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TETRAVALENT ONE-REGULAR GRAPHS OF ORDER 4p²

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Abstract. A graph is one-regular if its automorphism group acts regularly on the set of its arcs. In this paper tetravalent one-regular graphs of order $4p^2$, where *p* is a prime, are classified.

1. Introduction

A graph is *arc-transitive* if its automorphism group acts transitively on the set of its arcs. A graph is *one-regular* if its automorphism group acts regularly on the set of its arcs. Not surprisingly arc-transitive graphs - and one-regular graphs in particular - have received considerable attention over the years, the aim being to obtain structural results and possibly a classification of such graphs of particular orders or satisfying certain additional properties. Research in one-regular graphs is interesting for two reasons, the first being their connection to regular maps, a lively area of research. Namely, the underlying graphs of chiral maps admit one-regular group actions with a cyclic vertex stabilizers (see, for example, [8, 10–12]). Second, one may argue that one-regular graphs are interesting in their own right if one's goal is a description of arc-transitive graphs. For some classes of Cayley graphs, for example, circulants, this has been achieved, whereas for others, such as Cayley graphs of dihedral groups, all 2-arc-transitive graphs have been completely classified [16], but arc-transitivity remains an open problem.

Clearly, a one-regular graph with no isolated vertices is connected, and it is of valency 2 if and only if it is a cycle. The first example of a cubic one-regular graph was constructed by Frucht [21]. Further research in cubic one-regular graphs has been part of a more general project dealing with the investigation of cubic arc-transitive graphs (see [9, 15, 17–20, 31]). Tetravalent one-regular graphs have also received considerable attention. In [4] tetravalent one-regular graphs of prime order were constructed, and in [30] an infinite family of tetravalent one-regular Cayley graphs on alternating groups is given. Tetravalent one-regular circulant graphs were classified in [41], and tetravalent one-regular Cayley graphs on abelian groups were classified in [40]. Next, one may extract a classification of tetravalent one-regular Cayley graphs on dihedral

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groups from [26, 36, 38]. Let *p* and *q* be primes. Clearly every tetravalent one-regular graph of order *p* is a circulant graph. Also, by [7, 32, 34, 37, 40, 41], every tetravalent one-regular graph of order pq or p^2 is a circulant graph. Furthermore, the classification of tetravalent one-regular graphs of order 2pq is given in [43]. The aim of this paper is to classify tetravalent one-regular graphs of order $4p^2$, see Theorem 5.1. (For more results on tetravalent arc-transitive graphs, see [22, 23, 27, 33].)

In the next section we gather various concepts that are needed in the analysis of tetravalent one-regular graphs in Section 4 and in the proof of our main result in Section 5. In Section 3, we give examples of tetravalent one-regular graphs of order $4p^2$, where *p* is a prime.

2. Preliminaries

For a finite, simple and undirected graph *X*, we use *V*(*X*), *E*(*X*), *A*(*X*) and Aut(*X*) to denote its vertex set, its edge set, its arc-set and its full automorphism group, respectively. For $u, v \in V(X)$, denote by uv the edge incident to u and v in *X*. By C_n and K_n we denote the cycle of length n and the complete graph of order n, respectively.

A subgroup $G \le Aut(X)$ is said to be *vertex-transitive, edge-transitive* and *arc-transitive* provided it acts transitively on the sets of vertices, edges and arcs of X, respectively. The graph X is said to be *vertex-transitive, edge-transitive,* and *arc-transitive* if its automorphism group is vertex-transitive, edge-transitive and arc-transitive graph is also called a *symmetric* graph. An arc-transitive graph X is said to be *one-regular* if Aut(X) acts regularly on A(X). A subgroup $G \le Aut(X)$ is said to be *k-arc-transitive* if it acts transitively on the set of *k*-arcs, and it is said to be *k*-regular if it is *k*-arc-transitive and the stabilizer of a *k*-arc in *G* is trivial.

For a finite group *G* and a subset *S* of *G* such that $1 \notin S$ and $S = S^{-1}$, the *Cayley graph* Cay(*G*, *S*) on *G* with respect to *S* is defined to have vertex set *G* and edge set {{*g*,*sg*} | *g* \in *G*, *s* \in *S*}. Given *g* \in *G*, define the permutation *R*(*g*) on *G* by $x \mapsto xg$, $x \in G$. The permutation group *R*(*G*) = {*R*(*g*) | *g* \in *G*} on *G* is called the *right regular representation* of *G*. It is easy to see that *R*(*G*) is isomorphic to *G*, and it is a regular subgroup of the automorphism group Aut(Cay(*G*, *S*)). Furthermore, the group Aut(*G*, *S*) = { $\alpha \in$ Aut(*G*) | $S^{\alpha} = S$ } is a subgroup of Aut(Cay(*G*, *S*)). Actually, Aut(*G*, *S*) is a subgroup of Aut(Cay(*G*, *S*))₁, the stabilizer of the vertex 1 in Aut(Cay(*G*, *S*)). A Cayley graph Cay(*G*, *S*) is said to be *normal* if *R*(*G*) is normal in Aut(Cay(*G*, *S*)). Xu [42, Proposition 1.5] proved that Cay(*G*, *S*) is normal if and only if Aut(Cay(*G*, *S*))₁ = Aut(*G*, *S*).

Given a transitive group *G* acting on a set *V*, we say that a partition \mathcal{B} of *V* is *G*-invariant if the elements of *G* permute the parts, that is, *blocks* of \mathcal{B} , setwise. If the trivial partitions $\{V\}$ and $\{\{v\} : v \in V\}$ are the only *G*-invariant partitions of *V*, then *G* is said to be *primitive*, and is said to be *imprimitive* otherwise. In the latter case we shall refer to a corresponding *G*-invariant partition as to an *imprimitive block system* of *G*.

2.1. Group theoretic results

Throughout this paper we denote by \mathbb{Z}_n the cyclic group of order n as well as the ring of integers modulo n, and by \mathbb{Z}_n^* the multiplicative group of units of \mathbb{Z}_n . For two groups M and N, $N \leq M$ means that N is a subgroup of M and N < M means that N is a proper subgroup of M.

For a permutation group *G* on a set Ω and $\alpha \in \Omega$ we let G_{α} denote the stabilizer of α in *G*, that is, the subgroup of *G* fixing the element $\alpha \in \Omega$. The group *G* is said to be *semiregular* on Ω if $G_{\alpha} = 1$ for every $\alpha \in \Omega$, and it is said to be *regular* if it is both transitive and semiregular on Ω .

Below we gather various group-theoretic results that are needed in the subsequent sections of this paper. The first one is about transitive abelian permutation groups.

Proposition 2.1. [35, Proposition 4.4] *Every transitive abelian group G on a set* Ω *is regular.*

For a subgroup *H* of a group *G*, let $C_G(H)$ be the centralizer of *H* in *G*, and let $N_G(H)$ be the normalizer of *H* in *G*. Then $C_G(H)$ is normal in $N_G(H)$.

Proposition 2.2. [25, Chapter I, Theorem 4.5] Let G be a group and H a subgroup of G. Then the quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group Aut(H) of H.

The following result can be extracted from [13, P.285, summary].

Proposition 2.3. [13] Let G = PSL(2,7) and let A = PGL(2,7). Then Sylow 2-subgroups of G and A are, respectively, isomorphic to D_8 and D_{16} . Moreover, all involutions of G are conjugate, and G has no subgroup of order 14.

The following classical result is due to Wielandt [35, Theorems 3.4].

Proposition 2.4. [35] Let p be a prime and let P be a Sylow p-subgroup of a permutation group G acting on a set Ω . Let $w \in \Omega$. If p^m divides the length of the G-orbit containing ω , then p^m also divides the length of the P-orbit containing w.

2.2. Graph covers

A graph \widetilde{X} is called a *covering* of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\widetilde{v})} : N_{\widetilde{X}}(\widetilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in p^{-1}(v)$. The set fib_v = $p^{-1}(v)$ is a *fibre* of a vertex $v \in V(X)$. The subgroup K of all those automorphisms of \widetilde{X} which fix each of the fibres setwise is called the *group of covering transformations*. If the group of covering transformations is regular on the fibres of \widetilde{X} , we say that \widetilde{X} is a *regular K-covering*. We say that $\alpha \in Aut(X)$ *lifts* to an automorphism of \widetilde{X} if there exists $\widetilde{\alpha} \in Aut(\widetilde{X})$, called the *lift* of α , such that $\widetilde{\alpha}p = p\alpha$.

Let *X* be a graph and *K* a finite group. A *K*-voltage assignment of *X* is a function $\phi: A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$, where a^{-1} denotes the reverse arc of the arc a. The values of ϕ are called *voltages*, and *K* is the *voltage group*. The graph $X \times_{\phi} K$ derived from a voltage assignment $\phi: A(X) \to K$ has vertex set $V(X) \times K$ and edges of the form $(u, q)(v, q\phi(a))$ where $a = (u, v) \in A(X)$ and $q \in K$. Clearly, the derived graph $X \times_{\phi} K$ is a covering of X with the first coordinate projection $p: X \times_{\phi} K \to X$. By letting K act on $V(X \times_{\phi} K)$ as $(u, g')^g = (u, gg'), (u, g') \in V(X \times_{\phi} K)$, one obtains a semiregular subgroup of Aut($X \times_{\phi} K$), showing that $X \times_{\phi} K$ can in fact be viewed as a K-covering. Conversely, each regular covering X of X with a covering transformation group K can be derived from a K-voltage assignment. Moreover, Gross and Tucker [24] showed that every regular covering X of a graph X can in fact be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. (Given a spanning tree *T* of a graph *X*, a voltage assignment ϕ is said to be *T*-reduced if the voltages on the tree arcs are all equal to the identity of K.) If $X \times_{\phi} K \to X$ is a connected K-covering derived from a T-reduced voltage assignment ϕ then the problem whether an automorphism α of X lifts or not can be grasped in terms of voltages as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in Aut(X)$, we define a function $\overline{\alpha}$ from the set of voltages on fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by $(\phi(C))^{\overline{\alpha}} = \phi(C^{\alpha})$, where C ranges over all fundamental closed walks at *v*, and $\phi(C)$ and $\phi(C^{\alpha})$ are the voltages on *C* and C^{α} , respectively. Note that if *K* is abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X. The next proposition is a special case of [30, Theorem 4.2].

Proposition 2.5. [30] Let $X \times_{\phi} K \to X$ be a connected *K*-covering derived from a T-reduced voltage assignment ϕ . Then, an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of *K*.

For more results on graph covers we refer the reader to [1, 2, 14, 28, 29].

2.3. Tetravalent arc-transitive graphs

In this subsection we gather known results about tetravalent arc-transitive graphs that will be needed in subsequent sections. The first two propositions can be deduced from [40, Theorem 3.5].

Proposition 2.6. [40] Let p be a prime, and $G \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$ or $G \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$. Then there exists a tetravalent one-regular Cayley graph on G if and only if p - 1 is a multiple of 4. Moreover in each of these two cases exactly one such graph exists.

Proposition 2.7. [40] Let *p* be a prime and $G \cong \mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$. Then there is no tetravalent one-regular Cayley graph on *G*.

Let *X* be a connected symmetric graph and let $G \leq \text{Aut}(X)$ be an arc-transitive subgroup of Aut(*X*). For a normal subgroup *N* of *G*, the *quotient graph* X_N of *X* relative to the set of orbits of *N* is defined as the graph whose vertices are orbits of *N* on V(X) with two orbits being adjacent in X_N if there is an edge between these two orbits in *X*. The following proposition is a 'reduction' theorem which is deduced from [22, Theorem 1.1].

Proposition 2.8. [22, Theorem 1.1] Let X be a tetravalent connected symmetric graph and let $G \le Aut(X)$ be an arc-transitive subgroup of Aut(X). Then for each normal subgroup N of G one of the following holds:

- (1) N is transitive on V(X);
- (2) X is bipartite and N acts transitively on each of the two bipartition sets;
- (3) *N* has $r \ge 3$ orbits on *V*(*X*), the quotient graph X_N is a cycle of length *r*, and *G* induces the full automorphism group D_{2r} of X_N ;
- (4) N has $r \ge 5$ orbits on V(X), N acts semiregularly on V(X), the quotient graph X_N is a tetravalent connected *G*/N-symmetric graph and X is a regular cover of X_N .

To state the next result we need to introduce three families of tetravalent graphs that were first defined in [23]. First, let $C^{\pm 1}(p; 4, 2)$ be the graph with vertex set $\mathbb{Z}_p^2 \times \mathbb{Z}_4$, and adjacencies in $C^{\pm 1}(p; 4, 2)$ satisfying the following conditions: for $i, j \in \mathbb{Z}_p$ and $k \in \mathbb{Z}_4$

$$(i, j, k) \sim \begin{cases} (i \pm 1, j, k + 1) & \text{if } k \text{ is even} \\ (i, j \pm 1, k + 1) & \text{if } k \text{ is odd} \end{cases}$$

Second, for a prime $p \equiv \pm 1 \pmod{8}$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv 2 \pmod{p}$ the graph $NC_{4p^2}^0$ is defined to have vertex set and edge set

$$V(\mathcal{N}C_{4p^2}^0) = \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x, y, z) \mid x, y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\},$$

$$E(\mathcal{N}C_{4p^2}^0) = \{(x, y, 0)(x \pm 1, y, 1) \mid x, y \in \mathbb{Z}_p\} \cup \{(x, y, 1)(x, y \pm 1, 2) \mid x, y \in \mathbb{Z}_p\} \cup \{(x, y, 2)(x \mp 1, y \pm k, 3) \mid x, y \in \mathbb{Z}_p\} \cup \{(x, y, 3)(x \mp k, y \pm 1, 0) \mid x, y \in \mathbb{Z}_p\}.$$

And third, for a prime $p, p \equiv 1 \pmod{8}$ or $p \equiv 3 \pmod{8}$ and an element $k \in \mathbb{Z}_p^*$ such that $k^2 \equiv -2 \pmod{p}$ the graph $\mathcal{NC}_{4p^2}^1$ is defined to have vertex set and edge set

$$\begin{split} V(\mathcal{NC}_{4p^2}^1) &= \mathbb{Z}_p^2 \times \mathbb{Z}_4 = \{(x, y, z) \mid x, y \in \mathbb{Z}_p, z \in \mathbb{Z}_4\}, \\ E(\mathcal{NC}_{4p^2}^1) &= \{(x, y, 0)(x \pm 1, y, 1) \mid x, y \in \mathbb{Z}_p\} \cup \{(x, y, 1)(x, y \pm 1, 2) \mid x, y \in \mathbb{Z}_p\} \cup \{(x, y, 2)(x \pm 1, y \pm k, 3) \mid x, y \in \mathbb{Z}_p\} \cup \{(x, y, 3)(x \pm k, y \mp 1, 0) \mid x, y \in \mathbb{Z}_p\}. \end{split}$$

The graphs $NC_{4p^2}^0$ and $NC_{4p^2}^1$ are extracted from [23, Lemma 8.4, Lemma 8.7]. We can now state the result of Gardiner and Praeger [23, Theorem 1.2] about connected tetravalent graphs admitting arc-transitive subgroups of automorphisms with normal elementary abelian *p*-groups *N* such that the corresponding quotient graph X_N is a cycle.

Proposition 2.9. [23, Theorem 1.2] For an odd prime p let X be a connected, G-symmetric, tetravalent graph of order $4p^2$, let $N = \mathbb{Z}_p^2$ be a minimal normal subgroup of G with orbits of size p^2 , and let K be the kernel of the action of G on $V(X_N)$. If $X_N = C_4$ and $K_v = \mathbb{Z}_2$ then X is isomorphic to one of the following graphs: $C^{\pm 1}(p; 4, 2)$, $NC_{4p^2}^0$ and $NC_{4p^2}^1$.

In [23] it is proven that the three graphs in the above proposition all admit a one-regular subgroup of automorphisms. In the following two lemmas we improve this result by showing that $C^{\pm 1}(p; 4, 2)$ is not one-regular whereas $NC_{4p^2}^0$ and $NC_{4p^2}^1$ are.

Lemma 2.10. Let p be a prime. Then $C^{\pm 1}(p; 4, 2)$ is not one-regular.

Proof. First recall that the vertex set of $X = C^{\pm 1}(p, 4, 2)$ is equal to $V(X) = \{(i, j, k) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_q, k \in \mathbb{Z}_4\}$ and the edges are of the form

$$(i, j, 2l) \sim (i \pm 1, j, 2l + 1)$$
, where $i, j \in \mathbb{Z}_p$ and $l \in \{0, 1\}$
 $(i, j, 2l - 1) \sim (i, j \pm 1, 2l)$, where $i, j \in \mathbb{Z}_p$ and $l \in \{0, 1\}$.

Then the reader can check that a permutation α of V(X) defined by $(i, j, k)^{\alpha} = (-i, j, k)$ maps edges to edges, and hence α is an automorphism of X. Since α fixes the arc $(0, 0, 1)(0, 1, 2) \in A(X)$ it follows that X is not one-regular. \Box

Lemma 2.11. Let p be a prime. Then $NC_{4n^2}^0$ and $NC_{4n^2}^1$ are both one-regular graphs.

Proof. Let $X \in \{NC_{4p^2}^0, NC_{4p^2}^1\}$ and let X^2 be the distance-2-graph of X, that is, $V(X^2) = V(X)$ with two vertices being adjacent in X^2 if and only if they are at distance 2 in X. Let

$$\Delta_i = \{(x, y, i) \mid x, y \in \mathbb{Z}_p\}, \ i \in \mathbb{Z}_4.$$

Then for every $i \in \mathbb{Z}_4$ the subgraph $X^2[\Delta_i]$ of X^2 induced by the vertices in Δ_i is a 2-dimensional grid $C_p \times C_p$, whereas any edge uv in X^2 with endvertices $u \in \Delta_i$ and $v \in \Delta_j$, where $i \neq j$, is contained in an induced subgraph of X^2 isomorphic to the complete graph K_4 . Moreover this induced subgraph isomorphic to K_4 containing the edge uv is unique. Take four vertices $u_1, u_2, u_3, u_4 \in \Delta_i$ such that the subgraph Y of X^2 induced on these four vertices is isomorphic to a 4-cycle C_4 . Then Y^g for any $g \in \operatorname{Aut}(X^2)$ is an induced subgraph of X^2 isomorphic to C_4 . Since there is no set of four vertices containing vertices from different sets Δ_i such that the induced subgraph of X^2 is isomorphic to C_4 it follows that Y^g is a subgraph of $X^2[\Delta_j]$ for some $j \in \mathbb{Z}_4$. This shows that the sets $\Delta_i, i \in \mathbb{Z}_4$, are blocks of imprimitivity for $\operatorname{Aut}(X)$. Therefore every automorphism $g \in \operatorname{Aut}(X)$ that fixes the vertices (0, 0, 0) and (1, 0, 1), and thus the arc (0, 0, 0), (1, 0, 1), also fixes the vertices (2, 0, 0) and (-1, 0, 1). Now looking at the action of g on X^2 we get that g fixes both Δ_0 and Δ_1 pointwise. Since all the vertices in Δ_1 are fixed by g and the induced bipartite subgraph $X[\Delta_1, \Delta_2]$ is a disjoint union of p 2p-cycles it follows that also Δ_2 is fixed pointwise by g. Using the same argument for $X[\Delta_0, \Delta_3]$ one can see that g also fixes the vertices in Δ_3 and thus g = 1, which shows that X is one-regular. \Box

To state the next result we need to introduce two additional families of tetravalent graphs that were first defined in [23]. The graph $C^{\pm 1}(p; 4p, 1)$ is defined to have the vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ and the edge set $\{(i, j)(i \pm 1, j + 1) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_{4p}\}$. The graph $C^{\pm \varepsilon}(p; 4p, 1)$ is a graph with vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ with adjacencies in $C^{\pm \varepsilon}(p; 4p, 1)$ satisfying the following conditions:

$$(i, j) \sim \begin{cases} (i \pm \varepsilon, j + 1) & \text{if } j \text{ is odd} \\ (i \pm 1, j + 1) & \text{if } j \text{ is even} \end{cases}$$

where $i \in \mathbb{Z}_p$, $j \in \mathbb{Z}_{4p}$ and ε is an element of order 4 in \mathbb{Z}_p^* .

Proposition 2.12. [23, Theorem 1.1] Let p be an odd prime and let X be a connected, G-symmetric, tetravalent graph of order $4p^2$. Let $N = \mathbb{Z}_p$ be a minimal normal subgroup of G with orbits of size p and let K denote the kernel of the action of G on $V(X_N)$. If $X_N = C_{4p}$ and $K_v = \mathbb{Z}_2$ then X is isomorphic either to $C^{\pm 1}(p; 4p, 1)$ or to $C^{\pm \varepsilon}(p; 4p, 1)$.

We end this subsection with a result on tetravalent arc-transitive graphs of order 4*p*, where *p* is a prime. In order to state the result, first recall that the *lexicographic product* X[Y] (sometimes also called the *wreath product*) of two graphs *X* and *Y* has vertex set $V(X) \times V(Y)$, and two vertices (*a*, *u*) and (*b*, *v*) are adjacent in X[Y] if $ab \in E(X)$ or if a = b and $uv \in E(Y)$. Second, following [44], for a prime *p* congruent to 1 modulo 4, an element *w* of order 4 in \mathbb{Z}_p^* and the group $G = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$, we use notation $C\mathcal{H}_{4p}^0 = \text{Cay}(G, \{a, a^{-1}, a^{w^2}b, a^{-w^2}b\})$ and $C\mathcal{H}_{4p}^1 = \text{Cay}(G, \{a, a^{-1}, a^w b, a^{-w}b\})$. For the definition of the graph C(2, p, 2) stated in the sixth row of Table 1 see Section 4. Finally, by [44, Example 3.7], $g_{28} = \text{Cos}(G, T, TaT)$ is a coset graph of the group G = PGL(2, 7) with respect to a subgroup *T* isomorphic to A_4 and an involution *a* from the center of the normalizer of a Sylow 3-subgroup of *T* in *G*.

Proposition 2.13. [44, Theorem 4.1] Let *s* be a positive integer and let *p* be a prime. Then a connected tetravalent graph of order 4*p* is *s*-arc-transitive if and only if it is isomorphic to one of the graphs listed in Table 1. Furthermore, all graphs listed in Table 1 are pairwise non-isomorphic.

X	S	Aut(X)	comments
K _{4,4}	3	$\mathbb{Z}_2 \ltimes (S_4 \times S_4)$	<i>p</i> =2
$C_{2p}[2K_1]$	1	$D_{4p} \ltimes \mathbb{Z}_2^{2p}$	<i>p</i> > 2
$C\mathcal{R}^0_{4p}$	1	$\mathbb{Z}_2^2 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2),$	$p \equiv 1 \pmod{4}$
$C\mathcal{A}_{4p}^{1}$	1	$\mathbb{Z}_4 \ltimes (\mathbb{Z}_{2p} \times \mathbb{Z}_2),$	$p \equiv 1 \pmod{4}$
C(2, p, 2)	1	$D_{2p} \ltimes \mathbb{Z}_2^{2p}$	<i>p</i> > 2
\$ 28	3	$PGL(2,7) \times \mathbb{Z}_2$	<i>p</i> = 7

Table 1: Tetravalent s-arc-transitive graphs of order 4p.

3. Examples

In this section, we give examples of tetravalent one-regular graphs of order $4p^2$, where *p* is a prime. In this paper, the abbreviations *C* \mathcal{A} and *CN* will mean a Cayley graph on abelian group and a Cayley graph on non-abelian group, respectively.

Example 3.1. Introduced by Wilson [39] the *bicycle wheels* are defined in the following way. Given natural numbers n, a, r and s, the graph $X = \mathcal{BW}_n(a, r, s)$ is defined to be the graph of order 3n with vertex set $V(X) = \{A_i, B_i, C_i \mid i \in \mathbb{Z}_n\}$ and edge set

$$E(X) = \{A_i B_i, B_i A_{i+1}, B_i C_i, C_i B_{i+a}, A_i A_{i+r}, C_i C_{i+s} \mid i \in \mathbb{Z}_n\}$$

With the help of computer software package MAGMA [3] one can see that $\mathcal{BW}_{12}(5, 1, 5)$ is one-regular. In addition, it is a Cayley graph Cay(G_{36} , S) on the group $G_{36} = \langle a, b, c, d | a^2 = b^2 = c^3 = d^3 = 1 = [a, b] = [a, c] = [b, c] = [c, d], d^{-1}ad = b, d^{-1}bd = ab \rangle$ with respect to the generating set $S = \{ad, (ad)^{-1}, bdc, (bdc)^{-1}\}$, and Aut($C\mathcal{R}_{36}^2 \rangle \cong G_{36} \rtimes \mathbb{Z}_2^2$.

Remark: The automorphism group of the graph $\mathcal{BW}_{12}(5, 1, 5)$ has a non-normal Sylow 3-subgroup. Since, by Theorem 5.1, the automorphism groups of the graphs $C\mathcal{A}_{4p^2}^i$, $i \in \{0, 1, 2\}$, given in Examples 3.3 and 3.4 and Lemma 3.6, all have normal Sylow *p*-subgroups, the graph $\mathcal{BW}_{12}(5, 1, 5)$ is not isomorphic to any of these graphs.

Example 3.2. Given natural numbers *k* and *m*, and a 2 × 2 matrix *M* over \mathbb{Z}_n the 2-dimensional generalized power spidergraph $\mathcal{GPS2}(k, n, M)$ is defined to be the graph with vertex set $\mathbb{Z}_k \times \mathbb{Z}_n \times \mathbb{Z}_n$, and edge set $\{(i, x)(i + 1, x + a_i), (i, x)(i + 1, x + b_i) \mid i \in \mathbb{Z}_k, x \in \mathbb{Z}_n \times \mathbb{Z}_n\}$ where $a_i = (1, 0)M^i$ and $b_i = (-1, 0)M^i$ (see [39]). With the use of MAGMA [3] one can see that $\mathcal{GPS2}(4, 3, (0 \ 1) : (1 \ 2))$ is a one-regular graph. In addition, it is not a Cayley graph and the stabilizer of a vertex in the automorphism group is isomorphic to \mathbb{Z}_4 .

Example 3.3. Let $p \equiv 1 \pmod{4}$ be a prime and w an element of order 4 in \mathbb{Z}_p^* with $1 \le w \le p-1$. Let $G_{4p^2}^0 = \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$. Then, by [40, Proposition 3.3(iv)], the Cayley graph $C\mathcal{A}_{4p^2}^0 = \operatorname{Cay}(G_{4p^2}^0, \{a, a^{-1}, a^w b, a^{-w} b\})$ is a tetravalent one-regular graph. Furthermore, $\operatorname{Aut}(C\mathcal{A}_{4p^2}^0) \cong (\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2^2$.

Example 3.4. Let *p* be an odd prime and $G_{4p^2}^1 = \langle a, b | a^{4p} = b^p = 1, ab = ba \rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$. Then, by [40, Proposition 3.3], the Cayley graph $C\mathcal{A}_{4p^2}^1 = \text{Cay}(G_{4p^2}^1, \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\})$ is a tetravalent one-regular graph. Furthermore, $\text{Aut}(C\mathcal{A}_{4p^2}^1) \cong (\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$. The graph $\mathcal{DW}(12, 3)$ of order 36 given in [39] is the smallest example of such graphs.

For an odd prime p, the tetravalent graph $C^{\pm 1}(p; 4p, 1)$ is defined in the paragraph preceding Proposition 2.12. In the following lemma we prove that $C^{\pm 1}(p; 4p, 1)$ is isomorphic to $C\mathcal{R}^{1}_{4p^{2}}$, and thus it is one-regular in view of Example 3.4.

Lemma 3.5. Let *p* be an odd prime, let $G_{4p^2}^1 = \langle a, b | a^{4p} = b^p = 1, ab = ba \rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$, and let $S = \{ab, a^{-1}b, ab^{-1}, a^{-1}b^{-1}\}$. Then $C^{\pm 1}(p; 4p, 1) \cong \operatorname{Cay}(G_{4p^2}^1, S) = C\mathcal{A}_{4p^2}^1$.

Proof. Recall that $C^{\pm 1}(p; 4p, 1)$ has vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ and edge set $\{(i, j)(i \pm 1, j + 1) \mid i \in \mathbb{Z}_p, j \in \mathbb{Z}_{4p}\}$. The map defined by $(i, j) \mapsto a^j b^i$ is an isomorphism from $C^{\pm 1}(p; 4p, 1)$ to the Cayley graph $C\mathcal{A}_{4p^2}^1$. We leave the details to the reader. \Box

Let $p \equiv 1 \pmod{4}$ be a prime and let $\varepsilon \in \mathbb{Z}_p$ be such that $\varepsilon^2 \equiv -1 \pmod{p}$. The following lemma shows that $C^{\pm \varepsilon}(p; 4p, 1)$ is a Cayley graph.

Lemma 3.6. Let $p \equiv 1 \pmod{4}$ be a prime, let $\varepsilon \in \mathbb{Z}_p$ be such that $\varepsilon^2 \equiv -1 \pmod{p}$, let $G_{4p^2}^2 = \langle a, b | a^{4p} = b^p = 1, a^{-1}ba = b^{\varepsilon} \rangle$, and let $S = \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\}$. Then $CN_{4p^2}^2 = Cay(G_{4p^2}^2, S)$ is a symmetric graph isomorphic to $C^{\pm\varepsilon}(p; 4p, 1)$.

Proof. Recall that the graph $C^{\pm \varepsilon}(p; 4p, 1)$ has vertex set $\mathbb{Z}_p \times \mathbb{Z}_{4p}$ with adjacencies defined as follows:

$$(i, j) \sim \begin{cases} (i \pm \varepsilon, j + 1) & \text{if } j \text{ is odd} \\ (i \pm 1, j + 1) & \text{if } j \text{ is even} \end{cases}$$

where $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{4p}$.

Let $G = G_{4p^2}^2$ and X = Cay(G; S). Then the map defined by $(i, j) \mapsto a^j b^i$ is an isomorphism from $C^{\pm \varepsilon}(p; 4p, 1)$ to X. Since, by [23], the graph $C^{\pm \varepsilon}(p; 4p, 1)$ is symmetric, the lemma holds. \Box

4. Analysis of tetravalent one-regular graphs of order 4p²

Let *p* be an odd prime. Then define C(2, p, 2) to be a graph with $V(C(2, p, 2)) = \mathbb{Z}_4 \times \mathbb{Z}_p$ and adjacencies in C(2, p, 2) satisfying the following conditions:

$(0,i) \sim (0,j)$	$\iff j-i=\pm 1,$
$(0,i) \sim (1,j)$	$\iff j-i=-1,$
$(0,i)\sim(2,j)$	$\iff j-i=1,$
$(1,i)\sim(2,j)$	$\iff j-i=\pm 1,$
$(1,i)\sim(3,j)$	$\iff j-i=-1,$
$(2,i)\sim(3,j)$	$\iff j-i=1,$
$(3,i)\sim(3,j)$	$\iff j-i=\pm 1.$

Let X = C(2, p, 2) and let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$. Observe that for each $j \in \mathbb{Z}_p$, $j \neq i$, the subgraph $X[B_i, B_j]$ induced on the union $B_i \cup B_j$ is not an independent set of vertices if and only if $j = i \pm 1$. Moreover, for each such j we have that $X[B_j, B_{j+1}] \cong 2C_4$, see also Figure 1. The following lemma shows that there is no one-regular \mathbb{Z}_p -cover of C(2, p, 2).



Figure 1: A spanning tree in the base graph C(2, p, 2) for p = 7.

Lemma 4.1. Let Y be a tetravalent one-regular graph of order $4p^2$, p > 3 a prime, such that there exists a normal subgroup H of Aut(Y) of order p. Then Y is not a regular \mathbb{Z}_p -cover of the graph C(2, p, 2).

Proof. Let $\mathcal{K} = \{1, \tau_1, \tau_2, \tau_3\}$ be the Klein 4-group acting on \mathbb{Z}_4 so that $\tau_1 = (0\,1)(2\,3), \tau_2 = (0\,2)(1\,3)$ and $\tau_3 = (0\,3)(1\,2)$. Let X = C(2, p, 2), let $\mathcal{B} = \{B_i \mid i \in \mathbb{Z}_p\}$, where $B_i = \{(0, i), (1, i), (2, i), (3, i)\} \subseteq \mathbb{Z}_4 \times \mathbb{Z}_p$, and let K be the kernel of the action of Aut(X) on \mathcal{B} . We shall be sloppy and shall identify restrictions of elements of K to sets B_i by elements of \mathcal{K} . For instance, when we say that the restriction γ_i of $\gamma \in K$ to B_i is, for example, τ_1 , we mean that $\gamma_i = ((0, i)(1, i))((2, i)(3, i))$. Now, the structure of X indicated in Figure 1 implies that the restrictions γ_i must satisfy the following conditions:

$$\gamma_i \in \{1, \tau_1\} \iff \gamma_{i+1} \in \{1, \tau_2\} \quad \forall i \in \mathbb{Z}_p.$$

$$\tag{1}$$

Let the vertices of *X* be labeled in the following way: $a_i = (0, i)$, $b_i = (1, i)$, $c_i = (2, i)$ and $d_i = (3, i)$. Let $E = \langle \gamma_i | i \in \mathbb{Z}_p \rangle$. It is well known, see for instance [33, 44], that Aut(*X*) = $E \rtimes \langle \rho, \tau \rangle \cong \mathbb{Z}_2^p \rtimes D_{2p}$ where

$$\rho = (a_0 \ a_1 \ \dots \ a_{p-1})(b_0 \ b_1 \ \dots \ b_{p-1})(c_0 \ c_1 \ \dots \ c_{p-1})(d_0 \ d_1 \ \dots \ d_{p-1})$$

and

$$\tau = (a_0)(b_0 \ c_0)(d_0) \prod_{i=1}^{p-1} (a_i \ a_{-i})(b_i \ c_{-i})(c_i \ b_{-i})(d_i \ d_{-i}).$$

Now let *Y* be a tetravalent one-regular graph of order $4p^2$. Assume that Aut(*Y*) contains a normal subgroup *H* isomorphic to \mathbb{Z}_p such that the corresponding quotient graph Y_H is isomorphic to X = C(2, p, 2). Then, since the orbits of *H* form an Aut(*Y*)-invariant partition, the whole automorphism group Aut(*Y*) of *Y* projects to a subgroup of Aut(*X*). On the other hand, the graph *Y* can be viewed as an *H*-covering graph (that is, a \mathbb{Z}_p -covering) of *X*, and it can therefore be derived from *X* through a suitable voltage assignment ζ . To find this voltage assignment fix the spanning tree *T* of *X* as indicated on Figure 1.

Let *G* be the largest subgroup of Aut(*X*) which lifts with respect to the natural projection $X \times_{\zeta} \mathbb{Z}_p \cong Y \rightarrow Y_H \cong X$, where ζ is as given in Figure 1. Clearly, since *Y* is arc-transitive, we may assume that $\rho, \tau \in G$. Let *F* denote the largest subgroup of *E* which lifts. Then $G = F \rtimes \langle \rho, \tau \rangle$ and thus |G| = 2p|F|. We will show that |F| > 8. This will then imply that the lift \overline{G} of *G* is of order $|\overline{G}| = 2p^2|F| > 16p^2$, and consequently that *Y* is not one-regular.

Since $\rho, \tau \in G$, we have that

if
$$\phi \in F$$
 then $\phi^{\rho}, \phi^{\tau} \in F$.

It is convenient to view elements γ in *E* as vectors in \mathbb{Z}_{4}^{p} . Namely, we write $\gamma = (e_0, \ldots, e_{p-1})$ where $e_i = s$ if and only if $\gamma_i = \tau_s$ (where $e_i = 0$ means that $\gamma_i = \tau_0 = id$). Note that in this context (2) can be interpreted as follows: *F* is invariant under the "cyclic shift"

$$\phi = (f_0, f_1, \dots, f_{p-1}) \mapsto (f_{p-1}, f_0, \dots, f_{p-2}),$$

and under the "reflection around the first entry"

$$\phi = (f_0, f_1, \dots, f_{p-1}) \mapsto (f'_0, f'_{p-1}, f'_{p-2}, \dots, f'_2, f'_1),$$

where

$$f'_i = \begin{cases} 0 & , & \text{if } f_i = 0 \\ 1 & , & \text{if } f_i = 2 \\ 2 & , & \text{if } f_i = 1 \\ 3 & , & \text{if } f_i = 3 \end{cases}$$

Now choose $\phi \in F$. By (1) the first two components of ϕ can be one of the following pairs: $\phi = (0, 0, ...)$, $\phi = (0, 2, ...)$, $\phi = (1, 0, ...)$, $\phi = (1, 2, ...)$, $\phi = (2, 1, ...)$, $\phi = (2, 3, ...)$, $\phi = (3, 1, ...)$, or $\phi = (3, 3, ...)$. Since the lift of *G* acts arc-transitively on *Y* the group *G* must be of order $|G| = 2p|F| \ge 16p$ and thus $|F| \ne 1$.

Suppose first that there exist $\psi \in F$ such that $\psi \notin \{id, (3, 3, ..., 3)\}$. Since ρ is of prime order, the conjugacy class of ψ under $\langle \rho \rangle$ is of size p. But then, by (2), we have that |F| > 8, which implies that \overline{G} is not acting one-regularly on Y.

Suppose now that (3, 3, ..., 3) belongs to F. Then, since $\langle (3, 3, ..., 3) \rangle \leq F$ is of order 2 and |G| = 2p|F| = 16p, we have that there must also exist a non-identity automorphism $\psi \in F$ which is different from (3, 3, ..., 3). But then, as above, the conjugacy class of ψ is of size p, and consequently |F| > 8. This shows that \overline{G} is not acting one-regularly on Y, and the proof is completed. \Box

By the following lemma there are only two normal one-regular Cayley graphs on the group $G = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a \rangle$.

Lemma 4.2. Let *p* be a prime and $G = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1$, $a^g = b$, $b^g = a \rangle$. Then a tetravalent normal Cayley graph X of order $4p^2$ on G is one-regular if and only if it is either isomorphic to

$$CN_{4p^2}^3 = Cay(G, \{ag, bcg, b^{-1}g, a^{-1}cg\}) \text{ or to } CN_{4p^2}^4 = Cay(G, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\}).$$

Moreover, $\operatorname{Aut}(CN^3_{4p^2}) \cong G \rtimes \mathbb{Z}_2^2$ and $\operatorname{Aut}(CN^4_{4p^2}) \cong G \rtimes \mathbb{Z}_4$.

Proof. Let *X* be a tetravalent one-regular normal Cayley graph Cay(*G*, *S*) on the group *G* with respect to the generating set *S*. Since *X* is one-regular and normal, the stabilizer $A_1 = \text{Aut}(G, S)$ of the vertex $1 \in G$ is transitive on *S*, and either Aut(*G*, *S*) $\cong \mathbb{Z}_2^2$ or Aut(*G*, *S*) $\cong \mathbb{Z}_4$. This implies that elements in *S* are all of the same order.

Observe that *G* contains elements of order 2, *p* and 2*p*. In particular, elements of the form *c*, $a^i b^j g$ and $a^i b^j cg$, where $p \mid i + j$, are of order 2; elements of the form $a^i b^j$ are of order *p*; and elements of the form $a^i b^j c$, $a^m b^n g$ and $a^m b^n cg$, where $p \nmid m + n$, are of order 2*p*. In the following, we will show that up to isomorphism, there are only two generating sets of size 4 such that the corresponding Cayley graphs are normal and one-regular.

First, observe that neither four involutions nor two elements of order *p* can generate *G*. Moreover, *G* cannot be generated by the following pairs of elements of order 2*p*: $a^{i_1}b^{j_1}c$ and $a^{i_2}b^{j_2}c$, $a^{m_1}b^{n_1}g$ and $a^{m_2}b^{n_2}g$, $a^{m_1}b^{n_1}cg$ and $a^{m_2}b^{n_2}cg$, where $m_i + n_i \neq 0$ ($1 \leq i \leq 2$). Second, $Z(G) = \langle ab, c \rangle = \langle ab \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_2$, and thus $\langle c \rangle$ char *G*. Also, since Aut(*G*, *S*) is transitive on *S*, we have that $S \neq \{a^i b^j c, a^m b^n g, (a^i b^j c)^{-1}, (a^m b^n g)^{-1}\}$ and $S \neq \{a^i b^j c, a^m b^n cg, (a^i b^j c)^{-1}, (a^m b^n cg)^{-1}\}$, where $m + n \neq 0$. Now suppose that *G* is generated by

$$S_0 = \{a^i b^j g, a^{m'} b^{n'} cg, (a^i b^j g)^{-1}, (a^{m'} b^{n'} cg)^{-1}\},\$$

where $p \nmid i + j$ and $p \nmid m' + n'$.

CASE 1. Aut(*G*, *S*₀) = $\langle \alpha \rangle \times \langle \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, where α and β are such that $a^{\alpha} = a^{i_1}b^{j_1}$, $b^{\alpha} = a^{j_1}b^{i_1}$, $c^{\alpha} = c$, $g^{\alpha} = a^x b^{-x} cg$, $a^{\beta} = a^{i_2}b^{j_2}$, $b^{\beta} = a^{j_2}b^{i_2}$, $c^{\beta} = c$ and $g^{\beta} = a^y b^{-y}g$.

SUBCASE 1.1. Let i = j.

Since $ab \in Z(G)$, *G* can be generated by S_0 if and only if $m' \neq n'$. Now take an automorphism σ of *G* such that

$$a^{\sigma} = a^{\iota}, b^{\sigma} = b^{\iota}, c^{\sigma} = c, g^{\sigma} = g.$$

Then $(abg)^{\sigma} = a^i b^i g$, and hence

$$S = S_0^{\sigma^{-1}} = \{abg, a^m b^n cg, (abg)^{-1}, (a^m b^n cg)^{-1}\} = \{abg, a^m b^n cg, a^{-1}b^{-1}g, a^{-n}b^{-m}cg\}$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$. Moreover, it can be easily seen that $m \neq n$.

Suppose first that $(abg)^{\alpha} = a^{m}b^{n}cg$. Then $(a^{m}b^{n}cg)^{\alpha} = abg$, $(a^{-1}b^{-1}g)^{\alpha} = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^{\alpha} = a^{-1}b^{-1}g$. It follows that either m + n = 2 or m + n = -2. If m + n = 2 then, since $m \neq n$, we have that $m \neq 1$ and

$$a^{\alpha} = b, \ b^{\alpha} = a, \ c^{\alpha} = c, \ q^{\alpha} = a^{m-1}b^{1-m}cq.$$

If m + n = -2, then since $m \neq n$, we have $n \neq -1$ and

$$a^{\alpha} = a^{-1}, \ b^{\alpha} = b^{-1}, \ c^{\alpha} = c, \ g^{\alpha} = a^{-1-n}b^{1+n}cg$$

Suppose now that $(abg)^{\beta} = a^{-1}b^{-1}g$. Then $(a^{-1}b^{-1}g)^{\beta} = abg$, $(a^{m}b^{n}cg)^{\beta} = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^{\beta} = a^{m}b^{n}cg$. By a similar argument as above, one can get that

$$a^{eta} = b^{-1}, \ b^{eta} = a^{-1}, \ c^{eta} = c, \ g^{eta} = g$$

Consequently, either $S_0 = S_1 = \{abg, a^m b^{2-m} cg, a^{-1}b^{-1}g, a^{m-2}b^{-m}cg\}$, where $m \neq 1$, or

$$S_0 = S_2 = \{abg, a^{-2-n}b^n cg, a^{-1}b^{-1}g, a^{-n}b^{n+2}cg\},\$$

where $n \neq -1$. In addition, replacing -n with m, it can be seen that $S_2 = S_1$. Moreover, it can be easily seen that G can indeed be generated by S_1 . Namely, since $(abg)^p = g$ we have $g, ab \in \langle S_1 \rangle$. Then, since $a^m b^{2-m} cg \in \langle S_1 \rangle$, we get that $a^m b^{2-m} c \in \langle S_1 \rangle$. Further, since $(a^m b^{2-m} c)^p = c$, also $c, a^m b^{2-m} \in \langle S_1 \rangle$. Now, since $a^m b^{2-m} = a^m b^m b^{2-2m}$, $m \neq 1$, and $ab \in \langle S_1 \rangle$, we get that $b^{2-2m} \in \langle S_1 \rangle$. Finally, the fact that $b^g = a$ implies that $G = \langle S_1 \rangle$.

Subcase 1.2. Let $i \neq j$.

Take an automorphism σ of *G* such that $a^{\sigma} = a^i b^j$, $b^{\sigma} = a^j b^i$, $c^{\sigma} = c$, and $g^{\sigma} = g$. Then $(ag)^{\sigma} = a^i b^j g$ and

$$S = S_0^{\sigma^{-1}} = \{ag, a^m b^n cg, (ag)^{-1}, (a^m b^n cg)^{-1}\} = \{ag, a^m b^n cg, b^{-1}g, a^{-n}b^{-m}cg\}$$

where $a^{m}b^{n}cq = (a^{m'}b^{n'}cq)^{\sigma^{-1}}$.

Suppose first that $(ag)^{\alpha} = a^{m}b^{n}cg$. Then $(a^{m}b^{n}cg)^{\alpha} = ag$, $(b^{-1}g)^{\alpha} = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^{\alpha} = b^{-1}g$. In addition, either m + n = 1 or m + n = -1. If m + n = 1 then, since $\{ag, acg, b^{-1}g, b^{-1}cg\}$ cannot generate G, we have that $m \neq 1$. Thus α is mapping according to the rule: $a^{\alpha} = b$, $b^{\alpha} = a$, $c^{\alpha} = c$, and $g^{\alpha} = a^{m}b^{-m}cg$. If on the other hand m + n = -1 then, since $\{ag, b^{-1}cg, b^{-1}g, acg\}$ cannot generate G, we have that $n \neq -1$, and hence α is mapping according to the rule: $a^{\alpha} = a^{-1}$, $b^{\alpha} = b^{-1}$, $c^{\alpha} = c$, and $g^{\alpha} = a^{-n}b^{n}cg$.

Suppose now that $(ag)^{\beta} = b^{-1}g$. Then we have that $(b^{-1}g)^{\beta} = ag$, $(a^{m}b^{n}cg)^{\beta} = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^{\beta} = a^{m}b^{n}cg$. Whenever m + n = 1 or m + n = -1, we can get that β is mapping according to the rule: $a^{\beta} = b^{-1}$, $b^{\beta} = a^{-1}$, $c^{\beta} = c$, and $g^{\beta} = g$. Thus, we can conclude that either $S_0 = S_3 = \{ag, a^{m}b^{1-m}cg, b^{-1}g, a^{m-1}b^{-m}cg\}$, where $m \neq 1$, or $S_0 = S_4 = \{ag, a^{-n-1}b^{n}cg, b^{-1}g, a^{-n}b^{n+1}cg\}$, where $n \neq -1$. Moreover, replacing -n with m, it

can be easily seen that $S_4 = S_3$. Also, since $(ag)^2 = ab$ and $aga^m b^{1-m}cg = a^{2-m}b^m c$, we get that $c, a^{2-m}b^m \in \langle S_3 \rangle$. Further, the facts that $a^{2-m}b^m = a^{2-2m}a^m b^m$, $m \neq 1$ and $ab \in \langle S_3 \rangle$ combined together imply that $a^{2-2m} \in \langle S_3 \rangle$. Since $ag \in \langle S_3 \rangle$, it follows that $g \in \langle S_3 \rangle$. Finally, since $a^g = b$, G is indeed generated by S_3 .

Now considering the automorphism γ of G defined by $a^{\gamma} = a^{\frac{1}{2}}$, $b^{\gamma} = b^{\frac{1}{2}}$, $c^{\gamma} = c$, and $g^{\gamma} = a^{\frac{1}{2}}b^{-\frac{1}{2}}g$ we get that $S_1^{\gamma} = \{ag, a^{\frac{m+1}{2}}b^{1-\frac{m+1}{2}}cg, b^{-1}g, a^{\frac{m+1}{2}-1}b^{-\frac{m+1}{2}}cg\}$, where $m \neq 1$. Thus we only need to consider the generating set $S_3 = \{ag, a^m b^{1-m}cg, b^{-1}g, a^{m-1}b^{-m}cg\}$, where $m \neq 1$.

CASE 2. Aut(G, S_0) = $\langle \alpha \rangle \cong \mathbb{Z}_4$, where α is such that $a^{\alpha} = a^{i_1}b^{j_1}$, $b^{\alpha} = a^{j_1}b^{i_1}$, $c^{\alpha} = c$, and $g^{\alpha} = a^x b^{-x} cg$.

SUBCASE 2.1. Let i = j.

Since $ab \in Z(G)$, *G* can be generated by S_0 (where $p \nmid i$ and $p \nmid m' + n'$) if and only if $m' \neq n'$. Now take an automorphism σ of *G* such that $a^{\sigma} = a^i$, $b^{\sigma} = b^i$, $c^{\sigma} = c$, and $g^{\sigma} = g$. Then $(abg)^{\sigma} = a^i b^i g$, and consequently

$$S = S_0^{\sigma^{-1}} = \{abg, a^m b^n cg, (abg)^{-1}, (a^m b^n cg)^{-1}\} = \{abg, a^m b^n cg, a^{-1} b^{-1} g, a^{-n} b^{-m} cg\}$$

where $a^m b^n cg = (a^{m'} b^{n'} cg)^{\sigma^{-1}}$, and $m \neq n$.

Suppose first that $(abg)^{\alpha} = a^m b^n cg$. Then $(a^m b^n cg)^{\alpha} = a^{-1}b^{-1}g$, $(a^{-1}b^{-1}g)^{\alpha} = a^{-n}b^{-m}cg$, $(a^{-n}b^{-m}cg)^{\alpha} = abg$. Hence either $m + n = \omega$ or $m + n = -\omega$, where $\omega^2 = -4$. If $m + n = \omega$ then since $m \neq n$, we have that $m \neq \frac{\omega}{2}$. It follows that $a^{\alpha} = a^i b^{\frac{\omega}{2} - i}$, $b^{\alpha} = a^{\frac{\omega}{2} - i}b^i$, $c^{\alpha} = c$, and $g^{\alpha} = a^{m-\frac{\omega}{2}}b^{\frac{\omega}{2} - m}cg$, where $i = \frac{(m+1)\omega+2-2m}{2(2m-\omega)}$. If on the other hand $m + n = -\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^{\alpha} = a^i b^{-\frac{\omega}{2} - i}$, $b^{\alpha} = a^{-\frac{\omega}{2} - i}b^i$, $c^{\alpha} = c$, and $g^{\alpha} = a^{-\frac{\omega}{2} - n}b^{\frac{\omega}{2} + n}cg$, where $i = \frac{2-2n-(n+1)\omega}{2(2n+\omega)}$.

Suppose now that $(abg)^{\alpha} = a^{-n}b^{-m}cg$. Then $(a^{-n}b^{-m}cg)^{\alpha} = a^{-1}b^{-1}g$, $(a^{-1}b^{-1}g)^{\alpha} = a^{m}b^{n}cg$, and $(a^{m}b^{n}cg)^{\alpha} = abg$. Hence, either $m + n = \omega$ or $m + n = -\omega$, where $\omega^{2} = -4$. If $m + n = \omega$ then, since $m \neq n$, we have that $m \neq \frac{\omega}{2}$, and thus $a^{\alpha} = a^{i}b^{-\frac{\omega}{2}-i}$, $b^{\alpha} = a^{-\frac{\omega}{2}-i}b^{i}$, $c^{\alpha} = c$, and $g^{\alpha} = a^{m-\frac{\omega}{2}}b^{\frac{\omega}{2}-m}cg$, where $i = \frac{(1-m)\omega-2m-2}{2(2m-\omega)}$. If however $m + n = -\omega$ then, since $m \neq n$, we have that $n \neq -\frac{\omega}{2}$, and so $a^{\alpha} = a^{i}b^{\frac{\omega}{2}-i}$, $b^{\alpha} = a^{\frac{\omega}{2}-i}b^{i}$, $c^{\alpha} = c$, and $g^{\alpha} = a^{-\frac{\omega}{2}-n}b^{\frac{\omega}{2}+n}cg$, where $i = \frac{(n-1)\omega-2n-2}{2(2n+\omega)}$.

We can conclude that either $S_0 = S_5 = \{abg, a^m b^{\omega-m} cg, a^{-1}b^{-1}g, a^{m-\omega}b^{-m}cg\}$, where $m \neq \frac{\omega}{2}$, or $S_0 = S_6 = \{abg, a^{-\omega-n}b^n cg, a^{-1}b^{-1}g, a^{-n}b^{n+\omega}cg\}$, where $n \neq -\frac{\omega}{2}$. Moreover, replacing -n with m, it can be easily seen that $S_5 = S_6$. Also, the group G is indeed generated by S_5 . Namely, since $(abg)^p = g$ we have that $g, ab \in \langle S_5 \rangle$. Further, since $a^m b^{\omega-m} cg \in \langle S_5 \rangle$, also $a^m b^{\omega-m} c \in \langle S_5 \rangle$, and the fact that $(a^m b^{\omega-m} c)^p = c$ implies that $c, a^m b^{\omega-m} \in \langle S_5 \rangle$. Finally, since $a^m b^{\omega-m} = a^m b^m b^{\omega-2m}$, $m \neq \frac{\omega}{2}$, and $ab \in \langle S_5 \rangle$, it follows that $b^{\omega-2m} \in \langle S_5 \rangle$. Now this fact and $b^g = a$ combined together imply that $G = \langle S_5 \rangle$.

SUBCASE 2.2. Let $i \neq j$.

Take an automorphism σ of G such that $a^{\sigma} = a^{i}b^{j}$, $b^{\sigma} = a^{j}b^{i}$, $c^{\sigma} = c$, and $g^{\sigma} = g$. Then $(ag)^{\sigma} = a^{i}b^{j}g$, and consequently

$$S = S_0^{\sigma^{-1}} = \{ag, a^m b^n cg, (ag)^{-1}, (a^m b^n cg)^{-1}\} = \{ag, a^m b^n cg, b^{-1}g, a^{-n}b^{-m}cg\},\$$

where $a^{m}b^{n}cg = (a^{m'}b^{n'}cg)^{\sigma^{-1}}$.

Suppose first that $(ag)^{\alpha} = a^{m}b^{n}cg$. Then $(a^{m}b^{n}cg)^{\alpha} = b^{-1}g$, $(b^{-1}g)^{\alpha} = a^{-n}b^{-m}cg$, and $(a^{-n}b^{-m}cg)^{\alpha} = ag$. Also, either $m + n = \varepsilon$ or $m + n = -\varepsilon$, where $\varepsilon^{2} = -1$. If $m + n = \varepsilon$ then, since $\{ag, a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cg, b^{-1}g, a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cg\}$ cannot generate *G* (namely, for $\varphi \in \text{Aut}(G)$ such that $a^{\varphi} = a^{2}$, $b^{\varphi} = b^{2}$, $c^{\varphi} = c$, and $g^{\varphi} = a^{-1}bg$ we have $\{ag, a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cg, b^{-1}g, a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cg\}^{\varphi} = \{abg, a^{\varepsilon}b^{\varepsilon}cg, a^{-1}b^{-1}g, a^{-\varepsilon}b^{-\varepsilon}cg\}$), we have that $m \neq \frac{\varepsilon+1}{2}$. It follows that

$$a^{\alpha} = a^{i}b^{\varepsilon-i}$$
, $b^{\alpha} = a^{\varepsilon-i}b^{i}$, $c^{\alpha} = c$, and $g^{\alpha} = a^{m-i}b^{i-m}cg$,

where $i = \frac{m\varepsilon - m + 1}{2m - \varepsilon - 1}$. If on the other hand $m + n = -\varepsilon$ then, since *G* cannot be generated by

{
$$aq, a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cq, b^{-1}q, a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cq$$
},

we have that $n \neq -\frac{\varepsilon+1}{2}$, and so

$$a^{\alpha} = a^{i}b^{-\varepsilon-i}$$
, $b^{\alpha} = a^{-\varepsilon-i}b^{i}$, $c^{\alpha} = c$, and $g^{\alpha} = a^{-\varepsilon-i-n}b^{\varepsilon+i+n}cg$,

where $i = -\frac{(n+1)\varepsilon + n}{2n+\varepsilon+1}$.

Suppose now that $(ag)^{\alpha} = a^{-n}b^{-m}cg$. Then $(a^{-n}b^{-m}cg)^{\alpha} = b^{-1}g$, $(b^{-1}g)^{\alpha} = a^{m}b^{n}cg$, and $(a^{m}b^{n}cg)^{\alpha} = ag$. Also, either $m + n = \varepsilon$ or $m + n = -\varepsilon$, where $\varepsilon^{2} = -1$. If $m + n = \varepsilon$ then, since $\{ag, a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cg, b^{-1}g, a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cg\}$ cannot generate *G*, we have that $m \neq \frac{\varepsilon+1}{2}$, and thus

$$a^{\alpha} = a^{i}b^{-\varepsilon-i}, b^{\alpha} = a^{-\varepsilon-i}b^{i}, c^{\alpha} = c, \text{ and } g^{\alpha} = a^{m-\varepsilon-i}b^{\varepsilon+i-m}cg,$$

where $i = \frac{\varepsilon(1-m)-m}{2m-\varepsilon-1}$. If however $m + n = -\varepsilon$ then, since $\{ag, a^{\frac{1-\varepsilon}{2}}b^{-\frac{\varepsilon+1}{2}}cg, b^{-1}g, a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}cg\}$ cannot generate *G*, we have that $n \neq -\frac{\varepsilon+1}{2}$, and consequently

$$a^{\alpha} = a^{i}b^{\varepsilon-i}$$
, $b^{\alpha} = a^{\varepsilon-i}b^{i}$, $c^{\alpha} = c$, and $q^{\alpha} = a^{-i-n}b^{i+n}cq$,

where $i = \frac{n(\varepsilon-1)-1}{2n+\varepsilon+1}$.

We can conclude that either $S_0 = S_7 = \{ag, a^m b^{\varepsilon - m} cg, b^{-1}g, a^{m-\varepsilon} b^{-m} cg\}$, where $m \neq \frac{\varepsilon + 1}{2}$, or $S_0 = S_8 = \{ag, a^{-n-\varepsilon} b^n cg, b^{-1}g, a^{-n} b^{n+\varepsilon} cg\}$, where $n \neq -\frac{\varepsilon + 1}{2}$. Further, replacing -n with m, one can see that $S_8 = S_7$. That G is indeed generated by S_7 can be seen in the following way. Since $(ag)^2 = ab$ and $aga^m b^{\varepsilon - m} cg = a^{\varepsilon + 1 - m} b^m c$, we have that $c, a^{\varepsilon + 1 - m} b^m \in \langle S_7 \rangle$. Then, since $a^{\varepsilon + 1 - m} b^m = a^{\varepsilon + 1 - 2m} a^m b^m$, $m \neq \frac{\varepsilon + 1}{2}$, and $ab \in \langle S_7 \rangle$, we get that $a^{\varepsilon + 1 - 2m} \in \langle S_7 \rangle$. Finally, since $ag \in \langle S_7 \rangle$, it follows that also $g \in \langle S_7 \rangle$. Now the fact that $a^g = b$ implies that $G = \langle S_7 \rangle$.

Now considering the automorphism γ of *G* defined by

$$a^{\gamma} = a^{\frac{1}{2}}, b^{\gamma} = b^{\frac{1}{2}}, c^{\gamma} = c, \text{ and } g^{\gamma} = a^{\frac{1}{2}}b^{-\frac{1}{2}}g,$$

gives that $S_5^{\gamma} = \{ag, a^{\frac{m+1}{2}}b^{\frac{\omega}{2}-\frac{m+1}{2}}cg, b^{-1}g, a^{\frac{m+1}{2}-\frac{\omega}{2}}b^{-\frac{m+1}{2}}cg\}$, where $m \neq \frac{\omega}{2}$. So we only need to consider the generating set $S_7 = \{ag, a^m b^{\varepsilon-m}cg, b^{-1}g, a^{m-\varepsilon}b^{-m}cg\}$, where $m \neq \frac{\varepsilon+1}{2}$ and $\varepsilon^2 = -1$. Observe also, that this implies that $p \equiv 1 \pmod{4}$.

We have proved that when $\operatorname{Aut}(G, S_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ there always exists an automorphism σ of G such that $S_0^{\sigma} = S = \{ag, bcg, b^{-1}g, a^{-1}cg\}$. Moreover, $\operatorname{Aut}(G, S) = \langle \alpha, \beta \rangle$, where

$$a^{\alpha} = b, b^{\alpha} = a, c^{\alpha} = c, g^{\alpha} = cg a^{\beta} = b^{-1}, b^{\beta} = a^{-1}, c^{\beta} = c, \text{ and } g^{\beta} = g$$

One the other hand when $\operatorname{Aut}(G, S_0) \cong \mathbb{Z}_4$ there always exists an automorphism δ of G such that $S_0^{\delta} = S = \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\}$. Moreover, in this case $\operatorname{Aut}(G, S) = \langle \rho \rangle$, where

$$a^{\rho} = a^{\frac{\varepsilon-1}{2}}b^{\frac{\varepsilon+1}{2}}, b^{\rho} = a^{\frac{\varepsilon+1}{2}}b^{\frac{\varepsilon-1}{2}}, c^{\rho} = c, \text{ and } g^{\rho} = a^{\frac{1-\varepsilon}{2}}b^{\frac{\varepsilon-1}{2}}cg.$$

Observe also that the following hold:

- (1) If $\varepsilon^2 = -1$ then $\{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\}^{\tau} = \{ag, b^{-\varepsilon}cg, b^{-1}g, a^{\varepsilon}cg\}$, where τ is an automorphism of *G* mapping according to the rule $a^{\tau} = b^{-\varepsilon}, b^{\tau} = a^{-\varepsilon}, c^{\tau} = c$, and $g^{\tau} = cg$.
- (2) Since $agbcg = a^2c$, $(a^2c)^2 = a^4$, $(a^2c)^p = c$, $a^g = b$ and p is an odd prime, we can conclude that $\langle \{ag, bcg, b^{-1}g, a^{-1}cg\} \rangle = \langle ag, bcg \rangle = \langle a, b, c, g \rangle = G$.
- (3) Let $\varepsilon^2 = -1$. Then $agb^{\varepsilon}cg = a^{1+\varepsilon}c$, $(a^{1+\varepsilon}c)^2 = a^{2(1+\varepsilon)}$, and $(a^{1+\varepsilon}c)^p = c$. Since *p* is an odd prime and $a^g = b$, we can conclude that $\langle \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\} \rangle = \langle ag, b^{\varepsilon}cg \rangle = \langle a, b, c, g \rangle = G$.



Figure 2: A local structure of the graph $CN_{4n^2}^3$.

To finish the proof, it is sufficient to prove that the graphs

$$Cay(G, \{ag, bcg, b^{-1}g, a^{-1}cg\})$$
 and $Cay(G, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\})$

are normal Cayley graphs.

First, let $X = Cay(G, \{ag, bcg, b^{-1}g, a^{-1}cg\})$, let A = Aut(X) and let A_1^* be the subgroup of the stabilizer A_1 fixing the set $S = \{ag, bcg, b^{-1}g, a^{-1}cg\}$ pointwise. Then, since the 2-arc $(1, ag, a^{-1}bc)$ lies on a 6-cycle but the 2-arc (1, ag, ab) does not, one can see that A_1^* fixes every vertex at distance 2 from 1 in X (see also Figure 2). By connectivity of X and transitivity of A on V(X), A_1^* fixes every vertex in X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $Aut(G, S) = \mathbb{Z}_2^2 \leq A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_2^2, D_8, A_4, S_4\}$. If $A_1 \in \{A_4, S_4\}$ then there exists a permutation δ in A_1 of order 3. We can, without loss of generality, assume that δ fixes ag, and cyclically permutates the other three neighbors of 1. But, however, considering the images of the vertices at distance 2 from 1, one can see that this is impossible (see Figure 2). If $A_1 = D_8$ then we may, without loss of generality, assume that there exists an involution $\gamma \in A_1$ such that $\gamma \notin Aut(G, S)$, $(ag)^{\gamma} = ag$, $(b^{-1}g)^{\gamma} = b^{-1}g$, $(bcg)^{\gamma} = a^{-1}cg$ and $(a^{-1}cg)^{\gamma} = bcg$. However, ab is a common neighbor of ag and bcg in X, but there is no common neighbor of ag and $a^{-1}cg$, and thus this case cannot occur. It follows that $A_1 = Aut(G, S) = \mathbb{Z}_2^2$, and so X is a normal one-regular Cayley graph as claimed.

Now let $X = Cay(G, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\})$, let A = Aut(X) and let A_1^* be the subgroup of the stabilizer A_1 fixing S pointwise. Then considering 6-cycles passing through the vertex 1 one can see that A_1^* fixes all the vertices at distance 2 from 1 in X (see also Figure 3). Then, connectivity and vertex-transitivity of X combined together imply that A_1^* fixes every vertex of X and hence $A_1^* = 1$. It follows that $A_1 \cong A_1^S \leq S_4$. Since $Aut(G, S) \cong \mathbb{Z}_4 \leq A_1 \leq S_4$, we have that $A_1 \in \{\mathbb{Z}_4, D_8, S_4\}$. If $A_1 \in \{D_8, S_4\}$ then, without loss of generality, we may assume that there exists an involution $\zeta \in A_1$ such that $\zeta \notin Aut(G, S)$, $(ag)^{\zeta} = ag$, $(b^{-1}g)^{\zeta} = b^{-1}g, (b^{\varepsilon}cg)^{\zeta} = a^{-\varepsilon}cg$, and $(a^{-\varepsilon}cg)^{\zeta} = (b^{\varepsilon}cg)$. Since there is no 6-cycle passing through $b^{-1}g, 1, ag$ and ab, it follows that ζ fixes ab. On the other hand, since ζ normalizes a Sylow p-subgroup P of G ($P \leq A$, see Theorem 5.1), we have that $(xy)^{\zeta} = 1^{R(xy)\zeta} = 1^{\zeta^{-1}(R(x)R(y))\zeta} = 1^{R(x)^{\zeta}R(y)^{\zeta}} = R(x)^{\zeta}R(y)^{\zeta} = 1^{R(x)^{\zeta}}1^{R(y)^{\zeta}} = x^{\zeta}y^{\zeta}$, for every $x, y \in \langle a, b \rangle$. In other words, ζ induces an automorphism on $\langle a, b \rangle$. Thus, ζ fixes $\langle ab \rangle$ pointwise, and, in particular, ζ fixes both $a^{\varepsilon}b^{\varepsilon}$ and $a^{-\varepsilon}b^{-\varepsilon}$, a contradiction. This means that $A_1 = Aut(G, S) = \mathbb{Z}_4$, and thus X is a normal one-regular Cayley graph as claimed.



Figure 3: A local structure of the graph $CN_{4v^2}^4$.

Lemma 4.3. $C\mathcal{A}^1_{4p^2} \cong C\mathcal{N}^3_{4p^2}$.

Proof. Let $G_{4p^2}^1 = \langle a, b | a^{4p} = b^p = 1, ab = ba \rangle \cong \mathbb{Z}_{4p} \times \mathbb{Z}_p$ and let $G_{4p^2}^3 = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a \rangle$. Then the automorphism group of $CN_{4p^2}^3 = Cay(G_{4p^2}^3, \{ag, bcg, b^{-1}g, a^{-1}cg\})$, is equal to $Aut(CN_{4p^2}^3) = R(G_{4p^2}^3) \rtimes A_1 = R(G_{4p^2}^3) \rtimes \langle \alpha, \beta \rangle \cong G_{4p^2}^3 \rtimes \mathbb{Z}_2^2$, where $a^{\alpha} = b, b^{\alpha} = a, c^{\alpha} = c, g^{\alpha} = cg, a^{\beta} = b^{-1}, b^{\beta} = a^{-1}, c^{\beta} = c, g^{\beta} = g$.

Let $H = \langle R(ag)\alpha, R(b) \rangle$. Then it is easy to see that $H = \langle R(ag)\alpha \rangle \times \langle R(b) \rangle \cong G^1_{4p^2}$. Since $H_1 \leq A_1 = \langle \alpha, \beta \rangle \cong \mathbb{Z}^2_2$ and subgroups of order 4 in H are cyclic, we have that $H_1 < A_1$. Moreover, since $(R(ag)\alpha)^{2p}$ is a unique

element of order 2 in *H* and $1^{(R(ag)\alpha)^{2p}} \neq 1$, we have that $H_1 \notin \{\langle \alpha \rangle, \langle \beta \rangle, \langle \alpha \beta \rangle\}$. Thus $H_1 = 1$, that is, *H* is a regular subgroup of Aut($CN_{4p^2}^3$). Now Proposition 2.6 and Example 3.4 combined together imply that $C\mathcal{R}_{4p^2}^1 \cong CN_{4p^2}^3$. \Box

Lemma 4.4. $CN_{4p^2}^2 \cong CN_{4p^2}^4$.

Proof. Let $G_{4p^2}^2 = \langle a, b | a^{4p} = b^p = 1, a^{-1}ba = b^{\varepsilon}, \varepsilon^2 \equiv -1 \pmod{p}$, and let $G_{4p^2}^3 = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1, a^g = b, b^g = a$. Let 4^{-1} be the inverse of 4 in \mathbb{Z}_p and let $r = 4^{-1}(\varepsilon - 1)$. Observe that $8r(\varepsilon + 1) + 4 \equiv 0 \pmod{4p}$ and that $4r \neq (\varepsilon - 1)$ in \mathbb{Z}_{4p} .

Now define a map α from the vertex set of $CN_{4p^2}^4 = Cay(G_{4p^2}^3, \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\})$ to the vertex set of $CN_{4p^2}^2 = Cay(G_{4p^2}^2, \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\})$ in the following way:

 $\begin{array}{rcccc} a^i b^j & \mapsto & a^{4r(i-j)} b^{i+j} \\ a^i b^j c & \mapsto & a^{4r(i-j+\varepsilon+1)+2} b^{i+j} \\ a^i b^j g & \mapsto & a^{4r(j-i+1)+1} b^{i+j} \\ a^i b^j g c & \mapsto & a^{4r(j-i-\varepsilon)-1} b^{i+j} \end{array}$

where *c* and *g* are involutions in $G_{4n^2}^3$. Then

$$\begin{array}{lll} (a^{i}b^{j},ag\cdot a^{i}b^{j})^{\alpha} &=& (a^{i}b^{j},a^{j+1}b^{i}g)^{\alpha} = (a^{4r(i-j)}b^{i+j},a^{4r(i-j-1+1)+1}b^{i+j+1}) \\ &=& (a^{4r(i-j)}b^{i+j},a^{4r(i-j)+1}b^{i+j+1}) = (a^{4r(i-j)}b^{i+j},ab\cdot a^{4r(i-j)}b^{i+j}), \\ (a^{i}b^{j},b^{\varepsilon}cg\cdot a^{i}b^{j})^{\alpha} &=& (a^{i}b^{j},a^{j}b^{i+\varepsilon}gc)^{\alpha} = (a^{4r(i-j)}b^{i+j},a^{4r(i+\varepsilon-j-\varepsilon)-1}b^{i+j+\varepsilon}) \\ &=& (a^{4r(i-j)}b^{i+j},a^{4r(i-j)-1}b^{i+j+\varepsilon}) = (a^{4r(i-j)}b^{i+j},a^{-1}b^{\varepsilon}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^{i}b^{j},b^{-1}g\cdot a^{i}b^{j})^{\alpha} &=& (a^{i}b^{j},a^{j}b^{i-1}g)^{\alpha} = (a^{4r(i-j)}b^{i+j},a^{4r(i-1-j+1)+1}b^{i-1+j}) \\ &=& (a^{4r(i-j)}b^{i+j},a^{4r(i-j)+1}b^{i-1+j}) = (a^{4r(i-j)}b^{i+j},ab^{-1}\cdot a^{4r(i-j)}b^{i+j}), \\ (a^{i}b^{j},a^{-\varepsilon}cg\cdot a^{i}b^{j})^{\alpha} &=& (a^{i}b^{j},a^{j-\varepsilon}b^{i}gc)^{\alpha} = (a^{4r(i-j)}b^{i+j},a^{4r(i-j+\varepsilon-\varepsilon)-1}b^{i+j-\varepsilon}) \\ &=& (a^{4r(i-j)}b^{i+j},a^{4r(i-j)-1}b^{i+j-\varepsilon}) = (a^{4r(i-j)}b^{i+j},a^{-1}b^{-\varepsilon}\cdot a^{4r(i-j)}b^{i+j}). \end{array}$$

Similarly, it can be checked that for any edge $(u, s \cdot u)$, we have that $(u, s \cdot u)^{\alpha} = (v, \bar{s} \cdot v)$, where

$$u \in \{a^{i}b^{j}c, a^{i}b^{j}g, a^{i}b^{j}gc\},\$$

$$v \in \{a^{4r(i-j+\varepsilon+1)+2}b^{i+j}, a^{4r(j-i+1)+1}b^{i+j}, a^{4r(j-i-\varepsilon)-1}b^{i+j}\}\$$

$$s \in \{ag, b^{\varepsilon}cg, b^{-1}g, a^{-\varepsilon}cg\}, \text{ and }\$$

$$\bar{s} \in \{ab, a^{-1}b^{\varepsilon}, ab^{-1}, a^{-1}b^{-\varepsilon}\}.$$

From this it follows that α is an isomorphism from $CN_{4\nu^2}^2$ to $CN_{4\nu^2}^4$. The details are omitted. \Box

Lemma 4.5. The graphs $\mathcal{BW}_{12}(5,1,5)$, $\mathcal{GPS2}(4,3,(0\ 1):(1\ 2))$, $C\mathcal{A}_{4p^2}^i$, $i \in \{0,1\}$, $CN_{4p^2}^2$, $NC_{4p^2}^0$ and $NC_{4p^2}^1$, are pairwise non-isomorphic.

Proof. First, by the remark subsequent to Example 3.1, the graph $\mathcal{BW}_{12}(5, 1, 5)$ is not isomorphic to any of the other graphs listed in the lemma. Next, Example 3.2 shows that $\mathcal{GPS2}(4, 3, (0\,1) : (1\,2))$ is not isomorphic to any of the other graphs listed in the lemma. Then, since the automorphism group of $C\mathcal{A}_{4p^2}^0$ has a cyclic Sylow *p*-subgroup, $C\mathcal{A}_{4p^2}^0$ is not isomorphic to $C\mathcal{A}_{4p^2}^1$ and $CN_{4p^2}^2$. Also, Example 3.4 and Lemmas 4.3 and 4.4 combined together show that $C\mathcal{A}_{4p^2}^1$ and $C\mathcal{N}_{4p^2}^2$ are not isomorphic. Namely, the stabilizer of a vertex in $C\mathcal{A}_{4p^2}^1$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $C\mathcal{N}_{4p^2}^2$ is isomorphic to \mathbb{Z}_4^1 . Finally, since the automorphism groups of both $\mathcal{NC}_{4p^2}^0$ and $\mathcal{NC}_{4p^2}^1$ have a minimal normal Sylow *p*-subgroup and the automorphism groups of $C\mathcal{A}_{4p^2}^1$, $C\mathcal{N}_{4p^2}^2$, do not have a minimal normal Sylow *p*-subgroups, we have that

none of $NC_{4p^2}^0$ and $NC_{4p^2}^1$ is isomorphic to $C\mathcal{R}_{4p^2}^1$, $CN_{4p^2}^2$. Moreover, since the automorphism groups of both $NC_{4p^2}^0$ and $NC_{4p^2}^1$ have an elementary abelian Sylow *p*-subgroup and the automorphism group of $C\mathcal{R}_{4p^2}^0$ has a cyclic Sylow *p*-subgroup, which follows that none of $NC_{4p^2}^0$ and $NC_{4p^2}^1$ is isomorphic to $C\mathcal{R}_{4p^2}^0$. The result now follows from the fact that the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^1$ is isomorphic to \mathbb{Z}_2^1 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^1 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^1 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^1 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^1 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the stabilizer of a vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas the vertex in $NC_{4p^2}^0$ is isomorphic to \mathbb{Z}_2^2 whereas isomorphic isomorphic to $\mathbb{Z$

5. The classification

Х	V(X)	Aut(X)	References
$\mathcal{BW}_{12}(5,1,5)$	36	$G_{36} \rtimes \mathbb{Z}_2^2$	Example 3.1
<i>GPS</i> 2(4, 3, (0 1) : (1 2))	36	Aut(X) = 144	Example 3.2
$\mathcal{NC}^{0}_{4v^{2}}$	$4p^2, p > 7,$	given in	Lemma 2.11
1	$p \equiv \pm 1 \pmod{8}$	[23, Lemma 8.4]	
$\mathcal{NC}^1_{4p^2}$	$4p^2, p > 7,$	given in	Lemma 2.11
,	or $p \equiv 1 \text{ or } 3 \pmod{8}$	[23, Lemma 8.7]	
$C\mathcal{A}^0_{4p^2}$	$4p^2, p \equiv 1 \pmod{4}$	$(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) \rtimes \mathbb{Z}_4$	Example 3.3
$C\mathcal{A}^1_{4p^2}$	$4p^2, p > 2$	$(\mathbb{Z}_{4p} \times \mathbb{Z}_p) \rtimes \mathbb{Z}_2^2$	Example 3.4
$CN_{4p^2}^2$	$4p^2, p \equiv 1 \pmod{4}$	$G^3_{4p^2} \rtimes \mathbb{Z}_4$	Lemmas 4.2 and 3.6

Table 2: Tetravalent one-regular graphs of order $4p^2$.

We are now ready to state the main theorem of this paper.

Theorem 5.1. Let *p* be a prime. Then a tetravalent graph X of order $4p^2$ is one-regular if and only if it is isomorphic to one of the graphs listed in Table 2. Furthermore, all the graphs listed in Table 2 are pairwise non-isomorphic.

Proof. Let *X* be a tetravalent one-regular graph of order $4p^2$. Let A = Aut(X) and let A_v be the stabilizer of $v \in V(X)$ in *A*. By [39], there is no tetravalent one-regular graph of order 16, and $\mathcal{BW}_{12}(5, 1, 5)$, $\mathcal{GPS2}[4, 3, (0 1) : (1 2)]$ and $C\mathcal{A}^1_{36}$ are the only tetravalent one-regular graphs of order 36 (see also Examples 3.1, 3.2 and 3.4). Thus, we may assume that p > 3. Since *X* is one-regular we have that $|A| = 16p^2$, and thus *A* is a solvable group. Let *P* be a Sylow *p*-subgroup of *A*.

Claim: *P* is normal in *A*.

Since $|A| = 16p^2$ Sylow's theorems imply that the number of Sylow *p*-subgroups of *A* is equal to $|A : N_A(P)| = kp + 1$. In addition, this number divides 16. Hence, if p > 7 then we clearly have that *P* is normal in *A* as claimed. Now we will prove that *P* is normal in *A* also when $p \in \{5,7\}$.

Let $N = O_2(A)$ be the largest normal 2-subgroup of A. Suppose first that |N| = 16 and consider the quotient graph X_N . Then $N \le K$, where K is the kernel of A acting on $V(X_N)$, X_N is a symmetric graph of valency 2 or 4, and, by Proposition 2.8, A/K acts arc-transitively on X_N . But then $2 \mid |A/K|$, which is clearly impossible since $|A| = 16p^2$. Therefore $|N| \mid 8$. Now we distinguish three different cases depending on the order of N. Let T be a minimal normal subgroup of A.

Case 1. |N| = 1.

Then either $|T| = p^2$ or |T| = p. In the former case we have that T = P and thus $P \leq A$ as claimed. We may therefore assume that |T| = p. Let X_T be the quotient graph of X relative to the orbits of T, and let K be the kernel of A acting on $V(X_T)$. Then $T \leq K$ and A/K acts arc-transitively on X_T . If A/T is abelian then, since A/K is a quotient group of the group A/T, also A/K is abelian. But since A/K is vertex-transitive on X_T . Proposition 2.1 implies that it is regular on X_T , contradicting arc-transitivity of A/K on X_T . Thus A/T is a non-abelian group. Let $C = C_A(T)$. Then $T \leq C$ and, by Proposition 2.2, A/C is isomorphic to a subgroup of Aut(*T*) $\cong \mathbb{Z}_{p-1}$. It follows that *A*/*C* is abelian, and consequently *T* < *C*. Let *L*/*T* be a minimal normal subgroup of *A*/*T* contained in *C*/*T*. Then *L*/*T* $\cong \mathbb{Z}_p$, and therefore *P* = *L* \trianglelefteq *A*.

Case 2. |N| = 2.

Then $|T| \in \{p^2, p, 2\}$. If $|T| = p^2$ then $P \leq A$ as claimed. Suppose now that |T| = 2, and let $C = C_A(T)$. Then $T \leq C$ and, moreover, by Proposition 2.2, |A/C| = 1 which implies that T < C. Let L/T be a minimal normal subgroup of C/T. Then either $|L/T| = p^2$ or |L/T| = p. In the former case it follows that $|L| = 2p^2$, and consequently P char $L \leq A$, implying that $P \leq A$ as claimed. In the later case we have $L = \mathbb{Z}_2 \times \mathbb{Z}_p$. Suppose first that A/L is abelian and consider the quotient graph X_L of X relative to the orbits of L. Let K be the kernel of A acting on $V(X_L)$. Then $L \leq K$, A/K is a quotient group of A/L, and as such also abelian. But since A/K is vertex-transitive on X_L . Proposition 2.1 implies that A/K is regular on X_L , which is impossible since A/K acts arc-transitively on X_L . Thus, A/L is a non-abelian group. Let $C = C_A(L)$. Then $L \leq C$ and, by Proposition 2.2, $A/C \leq \operatorname{Aut}(L) \cong \mathbb{Z}_{p-1}$. It follows that A/C is abelian, and so L < C. Let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$ and thus $M \leq A$ and $|M| = 2p^2$. In addition, since P char $M \leq A$, we have that $P \leq A$ as claimed.

Assume now that |T| = p. Then an argument similar to the one used above shows that A/T is a nonabelian group. Let $C = C_A(T)$. Then, by Proposition 2.2, we have that $A/C \leq \operatorname{Aut}(T) \cong \mathbb{Z}_{p-1}$. Thus A/C is abelian, which implies that T < C. Let L/T be a minimal normal subgroup A/T contained in C/T. Then either $L/T \cong \mathbb{Z}_p$ or $L/T \cong \mathbb{Z}_2$. If $L/T \cong \mathbb{Z}_p$, then clearly $L = P \trianglelefteq A$. If however $L/T \cong \mathbb{Z}_2$, then $L \cong \mathbb{Z}_{2p}$ and, by Proposition 2.2, $A/C \leq \operatorname{Aut}(L) \cong \mathbb{Z}_{p-1}$ where $C = C_A(L)$. Hence A/C is abelian, and consequently L < C. Now let M/L be a minimal normal subgroup of A/L contained in C/L. Then $M/L \cong \mathbb{Z}_p$, and so $|M| = 2p^2$. But then P char $M \trianglelefteq A$, implying that $P \trianglelefteq A$ as claimed.

Case 3. $|N| \in \{4, 8\}$.

Then either $|A/N| = 2p^2$ or $|A/N| = 4p^2$. Clearly PN/N is a Sylow *p*-subgroup of A/N and by Sylow's theorems, $PN/N \leq A/N$. Moreover, $PN \leq A$. If |N| = 4 then for $p \in \{5,7\}$ we have that *P* is characteristic in *PN*, and hence normal in *A*. Also, if |N| = 8 and p = 5 then one can easily see that *P* is characteristic in *PN* and hence normal in *A*. Therefore we can now assume that |N| = 8 and p = 7. Then *N* is isomorphic to one of the following groups: D_8 , Q_8 (the quaternion group), \mathbb{Z}_8 , $\mathbb{Z}_4 \times \mathbb{Z}_2$ or \mathbb{Z}_2^3 . Let $C = C_A(N)$. By Proposition 2.2, we have that $A/C \leq \operatorname{Aut}(N)$. If $N \not\equiv \mathbb{Z}_2^3$ then $7 \nmid |\operatorname{Aut}(N)|$ and hence $7^2 \mid |C|$, which implies that $P \leq C$. It follows that *P* is characteristic in *PN* and hence normal in *A*. If however $N \cong \mathbb{Z}_2^3$ then $N \leq C$ and $\operatorname{Aut}(N) \cong \operatorname{PSL}(2,7)$. Observe that |A/N| = 98 and $A/C \leq \operatorname{Aut}(N) \cong \operatorname{PSL}(2,7)$. But $\operatorname{Aut}(N) = \operatorname{PSL}(2,7)$ has no subgroup of order 98 since $|\operatorname{PSL}(2,7)| = 168$, implying that $A/N \neq A/C$, and therefore N < C. Note also that |C| > 8, but $16 \nmid |C|$. Namely, if $16 \mid |C|$, the fact that A/K acts arc-transitively on X_C , where *K* is the kernel of *A* acting on $V(X_C)$, implies that $2 \mid |A/K|$. But this is impossible since $C \leq K$. Therefore $7 \mid |C|$. If $7^2 \nmid |C|$ then $|C| = 8 \cdot 7 = 56$. But then A/C is a group of order $|A/C| = 2 \cdot 7 = 14$ isomorphic to a subgroup of Aut(N) $\cong \operatorname{PSL}(2,7)$, which by Proposition 2.3 is impossible. Therefore $7^2 \mid |C|$, and consequently $P \leq C_A(N)$. It follows that *P* is characteristic in *PN*, and thus normal in *A*. This proves that *A* always has a normal Sylow *p*-subgroup as claimed.

Assume first that *P* is cyclic. Let X_P be the quotient graph of *X* relative to the orbits of *P* and let *K* be the kernel of *A* acting on $V(X_P)$. By Proposition 2.4, the orbits of *P* are of length p^2 . Thus $|V(X_P)| = 4$, $P \le K$ and A/K acts arc-transitively on X_P . By Proposition 2.8, we have that $X_P \cong C_4$ and hence $A/K \cong D_8$, forcing $|K| = 2p^2$. Since A/K is a quotient group of A/P, it follows that A/P is a non-abelian group. Moreover, $|K| = 2p^2$ and thus *K* is not semiregular on V(X). Then $K_v \cong \mathbb{Z}_2$ where $v \in V(X)$. By Proposition 2.2, $A/C \le \operatorname{Aut}(P) \cong \mathbb{Z}_{p(p-1)}$, where $C = C_A(P)$. Since A/P is not abelian, we have that *P* is a proper subgroup of *C*. If $C \cap K \neq P$ then $C \cap K = K$ ($|K| = 2p^2$). Since K_v is a Sylow 2-subgroup of *K*, K_v is characteristic in *K* and so normal in *A*, implying that $K_v = 1$, a contradiction. Thus, $C \cap K = P$ and $1 \neq C/P = C/(C \cap K) \cong CK/K \le A/K \cong D_8$. If $C/P \cong \mathbb{Z}_2$ then C/P is in the center of A/P and since $(A/P)/(C/P) \cong A/C$ is cyclic, A/P is abelian, a contradiction. It follows that $|C/P| \in \{4, 8\}$, and hence C/P has a characteristic subgroup of order 4, say H/P. Thus, $|H| = 4p^2$ and $H/P \le A/P$, implying that $H \le A$. In addition, since $H \le C = C_A(P)$, we have that *H* is abelian. Clearly, $|H_v| \in \{1, 2, 4\}$. First, suppose that $|H_v| = 4$. Then H_v is a Sylow 2-subgroup

of H, implying that H_v is characteristic in H. The normality of H in A implies that $H_v \leq A$, forcing $H_v = 1$, a contradiction. Second, suppose that $|H_v| = 2$, and let Q be a Sylow 2-subgroup of H. Then $Q \leq A$ and $Q_v = H_v$. Consider the quotient graph X_O of X relative to the orbits of Q. Since |Q| = 4 and $Q_v \cong \mathbb{Z}_2$, Proposition 2.8 implies that $X_Q \cong C_{2p^2}$ and hence $X \cong C_{2p^2}[2K_1]$, contradicting one-regularity of X. Thus, we have that $H_v = 1$, and since $|H| = 4p^2$, H is regular on V(X). It follows that X is a Cayley graph on an abelian group with a cyclic Sylow *p*-subgroup *P*. By elementary group theory, we know that up to isomorphism \mathbb{Z}_{4p^2} and $\mathbb{Z}_{2p^2} \times \mathbb{Z}_2$, where p > 3, are the only abelian groups with a cyclic Sylow *p*-subgroup. However, by Xu [41, Theorems 3], there is no tetravalent one-regular Cayley graph on \mathbb{Z}_{4p^2} , and so $H \cong \mathbb{Z}_{2p^2} \times \mathbb{Z}_2$. Proposition 2.6 and Example 3.3 combined together now imply that $X \cong C\mathcal{A}_{4n^2}^0$.

Now assume that *P* is elementary-abelian. Suppose first that *P* is a minimal normal subgroup of *A*, and consider the quotient graph X_P of X relative to the orbits of P. Let K be the kernel of A acting on $V(X_P)$. By Proposition 2.4, we have that the orbits of P are of length p^2 , and thus $|V(X_P)| = 4$. By Proposition 2.8, $X_P \cong C_4$, and hence $A/K \cong D_8$, forcing $|K| = 2p^2$ and thus $K_v = \mathbb{Z}_2$. Proposition 2.9 now implies that X is isomorphic to $C^{\pm 1}(p, 4, 2)$, $NC_{4p^2}^0$ or $NC_{4p^2}^1$. However, by Lemma 2.10, $C^{\pm 1}(p, 4, 2)$ is not one-regular whereas, by Lemma 2.11, $NC_{4p^2}^0$ and $NC_{4p^2}^1$ both are one-regular. Conditions on the prime *p* written in Table 2 follow from the definition of these graphs (see page 288).

Suppose now that P is not a minimal normal subgroup of A. Then a minimal normal subgroup N of Ais isomorphic to \mathbb{Z}_p . Let X_N be the quotient graph of X relative to the orbits of N and let K be the kernel of A acting on $V(X_N)$. Then $N \leq K$ and A/K is transitive on $V(X_N)$. Moreover, we have that $|V(X_N)| = 4p$. By Proposition 2.8, X_N is a cycle of length 4p, or N acts semiregularly on V(X), the quotient graph X_N is a tetravalent connected G/N-arc-transitive graph and X is a regular cover of X_N . If $X_N \cong C_{4p}$, and hence $A/K \cong D_{8p}$, then |K| = 2p and thus $K_v = \mathbb{Z}_2$. Applying Proposition 2.12 we get that X is either isomorphic to $C^{\pm 1}(p; 4p, 1)$ or to $C^{\pm \varepsilon}(p; 4p, 1)$. By Lemmas 3.5 and 3.6 and Example 3.4, these two graphs are both oneregular and they are, respectively, isomorphic to $C\mathcal{R}^0_{4p^2}$ and $C\mathcal{R}^1_{4p^2}$. If, however, X_N is a tetravalent connected G/N-symmetric graph, then, by Proposition 2.8, X is a covering graph of a symmetric graph of order 4p. By Proposition 2.13, there are six tetravalent symmetric graphs of order 4*p*: $K_{4,4}$, $C_{2p}[2K_1]$, $C\mathcal{A}_{4n}^0$, $C\mathcal{A}_{4n}^1$, C(2, p, 2)and g₂₈. But, since there is no tetravalent one-regular graph of order 16, the automorphism group of g₂₈ does not admit a one-regular subgroup, and since, by Lemma 4.1, there is no one-regular \mathbb{Z}_p -cover of C(2, p, 2), we only need to consider the covering graphs of $C_{2p}[2K_1]$, $C\mathcal{A}_{4p}^0$ and $C\mathcal{A}_{4p}^1$. Observe that in each of these three graphs a one-regular subgroup of automorphisms contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$. Let *H* be a one-regular subgroup of automorphisms of *X_N*. Since *X* is one-regular graph, *A* is the lift of *H*. Since *H* contains a normal regular subgroup isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_2$ also *A* contains a normal regular subgroup. Therefore X is a normal Cayley graph of order $4p^2$. Since $A/\mathbb{Z}_p \cong H$ and $\mathbb{Z}_{2p} \times \mathbb{Z}_2 \trianglelefteq H$, there exists a normal subgroup G of A such that $G/\mathbb{Z}_p \cong \mathbb{Z}_{2p} \times \mathbb{Z}_2$. The classification of groups of order $4p^2$, given in [5, 6], and a detail analysis of all these groups give that *G* is either isomorphic to $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$ or to $G = \langle a, b, c, g | a^p = b^p = c^2 = g^2 = [a, b] = [c, g] = [a, c] = [b, c] = 1$, $a^g = b$, $b^g = a \rangle \cong (\mathbb{Z}_p \times \mathbb{Z}_{2p}) \rtimes \mathbb{Z}_2$. However, by Proposition 2.7, there is no tetravalent one-regular graph on $\mathbb{Z}_{2p} \times \mathbb{Z}_{2p}$, whereas for the latter group, Lemmas 4.2, 4.3 and 4.4, combined together imply that X is either isomorphic to $C\mathcal{R}_{4v^2}^1$ or to $C\mathcal{N}_{4v^2}^2$. Since, by Lemma 4.3, graphs listed in Table 2 are pairwise non-isomorphic the proof is completed.

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