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Strongly 2-connected orientations of graphs

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Abstract

We prove that a graph admits a strongly 2-connected orientation if and only if it is 4-edge-connected, and every vertex-deleted subgraph is 2-edge-connected. In particular, every 4-connected graph has such an orientation while no cubic 3-connected graph has such an orientation.

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1 Introduction

Robbins [13] observed that every finite 2-edge-connected graph has a strongly connected orientation. Nash-Williams [12] proved the much stronger result that every $2k$-edge-connected graph has a strongly k-edge-connected orientation, and Mader [10] also obtained this result from his general lifting theorem. This motivated the following conjecture in [14], also discussed in [1] :

Conjecture 1 *For every positive integer k, there exists a smallest positive integer $f(k)$ such that every $f(k)$-connected graph has a strongly k-connected orientation.*

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The complete graphs show that $f(k) \geq 2k$. Robbins' theorem implies that $f(1) = 2$. Frank [6] made the following stronger conjecture.

Conjecture 2 *A graph has a strongly k-connected orientation if and only if the deletion of any j vertices, $0 \leq j < k$, leaves a $2(k-j)$-edge-connected graph.*

Berg and Jordán [2] verified Frank's conjecture for $k = 2$ when restricted to Eulerian graphs. The special case of 4-regular, 4-edge-connected graphs was also done by Gerards [7]. Jordán [8] combined the result of [2] with a result on the 2-dimensional rigidity matroid of a graph to prove that $f(2) \leq 18$. Cheriyan, Durand de Gevigney, and Szigeti [4] improved this to $f(2) \leq 14$. A referee points out that Frank's conjecture is false for each $k \geq 3$, and that, for each $k \geq 3$, it is NP-complete to decide if a graph has a strongly k-connected orientation, as proved recently by Durand de Gevigney [5].

The "only if" part in Frank's conjecture is trivial. In the present paper we verify the "if" part for $k = 2$ by establishing an appropriate technical extension. In particular, $f(2) = 4$. It is not known if $f(3)$ exists.

2 Orientations

The notation and terminology are essentially the same as in [3] and [11]. We repeat some of the most important concepts.

The graphs in this paper are not allowed to contain loops, but they may have multiple edges. If G is a graph and A is a set of vertices in G, then $G(A)$ is the subgraph of G induced by A, that is, $G(A)$ has vertex set A and contains all those edges of G which join two vertices in A. If A is a set of vertices in G, then the edges with precisely one end in A is called an *edge-cut*. An edge-cut with k edges is called a *k-edge-cut*. We call A and its complement the *sides* of the cut. If some edges of the graph have been given an orientation, and all edges leaving A have the same direction, then the cut is a *directed cut*. A *directed graph* is a graph where every edge has been given an orientation. If an edge $e = xy$ is directed from x to y, then x is the *tail* of e, and y is the *head* of e.

A directed graph is *strongly connected*, or just *strong*, if, for any ordered pair of vertices x, y, the graph has a directed path from x to y. A *strongly* *connected component*, or just *strong component*, is a maximal strong subgraph.

A graph (respectively directed graph) is k*-connected* (respectively *strongly* k*-connected*) if it has at least $k+1$ vertices and the deletion of fewer than k vertices always leaves a connected graph (respectively strongly connected directed graph).

We now review the ideas in the proof. We allow a vertex y_0 to have degree 3. This means that the graph is not 4-edge-connected (as required in Frank's conjecture) but we require it to be almost 4-edge-connected in that there is only one 3-edge-cut. In the conclusion of the theorem we shall require that any graph $G-v$ is strongly connected or almost strongly connected in that it may have two strongly connected components, one of which is y_0. The idea in the proof is simple: If y_0 has degree 3, we delete an edge yy_0, suppress y_0, that is, we replace the remaining two edges incident with y_0 by a single edge, and then we use induction where y now plays the role of y_0. When we have applied induction, there may be a multiple edge between x, z, say, such that all edges have the same direction. If that happens we reverse an orientation of one of those edges so that not all edges between x, z have the same direction. The resulting directed graph will still satisfy the conclusions of the theorem.

The proof has some resemblence with the proof of the weak 3-flow conjecture [15]. However, there is an important difference in the two results and their proofs: In [15] we may pre-orient all edges incident with a vertex of small degree. This is convenient when dealing with small edge-cuts, and such a pre-orientation might also be helpful in the present proof. However, this is not possible, even in the 4-regular case. For, if we pre-orient some edges, we must make sure that no vertex-deleted subgraph contains a directed edge-cut. And even if this additional condition is added, there are still counterexamples. So it is perhaps of independent interest to decide to which extent it is possible to pre-orient a few edges in the theorem below.

Theorem 1 *Let G be a graph with at least 3 vertices, and let y_0 be a vertex in G. Assume that G, y_0 satisfy the following assumptions.*

(a_1): G is 2-connected.

(a_2): G is 3-edge-connected.

(a_3): Either G is 4-edge-connected, or else y_0 has degree 3, and the only 3-edge-cut of G consists of the three edges incident with y_0.

(a_4): *For each vertex v of G, $G - v$ has at most one bridge. Moreover, if $G - v$ has a bridge e, then $G - v - e$ has precisely two components, one of which is y_0. (In other words, if $G - v$ has a bridge e, then y_0 has precisely two neighbors, namely v and and another neighbor which is joined to y_0 by precisely one edge, namely e.)*

Then there is an orientation D of G satisfying the following conclusions:
(c_1): *D is strong.*
(c_2): *For each vertex v in G, either $D-v$ is strong, or $D-v$ has precisely two strong components, one of which is y_0. In this case y_0 has either precisely two neighbors, one of which is joined to y_0 by a single edge, or else y_0 has precisely three neighbors each joined to y_0 by a single edge.*

Before we prove Theorem 1 we note that each 4-edge-connected graph G with the property that each vertex-deleted subgraph $G - v$ is 2-edge-connected also satisfies the assumptions in Theorem 1 with any vertex playing the role of y_0. Thus Theorem 1 proves the case $k = 2$ in Frank's conjecture and also proves that $f(2) = 4$.

Proof of Theorem 1: The proof is by induction on the number of vertices of G. The theorem is easily verified for graphs with 3 vertices: As G is 2-connected, there is an edge between any two vertices. As G is 3-edge-connected, there is at most one edge which is not part of a multiple edge. Now we orient the edges such that every edge (except possibly one) is part of a directed 2-cycle. So assume that G has at least 4 vertices.

We now assume that the theorem is false and let G be a counterexample with as few vertices as possible and (subject to this) with as few edges as possible.

We may assume that

(1): No two vertices of G are joined by 3 *or more edges.*

Proof of (1): Suppose (reductio ad absurdum) that x and y are joined by 3 or more edges, then we delete one, say e. Let $G' = G - e$. We claim that G' satisfies the assumption of the theorem and has the desired orientation, by the minimality of G. Clearly, G' is 2-connected. Also, $G' - v$ has the same bridges as $G - v$. Therefore, it suffices to consider the case where G' has a 3-edge-cut with sides A, B say, consisting of the edges e_1, e_2, e_3 which is not a 3-edge-cut in G. Then we can choose the notation such that e, e_1, e_2 join the same two vertices x, y where $x \in A$ and $y \in B$. Let x' be the end of e_3 in A, and let y' be the end of e_3 in B. As G is 2-connected, we may assume that e_3 is not incident with x, that is, $x' \neq x$. Also, $G(A)$ is connected. Then e_3 is a bridge of $G - x$. The assumption of the theorem now implies that y_0 is a component of $G - x - e_3$. Then either $y_0 = x'$ and A consists of x, x', or else $y_0 = y' = y$. The former cannot hold because otherwise, $y \neq y'$ (as G is 2-connected and has more than 3 vertices), and $G - y$ would have a bridge e_3, but y_0 would not be a component of $G - y - e_3$. Hence $y_0 = y' = y$. We have proved that any 3-edge-cut of G' consists of the edges incident with y_0. Hence G' satisfies the assumption of the theorem. This contradiction to the minimality of G shows that (1) holds.

(2): y_0 has at least three neighbors.

Proof of (2): As G is 2-connected, every vertex has at least two neighbors. Now suppose (reductio ad absurdum) that y_0 has only two neighbors x, y. By (1), and since G is 3-edge-connected, y_0 has degree 3 or 4. Choose the notation such that y_0 is joined to x by two edges. Then contract the edges between y_0 and x. The contracted vertex is called v', and the resulting graph is called G'. We claim that G' satisfies the assumption of the theorem and has the desired orientation, by the minimality of G.

As contraction of edges does not create new edge-cuts, it suffices to discuss the case where $G' - v' = G - y_0 - x$ has at least one bridge e. Then e is also a bridge in $G - x$. But this bridge is not incident with y_0, a contradiction. Also, if $y_0 y$ is a single edge, then x has degree at least 5, since otherwise, there would be a 3-edge-cut whose deletion leaves a graph with a component with vertices x, y_0. Hence G' satisfies the assumption of the theorem and has the desired orientation, by the minimality of G. We give the edges between y_0, x opposite directions, and the resulting orientation of G satisfies the conclusions of the theorem, a contradiction which proves (2).

It follows from Claim (2) and (a_4) that, for each vertex v of G, $G - v$ is 2-edge-connected. If G is 4-edge-connected, then every vertex can play the role of y_0. If G is not 4-edge-connected, then y_0 has degree 3, and y_0 has three distinct neighbors, by Claim (2).

(3): G is 3-connected.

Proof of (3): Suppose (reductio ad absurdum) that x, y are vertices such that $G - x - y$ is disconnected. Then G is the union of two graphs G_1, G_2 having only two vertices x, y and the edges joining x, y (if any) in common. Since $G - x$ is 2-edge-connected, there are at least two edges from y to each component of $G-x-y$. Similarly, there are at least two edges from x to each component of $G-x-y$. Using again the fact that $G-x$ is 2-edge-connected, we conclude that G_i has two edge-disjoint paths from two neighbors of x to two neighbors of y. (These neighbors need not be distinct.) If y_0 is one of x, y, then y_0 has degree at least 4 which implies that G is 4-edge-connected. But then every vertex can play the role of y_0. So we may assume that y_0 is in $G_2 - x - y$. By Claim (2), $G_2 - x - y$ has a vertex distinct from y_0.

Let G_i' denote G_i with two edges added between x, y. It is easy to see that G_i' satisfies the assumption of the theorem for $i = 1, 2$. (One may think of G_i' as a contraction of G.) It is possible that G_1' has only 3 vertices.

So, we apply induction to G_i' for $i = 1, 2$. G_1' is 4-edge-connected, so any vertex can play the role of y_0. Let D_i be the resulting directed graph for $i = 1, 2$. Then $D_1 \cup D_2$ satisfies the conclusion of the theorem. If G contains none or precisely one edge between x and y, then we delete both or one of the directed edges between x and y from $D_1 \cup D_2$, and the resulting orientation of G satisfies the conclusions of the theorem. To prove this it suffices to show that each of D_1, D_2 has directed paths of length at least 2 from x to y and from y to x. As $D_i - x$ is strong, there is a directed edge from y to $D_i - x - y$ and a directed edge to y from $D_i - x - y$ for $i = 1, 2$. Similarly for x. Using these edges and their ends combined with the fact that $D_i - x$ and $D_i - y$ are strong, it is easy to see that D_1 has directed paths of length at least 2 from x to y and from y to x. We now prove that D_2 has directed paths of length at least 2 from x to y. We let xz be a directed edge from x to $D_2 - x - y$. Now $D_2 - x$ has a directed path from z to y unless $z = y_0$ and y_0 is a strong component in $D_2 - x$. Since D_2 is strong, both edges incident with y_0 in $D_2 - x$ must leave y_0. As $D_2 - x$ has only two strong components, $D_2 - x$ has a directed path from z to y. This completes the proof of (3).

(4): If E is a 4-edge-cut with at least two vertices on each side, then one of the sides has precisely two vertices, and each of these two vertices has degree 4, and they are joined by two edges.

Proof of (4):

Assume that $V(G) = A \cup B$ such that E consists of the edges between A and B.

As G is 3-edge-connected and has at most one 3-edge-cut, it follows that each of $G(A), G(B)$ is 2-edge-connected. If one of A, B, say A, has precisely two vertices v_1, v_2, then v_1, v_2 are joined by a double edge because $G(A)$ is 2-edge-connected. By Claim (2), each of v_1, v_2 is incident with at least (and hence precisely) two edges in E. So Claim (4) holds if A has precisely two vertices.

Suppose (reductio ad absurdum) that $|A| \geq 3$, and $|B| \geq 3$. Put $E = \{e_1, e_2, e_3, e_4\}$.

No three edges of E can be incident with the same vertex because G is 3-connected.

We let G_A denote a graph obtained by adding two vertices a_1, a_2 to $G(A)$. We let two of the edges in E go to a_1 and the other two to a_2. We also add two edges between a_1, a_2. We let G_B be obtained in a similar way from $G(B)$ where the two new vertices are denoted b_1, b_2. Since each of $G(A), G(B)$ is 2-edge-connected, it is easy to see that each of G_A, G_B satisfies the induction hypothesis. We therefore apply induction to each of G_A, G_B. If all ends of the edges in E are distinct, then G_A can be chosen in three distinct ways. Similarly for G_B. There are three different ways of orienting the four edges of E (up to reversal of edges). As two of them can be realized in G_A and two of them can be realized in G_B, it is possible to apply induction to G_A, G_B in such a way that the orientations of E agree in both G_A and G_B. (If two of the edges in E, say e_1, e_2 are incident with the same vertex x in A, then G_A can be chosen in only two ways. However, if e_1, e_2 go from x to the same vertex a_1, then e_1, e_2 get distinct orientations, say e_1 from A to B and e_2 from B to A. But we may reverse these orientations such that e_1 goes from B to A and e_2 from A to B. So again, we can in G_A realize two of the three possible orientations of E.)

This implies that the two orientations can be combined to give an orientation of G. By deleting one of b_1, b_2 in G_B, we conclude that, for each head h in B of an edge in E and for each tail t in B of an edge in E, $G(B)$ has a directed path from h to t. A similar statement holds for $G(A)$. Using this observation, it follows that G satisfies the conclusion of the theorem.

This contradiction proves (4).

(5): If E is a 5-edge-cut containing a double edge joining two vertices x, y of degree 4, then one of the two sides of the cut E consists of y_0 and one of x, y. Moreover, y_0 has degree 3 and is incident with precisely two edges of E.

Proof of (5): As x, y are in distinct sides of E, each side has at least two vertices. If one of x, y, say x, is incident with three edges of E, then the remaining two edges of E together with the fourth edge incident with x form a 3-edge-cut, and that 3-edge-cut must consist of the edges incident with y_0. In that case one of the two sides of E consists of x, y_0.

Suppose now (reductio ad absurdum) that each of x, y is incident with only two edges in E. Let $V(G) = A \cup B$ such that E consists of the edges between A and B. Then $|A| \geq 3, |B| \geq 3$ because G is 3-connected. We now repeat the proof of (4). When we form G_A, then the two edges joining x, y will be incident with a_1. A third edge of E will also be incident with a_1. The last two edges will be incident with a_2. When we apply induction, these two edges will have opposite direction. As G_A can be constructed in three ways, we can realize two of the three possible orientations of the three edges in E not incident with x, y. As the same is possible for G_B, the two orientations can be combined to give an orientation of G. That orientation satisfies the conclusion of the theorem. To see this we delete one of a_1, a_2 to conclude that, for each head h in A of an edge in E and for each tail t in A of an edge in E, $G(A)$ has a directed path from h to t. A similar statement holds for $G(B)$.

This contradiction proves (5).

(6): If v is a vertex of G, and E is a 2-edge-cut of $G - v$, then one of the sides has precisely one vertex, and that vertex has degree 3 or 4 in G.

Proof of (6): Assume that $V(G) \setminus \{v\} = A \cup B$ such that E consists of the edges between A and B. Assume (reductio ad absurdum) that $|A| \geq 2, |B| \geq 2$. As $G - v$ has at most one bridge, it follows that $G(A)$ and $G(B)$ are connected. As G has at most one 3-edge-cut, namely the edges incident with y_0, it follows that G has at least two edges between v and A and at least two edges between v and B.

Now we contract B into a single vertex b and call the resulting graph G_b. We claim that G_b satisfies the assumption of the theorem. To verify this, we only need to prove that $G_b - b = G - V(B)$ has at most one bridge e, and (if this bridge e exists) $G_b - b - e$ has a component consisting of y_0 only. So assume that e is a bridge in $G_b - b = G - V(B)$ and that the two sides of the bridge are A', B' where A' contains v. Since G is 3-edge-connected, both edges of E join B with B'. Then $E \cup \{e\}$ is a 3-edge-cut in G and hence $B' = \{y_0\}$.

Now we apply induction to G_b. Then the edges in E get opposite direction because the orientation of $G_b - v$ is strong. Then we contract A into a vertex a and use induction.

By combining the two orientations, we obtain an orientation of G. The orientations between v and A are those in G_b. (The orientations between v and A in G_a are not important.)

We claim that this orientation satisfies the conclusion of the theorem, and obtain thereby a contradiction which proves (6). To prove this claim we first observe that $G_a - a$ and $G_b - b$ are strong. We next observe that $G_a - v$ has a directed cycle containing a, and $G_b - v$ has a directed cycle containing b because none of a, b is y_0. Using these observations it is now easy to verify that the orientation of G satisfies the conclusion of the theorem, a contradiction.

(7): Let x, y be two vertices of degree 4 joined by a double edge, and let G' denote the graph obtained from G minus this double edge by suppressing the two vertices x, y of degree 2. If y_0 has degree 3 we assume that y_0, y are neighbors. Then, for each vertex v in G', $G' - v$ has at most one bridge e. If e exists, then $G' - v - e$ has only two components, one of which is y_0. (In this case y_0 has only two neighbors in G' and one of these is v.)

Proof of (7): Suppose (reductio ad absurdum) that G' has a vertex v such that $G' - v$ has a bridge which does not satisfy the conclusion of (7).

Let $G_1, G_2, \ldots, G_k$ be the connected components of $G' - v$ minus its bridges. As $G - v$ has no bridge (because y_0 has at least three neighbors) the notation can be chosen such that $G' - v$ has a bridge e_i joining G_i and G_{i+1} for $i = 1, 2, \ldots, k-1$. We may also assume that the edge of G' corresponding to x is either in G_1 or joins v to G_1, and the edge of G' corresponding to y is either in G_k or joins v to G_k. By (6), each of $G_2, G_3, \ldots, G_{k-1}$ is a single vertex of degree 4 joined to v by a double edge. By (4), G' has at least two edges joining v to G_1 and at least two edges joining v to G_k. (For, if G' has only one edge e_0 from v to G_1, then the edges e_0, e_1 together with the two edges joining x, y, form a 4-edge-cut in G. As G is 3-connected, each of x, y has at least three neighbors, and hence the sides of the 4-edge-cut have a least three vertices each, a contradiction to (4).)

We contract (in G) $V(G_2) \cup V(G_3) \cup \ldots \cup V(G_k) \cup \{y\}$ into a single vertex y'. We call the resulting graph G'_1. We also contract $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{k-1}) \cup \{x\}$ into a single vertex x'. We call the resulting graph G'_k. We apply induction to each of G'_1, G'_k. This is possible because each of G'_1, G'_k is a contraction of G. In one of the graphs we may reverse e_1 or e_{k-1}. Therefore we may assume that e_1 is directed away from G_1 and that e_{k-1} is directed towards G_k. We orient the other bridges of $G' - v$ from G_i to G_{i+1}, and we let the two edges between v and G_i have opposite directions, for $i = 2, 3, \ldots, k-1$. (Note that none of $G_2, G_3, \ldots, G_{k-1}$ is y_0 because then $G - v$ would not satisfy (a_4) .) We claim that the resulting orientation of G satisfies the conclusions of the theorem, and we obtain thereby a contradiction.

To prove this claim, we first note that we may assume that none of v, x, y, x', y' is y_0 because y_0 has degree 3 or can otherwise be chosen at random. Let us denote the head of e_{k-1} by y''. If y_0 has degree 3, then it belongs to G'_k because y_0 is a neighbor of y. This implies that $G'_1 - v$ and $G'_1 - y'$ are strong. If also $G'_k - v$ and $G'_k - x'$ are strong, then it is easy to see that the orientation of G satisfies the conclusions of the theorem. In fact it is sufficient that v, y, y'' are in the same strong component of $G'_k - x'$ and that x', y, y'' are in the same strong component of $G'_k - v$. This holds unless $y'' = y_0$ and hence, y_0 has precisely three neighbors. So, let us assume that $y'' = y_0$ and y_0 has precisely three neighbors in G. Assume also that either $G'_k - x'$ or $G'_k - v$ has a strong component consisting of y_0 only. In the former case $y_0 = y''$ has outdegree 2, and hence $G'_k - x'$ has a directed path from y_0 to either of v, y which is sufficient to prove that G satisfies the conclusion of Theorem 1. So assume that $G'_k - v$ has a strong component consisting of y_0 only. Consider first the subcase where y is on an edge in G'_k. Then there is a directed edge from y_0 to v, and the edges e_{k-1}, yy_0 are directed towards y_0. But then $G'_k - v$ has a strong component which consists of y, x' only, a contradiction. So we are left with the subcase where G'_k has an edge y_0v, and y is inserted on that edge when we put y back. If G_k has more than one vertex, then G_k has precisely one edge e' incident with y_0. That edge e' is a bridge in $G - v$, and $G - v - e'$ has a component containing both of y_0, y, a contradiction. So, the only problem that remains in order to prove (7) is that G_k consists of y_0 only, G has an edge y_0v and also a path y_0yv. Now we choose the orientation of G'_k such that y_0yvy_0 is a directed cycle. Then each vertex-deleted subgraph of G is strong, except $G - y$.

This contradiction proves (7).

(8): y_0 has degree at least 4.

Proof of (8): Suppose (reductio ad absurdum) that y_0 has degree less than 4. By (2), y_0 has at least three neighbors. So, y_0 has precisely three neighbors, and each of these is joined to y_0 by a single edge. Let $e = y_0y$ be an edge incident with y_0. If possible, we choose y to have degree at least 5. If all neighbors of y_0 have degree 4, then we let y be a neighbor of smallest degree in the graph induced by the three neighbors of y. There are at most two edges in the neighborhood of y_0 since otherwise there would be a 3-edge-cut distinct from the edge set incident with y_0. Hence either y has degree 0 in the neighborhood or else it has degree 1 in the neighborhood and the two other neighbors are joined by an edge.

We now delete the edge e, and we suppress the vertex y_0 in $G - e$, that is, we replace the two remaining edges incident with y_0 by a single edge. The resulting graph is called G'. If G' satisfies the conditions of the theorem with y playing the role of y_0, then we apply induction to G'. We orient e such that y has indegree at least 2 and outdegree at least 2. We claim that the resulting orientation of G can be chosen such that it satisfies the conclusion of Theorem 1. This will be a contradiction which implies that G' does not satisfy all assumptions of Theorem 1. To prove the claim, we first observe that G is strong. Now let z be any vertex of G. If $z = y$, then $G - z$ is strong because $G' - z$ is strong. If z is distinct from y_0 and distinct from the neighbors of y_0, then $G' - z$ is strong or has precisely two strong components, one of which is y. Say y has outdegree 0 and indegree 2 in $G' - z$. But in $G - z$ we can walk (along a directed edge) from y to y_0 and then (along a directed path) to an in-neighbor of y showing that $G - z$ is strong. Assume next that z is a neighbor of y_0 distinct from y. We may also assume that z is a neighbor of y since otherwise $G' - z$ and hence also $G - z$ (or $G - z - y_0$) are strong. By the particular choice of y, there is an edge between z and the third neighbor w of y_0 (and no edge between y and w). This means that in G' there is a double edge between z and w. We may assume that these two edges have distinct orientations in G'. Suppose now that y has outdegree 0 in $G' - z$. Then in G the edges yz, yy_0 are the outgoing edges from x. Now we chose the orientation in G such that zy_0wz is a directed triangle which shows that $G - z$ is strong. Now it is easy to see that $G - w$ has precisely two strong components one of which is y_0, and from this it also follows that $G - y_0$ is strong. This contradiction shows that G' does not satisfy all assumptions of Theorem 1.

By (3), G' is 2-connected. Since G satisfies (a_3), G' is 3-edge-connected. By (6), G' satisfies (a_4). So we may assume that G' does not satisfy (a_3). Then (4) implies that y has degree 4 and is joined to a vertex x of degree 4 by a double edge. Each of the other two neighbors of y_0 has degree 4 in G, and, if those two neighbors are not joined by an edge, then each of them is joined to another vertex of degree 4 by a double edge.

Let G'' denote the graph obtained from G minus this double edge by suppressing the two vertices x, y of degree 2

By (5) and (7), G'' satisfies the conditions of the theorem. We apply the induction hypothesis to G''. We give the two edges between x, y opposite directions. We may assume that the resulting orientation of G does not satisfy the conclusion of Theorem 1. The only two possible ways in which this can happen are:

(i): x, y have a common out-neighbor (or common in-neighbor) z, in which case x, y are the vertices of a strong component of $G' - z$, or

(ii) G' has a vertex z distinct from y_0 such that y_0 is a strong component of $G' - z$, and x is on an edge in G' incident with z such that, again, x, y are the vertices of a strong component of $G' - z$.

Consider first (i). Then z has degree at least 5 in G because of (4). In particular, z is not a neighbor of y_0. We let H denote the graph obtained from $G - y$ by suppressing the two vertices of degree 2 (namely y_0, x). If H satisfies the conditions of the theorem, then we apply induction to H. We then direct the deleted edges such that $yzxy$ becomes a directed 3-cycle, and the resulting orientation of G satisfies the conclusions of the theorem, a contradiction. We may therefore assume that H does not satisfy the assumptions of the theorem. Hence H contains a vertex v such that $H - v$ has a bridge. So, we can divide $V(G) \setminus \{v, y\}$ into disjoint sets A, B such that G has precisely one edge between A and B. Then y has two edges going to A and two edges going to B because of (6). We may assume that x is in A and that y_0, z are in B and that the edge between A, B is xz. Then the fourth edge incident with x is a bridge in $G - v$, a contradiction.

Consider finally (ii). Since y_0 is a strong component of $G' - z$, it follows that z is a neighbor of y_0. In particular, z has degree 4 in G. If z has smallest degree in the graph induced by the three neighbors of y_0, then z is joined to a vertex of degree 4 by a double edge, and then the edges leaving the vertex set consisting of x, y, y_0, z form a 5-edge-cut contradicting (5). So z cannot replace y in the argument above. Since y is chosen to be a vertex of smallest degree in the graph induced by the three neighbors of y_0 (and that graph has at most two edges), it follows that z is also joined to the third neighbor y' of y_0.

Now we delete the edge zy_0, suppress y_0 and call the resulting graph

G'''. We apply induction to G''' with z playing the role of y_0. When we put the edge zy_0 back we orient it so that z gets indegree and outdegree 2. Then G satisfies the conclusion of Theorem 1, a contradiction. (The only problem that might occur is that $G''' - y'$ has two strong components, one of which is z such that z has only one outneighbor, namely y_0, and y_0 has no outneighbor in $G - y'$. But this cannot happen because otherwise y, x would be the vertices of a strong component in $G''' - y'$.)

This completes the proof of (8).

(9): Each vertex in G has degree precisely 4.

Proof of (9): Suppose reductio ad absurdum that a vertex z has degree at least 5. As y_0 has degree at least 4 we can let any vertex play the role of y_0. In particular, we may assume that $z = y_0$. Now we repeat the argument in the proof of (8) except that we do not suppress y_0 when we delete the edge $e = y_0y$. (The present proof is easier as we need not select y such that it has smallest degree in the graph induced by the neighbors of y_0. Also, the case (ii) in the proof of (8) cannot occur.) This proves (9).

Gerards [7] verified Frank's conjecture in the special case of 4-regular, 4-edge-connected graphs. We may use that result to complete the proof of Theorem 1. However, we wish to keep the proof self-contained.

(10): G has a vertex which is not incident with a double edge.

Proof of (10): If each vertex of G is incident with a double edge, then the double edges form a perfect matching. Now we orient the edges of G such that each double edge is a directed cycle and such that each vertex has indegree and outdegree 2. Then G is strong, and so is every vertex-deleted subgraph. This contradiction proves (10).

Let z be any vertex of G not incident with a double edge. Then we perform a lifting at z, that is, we delete z and add two edges between its neighbors such that the resulting graph G' is 4-edge-connected. This is possible by a lifting result of Lovász [9], which was generalized by Mader's lifting theorem [10]. However, as G is eulerian and z has degree 4, it is an easy exercise to prove that, if one of the three possible liftings at z creates a 2-edge-cut, then the other two liftings each results in a 4-edge-connected graph, see e.e. the proof of Theorem 2 in [16]. Since G' is 2-connected and 4-regular, every graph of the form $G' - v$ is 2-edge-connected. So we can apply induction to

G'. The resulting orientation of G' can be modified to an orientation of G satisfying the conclusions of the theorem, unless $G - z$ has a directed edge-cut. As every edge-cut in G' is balanced, the directed edge-cut in $G - z$ has precisely two edges. Hence $V(G - z)$ is the union of two disjoint sets A, B such that G has precisely two edges between A, B. Adding z to either A or B results in two 4-edge-cuts. As G has no double edges, this contradicts (4). So $G - z$ is strongly connected.

This contradiction finally proves the theorem.

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