

SYMMETRIES OF GENUS ZERO MODULAR OPERAD

Noémie C. Combe and Yuri I. Manin

ABSTRACT. In this article combining survey and certain research results, we introduce a categorical framework for description of symmetries of genus zero modular operad. This description merges the techniques of recent “persistence homology” studies and the classical formalism of groupoids. We show that the contravariant “poset in groupoids” embodying these symmetries, provides a new avatar of profinite Grothendieck–Teichmüller group acting upon this operad but seemingly not related with representations of the Galois group of all algebraic numbers.

1. Introduction and summary

Recently considerable attention was attracted by the studies of interaction of modular operad (playing the central role in quantum cohomology constructions) with the celebrated Grothendieck–Teichmüller group ([HaLoSch], [Ho], [BrHoRo], [Fr], [Lo]).

In our paper [CoMa] motivated by these studies, we started looking more closely upon symmetries of the genus zero components of this operad. We focused our attention there upon a very particular symmetry of order two and the respective quotient operad.

Here we take into account the whole family of automorphisms of $\overline{M}_{0,n}$, calculated in [BrMe] and combine them into two “locally finite” combinatorial objects (posets in groupoids) reflecting their compatibilities with operadic compositions (here results of [BrMe] are also crucial; their proofs are based upon Kapranov’s constructions from [Ka]).

This leads us to two versions of the notion of “full automorphism group” of the genus zero operad and opens a question about possibility to transfer in this combinatorial context also an action of the absolute Galois group ([I], [Sch]).

The point is that each study achieving this goal which we are aware of, passes through a homotopic environment, and we wonder whether it is necessary. The dendroid formalism used in [CiMoe] works with simplicial sets (or simplicial objects of other categories), and our formalism here may be considered as a simplified version of their constructions. We avoid in this way simplicial constructions and hopefully, can thus bypass the ∞ -versions of the classical Grothendieck–Teichmüller correspondence. Perhaps, a price we have to pay for this is an interaction with the Galois group which gets “lost in translation”.

Sections 2–5 of this article are short introductions to the combinatorics of components of the genus zero modular operad and of the relevant operadic compositions. Section 6 focusses on the main content of this note: a brief explanation of the structures of two "graphic skeleta" of the symmetries of this operad, one of which embodies covariant symmetries and another contravariant ones.

2. Cofinite combinatorics and operadic symmetries

Denote by \mathbb{N} the set of natural numbers $\{1, 2, 3, \dots\}$. Call a subset $S \subset \mathbb{N}$ *cofinite* one (cf for brevity), if its complement is finite. Call a map $f : \mathbb{N} \rightarrow \mathbb{N}$ *cf-map*, if it is identical on an appropriate cf-set (that may depend on f).

Claim. (a) *Composition of two cf-maps is a cf-map.*
 (b) *Bijjective cf-maps form a group wrt composition.*

Denote the latter group \mathbb{S}_∞ . Clearly, it coincides with the union $\bigcup_{n=1}^\infty \mathbb{S}_n$, in which every symmetric group \mathbb{S}_n is the full group of bijective cf-maps identical outside $\{1, \dots, n\}$.

Below, we work over a field of characteristic zero.

2.1. Theorem. (a) \mathbb{S}_∞ acts upon the family of operadic components $\overline{M}_{0,n}$, $n \geq 3$, in the following way.

Let $C_{0,n} \rightarrow \overline{M}_{0,n}$ be the universal family of stable curves of genus zero endowed with n structure sections $s_i : \overline{M}_{0,n} \rightarrow C_{0,n}$. For each $g \in \mathbb{S}_n$, produce from it another family $g^{-1}(C)_{0,n}$ by renumbering the sections: s_i acquires the new marking $s_{g(i)}$. By universality, we obtain for an appropriate automorphism $g_M : \overline{M}_{0,n} \rightarrow \overline{M}_{0,n}$ that the same renumbering can be obtained via g_M^* .

This map $\mathbb{S}_n \rightarrow \text{Aut } \overline{M}_{0,n}$ is surjective for all $n \geq 3$ and bijective for $n \geq 5$.

(b) This family of automorphisms $\{g_M\}$ can be naturally and uniquely extended to the family of all operadic compositions between the components.

Sketch of proof. (a) For a proof of the first statement, see [BrMe], Theorem 4.3.

(b) Regarding the second statement, we start with recalling that the structure of genus zero modular operad is given by a family of composition morphisms

$$\overline{M}_{0,n_1} \times \cdots \times \overline{M}_{0,n_r} \rightarrow \overline{M}_{0,n_0}.$$

On the level of geometric fibers of universal families of curves, each such morphism corresponds to a controlled degeneration of the respective fiber of C_{0,n_0} with marked

points by grafting to it fibers of $C_{0,n_1}, \dots, C_{0,n_r}$, and subsequent renumbering of all marked points, that have not been grafted.

The first part of this procedure is pure geometric one. Therefore it only remains to check the compatibility of these “operadic renumberings” with combinatorics of bijective renumberings used in the definition of $\mathbb{S}_{n_i} \rightarrow \text{Aut } \overline{M}_{0,n_i}$.

The interested reader will find more details in Sections 4 and 6.

Warning on boundary cases. For $n \geq 4$, $\overline{M}_{0,n}$ is a smooth projective variety of dimension $n - 3$; $\overline{M}_{0,3}$ is a point; and for $n = 0, 1, 2$ it is a stack, the classifying stack of the automorphism groups of P^1 with $0, 1, 2$ marked points respectively.

Studying componentwise symmetries of the modular operad we will usually omit an explicit discussion of special cases involving small values of n .

3. Stable curves of genus zero and their graphs

Below we will use essentially the language of genus zero quantum cohomology as it was introduced in [KoMa] and developed in various directions in many other papers including combinatorial formalisms of [BoMa].

Recall that a *stable* genus zero algebraic curve with marked points is a family (C, x_1, \dots, x_n) . Here C is an algebraic curve, possibly reducible, with smooth irreducible components of genus zero. If it has singular points, they are double points in which two different components intersect transversally. From the genus zero condition it follows that a sequence of irreducible components, in which every two consecutive curves intersect, cannot form a cycle. A finite set of smooth points (x_1, \dots, x_n) of C are marked. Stability means that the automorphism group of such a curve, fixing marked points, is finite. Equivalently, each irreducible component intersecting with other components at $k \geq 0$ points, carries at least $3 - k$ marked points.

If we forget precise positions of marked and singular points on their components, the remaining combinatorial structure of C is well encoded by the *dual graph* τ of this curve.

Such a graph consists of *vertices, flags and edges*. Vertices are points, bijectively corresponding to irreducible components of C , edges are segments of a line bijectively corresponding to singular points of C . Two vertices are connected by an edge iff the respective components of C intersect at the respective singular point. Finally, smooth marked points bijectively correspond to flags: such a flag is a “half-segment”, whose one end is attached to the vertex corresponding to the respective

component, and other end is free (we prefer not to include this free end into the set of vertices, unlike some other authors working with similar formalisms). An edge then might be imagined as the union of two flags whose free ends are now connected. If our curve C is irreducible, then its tree has only one vertex, to which several flags are attached. Such trees are called *corollas*.

In order not to mix free flags with halves of edges, one may call a free flag a *leaf*, or a *tail* as in [KoMa].

For wider contexts and more precise descriptions, generalizable to curves of higher genera, see [BoMa].

Among various types of labelings of sets, forming a graph, an important role is played by *orientations of flags*. If two flags form an edge, their orientations must agree. We often use oriented trees, in which one free flag is chosen as a *root*, and all other leaves are oriented in such a way, that from each leaf there exists a unique oriented path to the root.

If τ corresponds to a curve (C, x_1, \dots, x_n) on which marked points are labeled, say, by $1, \dots, n$, then it is convenient to use the same labels to mark tails of τ .

4. Stratifications of $\overline{M}_{0,n}$

Let τ be the (labeled) graph of a stable connected curve (C, x_1, \dots, x_n) . Denote by $M_{0,\tau} \subset \overline{M}_{0,n}$ the moduli submanifold (or generally, substack) parametrizing all curves having the same graph τ . Its closure will be denoted $\overline{M}_{0,\tau}$.

Examples. a) If τ is a corolla, $M_{0,\tau}$ is the maximal stratum, having dimension $n - 3 = \dim \overline{M}_{0,n}$.

b) If τ has two vertices, they must be connected by an edge. The possible distributions of leaves between vertices bijectively correspond to (unordered) partitions of $\{x_1, \dots, x_n\}$ into two subsets of cardinality ≥ 3 each, and in additional choices of one point in each part. Codimension of $\overline{M}_{0,\tau}$ is 1.

c) There are precisely $n+1$ strata that are *points* (codimension $n-3$). The respective trees are all isomorphic if one forgets labeling of leaves: they are represented by the sequence of vertices $\{v_1, \dots, v_{n-1}\}$ such that v_i and v_{i+1} are connected by an edge for $i \leq n-2$; besides, there is one leaf at each v_2, \dots, v_{n-3} and two leaves at v_1 and v_{n-1} each. All in all, we get n leaves, and \mathbb{S}_n acts simply transitive upon their ordering in the trees.

In order to state a general result in convenient form, we will remind the basic definitions of morphisms of stable trees, as they were stated in [KoMa], Sec. 6.6–6.8, but now in the language of their geometric realizations as above.

A morphism $f : \tau \rightarrow \sigma$ is determined by its covariant surjective action upon vertices $f_v : V_\tau \rightarrow V_\sigma$ and contravariant injective actions upon tails and edges:

$$f^t : T_\sigma \rightarrow T_\tau, \quad f^e : E_\sigma \rightarrow E_\tau.$$

Geometrically, f contracts edges from $E_\tau \setminus f^e(E_\sigma)$ and tails from $T_\tau \setminus f^t(T_\sigma)$, compatibly with its action upon vertices.

An important operation on graphs corresponding to operadic composition was called *gluing* in [KoMa], Sec. 6.6.4. Starting with two pairs consisting of a tree and its tail (τ_i, t_i) , $i = 1, 2$, we produce a new tree $(\tau_1, t_1) * (\tau_2, t_2)$ by connecting t_1 and t_2 into one new edge.

Then we have the following functoriality property of this operation: for any two morphisms $f_i : \tau_i \rightarrow \sigma_i$ not contracting t_i we have an obvious morphism

$$f_1 * f_2 : (\tau_1, t_1) * (\tau_2, t_2) \rightarrow (\sigma_1, (f_1^t)^{-1}(t_1)) * (\sigma_2, (f_2^t)^{-1}(t_2)).$$

The general result about geometry of stratification of $\overline{M}_{0,n}$ now can be stated as follows ([Ke], [KoMa]). Let $\overline{M}_{0,\tau}$ be the moduli space of stable curves of genus zero whose marked points are bijectively labeled by tails of τ . Let $M_{0,\tau}$ be the locally closed subspace of it where respective curves have combinatorial type exactly τ .

4.1. Theorem. *There is a unique family of stratifications of $\overline{M}_{0,\tau}$ by strata of the types $M_{0,\sigma}$ and behaving in the functorial way wrt a generating class of tree morphisms.*

5. Thin categories and groupoids

5.1. Thin categories and groupoids. From now on, our graphs may be infinite. To avoid set-theoretic complications, we will work in a fixed small universe: see [KashSch], Sec. 1.1.

We start with introducing some basic notions of “persistence formalism”. Here we rely upon [BuSSc]; for more details and basic sources and applications see Introduction and References in [MaMar].

A *proset* (preordered set) is a set P endowed with a binary relation \leq which is reflexive and transitive.

A *poset* (partially ordered set) is a set \mathcal{S} endowed with a binary relation \leq which is reflexive, transitive and anti-symmetric. A proset is a poset iff $X \leq Y$ and $Y \leq X$ imply $X = Y$. It follows that each proset has a canonically defined

quotient which is a poset: identify in P pair of objects X, Y for which we have simultaneously $X \leq Y$ and $Y \leq X$.

A poset \mathcal{S} defines an oriented graph (generally infinite) whose vertices are (marked by) \mathcal{S} , and edges connect pairs $X, Y \in \mathcal{S}$ with $X < Y$ oriented from X to Y .

A category \mathcal{C} is called *thin* if for any two objects X, Y , the set $\text{Hom}_{\mathcal{C}}(X, Y)$ consists of ≤ 1 element, and if both $\text{Hom}_{\mathcal{C}}(X, Y)$ and $\text{Hom}_{\mathcal{C}}(Y, X)$ are non-empty, then $X = Y$.

It follows that all automorphisms of X, Y act upon $\text{Hom}_{\mathcal{C}}(X, Y)$ as identity, so \mathcal{C} is equivalent to a category for which $\text{Hom}_{\mathcal{C}}(X, X) = \{id_X\}$ for any object X , which we will temporarily assume.

For such a category, $\text{Ob } \mathcal{C}$ has a canonical structure of a poset: $X \leq Y$ iff $\text{Hom}(X, Y)$ is nonempty. Conversely, each poset defines in this way a thin category in which morphisms are equivalence classes of *oriented paths* from X to Y . Hence, describing a thin category, one can restrict oneself to an explicit description of only some set of *generating morphisms* and keep in mind that each diagram in a thin category is automatically commutative.

In a sense, a complementary class of small categories is the class of *groupoids*. Recall that a category \mathcal{G} is a groupoid, if all morphisms in it are isomorphisms ([KashSch], p. 13).

These two classes can be naturally merged.

5.2. Definition. *We will call a poset in groupoids a category \mathcal{PG} , satisfying the following two conditions:*

- (a) *For any object X , the full subcategory of \mathcal{PG} consisting of all objects isomorphic to X is a groupoid.*
- (b) *If X and Y are not isomorphic and $\text{Hom}(X, Y)$ is non-empty, then $\text{Hom}(X, Y)$ has a single orbit with respect to the precomposition by the automorphism group of X and postcomposition by the automorphism group of Y .*

There is a natural universal functor from posets in groupoids to thin categories, identical on objects and identifying all morphisms in each non-empty $\text{Hom}(X, Y)$.

5.3. Thin category of stable trees. For two such trees π, τ we put $\pi < \tau$ if $\pi = (\sigma, t_1) * (\tau, t_2)$ for appropriate σ and t_1, t_2 , where we return to notations of Sec. 4. Passing to categories, we remind that this binary relation produces the generating morphisms of the respective category of trees: the remaining ones are obtained by composing them.

Theorem 4.1 offers another equivalent thin category whose objects are strata $\mathcal{M}_{0,\pi}$ of all components of genus zero modular operads.

There are versions of these posets that are posets in groupoids.

Namely, consider stable trees whose tails form themselves (or are bijectively labeled by) finite sets T of our small universe. Then trees of given topological type with fixed T form a groupoid whose morphisms can be identified with bijections $T \rightarrow T$.

6. Graphic skeleta of the symmetries of genus zero modular operad

The widest categorical context in which we can imagine symmetries of the genus zero operad would involve all forests τ in a small universe and respective spaces $\overline{M}_{0,\tau}$ and $M_{0,\tau}$.

As was explained above, in our “economy class version”, we will restrict ourselves to corollas, that is finite sets T marking points on stable curves, and moduli spaces $\overline{M}_{0,T}$. The automorphism group of $\overline{M}_{0,T}$ can be canonically identified with the group of bijections $T \rightarrow T$.

Now, in order to study the symmetries of the whole operad generated by $\overline{M}_{0,T}$ we must connect these finite permutation groups $Aut T$ by families of chosen morphisms with respect to which one could pass to some meaningful limits. This is what we did in Section 2, explaining the structure and action of the permutation group S_∞ .

This last and main Section performs this job producing another “infinite permutation group” **mGT** which is a combinatorial version of the (profinite) Grothendieck–Teichmüller group. We start with showing how to include groupoids of finite sets into a different poset of groupoids.

We return here to the (slightly changed) notations of Sec. 2. Consider the poset \mathcal{N}_* whose elements are subsets $\mathbf{n} := \{1, \dots, n\}$, $n = 1, 2, \dots \in \mathbb{N}$, with binary relation $\mathbf{m} \leq \mathbf{n}$ iff $\mathbf{m} \subseteq \mathbf{n}$. Obviously, it forms a thin category whose morphisms are cf–maps coinciding with usual embeddings.

Now, extend it to a larger category \mathcal{N}_*^{cf} having the same set of objects, but larger set of morphisms: $\text{Hom}_{\mathcal{N}_*^{cf}}(\mathbf{m}, \mathbf{n})$ consists of all cf–maps obtained by precomposition of a permutation of \mathbf{m} , standard embedding \mathbf{m} into \mathbf{n} and postcomposition with a permutation of \mathbf{n} .

6.1. Proposition. (a) \mathcal{N}_*^{cf} is a poset in groupoids.

(b) This poset (restricted to $n \geq 3$) acts termwise upon the poset of components of modular operad $\overline{M}_{0,n}$ whose morphisms are generated by standard embeddings of locally closed strata.

We will call this poset \mathcal{N}_*^{cf} “the covariant skeleton of symmetries” of our operad.

We pass now to the definition of the contravariant skeleton.

Denote by \mathcal{N}^* the poset with the same elements as \mathcal{N}_* but with different ordering: if $\mathbf{p} = \{1, \dots, p\}$ and $\mathbf{q} = \{1, \dots, q\}$, then in \mathcal{N}^* , $\mathbf{p} \leq \mathbf{q}$ means that p divides q .

Clearly, the map $\mathcal{N}^* \rightarrow \mathcal{N}_*$ identical on elements, is a bijection compatible with respective order relations, but the inverse map is not compatible.

It is convenient to introduce the family of commutative rings of residues $\mathbb{Z}/q\mathbb{Z}$, $q \geq 3$, related by the family of natural ring homomorphisms

$$t_{q,p} : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}, \quad a \bmod q \mapsto a \bmod p$$

for each pair of natural numbers p, q such that p divides q . It is clear that if p divides q, r which in turn divide s , then

$$t_{q,p} \circ t_{s,q} = t_{r,p} \circ t_{s,r} = t_{s,p}.$$

Now, residue classes of all $d \bmod q$ with $\text{g.c.d.}(d, q) = 1$, form the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^*$. Hence multiplications of $\mathbb{Z}/q\mathbb{Z}$ by them are permutations that also act compatibly with all $t_{q,p}$.

We pass now to the central definition of this Section.

6.2. Definition. The group \mathbf{mGT}_q is defined as the subgroup of permutations of $\mathbb{Z}/q\mathbb{Z}$, generated by the following maps:

- (i) multiplications by all elements $d \in (\mathbb{Z}/q\mathbb{Z})^*$;
- (ii) the involution $\theta_q : a \mapsto 1 - a$.

Notice that θ_q does not coincide with multiplication by any $d = d_0$ as above: $\theta_q(0) = 1$ whereas $d_0 \cdot 0 = 0$.

Moreover, \mathbf{mGT}_q is not commutative: $1 - da \neq d(1 - a)$ in $\mathbb{Z}/q\mathbb{Z}$ if $d \neq 1$.

6.3. Proposition. For each p, q with p/q , define the homomorphism $u_{q,p} : \mathbf{mGT}_q \rightarrow \mathbf{mGT}_p$ by the following prescription: each permutation of $\mathbf{q} \in \mathcal{N}^*$ belonging to \mathbf{mGT}_q is compatible with each map $t_{q,p}$ and after applying $t_{q,p}$ determines a group homomorphism

$$u_{q,p} : \mathbf{mGT}_q \rightarrow \mathbf{mGT}_p$$

These homomorphisms satisfy the following relations: if p divides q, r which in turn divide s , then

$$u_{q,p} \circ u_{s,q} = u_{r,p} \circ u_{s,r} = u_{s,p}.$$

This follows directly from the definitions.

6.4. Corollary. *There exists a well defined group \mathbf{mGT} , “modified profinite Grothendieck–Teichmüller group”, which is the projective limit of groups \mathbf{mGT}_q with respect to the homomorphisms $u_{q,p}$.*

It might be still possible to compare \mathbf{mGT} as an abstract group given by generators and relations, with the standard profinite GT–group ([I]) even bypassing a connection with the Galois group of $\overline{\mathbf{Q}}$ (embedded in \mathbf{C}).

But actually, such a connection is partially encoded already in our formalism. Namely, consider the field generated by roots of unity (i. e. the maximal abelian subextension of $\overline{\mathbf{Q}}$ embedded into \mathbf{C}). Then we can replace each group $\mathbb{Z}/q\mathbb{Z}$ by the group μ_q of roots of unity of degree q : $a \bmod q \mapsto e^{2\pi ia}$. The action of $(\mathbb{Z}/q\mathbb{Z})^*$ then becomes the action of the respective Galois group $e^{2\pi ia} \mapsto e^{2\pi ida}$. Finally, θ_q encodes the reflection with respect to 0 or ∞ , rather than 1 in [I].

On the operadic level, the role of indexing families μ_q in place of $\mathbb{Z}/q\mathbb{Z}$ also becomes more transparent: imagine \overline{M}_{0,μ_q} as moduli space of deformations of compactified \mathbf{G}_m with roots of unity, 0, and ∞ as marked points. And respective two reflections are related via conjugation with a very particular element of $PSL(2, \mathbb{Z})$, so the difference is not critical one.

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Noémie C. Combe, Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn

Yuri I. Manin, Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn