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Abstract. A simple method for estimation of parameters in first order systems with time delays is presented in this paper. The parameter estimation approach is based on a step response for the open loop system. It is shown that the estimation method does not require a complete step response, only a part of the response and the steady state value of the system before the step is applied. Further, for calculation of the time delay, it is also required that the time for the step is known.

Keywords:
First order systems, parameter estimation, time delay, step response.

1. Introduction

First order models with a time delay are used in a large number of different applications, where the main dynamic is reasonable damped. First order models are also the basic for a number of controller tuning methods as e.g. [4, 5, 7, 8, 10, 11]. Some of these tuning methods are also applied in a number of different auto-tuning methods for simple feedback controllers, see e.g. in [2, 12].

There exists a number of methods for estimation of the parameters in a first order system as described in e.g. [1, 3, 9, 11]. Some of these methods are standard methods in basic control text books, see e.g. [4, 6].

One of the most used methods for estimation of parameters in stable first order systems is to apply a step response. A step is shown in Fig. 1 for a system with a gain $k = 1$, a time constant $\tau = 1$, a time delay $\theta = 1$ and stepsize 1.

Based on the step response, it is easy to determine the gain, the time constant and the time delay, [4, 6]. However, the step response will not always be as nice and smooth as shown in Fig. 1. The steady value of the system before the step input might not be known exactly because disturbance might be included as well as other issues that might affect the steady state value. Further, the system will not in general exactly be a first order system, but a higher order system with a first order dynamic as the main dynamic. The result of this is that it might be difficult to calculate a tangent in the beginning of the step response. The tangent is one way to determine the time constant in the system. Further, when the exact starting point of the step response is unclear, it is also difficult to calculate the time delay in the system.

Another problem is that the steady state value after the step input might not necessarily occur or it might be unclear. The steady state output will not occur if the step input is not active for a time that is at least
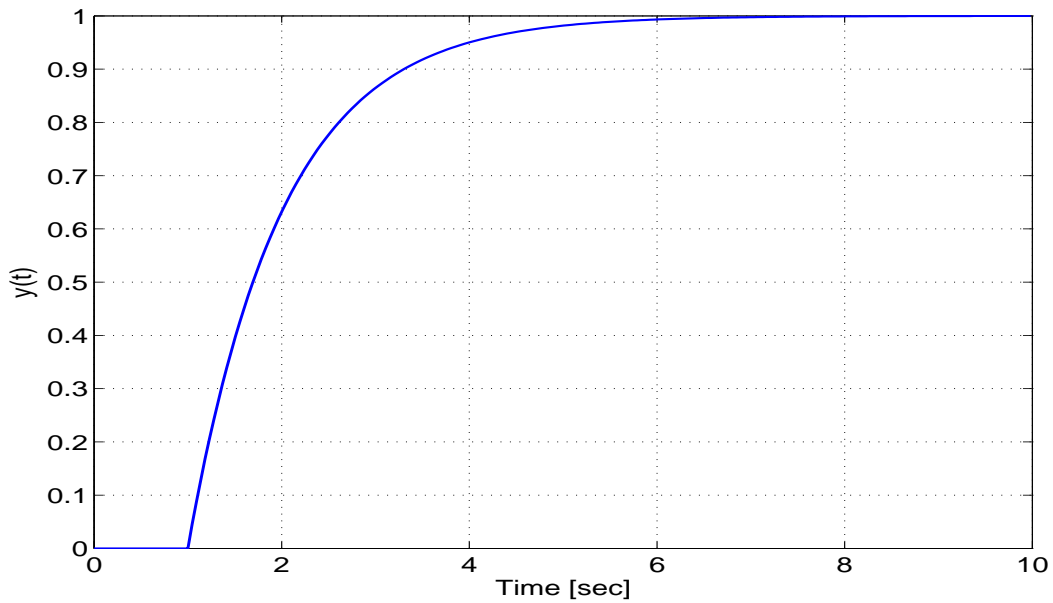
Figure 1. The step response for a first order system.

$5 - 6$ times the time constant τ. Another reason is that the input might not be a pure step input and therefore the steady state output might not be correct. If the steady state value occurs, it might again be disturbed by other known or unknown inputs to the system. A more realistic step response is shown in Fig. 2. The step response is from a dynamic system in an industrial environment. Considering the response, it is reasonable to describe the dynamic by using a first order model with a time delay.
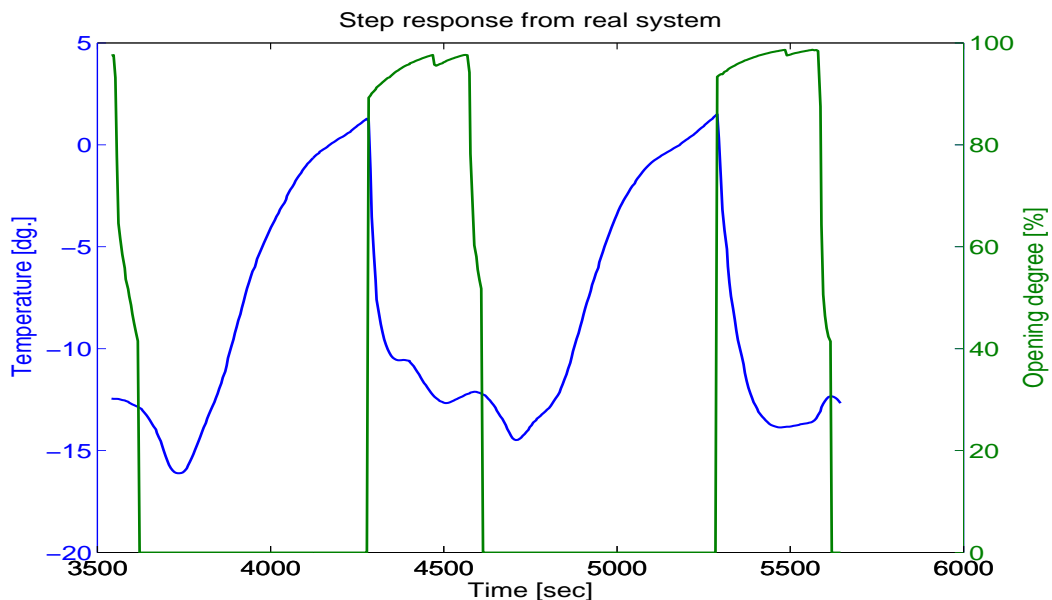
Figure 2. A step response from a refrigeration system. The opening degree is the input to the valve and temperature is the output of the system.

As shown in Fig. 2, we will typical have limited information about the step response. Therefore, we need

to base the estimation of the system parameters directly on the available information. In the following, it will be shown how it is possible to calculate the system parameters based on a part of a step response. The parameter estimation is based on the following information about the step response:

(i) A part of the step response is known.

(ii) The gain of the step input is known.

(iii) An estimate of the steady state value of the output before the step is needed, a mean value.

(iv) The time for the step input is applied, if the time delay is going to be estimated.

It will be shown in the following that based on the above, it is possible to estimate the parameters in a first order system including the time delay. It should be pointed out that it is possible to calculate the steady-state output value of the system as well as the time constant only based on a part of the step response.

The rest of this paper is organized as follows. In Section 2, the system is described together with some preliminary results. The main results are given in Section 3 followed by an example in Section 4. The paper is closed with a conclusion in Section 5.

2. System description

Let a first order system $G(s)$ be given by:

$$G(s) = k\frac{1}{\tau s + 1}e^{-\theta s} \tag{1}$$

where k is the gain, τ is the time constant and θ is the time delay in the system.

The problem is to estimate the three parameters in (1) based on a step response of the system.

Assume that a unit step is applied on the system at time $t = 0$.

It will be assumed that the time delay θ is zero in the following. A non-zero time delay is handled later in this paper.

The step response for the first order system with a unit step input is given by:

$$y(t) = k(1 - e^{-t/\tau}) \tag{2}$$

The steady state value is given by

$$y_{ss} = k \tag{3}$$

Further, we have some standard results with respect to the step response for the first order system, [4, 6].

$$y(\tau) = k(1 - e^{-1}) = k \times 0,632 \tag{4}$$

The derivative of the step response is given by:

$$y'(t) = k\frac{1}{\tau}e^{-t/\tau} \tag{5}$$

and at $t = 0$, we have that

$$y'(0) = k\frac{1}{\tau} \tag{6}$$

Multiplying the tangent in (6) by τ gives

$$y'(0)\tau = k = y_{ss} \tag{7}$$

i.e. it is equal to the steady state value.

The relation between the derivative given by (5) and the derivative for $t = 0$ given by (6) is:

$$\frac{y'(t)}{y'(0)} = e^{-t/\tau}$$

i.e. it depends only of the rate t/τ.

Based on these equations, there are some interesting observations that can be used in the following. First of all, we have that the value of the output is $63, 2\%$ of the steady state value after $t = \tau$, see (4). Further, the tangent slope times τ will be equal to the steady state value of the output, see (7). These two equations are normally used for the calculation of the parameters in the first order system. This is shown in Fig. 3, where the tangent given by (6) has been included. Further, both (4) and (7) can be seen from Fig. 3.
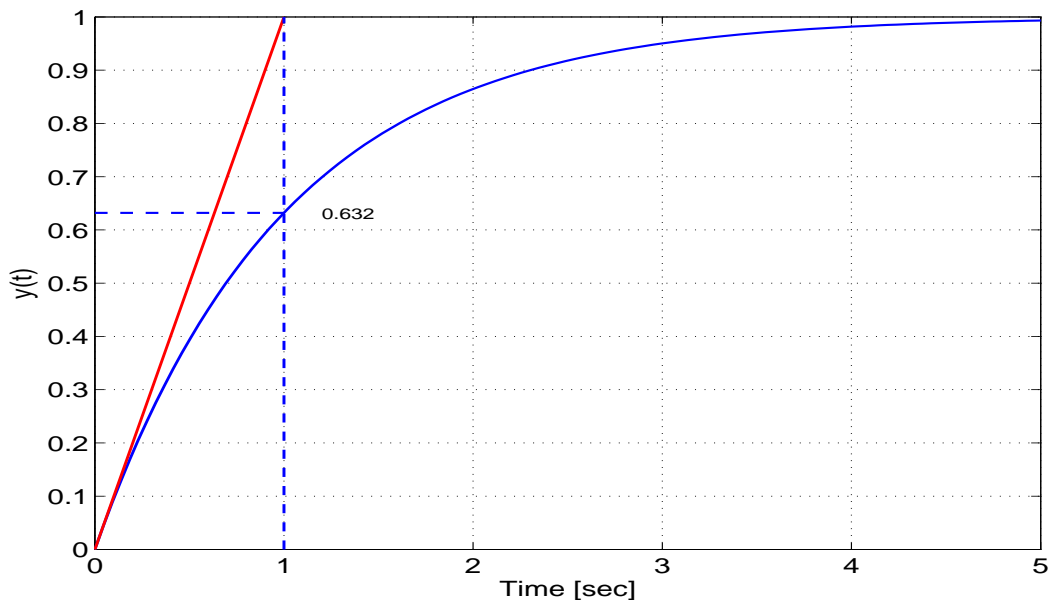
Figure 3. The step response for a first order system with $k = \tau = 1$.

3. Main Results

To be able to estimate the parameters in a first order system, we will have the following assumptions:

 (i) The step response is measure in a time interval that does not need to include the start of the response as well as the steady state value of the response.
 (ii) The step input is known, i.e. the gain of the step.
(iii) The value of the output $y(t)$ is known before the step is applied. This can be a mean value of the output in a given period before the step.

Note that it is not required that the time for the step input is known. If the system include a time delay that need to be estimated, then we need to know the time for the step input.

Based on these assumptions, we are able to estimate the system parameters.

3.1. Estimation of τ

It is assumed that the step is applied at time t_0. Let's consider the step response in the time interval $[t_1, t_2] = [t_0 + \Delta t_1, t_0 + \Delta t_2]$. The time interval will not necessarily include the beginning of the step response ($\Delta t_1 = 0$) as well as the steady state values of the output ($\Delta t_2 \to \infty$). The step response that is available is given by:

$$y(t) = y_0 + k(1 - e^{-(t-t_0)/\tau}), \ t \in [t_1, t_2] \tag{8}$$

where y_0 is the off-set or steady state value before the step input. The derivative of $y(t)$ is given by:

$$y'(t) = k\frac{1}{\tau}e^{-(t-t_0)/\tau}, \ t \in [t_1, t_2] \tag{9}$$

Consider the tangent slope given by (9) at time $t = t_1 = t_0 + \Delta t_1$, i.e. $y'(t_1)$ multiplied by the time interval Δt:

$$y'(t_1)\Delta t = k\frac{1}{\tau}e^{-(t_1-t_0)/\tau}\Delta t \tag{10}$$

As in the previous section, consider the tangent slope with a time interval $\Delta t = \tau$. This gives:

$$y'(t_1)\tau = ke^{-(t_1-t_0)/\tau} \tag{11}$$

This tangent start in $y(t_1) = y_0 + k(1 - e^{-(t_1-t_0)/\tau})$. If this is added to the tangent, we get:

$$\begin{aligned} y &= y(t_1) + y'(t_1)\tau \\ &= y_0 + k(1 - e^{-(t_1-t_0)/\tau}) + ke^{-(t_1-t_0)/\tau} \\ &= y_{ss} \end{aligned} \tag{12}$$

(12) together with (10) shows that any tangent at the time length τ will end at the steady state value y_{ss} of the output. This means that it is always possible to find the steady state value of the output if it is possible to calculate a tangent and the time constant τ is known.

At this point, τ is unknown, but it will be shown how τ can be calculated based on the part of the step response in the time interval $[t_1, t_2]$. Consider the difference in output for the time $t = t_1$ and $t = t_1 + \Delta t$. It is here assumed that $\Delta t \leq t_2 - t_1$. The difference is given by:

$$\begin{aligned} y(t_1 + \Delta t) - y(t_1) &= y_0 + k(1 - e^{-(t_1+\Delta t-t_0)/\tau}) - y_0 - k(1 - e^{-(t_1-t_0)/\tau}) \\ &= k(e^{-(\Delta t_1)/\tau} - e^{-(\Delta t_1+\Delta t)/\tau}) \\ &= k(1 - e^{-\Delta t/\tau})e^{-(\Delta t_1)/\tau} \end{aligned} \tag{13}$$

Further, consider the ratio between the tangent line and the real value of the output in the interval $[t_1, t_1 + \Delta t]$. Let the ratio be given by:

$$R_1(\Delta t) = \frac{y(t_1 + \Delta t) - y(t_1)}{y'(t_1)\Delta t} \tag{14}$$

Using (10) and (13) in (14) gives:

$$\begin{aligned} R_1(\Delta t) &= \frac{k(1-e^{-\Delta t/\tau})e^{-(\Delta t_1)/\tau}}{k\frac{1}{\tau}e^{-(\Delta t_1)/\tau}\Delta t} \\ &= \frac{\tau(1-e^{-\Delta t/\tau})}{\Delta t} \end{aligned} \tag{15}$$

Note that $R_1(\Delta t)$ is independent of t_0 and t_1. This mean that we do not need to know the time for the step injected in the system as well as we do not need to know the exact time for the t_1 and t_2. We only need to know the time difference Δt.

Using $\Delta t = \tau$ in (15) gives directly:

$$R_1(\tau) = (1 - e^{-1}) = 0.632 \tag{16}$$

This is the same result (with $k = 1$) as given by (4) when the calculation is based on the whole step response. (15) and (16) shows that we do not need to have the complete step response for calculation of the time constant τ.

A part of the step response in Fig. 3 is shown in Fig. 4. Here, the tangent has been calculated at $t = 1.5sec$. It is shown that the tangent is equal to the steady state value of the system for $t = 1.5sec + \tau = 2.5sec$. Further, we can also calculate $R_1(\tau)$ from Fig. 4. Based on the data in Fig. 4, we get:

$$\begin{aligned} R_1(\tau) &= \frac{y(1.5+\tau)-y(1.5)}{y'(1.5)\tau} \\ &= \frac{0.918-0.777}{0.223} \\ &= 0.632 \end{aligned}$$

where $y'(1.5) = 0.223$.
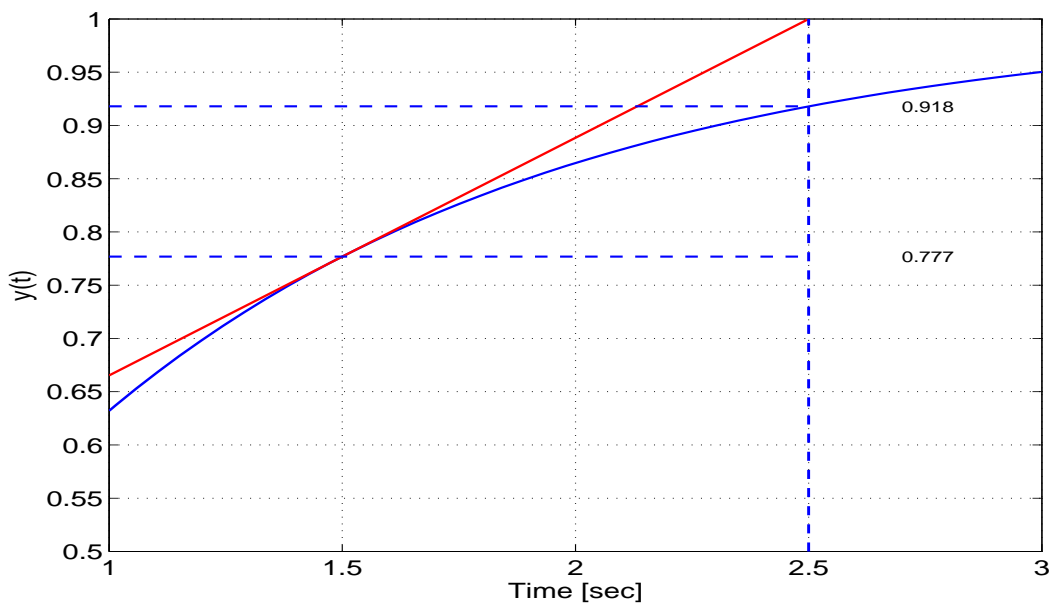
Figure 4. Part of step response for a first order system with $k = \tau = 1$.

In the above results, it is shown that it is easy to find τ where $R_1(\Delta t)$ is equal to $0,632$. However, τ might be larger than $t_2 - t_1$, so it is not possible to get the ratio directly given by (16). Instead, it is possible to calculate the ratio for different values of Δt. Now, let (15) be independent of the time by using
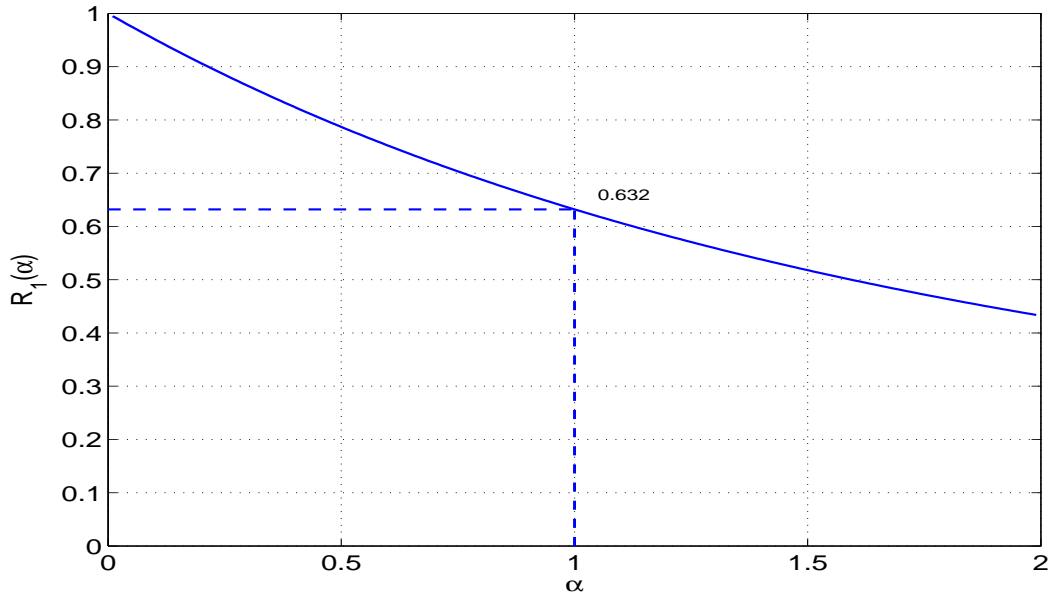
$$\Delta t = \alpha\tau$$

where α is a scaling factor with respect to the time constant τ. Including this in (15) gives directly:

$$R_1(\alpha\tau) = \frac{1 - e^{-\alpha}}{\alpha} \tag{17}$$

Note that the ratio R_1 is independent of τ but depend only on α, i.e. $R_1(\alpha\tau) = R_1(\alpha)$. The ratio $R_1(\alpha)$ as function of α is shown in Fig. 5.

Figure 5. The ratio $R_1(\alpha)$ is shown as function of of α.

Based on Fig. 5, the time constant τ is then given by:

$$\tau = \frac{\Delta t}{\alpha} \tag{18}$$

Again a part of the step response from Fig. 3 is shown in Fig. 6. The same tangent as shown in Fig. 4 has been shown, but the calculation of τ is now based on $0.5sec$. Based on the data from the figure, $R_1(\Delta t)$ in (14) is then given by:

$$
\begin{aligned}
R_1(0.5) &= \frac{y(2.0)-y(1.5)}{y'(1.5)0.5} \\
&= \frac{0.865-0.777}{0.223 \times 0.5} \\
&= 0.789
\end{aligned}
$$

From Fig. 5, we have that $R_1 = 0.789$ gives an α around 0.5. From (18) gives directly that $\tau = 1$.

Until now, we have only used a part of the step response to calculate the steady state value y_{ss} and the time constant τ. We still need to calculate the gain k and the time t_0 when the step is injected in the system. In general, we will know t_0, but we need to be able to calculate the start of the step response in the case when the system includes a time delay. It will then be possible to calculate the time delay as the difference between the time for the start of the step response and t_0. So based on this, assume that we know y_0 and calculate k and t_0 based on this information.

3.2. Estimation of gain k

It has been assumed that the amplitude of the input is 1. This gives directly that the gain k is given by:

$$k = y_{ss} - y_0 \tag{19}$$

where the steady state output y_{ss} is calculated from (12) when the time constant τ is known.
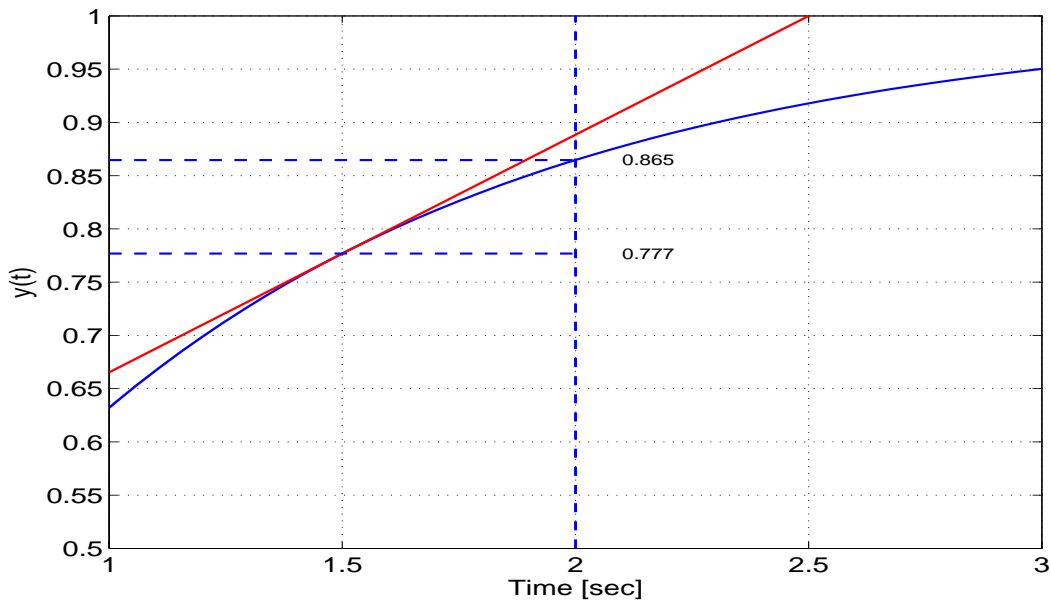
Figure 6. Part of step response for a first order system with $k = \tau = 1$.

If a step is applied with an amplitude different from 1, the term $y_{ss} - y_0$ need to be divided by this amplitude.

3.3. Calculation of the step time t_0

For calculation of t_0, consider the step response at time $t = t_1$. From (8) we have directly:

$$
\begin{aligned}
y(t_1) &= y_0 + k(1 - e^{-(t_1-t_0)/\tau}) \\
&= y_0 + k(1 - e^{-(\Delta t_1)/\tau})
\end{aligned}
\tag{20}
$$

Rewriting (20) gives the following index $R_2(\Delta t_1)$:

$$
R_2(\Delta t_1) = \frac{y(t_1) - y_0}{k} = (1 - e^{-(\Delta t_1)/\tau})
\tag{21}
$$

The index $R_2(\Delta t_1)$ can be calculated based on the output $y(t_1)$, the off-set y_0 and the gain k. The index $R_2(\Delta t_1)$ will always be in the interval from 0 to 1. $R_2(\Delta t_1)$ can now be used for the calculation of Δt_1. Again, let's introduce a scaling factor β:

$$
\Delta t_1 = \beta \tau
\tag{22}
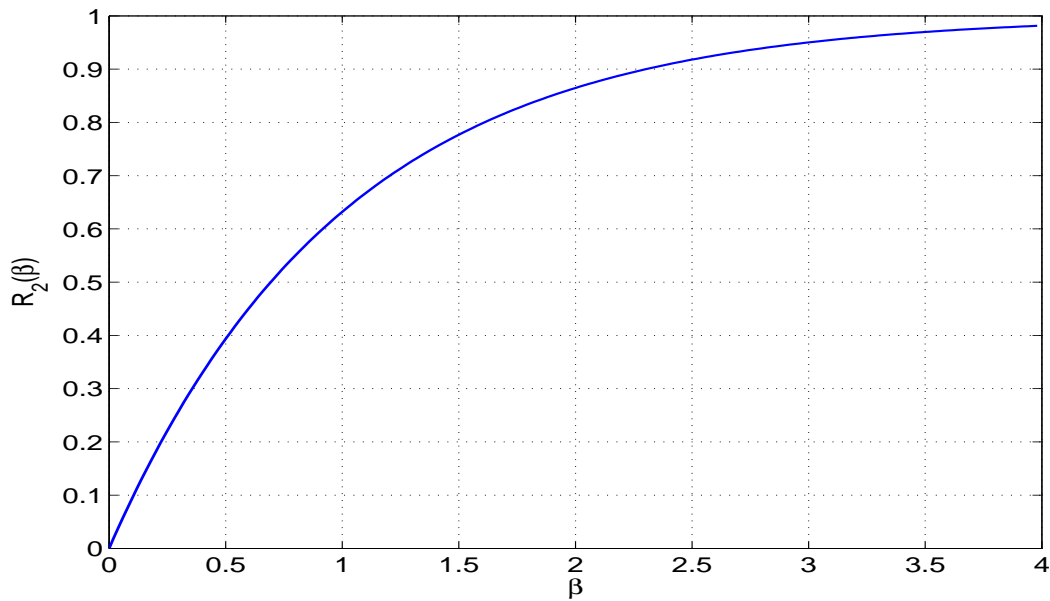$$

This gives the following index $R_2(\beta \tau)$:

$$
R_2(\beta \tau) = (1 - e^{-\beta})
\tag{23}
$$

As in connection with R_1, R_2 is also independent of τ.

The index $R_2(\beta)$ as function of β is shown in Fig. 7.

Based on Fig. 7, β can be found and then Δt_1 can be calculated by using (22). Let t_0 be the time when the response starts. With t_1 known, the time that the t_0 is given by:

$$
t_0 = t_1 - \Delta t_1
\tag{24}
$$

Figure 7. The ratio $R_2(\beta)$ is shown as function of of β.

Consider again the step response shown in Fig. 6. From the data in the figure, we can calculate R_2:

$$R_2 = \frac{y(2) - y_0}{k} = \frac{0.865 - 0}{1} = 0.865$$

From Fig. 7, β can then be found to 2. This gives directly t_0:

$$t_0 = t_1 - \Delta t_1 = t_1 - \beta\tau = 2 - 2 \times 1 = 0$$

3.4. System with time delay

In the case when the system includes a time delay as given by (1), the time delay θ also needs to be calculated. Again, it is assumed that the step is applied in the system at t_0. Let t_θ be the time when the system react. (24) take then the following form:

$$t_\theta = t_1 - \Delta t_1 \tag{25}$$

The system time delay is the difference between t_0 and t_θ, i.e.

$$\theta = t_\theta - t_0 = t_1 - t_0 - \Delta t_1 \tag{26}$$

4. Example

Let's consider again a step response from a refrigation system shown in Fig. 8.

As it can be seen from Fig 8, the system has different dynamics depending of the opening degree of the valve as expected. Further, it is also clear that it is not possible to describe the dynamic only by using first order systems. However, it is in general assumed that the main dynamic in refrigeration systems can be described by using first order dynamics.

In this example, let's consider the step response for an opening degree of 100%. In Fig. 9, a part of the step response from Fig. 8 for $t = 1430$ sec. to $t = 1670$ sec. is shown.
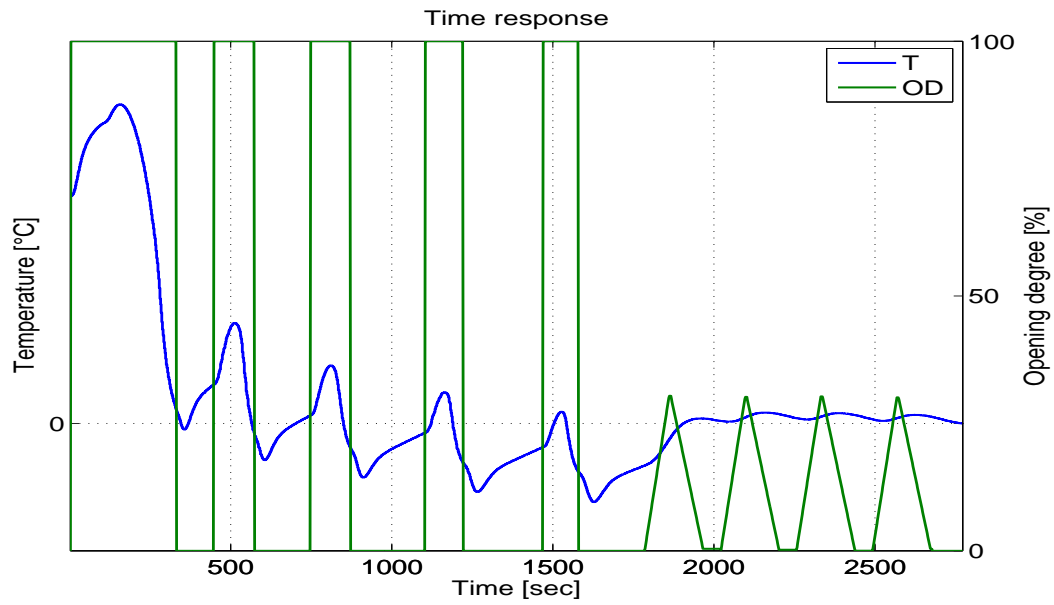
Figure 8. A step response from a refrigeration system. The opening degree is the input to the valve and temperature is the output of the system.
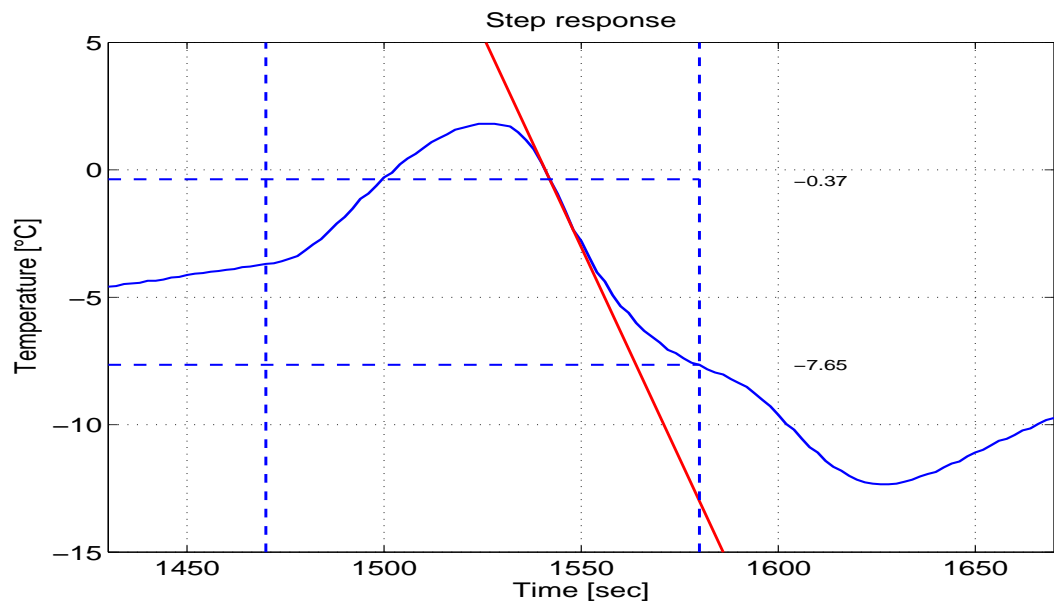
Figure 9. Part of the step response from Fig 8. The two vertical lines indicate the start and stop of the step input. Further, the calculation of the tangent around $t = 1542$ sec.

The step input starts at $t = 1470$ sec. and stops at $t = 1580$ sec. The opening degree is 100% in this time interval. The temperature is maximal at $t = 1524$ sec. with $T = 1.81^oC$. Further, it can also be seen from Fig. 9 that the response fits reasonable with a first order response when the temperature gets below zero degree around $t = 1542$ sec. to around $t = 1590$ sec. with $T = -8.36^oC$.

Calculating a tangent to the step response curve around $t = 1542$ sec. gives the equation for the tangent as:

$$T_{tangent}(t) = -0.32 * t + 493.1$$

The tangent is also shown in Fig. 9.

Based on the tangent in Fig. 9, the ratio $R_1(\Delta t)$ given by (14) can be calculated. From the data given in Fig. 9 together with $y'(t_1) = -0.32$ and $\Delta t = 38$ sec. gives

$$R_1(\Delta t) = \frac{y(t_1 + \Delta t) - y(t_1)}{y'(t_1)\Delta t} = \frac{-7.65 - (-0.37)}{-0.32 \times 38} = 0.599$$

From Fig. 5, α can be calculated. This gives (can also be calculated by using (17)):

$$\alpha = 1.13$$

The time constant is then given by:

$$\tau = \frac{\Delta t}{\alpha} = \frac{38}{1.13} = 33.6 \approx 34 \; sec.$$

The steady state value of the temperature can then be calculated based on the tangent. The tangent crosses the steady state value at $t_1 + \tau = 1542 + 34$ sec. $= 1576$ sec. Using the equation for the tangent gives that the steady state value for the temperature is:

$$T_{ss} = -11.25$$

From Fig. 9, it is clear that the system is not in steady state when the step starts, so we need to decide the steady state value before the step. We select the maximal temperature in the interval as the steady state value, i.e. $T_0 = 1.81^oC$. The reason is that at this point it starts decreasing due to the step input. From this, the gain of the system, k, is given by:

$$k = T_{ss} - T_0 = -11.25 - 1.81 = -13.06 \approx -13$$

The time when the system react t_θ can now be calculated. This is done by using the ratio $R_2(\Delta t)$ given by (21). Using $T(1542) = -0.37^oC$ gives:

$$R_2(\Delta t_1) = \frac{-0.368 - 1.81}{-13} = 0.17$$

Using Fig. 7, β is found to:

$$\beta \approx 0.2$$

which gives that

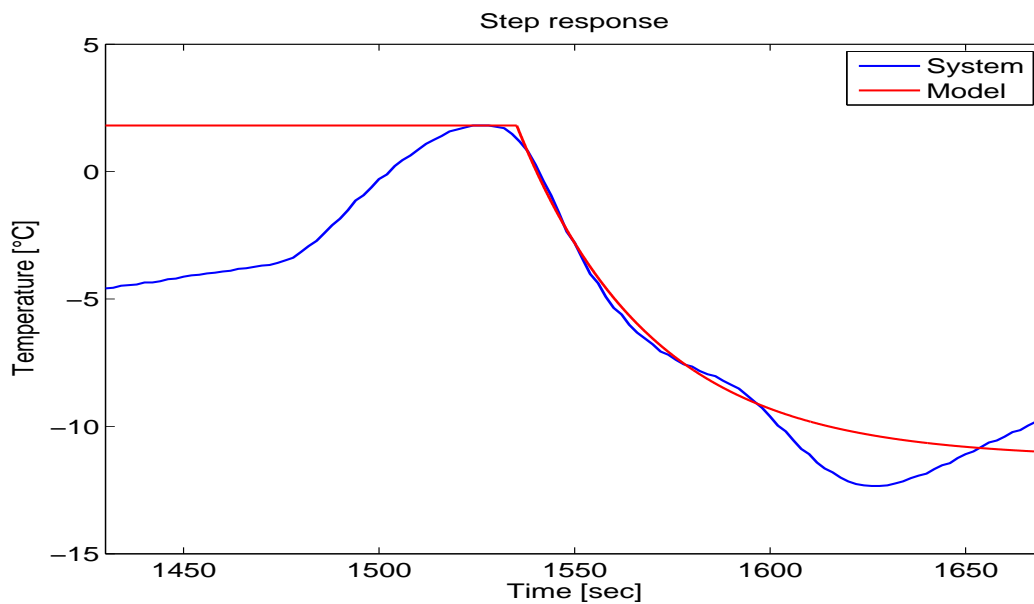$$t_\theta = t_1 - \beta\tau = 1542 - 0.2 \times 34 \approx 1535 sec.$$

The time delay is then given by:

$$\theta = t_\theta - t_0 = 1535 - 1470 = 65 sec.$$

The complete system is then given by

$$G(s) = -13\frac{1}{34s + 1}e^{-65s}$$

The step response from the system and the model is shown in Fig. 10. As expected, the step response from the model does not follow the system exactly, but it will in general be good enough.

Figure 10. Step response from the real system compared with a step response from the model.

5. Conclusion

A simple method for estimation of parameters in a first order system with time delay has been considered. It is shown that it is possible to estimate the parameters based only on a part of a step response. This is very relevant in connection with data from real systems, where it will not always be possible to get a complete step response, but only part of a step response will be available.

References

[1] S. Ahmed, B. Huang, and S.L. Shah. Novel identification method from step response. *Control Engineering practice*, 15:545 – 556, 2007.
[2] K.J. Åström and T. Hagglund. *PID controllers: Theory, Design and Tuning*. Instrument Society of America, 1995.
[3] A. Eisinberg, G. Fedele, and D. Frascino. An analytic optimization procedure to estimate a first-order plus time delay model from step response. In *16th Mediterranean Conference on Control and Automation*, pages 729 – 734, Congress Centre, Ajaccio, France, June 2008.
[4] G.F. Franklin, J.D. Powell, and A. Emami-Naeini. *Feedback control of dynamic systems*. Pearson, 2010.
[5] D.T. Korsane, V. Yadav, and K.H. Raut. Pid tuning rules for first order plus time delay system. *Int. Journal of Innovative Research in Electrical, Instrumentation and Control Engineering*, 2(1):582 – 586, January 2014.
[6] K. Ogata. *Modern systems engineering*. Pearson, 2010.
[7] D.D. Ruscio. On system pi controllers for integrating plus time delay systems. *Modelling, Identification and Control*, 31(4):145 – 164, 2010.
[8] F.A. Salem. New efficient model-based pid design method. *European Scientific Journal*, 9(15):181 – 199, May 2013.
[9] J.B.M. Santos and P.R. Barros. Simple identification techniques for first-order and second-order plus time-delay systems. In *XVIII Congresso Brasileiro de Automatica*, pages 1432 – 1439, Bonito-MS, September 2010.
[10] S. Skogestasd. Probably the best simple pid tuning rules in the world.
[11] S.W. Sung, J. Lee, and I.B. Lee. *Process identification and PID control*. John Wiley & Sons (Asia), 2009.
[12] Cheng-Ching Yu. *Autotuning of PID controllers - A relay feedback approach*. Springer Verlag, London, 2006.