THE KAUFFMAN BRACKET EXPANSION OF A GENERALIZED CROSSING

REBECCA SORSEN AND ALEXANDER ZUPAN

Abstract. We examine the Kauffman bracket expansion of the generalized crossing Δ_n, a half-twist on n parallel strands, as an element of the Temperley-Lieb algebra with coefficients in $\mathbb{Z}[A, A^{-1}]$. In particular, we determine the minimum and maximum degrees of all possible coefficients appearing in this expansion. Our main theorem shows that the maximum such degree is quadratic in n, while the minimum such degree is linear. We also include an appendix with explicit expansions for n at most six.

1. Introduction

The Jones polynomial, introduced in [Jon87], had a revolutionary impact on classical knot theory, fundamentally altering the fabric of low-dimensional topology. One well-known method of computing the Jones polynomial is via the Kauffman bracket $\langle \cdot \rangle$, which gives a set of rules for iteratively converting a knot diagram D into an element $\langle D \rangle$ of $\mathbb{Z}[A, A^{-1}]$ [Kau87]. Normalizing this polynomial $\langle D \rangle$ using the writhe of D yields the Jones polynomial of K. More generally, the Kauffman bracket can also be applied to any n-stranded tangle diagram $\mathcal{T}$. In this case, $\langle \mathcal{T} \rangle$ is an element of the Temperley-Lieb algebra TL_n (see [TL71]) with coefficients in $\mathbb{Z}[A, A^{-1}]$.

The main purpose of this paper is to elicit essential characteristics of the Kauffman bracket of a *generalized crossing* Δ_n, the tangle diagram obtained by performing a half twist on n parallel unknotted strands. As an element of the braid group, Δ_n is sometimes called the *Garside element.* In [MZ19], Jeffrey Meier and the second author produced a family

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of links $\{L_n\}$ such that each link L_n is slice – in fact, homotopy-ribbon – in the 4-ball B^4, and moreover, each L_n equipped with the 0-framing yields a handle decomposition of B^4. In view of Eisermann's work on the Jones polynomial of ribbon knots [Eis09], it may be useful to compute the Jones polynomials of the links in $\{L_n\}$. A subfamily $\{L'_n\}$, for example, contains links with unbounded crossing number; however, each link has a diagram with exactly six generalized crossings. It follows that an understanding of the Kauffman bracket of a generalized crossing could simplify the computation of Jones polynomials of links in this family, for instance. A motivating problem for our work is

Problem 1. *Find a closed form for $\langle \Delta_n \rangle$ as an element of TL_n*

Given an n-stranded tangle diagram $\mathcal{T}$, we denote the minimum and maximum degrees in A of any coefficient in $\langle \mathcal{T} \rangle$ by $\text{mindeg}_A\langle \mathcal{T} \rangle$ and $\text{maxdeg}_A\langle \mathcal{T} \rangle$, respectively. We prove that for the generalized crossing Δ_n, the minimum and maximum degrees have strikingly different behavior: $\text{mindeg}_A\langle \Delta_n \rangle$ is linearly related to n, whereas $\text{maxdeg}_A\langle \Delta_n \rangle$ is quadratically related to n. Specifically, we prove

Theorem 2. *For a generalized crossing Δ_n, the minimum and maximum degrees of the coefficients of $\langle \Delta_n \rangle$ satisfy*

$$maxdeg_A\langle \Delta_n \rangle = \frac{n(n-1)}{2}$$

while

$$mindeg_A\langle \Delta_n \rangle = \begin{cases} -\frac{n}{2} & \text{for } n \text{ even} \\ \frac{-n+1}{2} & \text{for } n \text{ odd} \end{cases}.$$

The proof of the theorem relies heavily on a closed formula for the Jones Polynomial of (n,n)-torus links given by Champanerkar and Kofman in [CK13], which also contains the bracket expansion of the full twist Δ_n^2. En route to proving the theorem, we establish a particular $\mathbb{Z}[A, A^{-1}]$ coefficient in Proposition 4. The expansions of $\langle \Delta_n \rangle$ appear to be rich with additional patterns; in the Appendix we have included explicit expansions of $\langle \Delta_n \rangle$ for $n \leq 6$ for further exploration.

2. Preliminaries

The Kauffman bracket polynomial is determined by the following three rules (where $\bigcirc$ represents an unknotted loop in a knot diagram):

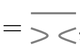
(1) $\langle \bigcirc \rangle = 1$
(2) $\langle \asymp\!\!\!\!\times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$
(3) $\langle \bigcirc \sqcup D \rangle = (-A^2 - A^{-2})\langle D \rangle$

The bracket polynomial can be converted into a knot invariant by taking into account the *writhe* $w(D)$ of a knot diagram D, the difference between the number of positive and negative crossings in D. For a given link L, the Kauffman polynomial $X_L(A)$ is defined to be

$$X_L(A) = \left(-A^3\right)^{-w(D)} \langle D \rangle,$$

where D is any diagram for L. Kauffman proved that $X_L(A)$ is an invariant of L, related to the Jones polynomial $V_L(t)$ by the rule $X_L(A) = V_L(A^{-4})$ [Kau87].

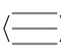
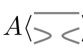
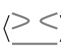
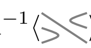
The *Temperley-Lieb algebra* TL_n is an algebra over $\mathbb{Z}[A, A^{-1}]$ whose elements can be thought of, for our purposes, as linear combinations of n-stranded planar tangle diagrams. As an algebra, TL_n is generated by $n-1$ elements, denoted $U_1, U_2, \ldots, U_{n-1}$. For example, as an algebra TL_3 is generated by

$U_1 =$  and $U_2 =$ ,

and multiplication corresponds to concatenating diagrams:

$U_1 \cdot U_2 =$  $\cdot$  $=$  $=$ 

As a module over $\mathbb{Z}[A, A^{-1}]$, on the other hand, the rank of TL_n is given by the Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$, where a basis is given by the C_{n+1} distinct planar tangle diagrams containing n strands. An example of the Kauffman bracket expansion of the generalized crossing of order three, Δ_3, with this basis is

$\langle \Delta_3 \rangle = A^3 \langle$  $\rangle + A \langle$  $\rangle + A \langle$  $\rangle + A^{-1} \langle$  $\rangle + A^{-1} \langle$  $\rangle$.

In this paper, we solve the following problem:

Problem 3. *Determine the minimum and maximum degrees in A of the coefficients of the expansion $\langle \Delta_n \rangle$.*

Formally, let $B_1, \ldots, B_{C_{n+1}}$ denote the basis for TL_n consisting of these planar tangle diagrams. For an arbitrary element $\sum_i p_i B_i \in TL_n$, we define the projection $P[B_i] : TL_n \to \mathbb{Z}[A, A^{-1}]$ by the rule $P[B_i]\left(\sum_i p_i B_i\right) = p_i$. With this notation, we have

$$\begin{aligned} \operatorname{maxdeg}_A \langle \Delta_n \rangle &= \max_i \{\deg P[B_i](\langle \Delta_n \rangle)\}; \\ \operatorname{mindeg}_A \langle \Delta_n \rangle &= \min_i \{\deg P[B_i](\langle \Delta_n \rangle)\}. \end{aligned}$$

Given a braid β, the *braid closure* $\overline{\beta}$ is the link obtained by connecting corresponding endpoints of β with crossing-less arcs. Similarly, given a diagram D obtained by multiplying a set of generators $U_1, \ldots, U_{n-1}$ in TL_n, the *closure* of D, denoted $\overline{D}$, is the diagram obtained by connecting opposite endpoints of D with crossing-less arcs. This concept and notation will be useful in the proof of the main theorem.

3. A computation

Before proving the main theorem, we demonstrate that at least one polynomial coefficient in the expansion $\langle \Delta_n \rangle$ – namely, the coefficient of the element $\langle \overline{\underline{\;\vdots\;}} \rangle$ – can always be predicted. The process by which a crossing $\times$ is replaced with $=$ is called the *A-smoothing*, whereas replacing $\times$ with $)($ is called the *B-smoothing*. As a convention, we will write Δ_n as the braid word $(\sigma_1 \dots \sigma_n)(\sigma_1 \dots \sigma_{n-1}) \dots (\sigma_1)$, with the braid drawn from right to left as shown for Δ_3 in Figure 1 below.

Proposition 4. *The polynomial coefficient of* $\langle \overline{\underline{\;\vdots\;}} \rangle$ *in* $\langle \Delta_n \rangle$ *is*

$$P[\overline{\underline{\;\vdots\;}}](\langle \Delta_n \rangle) = A^{\binom{n}{2}} = A^{\frac{1}{2}n^2 - \frac{1}{2}n}.$$

Proof. We prove the statement by induction on n. The base case $n = 2$ follows from the definition of the Kauffman bracket $\langle \cdot \rangle$. Thus, suppose by way of induction that $P[\overline{\underline{\;\vdots\;}}](\langle \Delta_{n-1} \rangle) = A^{\binom{n-1}{2}}$. Observe that the Kauffman bracket expansion computes $\langle D \rangle$ as a sum of $2^{c(D)}$ smoothings of the crossings in D. Strategically, we consider the strand of Δ_n that contains only overcrossings, which we call the *top strand*, and in particular, we consider the right-most crossing in the top strand, as shown below in Figure 1.

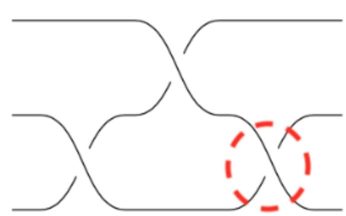


Figure 1. We first smooth along the circled crossing

Performing the B-smoothing on this crossing yields a planar arc which enters and exists the right side of the diagram, so that any smoothing yielding a non-zero coefficient on $\langle \overline{\underline{\;\vdots\;}} \rangle$ must include an A-smoothing on this right-most crossing. Perform the A-smoothing, and consider the next right-most crossing of the top strand. Once again, performing an B-smoothing on the next right-most crossing (assuming we have already performed an A-smoothing on the right-most crossing) yields a planar arc that enters and exits the right side of the diagram, yielding no $\langle \overline{\overline{\underline{\vdots}}} \rangle$ term.

Continuing in this manner, we see that any smoothing of $\langle \Delta_n \rangle$ producing a nonzero coefficient on $\langle \overline{\overline{\underline{\vdots}}} \rangle$ must contain all $n-1$ A-smoothings performed on the top strand, and these smoothings convert Δ_n to a tangle consisting of a straight line with no crossings above the generalized crossing Δ_{n-1}. In other words,

$$\langle \Delta_n \rangle = A^{n-1} \left\langle \overline{\Delta_{n-1}} \right\rangle + \dots$$

with the property that the only contribution to the coefficient on $\langle \overline{\overline{\underline{\vdots}}} \rangle$ arises from $A^{n-1} \left\langle \overline{\Delta_{n-1}} \right\rangle$. It follows using our inductive hypothesis that

$$P[\overline{\overline{\underline{\vdots}}}](\langle \Delta_n \rangle) = A^{n-1} \cdot P[\overline{\overline{\underline{\vdots}}}](\langle \Delta_{n-1} \rangle) = A^{n-1} \cdot A^{\binom{n-1}{2}} = A^{\binom{n}{2}}.$$

$\square$

In the next section, we will prove that this term has maximal degree among the coefficients of $\langle \Delta_n \rangle$.

4. Relating $\langle \Delta_n \rangle$ to $\langle T_{n,n} \rangle$

Let D be a tangle diagram considered as a collection of immersed arcs the plane $\mathbb{R}^2$, with crossing information at each double point, and such that the center of the diagram is the origin. In addition, let $r(D)$ denote the reflection of the diagram across the y-axis (corresponding to a 3-dimensional reflection of a tangle across the yz-plane), and let $\rho(D)$ denote the diagram obtained by reversing all crossings of $r(D)$ (corresponding to a 3-dimensional rotation of a tangle about the y-axis).

Lemma 5. *The tangle $\rho(\Delta_n)$ is identical to Δ_n. For each planar basis element B_i, we have $\rho(B_i) = r(B_i)$.*

Proof. The diagram $\rho(\Delta_n)$ is obtained by first reflecting D through the y-axis to get $r(D)$, followed by changing each crossing of $r(D)$. This composition sends Δ_n to itself. The second statement follows from the fact that B_i contains no crossings. $\square$

Patterns emerge from the explicit expansions of $\langle \Delta_n \rangle$ contained in the Appendix. One useful such pattern is proved in the next lemma.

Lemma 6. *For all basis elements B_i, we have*

$$P[r(B_i)](\langle \Delta_n \rangle) = P[B_i](\langle \Delta_n \rangle).$$

Proof. Let $\langle \Delta_n \rangle = \sum_i p_i B_i$. Observe that the transformation ρ respects crossing resolution, so that $\langle \rho(\Delta_n) \rangle = \sum_i p_i \rho(B_i)$. By Lemma 5, we have

$$\sum_i p_i B_i = \langle \Delta_n \rangle = \langle \rho(\Delta_n) \rangle = \sum_i p_i \cdot \rho(B_i) = \sum_i p_i \cdot r(B_i).$$

Since $\{B_i\}$ is a basis for TL_n, we conclude that the coefficient p_i for B_i appearing in the left sum above is identical to the coefficient p_i for $r(B_i)$ appearing in the right sum. We conclude that the desired equality holds. □

The (n, n)-torus link $T_{n,n}$ is an n-component link obtained by taking parallel copies of a $(1, 1)$-curve on an unknotted torus in S^3. Diagrammatically, we can obtain a diagram D_n for $T_{n,n}$ by taking the braid closure of the product of two copies of Δ_n; that is, $D_n = \overline{\Delta_n^2}$.

In Proposition 8 below, we will relate the maximum and minimum degrees of $\langle \Delta_n \rangle$ and $\langle \overline{\Delta_n^2} \rangle$. For this purpose, we need the next lemma. For a crossing-less link diagram D, we let $|D|$ denote the number of components in D.

Lemma 7. *For any two basis elements B_i and B_j, we have $|\overline{B_i \cdot B_j}| \leq n$, with equality if and only if $B_j = r(B_i)$.*

Proof. Label the endpoints of B_i as $x_1, \ldots, x_{2n}$ and the endpoints of B_j as $y_1, \ldots, y_{2n}$, with the convention that concatenation $B_i \cdot B_j$ connects x_k to y_{k-n} for $n + 1 \leq k \leq 2n$ and the closure $\overline{B_i \cdot B_j}$ connects x_k to y_{k+n} for $1 \leq k \leq n$. Observe that each component U of $\overline{B_i \cdot B_j}$ meets at least two points in $\{x_1, \ldots, x_{2n}\}$ and at least two points in $\{y_1, \ldots, y_{2n}\}$, so that $|\overline{B_i \cdot B_j}| \leq n$.

We have $|\overline{B_i \cdot B_j}| = n$ if and only if each component U meets $\{x_1, \ldots, x_{2n}\}$ in exactly two points and meets $\{y_1, \ldots, y_{2n}\}$ in exactly two points. Using the gluing conventions described above, suppose that U contains the endpoints x_k and x_l of B_i, so that U contains the endpoints y_{k+n} and y_{l+n} of B_j, with indices taken modulo $2n$. It follows that for every arc α in B_i, its image $r(\alpha)$ is contained in B_j, and we have $B_j = r(B_i)$, as desired. □

Observe that the braid closure $\overline{\Delta_n^2}$ is a link, so that $\langle \overline{\Delta_n^2} \rangle$ is a Laurent polynomial in A (instead of an element of TL_n). In the proposition below, we let $\text{maxdeg}_A \langle \overline{\Delta_n^2} \rangle$ and $\text{mindeg}_A \langle \overline{\Delta_n^2} \rangle$ denote the largest and smallest powers of A, respectively, appearing in this polynomial.

Proposition 8. *The following equalities hold:*

$$\text{maxdeg}_A \langle \overline{\Delta_n^2} \rangle = 2\, \text{maxdeg}_A \langle \Delta_n \rangle + 2n - 2 \tag{1}$$

$$\text{mindeg}_A \langle \overline{\Delta_n^2} \rangle = 2\, \text{mindeg}_A \langle \Delta_n \rangle - 2n + 2. \tag{2}$$

Proof. As above, let $\langle \Delta_n \rangle = \sum_i p_i B_i$. By splitting $\overline{\Delta_n^2}$ into $\overline{\Delta_n \cdot \Delta_n}$ and resolving each factor, we can see that the expansion of $\langle \overline{\Delta_n^2} \rangle$ can be computed from $\langle \Delta_n \rangle$ via the following formula:

$$\langle \overline{\Delta_n^2} \rangle = \sum_{i,j} (p_i \cdot p_j) \langle \overline{B_i \cdot B_j} \rangle = \sum_{i,j} (p_i \cdot p_j)(-A^2 - A^{-2})^{|\overline{B_i \cdot B_j}|-1}.$$

By Lemma 7, we have $|\overline{B_i \cdot B_j}| \leq n$. Since $\deg p_i \leq \operatorname{maxdeg}_A \langle \Delta_n \rangle$ for all i, we observe that

$$\begin{aligned} \operatorname{maxdeg}_A \langle \overline{\Delta_n^2} \rangle &\leq \max_{i,j} \deg \left(p_i \cdot p_j (-A^2 - A^{-2})^{|B_i \cdot B_j|-1} \right) \\ &\leq \max_{i,j} \left(\deg p_i + \deg p_j + \deg(-A^2 - A^{-2})^{n-1} \right) \\ &\leq 2 \operatorname{maxdeg}_A \langle \Delta_n \rangle + 2n - 2. \end{aligned}$$

To prove equality, we verify that the coefficient on $A^{2 \operatorname{maxdeg}_A \langle \Delta_n \rangle + 2n-2}$ in the expansion $\langle \overline{\Delta_n^2} \rangle$ is nonzero. Consider the largest degree terms in the expansion $\sum_{i,j} (p_i \cdot p_j)(-A^2 - A^{-2})^{|\overline{B_i \cdot B_j}|-1}$. A choice of indices i and j produces a maximum degree term precisely when $\deg p_i = \deg p_j = \operatorname{maxdeg}_A \langle \Delta_n \rangle$ and $|\overline{B_i \cdot B_j}| = n$. By Lemma 7, the third equality occurs if and only if $B_j = r(B_i)$, in which case Lemma 5 implies $p_i = p_j$. Let $\{q_1, \ldots, q_k\}$ be the set of maximal degree monomials appearing in any of the polynomials p_i. Then the largest degree term of $\langle \Delta_n \rangle$ is $(-A^2)^{n-1}(q_1^2 + \cdots + q_k^2) \neq 0$, verifying Equation (1) above.

An analogous argument shows that if $\{q_1', \ldots, q_l'\}$ is the set of minimal degree monomials appearing in any of the polynomials p_i, then the smallest degree term of $\langle \Delta_n \rangle$ is $(-A^2)^{-n+1}((q_1')^2 + \cdots + ((q_l')^2) \neq 0$, verifying Equation (2) above. $\square$

Following [CK13], we let $V_L(t)$ denote the Jones polynomial of a link L, and we recall that the Kauffman polynomial $X_L(A)$ satisfies $X_L(A) = V_L(A^{-4})$. As in the previous proof, we let the terms $\operatorname{maxdeg}_t V_{\overline{\Delta_n^2}}(t)$ and $\operatorname{mindeg}_t V_{\overline{\Delta_n^2}}(t)$ denote the largest and smallest powers of t appearing in $V_{\overline{\Delta_n^2}}(t)$.

Lemma 9. *The following equations hold:*

$$maxdeg_A \langle \overline{\Delta_n^2} \rangle = -4\, mindeg_t V_{\overline{\Delta_n^2}}(t) + 3n^2 - 3n \tag{3}$$

$$mindeg_A \langle \overline{\Delta_n^2} \rangle = -4\, maxdeg_t V_{\overline{\Delta_n^2}}(t) + 3n^2 - 3n. \tag{4}$$

Proof. Recall that $X_{\overline{\Delta_n^2}}(A) = (-A^3)^{-3w(\overline{\Delta_n^2})} \langle \overline{\Delta_n^2} \rangle$. The generalized crossing Δ_n contains $\binom{n}{2}$ positive crossings, which implies that $w(\overline{\Delta_n^2}) =$

$2 \cdot \binom{n}{2} = n^2 - n$. Thus,

$$\begin{aligned} \text{maxdeg}_A X_{\overline{\Delta_n^2}}(A) &= \text{maxdeg}_A \langle \overline{\Delta_n^2} \rangle - 3n^2 + 3n \\ \text{mindeg}_A X_{\overline{\Delta_n^2}}(A) &= \text{mindeg}_A \langle \overline{\Delta_n^2} \rangle - 3n^2 + 3n. \end{aligned}$$

Similarly, $V_{\overline{\Delta_n^2}}(t) = X_{\overline{\Delta_n^2}}(A^{-4})$; hence $-4\,\text{mindeg}_t V_{\overline{\Delta_n^2}}(t) = \text{maxdeg}_A \langle \overline{\Delta_n^2} \rangle$ and $-4\,\text{maxdeg}_t V_{\overline{\Delta_n^2}}(t) = \text{mindeg}_A \langle \overline{\Delta_n^2} \rangle$. Substituting and solving for $\text{maxdeg}_A \langle \overline{\Delta_n^2} \rangle$ and $\text{mindeg}_A \langle \overline{\Delta_n^2} \rangle$ yields the desired equations. □

Next, we analyze existing formulas for the Jones polynomial $V_{\overline{\Delta_n^2}}(t)$ of the (n, n)-torus link in order to find $\text{maxdeg}_t V_{\overline{\Delta_n^2}}(t)$ and $\text{mindeg}_t V_{\overline{\Delta_n^2}}(t)$. The following formula appears in [CK13], generalizing the formula for torus knots in [ILR93]:

$$V_{\overline{\Delta_n^2}}(t) = (-1)^{n+1} \frac{t^{\frac{1}{2}(n-1)^2}}{1 - t^2} \sum_{i=0}^{n} \binom{n}{i} \left(t^{(2+i)(n-i)} - t^{(1+i)(n-i+1)} \right). \tag{5}$$

Thus, for our purposes it is useful to compute $\text{maxdeg}_t V_{\overline{\Delta_n^2}}(t)$ and $\text{mindeg}_t V_{\overline{\Delta_n^2}}(t)$ from Equation (5) above.

Proposition 10. *For the (n, n)-torus link $\overline{\Delta_n^2}$, we have*

$$\text{mindeg}_t V_{\overline{\Delta_n^2}}(t) = \frac{1}{2}(n-1)^2 \tag{6}$$

$$\text{maxdeg}_t V_{\overline{\Delta_n^2}}(t) = \frac{3}{4}n^2 - \frac{1}{2} \qquad \text{if } n \text{ is even} \tag{7}$$

$$\text{maxdeg}_t V_{\overline{\Delta_n^2}}(t) = \frac{3}{4}n^2 - \frac{3}{4} \qquad \text{if } n \text{ is odd.} \tag{8}$$

Proof. It suffices to find the minimum and maximum values of the exponents $(2 + i)(n - i)$ and $(1 + i)(n - i + 1)$ for fixed values of n as i ranges from 0 to n. Both are quadratics in i, so that (for real $i \in [0, n]$), the function $(2+i)(n-i)$ will achieve its minimum value of 0 when $i = n$ and its maximum value at its vertex, a value of $(\frac{n}{2} + 1)^2$ when $i = \frac{n}{2} - 1$. On the other hand, the function $(1 + i)(n - i + 1)$ will achieve its minimum value of $1 + n$ when $i = 0$ or $i = n$ and its maximum value of $(\frac{n}{2} + 1)^2$ when $i = \frac{n}{2}$.

Thus, $\sum_{i=0}^{n} \binom{n}{i} \left(t^{(2+i)(n-i)} - t^{(1+i)(n-i+1)} \right)$ of $V_{\overline{\Delta_n^2}}(t)$ has lowest degree term 1, and when n is even, the highest degree term is

$$\left(\binom{n}{\frac{n}{2} - 1} - \binom{n}{\frac{n}{2}} \right) \cdot t^{(\frac{n}{2}+1)^2}.$$

In particular, the coefficient on $t^{(\frac{n}{2}+1)^2}$ is nonzero.

When n is odd, the maximum value of $(2+i)(n-i)$ for i an integer is $(\frac{n}{2}+1)^2 - \frac{1}{4}$ when $i = \frac{n-3}{2}$ and $i = \frac{n-1}{2}$, and the maximum value of $(1+i)(n-i+1)$ for i an integer is $(\frac{n}{2}+1)^2 - \frac{1}{4}$ when $i = \frac{n-1}{2}$ and $i = \frac{n+1}{2}$. It follows that for n odd, the highest degree term of the factor above is

$$\left(\binom{n}{\frac{n-3}{2}} + \binom{n}{\frac{n-1}{2}} - \binom{n}{\frac{n-1}{2}} - \binom{n}{\frac{n+1}{2}}\right) \cdot t^{(\frac{n}{2}+1)^2 - \frac{1}{4}},$$

and in particular, the coefficient on $t^{(\frac{n}{2}+1)^2-\frac{1}{4}}$ is nonzero.

Rearranging Equation (5) yields

$$(1-t^2)V_{\overline{\Delta_n^2}}(t) = (-1)^{n+1} t^{\frac{1}{2}(n-1)^2} \sum_{i=0}^{n} \binom{n}{i} \left(t^{(2+i)(n-i)} - t^{(1+i)(n-i+1)}\right).$$

The lowest degree term on the left has degree $\operatorname{mindeg}_t V_{\overline{\Delta_n^2}}(t)$, while the lowest degree term on the right has degree $\frac{1}{2}(n-1)^2$, yielding Equation (6). The highest degree term on the left has degree $\operatorname{maxdeg}_t V_{\overline{\Delta_n^2}}(t) + 2$, while for n even the highest degree term on the right has degree $\frac{1}{2}(n-1)^2 + (\frac{n}{2}+1)^2 = \frac{3}{4}n^2 + \frac{3}{2}$. For n odd, the highest degree term on the right has degree $\frac{1}{2}(n-1)^2 + (\frac{n}{2}+1)^2 - \frac{1}{4} = \frac{3}{4}n^2 + \frac{5}{4}$. In both cases, rearranging yields Equations (7) and (8), completing the proof. □

We have assembled all of the ingredients to prove the main theorem of the paper.

Proof of Theorem 2. Solving Equations (1) and (2) from Proposition 8 for $\operatorname{maxdeg}_A\langle\Delta_n\rangle$ and $\operatorname{mindeg}_A\langle\Delta_n\rangle$ yields

$$\operatorname{maxdeg}_A\langle\Delta_n\rangle = \frac{1}{2}\operatorname{maxdeg}_A\langle\overline{\Delta_n^2}\rangle - n + 1 \tag{9}$$

$$\operatorname{mindeg}_A\langle\Delta_n\rangle = \frac{1}{2}\operatorname{mindeg}_A\langle\overline{\Delta_n^2}\rangle + n - 1. \tag{10}$$

Equation (9) in conjunction with Equation (3) and Equation (6) from Proposition 10 gives

$$\begin{aligned}
\operatorname{maxdeg}_A\langle\Delta_n\rangle &= \frac{1}{2}\operatorname{maxdeg}_A\langle\overline{\Delta_n^2}\rangle - n + 1 \\
&= \frac{1}{2}\left(-4\operatorname{mindeg}_t V_{\overline{\Delta_n^2}}(t) + 3n^2 - 3n\right) - n + 1 \\
&= \frac{1}{2}\left(-2(n-1)^2 + 3n^2 - 3n\right) - n + 1 \\
&= \frac{n^2}{2} - \frac{n}{2} = \frac{n(n-1)}{2}.
\end{aligned}$$

For n even, we use Equations (4), (7), and (10) to calculate

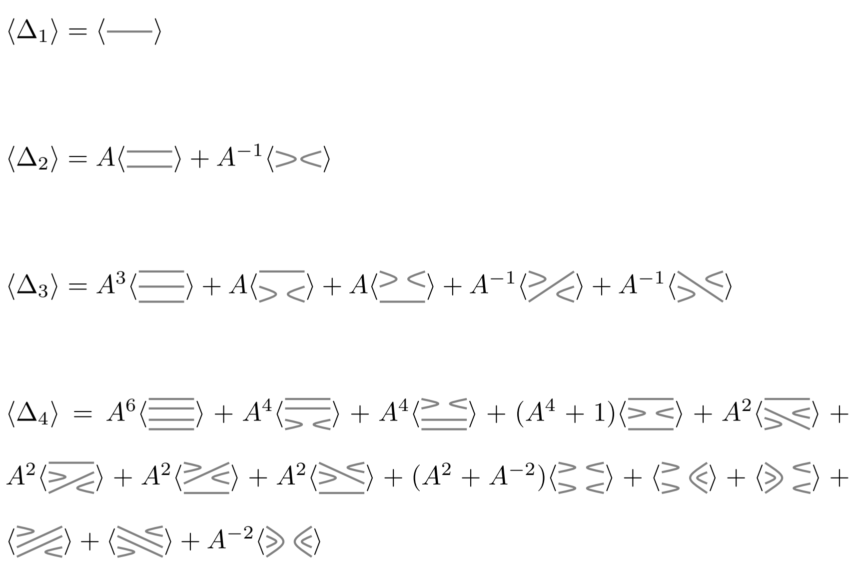
$$\begin{aligned}
\operatorname{mindeg}_A\langle\Delta_n\rangle &= \frac{1}{2}\operatorname{mindeg}_A\langle\overline{\Delta_n^2}\rangle + n - 1\\
&= \frac{1}{2}\left(-4\operatorname{maxdeg}_t V_{\overline{\Delta_n^2}}(t) + 3n^2 - 3n\right) + n - 1\\
&= \frac{1}{2}\left(-3n^2 + 2 + 3n^2 - 3n\right) + n - 1\\
&= -\frac{n}{2}.
\end{aligned}$$

For n odd, we use Equations (4), (8), and (10) to calculate

$$\begin{aligned}
\operatorname{mindeg}_A\langle\Delta_n\rangle &= \frac{1}{2}\operatorname{mindeg}_A\langle\overline{\Delta_n^2}\rangle + n - 1\\
&= \frac{1}{2}\left(-4\operatorname{maxdeg}_t V_{\overline{\Delta_n^2}}(t) + 3n^2 - 3n\right) + n - 1\\
&= \frac{1}{2}\left(-3n^2 + 3 + 3n^2 - 3n\right) + n - 1\\
&= \frac{-n+1}{2}.
\end{aligned}$$

□

Appendix



$$\langle\Delta_5\rangle = A^{10}\langle\;\rangle + A^{8}\langle\;\rangle + A^{8}\langle\;\rangle + (A^{8}+A^{4})\langle\;\rangle + (A^{8}+A^{4})\langle\;\rangle + A^{6}\langle\;\rangle + A^{6}\langle\;\rangle + A^{6}\langle\;\rangle + A^{6}\langle\;\rangle + (A^{6}+A^{2})\langle\;\rangle + (A^{6}+A^{2})\langle\;\rangle + (A^{6}+2A^{2})\langle\;\rangle + (A^{6}+2A^{2})\langle\;\rangle + (A^{6}+A^{-2})\langle\;\rangle + A^{4}\langle\;\rangle + A^{4}\langle\;\rangle + A^{4}\langle\;\rangle + A^{4}\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + (A^{4}+1)\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + A^{2}\langle\;\rangle + (A^{2}+A^{-2})\langle\;\rangle + (A^{2}+A^{-2})\langle\;\rangle + \langle\;\rangle + \langle\;\rangle + \langle\;\rangle + \langle\;\rangle + A^{-2}\langle\;\rangle + A^{-2}\langle\;\rangle$$

$$\langle\Delta_6\rangle = A^{15}\langle\;\rangle + A^{13}\langle\;\rangle + A^{13}\langle\;\rangle + (A^{13}+A^{9})\langle\;\rangle + (A^{13}+A^{9})\langle\;\rangle + (A^{13}+A^{9}+A^{5})\langle\;\rangle + (A^{11}+A^{7}+A^{3})\langle\;\rangle + (A^{11}+A^{7}+A^{3})\langle\;\rangle + (A^{11}+A^{-1})\langle\;\rangle + (A^{11}+2A^{7}+A^{3})\langle\;\rangle + (A^{11}+2A^{7}+A^{3})\langle\;\rangle + (A^{11}+3A^{7}+3A^{3}+A^{-1})\langle\;\rangle + A^{11}\langle\;\rangle + A^{11}\langle\;\rangle + A^{11}\langle\;\rangle + A^{11}\langle\;\rangle + (A^{11}+A^{7})\langle\;\rangle + (A^{11}+A^{7})\langle\;\rangle + (A^{11}+A^{7})\langle\;\rangle + (A^{11}+A^{7})\langle\;\rangle + A^{9}\langle\;\rangle + A^{9}\langle\;\rangle + A^{9}\langle\;\rangle + A^{9}\langle\;\rangle + (A^{9}+A^{5})\langle\;\rangle + (A^{9}+A^{5})\langle\;\rangle + (A^{9}+A)\langle\;\rangle + (A^{9}+A)\langle\;\rangle + (A^{9}+A)\langle\;\rangle + (A^{9}+A)\langle\;\rangle + (A^{9}+2A^{5}+A)\langle\;\rangle + (A^{9}+2A^{5}+A)\langle\;\rangle + (A^{9}+2A^{5}+A)\langle\;\rangle + (A^{9}+2A^{5}+A)\langle\;\rangle + (A^{9}+2A^{5})\langle\;\rangle + (A^{9}+2A^{5})\langle\;\rangle + (A^{9}+2A^{5})\langle\;\rangle + (A^{9}+2A^{5})\langle\;\rangle + (A^{9}+3A^{5}+3A+A^{-3})\langle\;\rangle + (A^{9}+2A^{5}+A)\langle\;\rangle + (A^{9}+2A^{5}+A)\langle\;\rangle + (A^{9}+A^{5})\langle\;\rangle + (A^{9}+A^{5})\langle\;\rangle + (A^{9}+A^{5})\langle\;\rangle + (A^{9}+A^{5})\langle\;\rangle + (A^{7}+A^{3}+$$

$A^{-1})\langle\,\rangle + (A^7+A^3+A^{-1})\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle +$
$(A^7+2A^3)\langle\,\rangle + (A^7+2A^3)\langle\,\rangle + (A^7+2A^3)\langle\,\rangle + (A^7+2A^3)\langle\,\rangle +$
$(A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle +$
$(A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle +$
$A^7\langle\,\rangle + A^7\langle\,\rangle + (A^7+A^3+A^{-1})\langle\,\rangle + (A^7+2A^3+A^{-1})\langle\,\rangle$
$+(A^7+2A^3+A^{-1})\langle\,\rangle + (A^7+2A^3+A^{-1})\langle\,\rangle + (A^7+2A^3+$
$A^{-1})\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle + (A^7+A^3)\langle\,\rangle +$
$(A^7+A^3)\langle\,\rangle + A^7\langle\,\rangle + A^7\langle\,\rangle + A^7\langle\,\rangle + A^7\langle\,\rangle + (A^5+$
$A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + A^5\langle\,\rangle +$
$A^5\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+$
$A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+$
$A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A+A^{-3})\langle\,\rangle + (A^5+$
$A+A^{-3})\langle\,\rangle + (A^5+2A)\langle\,\rangle + (A^5+2A)\langle\,\rangle + (A^5+A)\langle\,\rangle +$
$(A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + (A^5+A)\langle\,\rangle + A^5\langle\,\rangle + A^5\langle\,\rangle +$
$A^3\langle\,\rangle + A^3\langle\,\rangle + A^3\langle\,\rangle + A^3\langle\,\rangle + (A^3+A^{-1})\langle\,\rangle + (A^3+$
$A^{-1})\langle\,\rangle + (A^3+A^{-1})\langle\,\rangle + (A^3+A^{-1})\langle\,\rangle + (A^3+A^{-1})\langle\,\rangle +$
$(A^3+A^{-1})\langle\,\rangle + A^3\langle\,\rangle + A^3\langle\,\rangle + A^3\langle\,\rangle + A^3\langle\,\rangle + (A^3+$
$A^{-1})\langle\,\rangle + (A^3+A^{-1})\langle\,\rangle + A\langle\,\rangle + A\langle\,\rangle + A\langle\,\rangle + A\langle\,\rangle +$
$(A+A^{-3})\langle\,\rangle + A\langle\,\rangle + A\langle\,\rangle + A\langle\,\rangle + A\langle\,\rangle + A^{-1}\langle\,\rangle +$
$A^{-1}\langle\,\rangle + A^{-1}\langle\,\rangle + A^{-1}\langle\,\rangle + A^{-3}\langle\,\rangle$

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