# CONCORDANCE OF SURFACES IN 4-MANIFOLDS AND THE FREEDMAN-QUINN INVARIANT

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Abstract. We prove a concordance version of the 4-dimensional light bulb theorem for $\pi_1$-negligible compact orientable surfaces, where there is a framed but not necessarily embedded dual sphere. That is, we show that if $F_0$ and $F_1$ are such surfaces in a 4-manifold $X$ that are homotopic and there exists an immersed framed 2-sphere $G$ in $X$ intersecting $F_0$ geometrically once, then $F_0$ and $F_1$ are concordant if and only if their Freedman-Quinn invariant fq vanishes. The proof of the main result involves computing fq in terms of intersections in the universal covering space and then applying work of Sunukjian in the simply-connected case.

## 1. Introduction

The main goal of this paper is to prove a concordance analogue of Gabai's 4-dimensional light bulb theorem. The 4-dimensional light bulb theorem strengthens homotopy of embedded 2-spheres $R$ and $R'$ in a 4-manifold $X^4$ to isotopy, given a dual sphere $G$ which intersects both $R$ and $R'$ exactly once and a condition on how the homotopy interacts with 2-torsion in $\pi_1(X)$.

In this paper we work in the smooth category unless otherwise specified. All manifolds are smooth and oriented; all maps between manifolds are smooth. At the end of §6, we remark on how results extend to the topological category.

In §2 and §3 we discuss some context and motivation for our work. In §4 we discuss the Freedman-Quinn invariant in our context. In §5 we extend the work of Sunukjian and prove our main result for spheres (which follows from work of Stong as discussed below) and in §6 we extend this to higher genus surfaces. In §7 we give examples to illustrate the necessity of the various conditions in the statement of our theorem. In §8 we discuss some remaining questions.

**Definition 1.1.** Let $A$ and $B$ be $k$-dimensional submanifolds of an $n$-manifold $M^n$, $k \leq n$. Then $A$ and $B$ are *concordant* if there exists a $k$-manifold $\Sigma$ and a smooth embedding $f : \Sigma \times I \hookrightarrow M \times I$ so that $f(\Sigma \times 0) = A \times 0$ and $f(\Sigma \times 1) = -B \times 1$. Note that if $A$ and $B$ are ambiently isotopic, then they are concordant.

**Definition 1.2.** Given manifolds $V$ and $Z$, a map $f : V \to Z$ is called $\pi_1$-*negligible* if the resulting map on $\pi_1$ is trivial. A connected submanifold $A$ of $Z$ is called

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$\pi_1$-*negligible* if $\pi_1(A)$ maps trivially to $\pi_1(Z)$ with respect to the map induced by inclusion.

**Definition 1.3.** Given a surface $F$ embedded in a 4-manifold $X$, a *framed dual* to $F$ is an immersed sphere $G$ in $X$ that intersects $F$ in a single point, such that the normal bundle of $G$ has even Euler number. If the normal bundle of $G$ has odd Euler number, then we call $G$ an *unframed dual.*

Given a three-manifold $Y$ embedded in a 5-manifold $W$, a *framed dual* to $Y$ is an embedded sphere $G$ in $W$ that intersects $Y$ in a single point, such that the normal bundle of $G$ is trivial. If the normal bundle of $G$ is nontrivial, then we call $G$ an *unframed dual.*

These two definitions are related – if $G$ is a dual sphere for a surface $F$ in $X^4$ and $F$ is a boundary component of a 3-manifold $Y$ in $X \times I$, then after a small homotopy $G$ is a dual sphere for $Y$.

The following is the main result of this paper:

**Theorem 1.4.** *Let $F_0$ and $F_1$ be two homotopic embedded orientable genus-$g$ $\pi_1$-negligible surfaces in a 4-manifold $X^4$. Assume at there exists an immersed sphere $G$ in $X^4$ that is a framed dual to $F_0$. Then $F_0$ and $F_1$ are concordant if and only if* $\mathrm{fq}(F_0, F_1) = 0$.

We will first prove Theorem 1.4 in the case that $F_0$ and $F_1$ are 2-spheres (in §5) and then extend to positive-genus surfaces in §6. In §7, we give three explicit examples showing:

- The necessity of $\mathrm{fq}(F_0, F_1) = 0$ (Example 7.1, due to Schwartz [Sch19]),
- The sphere $G$ being framed (Example 7.2, constructed by the authors using Stong's [Sto93] km invariant),
- The conclusion of concordance rather than isotopy (Example 7.3, due to Sato [Sat91]).

We make use of recent work of Schneiderman and Teichner, who discuss the invariant fq associated to a pair of based-homotopic 2-spheres in a 4-manifold (this invariant originally appeared in work of Freedman-Quinn [FQ90] - hence the name). We define fq for based homotopic 2-spheres in §4 (and discuss free homotopy vs. based homotopy in §4.3). We extend to positive-genus surfaces whose fundamental groups include trivially into the ambient 4-manifold in §6.

The original version of this paper included the statement of Theorem 1.4 for spheres as the main theorem and the extension to higher genus surfaces as an application. Later, we were made aware of some relevant work of Freedman in Quinn in chapter 10 of [FQ90] - namely Theorem 10.9, and the subsequent correction and extension of this work by Stong in [Sto93]. In particular, Theorem 1.4 for spheres follows from the work of Stong in [Sto93] (which builds on and corrects [FQ90]). The authors are currently preparing an exposition/interpretation of Stong's work directly in the context of constructing/obstructing concordances of surfaces in 4-manifolds. A modification of Stong's proof also gives another proof of Theorem 1.4

– although using rather different techniques compared to what we use in this paper, as we will explain in our forthcoming work. In the context of spheres in a 4-manifold, Stong's work identifies two obstructions to two homotopic spheres being concordant: fq and a secondary obstruction which Stong calls the 5-dimensional Kervaire-Milnor invariant. Moreover, Stong shows that when these obstructions vanish, a concordance exists. It is useful to note that when either sphere has a framed immersed geometrically-dual sphere (as in Theorem 1.4), the secondary Kervaire-Milnor invariant automatically vanishes. More generally, this Kervaire-Milnor invariant vanishes if either sphere is not $s$-characteristic – i.e. intersects some immersed 2-sphere $R$ in $X^4$ with parity not equal to $R \cdot R \pmod 2$.

Theorem 1.4 has the following corollary (suggested by Sunukjian in [Sun15] and in personal communication).

**Corollary 1.5.** *Let $F_0$ and $F_1$ be homotopic genus-$g$ orientable surfaces embedded in a 4-manifold $X^4$, with $\pi_1(X^4)$ good in the sense of Freedman and Quinn* [FQ90]. *Assume all of the following:*

- *$\pi_1(F_i)$ maps trivially to $\pi_1(X)$ under inclusion for $i = 0, 1$,*
- *there exists a framed immersed sphere in $X^4$ that intersects $F_0$ geometrically once,*
- *there exists a (potentially unframed immersed sphere in $X^4$ that intersects $F_1$ geometrically once,*
- *$\mathrm{fq}(F_0, F_1) = 0$.*

*Then there exists a homeomorphism of pairs $(X, F_0) \cong (X, F_1)$.*

In this paper, Corollary 1.5 is the only time we leave the smooth category. A version of this corollary originally appeared in [Sun15] (for $\pi_1(X)$ with no 2-torsion) and later in [Mil19] (when $F_0$ is a 2-sphere and its dual is embedded). We restate the proof of Sunukjian [Sun15] here.

*Proof.* By Theorem 1.4, there is a concordance $C$ between $F_0$ and $F_1$ in $X \times I$. Let $\widetilde{X}$ denote the universal cover of $X$ and let $\widetilde{C}$ be the union of all lifts of $C$ in $\widetilde{X} \times I$. Let $\widetilde{F_i}$ denote the union of all lifts of $F_i$. Since the meridian of $F_i$ is nullhomotopic in $X - \mathring{N}(F_i)$, the cover $\widetilde{X} \times I - \mathring{N}(\widetilde{F_i})$ is simply-connected and hence is the universal cover of $X - \mathring{N}(F_i)$. Similarly, $\widetilde{X} \times I - \mathring{N}(\widetilde{C})$ is the universal cover of $X \times I - \mathring{N}(C)$. An application of the Mayer-Vietoris theorem yields that $\widetilde{X} \times I - \mathring{N}(\widetilde{C})$ is an $h$-cobordism from $\widetilde{X} - \mathring{N}(\widetilde{F_0})$ to $\widetilde{X} - \mathring{N}(\widetilde{F_1})$. Therefore, $X \times I - \mathring{N}(C)$ is an $h$-cobordism from $X - \mathring{N}(F_0)$ to $X - \mathring{N}(F_1)$. Since $\mathring{N}(C)$ and $(X \times I - \mathring{N}(C)) \cup \mathring{N}(C) = X \times I$ are both products and Whitehead torsion is additive, we conclude that $X \times I - \mathring{N}(C)$ has vanishing Whitehead torsion and hence is an $s$-cobordism. Finally, since $\pi_1(X \times I - \mathring{N}(C)) \cong \pi_1(X)$ is a good group, $X \times I - \mathring{N}(C)$ is topologically a product [FQ90]. This product structure yields a homeomorphism $(X, F_0) \cong (X, F_1)$. □

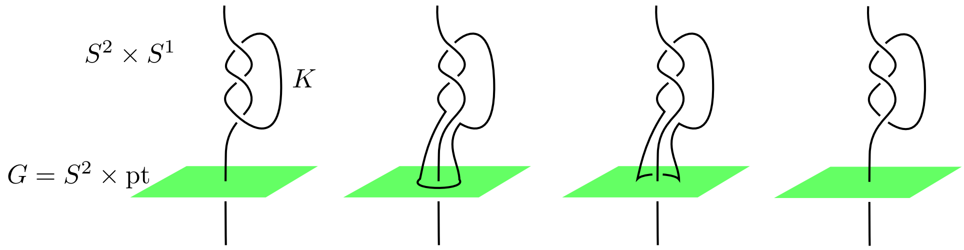


FIGURE 1. The 3-dimensional light bulb trick allows us to realize crossing changes of $K$ by isotopy.

## 2. Three-dimensional motivation

Both the light bulb theorem and the concordance analogue are motivated by 3-dimensional theorems in $S^2 \times S^1$. For notational purposes, let $J := \mathrm{pt} \times S^1$ and $G := S^2 \times \mathrm{pt}$ in $S^2 \times S^1$.

**Theorem 2.1** (3-dimensional light bulb trick (folklore))**.** *Let $K$ be a knot smoothly embedded in $S^2 \times S^1$ intersecting $G$ transversally exactly once. Then $K$ is isotopic to $J$.*

*Proof.* Isotope the knot $K$ to agree with $J$ near $G$. In the complement of $G$, $K$ and $J$ are homotopic arcs related by a sequence of crossing changes. Using the dual $G$, each of these crossing changes can be achieved as isotopy rather than homotopy. See Figure 1. □

**Theorem 2.2** ([Yil18], [DNPR18])**.** *Let $K$ be a knot smoothly embedded in $S^2 \times S^1$ in the homology class $[\mathrm{pt} \times S^1]$. Then $K$ is concordant to $J$.*

We summarize the proof of Yildiz [Yil18].

*Proof.* Since $\pi_1(S^2 \times S^1) \cong \mathbb{Z}$ is abelian, $K$ is homotopic to $J$ via some homotopy $H$. We show how to modify the track of the homotopy which we also call $H : S^1 \times I \to (S^2 \times S^1) \times I$ so that it is embedded. The levels where the track of the homotopy is not an embedding are the levels where crossing changes occur.

In Figure 2, we show how to change the track of the homotopy along 2-dimensional 1-handles (bands) together with 2-dimensional 2-handles (disk) to obtain an embedding of $S^1 \times I$. Each time a crossing change occurs in the track of $H$, we change the map so that a 1-handle (band) is added with the effect that the crossing change is achieved but a small meridian disk is also introduced. This small meridian disk is then carried throughout the rest of the new map. By considering the Euler characteristic, we see that this results in a new map that is a planar surface in $(S^2 \times S^1) \times I$ with boundary on one end the original knot $K$ and one the other end $J$ together with several small meridians of $J$. These meridians form an unlink and so by capping off the planar surface with disjoint disks bounding all of these meridians, we obtain the desired cobordism.

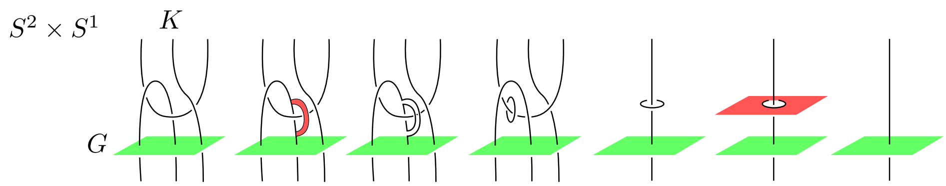


FIGURE 2. When $K \subset S^2 \times S^1$ intersects $G = S^2 \times \text{pt}$ algebraically once, we can achieve crossing changes of $K$ by concordance.

□

## 3. Four-dimensional motivation

In this section, we describe past results related to our main theorem to give context to our work. This section is not necessary to understand the proof of Theorem 1.4.

In 2017, Gabai [Gab17] proved the following theorem about isotopy of surfaces in 4-manifolds.

**Theorem 3.1** (4D light bulb theorem, [Gab17])**.** *Let $S_0, S_1$, and $G$ be 2-spheres smoothly embedded in a smooth 4-manifold $X^4$. Assume that $S_0$ and $S_1$ are homotopic, $G$ has trivial normal bundle, and $S_0$ and $S_1$ both transversally intersect $G$ in exactly one point. Finally, assume $\pi_1(X)$ has no 2-torsion. Then $S_0$ and $S_1$ are ambiently smoothly isotopic.*

In Theorem 3.1, the assumption that $\pi_1(X)$ has no 2-torsion is essential. Schwartz [Sch19] later gave explicit examples of triples $(S_0, S_1, G)$ of 2-spheres in a 4-manifold $X$ with $\pi_1(X) \cong \mathbb{Z}/2$ that satisfy the other hypotheses of Theorem 3.1, but yet $S_0$ and $S_1$ are not isotopic. We discuss these examples in §7.

The second author then proved a concordance version of the theorem, explicitly using the 4-dimensional light bulb theorem.

**Theorem 3.2** ([Mil19])**.** *Let $S_0, S_1$, and $G$ be 2-spheres smoothly embedded in a smooth 4-manifold $X^4$. Assume that $S_0$ and $S_1$ are homotopic, $G$ has trivial normal bundle, and $S_0$ transversally intersects $G$ in exactly one point. Finally, assume $\pi_1(X)$ has no 2-torsion. Then $S_0$ and $S_1$ are smoothly concordant.*

Note that Theorem 3.2 requires that one of the 2-spheres $(S_0)$ admit a dual sphere $G$. This assumption is essential; in general we do not expect arbitrary homotopic 2-spheres in a 4-manifold (even without 2-torsion in $\pi_1$) to be concordant. In Theorem 1.4, we weaken the condition that $S_0$ have a dual sphere to the meridian of $S_0$ being nullhomotopic in $X - S_0$. This weaker condition requires the presence of an immersed dual for $S_0$ rather than an embedded dual.

The conclusion of Theorem 3.2 (and Theorem 1.4) cannot be isotopy, rather than concordance. Sato [Sat91] constructs a 2-sphere $K$ in $S^2 \times S^2$ which is homotopic to $S^2 \times \mathrm{pt}$ but not isotopic to $S^2 \times \mathrm{pt}$. Theorem 3.2 (and indeed [Sun15, Theorem 6.1]) implies that $K$ is concordant to $S^2 \times \mathrm{pt}$. See Example 7.3.

Again, Schwartz's [Sch19] examples show that the 2-torsion assumption in Theorem 3.2 is necessary, as her counterexamples to the 4D light bulb theorem in the presence of 2-torsion are also counterexamples to an analog of Theorem 3.2 in the presence of 2-torsion. See Example 7.1.

The 4D lightbulb theorem was reproved and generalized by Schneiderman and Teichner in [ST19]. When $\pi_1(X)$ has 2-torsion, Schneiderman and Teichner show that it is necessary and sufficient for the *Freedman-Quinn invariant* $\mathrm{fq}(S_0, S_1)$ of the two spheres $S_0, S_1$ to vanish in order for there to exist an isotopy. We define this invariant in §4, but for now we discuss the full statement of the light bulb theorem.

**Theorem 3.3** (general 4D light bulb theorem, [Gab17] [ST19])**.** *Let $S_0, S_1$, and $G$ be 2-spheres smoothly embedded in a smooth 4-manifold $X^4$. Assume that $S_0$ and $S_1$ are homotopic, $G$ has trivial normal bundle, and $S_0$ and $S_1$ both transversally intersect $G$ in exactly one point. Then $S_0$ and $S_1$ are ambiently smoothly isotopic if and only if* $\mathrm{fq}(S_0, S_1) = 0$.

Gabai did not state Theorem 3.3 in terms of fq, but his full theorem statement is equivalent to the constructive direction of Theorem 3.3. Schneiderman and Teichner then noted that the invariant fq obstructs isotopy in the remaining cases.

The impetus of this current paper was a pair of subtle errors in an argument in [Sun15] involving 2-torsion and framings (see §5). Theorem 1.4 corrects this error by adding the assumption that the Freedman-Quinn invariant $\mathrm{fq}(S_0, S_1)$ vanishes and that the dual sphere $G$ is framed.

## 4. The Freedman-Quinn Invariant

We first review the definition of the Freedman-Quinn invariant fq as in [ST19]. We remind the reader that this invariant first appeared in work of Freedman-Quinn [FQ90], and indeed all of the results in this section are contained in [FQ90] (albeit phrased slightly differently). However, we will use the conventions of [ST19], and use them to give a new definition of fq via intersections of lifts in a universal cover. After reading the definition of the codomain of fq, the reader uninterested in the details can simply take Proposition 4.8 as the definition of fq (and Proposition 4.7 as the corresponding definition of the associated invariant $\mu_3$), and skip to Section 5. To arrive at this interpretation, we start with the 4-dimensional and 6-dimensional definitions of fq as in [ST19]. We include the 4-dimensional definition because we find it to be a natural starting point. The 6-dimensional definition is then interpreted in terms of intersections in the universal covering space.

To define the codomain of fq, one needs to define a certain homomorphism $\mu_3 : \pi_3(X) \to \mathbb{F}_2 T_X$. Each method of computing fq is accompanied by a corresponding method of computing $\mu_3$. We give the proofs of the different interpretations for both

$\mu_3$ and fq; however, the proofs are rather redundant and the reader might want to just look at the proofs for one of the two.

4.1. **Schneiderman-Teichner's interpretations.** Let $X$ be a smooth, oriented, based 4-manifold. We write

$$T_X := \{g \in \pi_1(X) : g^2 = 1, g \neq 1\}.$$

We call this the *2-torsion subset* of $\pi_1(X)$. In this section, we will describe Schneiderman and Teichner's [ST19] definition/method for computing the Freedman-Quinn invariant: the original definition is 6-dimensional, while the second (equivalent) definition is 5-dimensional. The 5-dimensional definition is most similar to the techniques that will be used to prove Theorem 1.4, but the 6-dimensional definition is easier to state and prove to be well-defined.

4.1.1. *A 6-dimensional definition of* fq. We write $\mathbb{F}_2 T_X$ to denote the $\mathbb{F}_2$-vector space with basis $T_X$. We have an embedding $\mathbb{F}_2 T_X \hookrightarrow \mathbb{Z}\pi_1(X)/\langle g + g^{-1}, 1\rangle$ given by $g \mapsto g$. We now describe a homomorphism $\mu_3 : \pi_3(X) \to \mathbb{F}_2 T_X$. Choose a basepoint for $S^3$. Given a based map $f : S^3 \to X$, cross the codomain with $\mathbb{R}^2$ to obtain a new map (we abuse notation) $f : S^3 \to X \times \mathbb{R}^2$, and then perturb $f$ so that it is generic and hence an embedding away from double points of intersection.

**Definition 4.1** ($\mu_3$, [ST19, Section 4])**.** For each double point $p$ in the image of $f$, choose an arc between the preimages $f^{-1}(p)$. This arc maps to a closed loop $\lambda_p$ in $f(S^3)$ based at $p$. Conjugate this loop by an arc contained in $f(S^3)$ from the basepoint to $p$ to obtain a based loop $\gamma_p$. The convention that the arc be contained in $f(S^3)$ ensures that the choice of the arc does not affect the resulting element of $\pi_1$. We write $g_p := [\gamma_p] \in \pi_1(X)$. See Figure 3. Thus, to every self-intersection of $f$, we associate an element of $\pi_1(X)$. Schneiderman–Teichner [ST19] show that this element is contained in $\{1\} \cup T_X$ (i.e. they show $g_p^2 = 1$).

We write

$$\mu_3(f) := \sum_{p \in f \cap f,\ g_p \neq 1} g_p \in \mathbb{F}_2 T_X.$$

As in [ST19], one can verify that $\mu_3 : \pi_3(X) \to \mathbb{F}_2 T_X$ is a homomorphism.

**Definition 4.2** (fq, [ST19, Section 4])**.** Let $S_0, S_1 : S^2 \hookrightarrow X$ be two based 2-spheres smoothly embedded in a compact, oriented, smooth based 4-manifold $X^4$ and assume that $S_0$ and $S_1$ are based-homotopic. Let $H : (S^2, *) \times I \to (X, *) \times I$ be an immersion with $H(S^2 \times 0) = S_0 \times 0$ and $H(S^2 \times 1) = -S_1 \times 1$ (e.g. the track of a based homotopy from $S_0$ to $S_1$). By taking the product of the codomain with $\mathbb{R}$, we have a map from a 3-manifold to a 6-manifold, $M := H(S^2 \times I) \subset X \times \{0\} \times I \subset X \times \mathbb{R} \times I$. By perturbing this map slightly, we obtain an immersion with isolated double points. As in the previous definition, for each point $p \subset M$ of

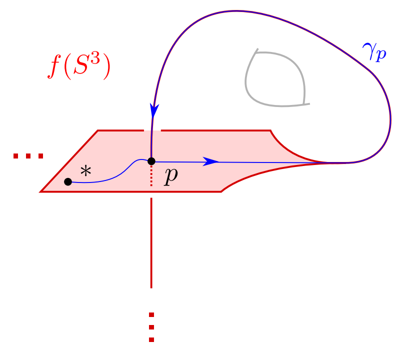


FIGURE 3. The image of a map $f : S^3 \to X^4 \times \mathbb{R}^2$ with isolated self-transverse self-intersections, one of which is point $p$. To associate an element of $\pi_1(X)$ to $p$, we choose an arc in $S^3$ between the two preimages of $p$. This arc maps to a loop in $f(S^3)$ through $p$. We then conjugate this loop by an arc contained in the image of $f$ between $p$ and the basepoint to obtain a based loop $\gamma_p$. We write $g_p := [\gamma_p] \in \pi_1(X)$.

self-intersection, we obtain a signed element $g_p$ of $T_X \cup \{1\}$. We define

$$\mathrm{fq}(S_0, S_1) := \sum_{p \in M \cap M, g_p \neq 1} g_p \in \mathbb{F}_2 T_X / \mu_3(\pi_3(X)).$$

The invariant fq is associated to the pair of spheres $(S_0, S_1)$, and is independent of the map $H$. Given an element $f$ of $\pi_3(X)$, we could surger $H$ along $f$ to obtain a new based map $H' : S^2 \times I \to X \times I$ where $\mathrm{fq}(H')$ and $\mathrm{fq}(H)$ differ by $\mu_3(f)$. We quotient the codomain of fq by $\mu_3(\pi_3(X))$ to ensure that fq is well-defined.

4.1.2. *A 5-dimensional construction of* fq. Now we discuss Schneiderman and Teichner's [ST19] 5-dimensional method of computing $\mu_3$ and fq. Our Proposition 4.4 is the content of Section 4.D of [ST19]. The following proposition is proved in the same way.

**Proposition 4.3** ([ST19]). *Fix a smooth based map $f : S^3 \to X \times I$. Perturb $f$ to have transverse self-intersections. The self-intersections of $f$ are circles $\gamma$ which are double-covered by $f^{-1}(\gamma)$. These double covers can either be connected or disconnected; see Figure 4. Let $C_f$ denote the set of circles of self-intersection of $f$ which have connected preimages under $f$. Given $\gamma \in C_f$, we may choose an arc in $f(S^3)$ from the basepoint to $\gamma$ to obtain an element $[\gamma] \in \pi_1(X)$. Since $f^{-1}(\gamma)$ is nullhomotopic in $S^3$, we must have $[\gamma] \in T_X$.*

*Then we have*

$$\mu_3(f) = \sum_{\gamma \in C_f, [\gamma] \neq 1} [\gamma] \in \mathbb{F}_2 T_X.$$

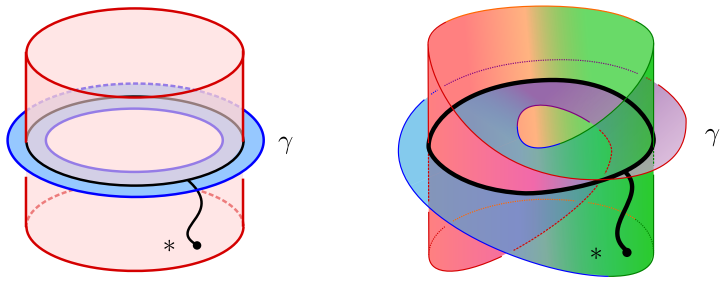


FIGURE 4. A circle $\gamma$ of self-intersection of an immersed $S^3$ or $S^2 \times I$ in $X \times I$. Since the 3-manifold is simply-connected, $\gamma$ represents an element $g$ of $\pi_1(X)$. **Left:** $\gamma$ has disconnected preimage under the immersion. We conclude $g = 1$ and $\gamma \notin C_f$. **Right:** $\gamma$ has connected preimage under the immersion. We conclude $g^2 = 1$ and $\gamma \in C_f$.

Note that the orientation of the circles $\gamma$ in the above does not affect the resulting element of $\pi_1$. Additionally, there is a similar alternative way of computing fq.

**Proposition 4.4** ([ST19])**.** *Let $S_0, S_1 : S^2 \to X$ be based 2-spheres smoothly embedded in a compact, oriented, smooth, based 4-manifold $X^4$. Let $H : (S^2, *) \times I \to (X, *) \times I$ be an immersion with self-transverse self-intersections and $H(S^2 \times 0) = S_0 \times 0$ and $H(S^2 \times 1) = -S_1 \times 1$ (e.g. the track of a based homotopy from $S_0$ to $S_1$). Let $C_H$ denote the set circles of self-intersection of $H$ which have connected preimage under $H$. Given $\gamma \in C_H$, we may choose an arc in $f(S^3)$ from the basepoint to $\gamma$ to obtain an element $[\gamma] \in \pi_1(X)$. Since $f^{-1}(\gamma)$ is nullhomotopic in $S^2 \times I$, we must have $[\gamma] \in T_X$. Then*

$$\mathrm{fq}(S_0, S_1) = \sum_{\gamma \in C_H, [\gamma] \neq 1} [\gamma] \in \mathbb{F}_2 T_X / \mu_3(\pi_3(X))$$

4.2. **Computing fq using intersections of lifts in covering spaces.** For the purposes of this paper, it will be useful to give new methods of computing $\mu_3$ and fq in terms of covering maps. We will again give 5-dimensional and 6-dimensional variations on the definition of both $\mu_3$ and fq.

4.2.1. *Computing* fq *using covering spaces: 6-dimensional version.* Let $X^4$ be a smooth based 4-manifold and choose a basepoint for the universal cover $\widetilde{X}$ above the basepoint of $X$.

**Proposition 4.5.** *Fix a based map $f : S^3 \to X$ and cross the codomain with $\mathbb{R}^2$ to obtain a new map $f : S^3 \to X \times \mathbb{R}^2$ and perturb $f$ to be generic and hence an embedding away from double points of intersection.*

*Let $Y$ indicate the union of images of all lifts of $f$ to $\widetilde{X} \times \mathbb{R}^2$, and let $Y_1$ be the image of the lift of $f$ based at the basepoint for $\widetilde{X}$. Perturb $Y$ equivariantly so that $Y_1$ intersects other components of $Y$ transversally in isolated points.*

*Now $\pi_1(X)$ acts on $\widetilde{X}$, and in particular on $Y$ by permuting the lifts of $f$. Let $Y_g = g \cdot Y_1$. Then*

$$\mu_3(f) = \sum_{g \in T_X} \frac{1}{2}|Y_1 \cap Y_g| \cdot g \in \mathbb{F}_2 T_X.$$

*Proof.* Let $\pi : \widetilde{X} \to X$ be the covering projection. Fix a point $p \in Y_1 \cap Y_g$. Let $q \in Y_1 \cap Y_g$ be the $g$-translate of $p$, where $g \in T_X$.

Let $A$ be an arc in $Y_1$ from the basepoint of $Y_1$ to $p$, and $B$ an arc in $Y_g$ from $p$ to $q$. Then $\gamma_p := \pi(\overline{A})\pi(B)\pi(A)$ is a based loop in $X$ with $[\gamma_p] = g_{\pi(p)}$, as in Definition 4.1.

We note that if $s$ is a self-intersection of $\pi(Y)$ with $g_s \neq 1$, then $s$ lifts to a point of self-intersection of $Y_1$ and $Y_{g_s}$. Since $g_s$ acts on $Y_1 \cap Y_{g_s}$ by permuting pairs of points, there are exactly two lifts of $s$ in $Y_1 \cap Y_{g_s}$.

We therefore conclude that for every two points in $Y_1 \cap Y_g$, we find one self-intersection $s$ of $\pi(Y)$ with $g_s = g$. Thus,

$$\mu_3(f) = \sum_{q \in \pi(Y) \cap \pi(Y), g_q \neq 1} g_q = \sum_{g \in T_X} \frac{1}{2}|Y_1 \cap Y_g| \cdot g.$$

□

The following result is proved similarly: we compare the proposed formula for fq to that in Proposition 4.2.

**Proposition 4.6.** *Let $S_0, S_1 : S^2 \to X^4$ be based embeddings of spheres in $X^4$ that are based homotopic in $X$. Let $Y$ be the image of an immersion $H : S^2 \times I \to \widetilde{X} \times \mathbb{R} \times I$ that has $H|_{S^2 \times 0} = S_0$ when we consider $S_0$ as a map into $X \times \{0\} \times \{0\}$ and similarly $H|_{S^2 \times 1} = -S_1$ when we consider $S_1$ as a map into $X \times \{0\} \times \{1\}$.*

*Given $g \in \pi_1(X)$, let $Y_g := g \cdot Y_1$. Perturb the $Y_g$ equivariantly so that $Y_1$ and $Y_g$ have isolated transverse intersections for all $g$. Then*

$$\mathrm{fq}(S_0, S_1) = \sum_{g \in T_X} \frac{1}{2}|Y_1 \cap Y_g| \cdot g \in \mathbb{F}_2 T_X / \mu_3(\pi_3(X)).$$

4.2.2. *Computing* fq *using covering spaces: 5-dimensional version.* Finally, we give yet another pair of definitions of $\mu_3$ and fq: 5-dimensional definitions involving coverings. These constructions will be essential in the proof of Theorem 1.4.

**Proposition 4.7.** *Fix a based map $f : S^3 \to X$ and cross the codomain with $\mathbb{R}$ to obtain a new map $f : S^3 \to X \times \mathbb{R}$ and perturb $f$ to be generic and hence an embedding away from circles of double points.*

*Let $Y$ indicate the union of images of all lifts of $f$ to $\widetilde{X} \times \mathbb{R}$, and let $Y_1$ be the image of the lift of $f$ based at the basepoint for $\widetilde{X}$. Perturb $Y$ equivariantly so that $Y_1$ intersects other components of $Y$ transversally in circles.*

*Now $\pi_1(X)$ acts on $\widetilde{X}$, and in particular on $Y$ by permuting the lifts of $f$. Let $Y_g = g \cdot Y_1$. Given $g \in \pi_1(X)$, let $c_g$ denote the number of components of $Y_1 \cap Y_g$ fixed by $g$. Note that if $g \notin T_X \cup \{1\}$, then $c_g = 0$. Then*

$$\mu_3(f) = \sum_{g \in T_X} c_g \cdot g \in \mathbb{F}_2 T_X.$$

*Proof.* We will prove this by showing that the given formula for $\mu_3$ is equivalent to that in Proposition 4.3.

Let $p : \widetilde{X} \to X$ denote the universal covering map. Fix $f \in \pi_3(X)$, a generic perturbation $f : S^3 \to X \times I$, and an element $g \in T_X$. Recall from Proposition 4.3 that every circle of self-intersection in the image of $f$ corresponds to an element of $\pi_1(X)$. We give bijections

$$\{\pi_0(Y_1 \cap Y_g) \mid g(C) = C\} \underset{\text{p}}{\overset{p^{-1} \cap (Y_1 \cap Y_g)}{\rightleftarrows}} \{\pi_0(f \cap f) \mid [C] = g,\ f^{-1}(C) \text{ connected}\}.$$

First, we show that these maps are well-defined and then we verify that they are inverses of one another. Let $C$ be a circle of intersection of $Y_1$ and $Y_g$ that is fixed setwise by $g$ (i.e. $g$ acts on $C$ by rotation through $\pi$). Then $p : C \to p(C)$ is a double cover of a circle $p(C)$ of self-intersection in the image of $f$, where $p(C)$ represents $g$. Since $C$ is connected, $f^{-1}(p(C))$ is connected.

On the other hand, suppose $C$ is a circle of self-intersection in the image of $f$ corresponding to $g \in T_X$ with $f^{-1}(C)$ connected. Then $\widetilde{C} := p^{-1}(C) \cap (Y_1 \cap Y_g)$ double covers $C$. Since $g$ is nontrivial, $\widetilde{C}$ is a connected circle formed by two arc lifts of $C$. The action of $g$ permutes these arcs, fixing $\widetilde{C}$ setwise.

Now consider the two defined maps, $p$ (projection) and $p^{-1} \cap (Y_1 \cap Y_g)$ (lifting to $Y_1 \cap Y_g$). The composition in either order is clearly the identity. This completes the proof. □

The following result follows by the same argument as in Proposition 4.7. We similarly compare the proposed formula for fq to that in Proposition 4.4.

**Proposition 4.8.** *Let $S_0, S_1 : S^2 \to X^4$ be based embeddings of spheres in $X$ that are based-homotopic in $X$. Let $H$ be an immersion $H : (S^2, *) \times I \to (X, *) \times I$ that has $H|_{S^2 \times 0} = S_0$ when we consider $S_0$ as a map into $X \times \{0\}$ and similarly $H|_{S^2 \times 1} = -S_1$ when we consider $S_1$ as a map into $X \times \{1\}$.*

*Let $Y$ indicate the union of images of all lifts of $H$ to $\widetilde{X} \times I$, and let $Y_1$ be the image of the lift of $f$ based at the basepoint for $\widetilde{X}$. Perturb $Y$ equivariantly so that $Y_1$ intersects other components of $Y$ transversally in circles.*

*Now $\pi_1(X)$ acts on $\widetilde{X}$, and in particular on $Y$ by permuting the lifts of $H$. Let $Y_g = g \cdot Y_1$. Given $g \in \pi_1(X)$, let $c_g$ denote the number of components of $Y_1 \cap Y_g$ fixed by $g$. Note that if $g \notin T_X \cup \{1\}$, then $c_g = 0$. Then*

$$\mathrm{fq}(S_0, S_1) := \sum_{g \in T_X} c_g \cdot g \in \mathbb{F}_2 T_X / \mu_3(\pi_3(X)).$$

4.3. **Based homotopy versus free homotopy.** In this section, we have given many definitions of $\mathrm{fq}(S_0, S_1)$ when $S_0, S_1$ are based-homotopic 2-spheres in a 4-manifold $X^4$. In the main theorem of this paper, we consider 2-spheres which are homotopic, but do not specify that this homotopy is based. Schneiderman and Teichner [ST19, Lemma 2.1] show that when $S_0$ and $S_1$ have a common geometric dual (and agree near the dual), then if $S_0$ and $S_1$ are homotopic then they are based-homotopic. We need an analogous lemma in order to deal with basepoints.

**Proposition 4.9.** *Let $S_0$ and $S_1$ be 2-spheres smoothly embedded in a smooth 4-manifold $X^4$. Let $G$ be a 2-sphere immersed in $X$. Assume that $S_0$ intersects $G$ transversally once at a point $z$ and that $S_0$ and $S_1$ are homotopic. Then after an isotopy of $S_1$, $S_0$ and $S_1$ are based-homotopic with basepoint $z$.*

*Proof.* Take $H : S^2 \times I \to X \times I$ to be a free homotopy of $S_0$ to $S_1$. Note $S_i$ represents an element $[S_i] \in \pi_2(X, z)$. Fix $z_0 \in S^2$ so that $H(S^2 \times 0) = z \times 0$. Then $H(z_0 \times I)$ represents an element $g$ of $\pi_1(X, z)$ with $g[S_0] = [S_1]$.

Take $H$ to be transverse to $G$ and consider $L = H^{-1}(G \times I)$, a properly embedded 1-dimensional submanifold of $S^2 \times I$. Note the boundary of $L$ consists of $z_0 \times 0$, and an odd number of points of the form $y_i \times 1$. Let $L_0$ be the component of $L$ containing $z_0 \times 0$ and assume the other endpoint of $L_0$ is $y_0 \times 1$. Compose $H$ with an isotopy of $S_1$ taking $H(y_0)$ to $z$ by isotopy in $G$ and set the resulting composed homotopy to be $H$. Now $\partial L_0 = z_0 \times \{0, 1\}$. Since $S^2 \times I$ is simply-connected, this implies $H(L_0)$ represents $g$. Since $H(L_0) \subset G \times I \cong S^2 \times I$ (away from self-intersections), $g \in \pi_1(X, z)$ is the identity and we conclude $[S_0] = [S_1] \in \pi_2(X, z)$. See Figure 5 for an illustration. □

In Theorem 1.4 (or in the 4D light bulb theorem), we assume that spheres $S_0$ and $S_1$ are homotopic. By Proposition 4.9, there exists another 2-sphere $S_1'$ isotopic to $S_1$ so that $S_0$ and $S_1'$ are based-homotopic. The sphere $S_0$ is concordant (resp. isotopic) to $S_1$ if and only if it is concordant (resp. isotopic) to $S_1'$. Thus, we may without loss of generality take $S_0$ to be based-homotopic to $S_1$.

## 5. Concordance of surfaces in 4-manifolds

In [Sun15], Sunukjian proves the following theorem that generalizes the result of Kervaire that all 2-knots are slice [Ker65]:

**Theorem 5.1** ([Sun15, Theorem 6.1])**.** *Let $X^4$ be a simply-connected 4-manifold. Let $\Sigma_0$ and $\Sigma_1$ be compact, oriented, embedded, homologous surfaces in $X^4$ of the same genus. Then $\Sigma_0$ and $\Sigma_1$ are concordant.*

Sunukjian then goes on to use Theorem 5.1 to study manifolds with nontrivial fundamental group [Sun15, Theorem 6.2]. The main strategy is to require $\pi_1(\Sigma_i)$ to

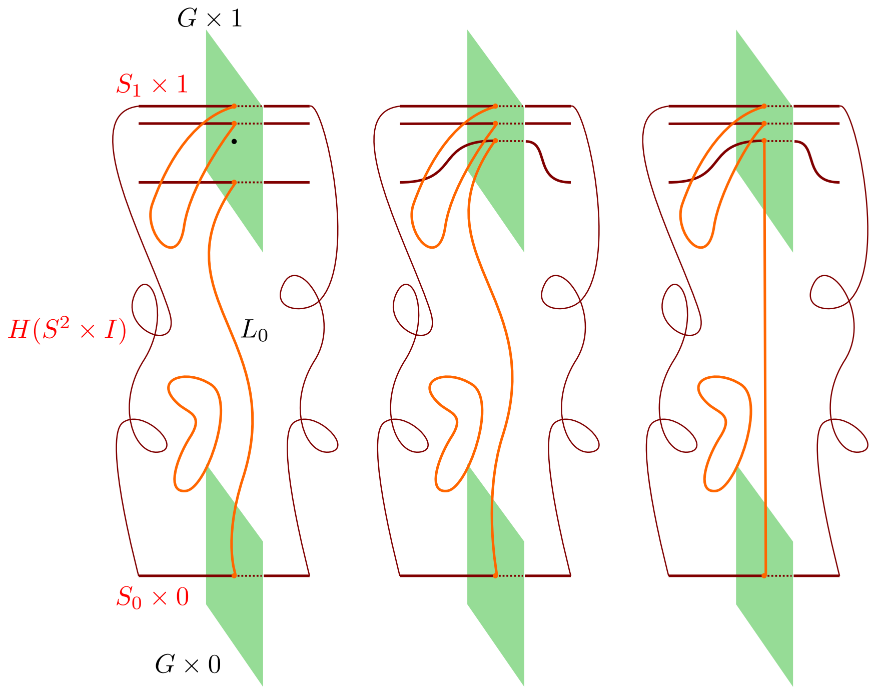


FIGURE 5. The proof of Lemma 4.9. **Left:** $H$ is a homotopy from $S_0$ to $S_1$. The arc $L_0$ is the component of $H^{-1}(G \times I)$ with $z \times 0$ as an endpoint. **Middle:** We isotope $S_1$ near $G$ and concatenate with $G$ to find a new homotopy where $L_0$ has boundary $z \times \{0, 1\}$. **Right:** Let $p : X \times I \to X$ be projection. Since $p \circ H(L_0)$ is contained in $G$ (away from self-intersections of $G$), $p \circ H(L_0)$ is a nullhomotopic loop. Contracting this loop yields a homotopy based at $z$.

include trivially into the 4-manifold so that $\Sigma_i$ can be lifted to the universal cover $\widetilde{X}$. In $\widetilde{X} \times I$, the various lifts of $\Sigma_0 \times \{0\}, -\Sigma_1 \times \{1\}$ then cobound embedded copies $Y_g, g \in \pi_1(X)$ of $\Sigma_i \times I$ which pairwise intersect. In the cover, one may attempt to equivariantly surger these 3-manifolds to force them to be disjoint, and then use the proof methods of the simply-connected case to do further equivariant surgery to obtain a disjoint collection of concordances. These concordances then project to a concordance in $X \times I$.

There is a problem in this approach, as pointed out in [Mil19, §1], that calls attention to 2-torsion in $\pi_1(X)$. A difficulty arises when some element $g$ of $\pi_1(X)$ fixes a circle $C$ of intersection in $Y_1 \cap Y_g$. If we attempt to surger $Y_1$ at $C$ to remove this intersection, then in order to do the surgery equivariantly we must then surger $Y_g$ at $C$ as well – potentially causing us to accidentally create a new circle of intersection near $C$ if we are not careful during this surgery operation. In fact, this obstacle is precisely the motivation for the definitions of $\mu_3$ and fq given in Propositions 4.7 and 4.8. There is an additional subtlety that requires us to assume that the dual sphere $G$ is framed; this is necessary in one of the constructive moves used by Sunukjian (Move 5.8).

Before proving Theorem 1.4, we review the techniques used in [Sun15] to modify embedded 3-manifolds in a 5-manifold. For our modification of the argument, we will also need to use these techniques on immersed 3-manifolds in a 5-manifold. The general principle is to surger the 3-manifold along 4-dimensional handles embedded in the ambient 5-manifold.

**Definition 5.2** (Ambient Dehn 1-surgery)**.** Let $Y^3$ be a 3-manifold embedded in a 5-manifold $W^5$. Suppose that we are given an arc $\alpha$ in $W$ with endpoints on $Y$ and interior away from $Y$. Frame $\alpha$; now the unit 3-ball bundle over $\alpha$ yields a $I \times B^3$, along the boundary of which we may surger $Y$ to obtain an embedded 3-manifold $Y'$. Choose the framing of $\alpha$ so that $Y' \cong Y \# S^1 \times S^2$. We say that $Y'$ is obtained from $Y$ by *ambient Dehn 1-surgery* along $\alpha$. The "1" refers to the fact that $Y$ is being surgered along a 4-dimensional 1-handle.

**Definition 5.3.** [Ambient Dehn 2-surgery] Let $Y^3$ be an oriented 3-manifold embedded in an oriented 5-manifold $W^5$. Suppose that we are given an oriented circle $\gamma \subset Y$ with an integral framing. If some pushoff of $\gamma$ into $W$ is null-homotopic in $W$, then there exists an embedded disk $\Delta \subset W$ with $Y \cap \Delta = \gamma$ ($\Delta$ is embedded since we are in dimension 5). If $\Delta$ has the property that there exists a 2-dimensional subbundle $B$ of its trivial 3-dimensional normal bundle $N_W(\Delta)$ such that $B$ has a trivialization extending the trivialization coming from the framing on the boundary circle in $N_Y(\gamma)$, then the unit disk bundle over $\Delta$ in $B$ yields a copy of $D^2 \times D^2$ in $W$ that intersects $Y$ in a tubular neighborhood of $\gamma$, such that $(Y - S^1 \times D^2) \cup (D^2 \times S^1)$ is the 3-manifold obtained by performing the Dehn surgery on $Y$ along $\gamma$ using the specified framing. We call this move *ambient Dehn 2-surgery*, where the "2" refers to the fact that $Y$ is being surgered along a 4-dimensional 2-handle.

For a framing $\phi$ of $N_Y\gamma$ and a given disk $\Delta$, we say that a framing is a $\Delta$-*admissible* framing if such a subbundle $B \subset N_W(\Delta)$ as above exists, with a framing on $B$ that extends $\phi$.

**Lemma 5.4.** *Fix a framing $\phi$ of $N_Y(\gamma)$ where the notation is as in Definition 5.3. Then the $\Delta$-admissible framings on $\gamma$ are exactly the result of of the even integers acting on $\phi$, or the result of all of the odd integers acting on $\phi$, where the free and transitive action of $\mathbb{Z}$ on the set of framings of $\gamma$ is as in Figure 6.*

*Proof.* Note that there is a unique framing, up to homotopy through framings $F : N_W(\Delta) \xrightarrow{\sim} D^2 \times \mathbb{R}^2$. By restricting $F$ to the bundle over $\gamma$, we have

$$F : N_W(\gamma) = N_Y(\gamma) \oplus \xi \xrightarrow{\sim} S^1 \times \mathbb{R}^3$$

where $\xi$ is the line bundle as is Figure 7. The framing $\phi$ gives another framing

$$\phi \oplus i : N_Y(\gamma) \oplus \xi \xrightarrow{\sim} S^1 \times \mathbb{R}^3$$

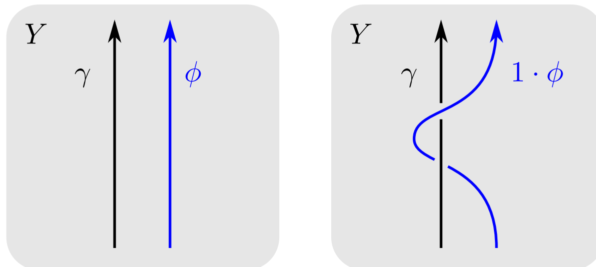


FIGURE 6. **Left:** A framing $\phi$ of $N_Y(\gamma)$, where $\gamma$ is a curve in 3-manifold $Y$. **Right:** There is an action of $\mathbb{Z}$ on the set of framings of $N_Y(\gamma)$. Here we draw $1 \cdot \phi$.

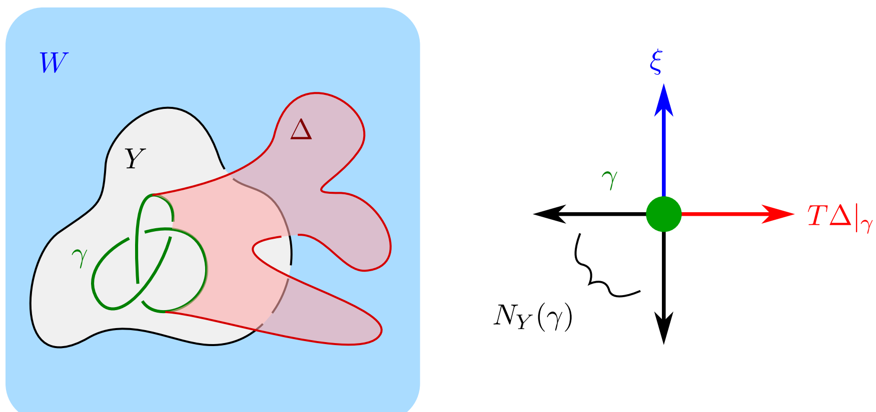


FIGURE 7. **Left:** A schematic of a disk $\Delta$ with boundary $\gamma \subset Y^3 \subset W^5$. **Right:** A cross-section of $\gamma$. The normal bundle $N_\gamma(W)$ is 4-dimensional. It contains the 2-dimensional subbundle $N_Y(\gamma)$ and the line bundles $T\Delta|_\gamma$ and $\xi$, all of which are normal to each other.

where $\xi$ is oriented as in Figure 7 and $i$ is the corresponding trivialization. Comparing these two trivializations, we have

$$(\phi \oplus i) \circ F^{-1} : S^1 \times \mathbb{R}^3 \xrightarrow{\simeq} S^1 \times \mathbb{R}^3$$

which can be thought of as a map $[\phi] \in \pi_1(\mathrm{GL}^+(\mathbb{R}^3)) \cong \pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}/2$.

Now $\phi$ is a $\Delta$-admissible framing if and only if there exists a 2-dimensional subbundle $B \subset N_W\Delta$ extending $N_Y(\gamma)$ together with a framing $\Phi : B \xrightarrow{\simeq} D^2 \times \mathbb{R}^2$ such that the following commutes:

$$\begin{tikzcd} B \arrow[r, "\Phi"] & D^2 \times \mathbb{R}^2 \\ N_Y(\gamma) \arrow[u, hook, "\partial"] \arrow[r, "\phi"] & S^1 \times \mathbb{R}^2 \arrow[u, hook, "\partial"'] \end{tikzcd}$$

or equivalently, if the trivialization $F$ can be chosen so that the following commutes:

$$\begin{array}{ccc} N_W(\Delta) & \xrightarrow{F} & D^2 \times \mathbb{R}^2 \\ \partial \big\uparrow & & \partial \big\uparrow \\ N_Y(\gamma) \oplus \xi & \xrightarrow{\phi} & S^1 \times \mathbb{R}^3 \end{array}$$

But this is equivalent to $[\phi] = 0$. To complete the proof, note that when $1 \in \mathbb{Z}$ acts on $\phi$ as in Figure 6, $[1 \cdot \phi] \neq [\phi]$. □

**Lemma 5.5.** *Let $\Delta$ be a disk in a 5-manifold with $\gamma := \partial\Delta$ contained in a 3-manifold $Y$ (and $\mathring{\Delta} \cap Y = \emptyset$). Let $R$ be a 2-sphere in $W$ with trivial normal bundle that is disjoint from $\Delta$ and $Y$. Let $\widetilde{\Delta}$ be a disk obtained from $\Delta$ by tubing $\Delta$ to $R$. Then a framing $\phi$ of $N_Y(\gamma)$ is $\Delta$-admissible if and only if it is $N_Y(\widetilde{\Delta})$ admissible.*

*Proof.* See Figure 8.

Note that $\Delta$ can be obtained from $\widetilde{\Delta}$ by tubing $\widetilde{\Delta}$ to a parallel copy of $R$ with opposite orientation, so it is sufficient to prove one direction of implication.

Let $\phi$ be a framing of $N_Y(\gamma)$, i.e. a choice of two non-vanishing vector fields $v_1, v_2$ on $\nu(\gamma)$ that span $N_Y(\gamma)|_p$ for each $p$ in $\gamma$. Assume $\phi$ is $\Delta$-admissible, so that $v_1$ and $v_2$ can be extended to non-vanishing vector fields $V_1$ and $V_2$ (respectively) on a neighborhood of $\Delta$ so that their restriction to $\Delta$ spans some 2-dimensional subbundle $B_\Delta$ of $N_W(\Delta)$.

Now $\widetilde{\Delta}$ is obtained by tubing $\Delta$ to $R$ along a framed arc $\delta$. Homtope $V_1, V_2$ so that $V_1(p)$ and $V_2(p)$ agree with the framing of $\delta$ at $p = \delta \cap \Delta$, and then use the framing of $\delta$ to extend $V_1, V_2$ to a neighborhood of $\delta$.

Since $R$ has trivial normal bundle, we can choose non-vanishing vector fields $\omega_1, \omega_2$ on near $R$ whose restriction to $R$ spans a 2-dimensional subbundle $B_R$ of $N_W(\Delta)$ whose total space is $R \times D^2$. Moreover, we can take $\omega_1, \omega_2$ to agree with $V_1, V_2$ near $\delta \cap R$. Therefore, we can combine $V_1$ with $\omega_1$ and $V_2$ with $\omega_2$ to obtain a framing of $\widetilde{\Delta}$ that extends $\phi$. We conclude that $\phi$ is $\widetilde{\Delta}$-admissible.

On the other hand, if $\phi$ is *not* $\Delta$-admissible, then $1 \cdot \phi$ is $\Delta$-admissible. By the above argument, $1 \cdot \phi$ is $\widetilde{\Delta}$-admissible. Then by Lemma 5.4, $\phi$ is not $\widetilde{\Delta}$-admissible. □

We can also perform ambient Dehn 2-surgeries along many disjoint disks simultaneously. That is, suppose that say $\pi_1(W - Y) = 1$, and we are given several disjoint curves $\gamma \subset Y$ along which we want to do an ambient Dehn surgery. The existence of disjoint embedded disks meeting $Y$ in $\gamma$ is guaranteed by our condition that $\pi_1(W - Y) = 1$ (again since we are in a 5-manifold surfaces will generically not intersect). Again, we may use the disks to perform ambient Dehn 2-surgery to $Y$ along $\gamma$. The choice of framing of the disks determines the framing of this Dehn surgery. If we initially specify a choice of framing of each curve in $\gamma$, then there is no guarantee that the disks will have the desired 2-dimensional subbundles of their normal bundles such that the framing trivializations extend.

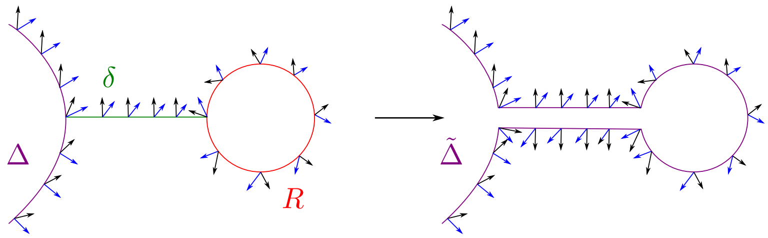


FIGURE 8. Here, $\widetilde{\Delta}$ is a disk obtained by tubing a disk $\Delta$ to a framed sphere $R$ along a framed arc $\delta$. Given a 2-dimensional framing $\phi$ of $\Delta$, we can amalgamate framings of $\delta$ and $R$ with $\phi$ to obtain a framing of $\widetilde{\Delta}$ that agrees with $\phi$ near $\partial\widetilde{\Delta} = \partial\Delta$.

In the proof of Theorem 5.1 in [Sun15], Sunukjian begins by taking some 3-manifold $Y$ in $X^4 \times I$ with boundary $\Sigma_0 \times \{0\} \sqcup -\Sigma_1 \times \{1\}$. The fundamental group conditions ensure the existence of disks bounding surgery curves in $Y$. The fact that all 3-manifolds are spin-cobordant is used to show that a framed link $\gamma$ that surgers $Y$ to $\Sigma_0 \times I$ can be chosen so that the specified surgery can be carried out ambiently, as described above. In our proof we will be using Theorem 5.1, or rather the following result which is proved implicitly in [Sun15]:

**Theorem 5.6** ([Sun15, Theorem 6.1])**.** *Let $Y^3 \subset W^5$ be a properly embedded submanifold, where $Y$ is compact, $W$ is not necessarily compact and is simply-connected, and $\pi_1(W - Y)$ is cyclic. Let $Y'$ be any other compact 3-manifold with $\partial Y' \cong \partial Y$. Then there is an ambient Dehn surgery that can be performed on $Y$ in $W^5$ that yields an embedded 3-manifold diffeomorphic to $Y'$.*

There are two situations in which we will use ambient Dehn 2-surgery. Together, these two moves will be used to eliminate double points of 3-manifolds immersed in 5-manifolds, assuming that the 3-manifolds have framed dual spheres.

**Move 5.7** (Remove an intersection)**.** Let $Y_1, Y_g$ be embedded 3-manifolds in $W^5$ with a single circle of self-intersection $\gamma$ so that $\gamma$ is an unknot in $Y_1$. Let $\Delta$ be a disk bounded by $\gamma$ contained in $Y_1$. Let $B$ be the 2-dimensional subbundle of $N_W(\Delta)$ that is normal to $Y_1$. Then by performing ambient Dehn 2-surgery on $Y_g$ along $\Delta \times D^2$ in $B$, we obtain a new 3-manifold $Y_g'$ that is disjoint from $Y_1$. Thus, this move, "eliminates double points." See Figure 9 (bottom row) or Figure 6 in [Sun15].

Move 5.7 may seem to have overly strong hypotheses on when it can be performed: there is generally no reason to suspect that $Y_1 \cap Y_g$ has any unlinked, unknotted components. However, we now consider another move that can simply the intersection link between $Y_1$ and $Y_g$.

**Move 5.8** (Unknot a self-intersection)**.** Take $Y_1$, $Y_g$ to be 3-manifolds immersed in $W$ and assume $Y_g$ has a framed dual sphere $G$ that does not intersect $Y_1$.

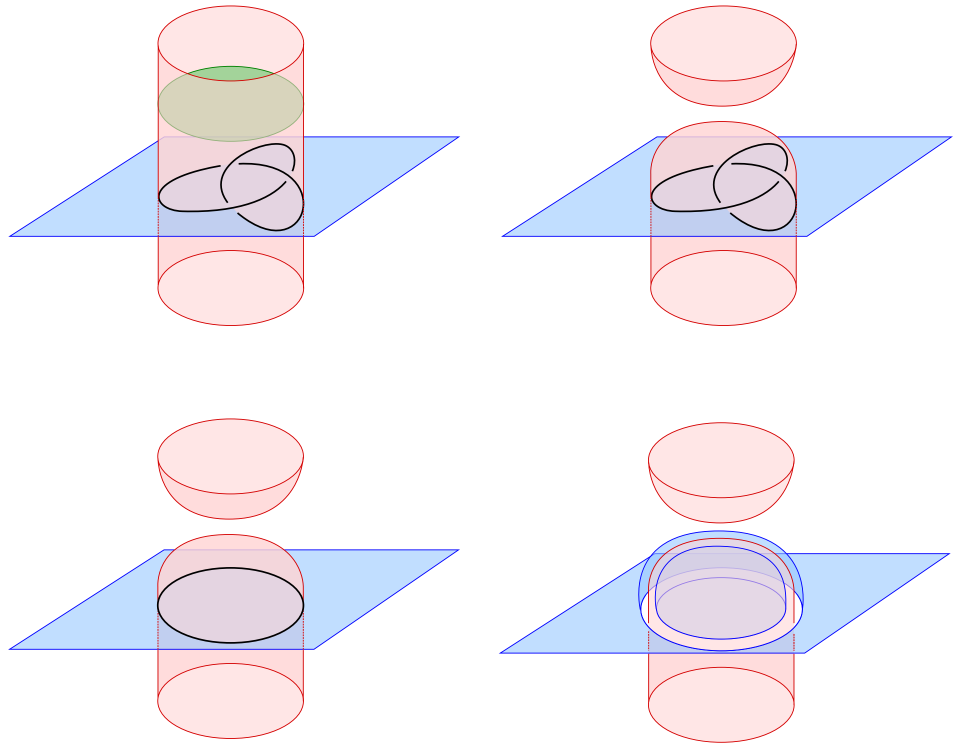

FIGURE 9. **Top Left:** Two 4-manifolds $Y_1$ and $Y_g$ intersect in a circle $C$. (If $Y_1 = Y_g$, then this is a self-intersection.) **Top Right:** We perform ambient Dehn 2-surgery on $Y_1$ at a solid torus parallel to $C$ (according to the surgery framing). Note that we require $Y_g$ to have a framed dual sphere in order to ensure that this framing is $\Delta$-admissible for some disk $\Delta$. Now the circle of intersection is an unknot in $Y_1$, but this changes the topology of $Y_1$. **Bottom Left:** The circle $C$ bounds a disk $D$ in $Y_1$. **Bottom Right:** We use $D$ perform ambient Dehn 2-surgery on $Y_g$. The chosen framing of $D$ is trivializable over the 2-dimensional subbundle of $N_W(D)$ that is normal to $Y_1$. Then after the surgery, we have removed the circle $C$ of intersection (without introducing any new intersections). This changes the topology of $Y_g$.

Note that $Y_1$ and $Y_g$ intersect in a 1-manifold. For simplicity, we will assume that this 1-manifold is just a single circle $C$. Fix a disk $D$ in $W$ with $D \cap Y_1 = \partial D = C$ and a parallel copy $\Delta$ of $D$ that meets $Y_1$ at some parallel curve $\gamma$ to $C$. Since $C$ and $\gamma$ are parallel, there is a natural choice of framing $\phi$ of $N_Y(\gamma)$, so that $C$ is isotopic in $\nu(\gamma) \setminus \gamma$ to a pushoff of $\gamma$ according to $\phi$.

Suppose $\phi$ is $\Delta$-admissible. Then by surgering $Y_1$ along $\Delta$ according to a framing extending $\phi$, we obtain a 3-manifold $Y_1$ in which $C$ is unknotted. On the other

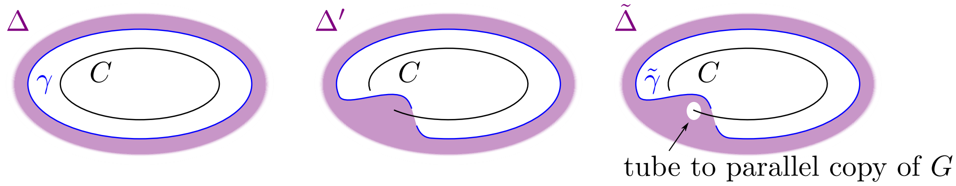


FIGURE 10. To obtain $\Delta'$ from $\Delta$, we isotope $\gamma = \partial\Delta$ once through $C$. This introduces an intersection point between $\Delta'$ and $Y_g$. By tubing $\Delta'$ to a parallel copy of $G$ at this intersection point, we obtain a disk $\widetilde{\Delta}$ that does not intersect $Y_g$. Moreover, since $G$ is framed, if a framing $\phi$ of $N_Y(\gamma)$ is not $\delta$-admissible, then it is also not $\widetilde{\Delta}$ admissible (where $N_Y(\gamma)$ and $N_Y(\widetilde{\gamma})$ are identified by the pictured isotopy).

hand, suppose that $\phi$ is *not* $\Delta$-admissible. Let $\Delta'$ be the result of isotoping $\Delta$ in a neighborhood of $C$ to achieve a crossing change of $C$ and $\gamma$, so that $\Delta'$ intersects $Y_g$ at one point in its interior. Then tube $\Delta'$ to $G$ along an arc in $Y_g$ ending at $\mathring{\Delta}' \cap Y_g$ to obtain a disk $\widetilde{\Delta}$ whose interior is disjoint from $Y_g$. By Lemma 5.5 (using the fact that $G$ is framed), $\phi$ is also not $\widetilde{\Delta}$-admissible. However, while $\widetilde{\gamma} := \partial\widetilde{\Delta}$ is parallel to $C$, the framing $\widetilde{\phi}$ that $C$ induces on $\widetilde{\gamma}$ is homotopic to $\pm 1 \cdot \phi$. By Lemma 5.4, $\widetilde{\phi}$ is $\widetilde{\Delta}$-admissible. Then by surgering $Y_1$ along $\widetilde{\Delta}$ according to a framing extending $\widetilde{\phi}$, we obtain a 3-manifold $Y_1'$ in which $C$ is unknotted.

In Figure 10, we give a schematic of the intuition for obtaining $\widetilde{\Delta}$. In Figure 11, we draw a more accurate picture of $\widetilde{\Delta}$. By using this move we can, "unknot the intersections" - see Figure 9 (top row). This is also pictured in Figure 5 in [Sun15].

Note that we used the framing on $G$ to unknot self-intersections of $Y$, as we needed to achieve surgery with a specific framing on a knot in $Y$. However, we do not need $G$ to be framed in order to remove intersections that are already unknotted, as then there is a natural disk with natural choice of framing.

*Proof of Theorem 1.4 when $F_0$ and $F_1$ are 2-spheres.* Let $S_0 := F_0$ and $S_1 := F_1$, and assume that each $F_i$ is a 2-sphere. (We rename the surfaces to stick to the convention that $S_i$ is a 2-sphere while $F_i$ is a surface of arbitrary genus.)

Let $\widetilde{X}$ denote the universal cover of $X$. Choose a preferred basepoint above the basepoint of $X$. By Lemma 4.9, we may take $S_0$ and $S_1$ to be based-homotopic with no loss of generality. (We implicitly used this fact to state Theorem 1.4 and make sense of whether $\mathrm{fq}(S_0, S_1)$ vanishes or not.)

Fix a based homotopy from $S_0$ to $S_1$ and look at its track $H : S^2 \times I \to X \times I$. Let $H_1$ denote the lift of $H$ to $\widetilde{X} \times I$ based at the basepoint of $\widetilde{X}$. Let $H_g$ denote the $g$-translate $gH_1$ of $H_1$, where $g \in \pi_1(X)$ and $\pi_1(X)$ acts on $\widetilde{X}$ as usual.

The meridian of $S_0$ in $X \times \{0\}$ lifts to a meridian of $H_1$. Seifert-van Kampen shows that $\pi_1(\widetilde{X} \times I - H_1)$ is normally generated by this meridian. Because the

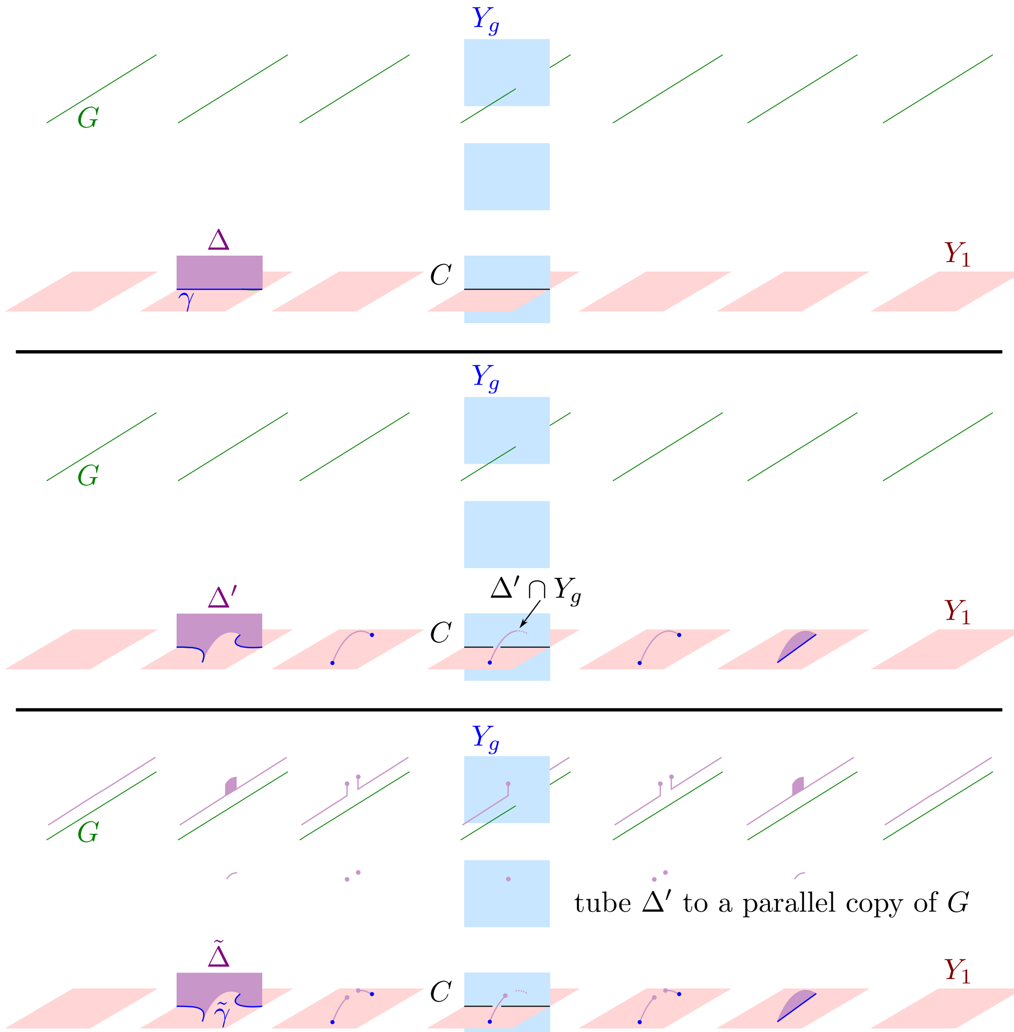


FIGURE 11. A more realistic depiction of the move described in Figure 10, which is part of Move 5.8. Each picture is 5-dimensional, consisting of a two-dimensional array of 3-dimensional drawings. **Top:** The disk $\Delta$. **Middle:** The disk $\Delta'$. **Bottom:** The disk $\widetilde{\Delta}$.

meridian of $S_0$ is null homotopic in $X - S_0$, the lift of the meridian is null-homotopic in $\widetilde{X} \times I - H_1$ and therefore $\pi_1(\widetilde{X} \times I - H_1) = 1$.

Additionally, $\pi_1(\widetilde{X} \times I - \cup_g H_g) = 1$. To see this, note that the group is normally generated by meridians of each of the $H_g$, and each of these meridians can be taken as a lift of the meridian of $S_0$. Additionally, the nullhomotopy of the meridian of $S_0$ in $X - S_0$ lifts to $\widetilde{X} - \widetilde{S}_0 \subset \widetilde{X} \times I - \cup_g Y_g$.

If $\mathrm{fq}(S_0, S_1) \neq 0$, then we know immediately that $S_0$ and $S_1$ are not concordant, since, by [ST19], a concordance could be used to compute $\mathrm{fq}(S_0, S_1) = 0$. We now assume $\mathrm{fq}(S_0, S_1) = 0$. We will use the interpretations of $\mu_3$ and fq that appear in §4.2.2 (Definitions 4.7 and 4.8), namely the 5-dimensional definitions through covering spaces.

Fix $\mathrm{fq}(H) \in \mathbb{F}_2 T_X$, where the coefficient of $g$ in $\mathrm{fq}(H)$ is the number of components of $H_1 \cap H_g$ fixed (setwise) by the action of $g$. Since $\mathrm{fq}(S_0, S_1) = 0$, there exists $f \in \pi_3(X)$ so that $\mu_3(f) = \mathrm{fq}(H)$. We alter $H$ near $X \times 0$ by surgering $H$ along $f$ near the basepoint. That is, delete a small ball $B$ in $H \cap f(S^3)$ from $H$ and reglue $\overline{f(S^3) - B}$. This yields a new homotopy $H'$ between $S_0$ and $S_1$ with the property that $\mathrm{fq}(H') = \mathrm{fq}(H) + \mu_3(f) = 2\,\mathrm{fq}(H) = 0$. Set $H := H'$. In words, for every $g \in \pi_1(X)$, the action of $g$ fixes an even number of components of $H_1 \cap H_g$ setwise.

**Proposition 5.9.** *We can perform equivariant ambient 1-surgeries to $\cup_g H_g$ to obtain a collection of immersed cobordisms $\{Y_g\}$ with $\partial Y_i = \partial H_i$ with the property that for each $g \in T_X$, $g$ fixes no components of $Y_1 \cap Y_g$ setwise.*

*Proof.* Refer to Figure 12. By assumption, there are an even number of circles $H_1 \cap H_g$ that are fixed by $g$, for each $g \in T_X$. Let $\gamma_1$ and $\gamma_2$ in $H_1 \cap H_g$ denote two such circles and let $\alpha$ denote an arc in $H_1$ connecting $\gamma_1$ and $\gamma_2$. Take the interior of $\alpha$ to avoid self-intersections of $H_1$ and intersections of $H_1$ with $H_{g'}$ for any $g' \in \pi_1(X)$. (This is possible because $H_1$ is connected and 3-dimensional while these intersections are 1-dimensional.)

Now for each $h \in \pi_1(X)$, we have an arc $h \cdot \alpha \subset H_h$ from $h \cdot \gamma_1$ to $h \cdot \gamma_2$, with $h \cdot \gamma_1, h \cdot \gamma_2 \in \pi_0(H_h \cap H_{hg})$. Use all of these arcs $h \cdot \alpha$ to equivariantly perform ambient Dehn 1-surgeries to $H_{hg}$. Let $Y_s$ denote the manifold obtained after performing these 1-surgeries to $H_s$. The effect of this is to replace the circles of intersection $\gamma_1, \gamma_2$, with new circles of intersection $\gamma_1', \gamma_2' \in \pi_0(Y_1 \cap Y_g)$ where now $g \cdot \gamma_1' = \gamma_2'$. (Translations of $\gamma_1$ and $\gamma_2$ are similarly altered, but these circles are not contained in $Y_1$.) Thus, in $Y_1 \cap Y_g$ there are now two fewer circles fixed by $g$. Since by assumption there are an even number of circles in $Y_1 \cap Y_g$ fixed by $g$, by repeating this process we can eliminate all of these circles. We take $\{Y_g\}$ to be the resulting cobordisms after performing all equivariant 1-surgeries.

Note that since $\{Y_g\}$ is obtained from $\{H_g\}$ by ambient 1-surgeries, $\pi_1(\widetilde{X} \times I - \cup_g Y_g) = 1$. □

**Proposition 5.10.** *We can perform equivariant ambient Dehn 2-surgeries to $\{Y_g\}$ to obtain a disjoint collection of embedded cobordisms $\{M_g\}$ between lifts of $S_0$ to lifts of $S_1$ with the property that $\pi_1(X - \cup_g M_g) = 1$.*

In Proposition 5.10, the manifolds $M_g$ are not necessarily products. We will invoke the proof of Theorem 5.1 in [Sun15] to conclude that we can further surger the results equivariantly to obtain disjoint $\pi_1(X)$-translates of $S^2 \times I$, as desired. This uses the moves outlined at the beginning of the section.

*Proof of Proposition 5.10.* We follow the strategy of [Sun15].

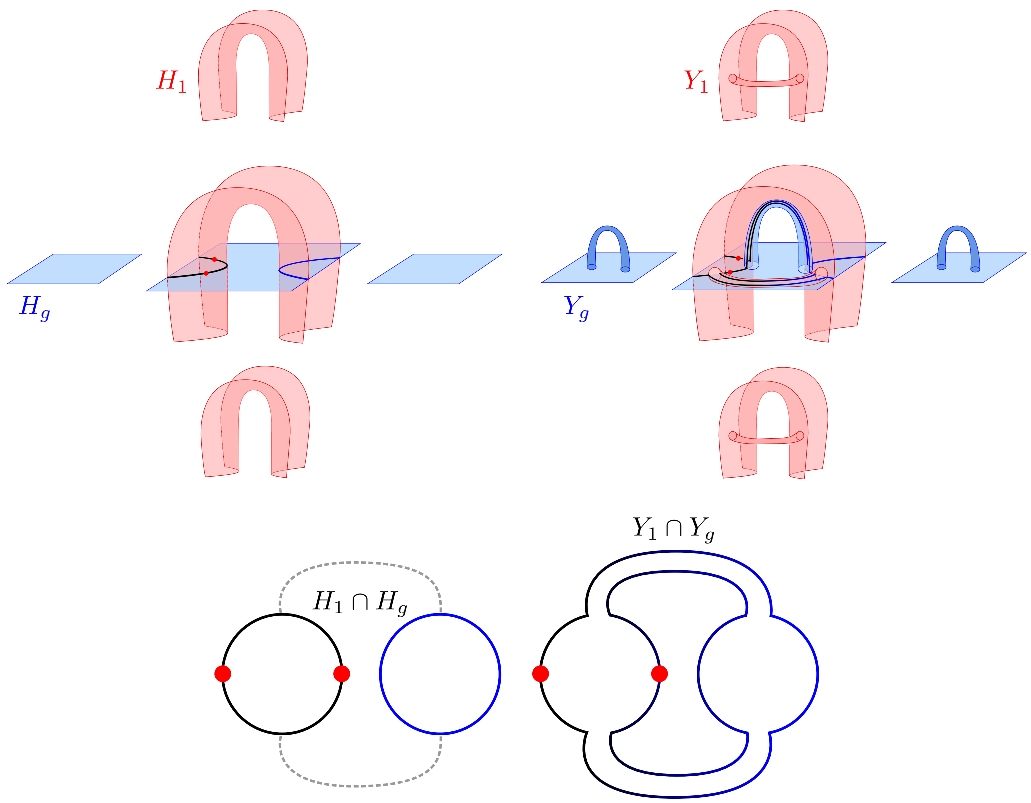


FIGURE 12. **Top row, left:** A schematic of $H_1$ and $H_g$, featuring arcs of two circles of $H_1 \cap H_g$ fixed componentwise by the action of $g$. **Top row, right:** We equivariantly perform ambient 1-surgeries to change these circles of intersection. The resulting manifolds are $Y_1$ and $Y_g$. **Bottom row:** A combinatorial description of the top row move; we indicate two points permuted by the action of $g$. Here we see that after the surgery, the two circles are permuted by $g$.

Note that the dual sphere $G$ of $S_0$ lifts to dual spheres of each lift of $S_0$ in $\widetilde{X}$. We can push these dual spheres into $\widetilde{X} \times I$ to obtain dual spheres for each $Y_g$, with the dual for $Y_g$ disjoint from $Y_h$ for $g \neq h$. This allows us to Perform 5.8.

First, we deal with self-intersections of $Y_1$. We perform Move 5.8 so that all self-intersections of $Y_1$ are unknotted in $Y_1$, at the cost of changing the topology of $Y_1$. For each ambient Dehn 2-surgery on a disk $\Delta$ with boundary in $Y_1$, we also surger each translate of $\Delta$ under the action of $\pi_1(X)$. (Generically, these framed disks are all disjoint.) That is, we perform equivariant ambient Dehn 2-surgery on each $Y_g$ using $g \cdot \Delta$. Abusing notation, we call the resulting manifold $Y_g$.

Now $\gamma$ is unknotted in $Y_1$ and unlinked with every other component of self-intersection of $Y_1$. (See Figure 9.) Repeat this procedure for every circle of self-intersection of $Y_1$ so that now the self-intersections of $Y_1$ form an unlink $L$ in $Y_1$.

Choose a collection of framed disks in $Y_1$ with boundary $L$. Perform equivariant ambient Dehn 2-surgery along the translates of each disk in this collection (this is Move 5.7), so that now $Y_g$ is embedded for every $g \in \pi_1(X)$.

We similarly perform equivariant ambient Dehn 2-surgeries on $\{Y_g\}$ to pairwise remove intersections as follows. Given $g \in \pi_1(X)$ with $g^2 \neq 1$, then we again equivariantly perform Move 5.8 at each curve in $Y_1 \cap Y_g$ so that $Y_1 \cap Y_g$ is now an unlink in $Y_1$. (Similarly, $Y_h \cap Y_{hg}$ is an unlink in $Y_h$ for all $h \in \pi_1(X)$.) We can then perform Move 5.7 on $Y_g$ once for each curve in $Y_1 \cap Y_g$ (and equivariantly on all other translates) so that $Y_1$ becomes disjoint from $Y_g$. (Similarly, $Y_h$ is disjoint from $Y_{hg}$ for all $h \in \pi_1(X)$.)

If $g \in T_X$, then the action of $g$ permutes the circles in $Y_1 \cap Y_g$ in pairs (here we use Proposition 5.9). Fix a circle $\gamma$ in $Y_1 \cap Y_g$. Then $g^2 \cdot \gamma = \gamma \neq g \cdot \gamma$. Pick one curve from each such pair $\{\gamma, g \cdot \gamma\}$, giving a set of curves $\{\gamma_1, ..., \gamma_k\}$ with $Y_1 \cap Y_g$ equal to the disjoint union of $\{\gamma_1, ..., \gamma_k\}$ and $\{g \cdot \gamma_1, ..., g \cdot \gamma_k\}$. Equivariantly perform Move 5.8 on $Y_1$ at $\gamma_1, \ldots, \gamma_k$, so that now $Y_h \cap Y_{hg}$ is an unlink in $Y_h$ for all $h \in \pi_1(X)$. Then equivariantly perform Move 5.7 on $Y_g$ at $\gamma_1, \ldots, \gamma_k$ so that $Y_h \cap Y_{hg} = \emptyset$ for all $h \in \pi_1(X)$. Repeat this for every $g \in T_X$.

Let $\{M_g\}$ denote these resulting mutually embedded manifolds. Since all surgeries were done equivariantly, $M_g = g \cdot M_1$. We conclude $M_g$ is embedded and $M_g \cap M_h = \emptyset$ for $g \neq h \in \pi_1(X)$.

All of the ambient Dehn 2-surgeries in this proposition took place in the interior of $\widetilde{X} \times I$, and thus far from the lifts of $S_0$ and the immersed dual of $S_0$. We thus conclude that a meridian of $M_h$ is nullhomotopic in $\widetilde{X} \times I \setminus \cup_g M_g$, so $\pi_1(X - \cup_g M_g) = 1$. □

Now by applying Theorem 5.6, we can equivariantly ambiently Dehn 2-surger the $\{M_g\}$ to be disjoint concordances. (That is, Theorem 5.6 gives instructions on how to ambiently Dehn 2-surger $M_1$ to obtain an embedded $S^2 \times I$. When performing each of these ambient Dehn 2-surgeries, we simultaneously perform all translates, which are generically along disjoint disks.) Projecting $M_1$ to $X \times I$ thus yields a concordance from $S_0$ to $S_1$, completing the proof of Theorem 1.4 in the case that $F_0 = S_0$ and $F_1 = S_1$ are 2-spheres. □

## 6. Higher genus surfaces

In this section, we will prove Theorem 1.4 when the surfaces are of arbitrary genus and also define $\mathrm{fq}(F_0, F_1)$ when $F_0, F_1$ are $\pi_1$-negligible based-homotopic positive-genus surfaces.

Let $F$ be a closed orientable surface and let $F_0, F_1 : F \hookrightarrow X$ be two based-homotopic embeddings of $F$. Further assume that $F_0$ and $F_1$ are $\pi_1$-negligible. Then we can define $\mu_3(H)$ for any generic map $H : (F, *) \times I \to (X, *) \times \mathbb{R} \times I$ with $H(F \times 0) = F_0 \times \{0\} \times \{0\}$ and $H(F \times 1) = -F_1 \times \{0\} \times \{1\}$ as in Definition 4.1 by again assigning an element of $\pi_1(X)$ to each self-intersection of $H$. The condition that $\pi_1(H)$ maps trivially to $\pi_1(X)$ ensures these group elements will be well-defined and be contained in $T_X \cup \{1\}$; we sum these elements to obtain $\mu_3(H) \in \mathbb{F}_2 T_X$.

To define $\mathrm{fq}(F_0, F_1)$ as we did when $F_0, F_1$ were 2-spheres, we need only to see that the choice of $H$ does not affect $\mu_3(H)$ (up to $\mu_3(\pi_3(X))$). We must reprove a lemma analogous to Lemma 4.4 in [ST19].

**Lemma 6.1.** *Let $H : F \times I \to X \times \mathbb{R} \times I$ be a generic map where the two ends are contained in $X \times \{0\} \times \{0\}$ and $X \times \{0\} \times \{1\}$ (respectively) and are identical (as submanifolds of $X \times \mathbb{R}$, with opposite induced orientations) and are $\pi_1$-negligible. Then $\mu_3(H) \in \mu_3(\pi_3(X))$.*

*Proof.* By [ST19, Lemma 4.4], it is sufficient to find a map $J : S^2 \times I \to X \times \mathbb{R} \times I$ with the above property of having identical ends and with $\mu_3(H) = \mu_3(J)$.

Suppose $F_i$ is genus $g$. Fix an essential loop $\alpha$ in $F_0$ away from the basepoint of $F_0$. Homotope $H$ rel boundary so that for all $t$, $H(\alpha \times \{t\})$ agrees with $H(\alpha \times \{0\})$ after projecting to $X \times \mathbb{R}$. (Here we are using the condition that $H(\alpha \times \{t\})$ is nullhomotopic in $X$.) Do further homotopy as necessary so that that all of the self-intersections of the homotopy occur away from $H(\alpha \times I)$.

Since $X \times \mathbb{R} \times \{0\}$ is 5-dimensional, $\alpha \times \{0\}$ bounds a framed embedded disk $D_0$ in $X \times \mathbb{R} \times \{0\}$ that is disjoint from the image of $H$ except at its boundary, where the disk is normal to $H$. Let $D_t$ be a copy of $D_0$ translated to $X \times \mathbb{R} \times \{1\}$. Perturb $D_0$ as necessary so each $D_t$ $(0 < t < 1)$ that intersects $H$ in its interior does so transversally. Now compress $H$ along the 4-dimensional 2-handle $\cup_{t \in [0,1]} D_t \times I$, where $D_t \times I \subset X \times \mathbb{R} \times \{t\}$. In words, we simultaneously compress each cross-section of $H$ along a copy of $D_t$. This yields a map $H' : \Sigma_{g-1} \times I \to X \times \mathbb{R} \times I$.

We claim $\mu_3(H') = \mu_3(H)$. (In this argument, we use Definition 4.1.) We note that $H'$ has exactly the same self-intersections as $H$, as well as two new self-intersections for each intersection of $H$ with the interior of some $D_t$. Each such pair of self-intersections contributes the same group element to $\mu_3(H')$; see Figure 13. Since the codomain $\mathbb{F}_2 T_X$ of $\mu_3$ is characteristic 2, we conclude that $\mu_3(H') = \mu_3(H)$.

Proceeding inductively on $g$, we eventually find a map $J : S^2 \times I \to X \times \mathbb{R} \times I$ as desired.

□

Thus, we can define $fq(F_0, F_1)$ for based-homotopic positive-genus surfaces. We now extend Lemma 4.9 to apply to positive-genus surfaces.

**Proposition 6.2.** *Let $F_0$ and $F_1$ be $\pi_1$-negligible smoothly embedded genus-$g$ in a smooth 4-manifold $X^4$. Let $G$ be a 2-sphere immersed in $X$. Assume that $F_0$ intersects $G$ transversally once at a point $z$ and that $F_0$ and $F_1$ are homotopic. Then after a isotopy of $F_1$, $F_0$ and $F_1$ are based-homotopic with basepoint $z$.*

The proof is essentially the same as for Lemma 4.9, but we restate the proof here for completeness.

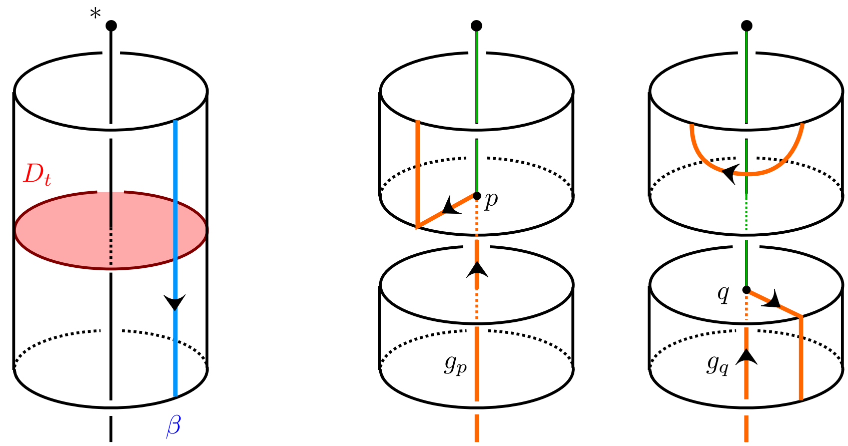


FIGURE 13. In Lemma 6.1, we compress cross-sections of a homotopy $H$ of genus-$g$ surfaces to obtain a homotopy $H'$ of genus-$(g-1)$ surfaces. **Left:** $D_t$ is a compression disk intersecting $H$ transversally. The boundary of $D_t$ is an essential curve $\alpha$ in $F_t$, so there exists a based curve $\beta$ in $F_t$ intersecting $\alpha$ exactly once. **Right:** We draw two copies of $H'$ near $D_t$. Each point of intersection $D_t \cap H$ yields two points of self-intersection of $H'$. The group elements $g_p$, $g_q$ contributed by these points to $\mu_3(H')$ are related by $q_q = [\beta]g_p$. By assumption, $[\beta] = 1$, so $g_p = g_q$.

*Proof.* Let $H : \Sigma_g \times I \to X \times I$ be a free homotopy of $F_0$ to $F_1$. Take $H$ to be transverse to $G$ and consider $L = H^{-1}(G \times I)$, a properly embedded 1-dimensional submanifold of $\Sigma_g \times I$. Note the boundary of $L$ consists of $z_0 \times 0$ and an odd number of points of the form $y_i \times 1$. Let $L_0$ be the component of $L$ containing $z_0 \times 0$ and assume the other endpoint of $L_0$ is $y_0 \times 1$. Compose $H$ with an isotopy of $F_1$ taking $H(y_0)$ to $z$ by isotopy in $G$ and set the resulting composed homotopy to be $H$. Now $\partial L_0 = z_0 \times \{0, 1\}$. Since $\pi_1(H(\Sigma_g \times I))$ maps trivially into $\pi_1(X \times I)$, $H(L_0)$ is homotopic rel boundary to $z \times I$. Moreover, we have $H(L_0) \subset G \times I$ away from self-intersections of $G \times I$, so $H(L_0)$ is a contractible based loop. Contracting this loop yields a based homotopy from $\Sigma_0$ to $\Sigma_1$. □

*Proof of Theorem 1.4.* As in §5, we implicitly define $\mathrm{fq}(F_0, F_1)$ to be $\mathrm{fq}(F_0, F_1')$, where $F_1'$ is isotopic to $F_1$ and based-homotopic to $F_0$ (via Lemma 6.2). Since isotoping $F_1$ does not affect whether $F_0$ and $F_1$ are concordant, the choice of $F_1'$ does not matter. The proof of Theorem 1.4 then follows exactly as in the case when $F_0, F_1$ are genus-zero as in §5. □

All of the above results also hold in the topological category - as mentioned also in [Sun15]. One way to see this is by smoothing any topological manifolds involved away from a point and then applying the above results.

## 7. Examples

In this section we give two examples from the literature and an example of our own that illustrate the necessity of the hypotheses in Theorem 1.4. Example 7.1 is due to Hannah Schwartz and it demonstrates the necessity of the condition that $fq = 0$. We use this example to illustrate our view of $fq$ in terms of lifts of covers from [Sch19].

We then construct Example 7.2: two homotopic spheres with vanishing Freedman-Quinn invariant and a common embedded unframed dual sphere that are not concordant. This illustrates the necessity of our condition that the dual sphere be framed as well as the necessity of the condition on the framing of the sphere in Gabai's lightbulb trick.

Finally, Example 7.2 highlights the relationship between Theorem 1.4 and a theorem in [ST19], namely, we are assuming that one of the two spheres has a null homotopic meridian and concluding that the surfaces are concordant, whereas in [ST19], the authors are assuming that both spheres have a common geometric dual and concluding that the surfaces are isotopic. The assumptions we make (and the conclusion we derive) are weaker than those in the theorem in [ST19] (the 4-dimensional light bulb theorem). The example we give, which appears in [Sat91], demonstrates that this weaker conclusion is strictly necessary and highlights the difference between concordance and isotopy.

**Example 7.1.** In [Sch19], Schwartz constructs two 2-spheres $S_0, S_1$ in a 4-manifold $X^4$ with $\pi_1(X) = \langle g \mid g^2 = 1\rangle$ containing a 2-sphere $G$ with trivial normal bundle intersecting $S_0$ and $S_1$ each transversally once. This example is interesting because Schwartz shows that $S_0$ and $S_1$ are not isotopic or even concordant. She then computes $\mathrm{fq}(S_0, S_1) = g$, illustrating that the Freedman-Quinn invariant can be used in this example to distinguish $S_0$ and $S_1$. In Figure 14, we draw a schematic of the universal cover of $X$ and the lifts of $S_0$ and $S_1$ in order to re-compute $\mathrm{fq}(S_0, S_1)$ using the definition of fq through covering spaces.

**Example 7.2.** We construct two 2-spheres $S_1, S_2$ that are based-homotopic, have embedded dual spheres and $\mathrm{fq}(S_1, S_2) = 0$, and yet $S_1$ and $S_2$ are not concordant. In this example, the dual spheres to $S_1$ and $S_2$ are necessarily not framed; this demonstrates the necessity of the word, "framed," in Theorem 1.4.

Our obstruction to concordance is Stong's Kervaire-Milnor invariant [Sto93], denoted km, which can be rephrased as secondary obstruction that exists for homotopic $s$-characteristic spheres in 4-manifolds that have vanishing Freedman-Quinn invariant. In a forthcoming work we will give a detailed exposition of this invariant in the context of concordance of surfaces, so here we just state its simplest properties that will be of use to us. The invariant km is valued in $H_1(X; \mathbb{Z}/2)$ and is computed from an immersion $H : S^2 \times I \to X \times I$ with ends two s-characteristic spheres $S_0 \subset X \times \{0\}$ and $S_1 \subset X \times \{1\}$. Assuming that the submanifold of self-intersections of $H$ is just two circles that double cover their image under $H$, and such that these

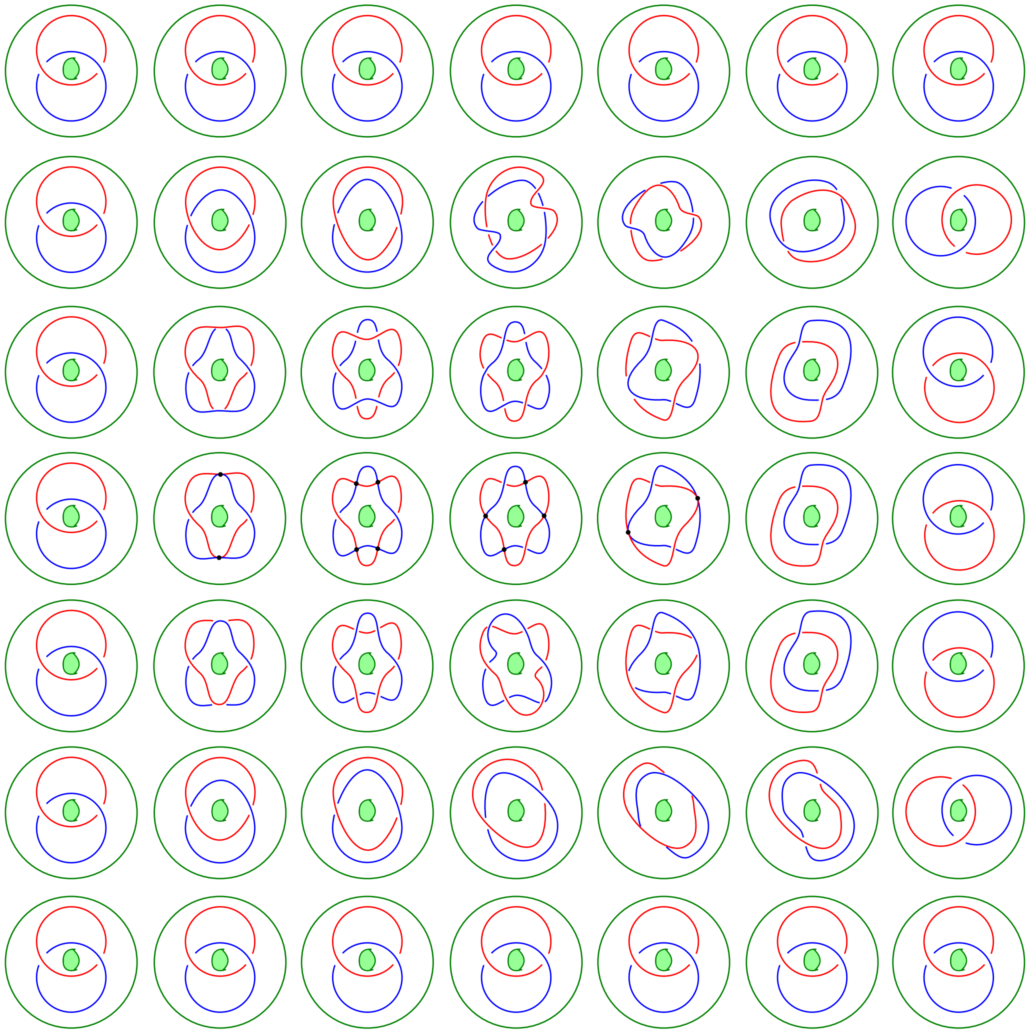

FIGURE 14. A schematic of $\widetilde{X} \times I$, where $X$ is the manifold constructed by Schwartz [Sch19]. Each column represents a copy of $\widetilde{X} \times t$, so that the $I$ axis is horizontal. The action of $g$ is to rotate each cell through an angle of $\pi$. **Left column:** A movie of the lift $\widetilde{S}_0$ of $S_0$ to $\widetilde{X}$. **Right column:** A movie of the lift $\widetilde{S}_1$ of $S_1$ to $\widetilde{X}$. **Whole diagram:** We find equivariant concordances between the components of $\widetilde{S}_0$ to the components of $\widetilde{S}_1$; one concordance is colored red and the other is colored blue. Even though these concordances are individually embedded, they intersect in a single circle $C$ which is preserved setwise by the action of $g$. We conclude that $\mathrm{fq}(S_0, S_1) = g \neq 0 \in \mathbb{F}_2 T_x/\mu_3(\pi_3(X)) = \mathbb{F}_2\{g\}$. Thus, as shown by Schwartz [Sch19], the spheres $S_0$ and $S_1$ are not concordant.

two circles form a Hopf link in $S^2 \times I$, then km is equal to the element of $H_1(X;\mathbb{Z}/2)$ that we obtain when applying $H$ to an arc in $S^2 \times I$ whose endpoints are points on the two different components of the Hopf link that get identified by $H$ and consider the resulting circle as an element of $H_1(X;\mathbb{Z}/2)$.

Let $S_1$ be the unknotted sphere in $S^1 \times S^3$. Let $S_1'$ be the immersed sphere obtained from $S_1$ by performing a finger move as in Figure 15. In Figure 15, we illustrate the framed Whitney disk $W$ along which a Whitney move would undo the finger move (thus yielding $S$ again). The boundary of $W$ consists of two arcs $\alpha, \beta$ in $S'$ between the self-intersections of $S'$. We also draw another pair of arcs $\alpha', \beta'$ between the self-intersections of $S'$, with the ends of $\alpha', \beta'$ contained in the same sheets as $\alpha, \beta$, respectively. However, $\alpha'$ intersects $\beta$ transversally once and $\beta'$ intersects $\alpha$ transversally once. If there exists a clean, framed Whitney disk $W'$ with boundary $\alpha' \cup \beta'$, then the homotopy from $S_1$ to a sphere $S_2$ that involves the pictured finger move followed by a Whitney move along $W'$ would have self-intersection a Hopf link. We could then conclude that $\mathrm{km}(S_1, S_2)$ is the generator of $H_1(S^1 \times S^3;\mathbb{Z}/2) \cong \mathbb{Z}/2$.

However, rather than a clean Whitney disk with boundary $\alpha' \cup \beta'$, in Figures 16 and 17 we obtain a framed Whitney disk $W'$ that intersects $S'$ twice. Now connect sum two copies of $\mathbb{C}P^2$ to $S^1 \times S^3$ and tube $S_1$ (and hence $S'$) to a copy of $\mathbb{C}P^1$ in each $\mathbb{C}P^2$. This yields two disjoint unframed dual spheres to $S'$. By tubing $W'$ at each intersection with $S'$ to a different dual, we can obtain an embedded Whitney disk $\widetilde{W}$ in $(S^1 \times S^3)\#2\mathbb{C}P^2$ – since we tube $W'$ to an even number of spheres with odd Euler number, $\widetilde{W}$ is still framed.

Now let $S_2$ be the 2-sphere in $(S^1 \times S^3)\#2\mathbb{C}P^2$ obtained from $S_1$ by a finger move about the $S^1$ factor in $S^1 \times S^3$ (as in Figure 15) followed by a Whitney move along $\widetilde{W}$. Note that $S_1$ and $S_2$ are based-homotopic, $S_1$ has an unframed dual, and $\mathrm{fq}(S_1, S_2) = 0$ automatically since $\pi_1((S^1 \times S^3)\#2\mathbb{C}P^2)$ has no 2-torsion. Moreover, $[S_i]$ is characteristic, so $\mathrm{km}(S_1, S_2)$ is defined. Then as desired, $\mathrm{km}(S_1, S_2) = 1$ in $H_1((S^1 \times S^3)\#2\mathbb{C}P^2;\mathbb{Z}/2) \cong \mathbb{Z}/2$, so $S_1$ and $S_2$ are not concordant. Note that this also gives an example where Gabai's lightbulb trick fails when the dual is not framed.

**Example 7.3.** The following appears in [Sat91]. We give two 2-spheres that are concordant but not isotopic in $S^2 \times S^2$. The first sphere is $S_0 := S^2 \times \mathrm{pt}$ which satisfies the hypothesis that the meridian circle is null homotopic in the complement. Any sphere in the same homotopy class of $S_0$ is concordant to $S_0$, by Theorem 5.1. We aim to construct a sphere $S_1$ with $[S_1] = [S_0]$ but $\pi_1(S^2 \times S^2 - S_0) = 1 \neq \pi_1(S^2 \times S^2 - S_1)$ and therefore conclude that $S_0$ and $S_1$ are not isotopic.

Let $K$ be a 2-knot (a knotted 2-sphere) in $S^4$ and let $C$ be a loop embedded in $S^4 - K$ that is homologous to a meridian of $K$ in $S^4 - K$. Surgery on $C$ yields $S^2 \times S^2$, in which we may view $K$ as the sphere $K(C)$. The assumption that $C$

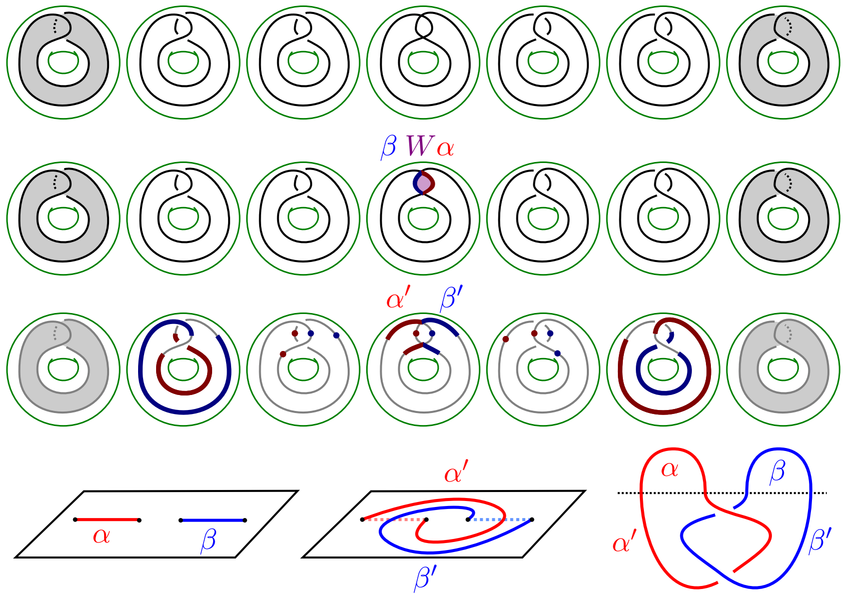


FIGURE 15. **Top row:** The 2-sphere $S_1' \subset S^1 \times S^3$, obtained from performing a finger move to the unknotted sphere. Each picture is 3-dimensional; the horizontal axis is taken to be the fourth dimension. **Second row:** The Whitney disk $W$. Performing a Whitney move to $S_1'$ along $W$ yields the unknotted sphere. **Third row:** A pair of arcs $\alpha', \beta'$ in $S_1'$. **Bottom row:** The arcs $\alpha, \beta, \alpha', \beta'$ in the domain of the immersion of $S'$. Performing a Whitney move using a disk with boundary $\alpha' \cup \beta'$ on $S'$ yields a homotopy whose self-intersection is a Hopf link.

is homologous to a meridian of $S^4 - K$ ensures that the resulting sphere $K(C)$ represents the homology class $[K(C)] = [S_0]$.

By Seifert-van Kampen, $\pi_1(S^2 \times S^2 - K(C)) = \pi_1(S^4 - K)/\langle\langle C \rangle\rangle$, where $\langle\langle C \rangle\rangle$ is the normal subgroup generated by $C$. Take $K$ to be the 5-twist spun trefoil knot [Zee65]. We have $\pi_1(S^4 - K) = G \times \mathbb{Z}$ where $G =< a, b; a^3 = b^5 = (ab)^2 >$ is a perfect group of order 120 (the binary dodecahedral group). Take $C$ to generate the $\mathbb{Z}$ factor and set $S_1 := K(C)$. Thus, $S_0$ and $S_1$ satisfy the hypotheses of Theorem 1.4 and are indeed concordant, but are not ambiently isotopic.

See Figure 18 for an illustration; we include this partly because extracting the generator of the $\mathbb{Z}$ summand from Zeeman's [Zee65] original description of $\pi_1(S^4 - K)$ takes some effort.

**Remark 7.4.** By connect summing the spheres in Example 7.1 with $g$ trivial tori, we may obtain genus-$g$ $\pi_1$-negligible surfaces $F_0$ and $F_1$ in $X^4$ that are based-homotopic and have dual spheres, but have nonvanishing Freedman-Quinn invariant and thus

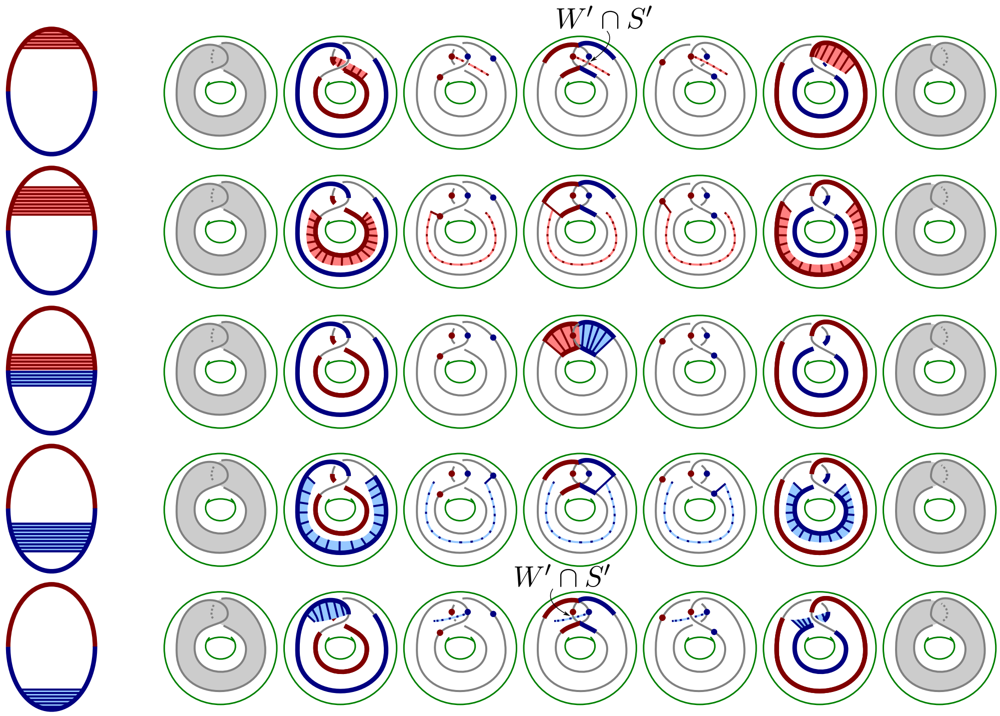


Figure 16. In each row, we draw a portion of a Whitney disk $W'$ for $S'$, as in Figure 15. On the left, we illustrate the corresponding portion of $W'$. We can explicitly see the two points in which $W'$ intersects $S'$. We illustrate the disk in pieces like this to give some intuition for its construction - a complete view is given in the following figure.

fail to be concordant. Similarly, by connect summing the spheres in Example 7.3 with $g$ trivial tori, we may obtain genus-$g$ surfaces $F_0$ and $F_1$ in $S^2 \times S^2$ that are based-homotopic with $\mathrm{fq}(F_0, F_1) = 0$ and so that $F_0$ has a dual sphere, but $F_0$ and $F_1$ are not isotopic (although they are, of course, concordant).

Connect summing the spheres in Example 7.2 with trivial tori likely gives examples of positive-genus $\pi_1$-negligible surfaces illustrating the necessity of the dual sphere in Theorem 1.4 being framed. However, the statements in Stong's work [Sto93] do not immediately apply in this setting, so these examples would require a more detailed explanation. We plan to discuss this in forthcoming work.

## 8. Further questions: nontriviality of $\pi_1(F_i) \hookrightarrow \pi_1(X)$

So far, we have required that positive genus surfaces $F_0, F_1$ be $\pi_1$-negligible. This condition ensures that $F_i$ and homotopies of $F_0$ to $F_1$ lift to $\widetilde{X}$ and $\widetilde{X} \times I$ respectively, and allows us to define the Freedman-Quinn invariant $\mathrm{fq}(F_0, F_1)$ if $F_0$ and

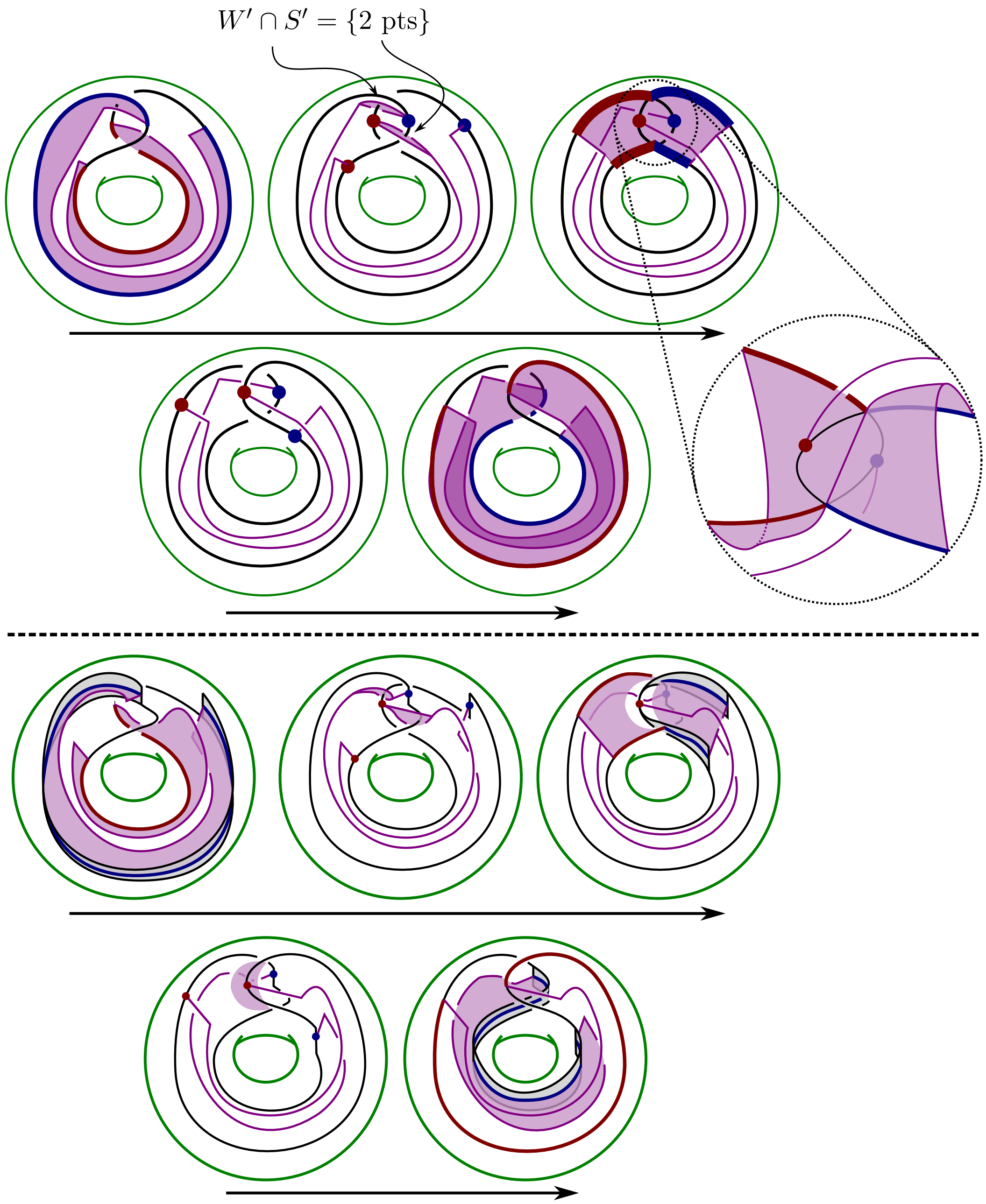


FIGURE 17. The Whitney disk $W'$ from Figure 16. In the top two rows, we draw the whole disk in $S^1 \times S^3$, perturbing slightly to avoid self-intersections. In the bottom two rows, we perturb $S'$ to make the framing of $W'$ clear. At every point in $W'$, consider a normal vector which points upward within one 3-dimensional cross-section. This induces a Whitney framing of $W'$.

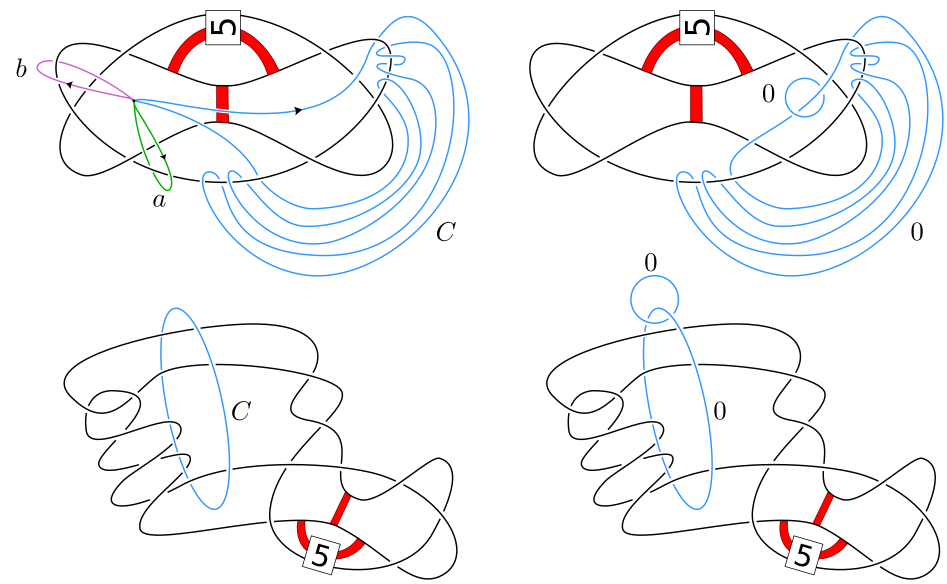


FIGURE 18. In this figure we use banded unlink diagrams to describe surfaces. We refer the reader to [HKM19] for exposition on these diagrams. **Top left:** the standard diagram of the 5-twist spun trefoil knot $K$. We indicate two meridians $a$ and $b$ which generate $\pi_1(S^4 - K)$ and a loop $C$ in the class $w = b^{-1}a^{-1}b^2a^{-1}b^{-1}a$, which generates the $\mathbb{Z}$ summand of $\pi_1(S^4 - K) = \langle a, b \mid baba^{-1}b^{-1}a^{-1} = 1, a^5b = ba^5\rangle = \langle x, y, w \mid x^5 = (xy)^3 = (xyx)^2, [x, w] = [y, w] = 1\rangle$ where $x = ab^{-1}$ and $y = b^{-1}a$ . **Top right:** Surgery along $C$ yields a sphere $K(C)$ in $S^2 \times S^2$ which is homotopic but not isotopic to $S^2 \times \text{pt}$. **Bottom left:** We isotope the diagram of $K$ to simplify $C$. **Bottom right:** An alternate picture of $K(C) \subset S^2 \times S^2$.

$F_1$ are based-homotopic. When $F_0$ and $F_1$ are based-homotopic but $\pi_1(F_i)$ maps nontrivially into $X^4$, then $\mathrm{fq}(F_0, F_1)$ is undefined.

**Question 8.1.** *Let $F_0$ and $F_1$ be genus-$g$ based-homotopic surfaces in a 4-manifold $X^4$. Assume that a meridian of $F_0$ is null-homotopic in $X - F_0$.*

(1) *What conditions on the homotopy between $F_0$ and $F_1$ ensure that $F_0$ and $F_1$ are concordant?*
(2) *What conditions obstruct $F_0$ and $F_1$ from being concordant?*
(3) *The same as Questions 1 and 2, if we assume $[F_0] = [F_1] \in H_2(X; \mathbb{Z}\pi_1(X))$.*

*Note that $[F_0] = [F_1] \in H_2(X; \mathbb{Z}\pi_1(X))$ implies that the lifts of $F_0$ and $F_1$ to $\widetilde{X}$ are componentwise homotopic. This condition holds $F_0$ and $F_1$ are $\pi_1$-negligible.*

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