SPRINGER MOTIVES

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Abstract. We show that the motive of a Springer fiber is pure Tate. We then consider a category of equivariant Springer motives on the nilpotent cone and construct an equivalence to the derived category of graded modules over the graded affine Hecke algebra.

Contents

1. Introduction

1.1. **Motive of the Springer Fiber.** Let G be a connected reductive algebraic group over an algebraically closed field k. Denote by $\mathcal{N} \subset \mathfrak{g} = \mathrm{Lie}(G)$ the associated nilpotent cone and by $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$ the Springer resolution. For $N \in \mathcal{N}$, denote by $\mathcal{B}_N = \mu^{-1}(N) \subset \tilde{\mathcal{N}}$ the Springer fiber.

Let Λ be some commutative ring of coefficients. Denote by $\mathrm{DM}(\mathsf{Spec}(k), \Lambda)$ the triangulated category of Voevodsky motives over the base field k with coefficients in Λ (see [MVW06]).

Theorem 1.1 (Springer Fiber is Pure Tate)**.** *The motive of the Springer fiber $M(\mathcal{B}_N) \in \mathrm{DM}(\mathsf{Spec}(k), \Lambda)$ is pure Tate, that is, a direct sum of Tate motives $\Lambda(n)[2n]$ for $n \geq 0$, if either $\mathrm{char}\, k = 0$ or if $p = \mathrm{char}\, k > 0$ and the following three conditions hold:*

(1) p is a good prime for every classical group appearing as a constituent of G.
(2) $p > 3(h + 1)$, where h denotes the maximum of all Coxeter numbers of exceptional constituents in G.
(3) p is invertible in Λ or $\mathsf{Spec}(k)$ admits resolutions of singularities.

Remark 1.2. (1) Springer fibers for classical groups admit an affine paving. This is shown in [DLP88, Theorem 3.9] for $\mathrm{char}\, k = 0$ and generalized in [Jan04, Chapter 11] to $p = \mathrm{char}\, k > 0$ for good primes p. The existence of an affine paving almost immediately implies that $M(\mathcal{B}_N)$ is Tate.

(2) For exceptional groups, the existence of an affine paving is not known. However, DeConcini–Lusztig–Procesi [DLP88] show a slightly weaker result, namely

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that for $k = \mathbb{C}$ the Borel–Moore-homology of $\mathcal{B}_N$ is torsion free, concentrated in even degrees and generated by algebraic cycles. We show how to adapt their arguments to prove that $M(\mathcal{B}_N)$ is pure Tate in this case, under the assumption on $\mathrm{char}(k)$.

(3) The last assumption on Λ and p ensures a good behavior of motives of singular varieties in $\mathrm{DM}(\mathsf{Spec}(k), \Lambda)$, see [Kel17]. For example, it guarantees the existence of the localization triangle.

1.2. **Equivariant Springer Motives.** Now let $k = \overline{\mathbb{F}}_p$. In [SVW18, Chapter II] Soergel–Virk–Wendt construct a mixed version of the Bernstein–Lunts equivariant derived category using motivic sheaves. To a linear group H acting on a variety $X \in \mathsf{Var}_k$ with finitely many orbits they associate a $\mathbb{Q}$-linear tensor triangulated category of *H-equivariant orbitwise mixed Tate motives* $\mathrm{MTDer}_H(X)$. We use this formalism to define the category of *H-equivariant Springer motives*

$$\mathrm{MTDer}_H^{Spr}(\mathcal{N}) \overset{def}{=} \langle \mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})(n) \mid n \in \mathbb{Z} \rangle_\Delta \subset \mathrm{MTDer}_H(\mathcal{N})$$

as full triangulated subcategory of $\mathrm{MTDer}_H(\mathcal{N})$ generated by the *Springer motive* $\mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})$ and its Tate twists. Here H denotes G or $G \times \mathbb{G}_m$, acting on $\mathcal{N}$ in the natural way.

We will show

Theorem 1.3 (Motivic Derived Springer Correspondence)**.** *There is an equivalence of categories*

$$\mathrm{MTDer}_H^{Spr}(\mathcal{N}) \cong \mathrm{Der}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}(\mathrm{CH}_H^\bullet(Z, \mathbb{Q}), \star))$$

between the category of H-equivariant Springer motives and the derived category of finitely generated graded right modules over $(\mathrm{CH}_H^\bullet(Z, \mathbb{Q}), \star)$.

Here $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ denotes the Steinberg variety. Furthermore, $(\mathrm{CH}_H^\bullet(Z, \mathbb{Q}), \star)$ denotes the H-equivariant Chow groups of Z equipped equipped with the convolution product

$$c \star c' = (p_{13})_* \left(p_{12}^*(c) \cap p_{23}^*(c')\right), \text{ for } c, c' \in \mathrm{CH}_H^\bullet(Z, \mathbb{Q}),$$

where p_{ij} denote the projection maps of the triple product $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Using the explicit description of $(\mathrm{CH}_H^\bullet(Z, \mathbb{Q}), \star)$ (see for example [ZZ17]) this yields

Corollary 1.4. *There are equivalences of categories*

$$\mathrm{MTDer}_G^{Spr}(\mathcal{N}) \cong \mathrm{Der}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}\,\mathbb{Q}[W] \# \mathrm{S}^\bullet(X(T))) \text{ and}$$
$$\mathrm{MTDer}_{G \times \mathbb{G}_m}^{Spr}(\mathcal{N}) \cong \mathrm{Der}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}\,\overline{\mathbb{H}}).$$

Here $X(T)$ denotes the character group of a maximal torus $T \subset G$, W the Weyl group of G and $\mathbb{Q}[W] \# \mathrm{S}^\bullet(X(T))$ the semidirect product of the group algebra of W with the symmetric algebra of $X(T)$. Furthermore $\overline{\mathbb{H}}$ denotes the graded affine Hecke algebra *associated to G as defined by Lusztig* [Lus89].

1.3. **Relation To Other Work.** The second half of this paper is a motivic version of the *derived Springer correspondence* as constructed by Rider [Rid13] in the context of equivariant mixed ℓ-adic sheaves. The motivic setup, as very recently introduced by Soergel–Wendt–Virk [SVW18], has certain advantages. First, there are no non-trivial extensions between Tate objects, corresponding to the vanishing of rational higher K-theory of finite fields and their algebraic closures. Using this, technical difficulties with ℓ-adic sheaves that necessitate to state the derived

Springer correspondence in terms of either dg-categories or the homotopy category of pure complex disappear. Second, Springer motives are defined rationally. Hence all statements are independent of ℓ.

1.4. **Future Work.** (1) In upcoming work with Shane Kelly, generalizing [EK19], we define a formalism of equivariant motives with coefficients in a finite field. This will allow us to prove a modular motivic derived Springer correspondence analogously.
(2) It would be interesting to also consider a generalized motivic derived Springer correspondence along the lines of Rider–Russell [RR16] and [RR17].
(3) Also, it would be interesting to obtain a K-theoretic version of the result using equivariant K-motives in the sense of Hoyois, see [Hoy17][Hoy16]. Then it would be possible to handle the affine Hecke algebra. Similar K-motivic statements have considered by the author in the case of flag varities [Ebe19].

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2. Motive of the Springer Fiber

In this section we show how to translate the results of [DLP88] and [Jan04, Chapter 11] to motives and prove Theorem 1.1. In the following, k denotes an algebraically closed field and Λ a commutative ring of coefficients, such that either

1. resolution of singularities holds over $\mathsf{Spec}(k)$ or
2. the exponential characteristic of k is invertible in Λ.

For all standard properties of motives we refer to [MVW06, Sections 14, 16] and [Kel17, Section 5.3]. While [MVW06] assumes resolution of singularities for many statements about motives of singular schemes, [Kel17] shows that requiring that the exponential characteristic of k is invertible in the coefficient ring Λ suffices. For a variety $X \in \mathsf{Var}_k$ over k we denote its motive by $M(X) \in \mathrm{DM}(\mathsf{Spec}(k), \Lambda)$ and its motive with compact support by $M^c(X) \in \mathrm{DM}(\mathsf{Spec}(k), \Lambda)$. Denote the Tate motive by $\Lambda(1) \in \mathrm{DM}(\mathsf{Spec}(k), \Lambda)$. By definition, the Tate motive is a shift of the reduced motive of $\mathbb{G}_m$, namely the cone of the natural morphism $M(\mathbb{G}_m)[-1] \to M(\mathsf{Spec}(k))[-1]$.

2.1. **Tate motives.** We state some general results on pure Tate motives.

Definition 2.1. *A motive $M \in \mathrm{DM}(\mathsf{Spec}(k), \Lambda)$ is called* pure Tate *if it is isomorphic to a finite direct sum of Tate motives of the form $\Lambda(n)[2n]$.*

Definition 2.2. *Let $X \in \mathsf{Var}_k$. An α-*partition *of X is a finite family of subvarieties $X_1, \dots, X_s$ of X such that $\bigcup_{i=1}^r X_i$ is closed in X for all $1 \leq r \leq s$. If each X_i is furthermore isomorphic to some affine space $\mathbb{A}^n$, we call the α-partition an* affine paving *of X.*

Lemma 2.3. *Let $X, Y \in \mathsf{Var}_k$ and $E \to X$ a vector bundle of rank r.*

(1) There is an isomorphism $\Lambda(p)[2p] \otimes \Lambda(q)[2q] \cong \Lambda(p+q)[2(p+q)]$.
(2) If X is proper, then $M^c(X) = M(X)$.
(3) There is an isomorphism $M(X \times Y) = M(X) \otimes M(Y)$ and if $M(X)$ and $M(Y)$ are pure Tate, then so is $M(X \times Y)$.

(4) *There is an isomorphism* $M^c(X \times Y) = M^c(X) \otimes M^c(Y)$ *and if* $M^c(X)$ *and* $M^c(Y)$ *are pure Tate, then so is* $M^c(X \times Y)$.
(5) *There is an isomorphism* $M^c(E) \cong M^c(X)(r)[2r]$ *and* $M^c(X)$ *is pure Tate if and only if* $M^c(E)$ *is pure Tate.*
(6) *There is an isomorphism*

$$M^c(\mathbb{P}(E)) \cong \bigoplus_{p=0}^{r-1} M^c(X)(p)[2p]$$

and $M^c(X)$ *is pure Tate if and only if* $M^c(\mathbb{P}(E))$ *is pure Tate.*
(7) *Let* X *be smooth and* $Z \subset X$ *be a closed smooth subvariety of codimension* c. *Denote the the blow-up of* X *along* Z *by* $\mathrm{Bl}_Z(X)$. *Then*

$$M(\mathrm{Bl}_Z(X)) \cong M(X) \oplus \bigoplus_{p=1}^{c-1} M(Z)(p)[2p]$$

and $M(X)$ *and* $M(Z)$ *are pure Tate if and only if* $M(\mathrm{Bl}_Z(X))$ *is pure Tate.*
(8) *If* $Z \subset X$ *is a closed subvariety and* $U = X \backslash Z$. *Then there is a distinguished triangle, called localisation triangle,*

$$M^c(Z) \longrightarrow M^c(X) \longrightarrow M^c(U) \xrightarrow{+1}$$

and $M^c(X)$ *is pure Tate if and only if* $M^c(Z)$ *and* $M^c(U)$ *are pure Tate.*
(9) *If* $X_1, \ldots, X_s$ *is an* α*-partition of* X, *then* $M^c(X)$ *is pure Tate if and only if* $M^c(X_i)$ *is pure Tate for all* i.
(10) *If* X *has an affine paving, then* $M^c(X)$ *is pure Tate.*

Proof. (1)-(7) Can be found in [MVW06, Sections 14-16].
(8) To show that $M^c(X)$ is pure Tate if $M^c(Z)$ and $M^c(U)$ are, we claim that the boundary map $M^c(U) \to M^c(Z)[1]$ in the localisation triangle vanishes. Let $\Lambda(q)[2q]$ and $\Lambda(p)[2p]$ be direct summands of $M^c(U)$ and $M^c(Z)[1]$, respectively. Then

$$\mathrm{Hom}_{\mathrm{DM}(\mathsf{Spec}(k),\Lambda)}\left(\Lambda(q)[2q], \Lambda(p)[2p+1]\right)) = \mathrm{CH}^{p-q}(\mathsf{Spec}(k), -1) \otimes \Lambda = 0$$

where the right hand side denotes a higher Chow group which vanishes since in general $\mathrm{CH}^\bullet(-,-1) = 0$. Hence $M^c(X) = M^c(Z) \oplus M^c(U)$ and the statement follows.
(9) Follows by (8) and induction.
(10) Follows from $M^c(\mathbb{A}^n) = \Lambda(n)[2n]$ and (9). □

As demonstrated in [Bro05], there is a Białynicki-Birula decomposition of motives of varieties with $\mathbb{G}_m$-actions. This can sometimes be used to show that a smooth projective variety is pure Tate.

Lemma 2.4. *Let* $X \in \mathsf{Var}_k$ *be a smooth projective variety equipped with an action of* $\mathbb{G}_m$. *Then* $M(X^{\mathbb{G}_m}) = M^c(X^{\mathbb{G}_m})$ *is pure Tate if and only if* $M(X) = M^c(X)$ *is.*

Proof. The Białynicki-Birula theorem gives an α-partition of X into vector bundles on the connected components of $X^{\mathbb{G}_m}$. The statement follows using the previous lemma. □

2.2. **Prehomogeneous Vector Spaces and Pure Tateness.** We show how the methods of [DLP88, Section 2] allow to prove that a certain variety associated to a prehomogeneous vector space is pure Tate. We recall some of their notation.

Let M be a connected linear algebraic group and V a prehomogeneous M-module, that is, V contains a dense M-orbit V^0. Fix a $v \in V^0$ and denote by M_v its stabilizer in M. Let H be a closed Borel-subgroup in M and U an H-stable linear subspace of V. Let

$$M_U = \{g \in M \mid g^{-1}v \in U\} \text{ and}$$
$$X_U = M_U/H.$$

We are interested in the motive of the varieties X_U. By [DLP88, Lemma 2.2(i)] and since H is a Borel subgroup, X_U is a smooth projective variety.

Let Γ be the set of H-stable subspaces of V. For $U \in \Gamma$, let P_U be the stabilizer of U in M and denote $\delta(U) = \dim(M/P_U) - \dim(V/U)$ and $\gamma(U) = \dim(M/H) - \dim(V/U)$. In [DLP88, Section 2.7], Γ is equipped with the structure of a directed graph, whose edges (U, U') have the property

(1) $U \subset U'$,
(2) $\dim U'/U = 1$ and
(3) there exists a parabolic subgroup $P \supset H$ of semisimple rank 1 and $U'' \in \Gamma$ such that $U'' \subset U$, $P \subset P_{U''}$, $P \not\subset P_U$, $P \subset P_{U'}$ and $\dim U/U'' = 1$.

Then, the M-module V is called *good* if for any $U \in \Gamma$ either $U \subset U'$ for some $U' \in \Gamma$ with $\delta(U') < 0$, or U lies in the same component of Γ as some $U' \in \Gamma$ with $\delta(U') \leq 0$.

In [DLP88, Proposition 2.12] it is then shown that under the condition that V is good the Borel–Moore-homology of X_U is torsion free, concentrated in even degrees and generated by algebraic cycles. We copy their arguments and show that $M(X_U)$ is pure Tate.

Proposition 2.5. *Assume that the M-module V is good. Then $M(X_U)$ is pure Tate for all $U \in \Gamma$.*

Proof. To translate the inductive argument used in [DLP88] to the world of motives, we will use the slice filtration ν for effective motives as studied in [HK06].

To any X_U, this associates a family of objects $\nu_{<n}M(X_U)$ for $n \geq 0$ and a family of compatible morphisms $\nu_{<n}M(X_U) \to \nu_{<m}M(X_U)$ for $n \geq m$. By definition $\nu_{<0}M(X_U) = 0$ and since X_U is a smooth projective variety by [HK06, Propositions 1.7, 1.8] we have $\nu_{<n}M(X_U) = M(X_U)$ for $n > \dim X_U$.

Hence it suffices to show that $\nu_{<n}M(X_U)$ is pure Tate for all n. For each $U \in \Gamma$ and $n \geq 0$ consider the following statement

(PT_n) $\nu_{<n}M(X_U)$ is pure Tate.

We prove this by induction on n. If $n = 0$, then $\nu_{<n}M(X_U) = 0$. Now let $n \geq 1$ and assume that (PT_{n-1}) holds for all $U \in \Gamma$.

If $U \in \Gamma$ is contained in $U' \in \Gamma$ with $\delta(U') < 0$, then X_U is empty and hence $M(X_U) = 0$ pure Tate. If $\delta(U) \leq 0$, then X_U is a finite disjoint union of flag varieties isomorphic to P_U/H by [DLP88, Paragraph 2.9 (a)]. The Bruhat decomposition provides an affine paving of P_U/H. Hence $M(X_U)$ is pure Tate by Lemma 2.3.

Since V is good, for each connected component of Γ there is hence some U for which (PT_n) holds. So the statement of the proposition reduces to the following lemma. □

Lemma 2.6. *Assume that $n \geq 1$ and (PT_{n-1}) holds for all $U_1 \in \Gamma$. Let $U \subset U'$ be an edge in Γ. Then (PT_n) holds for U if and only if (PT_n) holds for U'.*

Proof. Let U'' and P as in property (3) of edges of Γ. Let

$$Z = \{(gH, g'H) \in M/H \times M/H \,|\, g^{-1}v \in U \text{ and } g^{-1}g' \in P\}.$$

Then by [DLP88, Lemma 2.11] $Z \to X_U$ is the projectivization of a vector bundle of rank two $E \to X_U$ and furthermore $Z \cong \mathrm{Bl}_{X_{U''}}(X_{U'})$, where $X_{U''} \subset X_{U'}$ is a closed subvariety of codimension two. Hence by the projective bundle and blow-up formula we have

$$M(X_U) \oplus M(X_U)(1)[2] = M(Z) = M(X_{U'}) \oplus M(X_{U''})(1)[2].$$

Applying $\nu_{<n}$ yields that

$$\nu_{<n}\left(M(X_U) \oplus M(X_U)(1)[2]\right) = \nu_{<n}(M(X_U)) \oplus \nu_{<n-1}(M(X_U))(1)[2]$$

equals

$$\nu_{<n}\left(M(X_{U'}) \oplus M(X_{U''})(1)[2]\right) = \nu_{<n}(M(X_{U'})) \oplus \nu_{<n-1}(M(X_{U''}))(1)[2]$$

where we use that $\nu_{<n}(-(1)) = \nu_{<n-1}(-)(1)$, see [HK06, Corollary 1.4(v)]. Now $\nu_{<n-1}(M(X_{U''}))$ and $\nu_{<n-1}(M(X_U))$ are pure Tate by induction, and the Statement follows. □

2.3. **Springer Fiber is Pure Tate.** We prove Theorem 1.1 from the introduction. Let G be reductive algebraic group over k, $N \in \mathcal{N} \subset \mathfrak{g}$ be an element of the nilpotent cone in the Lie algebra of G and $\mathcal{B}_N = \mu^{-1}(N) \subset \tilde{\mathcal{N}}$ the Springer fiber in the Springer resolution $\mu : \tilde{\mathcal{N}} \to \mathcal{N}$. Let

$$\mathcal{B} = \{\mathfrak{b} \subset \mathfrak{g} \,|\, \mathfrak{b} \text{ is a Borel subalgebra}\}$$

denote the flag variety. Then we can identify

$$\mathcal{B}_N = \{\mathfrak{b} \subset \mathfrak{g} \,|\, N \in \mathfrak{b}\} \subset \mathcal{B}.$$

The goal is to prove that $M(\mathcal{B}_N) \in \mathrm{DM}(\mathsf{Spec}(k), \Lambda)$ is pure Tate. We note that the Springer fiber is proper and hence $M^c(\mathcal{B}_N) = M(\mathcal{B}_N)$.

As the Springer fiber only depends on the isogeny class of the semisimple part of G, we may assume that G is of adjoint type and hence a direct product of its simple constituents (see [Jan04, Section 2.7]). Furthermore, a Springer fiber of a direct product of groups decomposes into a direct product as well, and by Lemma 2.3(3) it suffices to consider each individual factor.

So we can assume that G is a simple algebraic group. If G is a classical group, that is, of type A, B, C or D, then $\mathcal{B}_N$ admits a paving by affine spaces by [DLP88] if $\mathrm{char}(k) = 0$ and more general by [Jan04, Theorem 11.22] if $p = \mathrm{char}(k)$ is good for G. Hence $M(\mathcal{B}_N)$ is pure Tate in this case by Lemma 2.3.

We can hence assume that G is a simple group of exceptional type E, F or G and assume that $p > 3(h+1)$, where h denotes the Coxeter-number of G. We proceed as in [DLP88, Section 3.4].

There exists a special cocharacter $\tau : \mathbb{G}_m \to G$ associated to N, see [Jan04, Section 5.2] for a definition and existence and uniqueness result, alternatively use the Morozov–Jacobson theorem, which holds since $p > 3(h-1)$. This cocharacter induces a decomposition of $\mathfrak{g}$ into even weight spaces

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(2n, \tau),$$

such that $N \in \mathfrak{g}(2,\tau)$. Let $G_0 \subset P \subset G$ be the Levi and parabolic subgroup with Lie algebra $\mathfrak{g}(0,\tau)$ and $\bigoplus_{n\geq 0} \mathfrak{g}(2n,\tau)$, respectively, and let $S = \tau(\mathbb{G}_m)$. Now $\mathcal{B}_N$ admits an α-filtration by intersecting it with the P-orbits $\mathcal{O}$ on $\mathcal{B}$. Each of those intersections $\mathcal{B}_{N,\mathcal{O}} = \mathcal{B}_N \cap \mathcal{O}$ is smooth projective.

So $M(\mathcal{B}_N)$ is pure Tate if and only if $M(\mathcal{B}_{N,\mathcal{O}})$ is pure Tate for each $\mathcal{O}$ by Lemma 2.3. Furthermore each $M(\mathcal{B}_{N,\mathcal{O}})$ is pure Tate if and only if $M(\mathcal{B}^S_{N,\mathcal{O}})$ is pure Tate by Lemma 2.4.

By a similar argument to [DLP88, Section 3.6] and using Lemmata 2.3, 2.4 again, we can reduce to the case that N is in fact a distinguished nilpotent element, so not already contained in the Lie algebra of any proper Levi subgroup of G.

Let $B_0 \subset G_0$ be a Borel subgroup. Denote its Lie algebra by $\mathfrak{b}_0$. Then there is a unique $\mathfrak{b}_{\mathcal{O}} \in \mathcal{O}^S$ with $\mathfrak{b}_{\mathcal{O}} \cap \mathfrak{g}(0,\tau) = \mathfrak{b}_0$. Let $U_{\mathcal{O}} = \mathfrak{b}_{\mathcal{O}} \cap \mathfrak{g}(2,\tau)$. This is a B_0-stable linear subspace of the prehomogeneous G_0-module $\mathfrak{g}(2,\tau)$. We can hence consider the variety $X_{U_{\mathcal{O}}}$ as defined in Section 2.2. In fact the map $X_{U_{\mathcal{O}}} \to \mathcal{B}^S_{N,\mathcal{O}}$, $gB_0 \mapsto g\mathfrak{b}_{\mathcal{O}}$ is an isomorphism.

Now [DLP88] show by an involved case by case computation that the G_0-modules $\mathfrak{g}(2,\tau)$ arising in this way from a distinguished nilpotent element for an exceptional group are *good.* This computation, as the whole paper, is carried out for $k = \mathbb{C}$ but also works as long as the Morozov–Jacobson theorem holds, so in particular if $p > 3(h-1)$. We thank George Lusztig for answering a question about that. We can hence use Proposition 2.5 to see that $M(X_{U_{\mathcal{O}}}) = M(\mathcal{B}^S_{N,\mathcal{O}})$ is pure Tate.

This concludes the proof of Theorem 1.1.

3. Equivariant Springer Motives

In this section we prove Theorem 1.3. We assume that $k = \overline{\mathbb{F}}_p$ and $\Lambda = \mathbb{Q}$. We denote a variety $X \in \mathsf{Var}_k$ with an action of a linear group H by $(H \looparrowright X) \in \mathsf{Var}_k$. Morphism between varieties with action are given by pairs

$$(\phi, f) : (H_1 \looparrowright X_1) \to (H_2 \looparrowright X_2)$$

of a morphism of linear groups and a morphism of varieties compatible with the actions. If $\phi = \mathrm{id}$ is the identity morphism, we will often drop it from the notation. Now [SVW18, Chapter I] associates to the the datum $(H \looparrowright X)$ the $\mathbb{Q}$-linear tensor triangulated category[1] of *H-equivariant $\mathbb{D}$-motives on X* denoted by $\mathbb{D}^+_H(X)$. We denote the tensor unit by $\mathbb{1} = \mathbb{1}_X$. The system of categories $\mathbb{D}^+_-(-)$ comes equipped with a six-functor-formalism and induction/restriction functors, very similar to the equivariant derived category of Bernstein–Lunts [BL94]. In the case that X has finitely many H-orbits, [SVW18, Chapter II] defines the category *H-equivariant orbitwise mixed Tate motives* $\mathrm{MTDer}_H(X) \subset \mathbb{D}^+_H(X)$ which are analogous to constructible equivariant sheaves. From now on we consider the case $H = G$ or $H = G \times \mathbb{G}_m$ and $X = \mathcal{N}$.

3.1. Orbitwise Pure Tateness of the Springer motive.

In the introduction we cheated a bit. A priori, it is not clear that the Springer motive $\mu_!(\mathbb{1}_{\tilde{\mathcal{N}}}) = \mu_*(\mathbb{1}_{\tilde{\mathcal{N}}}) \in \mathbb{D}^+_H(\mathcal{N})$ already lives in the subcategory $\mathrm{MTDer}_H(\mathcal{N})$. In this section we show how

[1]We work with $\mathbb{D}(-) = \mathrm{DA}_{\acute{e}t}(-,\mathbb{Q})$, the homotopical stable algebraic derivator of étale motives with rational coefficients over the category of varieties Var_k over k. We note that $\mathrm{DA}_{\acute{e}t}(\mathsf{Spec}(k),\mathbb{Q}) \cong \mathrm{DM}(\mathsf{Spec}(k),\mathbb{Q})$ which is the category of motives we considered in the first part of the paper.

the pure Tateness of the Springer fiber implies this and that $\mu_!(\mathbb{1}_{\tilde{\mathcal{N}}})$ is additionally pointwise pure.

Theorem 3.1. *Let $\mathbb{O}$ be an H-orbit on $\mathcal{N}$. Let N be a point in $\mathbb{O}$ and denote by $H_N \subset H$ its stabilizer. Denote the corresponding morphisms of varieties with group action by*

$$\begin{gathered}\mu = (\mathrm{id}, \mu) : (H \looparrowright \tilde{\mathcal{N}}) \to (H \looparrowright \mathcal{N}),\\ j = (\mathrm{id}, j) : (H \hookrightarrow \mathbb{O}) \hookrightarrow (H \looparrowright \mathcal{N}) \text{ and}\\ (\iota, i) : (H_N \looparrowright \{N\}) \hookrightarrow (H \looparrowright \mathbb{O}).\end{gathered}$$

Then for $? \in \{*, !\}$ we have a chain of functors

$$\mathbb{D}^+_H(\tilde{\mathcal{N}}) \xrightarrow{\mu_! = \mu_*} \mathbb{D}^+_H(\mathcal{N}) \xrightarrow{j^?} \mathbb{D}^+_H(\mathbb{O}) \xrightarrow{(\iota,i)^*} \mathbb{D}^+_{H_N}(\{N\}) \xrightarrow{\mathrm{For}} \mathbb{D}^+(\mathsf{Spec}(k))$$

where For *is the functor forgetting the H_N action. We claim that*

$$\mathrm{For}(\iota, i)^* j^? \mu_!(\mathbb{1}_{\tilde{\mathcal{N}}}) \in \mathbb{D}^+(\mathsf{Spec}(k))$$

is pure Tate, that is, a finite direct sum of Tate motives $\mathbb{1}(n)[2n]$.

Proof. Since the forgetful functor For commutes with all six functors, it suffices to show the corresponding statement where we forget about all group actions. Since μ is proper and hence $\mu_* = \mu_!$, it suffices to show the statement for $? = *$ by duality. Now we apply base change for the cartesian diagram

$$\begin{array}{ccccc} \mathcal{B}_N = \mu^{-1}(N) & \xrightarrow{k} & \mu^{-1}(\mathbb{O}) & \xrightarrow{l} & \tilde{\mathcal{N}} \\ \big\downarrow{\scriptstyle \mathrm{fin}_{\mathcal{B}_N}} & & \big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ \mathsf{Spec}(k) = \{N\} & \xrightarrow{i} & \mathbb{O} & \xrightarrow{j} & \mathcal{N} \end{array}$$

and see that

$$\mathrm{For}(\iota, i)^* j^* \mu_!(\mathbb{1}_{\tilde{\mathcal{N}}}) \cong \mathrm{fin}_{\mathcal{B}_N,!}(k^* l^* \mathbb{1}_{\tilde{\mathcal{N}}}) = \mathrm{fin}_{\mathcal{B}_N,!}(\mathbb{1}_{\mathcal{B}_N}) \in \mathbb{D}^+(\mathsf{Spec}(k)).$$

Since we are working with rational coefficients there is a natural equivalence

$$\mathbb{D}^+(\mathsf{Spec}(k)) = \mathrm{DA}_{\acute{e}t}(\mathsf{Spec}(k), \mathbb{Q}) \cong \mathrm{DM}(\mathsf{Spec}(k), \mathbb{Q}),$$

see [Ayo14], and the Verdier dual of $\mathrm{fin}_{\mathcal{B}_N,!}(\mathbb{1}_{\mathcal{B}_N}) \in \mathbb{D}^+(\mathsf{Spec}(k))$ corresponds to $M^c(\mathcal{B}_N) \in \mathrm{DM}(\mathsf{Spec}(k), \mathbb{Q})$, the motive of the Springer fiber. But $M^c(\mathcal{B}_N)$ is pure Tate by Theorem 1.1. □

By definition, $\mathrm{MTDer}_H(\mathcal{N})$ consists of motives M such that

$$\mathrm{For}(\iota, i)^* j^? M \in \langle \mathbb{1}(n) \,|\, n \in \mathbb{Z} \rangle_\Delta \subset \mathbb{D}^+(\mathsf{Spec}(k))$$

is mixed Tate, i.e. contained in the triangulated category generated by motives $\mathbb{1}(n) \in \mathbb{D}^+(\mathsf{Spec}(k))$, for each orbit $j : \mathbb{O} \hookrightarrow \mathcal{N}, ? \in \{*, !\}$ and $(\iota, i)^*$ defined as above. A motive M is called *pointwise ?-pure* if additionally $\mathrm{For}(\iota, i)^* j^? M$ is pure Tate (so a finite direct sum of motives $\mathbb{1}(n)[2n]$) for $? \in \{*, !\}$ and *pointwise pure* if it is pointwise $*$-pure and pointwise !-pure.

Corollary 3.2. *We have $\mu_*(\mathbb{1}_{\tilde{\mathcal{N}}}) \in \mathrm{MTDer}_H(\mathcal{N})$ and $\mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})$ is pointwise pure.*

3.2. **Tilting and Formality for Springer motives.** Denote by

$$\mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add} = \langle \mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})(n) \mid n \in \mathbb{Z} \rangle_{\oplus, \Subset} \subset \mathrm{MTDer}_H^{Spr}(\mathcal{N})$$

the idempotent closed additive subcategory of $\mathrm{MTDer}_H^{Spr}(\mathcal{N})$ generated by the Springer motive and its Tate twists.

Lemma 3.3. *The category* $\mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add}$ *is a* tilting subcategory *of* $\mathbb{D}_H^+(\mathcal{N})$, *meaning that it is*

(1) idempotent closed, additive and

(2) $\mathrm{Hom}_{\mathbb{D}_H^+(\mathcal{N})}(M, N[i]) = 0$ *for all* $M, N \in \mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add}$ *and* $i \neq 0$.

Proof. (1) Holds by construction.
(2) This is implied by the pointwise purity of the objects in $\mathrm{MTDer}_H(\mathcal{N})_{add}$ using an argument very similar to [SW16, Corollary 6.3]. More, generally we show that for all $M, N \in \mathrm{MTDer}_H^{Spr}(\mathcal{N})$ with M pointwise $*$-pure and N pointwise !-pure

$$\mathrm{Hom}_{\mathbb{D}_H^+(\mathcal{N})}(M, N[i]) = 0$$

for $i \neq 0$. For this, denote by $j : U \hookrightarrow \mathcal{N}$ and $i : Z = \mathcal{N} \backslash U \hookrightarrow \mathcal{N}$ the inclusion of the open orbit and its closed complement. Then the localisation sequence

$$j_! j^! M \longrightarrow M \longrightarrow i_* i^* M \xrightarrow{+1}$$

yields an exact sequence

$$\mathrm{Hom}_{\mathbb{D}_H^+(Z)}\left(i^* M, i^! N[i]\right) \longrightarrow \mathrm{Hom}_{\mathbb{D}_H^+(\mathcal{N})}(M, N[i]) \longrightarrow \mathrm{Hom}_{\mathbb{D}_H^+(U)}\left(j^* M, j^! N[i]\right).$$

Now $i^* M$ is pointwise $*$-pure and $i^! N$ is pointwise !-pure. Hence, the the left hand side vanishes for $i \neq 0$ by induction (here we use that there are only finitely many orbits). Also, the right hand side vanishes for $i \neq 0$ using the purity assumption, the induction equivalence and that $\mathrm{Hom}_{\mathbb{D}_{H_N}^+(\mathcal{N})}(\mathbb{Q}, \mathbb{Q}(p)[q]) = 0$ unless $p = 2q$, where H denotes the stabiliser of a point $N \in U$. □

Remark 3.4. The arguments in the proof of Lemma 3.3 (2) argument do not translate to the non-equivariant case immediately. Hence, it is not clear if a non-equivariant analogue of the motivic derived Springer correspondence is true. One would need to show that the higher Chow groups of nilpotent orbits vanish, which is not clear to the author.

We can now apply the tilting formalism and show

Theorem 3.5. *There is an equivalence of categories, called* tilting, *between*

$$\mathsf{tilt} : \mathrm{Hot}^b(\mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add}) \xrightarrow{\sim} \mathrm{MTDer}_H^{Spr}(\mathcal{N})$$

the homotopy category of bounded chain complexes $\mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add}$ *and the category of Springer motives* $\mathrm{MTDer}_H^{Spr}(\mathcal{N})$.

Proof. Since by Lemma 3.3 $\mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add}$ is a tilting subcategory of $\mathbb{D}_H^+(\mathcal{N})$ by [SVW18, Theorem B.3.1], which establishes a tilting formalism for stable derivators, there is a fully faithful functor

$$\mathsf{tilt} : \mathrm{Hot}^b(\mathrm{MTDer}_H^{Spr}(\mathcal{N})_{add}) \to \mathbb{D}_H^+(\mathcal{N}).$$

The essential image of tilt is the triangulated subcategory of $\mathbb{D}^+_H(\mathcal{N})$ generated by $\mathrm{MTDer}^{Spr}_H(\mathcal{N})_{add}$ which is $\mathrm{MTDer}^{Spr}_H(\mathcal{N})$ by definition. □

We name the $\mathbb{Z}$-graded "Ext"-algebra[2] of the Springer motive $\mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})$

$$E := \bigoplus_{n\in\mathbb{Z}} E^n, \text{ where } E^n = \mathrm{Hom}_{\mathbb{D}^+_H(\mathcal{N})}\left(\mu_*\left(\mathbb{1}_{\tilde{\mathcal{N}}}\right), \mu_*\left(\mathbb{1}_{\tilde{\mathcal{N}}}\right)(n)[2n]\right).$$

We denote the shift of grading functor for graded E-modules by $\langle - \rangle$. We can rephrase the last theorem as

Corollary 3.6. *There is an equivalence of categories*

$$\mathrm{Der}^b(\mathrm{mod}^{\mathbb{Z}}\text{-}E) \to \mathrm{MTDer}^{Spr}_H(\mathcal{N})$$

between the bounded derived category of finitely generated graded right modules over E and the category of Springer motives.

Proof. The category $\mathrm{MTDer}^{Spr}_H(\mathcal{N})_{add}$ can be identified with the full subcategory

$$\langle E\langle n\rangle \mid n \in \mathbb{Z}\rangle_{\oplus,\Subset} \subset \mathrm{mod}^{\mathbb{Z}}\text{-}E.$$

of graded right E-modules generated by finite direct sums and summand of shifts of E. By the explicit description of E, see the next section, E has finite cohomological dimension, and hence the homotopy category of bounded chain complexes in $\langle E\langle n\rangle \mid n \in \mathbb{Z}\rangle_{\oplus,\Subset}$ is equivalent to the bounded derived category of graded E-modules. □

3.3. **Description of the "Ext"-algebra E.** The last step in the proof of Theorem 1.3 is to explicitely describe the "Ext"-algebra E. Recall that $Z = \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ denotes the Steinberg variety and that H acts on Z by the diagonal action. We repeat some results from [CG10, Section 8.6], who give an explicit description of the sheaf-theoretic version of E in terms of the Borel–Moore homology of the Steinberg-variety. Their results translate to the motivic setting almost word by word, all that is needed is a six-functor formalism.

Lemma 3.7. *There is an isomorphism of graded algebras*

$$E \cong (\mathrm{CH}^\bullet_H(Z, \mathbb{Q}), \star)$$

between E and the H-equivariant Chow groups of Z equipped with the convolution product.

Proof. We just show that $E^n \cong \mathrm{CH}^n_H(Z, \mathbb{Q})$ as a vector space here. The statements about the algebra structure can be deduced as in [CG10, Section 8.6]. Consider the cartesian diagram of H-varieties:

$$\begin{array}{ccc} Z & \xrightarrow{p_2} & \tilde{\mathcal{N}} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \mu} \\ \tilde{\mathcal{N}} & \xrightarrow{\mu} & \mathcal{N} \end{array}$$

[2] We put the quotation marks here because of the Tate twists which is not part of the definition of the true Ext-algebra. In fact, the true Ext-algebra is concentrated in degree 0, by Lemma 3.3.

Denote for a variety $X \in \mathsf{Var}_k$ its structure map by $\mathrm{fin}_X : X \to \mathsf{Spec}(k) = pt$. Since $\tilde{\mathcal{N}}$ is smooth we have $\mathbb{1}_{\tilde{\mathcal{N}}} = \mathrm{fin}_{\tilde{\mathcal{N}}}^*(\mathbb{1}_{pt}) = \mathrm{fin}_{\tilde{\mathcal{N}}}^!(\mathbb{1}_{pt})(-d)[-2d]$ where $d = \dim(\tilde{\mathcal{N}})$. Furthermore note that μ is proper and hence $\mu_! = \mu_*$. Now consider

$$\begin{aligned} E^n &= \mathrm{Hom}_{\mathbb{D}^+_H(\mathcal{N})}\left(\mu_!(\mathbb{1}_{\tilde{\mathcal{N}}}), \mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})(n)[2n]\right) \\ &= \mathrm{Hom}_{\mathbb{D}^+_H(\tilde{\mathcal{N}})}\left(\mathbb{1}_{\tilde{\mathcal{N}}}, \mu^!\mu_*(\mathbb{1}_{\tilde{\mathcal{N}}})(n)[2n]\right) \\ &= \mathrm{Hom}_{\mathbb{D}^+_H(\tilde{\mathcal{N}})}\left(\mathbb{1}_{\tilde{\mathcal{N}}}, p_{1,*}p_2^!(\mathbb{1}_{\tilde{\mathcal{N}}})(n)[2n]\right) \\ &= \mathrm{Hom}_{\mathbb{D}^+_H(\tilde{\mathcal{N}})}\left(\mathrm{fin}_{\tilde{\mathcal{N}}}^*(\mathbb{1}_{pt}), p_{1,*}p_2^!\,\mathrm{fin}_{\tilde{\mathcal{N}}}^!(\mathbb{1}_{pt})(n-d)[2(n-d)]\right) \\ &= \mathrm{Hom}_{\mathbb{D}^+_H(pt)}\left(\mathbb{1}_{pt}, \mathrm{fin}_{Z,*}\,\mathrm{fin}_Z^!(\mathbb{1}_{pt})(n-d)[2(n-d)]\right) \\ &= \mathrm{CH}_H^{\dim(Z)+(n-d)}(Z,\mathbb{Q}) = \mathrm{CH}_H^n(Z,\mathbb{Q}) \end{aligned}$$

where in the last equality we used [SVW18, Theorem II.2.9.] (there is no resolution of singularities for $\mathsf{Spec}(k)$ necessary here, using [Kel17, Section 5.2] and that p is invertible in $\mathbb{Q}$) and that Z is of dimension $\dim(Z) = d$. □

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