aversion and the bequest motive. Because the optimal portfolio does not generate any fixed cash-flows during the first 10 years of retirement, we argue that the individual does not need an immediate term or whole life annuity—in the early retirement age she finances her consumption by rebalancing the portfolio.

In the presence of a bequest motive the optimal portfolio consists of significantly more stocks and bonds. The more risk averse individual allocates 40% to 60% of her savings to the non-life contingent assets; the less risk averse, 60% to 70%. The potential bequest amount decreases at each period as the individual consumes the savings. If the individual dies shortly after retirement, her dependants will inherit £30.4 ($\gamma = -5$) and £48.1 ($\gamma = -1$), corresponding to the value of the savings invested in the non-life contingent assets (for details see the last row of Table 6.A.1 in the

[9] In the CRRA utility maximization the level of the initial savings does not affect the optimal solution, therefore, for the ease of presentation we have chosen such a low value of w_0.

Appendix).



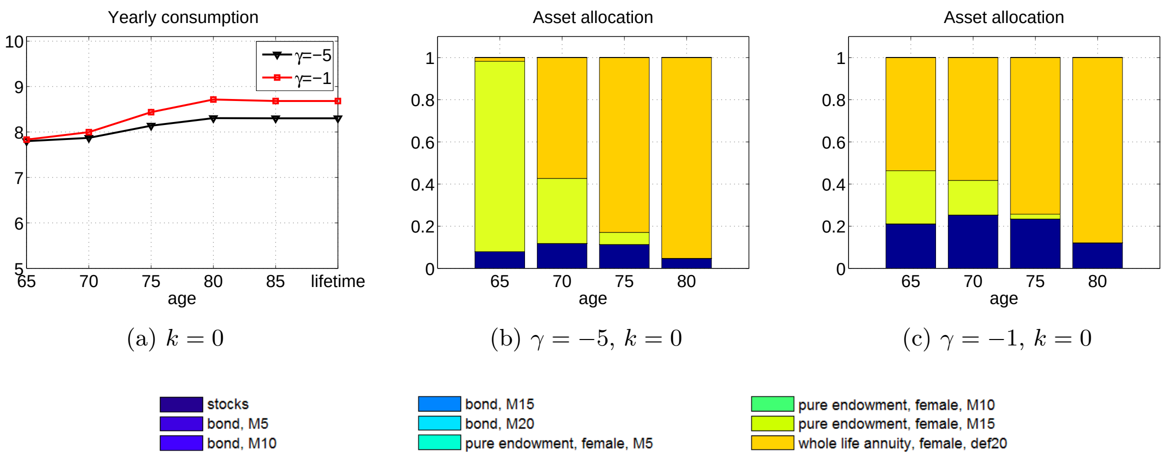
(a) $k = 0$ (b) $\gamma = -5$, $k = 0$ (c) $\gamma = -1$, $k = 0$

(d) $k = 3^{1-\gamma}$ (e) $\gamma = -5$, $k = 3^{1-\gamma} = 729$ (f) $\gamma = -1$, $k = 3^{1-\gamma} = 9$

Figure 6.3: Optimal consumption and asset allocation for a single individual with no access to variable pure endowments (average across 10,000 scenarios). For details see Table 6.A.1 in the Appendix.

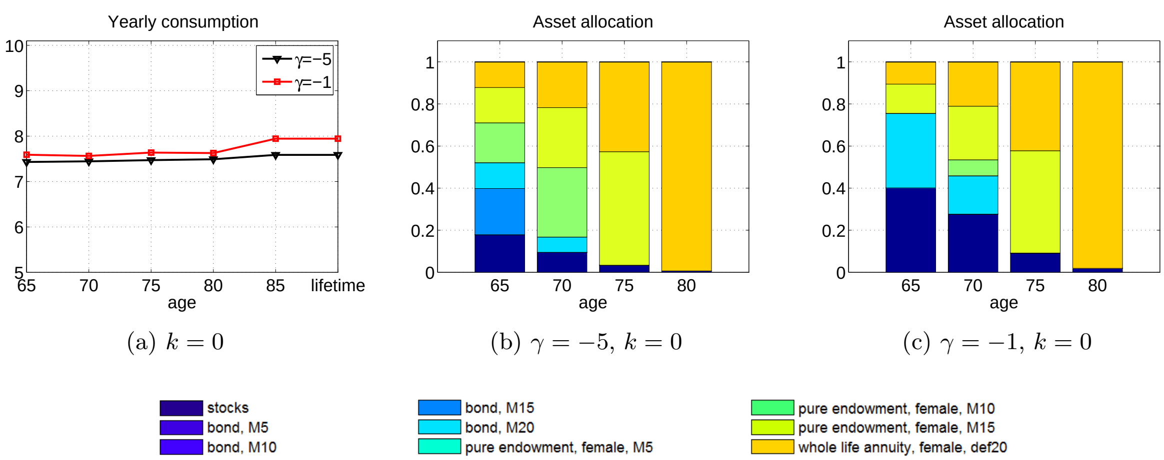
The impact of surrender charges One may wonder how much the surrender charges affect the optimal portfolio. Our model allows for selling pure endowments and annuities, however such a sale is subject to severe surrender charges, see Table 6.4. Not surprisingly, having removed the surrender charges (as well as other costs, i.e. $sc_t^a = 0$, $me^a = 0$, and $tc^b = tc^s = 0$), we find that the whole life annuities (which have the highest surrender charges) become more attractive, and the individual without a bequest motive prefers life contingent products rather than bonds, see Fig. 6.4. On the contrary, Fig. 6.5 shows the solution from an optimization problem including the "no sale" constraints on pure endowments and annuities, i.e. once these products are purchased, they can never be sold. The results indicate that irreversibility of life contingent assets affects the optimal solution in favor of the long term bonds. While the investment in stocks remains similar, the individual decreases the weight in the pure endowments with maturities 10 and 15 years in favor of the zero coupon bonds with maturities 15 and 20 years.

Variable pure endowments We expand the investment universe by adding variable pure endowments with maturities 5, 10, 15, and 20 years. We set the assumed interest rate, (6.2.8), to be equal



(a) $k = 0$ (b) $\gamma = -5$, $k = 0$ (c) $\gamma = -1$, $k = 0$

Figure 6.4: Optimal consumption and asset allocation for a single individual without a bequest motive and with no access to variable pure endowments. Assumed no surrender charges, M&E charges, and transaction costs, i.e. $sc_t^a = 0$, $me^a = 0$ and $tc^b = tc^s = 0$ (average across 10,000 scenarios). For details see Table 6.A.1 in the Appendix.



(a) $k = 0$ (b) $\gamma = -5$, $k = 0$ (c) $\gamma = -1$, $k = 0$

Figure 6.5: Optimal consumption and asset allocation for a one-person household without a bequest motive and with no access to variable pure endowments. The individual is not allowed to sell pure endowments and annuities, i.e. $fw^a = 0$ and $sc_t^a = 100\%$ (average across 10,000 scenarios). For details see Table 6.A.1 in the Appendix.

the interest rate (upon retirement) of a zero-coupon bond with maturity M, i.e. $\bar{r} = y(\boldsymbol{\beta}_{t_0,n_0}, M)$. Variable pure endowments have the same volatility as stocks, however, because they pay out conditionally on the individual's survival, their return is higher than the return on stocks. Especially in the absence of a bequest motive, the investor prioritizes the exceptionally high return on the variable pure endowments to more liquid stocks, and does not invest in stocks at all. If the individual without a bequest motive dies within the first 20 years of retirement, she almost does not leave any savings to the potential dependants, see Fig. 6.6 and Table 6.A.2 in the Appendix. In the presence of a bequest motive, the investor still keeps a significant amount in variable pure endowments, but she diversifies the portfolio with non-life contingent assets, i.e. stocks and long term bonds.

The individual invests in variable pure endowments with all possible maturities. In particular, she purchases the products with the shortest available maturity and spends the money provided by the cash-flows from these products on the purchase of other variable pure endowments (with the shortest available maturity). For example, the more risk averse individual without a bequest motive (Table 6.A.2) buys upon retirement only one asset that provides the cash-flow in the subsequent time period, namely, a variable pure endowment with a maturity of 5 years. Upon age 70, she consumes £7.7 and additionally purchases variable pure endowments with the maturities of 10 and 15 years, and possibly a pure endowment with maturity 15 and a whole life annuity deferred 20 years. The individual does not have any additional income, therefore finances the consumption and the purchase of new assets from the cash-flow provided by the variable pure endowment held in the portfolio and from the capital gains earned on selling different assets. Therefore, we argue that the individual does not purchase variable pure endowments to receive the specific cash-flows, but to earn a higher return than on stocks.

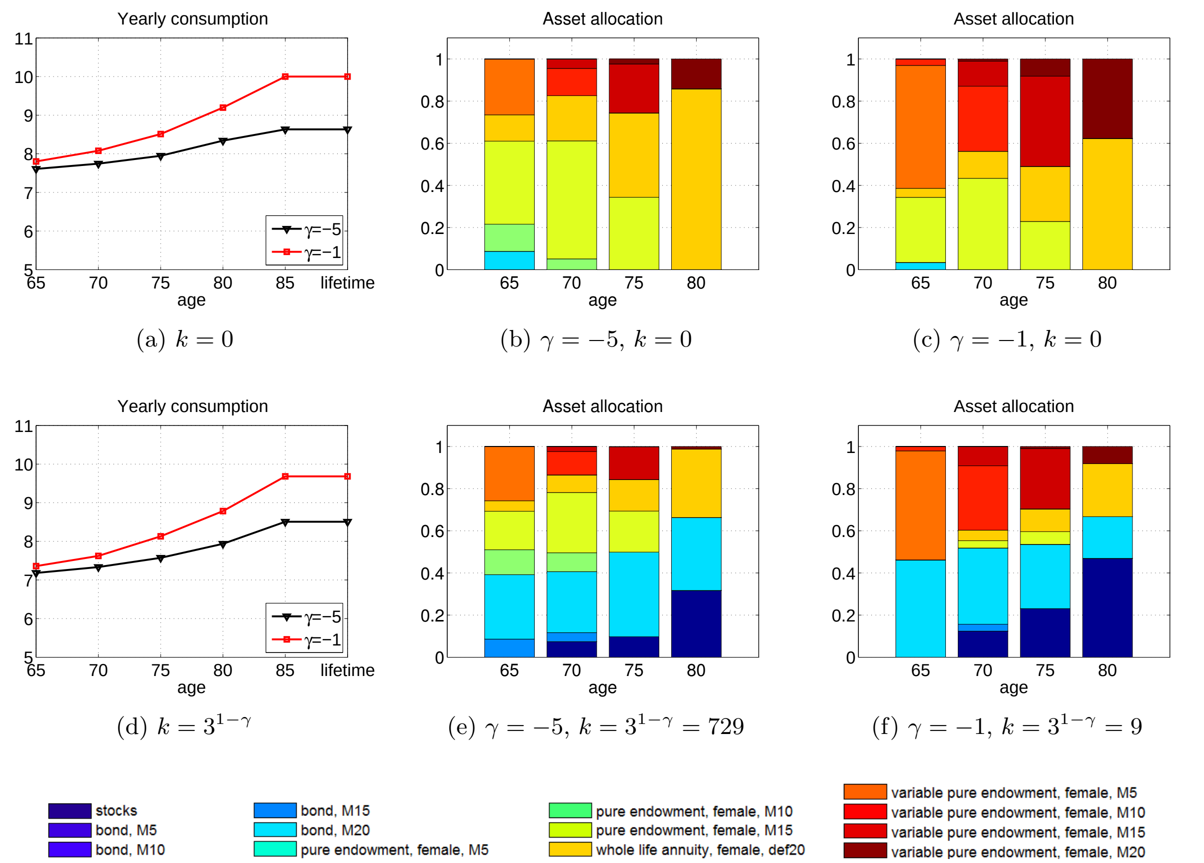


Figure 6.6: Optimal consumption and asset allocation for a single individual with access to variable pure endowments (average across 10,000 scenarios). For details see Table 6.A.2 in the Appendix.

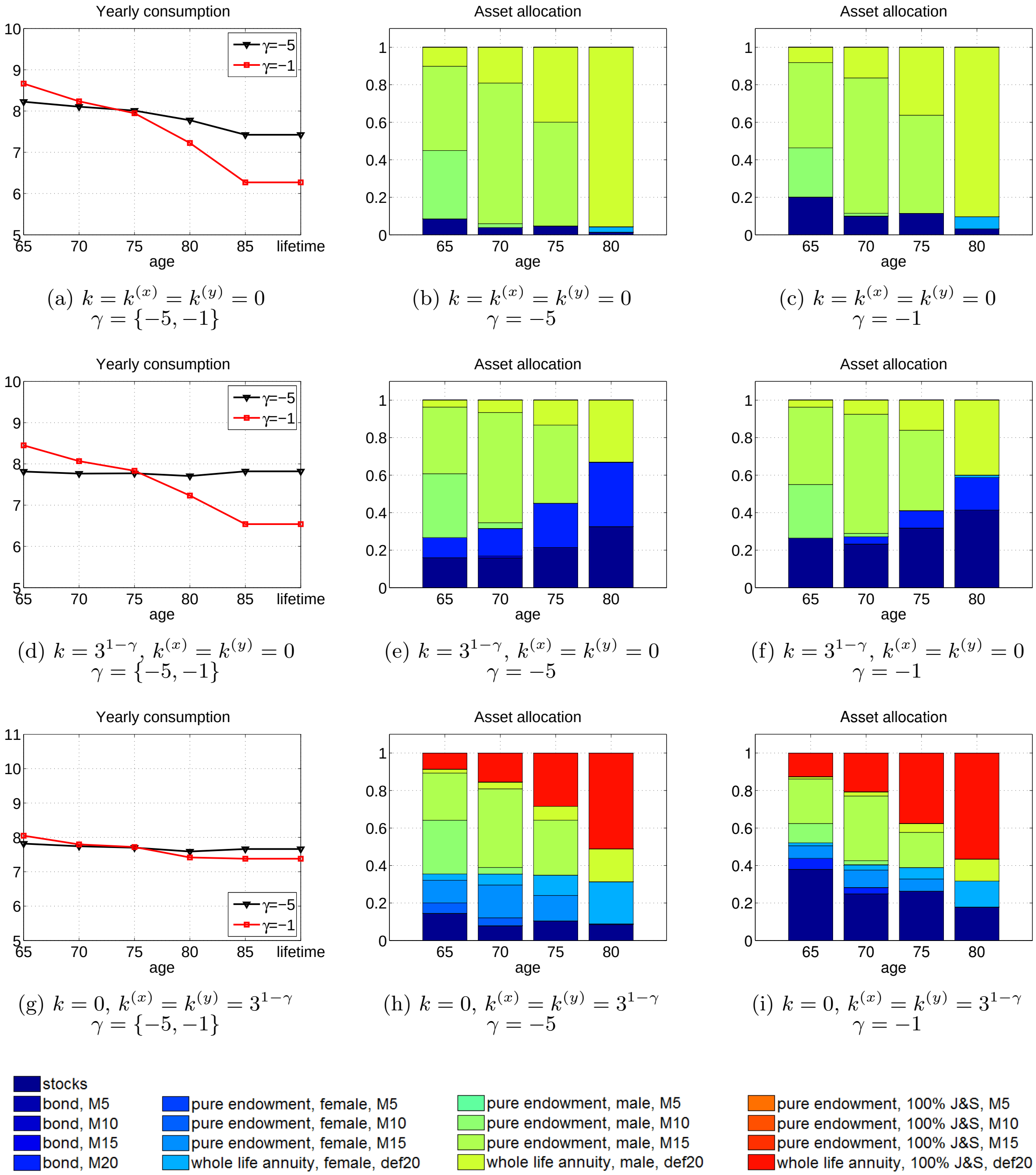
6.5.3 Optimal decisions for a two-person household

This section analyzes the optimal solution for a two-person household. For ease of presentation, we assume that both spouses are 65 years old upon retirement, have a time preference factor $\rho = 5\%$, and have joint savings of $w_0 = 100$ (£). We extend the investment universe by adding assets contingent on the husband's lifetime, i.e. pure endowments with maturities of 5, 10, and 15 years, a single whole life annuity deferred until age 85, and variable pure endowments with maturities of 5, 10, 15, and 20 years. We also add assets contingent on the joint lifetime of the spouses, specifically, the J&S (joint and survival) products that pay conditionally on the survival of at least one of the spouses. Similarly to assets contingent on a single lifetime, we add 100% J&S pure endowments with maturities 5, 10, and 15, a 100% J&S whole life annuity deferred 20 years, and 100% J&S variable pure endowments with maturities 5, 10, 15, and 20 years. Thus, we have in total 29 basic assets, which combined can replicate a wide variety of annuities. Upon retirement the individuals have to annuitize any wealth in the liquid assets, so that after age 85 the annuity payments for each spouse are equal.

Pure endowments contingent on the husband's lifetime The optimal portfolio for a two-person household consists of primarily the pure endowments contingent on the husband's lifetime, see Figs. 6.7 and 6.8. Because men live on average shorter than women, products contingent on the husband's lifetime provide higher returns (or, in other words, are less expensive) than the products contingent on the wife's lifetime or on the joint lifetime. Accordingly, the optimal portfolio for a two-person household without a bequest motive is heavily weighted by both the pure endowments and the variable pure endowments contingent on the husband's life with all available maturities, see Figs. 6.7b, 6.7c, 6.8b, and 6.8c. The only asset contingent on the wife's lifetime is the whole life annuity deferred 20 years. The household purchases it primarily upon the horizon to account for the terminal condition imposing even benefits for both spouses after age 85.

Bequest motive We further consider three different settings for a bequest motive. First, the couple has no bequest motive ($k = k^{(x)} = k^{(y)} = 0$); second, the couple has a moderate bequest motive for their children ($k = 3^{1-\gamma}$ and $k^{(x)} = k^{(y)} = 0$); and third, the couple has a moderate bequest motive for the spouse but not for other dependants ($k = 0$ and $k^{(x)} = k^{(y)} = 3^{1-\gamma}$).

The last three rows in each section of Tables 6.A.3 and 6.A.4 show how the bequest amount changes with different choices of k, $k^{(x)}$, and $k^{(y)}$. If the couple has no bequest motive, their objective is simply to maximize the expected utility of consumption. Especially, having access to variable pure endowments, e.g., Figs. 6.8b and 6.8c, the couple does not hold any stocks or bonds, and the value of the bequest for the children is equal to 0. The solution changes when the couple chooses to leave some money to the children—they allocate more savings to stocks and bonds, see e.g., Figs. 6.8e and 6.8f. For the cases with $k = k^{(x)} = k^{(y)} = 0$ and $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, the bequest amount for the wife and for the children is much lower than the bequest amount for the husband. Because the couple invests primarily in the assets contingent on the husband's lifetime, and upon the wife's death, the husband inherits almost the entire portfolio. This relation changes when the couple chooses to bequeath to the spouse. For example, for $k = 0$ and $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, Table 6.A.4 in the Appendix shows a much higher bequest for the wife than in the previous cases, and the optimal portfolio includes some products contingent on the wife's lifetime as well as the J&S assets. This case is particularly interesting because the optimal portfolio consists of even 15 different products.



(a) $k = k^{(x)} = k^{(y)} = 0$, $\gamma = \{-5, -1\}$

(b) $k = k^{(x)} = k^{(y)} = 0$, $\gamma = -5$

(c) $k = k^{(x)} = k^{(y)} = 0$, $\gamma = -1$

(d) $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, $\gamma = \{-5, -1\}$

(e) $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, $\gamma = -5$

(f) $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, $\gamma = -1$

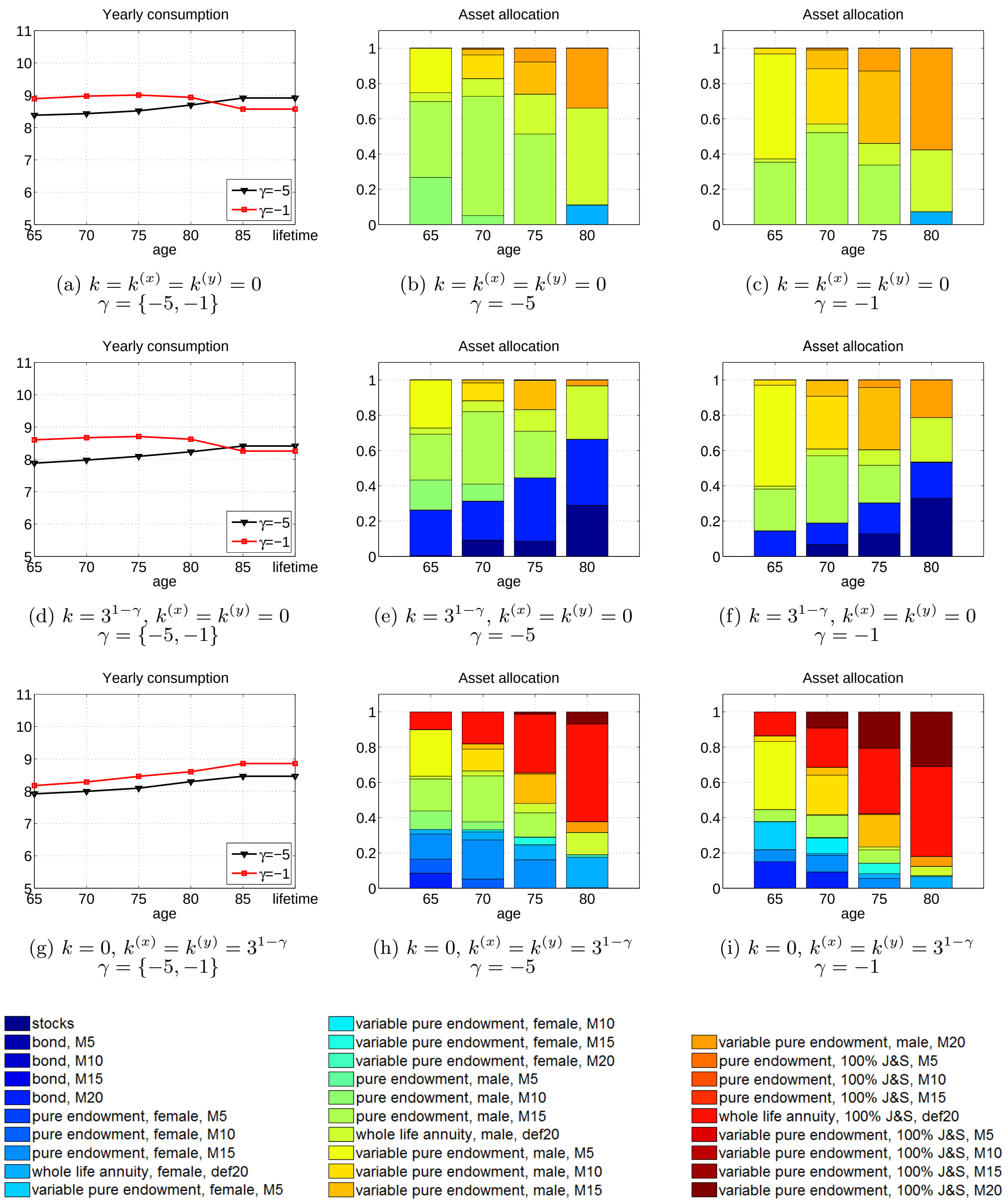
(g) $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, $\gamma = \{-5, -1\}$

(h) $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, $\gamma = -5$

(i) $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, $\gamma = -1$

Figure 6.7: Optimal consumption and asset allocation for a two-person household with no access to variable pure endowments (average across 10,000 scenarios). For details see Table 6.A.3 in the Appendix.

Joint and survivor products Y% J&S pure endowments and annuities are examples of the joint lifetime contingent assets that pay the cash-flows as long as at least one of the annuitants is alive. The parameter Y denotes the reduction of the original benefit upon the first death. To exemplify, 60% J&S pure endowment pays £1 conditionally on the survival of both annuitants, and £0.6 if only one individual is alive. Consequently, a Y% J&S pure endowment with maturity M is just a linear combination of two single life pure endowments, and a 100% J&S pure endowment, all three



(a) $k = k^{(x)} = k^{(y)} = 0$, $\gamma = \{-5, -1\}$
(b) $k = k^{(x)} = k^{(y)} = 0$, $\gamma = -5$
(c) $k = k^{(x)} = k^{(y)} = 0$, $\gamma = -1$
(d) $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, $\gamma = \{-5, -1\}$
(e) $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, $\gamma = -5$
(f) $k = 3^{1-\gamma}$, $k^{(x)} = k^{(y)} = 0$, $\gamma = -1$
(g) $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, $\gamma = \{-5, -1\}$
(h) $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, $\gamma = -5$
(i) $k = 0$, $k^{(x)} = k^{(y)} = 3^{1-\gamma}$, $\gamma = -1$

Figure 6.8: Optimal consumption and asset allocation for a two-person household with access to variable pure endowments (average across 10,000 scenarios). For details see Table 6.A.4 in the Appendix.

of them with maturity M and providing the same cash-flow, i.e.

$$price_t^{Y\%\ J\&S} = (1 - Y\%)\big(price_t^{(x)} + price_t^{(y)}\big) + (2Y\% - 1)price_t^{100\%\ J\&S},$$

where $price_t^{(x)}$ and $price_t^{(y)}$ are, respectively, the prices of annuities contingent on a female and on a male lifetime.

The results indicate that the household purchases 100% J&S pure endowments and annuities only if it has a bequest motive specifically for the spouse. To exemplify, the household with a bequest motive for the spouse, with a risk aversion $1-\gamma=2$, and without the access to variable pure endowment, invests in the 100% J&S whole life annuity deferred 20 years gradually during the entire retirement period so that upon the horizon 60% of the optimal portfolio consists of this annuity. The residual amount of wealth is allocated to two whole life annuities contingent on the wife's and the husband's lifetime. Whenever the cash-flows from these two annuities are even, we can show that a portfolio consisting of these annuities and of a 100% J&S annuity, is equivalent to a portfolio consisting of only one Y% J&S annuity, where Y is specified by the level of cash-flows from the particular annuities,

$$Y = \frac{\sum_{a\in\mathcal{A}^{(x)}} cf^a_{t,n} Hold^a_{t,n} + \sum_{a\in\mathcal{A}^{(100\% J\&S)}} cf^a_{t,n} Hold^a_{t,n}}{2\sum_{a\in\mathcal{A}^{(x)}} cf^a_{t,n} Hold^a_{t,n} + \sum_{a\in\mathcal{A}^{(100\% J\&S)}} cf^a_{t,n} Hold^a_{t,n}}. \tag{6.5.1}$$

Whenever the cash-flows from the single life annuities are not even (e.g., the portfolio includes more annuities contingent on the husband's lifetime), the portfolio consisting of two single life annuities and one 100% J&S annuity is equivalent to a single life annuity (contingent on the husband's lifetime) and a Y% J&S annuity with Y defined by (6.5.1). Thus, we can conclude that during the retirement period the optimal portfolio includes Y% J&S annuities (or pure endowments) with Y changing dynamically over time.

The portfolio of households that do not invest in 100% J&S (e.g., a household without a bequest motive, with a risk tolerance $1-\gamma=2$, and without access to variable pure endowment) consists upon the horizon of 59% and 41% invested in the whole life annuity contingent on the wife's and the husband's lifetime, respectively. The couple's total consumption is equal to £6.3, and in case of death of one of the spouses, the survivor would be receiving the yearly payments of £3.15 (due to the terminal condition (6.4.17)). This payout is equivalent to the payout from a 50% J&S annuity with the initial benefit of £6.3. Analogically to the case of a household with a bequest motive for the spouse, we conclude that during retirement the optimal portfolio includes 50% J&S annuities with time-dependent parameter Y.

6.5.4 Annuity product design

The purpose of this paper is to build a model that can be applied for advising the households on choosing the annuity product that best fits their needs. Our results indicate that there is no simple answer to this question. We observe that investing only in one product is never optimal, therefore retirement planning is not about choosing between a temporary single life annuity, a deferred 50% J&S annuity, or an annuity with a guaranteed period. The optimal solution suggests a product that is much more complex than any of those existing in the market. A product that provides the cash-flow equal to the optimal consumption as long as the individual or the couple is alive, and pays a death benefit equal to the optimal bequest amount. Because the model is based on the self-financing portfolio, the price of the optimal product is equal to the value of the savings upon retirement w_0.

To design a retirement product for, e.g., a couple with savings of £100, risk tolerance $\gamma = -1$, and a bequest motive for the spouse, i.e. $k = 0$ and $k^{(x)} = k^{(y)} = 3^{1-\gamma}$ (see Table 6.A.4 in the Appendix), we need an annuity with the expected payments increasing from £8.2 to £8.9 during the first 20 years of retirement. The payments are conditional on the survival of both spouses. Upon death of one of the spouses, the survivor receives a death benefit—primarily in instalments. The present value of the expected death benefit declines with time: the husband inherits {£47.4, £43.7, £36.8, £30.0, £14.8} if the wife dies between ages {65-70, 70-75, 75-80, 80-85, 85-110}, whereas if the husband dies during the same years, the wife inherits {£31.5, £32.7, £30.7, £28.4, £16.6}. Since the couple does not have a bequest motive for other dependants, in case of the couple's death, almost no savings are left for the dependants. Such a product is a special case of a whole life $Y\%$ joint and survivor annuity with variable and increasing payments, and with a life insurance policy providing the death sum that changes over time. All the product parameters such as the portfolio backing the annuities, the rate defining the level of the benefit after the death of the first spouse $Y\%$, and the level of the death sum, are dynamic, and sometimes also gender dependent.

6.6 Conclusions

In this paper we use a stochastic programming approach to help households choose the right annuity product. Specifically, we derive the optimal asset allocation, consumption and bequest amount, given investment opportunities in stocks, zero coupon bonds and pure endowments with different maturities, and whole life fixed annuities. The pure endowments offer either fixed or variable payouts, which, similarly to the whole life annuities, are contingent on either a single or a joint lifetime. We consider time-varying investment opportunities in the stock market returns and interest rates, and use the Nelson/Siegel approach to represent the whole term structure of interest rates with a few factors. We further include transaction costs on stocks and bonds, as well as mortality and expense charges, and surrender charges on pure endowments and annuities.

Among other findings, we show that independently of the bequest motive and the level of risk aversion, it is optimal to diversify the portfolio in a wide variety of products. Despite high surrender charges, life contingent products should be the primary assets in the portfolio. The households should invest in these products not necessarily to consume the cash-flows, but to be able to rebalance the portfolio during retirement, and to earn a higher return than on the non-life contingent assets. In particular, variable pure endowments with short maturities are an attractive replacement for stocks, especially for the individuals without a bequest motive. In addition, households should take advantage of the gender-specific pricing and invest primarily in the products contingent on the husband's lifetime and in the joint and survivor products. The households should invest differently depending whether they have a bequest motive for their children or for the spouse. In the first case, they should invest significant amount of savings in bonds and stocks; in the second, they should rather choose joint and survivor pure endowments and annuities. We argue that the right product for a household is much more complex than any of the annuity products existing in the market. The optimal product should be tailored to specific needs of a household in terms of the underlying portfolio, the expected cash-flow, and the level of death benefit.

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Appendix

6.A Additional numerical results

The appendix provides the details for the numerical results and figures in Sec. 6.5.

	$\gamma=-5$					$\gamma=-1$				
Assets/Age	65	70	75	80	85	65	70	75	80	85
Without a bequest motive, $k=0$										
Stocks	15	8	9	1	0	37	19	19	4	0
Pure endowment, female, M10	27	3	0	0	0	1	1	0	0	0
Pure endowment, female, M15	44	66	45	0	0	44	58	38	0	0
Whole life annuity, female, def20	14	24	45	99	100	12	22	43	96	100
Consumption	7.5	7.5	7.6	7.6	7.7	7.6	7.6	7.7	7.8	8.1
Bequest, W_t	9.8	4.2	4.0	0.3	0.0	27.0	10.2	8.1	1.0	0.0
Without a bequest motive, $k=0$, no transaction costs, M&E charges and surrender charges										
Stocks	8	12	11	5	0	21	25	23	12	0
Pure endowment, female, M15	90	31	6	0	0	25	16	2	0	0
Whole life annuity, female, def20	2	57	83	95	100	54	58	74	88	100
Consumption	7.8	7.9	8.1	8.3	8.3	7.8	8.0	8.4	8.7	8.7
Bequest, W_t	5.0	6.7	5.0	1.5	0.0	13.4	13.1	10.4	3.8	0.0
Without a bequest motive, $k=0$, no sales constraints on annuities and pure endowments										
Stocks	18	10	3	1	0	40	28	9	2	0
Bond, M15	22	0	0	0	0	0	0	0	0	0
Bond, M20	12	7	0	0	0	35	18	0	0	0
Pure endowment, female, M10	19	33	0	0	0	0	8	0	0	0
Pure endowment, female, M15	17	28	54	0	0	14	25	49	0	0
Whole life annuity, female, def20	12	22	43	99	100	11	21	42	98	100
Consumption	7.4	7.4	7.5	7.5	7.6	7.6	7.6	7.6	7.6	7.9
Bequest, W_t	33.8	8.8	1.1	0.1	0.0	48.4	23.4	3.0	0.4	0.0
With a moderate bequest motive, $k=3^{1-\gamma}$										
Stocks	19	18	22	33	0	41	37	44	50	0
Bond, M15	2	2	0	0	0	0	0	0	0	0
Bond, M20	25	22	29	33	0	32	25	21	19	0
Pure endowment, female, M10	20	7	0	0	0	1	1	0	0	0
Pure endowment, female, M15	28	44	33	0	0	21	29	20	0	0
Whole life annuity, female, def20	5	9	16	34	100	4	8	14	31	100
Consumption	7.1	7.2	7.4	7.6	8.1	7.3	7.3	7.6	7.9	8.6
Bequest, W_t	30.4	23.7	24.3	23.6	0.0	48.1	35.4	30.9	24.9	0.0

Table 6.A.1: Optimal consumption (in £), bequest (in £) and asset allocation (in %) for a one-person household (average across 10,000 scenarios). Assumed no access to variable pure endowments.

	$\gamma = -5$					$\gamma = -1$				
Assets/Age	65	70	75	80	85	65	70	75	80	85
Without a bequest motive, $k = 0$										
Pure endowment, female, M10	13	5	0	0	0	0	0	0	0	0
Pure endowment, female, M15	39	56	34	0	0	31	43	23	0	0
Whole life annuity, female, def20	12	21	40	86	100	4	13	26	62	100
Variable pure endowment, female, M5	27	0	0	0	0	58	0	0	0	0
Variable pure endowment, female, M10	0	13	0	0	0	3	31	0	0	0
Variable pure endowment, female, M15	0	4	23	0	0	0	12	43	0	0
Variable pure endowment, female, M20	0	0	2	14	0	0	1	8	38	0
Consumption	7.6	7.7	7.9	8.3	8.6	7.8	8.1	8.5	9.2	10.0
Bequest, W_t	5.6	0.0	0.0	0.0	0.0	2.2	0.0	0.0	0.0	0.0
With a moderate bequest motive, $k = 3^{1-\gamma}$										
Stocks	0	7	10	32	0	0	12	23	47	0
Bond, M15	9	4	0	0	0	0	3	0	0	0
Bond, M20	31	29	40	35	0	46	36	31	20	0
Pure endowment, female, M10	12	9	0	0	0	0	0	0	0	0
Pure endowment, female, M15	18	29	19	0	0	0	3	6	0	0
Whole life annuity, female, def20	5	8	15	33	100	0	5	11	25	100
Variable pure endowment, female, M5	26	0	0	0	0	52	0	0	0	0
Variable pure endowment, female, M10	0	11	0	0	0	2	31	0	0	0
Variable pure endowment, female, M15	0	2	16	0	0	0	9	29	0	0
Variable pure endowment, female, M20	0	0	0	1	0	0	0	1	8	0
Consumption	7.2	7.3	7.6	7.9	8.5	7.4	7.6	8.1	8.8	9.7
Bequest, W_t	25.8	25.7	24.5	25.2	0.0	30.1	31.6	27.0	26.5	0.0

Table 6.A.2: Optimal consumption (in £), bequest (in £) and asset allocation (in %) for a one-person household (average across 10,000 scenarios).

	$\gamma=-5$					$\gamma=-1$				
Assets/Age	65	70	75	80	85	65	70	75	80	85
Without a bequest motive, $k=k^{(x)}=k^{(y)}=0$										
Stocks	8	4	5	1	0	20	10	11	3	0
Whole life annuity, female, def20	0	0	0	3	59	0	0	0	6	59
Pure endowment, male, M10	36	2	0	0	0	26	1	0	0	0
Pure endowment, male, M15	45	75	55	0	0	45	72	52	0	0
Whole life annuity, male, def20	10	19	40	96	41	8	16	36	90	41
Consumption	8.2	8.1	8.0	7.8	7.4	8.7	8.2	7.9	7.2	6.3
Bequest for the husband, $W_t+A_t^{(y)}$	61.1	48.5	35.2	23.1	5.4	59.1	45.2	31.0	18.5	4.6
Bequest for the wife, $W_t+A_t^{(x)}$	5.2	2.1	1.7	1.0	8.0	11.9	4.9	3.6	1.9	6.7
Bequest for the children, W_t	5.2	2.1	1.7	0.3	0.0	11.9	4.9	3.6	0.5	0.0
With a moderate bequest motive, $k=3^{1-\gamma}$, $k^{(x)}=k^{(y)}=0$										
Stocks	16	16	21	33	0	26	23	32	41	0
Bond, M20	11	15	24	34	0	0	4	9	17	0
Whole life annuity, female, def20	0	0	0	0	59	0	0	0	1	59
Pure endowment, male, M10	34	3	0	0	0	29	2	0	0	0
Pure endowment, male, M15	36	59	42	0	0	41	64	43	0	0
Whole life annuity, male, def20	4	7	13	33	41	4	8	16	40	41
Consumption	7.8	7.8	7.8	7.7	7.8	8.5	8.1	7.8	7.2	6.5
Bequest for the husband, $W_t+A_t^{(y)}$	63.0	53.5	41.9	30.0	5.7	60.1	48.7	35.6	24.2	4.8
Bequest for the wife, $W_t+A_t^{(x)}$	16.8	18.2	20.0	20.8	8.4	15.9	13.8	15.2	15.5	7.0
Bequest for the children, W_t	16.8	18.2	20.0	20.8	0.0	15.9	13.8	15.2	15.1	0.0
With a moderate bequest motive for the spouse, $k=0$, $k^{(x)}=k^{(y)}=3^{1-\gamma}$										
Stocks	14	8	10	9	0	38	25	26	18	0
Bond, M20	0	0	0	0	0	6	3	0	0	0
Pure endowment, female, M10	6	4	0	0	0	0	0	0	0	0
Pure endowment, female, M15	12	17	14	0	0	7	9	7	0	0
Whole life annuity, female, def20	3	6	11	22	27	2	3	6	14	24
Pure endowment, male, M10	29	4	0	0	0	10	2	0	0	0
Pure endowment, male, M15	25	42	29	0	0	24	35	19	0	0
Whole life annuity, male, def20	2	4	7	18	19	1	2	5	12	16
Whole life annuity, 100% J&S, def20	9	15	28	51	54	13	21	38	57	60
Consumption	7.8	7.7	7.7	7.6	7.7	8.0	7.8	7.7	7.4	7.4
Bequest for the husband, $W_t+A_t^{(y)}$	49.8	38.1	31.0	23.0	13.0	56.8	46.1	35.6	25.0	13.5
Bequest for the wife, $W_t+A_t^{(x)}$	27.8	27.5	26.5	24.4	14.5	40.1	32.5	31.3	25.7	14.8
Bequest for the children, W_t	9.1	4.2	4.4	2.5	0.0	27.2	14.4	10.3	4.7	0.0

Table 6.A.3: Optimal consumption (in £), bequest (in £) and asset allocation (in %) for a two-person household (average across 10,000 scenarios). Assumed no access to variable pure endowments.

	$\gamma=-5$					$\gamma=-1$				
Assets/Age	65	70	75	80	85	65	70	75	80	85
Without a bequest motive, $k=k^{(x)}=k^{(y)}=0$										
Whole life annuity, female, def20	0	0	0	11	59	0	0	0	7	59
Pure endowment, male, M10	27	5	0	0	0	0	0	0	0	0
Pure endowment, male, M15	43	68	51	0	0	35	52	34	0	0
Whole life annuity, male, def20	5	10	23	55	41	2	5	12	35	41
Variable pure endowment, male, M5	25	0	0	0	0	60	0	0	0	0
Variable pure endowment, male, M10	0	13	0	0	0	3	31	0	0	0
Variable pure endowment, male, M15	0	3	18	0	0	0	11	41	0	0
Variable pure endowment, male, M20	0	1	8	34	0	0	1	13	58	0
Consumption	8.4	8.4	8.5	8.7	8.9	8.9	9.0	9.0	8.9	8.6
Bequest for the husband, $W_t+A_t^{(y)}$	60.4	49.2	36.4	22.9	6.5	58.1	46.1	33.4	21.3	6.3
Bequest for the wife, $W_t+A_t^{(x)}$	0.0	0.0	0.0	3.1	9.6	0.0	0.0	0.0	1.9	9.2
Bequest for the children, W_t	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
With a moderate bequest motive, $k=3^{1-\gamma}$, $k^{(x)}=k^{(y)}=0$										
Stocks	0	9	9	29	0	0	7	13	33	0
Bond, M15	1	1	0	0	0	0	0	0	0	0
Bond, M20	26	22	36	37	0	15	12	17	20	0
Whole life annuity, female, def20	0	0	0	0	59	0	0	0	0	59
Pure endowment, male, M10	17	10	0	0	0	0	0	0	0	0
Pure endowment, male, M15	26	41	26	0	0	24	38	21	0	0
Whole life annuity, male, def20	3	6	12	30	41	2	4	9	25	41
Variable pure endowment, male, M5	27	0	0	0	0	57	0	0	0	0
Variable pure endowment, male, M10	0	10	0	0	0	3	30	0	0	0
Variable pure endowment, male, M15	0	2	17	0	0	0	9	35	0	0
Variable pure endowment, male, M20	0	0	0	3	0	0	0	4	21	0
Consumption	7.9	8.0	8.1	8.2	8.4	8.6	8.7	8.7	8.6	8.3
Bequest for the husband, $W_t+A_t^{(y)}$	62.7	54.7	43.0	31.5	6.2	59.4	49.9	37.8	27.2	6.1
Bequest for the wife, $W_t+A_t^{(x)}$	16.5	18.4	19.3	21.5	9.1	8.6	10.2	11.7	14.8	8.9
Bequest for the children, W_t	16.5	18.4	19.3	21.5	0.0	8.6	10.2	11.7	14.7	0.0
With a moderate bequest motive for the spouse, $k=0$, $k^{(x)}=k^{(y)}=3^{1-\gamma}$										
Bond, M20	9	0	0	0	0	15	9	0	0	0
Pure endowment, female, M10	8	5	0	0	0	0	0	0	0	0
Pure endowment, female, M15	14	22	16	0	0	7	9	6	0	0
Whole life annuity, female, def20	3	5	9	17	26	0	1	3	6	28
Variable pure endowment, female, M5	0	0	0	0	0	16	0	0	0	0
Variable pure endowment, female, M10	0	1	0	0	0	0	9	0	0	0
Variable pure endowment, female, M15	0	0	4	0	0	0	0	6	0	0
Variable pure endowment, female, M20	0	0	0	1	0	0	0	0	1	0
Pure endowment, male, M15	18	26	14	0	0	7	12	8	0	0
Whole life annuity, male, def20	2	3	5	13	18	0	1	2	5	19
Variable pure endowment, male, M5	26	0	0	0	0	39	0	0	0	0
Variable pure endowment, male, M10	0	12	0	0	0	3	22	0	0	0
Variable pure endowment, male, M15	0	3	17	0	0	0	4	18	0	0
Variable pure endowment, male, M20	0	0	1	6	0	0	0	1	6	0
Whole life annuity, 100% J&S, def20	10	18	33	55	56	14	22	37	51	53
Variable pure endowment, 100% J&S, M20	0	0	1	7	0	0	9	20	31	0

Table 6.A.4: Optimal consumption (in £), bequest (in £) and asset allocation (%) for a two-person household (average across 10,000 scenarios).

	$\gamma = -5$					$\gamma = -1$				
Assets/Age	65	70	75	80	85	65	70	75	80	85
Consumption	7.9	8.0	8.1	8.3	8.5	8.2	8.3	8.5	8.6	8.9
Bequest for the husband, $W_t + A_t^{(y)}$	47.0	36.1	30.4	25.9	14.7	47.4	43.7	36.8	30.0	14.8
Bequest for the wife, $W_t + A_t^{(x)}$	27.1	27.3	26.8	25.6	16.3	31.5	32.7	30.7	28.4	16.6
Bequest for the children, W_t	5.3	0.0	0.0	0.1	0.0	9.3	5.4	0.0	0.0	0.0

Table 6.A.4: (continued)

Chapter 7

Do not pay for a Danish interest guarantee. The law of the triple blow

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Do not pay for a Danish interest guarantee. The law of the triple blow

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Abstract

We have investigated the performance of pension schemes of with-profit policies containing a guaranteed minimum rate of return and we have found that the price of the guarantee measured in terms of lost returns is enormous. We use simple simulations rather than complex pricing methods to illustrate that the price of an interest guarantee is high in pension products that are currently commercialised in the market. We have found that the customer loses up to about 0.75% yearly in the rate of return when an interest guarantee is purchased, compared to the return of an equivalent saving strategy with the same risk at the level 95%. This can explain why such arrangements are not widely popular. Our approach can be used to inform clients, who are not experts in modern financial models, the impact of paying for an interest guarantee.

Keywords

Retirement wealth; pension fund performance measurement; retirement saving schemes

1 Introduction

Risk transfer among generations through bonus-participating products with an underlying interest guarantee has been a widely popular approach to pension funding in many years in many countries. For example, in Denmark there are more than 200 billion Euros of savings in this type of product, e.g. more than 35.000 Euros per citizen. This paper analyses properties of with-profit interest guarantee products where the customer pays for the minimum guaranteed interest rate. We consider a simplified model approximating complicated non-transparent with-profit contracts with such interest guarantees.

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Our analysis shows that a customer receives three blows when paying for a minimum interest guarantee. The first blow comes from the with-profit mechanism itself. We find the performance of the with-profit mechanism unconvincing in the sense that it does not convincingly outperform a trivial fixed stock proportion benchmark product. The second blow comes from the payment of the risk premium. The third blow is an opportunity cost that follows any loss when saving: the money lost makes the customer more risk averse. However, this is an intrinsic feature of saving (if not invested in a risk free instrument) so does not differentiate between saving products carrying an interest rate guarantee and those that do not. In the Danish market of interest guarantee products the costs of the above three blows seem to be found in the neighbourhood of a loss of around 0.75 percent a year accumulating to a fortune over the life span of the individual pension saver.

To be able to quantify the above three blows we use the performance methodology recently developed by Guillén *et al.* (2011). This approach is indeed a pragmatic one and only a first step towards a scientifically based method to compare and evaluate pension products. However, the advantage of having a concrete idea of the magnitude of the customers losses is important to be able to identify and communicate the magnitude of the issue, especially to non-expert clients. For example, the approximative 0.75% a year loss of returns suggested by our results would imply that the total cost to savers could be approximated at about 1.5 billion Euros per year or even more.

Our contribution connects with the work by Gatzert *et al.* (2011) who found that subjective pricing of guarantees by customers of insurance products was below the true price. Customers' decisions in a market for products with guarantees may need simpler explanations than the ones that are currently given by underwriters. We believe that if customers are not well informed then their perceptions on the cost of guarantees may be distorted. These customers may think that guarantees are cheaper than they really are and may expect guaranteed minimum interest rates to have less impact on the long-term return than they do have. Nevertheless, we must remark that the cost of the guarantee is not completely taken by the firm as a profit as it can also be considered as a contribution to reserves that could somehow be paid back if not needed.

Our aim is to present a very simple approach to assess the impact of minimum interest guarantees and to find evidence with our illustration of the impact on return when buying an interest guarantee in pension funds that exist today in the Danish market. The paper is organised as follows. In section two we describe the pension products that we consider in our analysis and the investment strategy of our case study is presented. In section three we define our performance methodology. In section four we show the results of a Monte Carlo simulation study and section five summarises the main conclusions.

2 Description of the with-profit return mechanism

We model the company balance sheet at time t by three quantities. The market value of the company assets in total is denoted A_t. The liabilities of the company consists of the policyholders account balance P_t and a bonus reserve B_t. The latter represents the buffer whose function is to protect the policy reserve from the risks associated with the asset base and changes in actuarial variables, such as mortality. B_t is in other words the undistributed reserve, i.e. bonus reserve plus equity, and is defined as the difference $A_t - P_t$. Therefore the simplified pension company's balance sheet is given by assets A_t, and liabilities P_t and B_t.

The total value of the company investment evolves as in Guillén *et al.* (2011) according to a geometric Brownian motion:

$$dA(t) = \{r + \pi(t)(\mu - r)\}A(t)dt + \pi(t)\sigma A(t)dW(t) - r^* dP_t \qquad A(0) = A_0, \tag{1}$$

where r is the risk-free return, μ is the expected yearly return on stocks, σ is the volatility, and $\pi(t)$ is the proportion invested in risky assets which we specify later in equation (4). We denote by r^* the risk premium, which represents an indirect payment for the interest guarantee and it is subtracted from the company assets. Note that if the company offers some interest guarantee to its clients, this implies some liabilities to the equity and therefore a reduction in the company assets. This is the reason why the risk premium represents an indirect payment for the interest guarantee. We also assume that the net cash flow from and into the company is equal to zero.

On the liability side of the company's balance sheet we have:

$$\begin{aligned} P(t) &= (1 + r_P(t))P(t-1) \\ P(0) &= P_0. \end{aligned} \tag{2}$$

Note that the bonus distribution is made on a group level and not on the policy level. Therefore in the above equation we assumed that the net cash flow is equal to 0, so the policy rate $r_p(t)$ is independent of whether the pension saver has just purchased the contract and is paying the premiums, or retired and receiving annuities.

2.1 Bonus distribution mechanism

The bonus distribution mechanism is designed such that it captures the performance of the market and the solvency requirements the company has to fulfil, while providing a low-risk return. The pension saver receives some guaranteed return, but this promised rate may be topped off with a bonus depending on the degree of solvency in the company. We define the bonus by:

$$r_P(t; A(t-1), P(t-1)) = r_G + \max\left[0, \beta\left(\frac{B(t-1)}{P(t-1)} - \psi\right)\right], \tag{3}$$

where ψ is a critical buffer ratio, i.e. a constant ratio of bonus reserves to policy reserves $B(\cdot)/P(\cdot)$ which the company is willing to maintain, and β is a distribution ratio determining a positive fraction of excessive bonus reserve which is distributed to the pension saver's account. If the actual buffer relative to the policy account balance exceeds the critical level ψ, the company distributes a fraction β of the surplus. Policy interest $r_p(t)$ is determined at time $t-1$ (is F_{t-1}-measurable), which is motivated by practice where policy rates for year t are typically announced by the companies in mid-December of year $t-1$.

To determine the values for (β, ψ) we refer to Grosen & Jørgensen (2002). These authors estimate bonus distribution parameters $(\hat{\beta}, \hat{\psi})$ using the maximum likelihood technique (ML). The statistical model is based on a hand-collected set of solvency ratios reported by the largest Danish L&P companies during the period 1991–2000, and on the policy interest rates $r_p(\cdot)$ collected directly from the companies.

2.2 Investment strategy

The company assets consist of risky assets (stocks) and risk-free assets (bonds). The distribution between these two types is determined by the stock proportion function $\pi(t)$. The decision on π is made with the assumption that the company is willing to invest the largest possible stock share resulting in a probability greater or equal to $(1-\alpha^*)$ of being solvent at the end of the next year. In Denmark, as well as in the other EU countries, the capital requirement is connected to the liability side of the balance sheet. It consists of two elements: 4% of the policy reserves plus a small premium added to allow for the cost of hedging actuarial risks, approximated to be around 0.5% of the policy reserves. Therefore the relative buffer ratio should be at least 4.5%, i.e.

$$\frac{B(\cdot)}{P(\cdot)} \geq 4.5\%.$$

As a result, it is reasonable to define the stock proportion as follows:

$$\pi(t; A(t), P(t)) = \max_{0 \leq \pi \leq 100\%} \left\{ Prob\left(\frac{B(t+1)}{P(t+1)} < 4.5\% | A(t), P(t) \right) < \alpha^* \right\}, \tag{4}$$

where α^* is a risk level corresponding to solvency requirement. Indeed, the proportion invested in stocks would be maximised so that the buffer exceeds the solvency capital requirements with probability $(1-\alpha^*)$. We have chosen a risk level of $\alpha^* = 0.1\%$, so that solvency at the end of the subsequent year has a 99.9% probability. More details can be found in the Appendix.

2.3 Stable long term buffer levels

The stable long term buffer is higher than the target buffer. The reason follows from the relationship between the stock proportion function, the risk premium of stocks and the underlying guarantee. If the pensioner's interest guarantee contract starts at the target value, then on average the pensioner would in the long run have to contribute to raise the buffer from the target value to the stable value. Therefore, such a pensioner would indeed on average lose from this isolated effect. This argument would of course be reversed when the initial buffer value is above the long term stable value. At the current Danish market place most interest guarantee buffers are very low and we imagine that a current analysis of buffer values would lead to the overall conclusion that any pensioner entering the classical interest guarantee market would face a considerable loss from the effect of having to help raise buffer values to the stable level. We imagine that the recent financial credit crunch has left most developed economies in a similar situation as the Danish, with similar consequences for pensioners in such countries when entering the market place of classical interest guarantee products. For example, for the Codan strategy below with target buffer 10.85% we found through simulations that the long term stable value was 15.86% with a high guarantee level (and a resulting low average stock proportion) and 22.47% with a lower guarantee level (and a resulting higher average stock proportion). In our triple blow comparisons in the next sections we always start the pensioners contract in exactly the long term stable buffer implying that there will not be any long term effect of buffer accumulation in our calculations. In practice there will of course practically always be such a buffer accumulation effect, because either a pension contract is started above the long term stable value (implying an average advantage for the client) or below the long term stable value (implying an average loss for the client). As already indicated in the current financial climate the buffer accumulation probably gives any new client a considerable disadvantage and we could therefore with some right have subtitled this paper "The law of the quadruple blow" acknowledging this additional cause for concern for any young pensioner entering the current market. However, in the following, all analysis will not be confounded by the buffer effect and we start all our simulated contracts in the long term stable buffer value that we therefore also call the initial buffer value, see Table 1.

3 Performance measurement methodology

In this section we follow the performance measurement methodology of Guillén *et al.* (2011).

We use the simulated developments of the policy interest rate to generate $T = 60$ years development of the individual savings of a policyholder having the guaranteed product. In Table 12 in the Appendix we

Table 1. Initial buffer ratio

Company	Risk premium (r^*)	Initial buffer ratio (ψ_0) $r_G = -1.0\%$	$r_G = -2.5\%$
Codan			
$\beta = 26.28\%$, $\psi = 10.85\%$	0.00%	15.86%	22.47%
	0.25%	14.77%	21.59%
	0.50%	13.55%	20.70%
Danica			
$\beta = 12.25\%$, $\psi = 7.63\%$	0.00%	18.62%	32.34%
	0.25%	16.51%	30.56%
	0.50%	14.30%	28.75%
PFA			
$\beta = 20.66\%$, $\psi = 6.18\%$	0.00%	12.32%	20.53%
	0.25%	11.03%	19.41%
	0.50%	9.69%	18.23%
Tryg			
$\beta = 13.03\%$, $\psi = 7.44\%$	0.00%	17.69%	30.64%
	0.25%	15.72%	28.94%
	0.50%	13.62%	27.18%

also provide results for a contract with $T = 30$ years. As might be suspected the nominal values of the three blows are similar but smaller when $T = 30$ is considered instead of $T = 60$ years. We define the policyholder's payment stream $\Delta C(t)$ by the constant yearly premiums ($c > 0$) and annuities ($-c < 0$), i.e.

$$\Delta C(t) = \begin{cases} c & t = 0, \ldots, T/2-1, \\ -c & t = T/2, \ldots, T-1. \end{cases}$$

Then the policyholder's total savings Y_T develop as

$$Y_t = (1 + r_P(t))Y_{t-1} + \Delta C(t),$$
$$Y_0 = \Delta C(0).$$

Calculating the final wealth Y_T for each of the simulated developments of the policy interest rate, we get an estimate of the empirical distribution of Y_T.

The performance measurement methodology is a tool which allows us to compare the products with different risks. To evaluate the considered products with the guarantee we search for an equivalent benchmark strategy to each of the products. An equivalent strategy to a given product means that the strategy has the same risk as the product. In particular, we have chosen to work with the expected shortfall as risk measure, which we denote by CTE(95%). The benchmark strategy is the trivial unit-linked strategy with a constant stock proportion π^b for all $t \in [0,T]$. For technical details see Guillén *et al.* (2011).

For both the with-profit product and its equivalent strategy, the internal interest rate (r_{int}) is calculated, i.e.

$$\sum_{t=0}^{T-1} \Delta C(t)(1 + r_{int})^{T-t} - M_T = 0, \tag{5}$$

where M_T is the median of the final wealth distribution, i.e. the distribution of Y_T. The difference between the benchmark's and product's internal interest rates, $r_{int}^b - r_{int}^p$, is called the yearly financial loss (if positive) or gain (if negative). The difference indicates whether or not the product beats its equivalent benchmark strategy.

The parameters that we have chosen in our simulations are as follows. We assume that the risk free rate is equal to 0. Such a baseline has also been chosen in Guillén *et al*. (2011). The choice of the risk free rate does not affect the results. This has been investigated in Konicz (2010). We have estimated the yearly excess stock return to be equal to $\mu = 3.42\%$ and the volatility $\sigma = 15.44\%$. See the Appendix for details on the parameters estimation.

As mentioned above, we choose the initial buffer $\psi_0 = \frac{B_0}{P_0}$ to be equal to the long term stable buffer value. Let $P_0 = 1$, then $B_0 = \psi_0 P_0$, and the company assets which have to be equal to the liabilities are defined by $B_0 = P_0 + B_0$. Notice that the choice of P_0 does not matter. It is a scaling factor in the bonus distribution mechanism.

We set the short rate of interest equal to zero because we are only interested in seeing the return that exceeds the risk-free rate. This obviously affects the choice of the guaranteed yearly interest rate. No company would offer a guaranteed interest rate that is higher than the risk-free rate. We fix the risk-free rate equal to zero and therefore r_G must be some negative value, say $r_G = -2.5\%$, which means that the guaranteed rate will be 250 less basis points than the short risk-free rate. By looking at the difference between r and r_G, we can immediately see how generous the company is by offering the given guarantee. Finally, regarding the risk premium r^* in (1) we consider three possible values, 0.00%, 0.25% and 0.50%.

4 Quantifying the loss when paying for an interest guarantee

The quantification of the three blows does of course depend on the econometric model assumptions, the interest guarantee level and the risk measure. In this section we illustrate the calculation of the three blows. We give some sensitivity analyses illustrating how values vary with the assumptions made. Our approach is simple. The first blow is the lack of performance of the with-profit product itself when $r^* = 0.00\%$. The second and the third blows arise only when r^* is positive. The second blow is simply defined as to be equal to r^* To find the third blow we first calculate the total loss the pensioner faces from the interest guarantee product and we then subtract the first and the second blows. Therefore, by definition, the three blows aggregate to the total loss from the interest guarantee product compared to a simple benchmark product. We always assume that the with-profit return mechanism is initiated in the long term stable buffer. Therefore, we have not included the important extra risk for the individual pension saver that he might join the pension scheme at a buffer lower than the long term stable buffer. Our intuitive feeling would be that this is a very serious current risk for individuals joining now in an economic environment with extraordinary low buffers after our financial crisis.

4.1 Calculating the first, second and third blow

In Table 2 the three blows are calculated in respectively the Codan and the Danica with-profit scheme exemplified with a minus one percent interest guarantee (compared to the short rate) and expected shortfall at 95%. The first blow is the loss of the with-profit product compared to the equivalent trivial benchmark with $r^* = 0.00\%$. The second blow is given by r^* equal to 0.00%, 0.25% or 0.50%. The third blow is calculated as the total loss compared to the trivial benchmark product minus the first

Table 2. Calculating the blows. Parameters: *CTE* (95%), $r_G = -1.0\%$

	Risk premium (r^*)		
Company	0.00%	0.25%	0.50%
Codan			
first blow	0.02%	0.02%	0.02%
second blow	0.00%	0.25%	0.50%
third blow	0.00%	0.28%	0.51%
Total	0.02%	0.55%	1.03%
Danica			
first blow	0.03%	0.03%	0.03%
second blow	0.00%	0.25%	0.50%
third blow	0.00%	0.28%	0.52%
total	0.03%	0.56%	1.05%

This is part of Table 13 in the Appendix

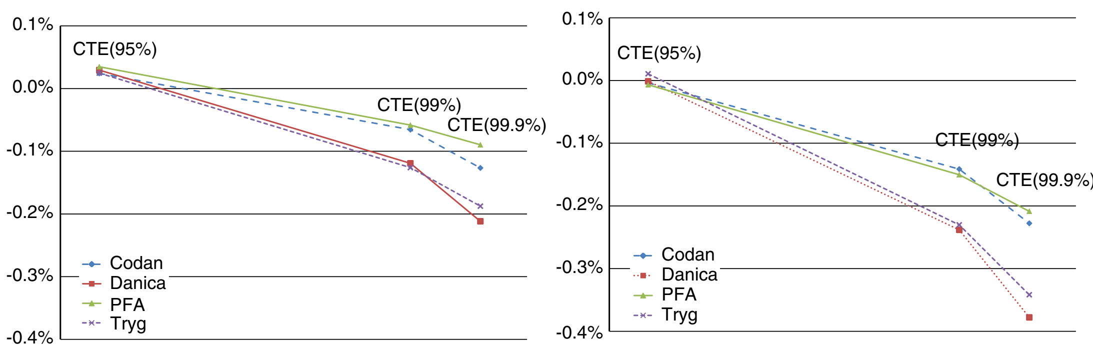

Fig. 1. (colour online) The first blow, when there is no risk premium, i.e. $r^* = 0.0\%$. The figure on the left shows the case for $r_G = -1.0\%$, and on the right for $r_G = -2.5\%$.

and the second blows. The numbers in Table 2 are taken from Tables in the Appendix, where more complementary calculations are available (see Tables 6, 13 and 14 in the Appendix).

4.2 Understanding the first blow

In Figure 1 we give a sensitivity analysis changing the level of the expected shortfall (see Tables 6 to 11 in the Appendix). We see that the first blow is negative for very high expected shortfall with magnitudes around -0.25% for the very high level of 99.9% indicating that the interest guarantee might be worth its cost (sometimes as low as -0.25%) for extraordinarily risk averse customers. Our point of view is, however, that a 99.9% risk level is inappropriate for the analysis at hand. There are many hidden risks in most with-profit schemes with guarantees. For example: Do you start your pension at a good initial buffer value? What credit risk is implied in the guarantee? Can you take the value of the interest guarantee with you when moving company? Does the non-transparency of the with-profit mechanism imply hidden costs? Are you unable to react if inflation kicks in?

All these underlying risks of the with-profit scheme with guarantees make expected shortfall levels of 99% and 99.9% inappropriately high in our view and we therefore prefer to use 95% as our benchmark expected shortfall in general.

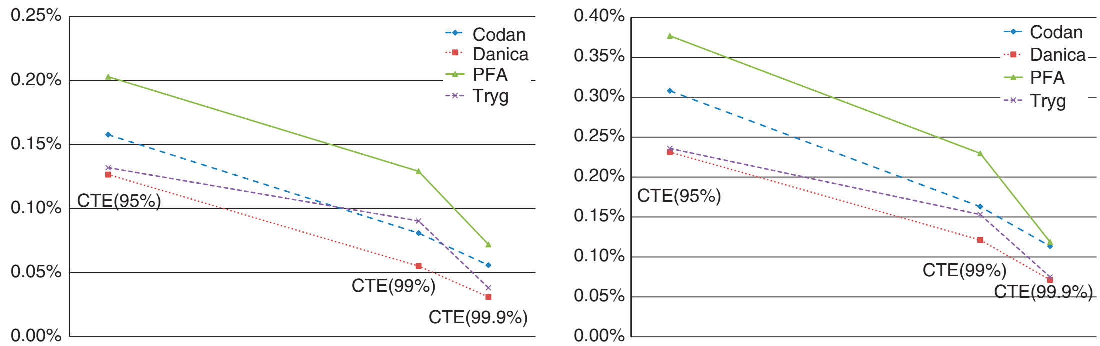

Fig. 2. (colour online) The third blow for $r_G = -2.5\%$. The figure on the left shows the case for $r^* = 0.25\%$, and on the right for $r^* = 0.5\%$.

4.3 Understanding the third blow

We also investigated the third blow as a function of the expected shortfall (see Figure 2). The third blow is very substantial at our preferred expected shortfall level of 95% but becomes less relevant for very high expected shortfall values (see also Figure 3 in the Appendix).

4.4 Comparing the with-profit products with the trivial benchmark strategy

We consider an individual entering the company when the buffer of the company is exactly on the long-term stable buffer target. Unfortunately, in the Danish market case in 2012 most commercial pension companies have initial buffers below this target and a new young customer entering the current Danish pension market through a with-profit interest guarantee scheme will be worse off than indicated by our results below.

Table 3 presents the results of our simulation study. The first column shows the pension fund mechanism that is being simulated over a period of 60 years using 10,000 simulations. Three possibilities for the risk premium r^* have been considered.

Every with-profit product in column one has been compared to the trivial strategy that has a fixed investment proportion in stocks and bonds over the whole length of the contract. The equivalent strategy, i.e. the proportion of wealth invested in stocks, is found as the one that would have exactly the same risk as the with-profit pension investment. The risk is measured as the CTE(95%) and it is shown in the second column of Table 3. The third column presents the corresponding proportion of investment in stocks. For instance, the first product, sold in Denmark by company Codan, has a risk equal to −42.6 units. Note that as monetary units are not relevant for our purposes, we just report this measure to show how the equivalent simple strategy is found. In the third column, we see that Codan's product with $r^* = 0\%$ has the same risk as the trivial benchmark product with 31% as a fixed percentage invested in stocks (69% in bonds) over the whole life span period.

Columns four and five of Table 3 show the internal yearly interest rate calculated as in (5) for the pension fund product and for the equivalent strategy. For example, for Codan's product with $r^* = 0.5\%$ we find that the internal yearly interest rate is 0.46% implying that, when considering the median of the final wealth distribution, this product has an internal interest rate that exceeds the yearly risk-free rate by 46 basis points. However, the equivalent trivial benchmark strategy with the same risk level obtains an excess yearly interest rate of 1.26% with respect to the risk-free rate

Table 3. Comparison of the with-profit products and the trivial strategy

Risk premium (r^*)	CTE(95%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate r^p_{int}	Internal interest rate r^b_{int}	Difference $(r^b_{int} - r^p_{int})$
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−42.6	31%	0.96%	0.95%	0.00%
0.25%	−53.2	37%	0.71%	1.11%	0.40%
0.50%	−62.5	43%	0.46%	1.26%	0.80%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−55.8	39%	1.16%	1.16%	0.00%
0.25%	−63.2	44%	0.91%	1.29%	0.38%
0.50%	−69.9	48%	0.65%	1.38%	0.73%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−37.4	27%	0.85%	0.84%	−0.01%
0.25%	−48.2	34%	0.58%	1.03%	0.45%
0.50%	−57.8	40%	0.32%	1.19%	0.87%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−53.6	38%	1.13%	1.14%	0.01%
0.25%	−61.4	43%	0.87%	1.26%	0.39%
0.50%	−68.5	47%	0.61%	1.35%	0.75%

For each product with the guarantee the expected shortfall CTE(95%) is calculated. Thereafter, by finding π^b the equivalent trivial strategy is determined. We compare all equivalent products by calculating the internal interest rates from the median of the final wealth. Parameters: $r = 0\%$, $r_G = -2.5\%$, $\alpha^* = 0.1\%$, $\mu = 3.42\%$, $\sigma = 15.44\%$. The difference $r^b_{int} - r^p_{int}$ compares the performance of the considered products versus the equivalent trivial benchmark.

implying a 0.80% loss for buying the expensive with-profit product compared to the trivial benchmark product. The difference between the internal yearly interest rates for each simulated product is found in column six of Table 3. The difference reported in column six means that, while having exactly the same risk, a customer would have a higher yearly internal interest rate if he or she would change from his or her market product to a simple strategy based on keeping a fixed proportion of wealth in stocks over the whole life span. The difference presented in column six indicates the yearly loss for having the with-profit scheme with guarantee assumed to be 250 basis points below the risk-free rate in this illustration.

The results shown in Table 3 indicate that all four versions of with-profit interest guarantee strategies lead to a financial loss of the customer. Note that when comparing results for $r^* = 0.25\%$ and $r^* = 0.50\%$ the loss in yearly interest rate reported in column six approximately doubles for all the funds considered here. On aggregate the total loss seems to be around 0.40% when $r^* = 0.25\%$ and around 0.75% when $r^* = 0.50\%$.[†]

[†] We also calculated the effect of paying for the interest guarantee directly instead of indirectly. Since indirect payment and direct payment gave almost identical results, we do not report the results from paying directly. However, the direct payment has a tendency to be slightly better for the customer than the indirect payment that makes the interest guarantee provider more risk averse.

In the Appendix we have included more simulation studies and show the results by replicating Table 3 for various levels of the CTE and guaranteed rates, and also a case for a shorter contract ($T = 30$ years). Interestingly, we observe that the magnitude of the three blows is higher for the higher guaranteed interest rates, i.e. the difference between the product and the corresponding benchmark is bigger when the company offers their clients a higher guarantee rate, such as $r_G = -1.0\%$. For instance, for risk premium $r^* = 0.5\%$, under $CTE(95\%)$ risk measure and for a lower guaranteed rate $r_G = -2.5\%$, the first and third blow for a Codan product, are equal respectively 0.00% and 0.31%. For a higher $r_G = -1.0\%$ these differences are bigger: 0.02% and 0.51%. See Tables 13 and 14 respectively.

5 Conclusion

We have modelled the performance of pension funds offering an interest guarantee that exist in the Danish market. We have shown that a customer who purchases a minimum interest guarantee has lower median returns compared to another customer that would have no interest guarantee, while having the same risk, measured as the expected shortfall at the 95% level. In our simulation study, we quantified the magnitude of that loss, which could be as big as 0.87% yearly in the rate of return.

The loss of returns is due to the mechanism behind with-profit pension products with a guaranteed interest rate. We have also identified the three components of that loss, which we call the three blows (the one coming from the with-profit mechanism itself, the risk premium and the opportunity cost). They have been calculated for different values of the guaranteed interest rate and for different levels of the expected shortfall.

We found that there are not important differences between the with-profit products, as they seem to underperform roughly equally to the corresponding benchmark strategy with the equivalent risk. We also observed that the magnitude of these three blows is higher for the higher guaranteed interest rates. Finally, the interest guarantee product might be worth its cost only for extraordinary risk averse customers, i.e. when we consider the expected shortfall at the 99.9% level, which is, from our point of view, an extreme risk aversion level for our analysis. Therefore, in the light of our Monte Carlo study results, we conclude that pension fund clients may perceive that minimum interest rate guarantees are too expensive, as an alternative product without an interest guarantee but having equivalent risk would provide higher median returns.

Acknowledgements

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Appendix

Investment strategy

Investment strategy defined in equation (4) can be calculated explicitly using the fact the Brownian motion is normally N(0,1) distributed and that α^* is reached by some $\pi(t)$.

$$Prob\left(\frac{A(t+1)-P(t+1)}{P(t+1)}<4.5\%\right)=\alpha^*$$

$$Prob\left(A(t)\exp\left\{r+\pi(t)(\mu-r)-\frac{\pi^2(t)\sigma^2}{2}+\pi(t)\sigma Z_i\right\}-r^*P(t)<(1+4.5\%)P(t+1)\right)=\alpha^*$$

$$Prob\left(r+\pi(t)(\mu-r)-\frac{\pi^2(t)\sigma^2}{2}+\pi(t)\sigma Z_i<\ln\frac{1.045P(t+1)+r^*P(t)}{A(t)}\right)=\alpha^*$$

$$Prob(Z_i<D(\pi(t)))=\alpha^*$$

where

$$D(\pi(t))=\frac{\ln(1.045P(t+1)+r^*P(t))-\ln A(t)-r-\pi(t)(\mu-r)+\frac{\pi^2(t)\sigma^2}{2}}{\pi(t)\sigma}.$$

For a given degree of the solvency risk α^*, we can read the value of $D\ (\pi(t))$ from the standard normal distribution table. Thereafter we can solve the above equation with respect to $\pi(t)$:

$$\begin{gathered}\pi^2(t)\frac{\sigma^2}{2}-\pi(t)(\sigma D+\mu-r)+\ln(1.045P(t+1)+r^*P(t))-\ln A(t)-r=0\\ \pi(t)=\frac{\sigma D+\mu-r\pm\sqrt{(\sigma D+\mu-r)^2-2\sigma^2(\ln(1.045P(t+1)+r^*P(t))-\ln A(t)-r)}}{\sigma^2},\end{gathered}\tag{6}$$

with a constraint

$$\pi(t)\in[0,100\%].$$

Note that by monotonicity of the probability function, definition (4) can be rewritten as

$$\pi(t;A(t),P(t))=\left\{\pi(t):Prob\left(\frac{B(t+1)}{P(t+1)}=4.5\%|A(t),P(t)\right)<\alpha^*\right\}$$

hence π(t) obtained in (6) is indeed the maximum π.

In the case where the value under the square root is negative $(\sigma D+\mu-r)^2-2\sigma^2(\ln(1.045P(t+1)+r^*P(t))-\ln A(t)-r)<0$, we set $\pi(t)=0$. This means that the solvency requirement cannot be fulfilled – the relative buffer ratio is smaller than 4.5% with the probability higher than $(1-\alpha^*)$. Investing only in bonds in this case is a natural decision here.

Parameters Estimation

To estimate parameter σ that we use in model (1) we refer to the data found in Dimson *et al.* (2002) that provide an overview of the long-term performance of individual market for sixteen countries, and estimate total returns on equities, bonds, bills, currencies, inflation and risk premia for 101 years from 1900 to 2000. The authors also discuss what the future holds and what the expectations of the future returns might be.

The statistics are based on sixteen countries worldwide from the perspective of a US investor. It is assumed that the worldwide return would have been received by a US citizen who bought foreign currency at the start of the period, invested it in the foreign market throughout the period, liquidated his position, and converted the proceeds back at the end of the period into US dollars. At the beginning of each period the investor buys a portfolio of sixteen such positions in each of the countries, weighting each country by its size.

For the purpose of the paper, where we assume risk free rate r to be fixed, the natural choice of data will be the risk premia relative to US bills (risk free rate is taken as the return on US treasury bills) because we will not have a volatility coming from the risk free rate. In Tables 4 and 5 we present this part of the data that is relevant for us.

We want to estimate the parameter σ based on historical volatility. Since we are provided only with expected excess returns over the periods of 1,2,…,10 decades, we cannot use the standard method to estimate historical volatility. In this situation we have to find another estimation method.

Table 4. World equity risk premia over various periods, 1900–2000. Annualised real returns. Source: Dimson *et al.* (2002).

from-to	Equity premium versus US bills
1900–1909	3.1
1910–1919	−1.2
1920–1929	6.9
1930–1939	−1.0
1940–1949	6.1
1950–1959	17.5
1960–1969	4.1
1970–1979	1.9
1980–1989	9.6
1990–2000	3.3

Table 5. World equity risk premia. A summary table based on the entire 101 years. Source: Dimson *et al.* (2002).

Mean return % per annum			Dispersion of annual returns		
Geometric	Arithmetic	SE	SD	Lowest return	Highest return
4.9	6.2	1.6	16.4	−39.8 (1931)	70.9 (1933)

From Table 5 we see that the standard deviation of annual excess returns on the entire 101 years (SD) is $\upsilon = 16.4\%$ and the arithmetic mean m is equal to 6.2%

$$E\left[\frac{\Delta S_t}{S_t}\right] = m \quad Var\left[\frac{\Delta S_t}{S_t}\right] = \upsilon^2.$$

At the same time we assume that the stock prices are lognormally distributed which means that the change in ln S is normally distributed with variance

$$Var\left[\ln\frac{S_{t+1}}{S_t}\right] = \sigma^2.$$

Hence our aim is to estimate the volatility parameter σ given $(m,\upsilon) = (6.2\%, 16.4\%)$.

Denote $\varepsilon = \frac{S_{t+1}-S_t}{S_t}$, then

$$\ln\frac{S_{t+1}}{S_t} = \ln\left(1 + \frac{S_{t+1}-S_t}{S_t}\right) = \ln(1+\varepsilon) =: f(\varepsilon).$$

Function $f(\varepsilon)$ is sufficiently differentiable, so we can approximate the moments using Taylor expansions. The second moment is given by

$$Var[f(\varepsilon)] \approx (f'(E[\varepsilon]))^2 Var[\varepsilon],$$

which in our case gives

$$Var[\ln(1+\varepsilon)] \approx \frac{\upsilon^2}{(1+m)^2},$$

which leads to

$$\sigma = \frac{\upsilon}{1+m} = 15.44\%.$$

To estimate the future excess stock return $\mu - r$ we cannot use historical data only. We let the expected excess log return $\hat{m} = 2.23\%$ implying $\mu = 3.42\%$, since $r = 0$ and $\hat{m} = \mu - \sigma^2/2$. The excess stock return - or equity premium - of the United States have historically been between 4% and 9% depending on the estimation period and estimation method, Constantinides (2002) and Dimson *et al.* (2002). The latter authors argue that this high level of excess stock return can not be expected in the future. We have therefore decided to follow Jørgensen & Nielsen (2002) and select an equity premium on the safe side.

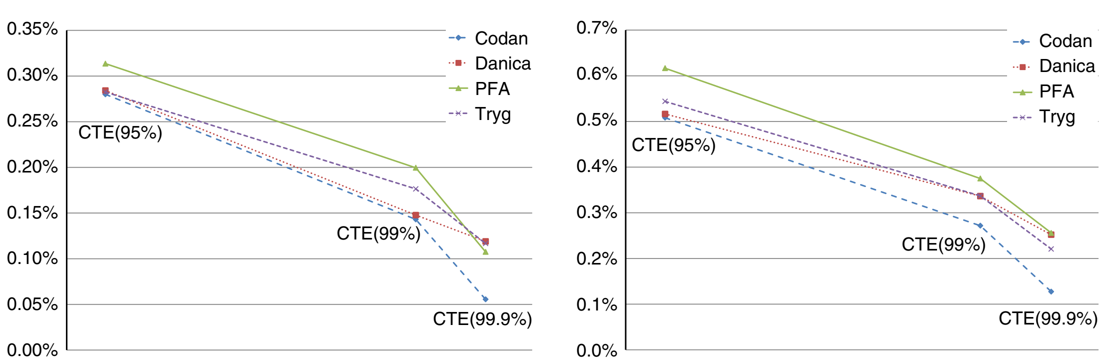

Fig. 3. (colour online) The third blow for $r_G = -1.0\%$. The figure on the left shows the case for $r^* = 0.25\%$, and on the right for $r^* = 0.5\%$.

Table 6. Parameters: $CTE(95\%)$, $r_G = -1.0\%$

Risk premium (r^*)	CTE(95%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate r^p_{int}	Internal interest rate r^b_{int}	Difference ($r^b_{int} r^p_{int}$)
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−28.4	21%	0.64%	0.67%	0.02%
0.25%	−41.4	30%	0.37%	0.92%	0.55%
0.50%	−52.8	37%	0.08%	1.11%	1.03%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−30.2	22%	0.67%	0.70%	0.03%
0.25%	−41.3	30%	0.36%	0.92%	0.56%
0.50%	−51.4	36%	0.04%	1.08%	1.05%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−18.9	15%	0.45%	0.49%	0.03%
0.25%	−33.1	24%	0.16%	0.76%	0.60%
0.50%	−45.9	33%	−0.15%	1.00%	1.15%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−28.8	21%	0.64%	0.67%	0.02%
0.25%	−40.3	29%	0.34%	0.90%	0.56%
0.50%	−50.8	36%	0.02%	1.08%	1.07%

Table 7. Parameters: $CTE(99\%)$, $r_G = -1.0\%$

Risk premium (r^*)	CTE(99%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate r^p_{int}	Internal interest rate r^b_{int}	Difference ($r^b_{int} r^p_{int}$)
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−42.8	18%	0.64%	0.58%	−0.07%
0.25%	−52.2	22%	0.37%	0.70%	0.33%
0.50%	−60.1	25%	0.08%	0.78%	0.71%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−41.4	17%	0.67%	0.55%	−0.12%
0.25%	−49.7	20%	0.36%	0.64%	0.28%
0.50%	−57.3	24%	0.04%	0.76%	0.72%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−29.4	12%	0.45%	0.39%	−0.06%
0.25%	−40.9	17%	0.16%	0.55%	0.39%
0.50%	−51.3	21%	−0.15%	0.67%	0.82%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−40.0	16%	0.64%	0.52%	−0.13%
0.25%	−48.8	20%	0.34%	0.64%	0.30%
0.50%	−56.7	23%	0.02%	0.73%	0.71%

Table 8. Parameters: $CTE(99.9\%)$, $r_G = -1.0\%$

Risk premium (r^*)	CTE(99.9%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate		Difference
			r_{int}^{p}	r_{int}^{b}	$(r_{int}^{b} r_{int}^{p})$
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−55.9	16%	0.64%	0.52%	−0.13%
0.25%	−61.4	17%	0.37%	0.55%	0.18%
0.50%	−64.8	18%	0.08%	0.58%	0.50%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−51.7	14%	0.67%	0.46%	−0.21%
0.25%	−57.6	16%	0.36%	0.52%	0.16%
0.50%	−62.5	18%	0.04%	0.58%	0.54%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−39.9	11%	0.45%	0.36%	−0.09%
0.25%	−48.9	13%	0.16%	0.43%	0.27%
0.50%	−57.1	16%	−0.15%	0.52%	0.67%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−50.5	14%	0.64%	0.46%	−0.19%
0.25%	−56.7	16%	0.34%	0.52%	0.18%
0.50%	−62.1	17%	0.02%	0.55%	0.53%

Table 9. Parameters: $CTE(95\%)$, $r_G = -2.5\%$

Risk premium (r^*)	CTE(95%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate		Difference
			r_{int}^{p}	r_{int}^{b}	$(r_{int}^{b} r_{int}^{p})$
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−42.6	31%	0.96%	0.95%	0.00%
0.25%	−53.2	37%	0.71%	1.11%	0.40%
0.50%	−62.5	43%	0.46%	1.26%	0.80%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−55.8	39%	1.16%	1.16%	0.00%
0.25%	−63.2	44%	0.91%	1.29%	0.38%
0.50%	−69.9	48%	0.65%	1.38%	0.73%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−37.4	27%	0.85%	0.84%	−0.01%
0.25%	−48.2	34%	0.58%	1.03%	0.45%
0.50%	−57.8	40%	0.32%	1.19%	0.87%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−53.6	38%	1.13%	1.14%	0.01%
0.25%	−61.4	43%	0.87%	1.26%	0.39%
0.50%	−68.5	47%	0.61%	1.35%	0.75%

Table 10. Parameters: $CTE(99\%)$, $r_G = -2.5\%$

Risk premium (r^*)	CTE(99%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate r^p_{int}	Internal interest rate r^b_{int}	Difference ($r^b_{int} r^p_{int}$)
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−62.6	26%	0.96%	0.81%	−0.14%
0.25%	−70.0	29%	0.71%	0.90%	0.19%
0.50%	−76.8	32%	0.46%	0.98%	0.52%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−71.5	30%	1.16%	0.92%	−0.24%
0.25%	−76.6	32%	0.91%	0.98%	0.07%
0.50%	−81.4	34%	0.65%	1.03%	0.38%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−54.3	22%	0.85%	0.70%	−0.15%
0.25%	−62.2	26%	0.58%	0.81%	0.23%
0.50%	−69.4	29%	0.32%	0.90%	0.58%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−69.7	29%	1.13%	0.90%	−0.23%
0.25%	−75.1	32%	0.87%	0.98%	0.11%
0.50%	−80.1	34%	0.61%	1.03%	0.42%

Table 11. Parameters: $CTE(99.9\%)$, $r_G = -2.5\%$

Risk premium (r^*)	CTE(99.9%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate r^p_{int}	Internal interest rate r^b_{int}	Difference ($r^b_{int} r^p_{int}$)
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−81.1	23%	0.96%	0.73%	−0.23%
0.25%	−86.1	25%	0.71%	0.78%	0.08%
0.50%	−90.6	27%	0.46%	0.84%	0.39%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−85.6	25%	1.16%	0.78%	−0.38%
0.25%	−88.9	26%	0.91%	0.81%	−0.10%
0.50%	−92.0	27%	0.65%	0.84%	0.19%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−70.3	20%	0.85%	0.64%	−0.21%
0.25%	−75.9	22%	0.58%	0.70%	0.11%
0.50%	−81.0	23%	0.32%	0.73%	0.41%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−84.1	25%	1.13%	0.78%	−0.34%
0.25%	−87.6	26%	0.87%	0.81%	−0.05%
0.50%	−90.9	27%	0.61%	0.84%	0.23%

Table 12. The case with a different payment process. Parameters: *CTE*(95%), $r_G = -2.5\%$

Risk premium (r^*)	CTE(95%)	Equivalent benchmark stock proportion (π^b)	Internal interest rate r^p_{int}	Internal interest rate r^b_{int}	Difference ($r^b_{int} r^p_{int}$)
Codan					
$\beta = 26.28\%$, $\psi = 10.85\%$					
0.00%	−26.4	27%	0.88%	0.87%	−0.01%
0.25%	−30.0	31%	0.63%	0.98%	0.35%
0.50%	−33.5	34%	0.39%	1.07%	0.68%
Danica					
$\beta = 12.25\%$, $\psi = 7.63\%$					
0.00%	−30.1	31%	0.93%	0.98%	0.05%
0.25%	−32.8	34%	0.69%	1.07%	0.37%
0.50%	−35.4	36%	0.44%	1.12%	0.68%
PFA					
$\beta = 20.66\%$, $\psi = 6.18\%$					
0.00%	−22.2	23%	0.75%	0.75%	0.00%
0.25%	−25.8	26%	0.49%	0.84%	0.34%
0.50%	−29.3	30%	0.23%	0.95%	0.72%
Tryg					
$\beta = 13.03\%$, $\psi = 7.44\%$					
0.00%	−29.2	30%	0.91%	0.95%	0.04%
0.25%	−32.0	33%	0.67%	1.04%	0.37%
0.50%	−34.7	36%	0.41%	1.12%	0.71%

The entire contract varies over 30 years. The pension saver pays the contribution of 10 dkk for 20 years, thereafter receives the benefits of 20 dkk for the remaining 10 years.

Table 13. Calculating the blows. Parameters: $r_G = -1.0\%$

	CTE(95%)			CTE(99%)			CTE(99.9%)		
Risk premium (r^*)	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
Codan									
first blow	0.02%	0.02%	0.02%	−0.07%	−0.07%	−0.07%	−0.13%	−0.13%	−0.13%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.28%	0.51%	0.00%	0.14%	0.27%	0.00%	0.06%	0.13%
total	0.02%	0.55%	1.03%	−0.07%	0.33%	0.71%	−0.13%	0.18%	0.50%
Danica									
first blow	0.03%	0.03%	0.03%	−0.12%	−0.12%	−0.12%	−0.21%	−0.21%	−0.21%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.28%	0.52%	0.00%	0.15%	0.34%	0.00%	0.12%	0.25%
total	0.03%	0.56%	1.05%	−0.12%	0.28%	0.72%	−0.21%	0.16%	0.54%
PFA									
first blow	0.03%	0.03%	0.03%	−0.06%	−0.06%	−0.06%	−0.09%	−0.09%	−0.09%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.31%	0.62%	0.00%	0.20%	0.38%	0.00%	0.11%	0.26%
total	0.03%	0.60%	1.15%	−0.06%	0.39%	0.82%	−0.09%	0.27%	0.67%
Tryg									
first blow	0.02%	0.02%	0.02%	−0.13%	−0.13%	−0.13%	−0.19%	−0.19%	−0.19%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.28%	0.54%	0.00%	0.18%	0.34%	0.00%	0.12%	0.22%
total	0.02%	0.56%	1.07%	−0.13%	0.30%	0.71%	−0.19%	0.18%	0.53%

Table 14. Calculating the blows. Parameters: $r_G = -2.5\%$

	CTE(95%)			CTE(99%)			CTE(99.9%)		
Risk premium (r^*)	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
Codan									
first blow	0.00%	0.00%	0.00%	−0.14%	−0.14%	−0.14%	−0.23%	−0.23%	−0.23%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.16%	0.31%	0.00%	0.08%	0.16%	0.00%	0.06%	0.11%
total	0.00%	0.40%	0.80%	−0.14%	0.19%	0.52%	−0.23%	0.08%	0.39%
Danica									
first blow	0.00%	0.00%	0.00%	−0.24%	−0.24%	−0.24%	−0.38%	−0.38%	−0.38%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.13%	0.23%	0.00%	0.05%	0.12%	0.00%	0.03%	0.07%
total	0.00%	0.38%	0.73%	−0.24%	0.07%	0.38%	−0.38%	−0.10%	0.19%
PFA									
first blow	−0.01%	−0.01%	−0.01%	−0.15%	−0.15%	−0.15%	−0.21%	−0.21%	−0.21%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.20%	0.38%	0.00%	0.13%	0.23%	0.00%	0.07%	0.12%
total	−0.01%	0.45%	0.87%	−0.15%	0.23%	0.58%	−0.21%	0.11%	0.41%
Tryg									
first blow	0.01%	0.01%	0.01%	−0.23%	−0.23%	−0.23%	−0.34%	−0.34%	−0.34%
second blow	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%	0.00%	0.25%	0.50%
third blow	0.00%	0.13%	0.24%	0.00%	0.09%	0.15%	0.00%	0.04%	0.07%
total	0.01%	0.39%	0.75%	−0.23%	0.11%	0.42%	−0.34%	−0.05%	0.23%

Erratum

The following is a list of typos found in the published version of "Do not pay for a Danish interest guarantee. The law of the triple blow".

Page	Wrong	Correct
191	$B_0 = \psi_0 P_0$	$B_0 = \psi_0$
191	$B_0 = P_0 + B_0$	$A_0 = P_0 + B_0$