Persistence in Lévy-flight anomalous diffusion

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The evolution of the number of persistent sites in a field governed by Lévy-flight anomalous diffusion is characterized. It is shown that, as in the case of ordinary diffusion, the number of persistent sites exhibits a long-time power-law decay. For the case of white-noise initial conditions, the exponent in this power-law decay can be numerically found from an algebraic equation as a function of the Lévy exponent γ. As expected, the decay is faster as the transport mechanism becomes more efficient, i.e., as γ decreases. Numerical simulations that validate the analytical results are also presented. [S1063-651X(97)04906-4]

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It has very recently been shown [1] that, rather surprisingly, ordinary diffusion processes exhibit some features whose evolution is not governed by the usual dynamical exponent of diffusion, which relates time with the mean square displacement, but by a new nontrivial exponent θ. This new exponent characterizes the temporal decay of the number of persistent sites from a noisy initial condition for the diffusion equation. Considering an initial diffusing field with zero mean value, persistent sites are the points where, up to a given time, the field has never changed its sign. The number of persistent points decays, for long times, as $N(t)\sim t^{-\theta}$. For white-noise initial conditions in a one-dimensional system, for instance, it has been found, both analytically and numerically, that $\theta\approx 0.12$. A closely related exponent appears in the spatial dependence of the correlation function of persistent sites, and therefore characterizes their spatial distribution [2]. In other works [3,4] it has been pointed out that nontrivial persistence exponents occur also in problems similar to diffusion, such as Glauber dynamics and other nonequilibrium phenomena.

In spite of the ubiquity of ordinary diffusion in natural processes, a variety of physical systems which have recently attracted a lot of attention are driven by a different class of transport mechanism, namely, by anomalous diffusion [5]. Like ordinary diffusion, this is an isotropic process, $\langle \mathbf{x}(t)-\mathbf{x}(0)\rangle=0$, but its mean square displacement is not necessarily linear with time, $\langle|\mathbf{x}(t)-\mathbf{x}(0)|^2\rangle=f(t)\neq Kt$. Anomalous diffusion has been identified as the leading transport mechanism, for instance, in fully developed turbulence [6], where it is more efficient than ordinary diffusion, $\lim_{t\to\infty}f(t)/t=\infty$, and in highy heterogeneous media such as porous substrates [5], where it is less efficient than ordinary diffusion, $\lim_{t\to\infty}f(t)/t=0$.

The case of hyperdiffusion, $\lim_{t\to\infty}f(t)/t=\infty$, can be modeled at the mesoscopic level by means of a generalization of discrete-time random walks, which allows for jumps with a long-tailed distribution $p(\mathbf{r})$. These so-called Lévy flights are defined by a characteristic function [namely, the Fourier transform of $p(\mathbf{r})$] of the form $p(\mathbf{q})=\exp(-bq^\gamma)$ ($0<\gamma<2$, b constant) [7]. Although the explicit form of $p(\mathbf{r})$ is not known, it can be shown that $p(\mathbf{r})\sim r^{-d-\gamma}$ for large r (d is the spatial dimension). This power-law tail, which implies that $\langle r^2\rangle\to\infty$, is the main ingredient for producing anomalous diffusion. In the limit of $\gamma=2$, the mean square displacement becomes finite and ordinary diffusion is recovered.

Recently, it has been shown [8,9] that Lévy-flight (LF) anomalous diffusion admits a macroscopic description in terms of a diffusionlike equation for a field $\phi(\mathbf{x},t)$ whose Fourier transform reads

$$\partial_t\hat{\phi}(\mathbf{k},t)+D_\gamma k^\gamma\hat{\phi}(\mathbf{k},t)=0, \tag{1}$$

where $D_\gamma\propto b/\tau$, with τ the (mean) time between jumps. This anomalous-diffusion equation can be expressed in the original $\mathbf{x}$ space in terms of fractional derivatives [10]. The main aim of this paper is to study the problem of persistence for Eq. (1). It can be expected that, as γ decreases and anomalous diffusion becomes more and more efficient, the number of persistent sites decays faster. In the following, we characterize this dependence of the persistence exponent on the Lévy exponent γ.

First, we briefly review the lines along which the exponent for persistence in ordinary diffusion was obtained in [1]. Consider a field $\phi(\mathbf{x},t)$ whose evolution is governed by the diffusion equation $\partial_t\phi=D\nabla^2\phi$ and whose initial state is a random function $\phi(\mathbf{x},0)$ with deltalike correlation, $\langle\phi(\mathbf{x}_1,0)\phi(\mathbf{x}_2,0)\rangle=\eta\delta(\mathbf{x}_1-\mathbf{x}_2)$. For some fixed point $\mathbf{x}$, define the variable $X(t)=\phi(\mathbf{x},t)/\langle[\phi(\mathbf{x},t)]^2\rangle$. The stochastic process $X(t)$ can be transformed into a stationary process by introducing the change of variable $T=\ln t$. In fact, it can be easily proven that the autocorrelation function of $X(t)$ is, in terms of the new variable,

$$a(t_1,t_2)=\langle X(t_1)X(t_2)\rangle=\left(\operatorname{sech}\frac{T_1-T_2}{2}\right)^{d/2}, \tag{2}$$

which depends on the difference $T=T_1-T_2$ only. Note, moreover, that it is independent of the diffusivity D and of the noise strength η. It can also be shown that the correlator of the process $\sigma(T)=\operatorname{sgn}[X(T)]$ is

$$A(T)=\langle\sigma(0)\sigma(T)\rangle=\frac{2}{\pi}\sin^{-1}a(T)=\frac{2}{\pi}\sin^{-1}\left(\operatorname{sech}\frac{T}{2}\right)^{d/2}, \tag{3}$$

which is, in turn, related to the probability distribution $P(T)$ of intervals between zeros of $X(T)$.

The calculation of the connection between the correlator of $\sigma(T)$ and $P(T)$ requires introducing an approximation, however, namely, that the intervals between successive zeros of $X(T)$ are statistically independent. This ''independent-interval approximation'' (IIA) neglects the fact that the evolution of $\phi(\mathbf{x},t)$ is being driven by the diffusion equation, which imposes a deterministic correlation between the values of ϕ at different points and times. Nevertheless, it has proven to be an excellent approximation in the case of ordinary diffusion [1]. The IIA makes it possible to show that the Laplace transforms of $P(T)$ and $A(T)$ are related according to

$$\tilde{P}(s)=\frac{2-\langle T\rangle s[1-s\tilde{A}(s)]}{2+\langle T\rangle s[1-s\tilde{A}(s)]}, \tag{4}$$

where $\langle T\rangle=2\pi\sqrt{2/d}$ is the mean interval size, which can be readily calculated from the explicit form of $A(T)$ [Eq. (3)]. Now, $\tilde{P}(s)$ has a simple pole at a real, negative value of its variable, say, at $s=-\theta$. This singularity implies that $P(T)\sim\exp(-\theta T)$ for large T, which gives in turn $P(t)\sim t^{-\theta}$. This power-law decay dominates, as a consequence, the asymptotics of the probability that $X(t)$ has not changed its sign, which is closely related to $P(t)$. The determination of the persistence exponent for ordinary diffusion reduces then to the calculation of the root of the denominator in the right-hand side of Eq. (4), namely, of

$$f(s)=2+2\pi s\sqrt{2/d}\left[1-\frac{2s}{\pi}\int_0^\infty\exp(-sT)\times\sin^{-1}\left(\operatorname{sech}\frac{T}{2}\right)^{d/2}dT\right]. \tag{5}$$

This can be readily done by numerical means. For $d=1$, for instance, this yields $\theta\approx 0.1203$.

In order to extend these results to the case of LF anomalous diffusion, where the field $\phi(\mathbf{x},t)$ is governed, through its Fourier transform, by Eq. (1), it is necessary to proceed entirely in the Fourier representation. In fact, the solution to Eq. (1),

$$\hat{\phi}(\mathbf{k},t)=\exp(-D_\gamma k^\gamma t)\hat{\phi}(\mathbf{k},0), \tag{6}$$

cannot be explicitly antitransformed. Taking into account that, in the Fourier representation, the white-noise correlation of the initial condition takes the form $\langle\hat{\phi}(\mathbf{k}_1,0)\hat{\phi}(\mathbf{k}_2,0)\rangle=\eta'\delta(\mathbf{k}_1+\mathbf{k}_2)$, the two-time correlation function for $\phi(\mathbf{x},t)$ is

$$\langle\phi(\mathbf{x},t_1)\phi(\mathbf{x},t_2)\rangle=\eta''\int\exp[-D_\gamma k^\gamma(t_1+t_2)]d\mathbf{k}=\eta'''(t_1+t_2)^{-d/\gamma}, \tag{7}$$

where η', η'', and η''' are constants whose explicit values are irrelevant in the following. From here, we immediately obtain $\langle[\phi(\mathbf{x},t)]^2\rangle=\eta'''(2t)^{-d/\gamma}$ and the correlator $a(t_1,t_2)$ for the variable $X(t)=\phi(\mathbf{x},t)/\langle[\phi(\mathbf{x},t)]^2\rangle$ at any point $\mathbf{x}$ reads

$$a_\gamma(t_1,t_2)=\langle X(t_1)X(t_2)\rangle=\left[\frac{4t_1t_2}{(t_1+t_2)^2}\right]^{d/2\gamma}. \tag{8}$$

Again, $X(t)$ can be converted into a stationary stochastic process with the change of variable $T=\ln t$. In terms of the new variable, we have $a_\gamma(T)=[\operatorname{sech}(T/2)]^{d/\gamma}$, with $T=T_1-T_2$, which generalizes Eq. (2). For large T, this correlation function behaves as $a_\gamma(T)\sim\exp(-Td/2\gamma)=t^{-d/2\gamma}$. Note that the decay of $a(T)$ is faster as the Lévy exponent γ decreases and anomalous diffusion becomes more efficient as a transport mechanism.

Now, we remark that the form of the correlator for $\sigma(T)=\operatorname{sgn}[X(T)]$ as a function of $a_\gamma(T)$ depends on the definition of σ only. Therefore, $A_\gamma(T)=\langle\sigma(0)\sigma(T)\rangle=(2/\pi)\sin^{-1}a_\gamma(T)$, and we have

$$A_\gamma(T)=\frac{2}{\pi}\sin^{-1}\left[\operatorname{sech}\frac{T}{2}\right]^{d/\gamma}. \tag{9}$$

This extends Eq. (3) to the case of LF anomalous diffusion. The next step in our derivation requires using the IIA approximation. As a matter of fact, we can argue that this approximation should be more accurate in the case of LF diffusion than for ordinary diffusion. This is due to the fact that LF diffusion acts over longer ranges and, therefore, correlations in space and time arising from the evolution are more widespread. As a consequence, successive values of the field $\phi(\mathbf{x},t)$ at a given point are relatively less correlated by the transport mechanism, and the statistical independence of successive zeros in $\phi(\mathbf{x},t)$ for fixed $\mathbf{x}$ is expected to hold, at least, as accurately as in ordinary diffusion. Within the IIA approximation for LF diffusion, we again encounter Eq. (4), now with $\langle T\rangle=2\pi\sqrt{\gamma/d}$. The function whose root gives the value of the persistence exponent is then

$$f_\gamma(s)=2+2\pi s\sqrt{\gamma/d}\left[1-\frac{2s}{\pi}\int_0^\infty\exp(-sT)\times\sin^{-1}\left(\operatorname{sech}\frac{T}{2}\right)^{d/\gamma}dT\right]. \tag{10}$$

This function diverges for $s=-d/2\gamma$, $f(-d/2\gamma)\to-\infty$, and grows monotonically as s approaches zero, with $f(0)=2$. It has, therefore, a root at some intermediate point, which, as in the case of ordinary diffusion, can easily be found by numerical means. The curve in Fig. 1 shows the result for the exponent θ as a function of γ with $d=1$. For $\gamma<2$, the value of θ is given by Eq. (10), whereas it is constant for $\gamma\geqslant 2$. For other spatial dimensions, the behavior of $\theta(\gamma)$ is qualitatively the same. Note, in fact, that for $\gamma<2$ the dependence on d can be eliminated by redefining $\gamma/d\to\gamma$. We see that θ grows as γ decreases, showing again that LF anomalous diffusion gives rise to faster evolution than ordinary diffusion.

In order to check our analytical results, we have carried out a series of numerical simulations of LF anomalous diffusion on a discrete one-dimensional L-site lattice with peri-

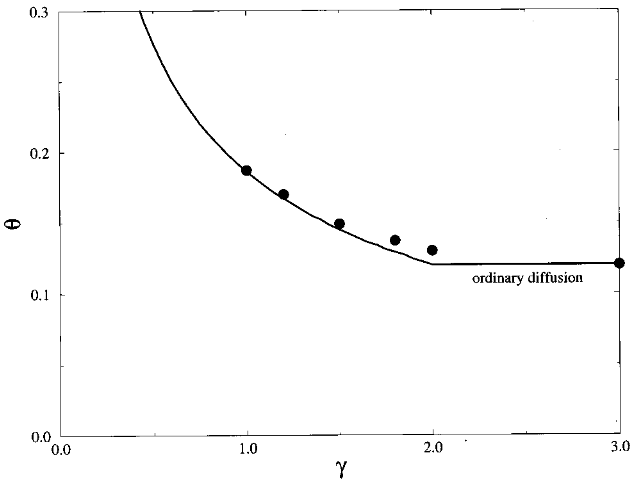

FIG. 1. The persistence exponent θ as a function of the Lévy exponent γ in one dimension. The curve corresponds, for $\gamma \leqslant 2$, to the numerical calculation of the roots of $f_\gamma(s)$ in Eq. (10). For $\gamma > 2$, the value of θ coincides with the one obtained for ordinary diffusion in [1]. Dots are results from numerical simulations of the Lévy-flight process.

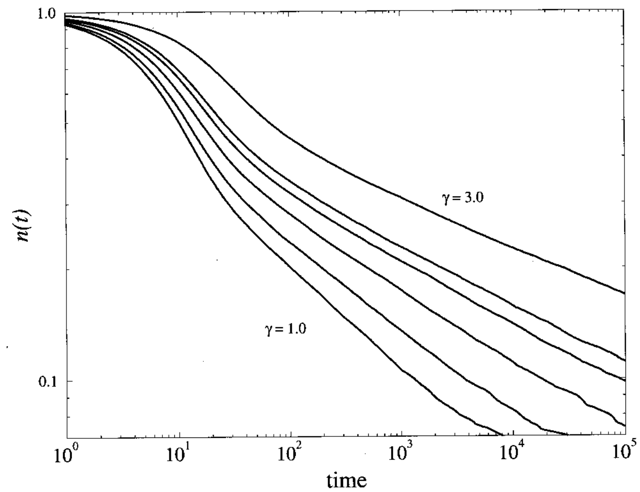

FIG. 2. The persistence index (normalized number of persistent sites) $n(t)$ as a function of time for some typical realizations in a one-dimensonal 10^5-site lattice. From bottom to top, $\gamma = 1.0$, 1.2, 1.5, 1.8, 2.0, and 3.0.

odic boundary conditions. As stated above, the jump probability associated with Lévy flights is defined through its Fourier transform, $p(\mathbf{q}) = \exp(-bq^\gamma)$, and the explicit form of $p(\mathbf{r})$ is not known. It is therefore usual to mimic Lévy flights taking a jump probability distribution with the same asymptotics for $r \to \infty$, namely, $p(\mathbf{r}) \sim r^{-1-\gamma}$. In our simulations in one dimension, we use $p(r) = (\gamma/2)(1+|r|)^{-1-\gamma}$ $(0 < \gamma < 2)$. For $\gamma \geqslant 2$, the distribution $p(r)$ has a finite second moment and the resulting transport process is ordinary diffusion.

At each time step, a site i is chosen at random. The field $\phi_i(t)$ at that site, which, within the computer precision, is a real number, is then decreased according to

$$\phi_i(t+\Delta t) = \frac{1}{2}\phi_i(t). \tag{11}$$

The ammount of field extracted from site i is equally distributed in two equidistant sites $i+[r]$ and $i-[r]$, where r is chosen with the probability distribution $p(r)$ and $[r]$ indicates its integer part. We have, therefore,

$$\phi_{i\pm[r]}(t+\Delta t) = \phi_{i\pm[r]}(t) + \frac{1}{4}\phi_i(t). \tag{12}$$

We take $\Delta t = L^{-1}$ so that, on average, each site is visited once at each time unit. The persistence index $n(t)$, namely, the fraction of sites where $\phi_i(t)$ has not changed its sign from the beginning of the evolution, is directly determined by simple counting.

Figure 2 shows some typical realizations with $L = 10^5$ and for several values of γ. The variation of the slopes in this log-log plot as the Lévy exponent changes is apparent. The temporal evolution of the slope for each curve can be numerically calculated, and its asymptotics can be evaluated from extrapolation of a suitably fitted function in the region where data are available. The result of this extrapolation, averaged over a few realizations, is shown as dots in Fig. 1. We find a very good agreement of numerical and analytical results, especially, for small and large γ. In the transition region, $\gamma \approx 2$, the agreement is not so good. The error, however, should be ascribed to the numerical data. In that region, where $p(r)$ passes from being a regular jump distribution to being an anomalous one, the asymptotic slope is in fact not so reliably defined. On the other hand, simulations for $\gamma < 1$, in which very long jumps have appreciable probability, exhibit noticeable finite-size effects.

We have shown here that the analysis of persistence features in ordinary diffusion can be readily generalized to the case of Lévy-flight anomalous diffusion. This at once extends some conclusions regarding the relevance of persistence in diffusion problems to Lévy flights. For instance, it has been pointed out [1] that persistence is closely related to some dynamical properties of systems of diffusing particles subject to bimolecular reactions, such as $A+B \to 0$, which have also been studied in the case of anomalous diffusion by other analytical means [11].

An interesting question regards the possibility of extending this generalization to other types of anomalous diffusion, namely, to fractional subdiffusion. In this transport process, jump probability densities are regular, but, on the other hand, waiting times have power-law distributions. The possibility of having long waiting times, in fact, gives rise to slower evolution than in ordinary diffusion. In principle, fractional subdiffusion also admits a macroscopic description by means of an equation of the type of Eq. (1), but in the Laplace representation with respect to the temporal variable [8,9]. This fact implies an operational drawback, since Laplace antitransformation of the solution will generally give origin to temporal nonlocality [12], which will, in turn, make the calculation of temporal correlators considerably more difficult. The possibility of this generalization, then, deserves further analysis.

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