RESOLUTIONS OF STANDARD MODULES OVER KLR ALGEBRAS OF TYPE A

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Abstract. Khovanov-Lauda-Rouquier algebras R_θ of finite Lie type are affine quasihereditary with standard modules $\Delta(\pi)$ labeled by Kostant partitions π of θ. In type A, we construct explicit projective resolutions of standard modules $\Delta(\pi)$.

1. Introduction

Let $R_{\theta,\mathbb{F}}$ be a Khovanov-Lauda-Rouquier (KLR) algebra of finite Lie type over a field $\mathbb{F}$ corresponding to $\theta \in Q_+$. It is known that $R_{\theta,\mathbb{F}}$ is affine quasihereditary [**1**, **4**, **7**, **8**]. In particular, it has finite global dimension and comes with a family of *standard modules* $\{\Delta(\pi)_\mathbb{F} \mid \pi \in \mathrm{KP}(\theta)\}$ and *proper standard modules* $\{\bar\Delta(\pi)_\mathbb{F} \mid \pi \in \mathrm{KP}(\theta)\}$, where $\mathrm{KP}(\theta)$ denotes the set of the Kostant partitions of θ. The (proper) standard modules have well-understood formal characters. Moreover, $L(\pi)_\mathbb{F} := \mathrm{head}\,\Delta(\pi)_\mathbb{F} \cong \mathrm{head}\,\bar\Delta(\pi)_\mathbb{F}$ is irreducible, and $\{L(\pi)_\mathbb{F} \mid \pi \in \mathrm{KP}(\theta)\}$ is a complete set of irreducible $R_{\theta,\mathbb{F}}$-modules up to isomorphism and degree shift. Finally, the projective cover $P(\pi)_\mathbb{F}$ of $L(\pi)_\mathbb{F}$ has a finite Δ-filtration and the (graded) decomposition number $d^\mathbb{F}_{\sigma,\pi} := [\bar\Delta(\sigma)_\mathbb{F} : L(\pi)_\mathbb{F}]_q$ equals the (well-defined) multiplicity $(P(\pi)_\mathbb{F} : \Delta(\sigma)_\mathbb{F})_q$, see [**1**, Corollary 3.14].

The KLR algebra is defined over the integers, so we have a $\mathbb{Z}$-algebra $R_\theta = R_{\theta,\mathbb{Z}}$ with $R_{\theta,\mathbb{F}} = R_\theta \otimes_\mathbb{Z} \mathbb{F}$. The standard modules have natural integral forms $\Delta(\pi)$ with $\Delta(\pi)_\mathbb{F} \cong \Delta(\pi) \otimes_\mathbb{Z} \mathbb{F}$, see [**9**, §4.2]. Moreover, as illustrated in [**9**, §4], understanding p-torsion in the $\mathbb{Z}$-module $\mathrm{Ext}^m_{R_\theta}(\Delta(\pi), \Delta(\sigma))$ is relevant for comparing decomposition numbers $d^\mathbb{C}_{\sigma,\pi}$ and $d^\mathbb{F}_{\sigma,\pi}$ for a field $\mathbb{F}$ of characteristic p. This motivates our interest in $\mathrm{Ext}^m_{R_\theta}(\Delta(\pi), \Delta(\sigma))$ and projective resolutions of (integral forms of) standard modules.

The problem of constructing a projective resolution of $\Delta(\pi)$ reduces easily to the semicuspidal case. To be more precise, let us fix a convex total order $\preceq$ on the set Φ_+ of positive roots. A *Kostant partition* of θ is a sequence $\pi = (\beta_1^{m_1}, \dots, \beta_t^{m_t})$ such that $m_1, \dots, m_t \in \mathbb{Z}_{>0}$, $\beta_1 \succ \cdots \succ \beta_t$ are positive roots, and $m_1\beta_1 + \cdots + m_t\beta_t = \theta$. Then $\Delta(\pi) \cong \Delta(\beta_1^{m_1}) \circ \cdots \circ \Delta(\beta_t^{m_t})$ where '$\circ$' stands for induction product. Since induction product preserves projective modules, it is enough to resolve the standard modules of the form $\Delta(\beta_s^{m_s})$, which are exactly the *semicuspidal standard modules*. Moreover, the case where β_s is a simple root is easy: the algebra $R_{m\alpha_i}$ is the nil-Hecke algebra of rank m and the standard module $\Delta(\alpha_i^m)$ is the projective

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indecomposable module $P(\alpha_i^m) = R_{m\alpha_i} 1_{i^{(m)}}$ for an explicit primitive idempotent $1_{i^{(m)}} \in R_{m\alpha_i}$, see [**5**, §2.2].

Let us now specialize to Lie type A_∞. Then every non-simple positive root is of the form $\alpha = \alpha_a + \alpha_{a+1} + \cdots + \alpha_{b+1}$ for integers $a \le b$, and we will work with the lexicographic convex order on the positive roots. Let $\theta := m\alpha$. We consider the set of compositions

$$\Lambda := \{\lambda = (\lambda_a, \dots, \lambda_b) \mid 0 \le \lambda_a, \dots, \lambda_b \le m\}.$$

Let $\Lambda(n) := \{\lambda \in \Lambda \mid \lambda_a + \cdots + \lambda_b = n\}$. In Section 3, for each $\lambda \in \Lambda$, we define explicit idempotents $e_\lambda \in R_\theta$ as concatenations of 'divided power idempotents' of the form $1_{i^{(m)}}$ mentioned above. We define projective modules $P_\lambda := q^{s_\lambda} R_\theta e_\lambda$, where q^{s_λ} stands for the grading shift by an explicitly defined integer s_λ. For every $\mu \in \Lambda(n+1)$ and $\mu \in \Lambda(n)$ we then define explicit elements $d_n^{\mu,\lambda} \in e_\mu R_\theta e_\lambda$ so that

$$P_\mu \mapsto P_\lambda,\ xe_\mu \mapsto xe_\mu d_n^{\mu,\lambda}$$

is an R_θ-module homomorphism. Taking direct sum over all such gives us a homomorhism

$$d_n : P_{n+1} := \bigoplus_{\mu \in \Lambda(n+1)} P_\mu \longrightarrow P_n := \bigoplus_{\lambda \in \Lambda(n)} P_\lambda.$$

We note that $P_n = 0$ for $n > m(b-a+1)$ and that $d_n^{\mu,\lambda} = 0$ unless the compositions μ and λ differ in just one part. We also have a natural map

$$\mathsf{p} : P_0 = q^{s_0} R_\theta e_0 \longrightarrow \Delta(\alpha^m),\ xe_0 \mapsto xv_{\alpha^m},$$

where v_{α^m} is the standard generator of $\Delta(\alpha^m)$ of weight $a^m(a+1)^m \cdots (b+1)^m$, see (3.33).

Theorem A. *We have that*

$$0 \longrightarrow P_{m(b-a+1)} \longrightarrow \cdots \longrightarrow P_{n+1} \xrightarrow{d_n} P_n \longrightarrow \cdots \xrightarrow{d_0} P_0 \xrightarrow{\mathsf{p}} \Delta(\alpha^m) \longrightarrow 0$$

is a projective resolution of the standard module $\Delta(\alpha^m)$.

When $m = 1$ this is a version of the resolution constructed in [**1**, Theorem 4.12], and our resolution can be thought of as the 'thick calculus generalization' (cf. [**6**]) of that resolution.

For an arbitrary θ, we denote $\Delta := \bigoplus_{\pi \in \mathrm{KP}(\theta)} \Delta(\pi)$. In [**2**], we will use the resolution from Theorem A to describe the algebra $\mathrm{Ext}^*_{R_\theta}(\Delta, \Delta)$ in two special cases: (1) θ is a positive root in type A and (2) Lie type is A_2, i.e. θ is of the form $r\alpha_1 + s\alpha_2$.

We now describe the contents of the paper and the main idea of the proof. After reviewing KLR algebras and standard modules in Section 2, we introduce the necessary combinatorial notation and define the resolution $P_\bullet$ in Section 3.

In order to prove that $P_\bullet$ is a resolution of $\Delta(\alpha^m)$, we want to show that it is a direct summand of a resolution $Q_\bullet$ of $q^{m(m-1)/2}\Delta(\alpha)^{\circ m}$ introduced in §3.3. The resolution $Q_\bullet$ is obtained by taking the mth induced power of the resolution constructed in [**1**, Theorem 4.12].

To check that $P_\bullet$ is a direct summand of $Q_\bullet$, in §3.4 we construct what will end up being a pair of chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$ with $g \circ f = \mathrm{id}$. The main difficulty is to verify that f and g are indeed chain maps. This verification occupies Sections 4 and 5. Modulo the fact that f and g are chain maps, Theorem A is proved in §3.7.

2. Preliminaries

2.1. **Basic notation.** Throughout, we work over an arbitrary commutative unital ground $\Bbbk$ (since everything is defined over $\mathbb{Z}$, one could just consider the case $\Bbbk = \mathbb{Z}$).

For $r, s \in \mathbb{Z}$, we use the segment notation $[r, s] := \{t \in \mathbb{Z} \mid r \leq t \leq s\}$, $[r, s) := \{t \in \mathbb{Z} \mid r \leq t < s\}$, etc.

Let q be a variable, and $\mathbb{Z}((q))$ be the ring of Laurent series. For a non-negative integer n we define $[n] := (q^n - q^{-n})/(q - q^{-1})$ and $[n]^! := [1][2] \cdots [n]$.

We denote by $\mathfrak{S}_d$ the symmetric group on d letters. It is a Coxeter group with generators $\{s_r := (r, r+1) \mid 1 \leq r < d\}$ and the corresponding length function ℓ. The longest element of $\mathfrak{S}_d$ is denoted w_0 or $w_{0,d}$. An element $w \in \mathfrak{S}_d$ is called fully commutative if it is possible to go between any two reduced decompositions of w using only the relations of the form $s_r s_t = s_t s_r$ for $|r-t| > 1$. By definition, $\mathfrak{S}_d$ acts on $[1, d]$ on the left. For a set I the d-tuples from I^d are written as words $\boldsymbol{i} = i_1 \cdots i_d$. The group $\mathfrak{S}_d$ acts on I^d via place permutations as $w \cdot \boldsymbol{i} = i_{w^{-1}(1)} \cdots i_{w^{-1}(d)}$.

Given a composition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of d, we have the corresponding standard parabolic subgroup $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_k} \leq \mathfrak{S}_d$. For compositions λ and μ of d, we denote by $\mathscr{D}^\lambda$ (resp. ${}^\mu\mathscr{D}$, resp. ${}^\mu\mathscr{D}^\lambda$) the set of the shortest coset representatives for $\mathfrak{S}_d/\mathfrak{S}_\lambda$ (resp. $\mathfrak{S}_\mu\backslash\mathfrak{S}_d$, resp. $\mathfrak{S}_\mu\backslash\mathfrak{S}_d/\mathfrak{S}_\lambda$). The following is well-known and can be deduced for example from [**3**, Lemma 1.6]:

Lemma 2.1. *Let λ, μ be compositions of d and $w \in {}^\mu\mathscr{D}$. Then there exist unique elements $x \in {}^\mu\mathscr{D}^\lambda$ and $y \in \mathfrak{S}_\lambda$ such that $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$.*

2.2. **KLR Algebras.** From now on, we set $I := \mathbb{Z}$. If $i, j \in I$ with $|i - j| = 1$ we set $\varepsilon_{i,j} := j - i \in \{1, -1\}$. For $i, j \in I$, we set

$$\mathsf{c}_{i,j} := \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We identify I with the set of vertices of the Dynkin diagram of type A_∞ so that the numbers $\mathsf{c}_{i,j}$ are the entries of the corresponding Cartan matrix. The corresponding simple roots are denoted $\{\alpha_i \mid i \in I\}$ and set of the positive roots is $\Phi_+ := \{\alpha_r + \cdots + \alpha_s \mid r \leq s\}$. The root lattice is $Q := \bigoplus_{i \in I} \mathbb{Z} \cdot \alpha_i$, and we set $Q_+ := \{\sum_i m_i \alpha_i \in Q \mid m_i \in \mathbb{Z}_{\geq 0} \text{ for all } i\}$. For $\theta = \sum_i m_i \alpha_i \in Q_+$, we define its height $\mathrm{ht}(\theta) := \sum_i m_i$. For $\theta \in Q_+$ of height d, we define $I^\theta := \{\boldsymbol{i} = i_1 \cdots i_d \in I^d \mid \alpha_{i_1} + \cdots + \alpha_{i_d} = \theta\}$. If $\boldsymbol{i} \in I^\theta$ and $\boldsymbol{j} \in I^\eta$ then the concatenation of words $\boldsymbol{i}\boldsymbol{j}$ is an element of $I^{\theta+\eta}$.

Let $\theta \in Q_+$ be of height d. The *KLR algebra* [**5**, **10**] is the unital $\Bbbk$-algebra R_θ (with identity denoted 1_θ) with generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in I^\theta\} \cup \{y_1, \ldots, y_d\} \cup \{\psi_1, \ldots, \psi_{d-1}\}$$

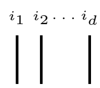
and defining relations

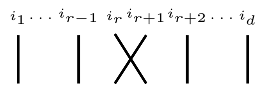
$$y_r y_s = y_s y_r;$$

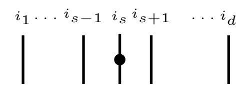
$$1_{\boldsymbol{i}}1_{\boldsymbol{j}} = \delta_{\boldsymbol{i},\boldsymbol{j}}1_{\boldsymbol{i}} \quad \text{and} \quad \sum_{\boldsymbol{i}\in I^\theta} 1_{\boldsymbol{i}} = 1_\theta;$$
$$y_r 1_{\boldsymbol{i}} = 1_{\boldsymbol{i}} y_r \quad \text{and} \quad \psi_r 1_{\boldsymbol{i}} = 1_{s_r\cdot \boldsymbol{i}}\psi_r;$$
$$(\psi_r y_t - y_{s_r(t)}\psi_r)1_{\boldsymbol{i}} = \delta_{i_r,i_{r+1}}(\delta_{t,r+1} - \delta_{t,r})1_{\boldsymbol{i}};$$
$$\psi_r^2 1_{\boldsymbol{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1},\\ \varepsilon_{i_r,i_{r+1}}(y_r - y_{r+1})1_{\boldsymbol{i}} & \text{if } |i_r - i_{r+1}| = 1,\\ 1_{\boldsymbol{i}} & \text{otherwise;}\end{cases}$$
$$\psi_r\psi_s = \psi_s\psi_r \text{ if } |r-s|>1;$$
$$(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)1_{\boldsymbol{i}} = \begin{cases} \varepsilon_{i_r,i_{r+1}}1_{\boldsymbol{i}} & \text{if } |i_r - i_{r+1}| = 1 \text{ and } i_r = i_{r+2},\\ 0 & \text{otherwise.}\end{cases}$$

The algebra R_θ is graded with $\deg 1_{\boldsymbol{i}} = 0$; $\deg(y_s) = 2$; $\deg(\psi_r 1_{\boldsymbol{i}}) = -\mathsf{c}_{i_r,i_{r+1}}$.

We will use the Khovanov-Lauda [**5**] diagrammatic notation for elements of R_θ. In particular, for $\boldsymbol{i} = i_1\cdots i_d \in I^\theta$, $1 \le r < d$ and $1 \le s \le d$, we denote

$1_{\boldsymbol{i}} =$ 

, $1_{\boldsymbol{i}}\psi_r =$ 

, $1_{\boldsymbol{i}}y_s =$ 

For each element $w \in \mathfrak{S}_n$, fix a reduced expression $w = s_{r_1}\cdots s_{r_l}$ which determines an element $\psi_w = \psi_{r_1}\cdots\psi_{r_l}$ (depending on the choice of a reduced expression).

Theorem 2.2. [**5**, Theorem 2.5], [**10**, Theorem 3.7] *Let $\theta \in Q_+$ and $d = \mathrm{ht}(\theta)$. Then the following sets are $\Bbbk$-bases of R_θ:*

$$\{\psi_w y_1^{k_1}\dots y_d^{k_d}1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d,\ k_1,\dots,k_d \in \mathbb{Z}_{\ge 0},\ \boldsymbol{i} \in I^\theta\},$$
$$\{y_1^{k_1}\dots y_d^{k_d}\psi_w 1_{\boldsymbol{i}} \mid w \in \mathfrak{S}_d,\ k_1,\dots,k_d \in \mathbb{Z}_{\ge 0},\ \boldsymbol{i} \in I^\theta\}.$$

2.3. **Parabolic subalgebras and divided power idempotents.** Let $\theta_1,\dots,\theta_t \in Q_+$ and set $\theta := \theta_1 + \dots + \theta_t$. Set

$$1_{\theta_1,\dots,\theta_t} := \sum_{\boldsymbol{i}^1\in I^{\theta_1},\dots,\boldsymbol{i}^t\in I^{\theta_t}} 1_{\boldsymbol{i}^1\dots\boldsymbol{i}^t} \in R_\theta.$$

Then we have an algebra embedding

$$\iota_{\theta_1,\dots,\theta_t} : R_{\theta_1}\otimes\dots\otimes R_{\theta_t} \hookrightarrow 1_{\theta_1,\dots,\theta_t}R_{\theta_1+\dots+\theta_t}1_{\theta_1,\dots,\theta_t} \tag{2.3}$$

obtained by horizontal concatenation of the Khovanov-Lauda diagrams. For $r_1 \in R_{\theta_1},\dots,r_t \in R_{\theta_t}$, when there is no confusion, we often write

$$r_1\circ\dots\circ r_t := \iota_{\theta_1,\dots,\theta_t}(r_1\otimes\dots\otimes r_t).$$

For example,

$$1_{\boldsymbol{i}^1}\circ\dots\circ 1_{\boldsymbol{i}^t} = 1_{\boldsymbol{i}^1\dots\boldsymbol{i}^t} \qquad (\boldsymbol{i}^1 \in I^{\theta_1},\dots,\boldsymbol{i}^t \in I^{\theta_t}). \tag{2.4}$$

We fix for the moment $i \in I$, $d \in \mathbb{Z}_{>0}$ and take $\theta = d\alpha_i$, in which case $R_{d\alpha_i}$ is isomorphic to the rank d nil-Hecke algebra $\mathcal{NH}_d$. Following [**5**], we consider the following elements of $R_{d\alpha_i}$:

$$y_{0,d} := \prod_{r=1}^{d} y_r^{r-1}, \quad 1_{i^{(d)}} := \psi_{w_{0,d}} y_{0,d}, \quad \text{and} \quad 1'_{i^{(d)}} := y_{0,d}\psi_{w_{0,d}}.$$

The following is well-known, see for example [**5**, §2.2]:

Lemma 2.5. *In $R_{d\alpha_i}$, the elements $1_{i^{(d)}}$ and $1'_{i^{(d)}}$ are idempotents. Moreover,*

(i) *$\psi_{w_{0,d}} f \psi_{w_{0,d}} = 0$ for any polynomial f in $y_1, \dots, y_d$ of degree less than $d(d-1)/2$.*
(ii) *$\psi_{w_{0,d}} y_{0,d} \psi_{w_{0,d}} = \psi_{w_{0,d}}$.*

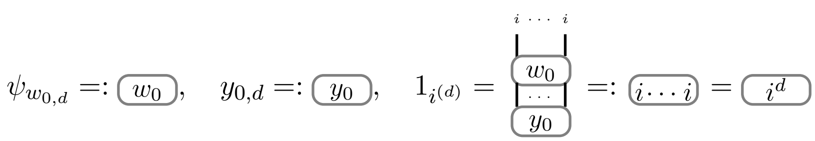
Now, let $\theta \in Q_+$ be arbitrary. We define I^θ_{div} to be the set of all expressions of the form $i_1^{(d_1)} \cdots i_r^{(d_r)}$ with $d_1, \dots, d_r \in \mathbb{Z}_{\geq 0}$, $i_1, \dots, i_r \in I$ and $d_1\alpha_{i_1} + \cdots + d_r\alpha_{i_r} = \theta$. We refer to such expressions as *divided power words*. We identify I^θ with the subset of I^θ_{div} which consists of all divided power words as above with all $d_k = 1$. We use the same notation for concatenation of divided power words as for concatenation of words. For $\boldsymbol{i} = i_1^{(d_1)} \cdots i_r^{(d_r)} \in I^\theta_{\mathrm{div}}$, we define the *divided power idempotent*

$$1_{\boldsymbol{i}} = 1_{i_1^{(d_1)} \cdots i_r^{(d_r)}} := 1_{i_1^{(d_1)}} \circ \cdots \circ 1_{i_r^{(d_r)}} \in R_\theta.$$

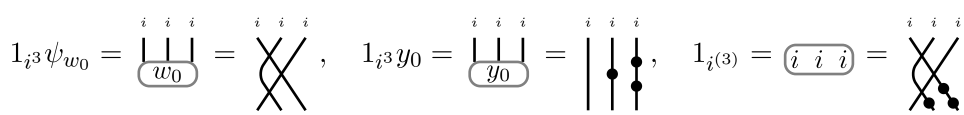
Then we have the following analogue of (2.4):

$$1_{\boldsymbol{i}^1} \circ \cdots \circ 1_{\boldsymbol{i}^t} = 1_{\boldsymbol{i}^1 \cdots \boldsymbol{i}^t} \qquad (\boldsymbol{i}^1 \in I^{\theta_1}_{\mathrm{div}}, \dots, \boldsymbol{i}^t \in I^{\theta_t}_{\mathrm{div}}). \tag{2.6}$$

To be used as part of the Khovanov-Lauda diagrammatics, we denote



For example, if $d = 3$, we have



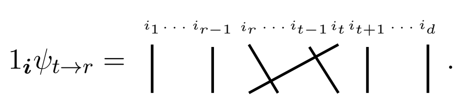
More generally, we denote

$$1_{i_1^{(d_1)} \cdots i_r^{(d_r)}} =: \boxed{i_1^{d_1}} \cdots \boxed{i_r^{d_r}}$$

For any $1 \leq r < t \leq d$, we denote the cycle $(t, t-1, \dots, r) \in \mathfrak{S}_d$ by $(t \to r)$. Note that $(t \to r) = s_{t-1}s_{t-2}\cdots s_r$ is a unique reduced decomposition. So

$$\psi_{t\to r} := \psi_{(t\to r)} = \psi_{t-1}\psi_{t-2}\dots\psi_r \in R_\theta, \tag{2.7}$$

In terms of the diagrammatic notation we have



Lemma 2.8. *In the algebra $R_{d\alpha_i}$, we have*

(i) *If $r_1 + \cdots + r_t = d$ then $1_{i^{(r_1)} \dots i^{(r_t)}} \psi_{w_{0,d}} = \psi_{w_{0,d}}$ and $1_{i^{(r_1)} \dots i^{(r_t)}} 1_{i^{(d)}} = 1_{i^{(d)}}$.*
(ii) *$1_{i^{(d)}} \psi_{d\to 1} 1_{i\, i^{(d-1)}} = \psi_{d\to 1} 1_{i\, i^{(d-1)}}$.*

Proof. (i) Write $\psi_{w_{0,d}} = (\psi_{w_{0,r_1}} \circ \cdots \circ \psi_{w_{0,r_t}})\psi_u$ for some $u \in \mathfrak{S}_d$ and use Lemma 2.5.

(ii) We have

$$\begin{aligned} 1_{i^{(d)}}\psi_{d-1}\psi_{d-2}\dots\psi_1 1_{i\,i^{(d-1)}} &= 1_{i^{(d)}}\psi_{d-1}\psi_{d-2}\dots\psi_1(1_{\alpha_i} \circ \psi_{w_{0,d-1}}y_{0,d-1}) \\ &= 1_{i^{(d)}}\psi_{w_{0,d}}(1_{\alpha_i} \circ y_{0,d-1}) \\ &= \psi_{w_{0,d}}(1_{\alpha_i} \circ y_{0,d-1}) \\ &= \psi_{d-1}\psi_{d-2}\dots\psi_1(1_{\alpha_i} \circ \psi_{w_{0,d-1}}y_{0,d-1}) \\ &= \psi_{d\to 1}1_{i\,i^{(d-1)}}, \end{aligned}$$

where we have used Lemma 2.5(ii) for the third equality. □

The following lemma easily follows from the defining relations of R_θ:

Lemma 2.9. *Let $1 \le r < s \le d$, $t \in (r,s)$, $u \in [r,s)$, and $\boldsymbol{i} \in I^\theta$. Then in R_θ we have:*

(i) $1_{\boldsymbol{i}}\psi_{s\to r}\psi_t = 1_{\boldsymbol{i}}\psi_{t-1}\psi_{s\to r}$ *unless* $i_s = i_{t-1} = i_t \pm 1$.
(ii) $1_{\boldsymbol{i}}\psi_{s\to r}y_{u+1} = 1_{\boldsymbol{i}}y_u\psi_{s\to r}$ *unless* $i_s = i_u$.

2.4. **Modules over R_θ.** Let $\theta \in Q_+$. We denote by R_θ-Mod the category of graded left R_θ-modules. The morphisms in this category are all homogeneous degree zero R_θ-homomorphisms, which we denote $\hom_{R_\theta}(-,-)$. For $V \in R_\theta$-Mod, let q^dV denote its grading shift by d, so if V_m is the degree m component of V, then $(q^dV)_m = V_{m-d}$. More generally, for a Laurent series $a = a(q) = \sum_n a_nq^n \in \mathbb{Z}[q,q^{-1}]$ with non-negative coefficients, we set $aV := \bigoplus_n (q^nV)^{\oplus a_n}$. For $U, V \in R_\theta$-Mod, we set $\mathrm{Hom}_{R_\theta}(U,V) := \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_{R_\theta}(U,V)_d$, where

$$\mathrm{Hom}_{R_\theta}(U,V)_d := \hom_{R_\theta}(q^dU,V) = \hom_{R_\theta}(U,q^{-d}V).$$

We define $\mathrm{Ext}^m_{R_\theta}(U,V)$ similarly in terms of $\mathrm{ext}^m_{R_\theta}(U,V)$.

For $\theta_1,\dots,\theta_t \in Q_+$ and $\theta := \theta_1 + \cdots + \theta_t$, recalling (2.3), we have a functor

$$\mathrm{Ind}_{\theta_1,\dots,\theta_t} = R_\theta 1_{\theta_1,\dots,\theta_t} \otimes_{R_{\theta_1}\otimes\cdots\otimes R_{\theta_t}} - : (R_{\theta_1} \otimes \cdots \otimes R_{\theta_t})\text{-Mod} \to R_\theta\text{-Mod}\,.$$

For $V_1 \in R_{\theta_1}$-Mod$,\dots, V_t \in R_{\theta_t}$-Mod, we denote by $V_1 \boxtimes \cdots \boxtimes V_t$ the $\Bbbk$-module $V_1 \otimes \cdots \otimes V_t$, considered naturally as an $(R_{\theta_1} \otimes \cdots \otimes R_{\theta_t})$-module, and set

$$V_1 \circ \cdots \circ V_t := \mathrm{Ind}_{\theta_1,\dots,\theta_t} V_1 \boxtimes \cdots \boxtimes V_t.$$

For $v_1 \in V_1, \dots, v_t \in V_t$, we denote

$$v_1 \circ \cdots \circ v_t := 1_{\theta_1,\dots,\theta_t} \otimes v_1 \otimes \cdots \otimes v_t \in V_1 \circ \cdots \circ V_t.$$

Since $R_\theta 1_{\theta_1,\dots,\theta_t}$ is a free $R_{\theta_1} \otimes \cdots \otimes R_{\theta_t}$-module of finite rank by Theorem 2.2, we get the following well-known properties:

Lemma 2.10. *The functor* $\mathrm{Ind}_{\theta_1,\dots,\theta_t}$ *is exact and sends finitely generated projectives to finitely generated projectives.*

The following lemma is easy to check using the fact that $1_{\boldsymbol{k}}$ is an idempotent in R_χ for $\boldsymbol{k} \in I^\chi_{\mathrm{div}}$:

Lemma 2.11. *For $\boldsymbol{i} \in I^\theta_{\mathrm{div}}$ and $\boldsymbol{j} \in I^\eta_{\mathrm{div}}$, we have $R_\theta 1_{\boldsymbol{i}} \circ R_\eta 1_{\boldsymbol{j}} \cong R_{\theta+\eta}1_{\boldsymbol{ij}}$.*

2.5. **Standard modules.** The algebra R_θ is affine quasihereditary in the sense of [**7**]. In particular, it comes with an important class of *standard modules*, which we now describe explicitly, referring to [**1**, §3]. We are going to work with the lexicographic convex order $\preceq$ on Φ_+, i.e. for $\alpha = \alpha_r + \cdots + \alpha_s \in \Phi_+$ and $\alpha' = \alpha_{r'} + \cdots + \alpha_{s'} \in \Phi_+$, we have $\alpha \prec \alpha'$ if and only if either $r < r'$ or $r = r'$ and $s < s'$.

Fix $\alpha = \alpha_r + \cdots + \alpha_s \in \Phi_+$ of height $l := s - r + 1$, and set

$$\boldsymbol{i}_\alpha := r\,(r+1)\cdots s \in I^\alpha.$$

We define the R_α-module $\Delta(\alpha)$ to be the cyclic R_α-module generated by a vector v_α of degree 0 with defining relations

- $1_{\boldsymbol{i}} v_\alpha = \delta_{\boldsymbol{i},\boldsymbol{i}_\alpha} v_\alpha$ for all $\boldsymbol{i} \in I^\alpha$;
- $\psi_t v_\alpha = 0$ for all $1 \le t < l$;
- $y_t v_\alpha = y_u v_\alpha$ for all $1 \le t, u \le l$.

By [**1**, Corollary 3.5], $\Delta(\alpha)$ is indeed the standard module corresponding to α. In fact, if $\Bbbk = \mathbb{Z}_p$, by [**9**, (4.11)], $\Delta(\alpha)$ is a universal $\Bbbk$-form of the standard module in the sense of [**9**, §4.2].

The module $\Delta(\alpha)$ can be considered as an $(R_\alpha, \Bbbk[x])$-bimodule with the right action given by $v_\alpha x := y_1 v_\alpha$. Then it is easy to check that there is an isomorphism of graded $\Bbbk$-modules $\Bbbk[x] \to \Delta(\alpha),\ f \mapsto v_\alpha f$.

Fix an integer $m \in \mathbb{Z}_{>0}$. Let $\mathcal{NH}_m$ be the rank m nil-Hecke algebra with standard generators $x_1, \dots, x_m, \tau_1, \dots, \tau_{m-1}$. For any $i \in I$, there is an isomorphism $\varphi : R_{m\alpha_i} \xrightarrow{\sim} \mathcal{NH}_m,\ y_t \mapsto x_t,\ \psi_u \mapsto \tau_u$, and we have the idempotent

$$e_m := \varphi(1'_{i^{(d)}}) \in \mathcal{NH}_m. \tag{2.12}$$

The $R_{m\alpha}$-module $\Delta(\alpha)^{\circ m}$ is cyclicly generated by $v_\alpha^{\circ m}$. As explained in [**1**, §3.2], $\mathcal{NH}_m$ acts on $\Delta(\alpha)^{\circ m}$ on the right so that

$$\begin{aligned} v_\alpha^{\circ m} x_t &= v_\alpha^{\circ(t-1)} \circ (v_\alpha x) \circ v_\alpha^{\circ(m-t)} \qquad (1 \le t \le m),\\ v_\alpha^{\circ m} \tau_u &= v_\alpha^{\circ(u-1)} \circ (\psi_z(v_\alpha \circ v_\alpha)) \circ v_\alpha^{\circ(m-u-1)} \qquad (1 \le u < m), \end{aligned}$$

where z is the longest element of $\mathscr{D}^{(l,l)}$.

Define

$$\Delta(\alpha^m) := q^{\binom{m}{2}} \Delta(\alpha)^{\circ m} e_m. \tag{2.13}$$

As in [**1**, Lemma 3.10], we have an isomorphism of $R_{m\alpha}$-modules

$$\Delta(\alpha)^{\circ m} \cong [m]^! \Delta(\alpha^m). \tag{2.14}$$

Given $\theta \in Q_+$ and a Kostant partition $\pi = (\beta_1^{m_1}, \dots, \beta_t^{m_t}) \in \mathrm{KP}(\theta)$ as in the Introduction, we define the corresponding *standard module*

$$\Delta(\pi) := \Delta(\beta_1^{m_1}) \circ \cdots \circ \Delta(\beta_t^{m_t}).$$

If $\Bbbk$ is a field, the modules $\{\Delta(\pi) \mid \pi \in \mathrm{KP}(\theta)\}$ are the standard modules for an affine quasihereditary structure on the algebra R_θ, see [**1**, **7**]. If $\Bbbk = \mathbb{Z}$ or $\mathbb{Z}_p$, they can be thought of as integral forms of the standard modules, see [**9**, §4].

3. Semicuspidal resolution

Throughout the subsection, we fix $m \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{Z}$ with $a \leq b$, and set

$$l := b + 2 - a, \quad d := lm.$$

We denote

$$\alpha := \alpha_a + \dots + \alpha_{b+1} \in \Phi_+$$

and $\theta := m\alpha$. Note that $l = \mathrm{ht}(\alpha)$ and $d = \mathrm{ht}(\theta)$. Our goal is to construct a resolution $P_\bullet = P_\bullet^{\alpha^m}$ of the semicuspidal standard module $\Delta(\alpha^m)$.

3.1. **Combinatorics.** We consider the set of compositions

$$\Lambda = \Lambda^{\alpha^m} := \{\lambda = (\lambda_a, \dots, \lambda_b) \mid \lambda_a, \dots, \lambda_b \in [0, m]\}.$$

For $\lambda \in \Lambda$, we denote $|\lambda| := \lambda_a + \dots + \lambda_b$, and for $n \in \mathbb{Z}_{\geq 0}$, we set

$$\Lambda(n) = \Lambda^{\alpha^m}(n) := \{\lambda \in \Lambda \mid |\lambda| = n\}.$$

Let $a \leq i \leq b$. We set

$$\mathsf{e}_i := (0, \dots, 0, 1, 0, \dots, 0) \in \Lambda(1),$$

with 1 in the ith position.

Let $\lambda \in \Lambda$. Set

$$s_\lambda := -\frac{lm(m-1)}{2} + (m+1)n - \sum_{i=a}^{b} \lambda_i^2 \in \mathbb{Z},$$
$$\boldsymbol{j}^\lambda := a^{m-\lambda_a}(a+1)^{m-\lambda_{a+1}} \cdots b^{m-\lambda_b}(b+1)^m b^{\lambda_b}(b-1)^{\lambda_{b-1}} \cdots a^{\lambda_a} \in I^\theta,$$
$$\boldsymbol{i}^\lambda := a^{(m-\lambda_a)}(a+1)^{(m-\lambda_{a+1})} \dots b^{(m-\lambda_b)}(b+1)^{(m)} b^{(\lambda_b)}(b-1)^{(\lambda_{b-1})} \dots a^{(\lambda_a)} \in I^\theta_{\mathrm{div}}.$$

We also associate to λ a composition ω_λ of d with $2n+1$ non-negative parts:

$$\omega_\lambda := (m - \lambda_a, m - \lambda_{a+1}, \dots, m - \lambda_b, m, \lambda_b, \lambda_{b-1}, \dots, \lambda_a).$$

Let $i \in [a, b]$. We denote

$$r_i^-(\lambda) := \sum_{s=a}^{i} (m - \lambda_s), \quad r_i^+(\lambda) := d - \sum_{s=a}^{i-1} \lambda_s, \quad l_i^\pm(\lambda) := r_{i\pm 1}^\pm(\lambda) + 1,$$

where $r_{a-1}^-(\lambda)$ is interpreted as 0, and $r_{b+1}^+(\lambda)$ is interpreted as $d - \sum_{s=a}^b \lambda_s$. Moreover, denote

$$r_{b+1}(\lambda) := d - \sum_{s=a}^{b} \lambda_s, \quad l_{b+1}(\lambda) := r_{b+1}(\lambda) - m + 1.$$

Define

$$U_i^\pm(\lambda) := [l_i^\pm(\lambda), r_i^\pm(\lambda)], \ U_i(\lambda) := U_i^-(\lambda) \sqcup U_i^+(\lambda), \ U_{b+1}(\lambda) \ := [l_{b+1}(\lambda), r_{b+1}(\lambda)].$$

Observe that for all $j \in [a, b+1]$, we have

$$U_j(\lambda) = \{s \in [1, d] \mid \boldsymbol{j}_s^\lambda = j\}.$$

We also consider the sets of multicompositions

$$\boldsymbol{\Lambda} := (\Lambda^\alpha)^m = \{\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \mid \delta^{(1)}, \dots, \delta^{(m)} \in \Lambda^\alpha\},$$
$$\boldsymbol{\Lambda}(n) := \{\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda} \mid |\delta^{(1)}| + \dots + |\delta^{(m)}| = n\}.$$

Note that by definition all $\delta_i^{(r)} \in \{0,1\}$. For $1 \le r \le m$ and $a \le i \le b$, we define $\mathbf{e}_i^r \in \boldsymbol{\Lambda}(1)$ to be the multicomposition whose rth component is $\mathtt{e}_i$ and whose other components are zero. For $\boldsymbol{\delta} \in \boldsymbol{\Lambda}$, denote

$$\lambda^{\boldsymbol{\delta}} := \delta^{(1)} + \cdots + \delta^{(m)} \in \Lambda.$$

Fix $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}$. Define

$$\boldsymbol{j}^{\boldsymbol{\delta}} := \boldsymbol{j}^{\delta^{(1)}} \dots \boldsymbol{j}^{\delta^{(m)}} \in I^\theta.$$

For $i \in [a,b]$ we define

$$U_i^{(\pm)}(\boldsymbol{\delta}) := \{l(r-1)+u \mid r \in [1,m],\, u \in U_i^{(\pm)}(\delta^{(r)})\}, \quad U_i(\boldsymbol{\delta}) := U_i^+(\boldsymbol{\delta}) \sqcup U_i^-(\boldsymbol{\delta}),$$

$$U_{b+1}(\boldsymbol{\delta}) := \{l(r-1)+u \mid r \in [1,m],\, u \in U_{b+1}(\delta^{(r)})\}.$$

Observe that for any $j \in [a,b+1]$, we have

$$U_j^{(\boldsymbol{\delta})} = \{s \in [1,d] \mid \boldsymbol{j}_s^{\boldsymbol{\delta}} = j\} \quad \text{and} \quad |U_j^{(\pm)}(\lambda^{\boldsymbol{\delta}})| = |U_j^{(\pm)}(\boldsymbol{\delta})|.$$

For $\lambda \in \Lambda$, $\boldsymbol{\delta} \in \boldsymbol{\Lambda}$, and $i \in [a,b]$, we define some signs:

$$\mathtt{sgn}_{\lambda;i} := (-1)^{\sum_{j=a}^{i-1}\lambda_j}, \qquad \mathtt{sgn}_{\boldsymbol{\delta};r,i} := (-1)^{\sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)}},$$

$$t_{\boldsymbol{\delta}} := \sum_{\substack{1\le r<s\le m\\ a\le j<i\le b}} \delta_i^{(r)}\delta_j^{(s)}, \qquad \sigma_{\boldsymbol{\delta}} := (-1)^{t_{\boldsymbol{\delta}}},$$

$$\tau_\lambda := (-1)^{\sum_{i=a}^{b}\lambda_i(\lambda_i-1)/2}, \qquad \tau_{\boldsymbol{\delta}} := \sigma_{\boldsymbol{\delta}}\tau_{\lambda^{\boldsymbol{\delta}}}.$$

($\tau_\lambda, \tau_{\boldsymbol{\delta}}$ are not to be confused with $\tau_w \in \mathcal{NH}_d$ which will not be used again).

Lemma 3.1. *Let $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}$, $i \in [a,b]$ and $r \in [1,m]$. If $\delta_i^{(r)} = 0$, then*

$$\mathtt{sgn}_{\lambda^{\boldsymbol{\delta}};i}\sigma_{\boldsymbol{\delta}} = \sigma_{\boldsymbol{\delta}+\mathbf{e}_i^r}\mathtt{sgn}_{\boldsymbol{\delta};r,i}(-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}.$$

Proof. Let $\lambda := \lambda^{\boldsymbol{\delta}}$ and $\boldsymbol{\gamma} := \boldsymbol{\delta} + \mathbf{e}_i^r$. Writing '$\equiv$' for '$\equiv \pmod 2$', we have to prove

$$\sum_{j=a}^{i-1}\lambda_j + \sum_{t<s,\ k>j}\delta_k^{(t)}\delta_j^{(s)} \equiv \sum_{t<s,\ k>j}\gamma_k^{(t)}\gamma_j^{(s)} + \sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)} + \sum_{s=1}^{r-1}\delta_i^{(s)}.$$

Note that

$$\sum_{t<s,\ k>j}\gamma_k^{(t)}\gamma_j^{(s)} = \sum_{t<s,\ k>j}\delta_k^{(t)}\delta_j^{(s)} + \sum_{s>r,\ j<i}\delta_j^{(s)} + \sum_{s<r,\ j>i}\delta_j^{(s)}, \tag{3.2}$$

so the required comparison boils down to

$$\sum_{j=a}^{i-1}\lambda_j \equiv \sum_{s>r,\ j<i}\delta_j^{(s)} + \sum_{s<r,\ j>i}\delta_j^{(s)} + \sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)} + \sum_{s=1}^{r-1}\delta_i^{(s)},$$

which is easy to see. □

Lemma 3.3. *Let $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(m)}) \in \boldsymbol{\Lambda}$, $i \in [a,b]$ and $r \in [1,m]$. If $\gamma_i^{(r)} = 1$, then*

$$\tau_{\boldsymbol{\gamma}}\mathtt{sgn}_{\lambda^{\boldsymbol{\gamma}};i} = \mathtt{sgn}_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i}\tau_{\boldsymbol{\gamma}-\mathbf{e}_i^r}(-1)^{\sum_{s=r+1}^{m}\gamma_i^{(s)}}.$$

Proof. Let $\mu := \lambda^{\boldsymbol{\gamma}}$, $\boldsymbol{\delta} := \boldsymbol{\gamma} - \mathbf{e}_i^r$, and $\lambda := \lambda^{\boldsymbol{\delta}}$. Writing '$\equiv$' for '$\equiv \pmod 2$', we have to prove the comparison

$$\sum_{t<s,\ k>j} \gamma_k^{(t)}\gamma_j^{(s)} + \sum_{j=a}^{b} \mu_j(\mu_j-1)/2 + \sum_{j=a}^{i-1}\mu_j$$
$$\equiv \sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)} + \sum_{t<s,\ k>j}\delta_k^{(t)}\delta_j^{(s)} + \sum_{j=a}^{b}\lambda_j(\lambda_j-1)/2 + \sum_{s=r+1}^{m}\gamma_i^{(s)}.$$

Note that

$$\sum_{j=a}^{b}\mu_j(\mu_j-1)/2 - \sum_{j=a}^{b}\lambda_j(\lambda_j-1)/2 = \lambda_i.$$

So, using also (3.2), the required comparison boils down to

$$\sum_{s>r,\ j<i}\delta_j^{(s)} + \sum_{s<r,\ j>i}\delta_j^{(s)} + \sum_{j=a}^{i}\lambda_j \equiv \sum_{s=1}^{r-1}|\delta^{(s)}| + \sum_{j=a}^{i-1}\delta_j^{(r)} + \sum_{s=r+1}^{m}\delta_i^{(s)},$$

which is easy to see. □

3.2. **The resolutions $P_\bullet^{\alpha^m}$ and $P_\bullet^{\pi}$.** Let $\lambda \in \Lambda$. Recalling the divided power word $\boldsymbol{i}^\lambda \in I^\theta_{\mathrm{div}}$ and the integer s_λ from §3.1, we set

$$e_\lambda := 1_{\boldsymbol{i}^\lambda} \in R_\theta \qquad \text{and} \qquad P_\lambda = P_\lambda^{\alpha^m} := q^{s_\lambda} R_\theta e_\lambda.$$

In particular, P_λ is a projective left R_θ-module. Note that $1_{\boldsymbol{j}^\lambda} e_\lambda = e_\lambda 1_{\boldsymbol{j}^\lambda} = e_\lambda$. Further, set for any $n \in \mathbb{Z}_{\geq 0}$:

$$P_n = P_n^{\alpha^m} := \bigoplus_{\lambda\in\Lambda(n)} P_\lambda.$$

Note that $P_n = 0$ for $n > d-m$. The projective resolution $P_\bullet = P_\bullet^{\alpha^m}$ of $\Delta(\alpha^m)$ will be of the form

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_n} P_n \longrightarrow \cdots \xrightarrow{d_0} P_0.$$

To describe the boundary maps d_n, we first consider a more general situation. Suppose we are given two sets of idempotents $\{e_a \mid a \in A\}$ and $\{f_b \mid b \in B\}$ in an algebra R. An $A \times B$ matrix $D := (d^{a,b})_{a\in A, b\in B}$ with every $d^{a,b} \in e_a R f_b$ then yields the homomorphism between the projective R-modules

$$\rho_D : \bigoplus_{a\in A} Re_a \to \bigoplus_{b\in B} Rf_b,\ (r_a e_a)_{a\in A} \mapsto \Big(\sum_{a\in A} r_a d^{a,b}\Big)_{b\in B}, \tag{3.4}$$

which we refer to as the *right multiplication with D*.

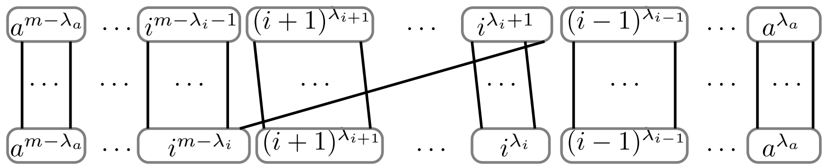
We now define a $\Lambda(n+1)\times\Lambda(n)$ matrix D_n with entries $d_n^{\mu,\lambda} \in e_\mu R_\theta e_\lambda$. Let $\lambda \in \Lambda(n)$ and $a \leq i \leq b$ be such that $\lambda_i < m$. Recalling (2.7), define

$$\psi_{\lambda;i} := \psi_{r_i^+(\lambda+\mathsf{e}_i)\to r_i^-(\lambda)}.$$

Note that $\psi_{\lambda;i} 1_{\boldsymbol{j}^\lambda} = 1_{\boldsymbol{j}^{\lambda+\mathsf{e}_i}}\psi_{\lambda;i}$. Recalling the sign $\mathtt{sgn}_{\lambda;i}$ from §3.1, we now set

$$d_n^{\mu,\lambda} := \begin{cases} \mathtt{sgn}_{\lambda;i} e_\mu \psi_{\lambda;i} e_\lambda & \text{if } \mu = \lambda + \mathsf{e}_i \text{ for some } a \leq i \leq b, \\ 0 & \text{otherwise.}\end{cases}$$

Diagrammatically, for $\mu = \lambda + \mathsf{e}_i$ as above, we have

$d_n^{\mu,\lambda} = \pm$ 

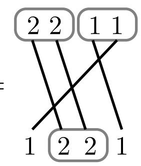
We now set the boundary map d_n to be the right multiplication with D_n:

$$d_n := \rho_{D_n}.$$

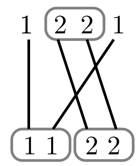
Example 3.5. Let $a = b = 1$ and $m = 2$. Then the resolution $P_\bullet$ is

$$0 \to R_\theta 1_{2^{(2)}1^{(2)}} \xrightarrow{d_1} R_\theta 1_{12^{(2)}1} \xrightarrow{d_0} q^{-2} R_\theta 1_{1^{(2)}2^{(2)}} \longrightarrow \Delta((\alpha_1+\alpha_2)^2) \longrightarrow 0,$$

where d_1 is a right multiplication with

$1_{2^{(2)}1^{(2)}}\psi_3\psi_2\psi_1 1_{12^{(2)}1} =$ 

and d_0 is a right multiplication with

$1_{12^{(2)}1}\psi_3\psi_2 1_{1^{(2)}2^{(2)}} =$ 

It is far from clear that $\ker d_n = \operatorname{im} d_{n+1}$ but at least the following is easy to see:

Lemma 3.6. *The homomorphisms d_n are homogeneous of degree 0 for all n.*

Proof. Let $\lambda \in \Lambda(n)$ be such that $\lambda_i < m$ for some $a \le i \le b$, so that $\mu := \lambda + \mathsf{e}_i \in \Lambda(n+1)$. The homomorphism $d_n = \rho_{D_n}$ is a right multiplication with the matrix D_n. Its (μ, λ)-component is a homomorphism $P_\mu \to P_\lambda$ obtained by the right multiplication with $\pm e_\mu \psi_{\lambda;i} e_\lambda$. Recall that $P_\mu = q^{s_\mu} R_\theta e_\mu$ and $P_\lambda = q^{s_\lambda} R_\theta e_\lambda$. So we just need to show that $s_\mu = s_\lambda + \deg(e_\mu \psi_{\lambda;i} e_\lambda)$. This is an easy computation using the fact that by definition we have $\deg(e_\mu \psi_{\lambda;i} e_\lambda) = m - 2\lambda_i$. $\square$

Let $\eta \in Q_+$ be arbitrary and $\pi = (\beta_1^{m_1}, \dots, \beta_t^{m_t}) \in \mathrm{KP}(\eta)$. We will define the resolution $P_\bullet^\pi$ of $\Delta(\pi) = \Delta(\beta_1^{m_1}) \circ \cdots \circ \Delta(\beta_t^{m_t})$ using the general notion of the induced product of chain complexes.

Let $(C_\bullet, d_C)$ and $(D_\bullet, d_D)$ be chain complexes of left $R_{\eta'}$-modules and $R_{\eta''}$-modules, respectively. We define a complex of $R_{\eta'+\eta''}$-modules

$$(C_\bullet \circ D_\bullet)_n := \bigoplus_{p+q=n} C_p \circ D_q \tag{3.7}$$

with differential given by

$$d_{C\circ D} : C_\bullet \circ D_\bullet \to C_\bullet \circ D_\bullet,\ x \circ y \mapsto x \circ d_D(y) + (-1)^q d_C(x) \circ y.$$

Lemma 3.8. *If $C_\bullet$ and $D_\bullet$ are projective resolutions of modules M and N, respectively, then $C_\bullet \circ D_\bullet$ is a projective resolution of $M \circ N$.*

Proof. This follows from Lemma 2.10 and [**11**, Lemma 2.7.3]. □

We now define

$$P_\bullet^\pi := P_\bullet^{\beta_1^{m_1}} \circ \cdots \circ P_\bullet^{\beta_t^{m_t}} \tag{3.9}$$

which, by Lemma 3.8, will turn out to be a projective resolution of $\Delta(\pi)$ in view of Theorem 3.34.

3.3. **The resolution $Q_\bullet$.** In order to check that $P_\bullet$ is a resolution of $\Delta(\alpha^m)$, we show that it is a direct summand of a known resolution $Q_\bullet$ of $q^{m(m-1)/2}\Delta(\alpha)^{\circ m}$. To describe the latter resolution, let us first consider the special case $m = 1$.

Lemma 3.10. *We have that $P_\bullet^\alpha$ is a resolution of $\Delta(\alpha)$.*

Proof. This is a special case of [**1**, Theorem 4.12], corresponding to the standard choice of $(\alpha_{i+1} + \cdots + \alpha_j, \alpha_i)$ as the minimal pair for an arbitrary positive root $\alpha_i + \cdots + \alpha_j$ in the definition of $\boldsymbol{i}_{\alpha,\sigma}$, see [**1**, §4.5]. □

Let $Q_\bullet$ be the resolution $q^{m(m-1)/2}(P_\bullet^\alpha)^{\circ m}$, cf. (3.7). To describe $Q_\bullet$ more explicitly, let $n \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{\delta} = (\delta^{(1)}, \ldots, \delta^{(m)}) \in \boldsymbol{\Lambda}(n)$. Recalling the definitions of §3.1, we set

$$e_{\boldsymbol{\delta}} := 1_{\boldsymbol{j}^{\boldsymbol{\delta}}} \in R_\theta, \quad Q_{\boldsymbol{\delta}} := q^{n+m(m-1)/2} R_\theta e_{\boldsymbol{\delta}}.$$

For example taking each $\delta^{(r)}$ to be $0 \in \Lambda(0)$, we get

$$\boldsymbol{\delta} = \mathbf{0} := (0, \ldots, 0) \in \boldsymbol{\Lambda}(0) \tag{3.11}$$

and $e_{\mathbf{0}} = 1_{(a\,(a+1)\cdots(b+1))^m}$. Further for any $n \in \mathbb{Z}_{\geq 0}$, we set

$$Q_n := \bigoplus_{\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)} Q_{\boldsymbol{\delta}}.$$

The projective resolution $Q_\bullet$ is

$$\cdots \longrightarrow Q_{n+1} \xrightarrow{c_n} Q_n \longrightarrow \cdots \xrightarrow{c_0} Q_0 \xrightarrow{\mathsf{q}} q^{m(m-1)/2}\Delta(\alpha)^{\circ m} \longrightarrow 0,$$

with the augmentation map

$$\mathsf{q} : Q_0 \to q^{m(m-1)/2}\Delta(\alpha)^{\circ m}, \ x e_{\mathbf{0}} \mapsto x v_\alpha^{\circ m}, \tag{3.12}$$

see §2.5, and c_n being right multiplication with the $\boldsymbol{\Lambda}(n+1) \times \boldsymbol{\Lambda}(n)$ matrix $C = (c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}})$ defined as follows. If $\boldsymbol{\delta} + \mathbf{e}_i^r \in \boldsymbol{\Lambda}$ for some $r \in [1, m]$ and $i \in [a, b]$, i.e. $\delta_i^{(r)} = 0$, we set

$$\psi_{\boldsymbol{\delta};r,i} := \iota_{\alpha,\ldots,\alpha}(1^{\otimes(r-1)} \otimes \psi_{\delta^{(r)};i} \otimes 1^{\otimes(m-r)}) = 1^{\circ(r-1)} \circ \psi_{\delta^{(r)};i} \circ 1^{\circ(m-r)}. \tag{3.13}$$

Recalling the signs defined in §3.1, for $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and $\boldsymbol{\gamma} \in \boldsymbol{\Lambda}(n+1)$, we now define

$$c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}} = \begin{cases} \mathtt{sgn}_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\gamma}} \psi_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\delta}} & \text{if } \boldsymbol{\gamma} = \boldsymbol{\delta} + \mathbf{e}_i^r \text{ for some } 1 \leq r \leq m \text{ and } a \leq i \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

The fact that $Q_\bullet$ is indeed isomorphic to the resolution $q^{m(m-1)/2}(P_\bullet^\alpha)^{\circ m}$ is easily checked using the isomorphism $R_\theta e_{\boldsymbol{\delta}} \cong R_\theta 1_{\boldsymbol{j}^{\delta^{(1)}}} \circ \cdots \circ R_\theta 1_{\boldsymbol{j}^{\delta^{(m)}}}$, which comes from Lemma 2.11.

Example 3.14. Let $a = b = 1$ and $m = 2$. Then the resolution $Q_\bullet$ is

$$0 \to q^3 R_\theta 1_{2121} \xrightarrow{c_1} q^2 R_\theta 1_{2112} \oplus q^2 R_\theta 1_{1221} \xrightarrow{c_0} q R_\theta 1_{1212} \xrightarrow{\mathsf{q}} q\Delta(\alpha_1 + \alpha_2)^{\circ 2} \longrightarrow 0,$$

where c_1 is a right multiplication with the matrix $(-1_{2121}\psi_3 \;\; 1_{2121}\psi_1)$, and c_0 is a right multiplication with the matrix $\begin{pmatrix} 1_{2112}\psi_1 \\ 1_{1221}\psi_3 \end{pmatrix}$.

3.4. **Comparison maps.** We now construct what will end up being a pair of chain maps $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$ with $g \circ f = \mathrm{id}$. As usual, f_n and g_n will be given as right multiplications with certain matrices F_n and G_n, respectively.

Let $\lambda \in \Lambda$. Recall the definitions of §3.1. We denote by w_0^λ the longest element of the parabolic subgroup $\mathfrak{S}_{\omega_\lambda} \leq \mathfrak{S}_d$. We also denote

$$y^\lambda := 1_{a^{m-\lambda_a} \cdots b^{m-\lambda_b}} \circ y_{0,m} \circ 1_{b^{\lambda_b} \cdots a^{\lambda_a}}.$$

Let $\boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}$. We define $u(\boldsymbol{\delta}) \in \mathfrak{S}_d$ as follows: for all $i = a, \dots, b$, the permutation $u(\boldsymbol{\delta})$ maps:

- the elements of $U_i^{\pm}(\lambda^{\boldsymbol{\delta}})$ increasingly to the elements of $U_i^{\pm}(\boldsymbol{\delta})$;
- the elements of $U_{b+1}(\lambda^{\boldsymbol{\delta}})$ increasingly to the elements of $U_{b+1}(\boldsymbol{\delta})$.

Set $w(\boldsymbol{\delta}) := u(\boldsymbol{\delta})^{-1}$. Then $w(\boldsymbol{\delta})$ can also be characterized as the element of $\mathfrak{S}_d$ which for all $i = a, \dots, b$, maps the elements of $U_i^{(\pm)}(\boldsymbol{\delta})$ increasingly to the elements of $U_i^{(\pm)}(\lambda^{\boldsymbol{\delta}})$ and the elements of $U_{b+1}(\boldsymbol{\delta})$ increasingly to the elements of $U_{b+1}(\lambda^{\boldsymbol{\delta}})$.

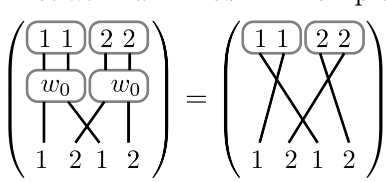
Recall the signs $\sigma_{\boldsymbol{\delta}}$ and $\tau_{\boldsymbol{\delta}}$ defined in §3.1. We now define F_n as the $\Lambda(n) \times \boldsymbol{\Lambda}(n)$-matrix with the entries $f_n^{\lambda,\boldsymbol{\delta}}$ defined for any $\lambda \in \Lambda(n), \boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ as follows:

$$f_n^{\lambda,\boldsymbol{\delta}} := \begin{cases} \sigma_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}} & \text{if } \lambda = \lambda^{\boldsymbol{\delta}}, \\ 0 & \text{otherwise.} \end{cases}$$

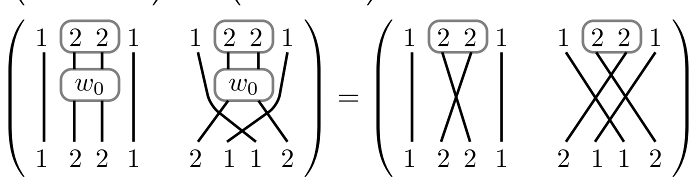
We define G_n as the $\boldsymbol{\Lambda}(n) \times \Lambda(n)$-matrix with the entries $g_n^{\boldsymbol{\delta},\lambda}$ defined for any $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n), \lambda \in \Lambda(n)$ as follows:

$$g_n^{\boldsymbol{\delta},\lambda} := \begin{cases} \tau_{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda & \text{if } \lambda = \lambda^{\boldsymbol{\delta}}, \\ 0 & \text{otherwise.} \end{cases}$$

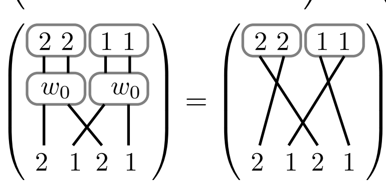
Example 3.15. Let $m = d = 2$ as in Examples 3.5 and 3.14. Then:

$F_0 =$ 

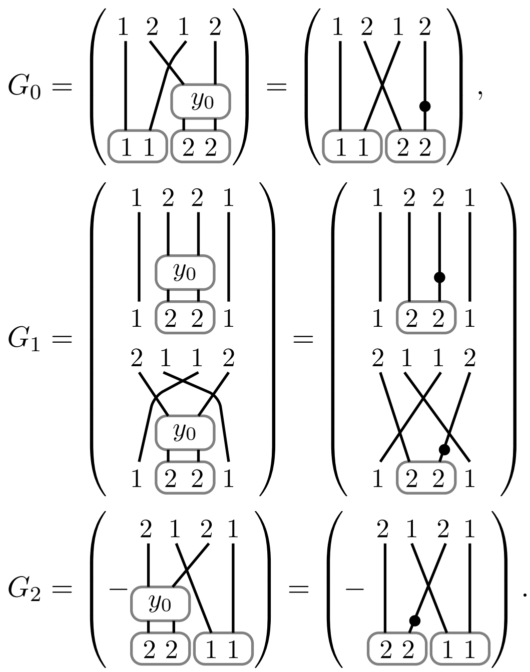
,

$F_1 =$ 

,

$F_2 =$ 

,



Lemma 3.16. *Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and $\lambda = \lambda^{\boldsymbol{\delta}}$. Then:*

(i) $\deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}) = \frac{m(m-1)(l-1)}{2} - mn + \sum_{i=a}^{b} \lambda_i^2$,
(ii) $\deg(f_n^{\lambda,\boldsymbol{\delta}}) = -\frac{m(m-1)(l+1)}{2} + mn - \sum_{i=a}^{b} \lambda_i^2$,
(iii) $\deg(g_n^{\boldsymbol{\delta},\lambda}) = \frac{m(m-1)(l+1)}{2} - mn + \sum_{i=a}^{b} \lambda_i^2$.

Proof. (i) We prove this by induction on m. Denote the right hand side by $R(m)$ and the left hand side by $L(m)$. If $m = 1$ then $w(\boldsymbol{\delta}) = 1$, so $L(1) = 0$. Moreover,

$$R(1) = -n + \sum_{i=a}^{b} \lambda_i^2 = -n + \sum_{i=a}^{b} (\delta_i^{(1)})^2 = -n + \sum_{i=a}^{b} \delta_i^{(1)} = 0.$$

Let $m > 1$. It suffices to prove that $R(m) - R(m-1) = L(m) - L(m-1)$. Let $\delta^{(m)} = (\varepsilon_a, \dots, \varepsilon_b)$. Then, since all ε_i are 0 or 1, we have

$$\begin{aligned} R(m) - R(m-1) &= \frac{m(m-1)(l-1)}{2} - mn + \sum_{i=a}^{b} \lambda_i^2 - \frac{(m-1)(m-2)(l-1)}{2} \\ &\quad + (m-1)(n - \sum_{i=a}^{b} \varepsilon_i) - \sum_{i=a}^{b} (\lambda_i - \varepsilon_i)^2 \\ &= (m-1)(l-1) - n - m \sum_{i=a}^{b} \varepsilon_i + 2 \sum_{i=a}^{b} \lambda_i \varepsilon_i. \end{aligned}$$

On the other hand, consider the Khovanov-Lauda diagram of $\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$. The bottom positions of the diagram correspond to to the letters if the word $\boldsymbol{j}^{\boldsymbol{\delta}}$, and so the rightmost l bottom positions of this diagram correspond to the letters of $\boldsymbol{j}^{\delta^{(m)}}$. In other words, counting from the right, the sequence of colors of these positions is $a^{\varepsilon_a}, \dots, b^{\varepsilon_b}, b+1, b^{1-\varepsilon_b}, \dots, a^{1-\varepsilon_a}$. Note that the strings which originate in these positions do not intersect each other, so $L(m)-L(m-1)$ equals the sum of the degrees of the intersections of these strings with the other strings of the diagram, i.e.

$$\begin{aligned} L(m)-L(m-1) = &\sum_{i=a+1}^{b} \varepsilon_i(\lambda_{i-1}-\varepsilon_{i-1}) + \lambda_b - \varepsilon_b \\ &+ \sum_{i=a+1}^{b} (1-\varepsilon_i)(m-1-2(\lambda_i-\varepsilon_i)+\lambda_{i-1}-\varepsilon_{i-1}) \\ &+ (1-\varepsilon_a)(m-1-2(\lambda_a-\varepsilon_a)), \end{aligned}$$

which is easily seen to equal the expression for $R(m)-R(m-1)$ obtained above.

(ii) This follows from (i) since $\deg(f_n^{\lambda,\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{d}}) + \deg(e_\lambda\psi_{w_0^\lambda})$ and

$$\deg(e_\lambda\psi_{w_0^\lambda}) = -m(m-1) - \sum_{i=a}^{b}(\lambda_i(\lambda_i-1)+(m-\lambda_i)(m-\lambda_i-1)).$$

(iii) This follows from (i) since

$$\deg(g_n^{\boldsymbol{\delta},\lambda}) = \deg(e_{\boldsymbol{\delta}}\psi_{u(\boldsymbol{\delta})}) + \deg(y^\lambda) = \deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}) + m(m-1).$$

□

Corollary 3.17. *The homomorphisms f_n and g_n are homogeneous of degree 0 for all n.*

Proof. Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and $\lambda = \lambda^{\boldsymbol{\delta}}$. The homomorphism f_n is a right multiplication with the matrix F_n. Its $(\lambda,\boldsymbol{\delta})$-component is a homomorphism $P_\lambda \to Q_{\boldsymbol{\delta}}$ obtained by the right multiplication with $f_n^{\lambda,\boldsymbol{\delta}}$. Recall that $P_\lambda = q^{s_\lambda}R_\theta e_\lambda$ and $Q_{\boldsymbol{\delta}} = q^{n+m(m-1)/2}R_\theta e_{\boldsymbol{\delta}}$. So we just need to show that $s_\lambda = n+m(m-1)/2+\deg(f_n^{\lambda,\boldsymbol{\delta}})$, which easily follows from Lemma 3.16(ii).

The homomorphism g_n is a right multiplication with the matrix G_n. Its $(\boldsymbol{\delta},\lambda)$-component is a homomorphism $Q_{\boldsymbol{\delta}} \to P_\lambda$ obtained by the right multiplication with $g_n^{\boldsymbol{\delta},\lambda}$. So we just need to show that $n+m(m-1)/2 = s_\lambda + \deg(g_n^{\boldsymbol{\delta},\lambda})$, which easily follows from Lemma 3.16(iii). □

Corollary 3.18. *Suppose $\boldsymbol{\delta},\boldsymbol{\varepsilon} \in \boldsymbol{\Lambda}(n)$ are such that $\lambda^{\boldsymbol{\delta}} = \lambda^{\boldsymbol{\varepsilon}}$. Then $\deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\varepsilon})}e_{\boldsymbol{\varepsilon}})$.*

3.5. **Independence of reduced decompositions.** Throughout this subsection we fix $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ and set $\lambda := \lambda^{\boldsymbol{\delta}}$.

Recall that in general the element $\psi_w \in R_\theta$ depends on a choice of a reduced decomposition of $w \in \mathfrak{S}_d$. While it is clear from the form of the braid relations in the KLR algebra that $e_\lambda\psi_{w_0^\lambda}$ does not depend on a choice of a reduced decomposition of w_0^λ, it is not obvious that a similar statement is true for $\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}$ and $e_{\boldsymbol{\delta}}\psi_{u(\boldsymbol{\delta})}$. So a priori the elements $f_n^{\lambda,\boldsymbol{\delta}} = \pm e_\lambda\psi_{w_0^\lambda}\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}$ and $g_n^{\boldsymbol{\delta},\lambda} = \pm e_{\boldsymbol{\delta}}\psi_{u(\boldsymbol{\delta})}y^\lambda e_\lambda$ might depend on choices of reduced decompositions of $w(\boldsymbol{\delta})$ and $u(\boldsymbol{\delta})$. In this subsection we will prove that this is not the case, and so in a sense the maps f_n and g_n are canonical.

Recall the composition ω_λ and the words $\boldsymbol{j}^\lambda, \boldsymbol{j}^{\boldsymbol{\delta}}$ from §3.1.

Lemma 3.19. *The element $w(\boldsymbol{\delta})$ is the unique element of ${}^{\omega_\lambda}\mathscr{D}^{(l^m)}$ with $w(\boldsymbol{\delta})\cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$.*

Proof. That $w(\boldsymbol{\delta}) \in {}^{\omega_\lambda}\mathscr{D}^{(l^m)}$ and $w(\boldsymbol{\delta}) \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$ follows from the definitions. To prove the uniqueness statement, let $w \in {}^{\omega_\lambda}\mathscr{D}^{(l^m)}$ and $w\cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$. Since $w \in {}^{\omega_\lambda}\mathscr{D}$, it maps the elements of $U_{b+1}(\boldsymbol{\delta})$ increasingly to the elements of $U_{b+1}(\lambda)$. By definition, $\boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^{\delta^{(1)}} \dots \boldsymbol{j}^{\delta^{(m)}}$. For every $r \in [1,m]$, the entries of $\boldsymbol{j}^{\delta^{(r)}}$ have the following properties: (1) each $i \in [a, b+1]$ appears among them exactly once; (2) the entries that precede $b+1$ appear in the increasing order; (3) the entries that succeed $b+1$ appear in the decreasing order. Since $w \in \mathscr{D}^{(l^m)}$, it maps the positions corresponding to the entries in (2) to the positions which are to the left of the positions occupied with $b+1$ in $\boldsymbol{j}^\lambda$, and it maps the positions corresponding to the entries in (3) to the positions which are to the right of the positions occupied with $b+1$ in $\boldsymbol{j}^\lambda$. In other words, for all $i \in [a, b]$, the permutation w maps the elements of $U_i^\pm(\boldsymbol{\delta})$ to the elements of $U_i^\pm(\lambda)$. As $w \in {}^{\omega_\lambda}\mathscr{D}$, it now follows that for every $i \in [a, b]$, the permutation w maps the elements of $U_i^\pm(\boldsymbol{\delta})$ to the elements of $U_i^\pm(\lambda)$ increasingly. We have shown that $w = w(\boldsymbol{\delta})$. □

Lemma 3.20. *Let $\varepsilon, \delta \in \Lambda^\alpha$, and $\boldsymbol{j}^\varepsilon = w \cdot \boldsymbol{j}^\delta$ for some $w \in \mathfrak{S}_l$. Then either $\varepsilon = \delta$ and $w = 1$, or* $\deg(\psi_w 1_{\boldsymbol{j}^\delta}) > 0$.

Proof. Since every $i \in [a, b+1]$ appears in $\boldsymbol{j}^\delta$ exactly once, $\varepsilon = \delta$ implies $w = 1$. On the other hand, if $\varepsilon \neq \delta$, let i be maximal with $\varepsilon_i \neq \delta_i$. Then the strings colored i and $i+1$ in the Khovanov-Lauda diagram D for $\psi_w 1_{\boldsymbol{j}^\delta}$ intersect (for any choice of a reduced decomposition of w), which contributes a degree 1 crossing into D. On the other hand, since every $j \in [a, b+1]$ appears in $\boldsymbol{j}^\delta$ exactly once, D has no same color crossings, which are the only possible crossings of negative degree. The lemma follows. □

Lemma 3.21. *Suppose that $w \in {}^{(l^m)}\mathscr{D}^{\omega_\lambda}$ and $w\cdot \boldsymbol{j}^\lambda$ is of the form $\boldsymbol{i}^{(1)} \dots \boldsymbol{i}^{(m)}$ with $\boldsymbol{i}^{(1)}, \dots, \boldsymbol{i}^{(m)} \in I^\alpha$. Then $w \cdot \boldsymbol{j}^\lambda = \boldsymbol{j}^{\boldsymbol{\varepsilon}}$ for some $\boldsymbol{\varepsilon} \in \boldsymbol{\Lambda}$ with $\lambda^{\boldsymbol{\varepsilon}} = \lambda$.*

Proof. Ler $r \in [1, m]$. By assumption, the entries $i_1^{(r)}, \dots, i_l^{(r)}$ of $\boldsymbol{i}^{(r)}$ have the following properties: (1) each $i \in [a, b+1]$ appears among them exactly once; (2) the entries that precede $b+1$ appear in the increasing order; (3) the entries that succeed $b+1$ appear in the decreasing order. The result follows. □

Lemma 3.22. *Let $P = \{w \in {}^{\omega_\lambda}\mathscr{D} \mid w \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda\}$. Then $w(\boldsymbol{\delta}) \in P$ and* $\deg(\psi_{w(\boldsymbol{\delta})}e_{\boldsymbol{\delta}}) < \deg(\psi_w e_{\boldsymbol{\delta}})$ *for any $w \in P \setminus \{w(\boldsymbol{\delta})\}$.*

Proof. It is clear that $w(\boldsymbol{\delta}) \in P$. On the other hand, by Lemma 2.1, an arbitrary $w \in P$ can be written uniquely in the form $w = xy$ with $x \in {}^{\omega_\lambda}\mathscr{D}^{(l^m)}$, $y \in \mathfrak{S}_{(l^m)}$ and $\ell(xy) = \ell(x) + \ell(y)$.

Since $xy \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$, we have $y \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = x^{-1} \cdot \boldsymbol{j}^\lambda$. As $x^{-1} \in {}^{(l^m)}\mathscr{D}^{\omega_\lambda}$, it follows from Lemma 3.21, that $y \cdot \boldsymbol{j}^{\boldsymbol{\delta}} = x^{-1} \cdot \boldsymbol{j}^\lambda$ is of the form $\boldsymbol{j}^{\boldsymbol{\varepsilon}}$ for some $\boldsymbol{\varepsilon} \in \boldsymbol{\Lambda}$ with $\lambda^{\boldsymbol{\varepsilon}} = \lambda$. By Lemma 3.19, $x = w(\boldsymbol{\varepsilon})$. If $w \neq w(\boldsymbol{\delta})$, then $y \neq 1$ and we have

$$\deg(\psi_w e_{\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\varepsilon})} e_{\boldsymbol{\varepsilon}}) + \deg(\psi_y e_{\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}) + \deg(\psi_y e_{\boldsymbol{\delta}}) > \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}),$$

where we have used Corollary 3.18 for the second equality and Lemma 3.20 for the inequality. □

Lemma 3.23. *The element* $f_n^{\lambda,\boldsymbol{\delta}} = \sigma_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ *is independent of the choice of reduced expressions for* w_0^λ *and* $w(\boldsymbol{\delta})$.

Proof. It is clear from the form of the braid relations in the KLR algebra that $e_\lambda \psi_{w_0^\lambda}$ is independent of the choice of a reduced expression for w_0^λ. On the other hand, if $\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ and $\psi'_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ correspond to different reduced expressions of $w(\boldsymbol{\delta})$, it follows from the defining relations of the KLR algebra and Theorem 2.2 that $\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}} - \psi'_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}}$ is a linear combination of elements of the form $\psi_u y e_{\boldsymbol{\delta}}$ with $u \in \mathfrak{S}_d$, $y \in \Bbbk[y_1, \dots, y_d]$ such that, $\deg(\psi_u y e_{\boldsymbol{\delta}}) = \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}})$ and $u\boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$. We have to prove $e_\lambda \psi_{w_0^\lambda} \psi_u y e_{\boldsymbol{\delta}} = 0$. Suppose otherwise.

Since we are using any preferred reduced decompositions for u, we may assume in addition that $u \in {}^{\omega_\lambda}\mathscr{D}$, since otherwise $e_\lambda \psi_{w_0^\lambda} \psi_u = 0$. Now by Lemma 3.22, $\deg(\psi_u e_{\boldsymbol{\delta}}) > \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}})$, whence $\deg(\psi_u y e_{\boldsymbol{\delta}}) > \deg(\psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}})$, giving a contradiction. □

Lemma 3.24. *The element* $g_n^{\boldsymbol{\delta},\lambda} = \tau_{\boldsymbol{\delta}} e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda$ *is independent of the choice of a reduced expression for* $u(\boldsymbol{\delta})$.

Proof. The argument is similar to that of the previous lemma. If $e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})}$ and $e_{\boldsymbol{\delta}} \psi'_{u(\boldsymbol{\delta})}$ correspond to different reduced expressions of $u(\boldsymbol{\delta})$, then $e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})} - e_{\boldsymbol{\delta}} \psi'_{u(\boldsymbol{\delta})}$ is a linear combination of elements of the form $e_{\boldsymbol{\delta}} y \psi_w$ with $w \in \mathfrak{S}_d$, $y \in \Bbbk[y_1, \dots, y_d]$ such that $\deg(e_{\boldsymbol{\delta}} y \psi_w) = \deg(e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})})$ and $w^{-1}\boldsymbol{j}^{\boldsymbol{\delta}} = \boldsymbol{j}^\lambda$. Moreover, in the KL dialgram of $\psi_w 1_{\boldsymbol{j}^\lambda}$ the strings colored $b+1$ do not cross each other, since this was the case for the KL diagram of $\psi_{u(\boldsymbol{\delta})} 1_{\boldsymbol{j}^\lambda}$. Hence, if $e_{\boldsymbol{\delta}} y \psi_w y^\lambda e_\lambda \neq 0$, we may assume in addition that $w^{-1} \in {}^{\omega_\lambda}\mathscr{D}$. Now, using Lemma 3.22, we conclude that $\deg(e_{\boldsymbol{\delta}} y \psi_w) > \deg(e_{\boldsymbol{\delta}} \psi_{u(\boldsymbol{\delta})})$, getting a contradiction. □

3.6. **Splitting.** In this subsection, we aim to show that $g \circ f = \mathrm{id}$. We fix $\lambda \in \Lambda(n)$ throughout the subsection. We need to prove $\sum_{\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)} f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = e_\lambda$. Since $f_n^{\lambda,\boldsymbol{\delta}} = 0$ unless $\lambda^{\boldsymbol{\delta}} = \lambda$, this is equivalent to

$$\sum_{\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n),\ \lambda^{\boldsymbol{\delta}} = \lambda} f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = e_\lambda.$$

Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ with $\lambda^{\boldsymbol{\delta}} = \lambda$. We say that $\boldsymbol{\delta}$ is *initial* if a preceeds $a+1$ in $\boldsymbol{j}^{\delta^{(r)}}$ for $r \in [1, m - \lambda_a]$ and a succeeds $a+1$ in $\boldsymbol{j}^{\delta^{(r)}}$ for $r \in (m - \lambda_a, m]$. In other words, $\boldsymbol{\delta}$ is initial if $\delta_a^{(r)} = 0$ for $r \in [1, m - \lambda_a]$ and $\delta_a^{(r)} = 1$ for $r \in (m - \lambda_a, m]$.

Let $w \in \mathfrak{S}_d$ and $1 \leq r, s \leq d$. We say that (r, s) is an *inversion pair for* w if $r < s$, $w(r) > w(s)$, and $\boldsymbol{j}_s^\lambda - \boldsymbol{j}_r^\lambda = \pm 1$.

Lemma 3.25. *Let* $\boldsymbol{\delta} \in \boldsymbol{\Lambda}(n)$ *be initial with* $\lambda^{\boldsymbol{\delta}} = \lambda$. *Set* $\bar{\alpha} = \alpha_{a+1} + \cdots + \alpha_b$, $\bar{\theta} = m\bar{\alpha}$, $\bar{\lambda} = (\lambda_{a+1}, \dots, \lambda_b)$, $\bar{n} := \lambda_{a+1} + \cdots + \lambda_b$, *and* $\bar{\boldsymbol{\delta}} = (\bar{\delta}^{(1)}, \dots, \bar{\delta}^{(m)})$, *where* $\bar{\delta}^{(r)} = (\delta_{a+1}^{(r)}, \dots, \delta_b^{(r)})$ *for all* $r \in [1, m]$. *Then*

$$f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = 1_{a^{(m-\lambda_a)}} \circ f_{\bar{n}}^{\bar{\lambda},\bar{\boldsymbol{\delta}}} g_{\bar{n}}^{\bar{\boldsymbol{\delta}},\bar{\lambda}} \circ 1_{a^{(\lambda_a)}}. \tag{3.26}$$

Proof. By definition,

$$f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = (-1)^{\sum_{i=a}^b \lambda_i(\lambda_i - 1)/2} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda.$$

Throughout the proof, 'inversion pair' means 'inversion pair for $w(\boldsymbol{\delta})$'. Recall that $w(\boldsymbol{\delta}) = u(\boldsymbol{\delta})^{-1}$. Since $\boldsymbol{\delta}$ is initial, in the Khovanov-Lauda diagram for $\psi_{w(\boldsymbol{\delta})}$ (for any choice of reduced expression) no strings of color a cross each other. We want to apply quadratic relations on pairs of strings one of which has color a and the other has color $a+1$. These correspond to inversion pairs (r, s) with $r \in U_a^-(\lambda), s \notin U_a^-(\lambda)$ or $s \in U_a^+(\lambda), r \notin U_a^+(\lambda)$.

Note that there are exactly $r - 1$ inversion pairs of the form (r, s) when $r \in U_a^-(\lambda)$ and $d - s$ inversion pairs of the form (r, s) when $s \in U_a^+(\lambda)$. Applying the corresponding quadratic relations, we see that $f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda}$ equals

$$(-1)^{\lambda_a(\lambda_a-1)/2} e_\lambda \psi_{w_0^\lambda}(y_{0,m-\lambda_a} \circ f_{\bar{n}}^{\bar{\lambda},\bar{\boldsymbol{\delta}}} g_{\bar{n}}^{\bar{\boldsymbol{\delta}},\bar{\lambda}} \circ y'_{0,\lambda_a}) y^\lambda e_\lambda + (*), \tag{3.27}$$

where $(*)$ a sum of elements of the form

$$e_\lambda \psi_{w_0^\lambda} \iota_{(m-\lambda_a)\alpha_a,\bar{\theta},\lambda_a\alpha_a}(Y^- \otimes X \otimes Y'^+) y^\lambda e_\lambda,$$

with $X \in R_{\bar{\theta}}$, $Y^\pm$ a polynomial in the variables y_r with $r \in U_a^\pm(\lambda)$, and $\deg Y^- + \deg Y^+ < \deg y_{0,m-\lambda_a} + \deg y'_{0,\lambda_a}$. By Lemma 2.5(i), we have $(*) = 0$. So by Lemma 2.5(ii), the expression (3.27) equals the right hand side of (3.26). $\square$

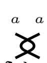
Define $\boldsymbol{\delta}_\lambda = (\delta_\lambda^{(1)}, \cdots, \delta_\lambda^{(m)})$ to be the unique element of $\boldsymbol{\Lambda}(n)$ such that for each $a \leq i \leq b$ we have:

- i precedes $b+1$ in $\boldsymbol{j}^{\delta_\lambda^{(r)}}$ for $1 \leq r \leq m - \lambda_i$;
- i succeeds $b+1$ in $\boldsymbol{j}^{\delta_\lambda^{(r)}}$ for $m - \lambda_i < r \leq m$.

Note that $\lambda^{\boldsymbol{\delta}_\lambda} = \lambda$ but $\boldsymbol{\delta}_{\lambda^{\boldsymbol{\delta}}}$ in general differs from $\boldsymbol{\delta}$.

Lemma 3.28. *Let $\boldsymbol{\delta} \in \boldsymbol{\Lambda}$ satisfy $\lambda^{\boldsymbol{\delta}} = \lambda$. Then*

$$f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = \begin{cases} e_\lambda & \textit{if } \boldsymbol{\delta} = \boldsymbol{\delta}_\lambda, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If $\boldsymbol{\delta} = \boldsymbol{\delta}_\lambda$, the result follows by induction on $\mathrm{ht}(\alpha)$ using from Lemma 3.25. If $\boldsymbol{\delta} \neq \boldsymbol{\delta}_\lambda$, we may assume using Lemma 3.25 that $\boldsymbol{\delta}$ is not initial. This implies that for some $r \in [1, m)$, we have $\delta_a^{(r)} = 1$ and $\delta_a^{(r+1)} = 0$, i.e. the last entry of the word $\boldsymbol{j}^{\delta^{(r)}}$ and the first entry of the word $\boldsymbol{j}^{\delta^{(r+1)}}$ are both equal to a. It follows that  is a sub-diagram of a Khovanov-Lauda diagram for $\psi_{w(\boldsymbol{\delta})}\psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda$, so $f_n^{\lambda,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda} = \pm e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} \psi_{u(\boldsymbol{\delta})} y^\lambda e_\lambda = 0$. $\square$

Corollary 3.29. *For any $n \in \mathbb{Z}_{\geq 0}$, we have $g_n \circ f_n = \mathrm{id}$.*

3.7. **Proof of Theorem A, assuming f and g are chain maps.** In Sections 4 and 5, we will prove that f and g constructed above are chain maps. The goal of this subsection is to demonstrate that this is sufficient to establish our main result.

Let

$$Y_0 := \prod_{k=1}^{m} y_{kl}^{k-1} \in \Bbbk[y_1, \dots, y_d].$$

We also define W_0 to be the longest element of $\mathscr{D}^{(l^m)}$. For $1 \le r < m$, we have a fully commutative element

$$z_r := \prod_{k=1}^{l} ((r-1)l + k, rl + k) \in \mathfrak{S}_d.$$

In fact, if $w_{0,m} = s_{r_1} \dots s_{r_N}$ is a reduced decomposition in $\mathfrak{S}_m$, then $W_0 = z_{r_1} \dots z_{r_N}$ in $\mathfrak{S}_d$ with $\ell(W_0) = \ell(z_{r_1}) + \dots + \ell(z_{r_N})$. Recalling (3.11) we have

$$\boldsymbol{j}^{\mathbf{0}} = (a\,(a+1)\cdots(b+1))^m \in I^d.$$

Then the following is easy to check:

Lemma 3.30. *Let $P = \{w \in \mathscr{D}^{(l^m)} \mid w \cdot \boldsymbol{j}^{\mathbf{0}} = \boldsymbol{j}^{\mathbf{0}}\}$. Then $W_0 \in P$ and $\deg(\psi_{W_0} e_{\mathbf{0}}) < \deg(\psi_w e_{\mathbf{0}})$ for any $w \in P \setminus \{W_0\}$*

Lemma 3.31. *Let ψ_{W_0} and ψ'_{W_0} in R_θ correspond to different reduced expressions of $W_0 \in \mathfrak{S}_d$. Then $\psi_{W_0} v_\alpha^{\circ m} = \psi'_{W_0} v_\alpha^{\circ m}$ in $\Delta(\alpha)^{\circ m}$.*

Proof. It follows from the defining relations of the KLR algebra and Theorem 2.2 that $\psi_{W_0} e_{\mathbf{0}} - \psi'_{W_0} e_{\mathbf{0}}$ is a linear combination of elements of the form $\psi_w e_{\mathbf{0}}$ where $w \cdot \boldsymbol{j}^{\mathbf{0}} = \boldsymbol{j}^{\mathbf{0}}$, $w \neq W_0$, and $\deg(\psi_w e_{\mathbf{0}}) = \deg(\psi_{W_0} e_{\mathbf{0}})$. Moreover, for each such w, by Lemma 2.1, there exist (unique) elements $x \in \mathscr{D}^{(l^m)}$ and $y \in \mathfrak{S}_{(l^m)}$ such that $w = xy$ and $\ell(w) = \ell(x) + \ell(y)$. Since $\psi_y v_\alpha^{\circ m} = 0$ if $y \neq 1$, we may assume $w \in \mathscr{D}^{(l^m)}$. We now have $w \in \mathscr{D}^{(l^m)}$, $w \cdot \boldsymbol{j}^{\mathbf{0}} = \boldsymbol{j}^{\mathbf{0}}$, and $w \neq W_0$. By Lemma 3.30, no such element w can exist, so $(\psi_{W_0} e_{\mathbf{0}} - \psi'_{W_0} e_{\mathbf{0}}) v_\alpha^{\circ m} = 0$. □

Lemma 3.32. *There is a choice of reduced expression of W_0 for which $g_0^{\mathbf{0},0} f_0^{0,\mathbf{0}} = e_{\mathbf{0}} Y_0 \psi_{W_0} e_{\mathbf{0}}$.*

Proof. Note that $\ell(u(\mathbf{0}) w_0^0 w(\mathbf{0})) = \ell(u(\mathbf{0})) + \ell(w_0^0) + \ell(w(\mathbf{0}))$ and $u(\mathbf{0}) w_0^0 w(\mathbf{0}) = W_0$. So for an appropriate choice of reduced expression for W_0, we have $\psi_{u(\mathbf{0})} \psi_{w_0^0} \psi_{w(\mathbf{0})} = \psi_{W_0}$. We now compute:

$$\begin{aligned} g_0^{\mathbf{0},0} f_0^{0,\mathbf{0}} &= e_{\mathbf{0}} \psi_{u(\mathbf{0})} y^0 e_0 \psi_{w_0^0} \psi_{w(\mathbf{0})} e_{\mathbf{0}} \\ &= e_{\mathbf{0}} \psi_{u(\mathbf{0})} y^0 \psi_{w_0^0} \psi_{w(\mathbf{0})} e_{\mathbf{0}} \\ &= e_{\mathbf{0}} Y_0 \psi_{u(\mathbf{0})} \psi_{w_0^0} \psi_{w(\mathbf{0})} e_{\mathbf{0}} \\ &= e_{\mathbf{0}} Y_0 \psi_{W_0} e_{\mathbf{0}}, \end{aligned}$$

where the first equality is by definition of $g_0^{\mathbf{0},0}$ and $f_0^{0,\mathbf{0}}$, the second follows from Lemma 2.5(ii), the third by the defining relations of R_θ and the observation that $Y_0 = u(\mathbf{0}) \cdot y^0$, and the fourth from the remark at the beginning of the proof. □

Note that

$$f_0^{0,\mathbf{0}} = \sigma_{\mathbf{0}} e_0 \psi_{w_0^0} \psi_{w(\mathbf{0})} e_{\mathbf{0}} = \psi_g 1_{\boldsymbol{i}_\alpha^m},$$

where g is the longest element of $\mathfrak{S}_d$ with $g \cdot (\boldsymbol{i}_\alpha^m) = a^m (a+1)^m \cdots (b+1)^m$. Let

$$v_{\alpha^m} := f_0^{0,\mathbf{0}} v_\alpha^{\circ m} e_m = \psi_g v_\alpha^{\circ m} e_m \in \Delta(\alpha^m). \tag{3.33}$$

We refer to v_{α^m} as the *standard generator* of $\Delta(\alpha^m)$. It can be checked directly but also follows from Theorem 3.34 that it does generate $\Delta(\alpha^m)$. Note that $e_0 v_{\alpha^m} = v_{\alpha^m}$, so there is a homomorphism

$$\mathsf{p} : P_0 = q^{-lm(m-1)/2} R_\theta e_0 \longrightarrow \Delta(\alpha^m),\ xe_0 \mapsto xv_{\alpha^m},$$

where v_{α^m} is the standard generator of $\Delta(\alpha^m)$, see (3.33).

Theorem 3.34. *If f and g are chain maps then $P_\bullet = P_\bullet^{\alpha^m}$ is a projective resolution of $\Delta(\alpha^m)$, with the augmentation map p.*

Proof. The modules P_n are projective by construction. By Corollary 3.29, $P_\bullet$ is a complex, isomorphic to a direct summand of the complex $Q_\bullet$. Since $Q_\bullet$ is a resolution of $q^{m(m-1)/2}\Delta(\alpha)^{\circ m}$, it follows from the assumptions that $P_\bullet$ is exact in degrees > 0 and its 0th cohomology is given by $\mathsf{q} f_0 g_0(Q_0)$ where $\mathsf{q} : Q_0 \to q^{m(m-1)/2}\Delta(\alpha)^{\circ m}$ is the augmentation map (3.12).

Recalling the $(R_\theta, \mathcal{NH}_m)$-bimodule structure of $q^{m(m-1)/2}\Delta(\alpha)^{\circ m}$ from §2.5 and the idempotent $e_m \in \mathcal{NH}_m$ defined in (2.12), using Lemma 3.31, we have

$$Y_0 \psi_{W_0} v_\alpha^{\circ m} = Y_0 \psi_{z_{r_1}} \cdots \psi_{z_{r_N}} v_\alpha^{\circ m} = v_\alpha^{\circ m} e_m, \tag{3.35}$$

where $w_{0,m} = s_{r_1} \cdots s_{r_N}$ is a reduced decomposition in $\mathfrak{S}_m$. We now have:

$$\begin{aligned}
\mathsf{q} f_0 g_0(Q_0) &= \mathsf{q}(q^{m(m-1)/2} R_\theta e_{\mathbf{0}} Y_0 \psi_{W_0} e_{\mathbf{0}}) \\
&= q^{m(m-1)/2} R_\theta Y_0 \psi_{W_0} v_\alpha^{\circ m} \\
&= q^{m(m-1)/2} R_\theta v_\alpha^{\circ m} e_m \\
&= q^{m(m-1)/2} \Delta(\alpha)^{\circ m} e_m \\
&= \Delta(\alpha^m),
\end{aligned}$$

where the first equality follows from Lemma 3.32, the second from the definition of q, the third from (3.35), and the last two from the definitions.

It remains to note that $\mathsf{q}(f_0(e_0)) = v_{\alpha^m}$. For, note that $\mathsf{q}(f_0(e_0)) = f_0 v_\alpha^{\circ m}$, and $\mathsf{q}(f_0(e_0)) \in \mathsf{q}(f_0(P_0)) = \mathsf{q}(f_0(g_0(Q_0))) = \Delta(\alpha^m)$. So $f_0 v_\alpha^{\circ m} = f_0 v_\alpha^{\circ m} e_m = v_{\alpha^m}$. □

4. Verification that f is a chain map

We continue with the running assumptions of the previous section. In addition, throughout the section we fix

$$n \in \mathbb{Z}_{\geq 0}, \quad \mu = (\mu_a, \dots, \mu_b) \in \Lambda(n+1) \quad \text{and} \quad \boldsymbol{\delta} = (\delta^{(1)}, \dots, \delta^{(m)}) \in \boldsymbol{\Lambda}(n).$$

4.1. **Special reduced expressions.** Recall the notation of §3.1. Let $\lambda \in \Lambda^{\alpha^m}$ and $\delta \in \Lambda^\alpha$ be such that $\bar{\lambda} := \lambda - \delta \in \Lambda^{\alpha^{m-1}}$. For $i \in [a, b+1]$, we denote by $p_i = p_i(\delta) \in [1, l]$ the position occupied by i in $\boldsymbol{j}^\delta$, and set

$$q_i = q_i(\lambda, \delta) := \begin{cases} l_i^-(\lambda) & \text{if } i \neq b+1 \text{ and } \delta_i = 0, \\ l_i^+(\lambda) & \text{if } i \neq b+1 \text{ and } \delta_i = 1, \\ l_{b+1}(\lambda) & \text{if } i = b+1. \end{cases} \tag{4.1}$$

Let

$$Q := \{q_a, \dots, q_{b+1}\} \subseteq [1, d].$$

Note that $\{p_a, \dots, p_{b+1}\} = [1, l]$. Define $x_\delta^\lambda \in \mathfrak{S}_d$ to be the permutation which maps p_i to q_i for all $i \in [a, b+1]$, and maps the elements of $[l+1, d]$ increasingly to the elements of $[1,d] \setminus Q$. It is easy to see that x^λ_δ is fully commutative, so $\psi_{x^\lambda_\delta}$ is well defined, and $1_{\boldsymbol{j}^\lambda}\psi_{x^\lambda_\delta} = \psi_{x^\lambda_\delta} 1_{\boldsymbol{j}^\delta \boldsymbol{j}^{\bar\lambda}}$.

Now let $\lambda = \lambda^{\boldsymbol{\delta}}$, and $r \in [1,m]$. Define

$$\boldsymbol{\delta}^{\geq r} := (\delta^{(r)}, \dots, \delta^{(m)}) \in (\Lambda^\alpha)^{m-r+1}, \quad \lambda^{\geq r} := \lambda^{\boldsymbol{\delta}^{\geq r}} \in \Lambda^{\alpha^{m-r+1}}.$$

Define $x(\boldsymbol{\delta},1) := x^\lambda_{\delta^{(1)}} \in \mathfrak{S}_d$. More generally, define for all $r = 1, \dots, m-1$, the permutations $x(\boldsymbol{\delta}, r) \in \mathfrak{S}_d$ so that $x(\boldsymbol{\delta}, r)$ is the image of

$$(1_{\mathfrak{S}_{(r-1)l}}, x^{\lambda^{\geq r}}_{\delta^{(r)}}) \in \mathfrak{S}_{(r-1)l} \times \mathfrak{S}_{(m-r+1)l}$$

under the natural embedding $\mathfrak{S}_{(r-1)l} \times \mathfrak{S}_{(m-r+1)l} \hookrightarrow \mathfrak{S}_d$.

Recall the element $w(\boldsymbol{\delta}) \in \mathfrak{S}_d$ defined in §3.4. The following lemma follows from definitions:

Lemma 4.2. *We have* $w(\boldsymbol{\delta}) = x(\boldsymbol{\delta},1)\cdots x(\boldsymbol{\delta}, m-1)$ *and* $\ell(w(\boldsymbol{\delta})) = \ell(x(\boldsymbol{\delta},1)) + \cdots + \ell(x(\boldsymbol{\delta}, m-1))$.

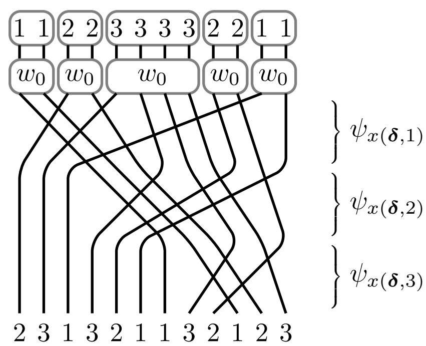
In view of the lemma, when convenient, we will always choose reduced decompositions so that

$$\psi_{w(\boldsymbol{\delta})} = \psi_{x(\boldsymbol{\delta},1)} \cdots \psi_{x(\boldsymbol{\delta},m-1)}. \tag{4.3}$$

By Lemma 3.23, we then have

$$f_n^{\lambda,\boldsymbol{\delta}} = \sigma_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{x(\boldsymbol{\delta},1)} \cdots \psi_{x(\boldsymbol{\delta},m-1)}. \tag{4.4}$$

Example 4.5. Let $a = 1$, $b = 2$, $\lambda = (2,2)$, and $\boldsymbol{\delta} = \big((1,0),(1,1),(0,1),(0,0)\big)$, so that $m = 4$ and $n = 4$. Then $f_n^{\lambda,\boldsymbol{\delta}}$ is



4.2. **A commutation lemma.** It will be convenient to use the following notation. Let $\lambda \in \Lambda(n)$. Consider the parabolic (non-unital) subalgebra

$$R^\lambda := R_{(m-\lambda_a)\alpha_a, \dots, (m-\lambda_b)\alpha_b, m\alpha_{b+1}, \lambda_b\alpha_b, \dots, \lambda_a\alpha_a} \subseteq R_\theta.$$

We have

$$R^\lambda \cong R_{(m-\lambda_a)\alpha_a} \otimes \cdots \otimes R_{(m-\lambda_b)\alpha_b} \otimes R_{m\alpha_{b+1}} \otimes R_{\lambda_b\alpha_b} \otimes \cdots \otimes R_{\lambda_a\alpha_a}.$$

The natural (unital) embeddings of the algebras

$$R_{(m-\lambda_a)\alpha_a}, \dots, R_{(m-\lambda_b)\alpha_b}, R_{m\alpha_{b+1}}, R_{\lambda_b\alpha_b}, \dots, R_{\lambda_a\alpha_a}$$

into R^λ, allow us to consider them as (non-unital) subalgebras of R_θ. We denote the corresponding (non-unital) algebra embeddings by

$$\iota^\lambda_{a;-}, \dots, \iota^\lambda_{b;-}, \iota^\lambda_{b+1}, \iota^\lambda_{b;+}, \dots, \iota^\lambda_{a;+}.$$

For example, setting

$$\psi^\lambda_{w_0}(i;-) := \iota^\lambda_{i;-}(\psi_{w_{0,m-\lambda_i}}),\ \psi^\lambda_{w_0}(i;+) := \iota^\lambda_{i;+}(\psi_{w_{0,\lambda_i}}),\ \psi^\lambda_{w_0}(b+1) := \iota^\lambda_{b+1}(\psi_{w_{0,m}}),$$

for all $i \in [a,b]$, can write $\psi_{w_0^\lambda} \in R_\theta$ as a commuting product

$$\psi_{w_0^\lambda} = \psi^\lambda_{w_0}(a;-)\dots\psi^\lambda_{w_0}(b;-)\psi^\lambda_{w_0}(b+1)\psi^\lambda_{w_0}(b;+)\dots\psi^\lambda_{w_0}(a;+). \tag{4.6}$$

Lemma 4.7. *Let $i \in [a,b]$ with $\mu_i > 0$, and $\lambda := \mu - \mathsf{e}_i$. Then*

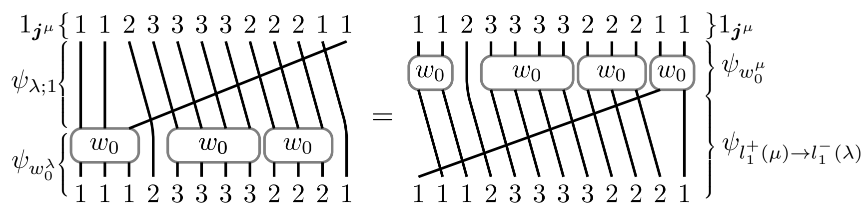
$$1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}\psi_{w_0^\lambda} = 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}.$$

Proof. We have

$$\begin{aligned}
1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}\psi_{w_0^\lambda} &= 1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}1_{\boldsymbol{j}^\lambda}\psi_{w_0^\lambda}\\
&= 1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}\psi^\lambda_{w_0}(b+1)\prod_{j\in[a,b]}\psi^\lambda_{w_0}(j;-)\psi^\lambda_{w_0}(j;+)\\
&= 1_{\boldsymbol{j}^\mu}\psi^{\boldsymbol{\mu}}_{w_0}(b+1)\Big[\prod_{j\in[a,b]\setminus\{i\}}\psi^{\boldsymbol{\mu}}_{w_0}(j;-)\psi^{\boldsymbol{\mu}}_{w_0}(j;+)\Big]\psi_{\lambda;i}\psi^\lambda_{w_0}(i;-)\psi^\lambda_{w_0}(i;+)\\
&= 1_{\boldsymbol{j}^\mu}\psi^{\boldsymbol{\mu}}_{w_0}(b+1)\Big[\prod_{j\in[a,b]}\psi^{\boldsymbol{\mu}}_{w_0}(j;-)\psi^{\boldsymbol{\mu}}_{w_0}(j;+)\Big]\psi_{l_i^+(\mu)\to l_i^-(\lambda)}\\
&= 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)},
\end{aligned}$$

where we have used (4.6) for the second equality, Lemma 2.9(i) for the third equality, and the definition of $\psi_{\lambda,i}$ as an explicit cycle element for the fourth equality. □

Example 4.8. We illustrate Lemma 4.7 diagramatically. Let $a = 1$, $b = 2$, $m = 4$ and $\mu = (2,3)$. Then, Lemma 4.7 claims the following equality:



4.3. f **is a chain map.** Let $\lambda = \lambda^{\boldsymbol{\delta}}$. For $i \in [a, b+1]$, let $q_i = q_i(\lambda, \delta^{(1)})$ be defined as in (4.1), and consider the cycle

$$c_i = \begin{cases} r_i^-(\mu) \to l_i^-(\mu) & \text{if } i \neq b+1 \text{ and } \delta_i^{(1)} = 0,\\ r_i^+(\mu) \to l_i^+(\mu) & \text{if } i \neq b+1 \text{ and } \delta_i^{(1)} = 1,\\ r_{b+1}(\mu) \to l_{b+1}(\mu) & \text{if } i = b+1.\end{cases}$$

Let c be the commuting product of cycles:

$$c := c_a c_{a+1}\dots c_{b+1}.$$

Recall also the elements defined in (3.13).

Lemma 4.9. *Suppose $i \in [a,b]$ is such that $\mu_i > 0$ and $\lambda := \mu - \mathtt{e}_i = \lambda^{\boldsymbol{\delta}}$. Set $\bar\theta = (m-1)\alpha$, $\bar\mu := \mu - \delta^{(1)}$, and $\bar\lambda := \lambda - \delta^{(1)}$. Then $1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}e_\lambda\psi_{w_0^\lambda}\psi_{x(\boldsymbol{\delta},1)}$ equals*

$$\begin{cases} 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i} + 1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}(1_\alpha \circ 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}) & \text{if } \delta_i^{(1)} = 0, \\ -1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}(1_\alpha \circ 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}) & \text{if } \delta_i^{(1)} = 1. \end{cases}$$

Proof. By Lemma 2.5(ii), we have $e_\lambda\psi_{w_0^\lambda} = 1_{\boldsymbol{j}^\lambda}\psi_{w_0^\lambda}$. So using also Lemma 4.7, we get

$$1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}e_\lambda\psi_{w_0^\lambda}\psi_{x(\boldsymbol{\delta},1)} = 1_{\boldsymbol{j}^\mu}\psi_{\lambda;i}1_{\boldsymbol{j}^\lambda}\psi_{w_0^\lambda}\psi_{x(\boldsymbol{\delta},1)} = 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}\psi_{x(\boldsymbol{\delta},1)}.$$

As usual, we denote by p_j the position occupied by j in $\delta^{(1)}$ and q_j be defined as in (4.1). By Lemma 4.2, the permutation $x(\boldsymbol{\delta},1)$ maps p_j to q_j for all $j \in [a, b+1]$, and the elements of $[l+1,d]$ increasingly to the elements of $[1,d] \setminus Q$.

Case 1: $\delta_i^{(1)} = 0$. In this case we have $q_i = l_i^-(\lambda)$. So the KLR diagram D of $1_{\boldsymbol{j}^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}\psi_{x(\boldsymbol{\delta},1)}$ has an i-string S from the position p_i in the bottom to the position $l_i^+(\mu)$ in the top, and the only (i,i)-crossings in D will be the crossings of the string S with the $m-\mu_i$ strings which originate in the positions $L_i^-(\mu)$ in the top. The $(i+1)$-string T in D originating in position p_{i+1} in the bottom is to the right of all (i,i) crossings. Pulling T to the left produces error terms, which arise from opening (i,i)-crossings, but all of them, except the last one, amount to zero, when multiplied on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$. The last error term is equal to $\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha \otimes \psi_{l_i^+(\bar\mu)\to l_i^-(\bar\lambda)}))$, and the result of pulling T past all (i,i)-crossings gives $\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i}$. Multiplying on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$, gives

$$1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)}\psi_{\boldsymbol{\delta};1,i} + 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha \otimes \psi_{l_i^+(\bar\mu)\to l_i^i(\bar\lambda)}),$$

and it remains to observe using Lemma 2.9 that

$$\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha \otimes \psi_{l_i^+(\bar\mu)\to l_i^i(\bar\lambda)}) = 1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}(1_\alpha \circ 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}).$$

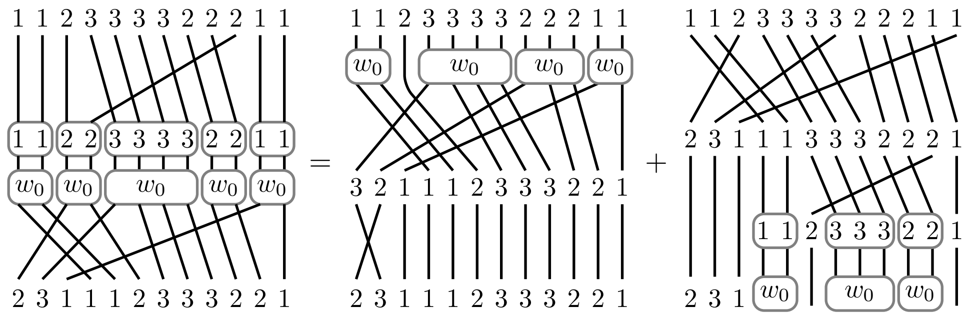
Case 2: $\delta_i^{(1)} = 1$. Let D be the KLR diagram of $1_{\boldsymbol{j}^\mu}\psi_{l_i^+(\mu)\to l_i^-(\lambda)}\psi_{x(\boldsymbol{\delta},1)}$. Let S be the i-string originating in the position $l_i^+(\mu)$ in the top row of D, and T be the $(i+1)$-string originating in the position p_{i+1} in the bottom row of D. The quadratic relation on these strings produces a difference of two terms, one with a dot on S and the other with a dot on T. The term with a dot on T equals 0 after multiplying on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$. The term with a dot on S, when multiplied on the left by $1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}$, yields

$$-1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha \otimes \psi_{l_i^+(\bar\mu)\to l_i^-(\bar\lambda)}) = -1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}(1_\alpha \circ 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}),$$

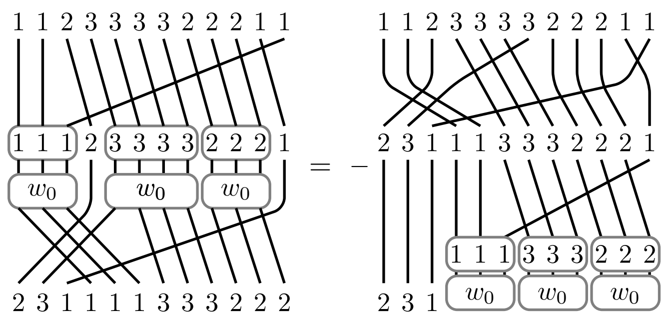
where we have used Lemma 2.9 for the last equality. □

Example 4.10. We illustrate Lemma 4.9 diagramatically. Let $a=1$, $b=2$, $m=4$, $\mu=(2,3)$ and $\boldsymbol{\delta} = ((1,0),(0,1),(1,1),(0,1))$.

First, take $i = 2$. Then $\delta_1^{(2)} = 0$ and Lemma 4.9 claims the following:



On the other hand, if $i = 1$, we have $\delta_1^{(1)} = 1$, and Lemma 4.9 posits the following equality.



Corollary 4.11. *If $\mu_i > 0$ for some $i \in [a, b]$ and $\lambda := \mu - \mathsf{e}_i = \lambda^{\boldsymbol{\delta}}$, then*

$$1_{\boldsymbol{j}^\mu} \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} = \sum_{r \in [1,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1} \delta_i^{(s)}} 1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)} \psi_{\boldsymbol{\delta};r,i}. \qquad (4.12)$$

Proof. The proof is by induction on m, the induction base $m = 1$ being obvious. Let $\bar{\boldsymbol{\delta}} := (\delta^{(2)}, \dots, \delta^{(m)})$, $\bar{\theta} = (m-1)\alpha$, $\bar{\mu} := \mu - \delta^{(1)}$, and $\bar{\lambda} := \lambda - \delta^{(1)}$. By (4.3), we have

$$1_{\boldsymbol{j}^\mu} \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} = 1_{\boldsymbol{j}^\mu} \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{x(\boldsymbol{\delta},1)} \psi_{x(\boldsymbol{\delta},2)} \cdots \psi_{x(\boldsymbol{\delta},m-1)}. \qquad (4.13)$$

Now we apply Lemma 4.9. We consider the case $\delta_i^{(1)} = 0$, the case $\delta_i^{(1)} = 1$ being similar. Then we get the following expression for the right hand side of (4.13):

$$\big(1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)} \psi_{\boldsymbol{\delta};1,i} + 1_{\boldsymbol{j}^\mu} \psi_c \psi_{x^\mu_{\delta^{(1)}}} (1_\alpha \circ 1_{\boldsymbol{j}^{\bar{\mu}}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \psi_{w_0^{\bar{\lambda}}})\big) \psi_{x(\boldsymbol{\delta},2)} \cdots \psi_{x(\boldsymbol{\delta},m-1)}.$$

Opening parentheses, we get two summands $S_1 + S_2$. Note that

$$\begin{aligned} S_1 &= 1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)} \psi_{\boldsymbol{\delta};1,i} \psi_{x(\boldsymbol{\delta},2)} \cdots \psi_{x(\boldsymbol{\delta},m-1)} \\ &= 1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^1,1)} \psi_{x(\boldsymbol{\delta},2)} \cdots \psi_{x(\boldsymbol{\delta},m-1)} \psi_{\boldsymbol{\delta};1,i} \\ &= 1_{\boldsymbol{j}^\mu} \psi_{w_0^\mu} \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^1)} \psi_{\boldsymbol{\delta};1,i}. \end{aligned}$$

Moreover, using the inductive assumption for the third equality below, we get that S_2 equals

$$\begin{aligned} &1_{\boldsymbol{j}^\mu} \psi_c \psi_{x^\mu_{\delta^{(1)}}} (1_\alpha \circ 1_{\boldsymbol{j}^{\bar{\mu}}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \psi_{w_0^{\bar{\lambda}}}) \psi_{x(\boldsymbol{\delta},2)} \cdots \psi_{x(\boldsymbol{\delta},m-1)} \\ =&1_{\boldsymbol{j}^\mu} \psi_c \psi_{x^\mu_{\delta^{(1)}}} (1_\alpha \circ 1_{\boldsymbol{j}^{\bar{\mu}}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \psi_{w_0^{\bar{\lambda}}} \psi_{x(\bar{\boldsymbol{\delta}},1)} \cdots \psi_{x(\bar{\boldsymbol{\delta}},m-2)}) \end{aligned}$$

$$
\begin{aligned}
&= 1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}\big(1_\alpha \circ 1_{\boldsymbol{j}^{\bar\mu}}\psi_{\bar\lambda;i}e_{\bar\lambda}\psi_{w_0^{\bar\lambda}}\psi_{w(\bar{\boldsymbol{\delta}})}\big)\\
&= 1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}\Big(1_\alpha \circ \sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=2}^{r-1}\delta_i^{(s)}} 1_{\boldsymbol{j}^{\bar\mu}}\psi_{w_0^{\bar\mu}}\psi_{w(\bar{\boldsymbol{\delta}}+\mathbf{e}_i^{r-1})}\psi_{\bar{\boldsymbol{\delta}};r-1,i}\Big).
\end{aligned}
$$

As $\delta_i^{(1)}=0$, we have $\sum_{s=2}^{r-1}\delta_i^{(s)} = \sum_{s=1}^{r-1}\delta_i^{(s)}$. So

$$
\begin{aligned}
&\iota_{\alpha,\bar\theta}\Big(1_\alpha \otimes \sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=2}^{r-1}\delta_i^{(s)}} 1_{\boldsymbol{j}^{\bar\mu}}\psi_{w_0^{\bar\mu}}\psi_{w(\bar{\boldsymbol{\delta}}+\mathbf{e}_i^{r-1})}\psi_{\bar{\boldsymbol{\delta}};r-1,i}\Big)\\
&= \sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}\iota_{\alpha,\bar\theta}\Big(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{w_0^{\bar\mu}}\Big(\prod_{t=1}^{m-2}\psi_{x(\bar{\boldsymbol{\delta}}+\mathbf{e}_i^{r-1},t)}\Big)\psi_{\bar{\boldsymbol{\delta}};r-1,i}\Big)\\
&= \sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}}\iota_{\alpha,\bar\theta}\big(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{w_0^{\bar\mu}}\big)\Big(\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^{r},t)}\Big)\psi_{\boldsymbol{\delta};r,i}.
\end{aligned}
$$

Moreover,

$$
\psi_{x^\mu_{\delta^{(1)}}}\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^r,t)} = \psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^r,1)}\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^r,t)} = \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)}.
$$

So S_2 equals

$$
\begin{aligned}
&\sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}} 1_{\boldsymbol{j}^\mu}\psi_c\psi_{x^\mu_{\delta^{(1)}}}\iota_{\alpha,\bar\theta}(1_\alpha\otimes 1_{\boldsymbol{j}^{\bar\mu}}\psi_{w_0^{\bar\mu}})\Big(\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^r,t)}\Big)\psi_{\boldsymbol{\delta};r,i}\\
&= \sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}} 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{x^\mu_{\delta^{(1)}}}\Big(\prod_{t=2}^{m-1}\psi_{x(\boldsymbol{\delta}+\mathbf{e}_i^r,t)}\Big)\psi_{\boldsymbol{\delta};r,i}\\
&= \sum_{r\in[2,m]:\ \delta_i^{(r)}=0} (-1)^{\sum_{s=1}^{r-1}\delta_i^{(s)}} 1_{\boldsymbol{j}^\mu}\psi_{w_0^\mu}\psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)}\psi_{\boldsymbol{\delta};r,i},
\end{aligned}
$$

where we have used Lemma 2.9(i) to see the first equality. Thus S_1+S_2 equals the right hand side of (4.12). □

The following statement means that f is chain map:

Proposition 4.14. *Let $\mu\in\Lambda(n+1)$ and $\boldsymbol{\delta}\in\boldsymbol{\Lambda}(n)$. Then*

$$
\sum_{\lambda\in\Lambda(n)} d_n^{\mu,\lambda} f_n^{\lambda,\boldsymbol{\delta}} = \sum_{\boldsymbol{\gamma}\in\boldsymbol{\Lambda}(n+1)} f_{n+1}^{\mu,\boldsymbol{\gamma}} c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}}.
$$

Proof. By definition, $d_n^{\mu,\lambda}=0$ unless $\lambda=\mu-\mathtt{e}_i$ for some $i\in[a,b]$, and $f_n^{\lambda,\boldsymbol{\delta}}=0$ unless $\lambda=\lambda^{\boldsymbol{\delta}}$. On the other hand, $f_n^{\mu,\boldsymbol{\gamma}}=0$ unless $\mu=\lambda^{\boldsymbol{\gamma}}$, and $c_n^{\boldsymbol{\gamma},\boldsymbol{\delta}}=0$, unless $\boldsymbol{\delta}=\boldsymbol{\gamma}-\mathbf{e}_i^r$ for some $i\in[a,b]$ and $r\in[1,m]$. So we may assume that $\mu=\lambda^{\boldsymbol{\delta}+\mathbf{e}_i^r}$ for some $i\in[a,b]$ and $r\in[1,m]$ such that $\delta_i^{(r)}=0$. In this case, letting $\lambda:=\lambda^{\boldsymbol{\delta}}$, we have to prove

$$
d_n^{\mu,\lambda} f_n^{\lambda,\boldsymbol{\delta}} = \sum_{r\in[1,m]:\ \delta_i^{(r)}=0} f_{n+1}^{\mu,\boldsymbol{\delta}+\mathbf{e}_i^r} c_n^{\boldsymbol{\delta}+\mathbf{e}_i^r,\boldsymbol{\delta}}.
$$

By definition of the elements involved, this means

$$(\mathtt{sgn}_{\lambda;i} e_\mu \psi_{\lambda;i} e_\lambda)(\sigma_{\boldsymbol{\delta}} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} e_{\boldsymbol{\delta}})$$
$$= \sum_{r\in[1,m]:\ \delta_i^{(r)}=0} (\sigma_{\boldsymbol{\delta}+\mathbf{e}_i^r} e_\mu \psi_{w_0^\mu} \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)} e_{\boldsymbol{\delta}+\mathbf{e}_i^r})(\mathtt{sgn}_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\delta}+\mathbf{e}_i^r} \psi_{\boldsymbol{\delta};r,i} e_{\boldsymbol{\delta}}).$$

Equivalently, we need to prove

$$\mathtt{sgn}_{\lambda;i} \sigma_{\boldsymbol{\delta}} e_\mu \psi_{\lambda;i} e_\lambda \psi_{w_0^\lambda} \psi_{w(\boldsymbol{\delta})} = \sum_{r\in[1,m]:\ \delta_i^{(r)}=0} \sigma_{\boldsymbol{\delta}+\mathbf{e}_i^r} \mathtt{sgn}_{\boldsymbol{\delta};r,i} e_\mu \psi_{w_0^\mu} \psi_{w(\boldsymbol{\delta}+\mathbf{e}_i^r)} \psi_{\boldsymbol{\delta};r,i},$$

which, in view of Corollary 4.11, is equivalent to the statement that

$$\mathtt{sgn}_{\lambda;i} \sigma_{\boldsymbol{\delta}} = \sigma_{\boldsymbol{\delta}+\mathbf{e}_i^r} \mathtt{sgn}_{\boldsymbol{\delta};r,i} (-1)^{\sum_{s=1}^{r-1} \delta_i^{(s)}}$$

for all $r \in [1,m]$ such that $\delta_i^{(r)} = 0$. But this is Lemma 3.1. □

5. Verification that g is a chain map

We continue with the running assumptions of Section 3. In addition, throughout the section we fix $n \in \mathbb{Z}_{\geq 0}$, $\lambda \in \Lambda(n)$ and $\boldsymbol{\gamma} = (\gamma^{(1)}, \dots, \gamma^{(m)}) \in \boldsymbol{\Lambda}(n+1)$.

5.1. **Special reduced expressions and a commutation lemma.** Recall the notation of §3.1. Let $\mu \in \Lambda^{\alpha^m}$ and $\gamma \in \Lambda^\alpha$ be such that $\bar{\mu} := \mu - \gamma \in \Lambda^{\alpha^{m-1}}$. For $i \in [a, b+1]$, we denote

$$p^i \;:=\; (m-1)l + (\text{the position occupied by } i \text{ in } \boldsymbol{j}^\gamma), \tag{5.1}$$

$$q^i \;:=\; \begin{cases} r_i^-(\mu) & \text{if } i \neq b+1 \text{ and } \gamma_i^{(m)} = 0, \\ r_i^+(\mu) & \text{if } i \neq b+1 \text{ and } \gamma_i^{(m)} = 1, \\ r_{b+1}(\mu) & \text{if } i = b+1. \end{cases} \tag{5.2}$$

Let $Q := \{q^a, \dots, q^{b+1}\}$. Note that $\{p^a, \dots, p^{b+1}\} = (d-l, d]$.

Define $z_\mu^\gamma \in \mathfrak{S}_d$ to be the permutation which maps q^i to p^i for all $i \in [a, b+1]$, and maps the elements of $[1,d]\backslash Q$ increasingly to the elements of $[1, d-l]$. It is easy to see that z_μ^γ is fully commutative, so $\psi_{z_\mu^\gamma}$ is well defined, and $\psi_{z_\mu^\gamma} 1_{\boldsymbol{j}^\mu} = 1_{\boldsymbol{j}^{\bar{\mu}} \boldsymbol{j}^\gamma} \psi_{z_\mu^\gamma}$.

Now let $\mu := \lambda^{\boldsymbol{\gamma}}$, and $r \in [1, m]$. Define

$$\boldsymbol{\gamma}^{\leq r} := (\gamma^{(1)}, \dots, \gamma^{(r)}) \in \boldsymbol{\Lambda}^{\alpha^r}, \quad \mu^{\leq r} := \lambda^{\boldsymbol{\gamma}^{\leq r}} \in \Lambda^{\alpha^r}.$$

Define $z(\boldsymbol{\gamma}, m) := z_\mu^{\gamma^{(m)}} \in \mathfrak{S}_d$. More generally, define for all $r = 2, \dots, m$, the permutations

$$z(\boldsymbol{\gamma}, r) := (x_{\mu^{\leq r}}^{\gamma^{(r)}}, 1_{\mathfrak{S}_{(m-r)l}}) \in \mathfrak{S}_{rl} \times \mathfrak{S}_{(m-r)l} \leq \mathfrak{S}_d.$$

Recall the element $u(\boldsymbol{\gamma}) \in \mathfrak{S}_d$ defined in §3.4. The following lemma follows from definitions:

Lemma 5.3. *We have $u(\boldsymbol{\gamma}) = z(\boldsymbol{\gamma}, 2) \cdots z(\boldsymbol{\gamma}, m)$ and $\ell(u(\boldsymbol{\gamma})) = \ell(z(\boldsymbol{\gamma}, 1)) + \cdots + \ell(z(\boldsymbol{\gamma}, m))$.*

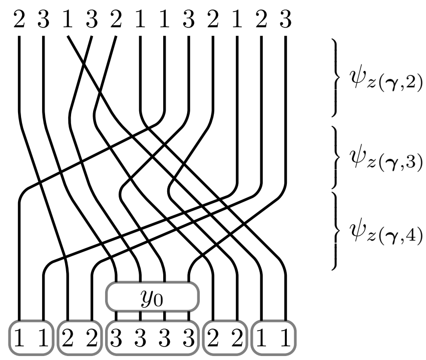
In view of the lemma, when convenient, we will always choose reduced decompositions so that

$$\psi_{u(\gamma)} = \psi_{z(\gamma,2)} \dots \psi_{z(\gamma,m)}. \tag{5.4}$$

In view of Lemma 3.23, we have

$$g_{n+1}^{\gamma,\mu} = \tau_\gamma \psi_{z(\gamma,2)} \dots \psi_{z(\gamma,m)} y^\mu e_\mu. \tag{5.5}$$

Example 5.6. Let $a = 1$, $b = 2$, $\mu = (2,2)$, and $\gamma = \big((1,0),(1,1),(0,1),(0,0)\big)$, so that $m = 4$ and $n = 4$. Then $g_n^{\lambda,\gamma}$ is



Lemma 5.7. *Let $i \in [a,b]$ with $\lambda_i < m$, and $\mu := \lambda + \mathsf{e}_i$. Then*

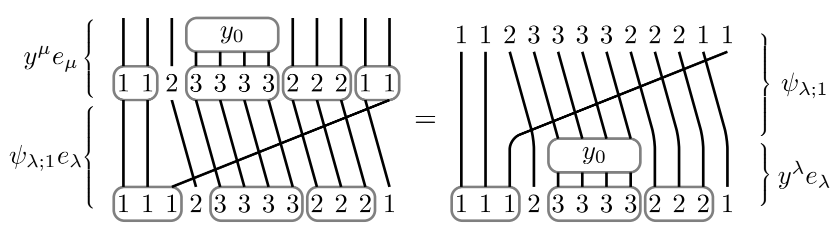
$$y^\mu e_\mu \psi_{\lambda;i} e_\lambda = \psi_{\lambda;i} y^\lambda e_\lambda.$$

Proof. Recall the notation of §4.2, we have that $y^\mu e_\mu \psi_{\lambda;i} e_\lambda$ equals

$$\begin{aligned}
&\Big[\prod_{j\in[a,b]} \iota^\mu_{j,-}(1_{j^{(m-\mu_j)}})\iota^\mu_{j,+}(1_{j^{(\mu_j)}})\Big]\iota^\mu_{b+1}(y_{0,m}1_{(b+1)^{(m)}})\psi_{\lambda;i}e_\lambda \\
=&\iota^\mu_{i,-}(1_{i^{(m-\mu_i)}})\iota^\mu_{i,+}(1_{i^{(\mu_i)}})\Big[\prod_{j\in[a,b]\setminus\{i\}} \iota^\mu_{j,-}(1_{j^{(m-\mu_j)}})\iota^\mu_{j,+}(1_{j^{(\mu_j)}})\Big] \\
&\times \iota^\mu_{b+1}(y_{0,m}1_{(b+1)^{(m)}})\psi_{\lambda;i}e_\lambda \\
=&\iota^\mu_{i,-}(1_{i^{(m-\mu_i)}})\iota^\mu_{i,+}(1_{i^{(\mu_i)}})\psi_{\lambda;i}\Big[\prod_{j\in[a,b]\setminus\{i\}} \iota^\lambda_{j,-}(1_{j^{(m-\lambda_j)}})\iota^\lambda_{j,+}(1_{j^{(\lambda_j)}})\Big] \\
&\times \iota^\lambda_{b+1}(y_{0,m})\iota^\lambda_{b+1}(1_{(b+1)^{(m)}})e_\lambda \\
=&\iota^\mu_{i,-}(1_{i^{(m-\mu_i)}})\iota^\mu_{j,+}(1_{j^{(\mu_j)}})\psi_{\lambda;i}y^\lambda e_\lambda = \psi_{\lambda;i}y^\lambda e_\lambda,
\end{aligned}$$

where we have used Lemma 2.9 for the second equality and Lemma 2.8 for the last equality. □

Example 5.8. We illustrate Lemma 5.7 diagramatically. Let $a = 1$, $b = 2$, $m = 4$, $\mu = (2,3)$ and $\lambda = (1,3)$ so that $i = 1$ using the notation of Lemma 5.7. Then, the lemma claims the following equality.



5.2. g **is a chain map.** Recall that we have fixed $\lambda \in \Lambda(n)$ and $\boldsymbol{\gamma} \in \boldsymbol{\Lambda}(n+1)$.

Lemma 5.9. *Suppose that $i \in [a, b]$ is such that $\lambda_i < m$ and $\mu := \lambda + \mathsf{e}_i = \lambda^{\boldsymbol{\gamma}}$. Set $\bar{\theta} = (m-1)\alpha$, $\bar{\mu} := \mu - \gamma^{(m)}$, and $\bar{\lambda} := \lambda - \gamma^{(m)}$. Then $\psi_{z(\boldsymbol{\gamma},m)} y^\mu e_\mu \psi_{\lambda;i} e_\lambda$ equals*

$$\begin{cases} \begin{array}{l} \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^\lambda e_\lambda \\ -(y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda \end{array} & \text{if } \gamma_i^{(m)} = 1, \\ (y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda & \text{if } \gamma_i^{(m)} = 0. \end{cases}$$

Proof. Using Lemma 5.7, we get

$$\psi_{z(\boldsymbol{\gamma},m)} y^\mu e_\mu \psi_{\lambda;i} e_\lambda = \psi_{z(\boldsymbol{\gamma},m)} \psi_{\lambda;i} y^\lambda e_\lambda$$

Recalling (5.1),(5.2), the permutation $z(\boldsymbol{\gamma}, m)$ maps q^j to p^j for all $j \in [a, b+1]$, and the elements of $[1, d] \setminus Q$ increasingly to the elements of $[1, d-l]$.

Case 1: $\gamma_i^{(m)} = 1$. In this case we have $q^i = r_i^+(\lambda)$. So the KLR diagram D of $\psi_{z(\boldsymbol{\gamma},m)}\psi_{\lambda;i} 1^{\boldsymbol{j}^\lambda}$ has an i-string S from the position $r_i^-(\lambda)$ in the bottom to the position p^i in the top, and the only (i,i)-crossings in D will be the crossings of the string S with the λ_i strings which originate in the positions $R_i^+(\lambda)$ in the bottom. The $(i+1)$-string T in D originating in the position p_{i+1} in the top is to the left of all (i,i) crossings. Pulling T to the right produces error terms, which arise from opening (i,i)-crossings, but all of them, except the last one, amount to zero, when multiplied on the right by $y^\lambda e_\lambda$. The last error term is equal to $-\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}}$, and the result of pulling T past all (i,i)-crossings gives $\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)}$. Multiplying on the left by $y^\lambda e_\lambda$, gives

$$-\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^\lambda e_\lambda + \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i}\psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^\lambda e_\lambda,$$

and it remains to observe using Lemma 2.9 that

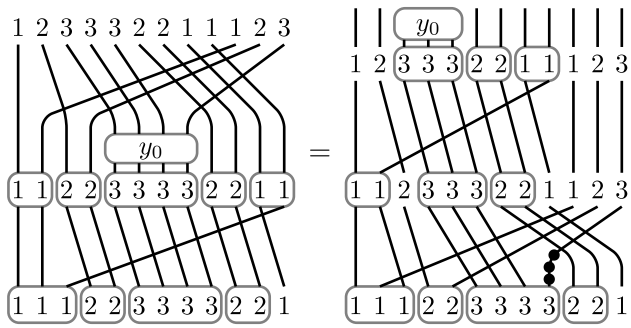
$$-\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^\lambda e_\lambda = -(y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda.$$

Case 2: $\gamma_i^{(m)} = 0$. Let D be the KLR diagram of $\psi_{z(\boldsymbol{\gamma},m)}\psi_{\lambda;i} 1_{\boldsymbol{j}^\lambda}$. Let S be the i-string originating in the position $r_i^-(\lambda)$ in the bottom row of D, and T be the $(i+1)$-string originating in the position p^{i+1} in the top row of D. The quadratic relation on these strings produces a linear combination of two diagrams, one with a dot on S and the other with a dot on T. The term containing a dot on T produces a term which is zero after multiplying on the right by $y^\lambda e^\lambda$. The term containing a dot on S, when multiplied on the right by $y^\lambda e^\lambda$, yields

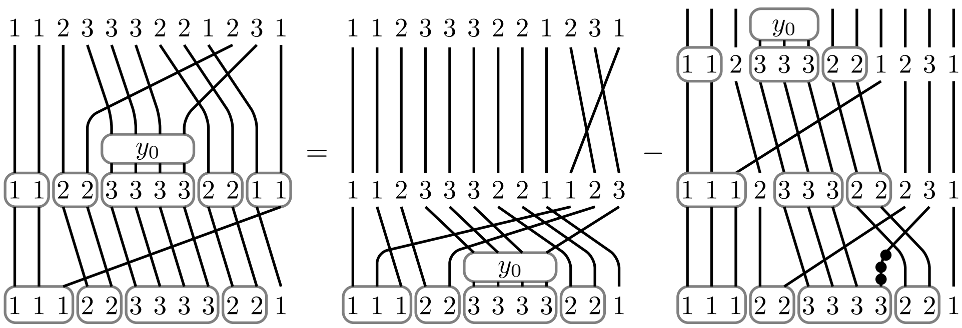
$$\iota_{\bar{\theta},\alpha}(\psi_{\bar{\lambda};i} \otimes 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^\lambda e_\lambda = (y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_\alpha)\psi_{z_\lambda^{\gamma^{(m)}}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda,$$

where we have used Lemmas 2.9 and 2.8 to deduce the last equality. ☐

Example 5.10. We illustrate Lemma 5.9 diagramatically. Using the notation of the lemma, let $a = 1$, $b = 2$, $\mu = (2, 2)$, $\lambda = (1, 2)$, $m = 4$ and $i = 1$. Then, if $\boldsymbol{\gamma} = \big((1,0),(1,1),(0,1),(0,0)\big)$, we are in the $\gamma_i^{(m)} = 0$ case and the lemma claims the following equality.



If instead we let $\boldsymbol{\gamma} = \big((0,0),(1,1),(0,1),(1,0)\big)$, we are in the $\gamma_i^{(m)} = 1$ case and the lemma claims the following equality.



Corollary 5.11. *Suppose that $i \in [a,b]$ is such that $\lambda_i < m$ and $\mu := \lambda + \mathsf{e}_i = \lambda^{\boldsymbol{\gamma}}$. Then*

$$\psi_{u(\boldsymbol{\gamma})} y^{\mu} e_{\mu} \psi_{\lambda;i} e_{\lambda} = \sum_{r\in[1,m]:\ \gamma_i^{(r)}=1} (-1)^{\sum_{s=r+1}^{m} \gamma_i^{(s)}} \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^r;r,i} \psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^r)} y^{\lambda} e_{\lambda}.$$

Proof. The proof is by induction on m, the induction base $m=1$ being obvious. Let $\bar{\gamma} := (\gamma^{(1)}, \dots, \gamma^{(m-1)})$, $\bar{\theta} = (m-1)\alpha$, $\bar{\mu} := \mu - \gamma^{(m)}$, and $\bar{\lambda} := \lambda - \gamma^{(m)}$. By (5.4), we have

$$\psi_{u(\boldsymbol{\gamma})} y^{\mu} e_{\mu} \psi_{\lambda;i} e_{\lambda} = \psi_{z(\boldsymbol{\gamma},2)} \cdots \psi_{z(\boldsymbol{\gamma},m)} y^{\mu} e_{\mu} \psi_{\lambda;i} e_{\lambda}. \tag{5.12}$$

Now we apply Lemma 5.9. We consider the case $\gamma_i^{(m)} = 1$, the case $\gamma_i^{(m)} = 0$ being similar. Then we get the following expression for the right hand side of (5.12):

$$\begin{aligned} &\psi_{z(\boldsymbol{\gamma},2)} \cdots \psi_{z(\boldsymbol{\gamma},m-1)} \big(\psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^{\lambda} e_{\lambda} \\ &-(y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_{\alpha}) \psi_{z_{\lambda}^{\gamma^{(m)}}} y^{m-1}_{r_{b+1}(\lambda)} e_{\lambda}\big). \end{aligned}$$

Opening parentheses, we get two summands $S_1 + S_2$. Note that

$$\begin{aligned} S_1 &= \psi_{z(\boldsymbol{\gamma},2)} \cdots \psi_{z(\boldsymbol{\gamma},m-1)} \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^{\lambda} e_{\lambda} \\ &= \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{z(\boldsymbol{\gamma},2)} \cdots \psi_{z(\boldsymbol{\gamma},m-1)} \psi_{z(\boldsymbol{\gamma}-\mathbf{e}_i^m,m)} y^{\lambda} e_{\lambda} \\ &= \psi_{\boldsymbol{\gamma}-\mathbf{e}_i^m;m,i} \psi_{u(\boldsymbol{\gamma}-\mathbf{e}_i^m)} y^{\lambda} e_{\lambda}. \end{aligned}$$

Moreover, using the inductive assumption, for the third equality below, S_2 equals

$$- \psi_{z(\boldsymbol{\gamma},2)} \cdots \psi_{z(\boldsymbol{\gamma},m-1)} (y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_{\alpha}) \psi_{z_{\lambda}^{\gamma^{(m)}}} y^{m-1}_{r_{b+1}(\lambda)} e_{\lambda}$$

$$
\begin{aligned}
&= - \left(\psi_{z(\bar{\gamma},2)} \dots \psi_{z(\bar{\gamma},m-1)} y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_\alpha\right) \psi_{z_\lambda^{\gamma(m)}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda \\
&= - \left(\psi_{u(\bar{\gamma})} y^{\bar{\mu}} e_{\bar{\mu}} \psi_{\bar{\lambda};i} e_{\bar{\lambda}} \circ 1_\alpha\right) \psi_{z_\lambda^{\gamma(m)}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda \\
&= - \Big[\sum_{r \in [1,m-1]:\ \gamma_i^{(r)}=1} (-1)^{\sum_{s=r+1}^{m-1} \gamma_i^{(s)}} \psi_{\bar{\gamma}-\mathbf{e}_i^r;r,i} \psi_{u(\bar{\gamma}-\mathbf{e}_i^r)} y^{\bar{\lambda}} e_{\bar{\lambda}} \circ 1_\alpha \Big] \psi_{z_\lambda^{\gamma(m)}} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda \\
&= \sum_{r \in [1,m-1]:\ \gamma_i^{(r)}=1} (-1)^{\sum_{s=r+1}^{m} \gamma_i^{(s)}} \psi_{\gamma-\mathbf{e}_i^r;r,i} \psi_{z(\gamma-\mathbf{e}_i^r,2)} \dots \psi_{z(\gamma-\mathbf{e}_i^r,m-1)} \\
&\quad \times \Big[y^{\bar{\lambda}} e_{\bar{\lambda}} \circ 1_\alpha \Big] \psi_{z(\gamma-\mathbf{e}_i^r,m)} y^{m-1}_{r_{b+1}(\lambda)} e_\lambda \\
&= \sum_{r \in [1,m-1]:\ \gamma_i^{(r)}=1} (-1)^{\sum_{s=r+1}^{m} \gamma_i^{(s)}} \psi_{\gamma-\mathbf{e}_i^r;r,i} \psi_{z(\gamma-\mathbf{e}_i^r,2)} \dots \psi_{z(\gamma-\mathbf{e}_i^r,m)} \\
&\quad \times \Big[y^{\bar{\lambda}} e_{\bar{\lambda}} \circ 1_\alpha \Big] y^{m-1}_{r_{b+1}(\lambda)} e_\lambda \\
&= \sum_{r \in [1,m-1]:\ \gamma_i^{(r)}=1} (-1)^{\sum_{s=r+1}^{m} \gamma_i^{(s)}} \psi_{\gamma-\mathbf{e}_i^r;r,i} \psi_{u(\gamma-\mathbf{e}_i^r)} y^{\lambda} e_\lambda.
\end{aligned}
$$

Thus $S_1 + S_2$ equals the right hand side of the expression in the corollary. □

The following statement means that g is chain map:

Proposition 5.13. *Let $\lambda \in \Lambda(n)$ and $\boldsymbol{\gamma} \in \mathbf{\Lambda}(n+1)$. Then*

$$\sum_{\mu \in \Lambda(n+1)} g_{n+1}^{\gamma,\mu} d_n^{\mu,\lambda} = \sum_{\boldsymbol{\delta} \in \mathbf{\Lambda}(n)} c_n^{\gamma,\boldsymbol{\delta}} g_n^{\boldsymbol{\delta},\lambda}.$$

Proof. By definition, $d_n^{\mu,\lambda} = 0$ unless $\mu = \lambda + \mathtt{e}_i$ for some $i \in [a,b]$, and $g_{n+1}^{\gamma,\mu} = 0$ unless $\mu = \lambda^{\gamma}$. On the other hand, $g_n^{\boldsymbol{\delta},\lambda} = 0$ unless $\lambda = \lambda^{\boldsymbol{\delta}}$, and $c_n^{\gamma,\boldsymbol{\delta}} = 0$, unless $\boldsymbol{\delta} = \boldsymbol{\gamma} - \mathbf{e}_i^r$ for some $i \in [a,b]$ and $r \in [1,m]$. So we may assume that there exists $i \in [a,b]$ such that $\lambda^{\gamma} = \lambda + \mathtt{e}_i$, in which case, setting $\mu := \lambda^{\gamma}$, we have to prove

$$g_{n+1}^{\gamma,\mu} d_n^{\mu,\lambda} = \sum_{r \in [1,m]:\ \gamma_i^{(r)}=1} c_n^{\gamma,\gamma-\mathbf{e}_i^r} g_n^{\gamma-\mathbf{e}_i^r,\lambda}.$$

By definition of the elements involved, this means

$$
\begin{aligned}
&(\tau_{\gamma} e_{\gamma} \psi_{u(\gamma)} y^{\mu} e_\mu)(\mathtt{sgn}_{\lambda;i} e_\mu \psi_{\lambda;i} e_\lambda) \\
&= \sum_{r \in [1,m]:\ \gamma_i^{(r)}=1} (\mathtt{sgn}_{\gamma-\mathbf{e}_i^r;r,i} e_{\gamma} \psi_{\gamma-\mathbf{e}_i^r;r,i} e_{\gamma-\mathbf{e}_i^r})(\tau_{\gamma-\mathbf{e}_i^r} e_{\gamma-\mathbf{e}_i^r} \psi_{u(\gamma-\mathbf{e}_i^r)} y^{\lambda} e_\lambda).
\end{aligned}
$$

Equivalently, we need to prove

$$\tau_{\gamma} \mathtt{sgn}_{\lambda;i} \psi_{u(\gamma)} y^{\mu} e_\mu \psi_{\lambda;i} e_\lambda = \sum_{r \in [1,m]:\ \gamma_i^{(r)}=1} \mathtt{sgn}_{\gamma-\mathbf{e}_i^r;r,i} \tau_{\gamma-\mathbf{e}_i^r} \psi_{\gamma-\mathbf{e}_i^r;r,i} \psi_{u(\gamma-\mathbf{e}_i^r)} y^{\lambda} e_\lambda,$$

which, in view of Corollary 5.11, is equivalent to the statement that

$$\tau_{\gamma} \mathtt{sgn}_{\lambda;i} = \mathtt{sgn}_{\gamma-\mathbf{e}_i^r;r,i} \tau_{\gamma-\mathbf{e}_i^r} (-1)^{\sum_{s=r+1}^{m} \gamma_i^{(s)}}$$

for all $r \in [1,m]$ such that $\gamma_i^{(r)} = 1$. But this is Lemma 3.3. □

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