THE QUEENS DOMINATION PROBLEM

by

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Summary

The queens graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column or diagonal. A set D of squares of Q_n is a *dominating set* for Q_n if every square of Q_n is either in D or adjacent to a square in D. If no two squares of a set I are adjacent then I is an *independent set*. Let $\gamma(Q_n)$ denote the minimum size of a dominating set of Q_n and let $i(Q_n)$ denote the minimum size of an independent dominating set of Q_n. The main purpose of this thesis is to determine new values for $\gamma(Q_n)$. We begin by discussing the most important known lower bounds for $\gamma(Q_n)$ in Chapter 2. In Chapter 3 we state the hitherto known values of $\gamma(Q_n)$ and explain how they were determined. We briefly explain how to obtain all non-isomorphic minimum dominating sets for Q_8 (listed in Appendix A). It is often useful to study these small dominating sets to look for patterns and possible generalisations. In Chapter 4 we determine new values for $\gamma(Q_{69})$, $\gamma(Q_{77})$, $\gamma(Q_{30})$ and $i(Q_{45})$ by considering asymmetric and symmetric dominating sets for the case $n = 4k + 1$ and in Chapter 5 we search for dominating sets for the case $n = 4k + 3$, thus determining the values of $\gamma(Q_{19})$ and $\gamma(Q_{31})$. In Chapter 6 we prove the upper bound $\gamma(Q_n) \leq \frac{8}{15}n + \mathcal{O}(1)$, which is better than known bounds in the literature and in Chapter 7 we consider dominating sets on hexagonal boards. Finally, in Chapter 8 we determine the irredundance number for the hexagonal boards H_5 and H_7, as well as for Q_5 and Q_6.

Key terms: chessboards, queens graph, queens domination problem, domination, irredundance, hexagonal boards.

Chapter 1

Introduction

As far as could be established, the earliest ideas of dominating sets date back to the origin of the game of chess in India over 400 years ago. Chess, of course, is much more than merely a mathematical activity; nevertheless, in chess one studies sets of chess pieces which cover, or dominate, various opposing pieces or various squares of the chessboard. Chessboard domination problems thus initiated the study of dominating sets of graphs, at first rather informally until the topic of domination was given formal mathematical definition with the publication of the books by Berge [2] and Ore [14] in 1962.

For a given graph $G = (V, E)$ and a vertex $v \in V$, we denote the *open neighbourhood* of v by $N(v)$ and the *closed neighbourhood* by $N[v]$, that is, $N(v) = \{u \in V \mid uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. For $S \subseteq V$ we define $N[S] = \bigcup_{s \in S} N[s]$. We further define the *private neighbourhood* of $v \in S$ as $pn[v, S] = N[v] - N[S - \{v\}]$. If $pn[v, S] \neq \emptyset$ for some vertex v, then every vertex in $pn[v, S]$ is called a *private neighbour* of v (relative to S). Note that a vertex can be its own private neighbour. A set $S \subseteq V$ is called a *dominating set* of G if each vertex of G which is not in S, is adjacent to a vertex in S, that is, $N[S] = V$. Further, S is an *independent set* if no two vertices in S are adjacent in G. A vertex $v \in S$ is *irredundant in* S if it has at least one private neighbour relative to S, and the set S is *irredundant* if every vertex $v \in S$

is irredundant in S. An irredundant set S is *maximal irredundant* if for every vertex $u \in V - S$, the set $S \cup \{u\}$ is not irredundant, which means that there exists at least one vertex $w \in S \cup \{u\}$ which does not have a private neighbour.

The *domination number* $\gamma(G)$ (the *independent domination number* $i(G)$, respectively) of the graph G is the smallest number of vertices in a dominating set (an independent dominating set) of G. The minimum cardinality of a maximal irredundant set in G is called the *irredundance number* and is denoted by $ir(G)$. As shown in [7] , a minimal dominating set is also a maximal irredundant set. Since an independent dominating set is a dominating set by definition, it follows that $ir(G) \leq \gamma(G) \leq i(G)$ for any graph G.

That even the original chessboard domination problems are astonishingly difficult is apparent in view of the fact that so few of these problems have been solved completely. The unsolved classical problems were important in motivating the revival of the study of dominating sets in graphs in the early 1970's. One of the most interesting – and most difficult – chessboard problems is the queen domination problem in which one has to determine the minimum number of queens necessary to cover (or dominate) all squares on an $n \times n$ chessboard. This can also be considered as a graph domination problem, in the following way:

The queens graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if a queen placed on one square covers the other square, that is, if the squares are in the same row, column or diagonal. A set D of squares of Q_n is a *dominating set* of Q_n if every square of Q_n is either in D or adjacent to a square in D. If no two squares of a set I are adjacent, then I is an *independent set*. If each queen on a set X of squares covers a square which is not covered by a queen on any other square in X, then X is an *irredundant set* of Q_n. As for graphs in general, $\gamma(Q_n)$ denotes the minimum size of a dominating set for Q_n, $i(Q_n)$ denotes the minimum size of an independent dominating set of Q_n and $ir(Q_n)$ denotes the minimum size of a maximal irredundant set of Q_n. We emphasize that for any n, $ir(Q_n) \leq \gamma(Q_n) \leq i(Q_n)$.

In 1862, C. F. de Jaenisch [8] considered the problem of determining values of $\gamma(Q_n)$, and in 1892, W. W. Rouse Ball [1] gave values of $\gamma(Q_n)$ up to $n = 8$. Much more recently P. H. Spencer, as cited in [6, 17] , proved the lower bound $\gamma(Q_n) \geq \frac{1}{2}(n-1)$, $n \geq 1$. Several researchers (see [6, 9, 10, 4]) established upper bounds. W. D. Weakley [17] refined the lower bound by proving $\gamma(Q_{4k+1}) \geq 2k+1$ for all $k \geq 0$. He also showed that $\gamma(Q_{4k+1}) = 2k+1$ for $k = 3, 4, 5, 6$ and 8 by constructing dominating sets of order $2k+1$. A. P. Burger [5] added $k = 9, 12, 13$ and 15 to the list and P. B. Gibbons and J. A. Webb [11] filled in the gaps by finding sets for $k = 7, 10, 11$ and 14 so that $\gamma(Q_{4k+1}) = 2k+1$ for $0 \leq k \leq 15$. For surveys of this and other chessboard problems see [2, 9, 13] .

The main purpose of this thesis is to determine new values for $\gamma(Q_n)$. We begin by discussing the most important known lower bounds for $\gamma(Q_n)$ in Chapter 2. In Chapter 3 we state the hitherto known values of $\gamma(Q_n)$ and explain how they were determined. We briefly explain how to obtain all non-isomorphic minimum dominating sets for Q_8 (listed in Appendix A). It is often useful to study these small dominating sets to look for patterns and possible generalisations. In Chapter 4 we determine new values for $\gamma(Q_{69})$, $\gamma(Q_{77})$, $\gamma(Q_{30})$ and $i(Q_{45})$ by considering asymmetric and symmetric dominating sets for the case $n = 4k+1$ and in Chapter 5 we search for dominating sets for the case $n = 4k+3$, thus determining the values of $\gamma(Q_{19})$ and $\gamma(Q_{31})$. In Chapter 6 we prove the upper bound $\gamma(Q_n) \leq \frac{8}{15}n + \mathcal{O}(1)$, which is better than known bounds in the literature and in Chapter 7 we consider dominating sets on hexagonal boards. Finally, in Chapter 8 we determine the irredundance number for the hexagonal boards H_5 and H_7, as well as for Q_5 and Q_6.

Chapter 2

Lower bounds for the domination number of Q_n

In this chapter we discuss lower bounds for $\gamma(Q_n)$. We begin with the proof of the bound $\gamma(Q_n) \geq \frac{1}{2}(n-1)$ found by Spencer. We then give some properties of dominating sets, discovered by Weakley, which attain this bound, and which lead to a refinement of this bound.

Note that any one queen can attack at most $4n-3$ squares on an $n \times n$ chessboard: A queen closest to the centre of the board dominates the most squares ($4n-3$ if n is odd) and all other queens dominate fewer squares. Therefore $\gamma(Q_n)$ is bounded below by $\frac{1}{4}n$. Until quite recently, no non-trivial lower bounds were known, in spite of the fact that this problem dates as far back as 1862.

We will identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, having sides parallel to the coordinate axes. We usually place the board so that the centre of the lower left corner square has coordinates $(1,1)$, and refer to board squares by the coordinates of their centres. In some cases we place the board so that the centre square has coordinates (0,0). The square (x, y) is in the column x and row y. A square is called *even* (respectively *odd*) if the sum of its coordinates is even (respectively odd). Notice that diagonals consist of squares with the same parity. *Positive* and *negative diagonals* are sets of squares whose centres lie on lines of slope 1 and -1, respectively. The *main diagonals* are the two diagonals from corner square to corner square. The *edge squares* are the $4n-4$ squares on the edge of an $n \times n$ board. The set of all edge squares is also referred to as the *edge*. A set of squares (rows, diagonals, etc.) *meet* another set of squares in the intersection of the two sets. A square (row, column, diagonal) is said to be *occupied* if there is a queen on that square (or in the row, column, diagonal); otherwise it is *unoccupied*.

The following theorem by P. H. Spencer gives a lower bound for $\gamma(Q_n)$. The proof given can be found in [17] .

Theorem 2.1 *For each positive integer n,* $\gamma(Q_n) \geq \frac{1}{2}(n-1)$.

Proof. It is easy to see the theorem is true for $n < 3$, so we assume $n \geq 3$. Note that placing queens at (i, i) for all i but 1 and 3 gives a dominating set of $n-2$ queens, so $\gamma(Q_n) \leq n-2$. Thus any dominating set of minimal size leaves at least two rows and two columns empty.

Assume we have a dominating set for Q_n, $n \geq 3$, with $\gamma = \gamma(Q_n)$ queens. Let a be the number of the leftmost empty column, b the number of the rightmost empty column, c the number of the lowest empty row, d the number of the highest empty row. By symmetry we may assume $d - c \leq b - a$. This inequality implies that we can find $b-a$ consecutive rows including those rows lying strictly between rows c and d. Thus there exist an integer m and rows

$$m, m+1, m+2, \ldots, m+b-a-1, \tag{2.1}$$

with

$$1 \leq m \leq c+1 \text{ and } d-1 \leq m+b-a-1 \leq n. \tag{2.2}$$

For such an m, let S_m be those squares in columns a and b which are in the rows (2.1) and let P_m be all the squares below row m or above row $m+b-a-1$ which contain queens. Since all rows above d or below c contain queens and at most one of the rows below or above S_m can be empty, namely row c or d (remember $b-a \geq d-c$), we have

$$|P_m| \geq n-(b-a)-1. \tag{2.3}$$

No diagonal contains more than one square of S_m, so no queen diagonally attacks more than two squares of S_m. Queens of P_m do not attack any squares of S_m by row or column, so each attacks at most two squares of S_m. Other queens attack at most four squares of S_m. Each of the $2(b-a)$ squares of S_m is attacked, so

$$\begin{aligned} 2(b-a) &\leq 2|P_m| + 4(\gamma - |P_m|) \\ &= 4\gamma - 2|P_m| \\ &\leq 4\gamma - 2[n-(b-a)-1], \end{aligned} \tag{2.4}$$

which simplifies to $\gamma \geq \frac{1}{2}(n-1)$. □

Placing a queen on the centre square of Q_3 shows $\gamma(Q_3) = 1$ and Figure 2.1 shows a placement, establishing $\gamma(Q_{11}) = 5$. No other cases are known in which the bound of Theorem 2.1 holds. In Chapter 5 we investigate this matter further. The next theorem by W. D. Weakley [17] gives some idea of why the bound of Theorem 2.1 is rarely attained. This result also yields a refinement of the bound in Theorem 2.1, namely $\gamma(Q_{4k+1}) \geq 2k+1$.

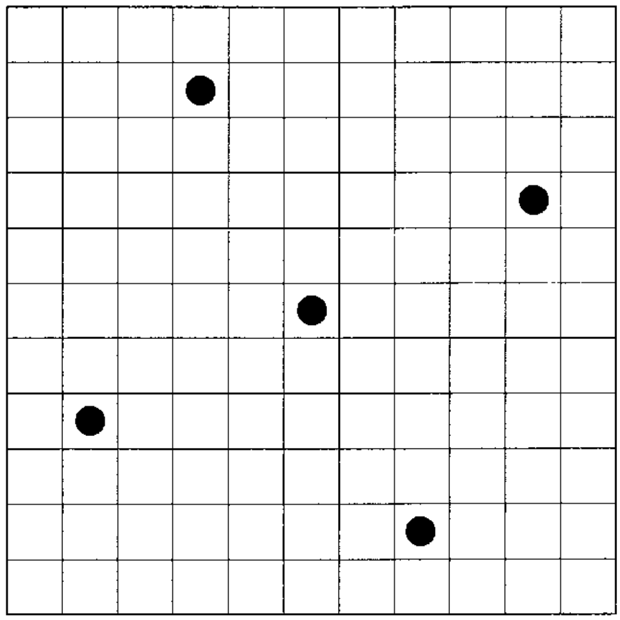

Figure 2.1 $\gamma\,(Q_{11}) = 5$

Theorem 2.2 *Let R be a dominating set of Q_n such that $|R| = \frac{1}{2}(n-1)$. Then*

1. $n \equiv 3 \pmod 4$.
2. *R is independent.*
3. *There is an odd integer j, $\frac{3}{4}(n+1) \leq j \leq n$, such that there is a $j \times j$ sub-board U of Q_n satisfying:*
   (a) *each edge square of U is attacked exactly once;*
   (b) *each row or column of Q_n that does not meet U contains exactly one queen, as does each main diagonal of U.*

Proof. We use the notation and definitions of the proof of Theorem 2.1 and assume that R is a dominating set of size $\frac{1}{2}(n-1)$ of Q_n, oriented so that $d - c \leq b - a$. Let m be any integer satisfying (2.2). From (2.4) we have $2(b-a) \leq 4\gamma - 2|P_m|$, and together with $|R| = \frac{1}{2}(n-1)$ we obtain $|P_m| \leq n - (b-a) - 1$, which with (2.3) implies $|P_m| = n - (b-a) - 1$.

If $d - c < b - a$ then we can choose m satisfying (2.2) so that both rows c and d meet S_m. Then each row not meeting S_m contains at least one queen, giving $|P_m| \geq n - (b - a)$, a contradiction. Therefore $d - c = b - a$.

Let $j = b - a + 1$ and let U be the $j \times j$ sub-board having corners $(a, c), (a, d), (b, c)$ and (b, d). Let E denote the set of edge squares of U.

The only values of m satisfying (2.2) are $m = c, c + 1$. Since $|P_c| = |P_{c+1}| = n - (b - a) - 1$, each row that does not meet U contains exactly one queen. Similarly each column not meeting U contains exactly one queen.

Since the inequalities in (2.4) are equations when $|R| = \gamma = \frac{1}{2}(n - 1)$, for both $m = c$ and $m = c + 1$ each queen of P_m must attack two squares of S_m diagonally, necessarily one along each diagonal. Therefore every queen of R lies strictly between the positive diagonals through (a, d) and (b, c), and strictly between the negative diagonals through (a, c) and (b, d). From this fact we draw two conclusions.

First, no queen has the property that both its row and its column miss U, so there are $2(n - j)$ queens outside U, each attacking six squares of E. The remaining $\frac{1}{2}(n - 1) - 2(n - j)$ queens each attack eight squares of E, so the number of squares of E attacked is at most

$$6.2(n - j) + 8[+(n - 1) - 2(n - j)],$$

which equals $4(j - 1)$. Since R is a dominating set and E contains $4(j - 1)$ squares, each square of E is attacked exactly once. This establishes (a).

Second, the corner squares of U are not attacked diagonally from outside U. Since these squares lie in unoccupied rows and columns, they must be diagonally attacked from inside U. Thus the long diagonals of U are occupied. This shows there is at least one queen inside U, and since there are $2(n - j)$ queens outside U we have $1 + 2(n - j) \leq \frac{1}{2}(n - 1)$, which reduces to $\frac{3}{4}(n + 1) \leq j$.

If R is dependent then there are two queens in some line (row, column, or diagonal); since each square of E is attacked only once, that line does not meet E. All diagonals from occupied squares meet E, so the line is a row or a column. But a row or column that does not meet E also does not meet U, and we have shown that the rows and columns not meeting U each contain only one queen, so no line contains two queens. Therefore R is independent, and (b) is also established.

We now show $n \equiv 3 \pmod 4$. Let D denote the set of squares of E that lie below the long negative diagonal of U. Let L denote the set of occupied squares that lie below the extension of the same diagonal to the $n \times n$ board. Each queen of R attacks exactly one square of D along its positive diagonal. Each of the $2(n-j)$ queens outside U attacks exactly one square of D by row or column. Each of the queens inside U attacks exactly two squares of D by row or column. Finally, each queen of L attacks exactly two squares of D along its negative diagonal, and other queens do not attack squares of D along their negative diagonals. Since each of the $2j-3$ squares in D is attacked exactly once, we have

$$\frac{1}{2}(n-1) + 2(n-j) + 2[\frac{1}{2}(n-1) - 2(n-j)] + 2|L| = 2j - 3.$$

This reduces to $n = 4|L| + 3$, so $n \equiv 3 \pmod 4$ and $\frac{1}{2}(n-1)$ is odd.

Suppose that j is even. Then the set E contains equal numbers of even and odd squares. Since j is even, any occupied row or column that meets E does so at one even and one odd square. Thus the subset of E consisting of those squares that are attacked diagonally must contain equal numbers of even squares and odd squares. This implies that the number of queens on even squares equals the number of queens on odd squares, and thus the total number of queens is even. But the total number of queens is $\frac{1}{2}(n-1)$, which is odd. This contradiction implies that j is odd. $\square$

Corollary 2.3 *For each non-negative integer k, $\gamma(Q_{4k+1}) \geq 2k+1$.*

Proof. From Theorem 2.1 we have $\gamma(Q_{4k+1}) \geq 2k$, but from Theorem 2.2 (1) equality cannot hold, therefore $\gamma(Q_{4k+1}) \geq 2k+1$. □

Chapter 3

Known values of the domination and independent domination numbers of Q_n

In this chapter we list the known values of γ and i and describe how they are determined. We also describe how to obtain all non-isomorphic minimum dominating sets for Q_8.

Tables 3.1 and 3.2 give the known values for γ and i. For $n \leq 3$ we have $\gamma(Q_n) = i(Q_n) = 1$. For $n < 5$, optimal placements are easily discovered by trial.

Most of the values of γ are established by a placement which attains some lower bound. With Spencer's bound, $i(Q_n) \geq \gamma(Q_n) \geq \frac{1}{2}(n-1)$, $\gamma(Q_{11}) = i(Q_{11}) = 5$ is established. However, $n = 3$ and $n = 11$ are the only values known for which this bound holds exactly. If n is even Spencer's bound becomes $\gamma(Q_{2m}) \geq m$, and this together with dominating sets of the rquired size establishes values for γ for $n = 4, 6, 10, 12, 18$ and 30. The value $\gamma(Q_{18}) = 9$ was determined by A. McRae [10] , and we will explain in Chapter 4 how the value $\gamma(Q_{30}) = 15$ is obtained. The bound $\gamma(Q_{4k+1}) \geq 2k + 1$ together with dominating sets of this size establish values for

γ for $1 \leq k \leq 15$ (the case $\gamma(Q_1) = 1$ being trivial) as well as $k = 17$ and 19. Weakley [17] claimed $k = 3, 4, 5, 6$ and 8, the values for $k = 9, 12, 13$ and 15 were determined in [5] while Gibbons and Webb [11] claimed $k = 7, 10, 11$ and 14. We will explain in Chapter 4 how these values, including the new values for $k = 17$ and 19, are determined. The exceptional value $\gamma(Q_8) = 5$ is claimed by W. W. Rouse Ball [1], and Weakley [17] proved $\gamma(Q_7) = 4$ of which we will give an alternative proof in Chapter 5. The values $\gamma(Q_{19}) = 10$ and $\gamma(Q_{31}) = 16$ are also explained in Chapter 5.

n	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	30	31
γ	2	3	3	4	5	5	5	5	6	7	7/8	8/9	8/9	9	9	10	15	16
i	3	3	4	4	5	5	5	5	7	7	8	9	9	9				

Table 3.1 Known values for γ and i

k	5	6	7	8	9	10	11	12	13	14	15	17	19
n	21	25	29	33	37	41	45	49	53	57	61	69	77
γ	11	13	15	17	19	21	23	25	27	29	31	35	39
i	11	13		17			23						

Table 3.2 Known values for γ and i

From Table 3.1 we can see that there are only three values of n for which it is known that $\gamma(Q_n) < i(Q_n)$. The value $i(Q_{12}) = 7$ (see [3]) is obtained by doing an exhaustive search on all independent sets of six queens, showing that $i(Q_{12}) \neq 6$. Gibbons and Webb [11] did the same to establish the values of i for $n = 14, 15$ and 16.

To confirm the values of γ in Tables 3.1 and 3.2 we need to give some placements of dominating sets. All possible minimum dominating sets of Q_n up to n = 11 can be found by computer, simply by checking all possible placements of queens. For $n \geq 5$, all possible solutions, up to symmetry, are listed in Appendix A. Gibbons and Webb

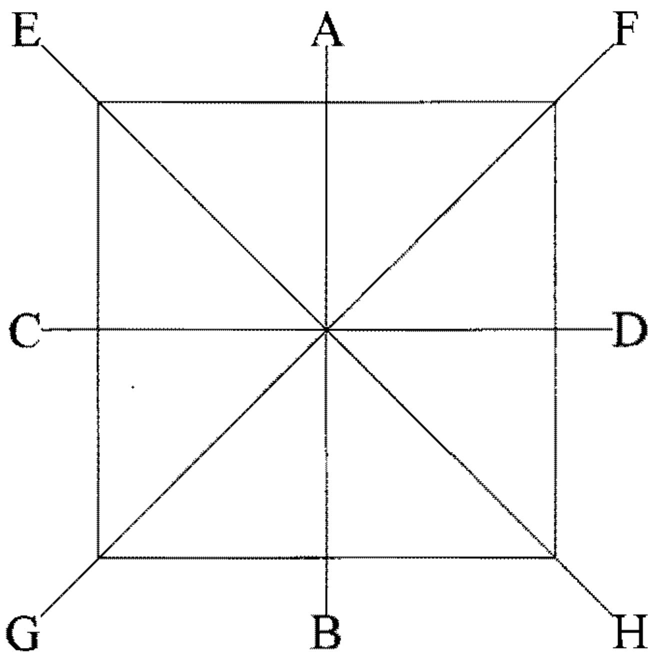

Figure 3.1 Reflections of the chessboard

determined all possible placements of minimum independent dominating sets up to $n = 15$. If the number of dominating sets is small, reflections and rotations can be eliminated by hand. However, for $n = 8$, there are 638 different optimal dominating sets. To ensure that no reflections or rotations of dominating sets are repeated, dominating sets for $n = 8$ are oriented so that:

(1) most (three or more) of the queens are in the lower half of the board;
(2) most of the queens are in the left half of the board;
(3) most of the queens are on or below the positive main diagonal.

Note that (1) ensures that reflections about CD (see Figure 3.1) and 180° rotations are not repeated and (2) ensures that reflections about AB are not repeated. (1) combined with (2) ensures that 90° and 270° rotations as well as reflections about EH are not repeated. Note that all these symmetries are eliminated, because n is even and the number of queens is odd. The only symmetry left, is the reflection about FG, most of which are eliminated by (3). The reflections which (3) does not eliminate are the dominating sets with an odd number of queens on the positive diagonal with the same number of queens below and above the concerned diagonal. In this case we listed the first dominating set according to the lexicographic ordering of ordered pairs.

We now give dominating sets of Q_n for those values of n listed in Table 3.1, where $13 \leq n \leq 25$. Where sets are symmetric, *i.e.* for Q_{13}, Q_{15}, Q_{17}, Q_{21} and Q_{25}, the centre square has coordinates $(0,0)$, otherwise the lower left corner square has coordinates $(1,1)$. Where $\gamma(Q_n) = i(Q_n)$, we list the independent set. The dominating and independent dominating sets for Q_{12} can be obtained by simply adding a queen (respectively two independent queens) to the minimum dominating set for Q_{11} given in Appendix A. Dominating sets for $n \geq 29$ are listed in Chapter 4.

Q_{13}. $\pm(1,-5), \pm(3,-1), \pm(5,3), (0,0)$.

Q_{14}. $(2,6), (4,14), (6,4), (8,10), (10,2), (12,8), (13,1), (14,14)$.

Q_{15}. $\pm(1,3), \pm(3,7), \pm(5,1), \pm(7,-5), (0,0)$.

Q_{16}. $(1,13), (3,7), (5,1), (7,5), (8,8), (9,11), (11,15), (13,9), (15,3)$.

Q_{17}. $\pm(2,4), \pm(4,-8), \pm(6,2), \pm(8,-6), (0,0)$.

Q_{18}. $(2,10), (4,16), (6,2), (8,12), (10,8), (12,4), (14,14), (16,18), (18,6)$.

Q_{21}. $\pm(1,3), \pm(3,9), \pm(5,-7), \pm(7,-1), \pm(9,5), (0,0)$.

Q_{25}. $\pm(1,5), \pm(3,11), \pm(5,-1), \pm(7,9), \pm(9,-7), \pm(11,3), (0,0)$.

Chapter 4

Domination on Q_{4k+1}

In this chapter we consider dominating sets for Q_{4k+1} with one queen in every second row and column. Figure 4.1 is an example of such a set. Such dominating sets have $2k+1$ queens, therefore reach the bound $\gamma(Q_{4k+1}) \geq 2k+1$. Thus finding such sets for a specific k establishes $\gamma(Q_{4k+1}) = 2k+1$ for that k. Since the case $k=0$ is trivial, we will assume henceforth that $k \geq 1$.

We label the rows and columns of the chessboard as illustrated in Figure 4.1. A row or column is called *even* (respectively *odd*) if it has an even (respectively odd) label. A square of the chessboard is called *even-even, even-odd, odd-even*, or *odd-odd* according to the labels of its row and column. With queens on every even row and column all the squares in one of these rows or columns are dominated. Hence the only squares that need to be considered are the odd-odd squares (shaded in Figure 4.1), that must be dominated diagonally. We can simplify the representation by drawing only the odd-odd squares. (See Figure 4.2). Imagine that the even rows and even columns are squeezed to be only lines. Place the simplified board on the x-y-plane with the centre of the board at coordinates $(0,0)$ and the lines formed by the even rows and columns at unit lengths from each other. The diagonals (of squares) that rise from left to right correspond to the straight lines with equations $y = x + d$, where $d \in \{-(2k-1), \ldots, -1, 0, 1, \ldots, 2k-1\}$. These diagonals or lines, which we use interchangeably,

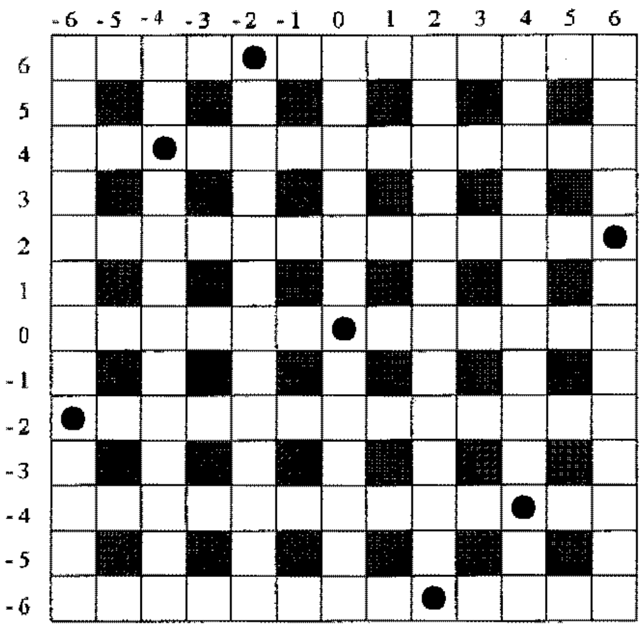

Figure 4.1 A dominating set for Q_{13}

are called *d-diagonals* and are labelled $d = -(2k-1), \dots, d = -1, d = 0, d = 1, \dots, d = 2k-1$ according to their intersections with the y-axis. Similarly, the *s-diagonals* fall from left to right, and correspond to the straight lines with equations $y = -x + s$, $s \in \{-(2k-1), \dots, -1, 0, 1, \dots, 2k-1\}$ and are also labelled according to their intersections with the y-axis. An *even (odd) diagonal* is a diagonal with an even (odd) intersection with the y-axis. Notice that a queen that lies on an odd (even) d-diagonal, also lies on an odd (even) s-diagonal and *vice versa*. Sometimes we will refer to queens on odd (even) diagonals as *odd (even) queens*. As in the case of squares of the chessboard, a queen in the simplified representation is called *even-even, even-odd, odd-even or odd-odd* according to the parity of its coordinates. By a *point on a d-diagonal* (or *s-diagonal*) we mean an intersection point of its corresponding line and a line formed by an even row or column.

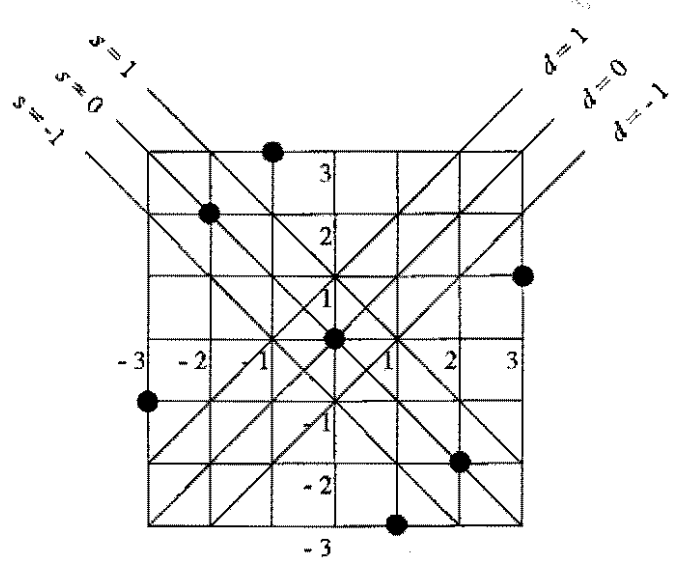

Figure 4.2 Simplified representation of Q_{13}

Notice that the difference between the y and x coordinates of any point on a d-diagonal is equal to its label. Similarly the sum of the coordinates of any point on the s-diagonals is equal to its label. Figure 4.3 illustrates these concepts for a general board.

Henceforth, when we refer to a dominating set D of Q_{4k+1}, we assume that, unless stated otherwise, $|D| = 2k + 1$ and that there is one queen on each even row and each even column. According to the representation of Q_{4k+1} described above, we consider these queens to be placed on points on the diagonals of Q_{4k+1}. The coordinates of the queens are then the coordinates of the points (in the plane) on which they are placed. Notice that a queen with coordinates $(2x, 2y)$ on the normal chessboard has coordinates (x, y) on the simplified representation. Henceforth, when we refer to coordinates, unless stated otherwise, it will be of the simplified representation. We now prove some properties of such dominating sets of Q_{4k+1}. Some of these results were also reported

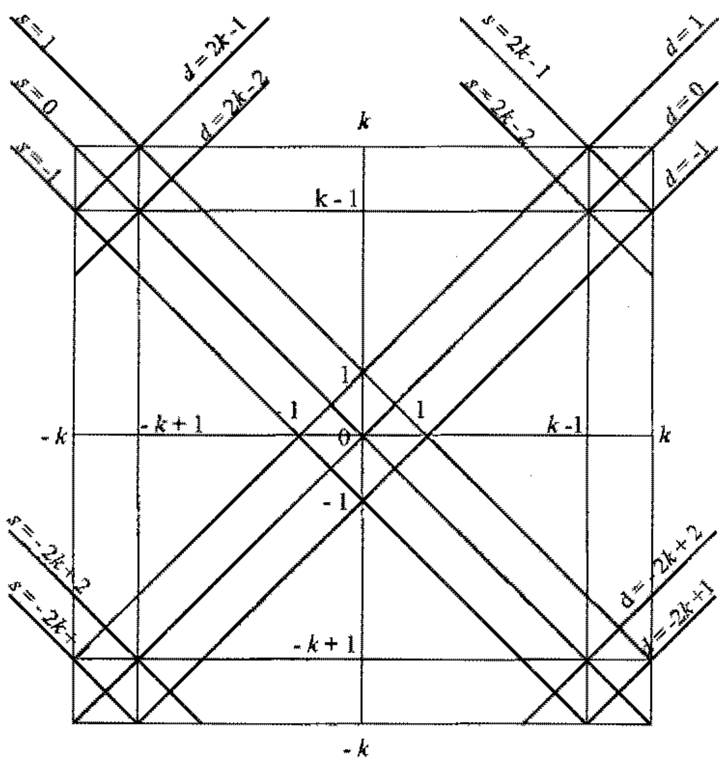

Figure 4.3 Numbering of diagonals

in [5] .

4.1 Properties of dominating sets

Lemma 4.1 [5] *If D is a dominating set of Q_{4k+1} and there are no queens on $d = i$ (respectively $s = i$), then there must be queens on:*

$$s(d) = 0, \pm 2, \pm 4, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k - |i| - 1), \qquad i \text{ odd}$$

$$s(d) = \pm 1, \pm 3, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k - |i| - 1), \qquad i \text{ even.}$$

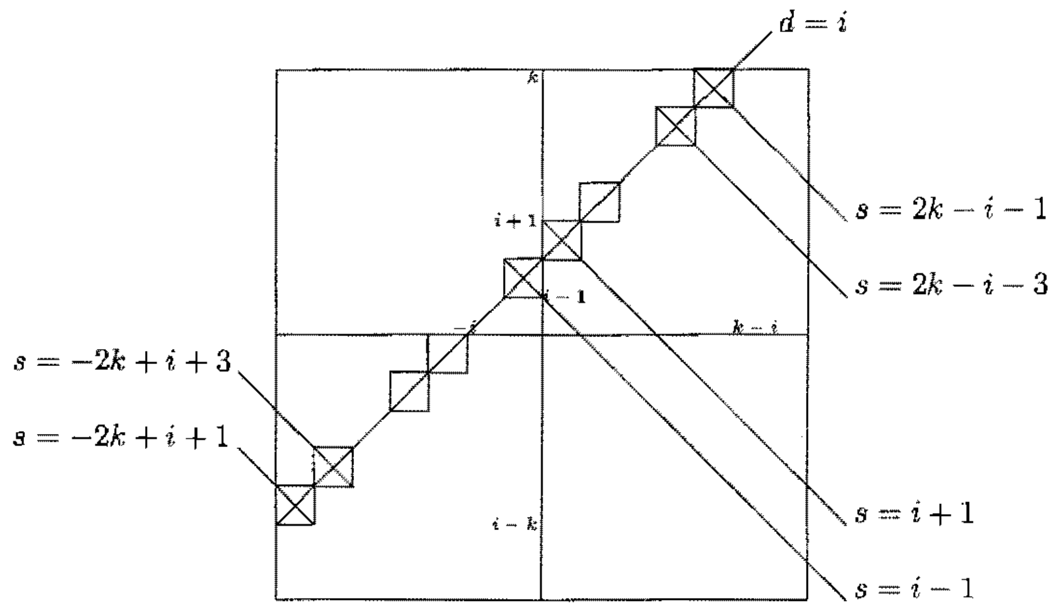

Figure 4.4 Squares not dominated by d-diagonal must be dominated by s-diagonal

Proof. All squares concerned must be dominated diagonally. Thus if a square is not dominated by a d-diagonal, it must be dominated by an s-diagonal and *vice versa*. (See Figure 4.4) $\square$

A diagonal which does not contain a queen is called an *empty diagonal*. Beginning with the diagonals $s = 0$ and $d = 0$, a dominating set with the first empty s-diagonal $s = i$ or $s = -i$ and the first empty d-diagonal $d = j$ or $d = -j$ is called an (i, j)-*dominating set*, $i, j \geq 0$. We now show that with respect to the first empty diagonals there are only two types of dominating sets.

Theorem 4.2 [5] *There are precisely two types of dominating sets:*

(a) (i, i)*-dominating sets with queens on the diagonals*

$$s, d = 0, \pm 1, \pm 2, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k - |i| - 1). \tag{4.1}$$

(b) $(i, i+2)$*-dominating sets with queens on the diagonals*

$$d = 0, \pm 1, \pm 2, \ldots, \pm i, \pm(i+1), \pm(i+3), \ldots, \pm(2k - |i| - 1). \tag{4.2}$$

$$s = 0, \pm 1, \pm 2, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k-|i|-3). \tag{4.3}$$

Proof. Without loss of generality we can assume that the first empty diagonals are positive, because the set of queens can be rotated and/or "flipped". If D is an (i,i)-dominating set, it follows from Lemma 4.1 and the fact that $s = i$ and $d = i$ are the first empty diagonals that there are queens on the diagonals in (4.1). (See Figure 4.2 for an example of a (1,1)-dominating set.)

Now suppose D is an (i,j)-dominating set with $j > i$. Because $d = j > i$ is the first empty d-diagonal, there are queens on $d = 0, \pm 1, \pm 2, \ldots, \pm i$. Also, because $s = i$ is empty it follows from Lemma 4.1 that there are queens on $d = \pm(i+1), \pm(i+3), \ldots, \pm(2k-|i|-1)$. This gives $(1+2i)+(2k-2i) = 2k+1$ d-diagonals containing queens. But there are only $2k+1$ queens available. Thus (4.2) are the only d-diagonals containing queens, so that $j = i+2$. Since $d = i+2$ is empty and $s = i$ is the first empty s-diagonal it follows from Lemma 4.1 that there are queens on the diagonals listed in (4.3). (See Figure 4.5 for an example of an (1,3)-dominating set.) □

Lemma 4.3 [5] *If D is a dominating set of Q_{4k+1}, then there is a one-to-one correspondence between even-odd and odd-even queens.*

Proof. Consider all the x-coordinates and all the y-coordinates of queens in D. Since each integer in the set $\{-k, \ldots, -1, 0, 1, \ldots, k\}$ occurs as x-coordinate of some queen and as y-coordinate of a (possibly different) queen, there are just as many even (odd) x-coordinates as even (odd) y-coordinates. Thus there is a one-to-one correspondence between even-odd and odd-even queens. □

Lemma 4.4 *If D is a dominating set of Q_{4k+1}, the numbers of even-even and odd-odd queens differ by one.*

Proof. Consider all the x-coordinates and all the y-coordinates of queens in D, namely $\{-k, \ldots, -1, 0, 1, \ldots, k\}$. Depending on the parity of k, there are either one more

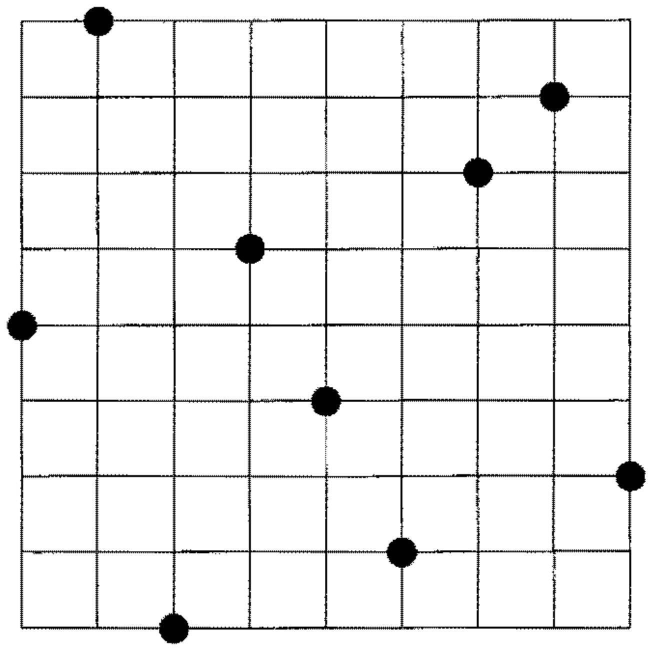

Figure 4.5 Simplified representation of Q_{17}

even coordinate than odd coordinates or *vice versa*. Thus because there are just as many even-odd queens as odd-even queens (Lemma 4.3), the number of even-even and odd-odd queens must differ by one. □

Depending on the parity of i, there are either more odd queens or more even queens in a dominating set D of Q_{4k+1} (See Theorem 4.2). Call the smaller of these sets the *core* of D, and the bigger one the *body* of D. The *core diagonals* (respectively *body diagonals*) are the diagonals listed in Theorem 4.2 with core (respectively body) queens on it. Note that there can be core (body) queens that are not on the core (body) diagonals.

Lemma 4.5 *If D is an (i,i)-dominating set, then there are $i-1$ core $s\,(d)$-diagonals and $2k-i$ body $s\,(d)$-diagonals.*

Proof. There must be queens on $s(d) = 0, \pm1, \pm2, \ldots, \pm(i-1), \pm(i+1), \ldots,$

$\pm(2k-i-1)$. If i is even the core diagonals are $0, \pm 2, \pm 4, \ldots, \pm(i-2)$ and the body diagonals are $\pm 1, \pm 3, \pm 5, \ldots, \pm(2k-i-1)$. This gives $i+1$ and $2k-i$ core and body diagonals respectively. Similarly, if i is odd we get the same numbers of diagonals. □

Consider an (i,i)-dominating set D. If there are only $i-1$ queens in the body, there can only be one queen on each core diagonal. We then say the *core is restricted*. Similarly, if there are only $2k-i$ queens in the body we say the *body is restricted*. If the body (respectively core) has more than $2k-i$ (respectively $i-1$) queens we say the *body* (respectively *core*) *is relaxed*.

Lemma 4.6 *If D is an (i,i)-dominating set then either the core or the body is relaxed.*

Proof. There are $2k+1$ queens and $2k-1$ s (d)-diagonals that must contain queens. Therefore there are two extra queens. If i is even, there is an even number of body diagonals, namely $2k-i$ odd s (d)-diagonals. Because there must be an even number of odd queens (Lemma 4.3), both or none of the two extra queens must be in the body.

If i is odd, the number of odd s (d)-diagonals is also even, namely $i-1$ (in the core). Again to keep the number of odd queens even, both or none of the two extra queens must be in the core. □

4.2 Description of Algorithm

For each k, i and type of dominating set D of Q_{4k+1} we run a different programme. The algorithm considers all possibilities of placements of queens on s-diagonals, eliminating non-dominating sets as soon as possible. The fact that there is only one queen per s-diagonal and d-diagonal speeds up the programme considerably. We treat (i,i)

and $(i, i+2)$ sets slightly differently:

$(i, i+2)$**-dominating sets:** We have queens on the diagonals listed in (4.2) and (4.3). Since there are $2k+1$ d-diagonals containing queens, there is only one queen on each d-diagonal listed and no queens on any other d-diagonals. There are $2k-3$ s-diagonals that must contain queens. That leaves four extra queens that can be either on diagonals listed in (4.3) or on other s-diagonals. The algorithm considers all possibilities of one queen on each s-diagonal listed in (4.3), making sure there are not more than one queen on any d-diagonal, row or column. That will leave four d-diagonals (two in the body and two in the core) listed in (4.2) without queens. We then try to place the remaining four queens on these d-diagonals. See Programme 1 in Appendix B for an example programme.

(i, i)**-dominating sets:** Either the core or the body must be relaxed. The algorithm considers all possibilities of one queen on each s-diagonal listed in (4.1), making sure there are not more than one queen on any d-diagonal, row or column. However, in the relaxed part (body or core) we allow on two occasions any one of the following relaxations: (1) a queen on the same d-diagonal than a queen already placed or (2) a queen that is not on a d-diagonal listed in (4.1). We then try to place the remaining two queens. If the core is restricted we can determine first if such a core is possible. (For example, there are no restricted core sets for $i = 7$.)

4.3 Results

We did a computer search for $(i, i+2)$ and (i, i)-dominating sets and found dominating sets for all k up to $k = 14$. Several of these are asymmetric sets not found before, including independent sets for Q_{45}. Tables 4.1 and 4.2 give the number of $(i, i+2)$ and (i, i) dominating sets found.

With a computer search we found that for $k = 8$ and $k = 11$ some sets are independent, establishing $i(Q_{33}) = 17$ and $i(Q_{45}) = 23$. We also searched for sets that dominate an additional row and column, and found two such sets for $k = 7$, establishing $\gamma(Q_{30}) = 15$. We now give a list of coordinates of minimum dominating sets for $7 \leq k \leq 14$, confirming some of the results in Tables 3.1 and 3.2. For Q_{33} and Q_{45} we give independent sets. We give only the corresponding y-coordinates of the x-coordinates $(-k, \ldots, -1, 0, 1, \ldots, k)$, where the coordinates correspond to those of the simplified representation as illustrated in Figure 4.2.

Q_{29}. (0, 5, -4, 1, -3, 9, -6, 2, -1, -7, 8, 3, -2, -5, 4). This also dominates Q_{30}.

Q_{33}. (4, -3, -8, 3, 6, -2, -6, 2, -1, -7, 8, 5, 1, -5, 0, 7, -4).

Q_{37}. (-2, 3, 6, 9, -8, 5, -4, -7, 1, 0, -1, 7, 4, -5, 8, -9, -6, -3, 2).

Q_{41}. (-2, -5, 8, 5, -6, 9, 6, -7, 1, -3, -8, 0, 3, -9, 10, 7, -10, 4, 2, -1, -4).

Q_{45}. (6, -5, 0, 5, -6, -11, 10, 7, -3, 2, -10, -2, 3, -9, -1, 11, 8, -7, 4, 1, -8, 9, -4).

Q_{49}. (-5, 6, -1, 10, -7, 4, -8, -2, -9, 12, 11, 3, 0, -3, -11, -12, 9, 2, 8 ,-4, 7, -10, 1, -6, 5).

Q_{53}. (-6, 12, -2, -5, -8, 9, 4, 13, 10, 7, -1, 11, 3, 0, -3, -11, 1, -7, -10, -13, -4, -9, 8, 5, 2, -12, 6).

Q_{57}. (6, -7, 10, 7, -8, 3, 8, -13, -10, 9, -6, -11, -1, 4, -12, 0, -3, 13, 1, 5, 14, 11, -14, 12, -4, -9, 2, -5, -2).

We note that many of the dominating sets listed in Tables 4.1 and 4.2 are 180°-symmetric. Much bigger sets can be searched for if we restrict our search to 180°-symmetric sets and even bigger sets if we restrict our search to 90°-symmetric sets, which will be considered in the next two sections.

4.4 180°-symmetric dominating sets

The following results concern 180°-symmetric sets, that is, placements of queens that

k	n	(1,3)	(2,4)	(3,5)	(4,6)	(5,7)
2	9	0+1				
3	13	0+2	1+0			
4	17	0+4	none	1+0		
5	21	0+3	1+2	0+1	none	
6	25	0+2	0+2	none	1+0	none
7	29	none	0+3	0+3	none	none
8	33	none	0+11	2+2	none	none
9	37	none	12+0	4+48	none	none
10	41		none	0+109	0+25	none
11	45		none	0+165	0+68	0+50
12	49		none	none	36+672	0+85
14	57					0+30*

*Number of sets found in first two hours

Table 4.1 Number of symmetric + asymmetric $(i, i+2)$-dominating sets found

k	n	(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)
2	9		1+0				
3	13	1+0	1+0	none			
4	17	3+0	4+0	none			
5	21	none	2+1	2+0	none		
6	25	none	3+0	3+1	none		
13	53						164+196

Table 4.2 Number of symmetric + asymmetric (i, i)-dominating sets found

are invariant under a rotation of 180° of the board. We note that since there is only one queen on each row and column, it follows that a 180°-symmetric dominating set must have a queen on (0,0).

Lemma 4.7 *If D is a 180°-symmetric dominating set of Q_{4k+1}, the number of pairs of queens on odd diagonals is even.*

Proof. By Lemma 4.3 there is a one-to-one correspondence between even-odd and odd-even queens. Because D is symmetric, for each even-odd queen there is another even-odd queen. The same holds for each odd-even queen. But the even-odd and odd-even queens lie on the odd diagonals and it follows that there is an even number of queens on odd diagonals. □

Theorem 4.8 *If D is a 180°-symmetric $(i, i+2)$-dominating set with*

(a) *i even, then $k - \frac{1}{2}i$ is even*

(b) *i odd, then $\frac{1}{2}(i+1)$ is even.*

Proof. Consider an $(i, i+2)$-dominating set:

(a) i even: By Theorem 4.2 the number of pairs of queens on odd d-diagonals is $k - \frac{1}{2}i$. By the labelling of the diagonals, if a queen is on an odd d-diagonal it is also on an odd s-diagonal. Therefore the total number of pairs of queens on odd diagonals is $k - \frac{1}{2}i$. By Lemma 4.7, $k - \frac{1}{2}i$ is even.

(b) i odd: Again, by Theorem 4.2, the number of pairs of queens on odd diagonals is $\frac{1}{2}(i+1)$. Thus the total number of pairs of queens on odd diagonals is $\frac{1}{2}(i+1)$ and it follows from Lemma 4.7 that $\frac{1}{2}(i+1)$ is even. □

Theorem 4.9 *If D is an 180°-symmetric (i, i)-dominating set, then:*

1. *The body is restricted if:*
   (a) *i is odd and $\frac{1}{2}(i-1)$ is odd*
   (b) *i is even and $k - \frac{1}{2}i$ is even.*

k	n	γ	type	No of sets
15	61	31	(6,6)	358
17	69	35	(6,8)	3282
19	77	39	(8,8)	14552

Table 4.3 Symmetric dominating sets found

2. *The core is restricted if:*
   (b) *i is odd and $\frac{1}{2}(i-1)$ is even*
   (c) *i is even and $k - \frac{1}{2}i$ is odd.*

Proof. By Lemma 4.6 either the body or the core is relaxed.

If i is odd, the core diagonals are odd. By Lemma 4.5 there are $i-1$ core diagonals. Since there must be an even number of pairs of queens on odd diagonals (Lemma 4.7), the core is restricted if $\frac{1}{2}(i-1)$ is even, and the core is relaxed (*i.e.* the body is restricted) if $\frac{1}{2}(i-1)$ is odd.

If i is even, the body diagonals are odd. By Lemma 4.5, there are $2k-i$ body diagonals. Again, since there must be an even number of pairs of queens on odd diagonals, the body is restricted if $k-\frac{1}{2}i$ is even, and the body is relaxed (*i.e.* the core is restricted) if $k-\frac{1}{2}i$ is odd. □

4.5 Results for 180°-symmetric dominating sets

The algorithm for finding symmetric dominating sets is the same as for asymmetric sets, except that only half the queens need to be considered and there are restrictions on the values of k and i, as seen in the theorems above. Table 4.3 shows values of k for which minimum dominating sets were found that were not found while searching for asymmetric dominating sets. None of these sets are independent, or cover a bigger board (*i.e.* two or more extra edges).

We now give a list of coordinates of queens for the cases $k = 15, 17$ and 19, confirming the results in Tables 3.2 and 4.3. Note that the sets found for Q_{69} and Q_{77} determine the new domination numbers $\gamma(Q_{69}) = 35$ and $\gamma(Q_{77}) = 39$. Since the dominating sets are symmetric, we give only the corresponding y-coordinates of the x-coordinates $(0, 1, 2, \ldots, k)$.

Q_{61}. (0, -3, -11, 1, 15, 12, -13, 8, 5, 14, -7, -10, -4, -2, -9, 6).

Q_{69}. (0, 3, 15, -1, 10, 16, 13, 8, 17, 14, 7, -4, -9, -12, -5, -2, -11, -6)

Q_{77}. (0, -3, 17, 14, 2, -1, 19, 16, -9, 12, 5, 18, -15, -6, 13, -8, -11, -4, -7, -10)

4.6 90°-symmetric dominating sets

We now consider 90°-symmetric sets, that is, placements of queens that are invariant under a 90° rotation (say anti-clockwise) of the board.

Lemma 4.10 *If D is a 90°-symmetric (i,i)-dominating set, then k is even.*

Proof. There are $2k+1$ queens in D. There is one queen on (0,0), and the remaining $2k$ queens must be a multiple of four. Therefore k must be even. □

Lemma 4.11 *If D is a 90°-symmetric dominating set, then (queen on $(0,0)$ excluded):*

(a) *The number of odd-odd queens equals the number of even-even queens and is a multiple of four.*

(b) *The number of even-odd queens equals the number of odd-even queens and is even.*

Proof. Because D is 90°-symmetric, each queen (except (0,0)) has three other corresponding queens, *i.e.* if there is a queen on (x, y), there must also be queens on

$(-x,-y)$, $(-y,x)$ and $(y,-x)$. Thus it follows that the number of even-even and odd-odd queens is a multiple of four. Also, the number of even-odd queens equals the number of odd-even queens. Consider all the coordinates ((0,0) excluded) of D. If k is even we notice that the number of even x, odd x, even y and odd y coordinates are all equal (to k). Thus if the number of even-odd queens equals the number of odd-even queens, the number of even-even and odd-odd queens must also be the same. □

Lemma 4.12 *If D is a 90°-symmetric dominating set, then (queen on $(0,0)$ excluded) the number of even queens is a multiple of eight and the number of odd queens is a multiple of four.*

Proof. The result follows directly from Lemma 4.11. □

Lemma 4.13 *If D is a 90°-symmetric dominating set, then Table 4.4 sums up the possible values of k for different values of i, and states whether the body or core is relaxed.*

i	k	relax
$8m-3$	$4n+2$	body
$8m-2$	not possible	
$8m-1$	$4n$	core
$8m$	$2n$	core
$8m+1$	$4n$	body
$8m+2$	$2n$	body
$8m+3$	$4n+2$	core
$8m+4$	not possible	

Table 4.4 Values for k for different i

Proof. *i even:* If i is even, then there are $i-2$ even s (d)-diagonals (0-diagonal excluded) in the core and $2k-i$ s (d)-diagonals in the body (Lemma 4.5). Either the core or the body must be relaxed (Lemma 4.6). If we relax the core there are i queens in the core ((0,0) queen excluded) and $2k-i$ queens in the body. By Lemma 4.12, i

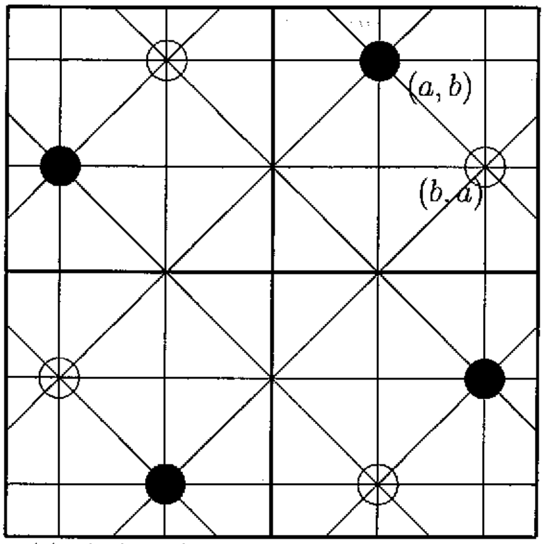

Figure 4.6 The black dots dominate the same squares as the white dots

must be a multiple of eight, and $2k - i$ a multiple of four, proving the case $i = 8m$. If we relax the body, $i - 2$ must be a multiple of eight and $2k - i + 2$ a multiple of four, proving the case $i = 8m + 2$. All other even values of i are impossible.

i odd: For i odd there are $i - 1$ odd s (d)-diagonals in the core and $2k - i - 1$ even s (d)-diagonals in the body (0-diagonal excluded). If we relax the core, $i + 1$ must be a multiple of four and $2k - (i + 1)$ a multiple of eight. There are two possibilities: (1) If $i + 1 = 8m$, then k must be a multiple of four, proving the case $i = 8m - 1$. (2) If $i + 1 = 8m + 4$, then $2k - 8m + 4$ must be a multiple of eight, *i.e.* $k = 4n + 2$, proving the case $i = 8m + 3$. Similarly, if we relax the body, $i - 1$ must be a multiple of four and $2k - (i - 1)$ a multiple of eight. Again there are two possibilities, namely $i - 1 = 8m$ and $i - 1 = 8m - 4$, giving $k = 4n$ and $k = 4n + 2$ respectively, which prove the cases $i = 8m - 1$ and $i = 8m - 3$. □

From Figure 4.6 we see that a set of four queens which is 90°-symmetric can be placed in two different ways and still dominate the same squares. Therefore, we only need to consider one eighth of the board, because if we, for example, consider the positive quadrant we can always choose the queens so that the x-coordinates are bigger than the y-coordinates in the positive quadrant.

4.7 Results for 90°-symmetric dominating sets

Unfortunately, we were unable to find such dominating sets. It may be possible to find solutions for larger values of n by using faster computers.

Chapter 5

Domination on Q_{4k+3}

The only values known for n up to now for which $\gamma(Q_n) = \frac{1}{2}(n-1)$ are 3 and 11. Theorem 2.2 by Weakley describes some properties of such sets. Firstly, $n \equiv 3 \pmod 4$. Thus we only have to search for sets satisfying $\gamma(Q_{4k+3}) = 2k+1$. In this chapter we try to determine whether there exist more such dominating sets. We begin by showing $\gamma(Q_{4k+3}) > 2k+1$ for $3 \le k \le 7$. We then consider sets that dominate only the edge. Finally, we introduce the concept of the radius of a queen to study the distribution of dominating sets.

Another property of the concerned dominating sets proved in Theorem 2.2 is that there exists a $j \times j$ sub-board U for some odd integer j ($\frac{3}{4}(n+1) \le j \le n$) so that each edge square of U is dominated exactly once. Thus for each board we consider the possible $j \times j$ sub-boards separately. In order to reduce computer time, we will first determine on which sets of rows (columns) dominating sets could possibly be found. Place the board on the x-y-plane with the centre of the $j \times j$ sub-board at the coordinates $(0,0)$. As before we will refer to the board squares by the coordinates of their centres. Notice that the centre of the $j \times j$ sub-board and the centre of the entire board do not have to be the same square. Let $l = \frac{1}{2}(j-1)$; then the edge squares of the sub-board are on squares with their x or y coordinates equal to l. The following lemmas and theorems concern rows and columns intersecting the $j \times j$ sub-board.

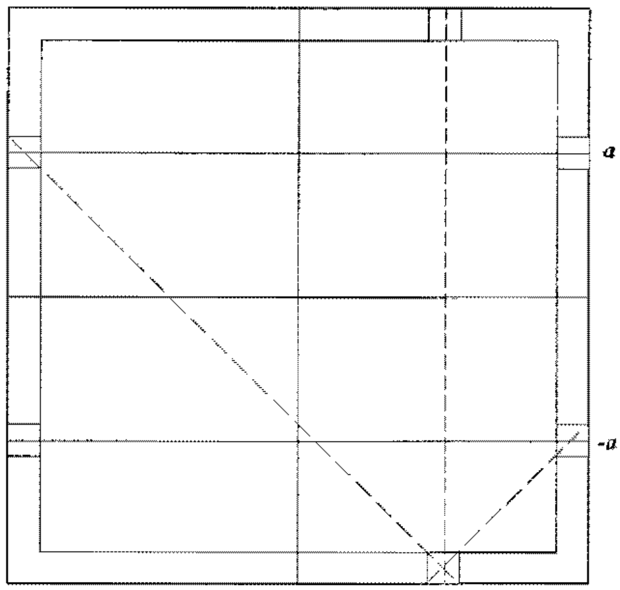

Figure 5.1 If $x = a$ is empty, then $y = a$ or $y = -a$ is also empty.

Thus for all a and b mentioned, $-l + 1 \leq a, b \leq l - 1$. Also, when we refer to a set S, we mean a set of queens with cardinality $2k + 1$ on Q_{4k+3}, $k \geq 3$, which dominates the edge of the $j \times j$ sub-board ($j = 2l + 1$).

Lemma 5.1 *For a set S, if $x = a$ (respectively $y = a$) is empty, then $y = a$ or $y = -a$ (respectively $x = a$ or $x = -a$) must also be empty.*

Proof. Suppose $y = a$ and $y = -a$ are both occupied. Then $d = -l - a$ and $s = -l + a$ must be empty (see Figure 5.1). But then $(a, -l)$ is not dominated. Thus $y = a$ or $y = -a$ must be empty. □

Lemma 5.2 *For a set S, if $x = a$ is occupied, then $y = a$ or $y = -a$ is also occupied.*

Proof. Suppose $y = a$ and $y = -a$ are both empty. Because $x = a$ is occupied both $s = a + l$ and $d = l - a$ must be empty (see Figure 5.2). This means $d = a - l$ and

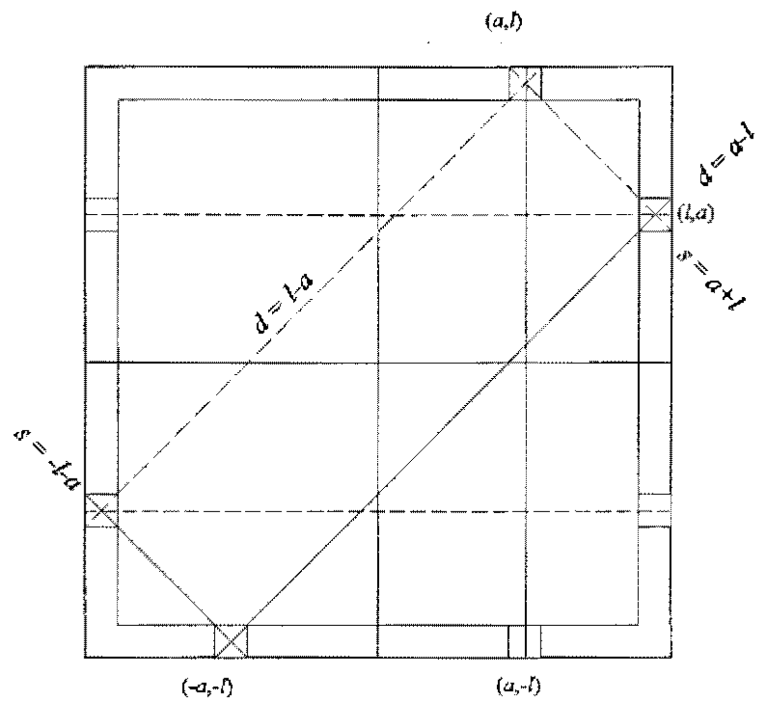

Figure 5.2 If $x = a$ is occupied, then $y = a$ or $y = -a$ is also occupied.

$s = -a - l$ must be occupied. But then $(-a, -l)$ is dominated twice. Thus $y = a$ or $y = -a$ is occupied. □

Lemma 5.3 *For a set S, if $x = \pm a$ is occupied, then $y = \pm a$ is occupied.*

Proof. Suppose $x = \pm a$ is occupied and, without loss of generality, $y = a$ is empty. But if $y = a$ is empty, then from Lemma 5.1 $x = a$ or $x = -a$ must be empty. This is a contradiction, thus $y = \pm a$ must be occupied.. □

Lemma 5.4 *If for a set S, the set of rows is symmetric, then the set of columns is the same as the rows.*

Proof. Because there are just as many rows containing queens as columns containing queens, we only have to show that for each $x = \pm a$ that is occupied, $y = \pm a$ is also

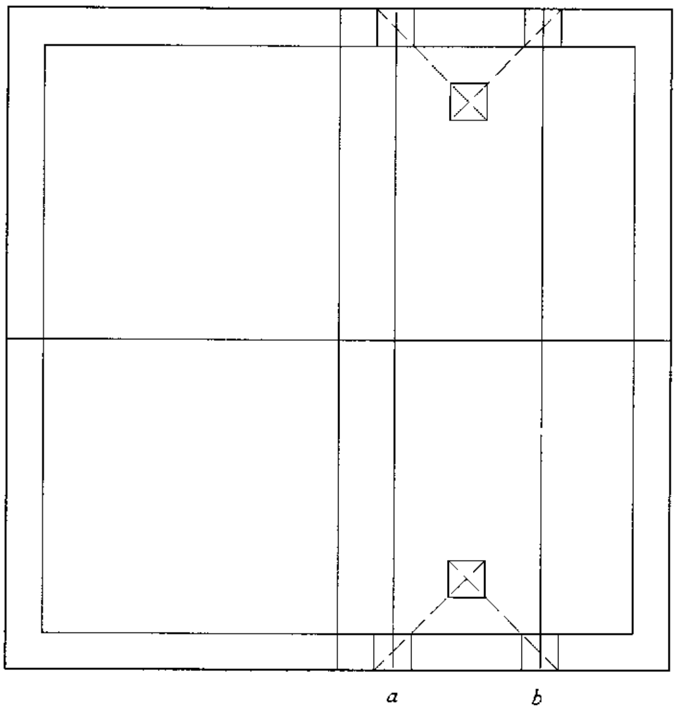

Figure 5.3 $\left(\frac{a+b}{2}, l-\frac{b-a}{2}\right)$ and $\left(\frac{a+b}{2}, -l+\frac{b-a}{2}\right)$ must be dominated by row or column.

occupied. This follows directly from Lemma 5.3, thus the lemma is proved. □

Theorem 5.5 *If for a set S, $x=a$ and $x=b$ are occupied, with $a<b$ and $a+b$ even, then if :*

(1) $x=\frac{a+b}{2}$ *is empty, and*

(2) $x=l-\frac{b-a}{2}$ *or* $x=-l+\frac{b-a}{2}$ *is empty,*

then S is not dominating.

Proof. If $x=a$ and $x=b$ are occupied, then the squares $(\frac{a+b}{2}, l-\frac{b-a}{2})$ and $(\frac{a+b}{2}, -l+\frac{b-a}{2})$ must be dominated by row or column (see Figure 5.3). If $x=l-\frac{b-a}{2}$ or $x=-l+\frac{b-a}{2}$ are empty, then from Lemma 5.1 $y=l-\frac{b-a}{2}$ or $y=-l+\frac{b-a}{2}$ is also empty. Thus if $x=\frac{a+b}{2}$ is empty and $x=l-\frac{b-a}{2}$ or $x=-l+\frac{b-a}{2}$ are empty, $(\frac{a+b}{2}, l-\frac{b-a}{2})$ and $(\frac{a+b}{2}, -l+\frac{b-a}{2})$ can not be dominated. □

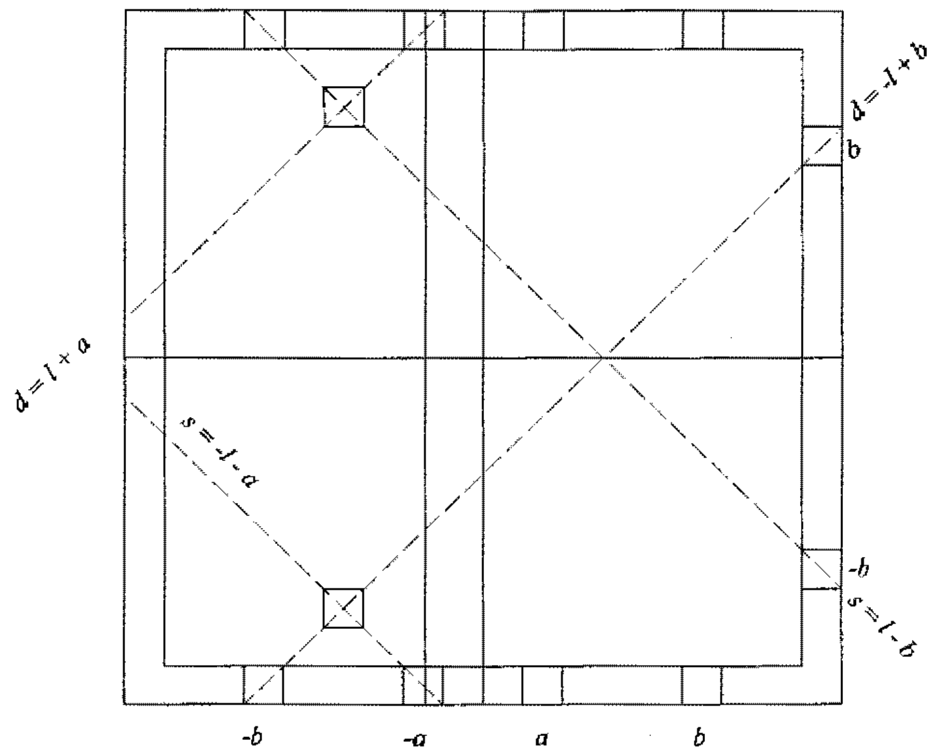

Figure 5.4 $\left(-\frac{a+b}{2}, l-\frac{b-a}{2}\right)$ and $\left(-\frac{a+b}{2}, -l+\frac{b-a}{2}\right)$ must be dominated by row or column.

Theorem 5.6 *If for a set S, $x = a$ and $x = b$ are occupied, with $a < b$ and $a + b$ even, then if:*

(1) $x = -a$ or $x = -b$ *is occupied, and*

(2) $x = -\frac{a+b}{2}$ *is empty, and*

(3) $x = -l + \frac{b-a}{2}$ or $x = l - \frac{b-a}{2}$ *is empty*

then S is not dominating.

Proof. Suppose there are queens on $x = a, x = b$ and, without loss of generality, $x = -a$. Then $s = -l - a$ and $d = l + a$ are empty (see Figure 5.4). Because $x = b$ is occupied, $y = b$ or $y = -b$ must be occupied (Lemma 5.2). Thus $d = -l + b$

$j = 31$	0,3,5,6,8,11,13,14
	0,3,4,6,7,10,13,14
	0,2,5,7,9,11,12,14
	0,2,4,6,8,10,12,14
	0,1,5,6,7,11,12,13
	0,1,4,5,8,9,12,13
	0,1,2,8,9,10,11,12
	0,1,2,3,4,5,6,7
$j = 29$	0,3,6,9,10,12,13
	0,2,4,6,9,11,13
	0,2,3,5,8,10,13
$j = 27$	0,3,6,9,11,12
	0,2,5,7,9,12
	0,3,4,7,8,11
$j = 25$	0,2,4,9,11
	0,3,5,8,11
	0,3,4,7,10

Table 5.1 Possible rows containing queens for different subboards of Q_{31}

or $s = l - b$ must be empty. This means $(-\frac{a+b}{2}, l - \frac{b-a}{2})$ or $(-\frac{a+b}{2}, -l + \frac{b-a}{2})$ is not dominated by diagonal, but by row or column. Thus if $x = -\frac{b+a}{2}$ is empty and $x = -l + \frac{b-a}{2}$ or $x = l - \frac{b-a}{2}$ is empty, the set can not be dominating. □

By using the above theorems, we can use a computer to find permissible sets of rows for dominating sets. For a specific board we run a different programme for each permissible sub-board (Theorem 2.2). The algorithm simply generates all possible sets of rows and eliminates sets that cannot be dominating sets according to Theorems 5.5 and 5.6. As expected, the row pattern with queens in every second row and column was found when $j = n$. As an example, Table 5.1 shows the possible sets of rows (columns) for different sub-boards for Q_{31}. In this case the sets are symmetric, thus we only give the non-negative rows (columns).

For those sets that are symmetric, it follows from Lemma 5.4 that the set of columns is the same as the rows. The following observation eliminates even more sets of rows (columns): Consider a queen directly outside the $j \times j$ sub-board. It dominates three consecutive edge squares. The middle one by row (column) and the other two diagonally. Thus if $j < n$, then there must exist at least two rows and columns with queens on them, but with no queens on the rows(columns) directly next to them. Similar eliminations can be made by considering queens inside the sub-board.

If we know all the rows (columns), it reduces the time for the computer search for dominating sets considerably, especially because we also know which edge squares of the sub-board must be dominated diagonally. Thus for each occupied row, there are only a few possible positions for a queen. Programme 3 in Appendix B is an example programme for the case $n = 31$ and sub-board $j = 29$ (with the sub-board placed in the centre) for a specific row pattern. Note that it must also be taken into account that the sub-board can be off-centre. The algorithm considers all possible placements of queens (on permissible rows and columns) on the board by first finding placements of the queens outside the sub-board (if any) and then the queens inside. It then eliminates sets that dominate some edge square of the sub-board more than once as soon as possible. (Such sets cannot be dominating, because by Theorem 2.2, each edge square of the sub-board must be dominated exactly once.) If a set that dominates the edge of the sub-board is found, it can easily be checked whether it also dominates the rest of the board. As expected, the algorithm found all the so-called "edge dominating sets" with queens in every second row and column, where $2 \leq k \leq 6$, discussed later in this chapter (see Table 5.2).

We only checked boards up to $n = 31$, and found no dominating sets. We can now state the following.

Theorem 5.7 *For* $3 \leq k \leq 7$, $\gamma(Q_{4k+3}) > 2k + 1$. □

k	$2k+1$	$4k+3$	no of sets	$\frac{1}{2}\sum R$
1	3	7	none	
2	5	11	1	8
3	7	15	1	16
4	9	19	none	
5	11	23	2	40
6	13	27	40	56
7	15	31	none	
8	17	35	90	96
9	19	39	272	120
10	21	43	none	
11	23	47	3728	176
12	25	51	many	208

Table 5.2 Number of edge dominating sets found and their $\frac{1}{2}\sum R$

Theorem 5.8 $\gamma(Q_{19}) = 10$ *and* $\gamma(Q_{31}) = 16$.

Proof. From Theorem 5.7 $\gamma(Q_{19}) > 9$ and $\gamma(Q_{31}) > 15$. From Chapters 3 and 4 we have $\gamma(Q_{18}) = 9$ and $\gamma(Q_{30}) = 15$. Thus adding a queen to these sets we obtain dominating sets for Q_{19} and Q_{31}, which proves the theorem. □

For both $n = 3$ and $n = 11$ we have $j = n$, *i.e.* no queens on the edge of the board. We suspect this is true for all cases where $\gamma(Q_{4k+3}) = 2k+1$, because a queen on the edge dominates fewer squares than one closer to the centre. We therefore state the following conjecture:

Conjecture 1: *If* $\gamma(Q_{4k+3}) = 2k+1$, *then the edge is empty.*

We will restrict our search to dominating sets with no queens on the edge and each edge square dominated once. We define an *edge dominating set* as a set of queens that dominates each edge square exactly once (not necessarily dominating the rest of the board). Note that an edge dominating set on Q_{4k+3} has $2k+1$ queens and that none of

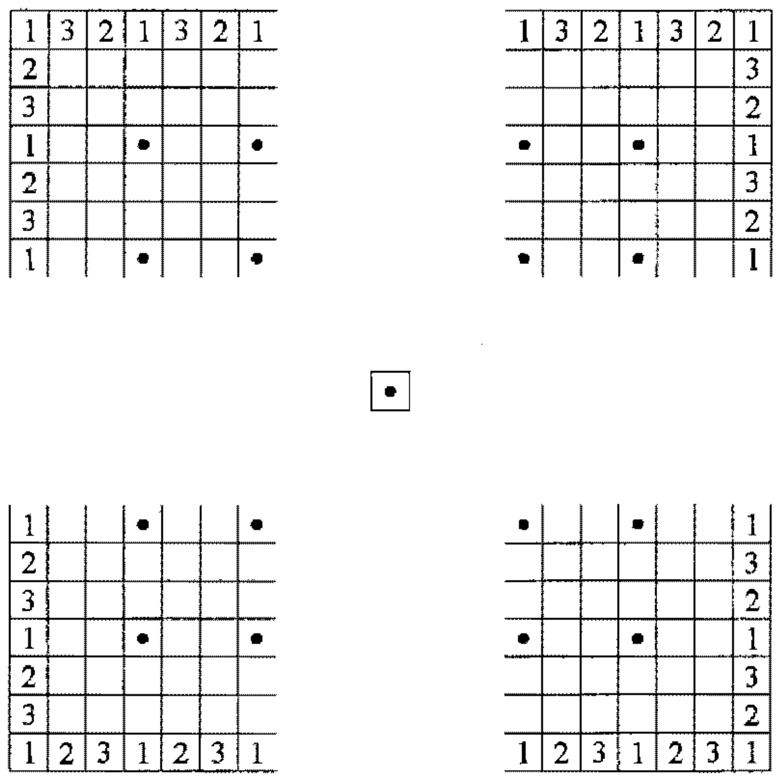

Figure 5.5 Colouring of the edge squares

these queens can be on the edge, since such a queen will cause several edge squares to be dominated more than once.

We searched for edge dominating sets by computer and found several such sets. (See Table 5.2). However, we did not find any such sets when the number of queens is a multiple of three, which leads to our next conjecture:

Conjecture 2: *There are no edge dominating sets if* $2k+1$ *(the number of queens) is a multiple of three.*

If the above two conjectures are true then the lower bound for $\gamma(Q_{4k+3})$ can be improved for the cases where $2k+1$ is a multiple of 3. We now prove why there cannot be edge dominating sets for some of the cases.

Theorem 5.9 *Consider a set of* $2k+1$ *queens on* Q_{4k+3} *with* $2k+1$ *a multiple of three. Then if* $2k+1$ *is not a multiple of nine, the set cannot be edge dominating.*

Proof. Colour the edge squares with 1, 2 and 3 repeatedly so that every third square is the same colour, starting with 1 in a corner (see Figure 5.5). Edge squares on the same diagonal, row or column are either both coloured 1 or coloured differently, namely 2 and 3. From Figure 5.5 we see that a queen can dominate two possible combinations of colours on the edge: Firstly, queens with both coordinates a multiple of three (indicated by dots in the figure) dominate only edge squares of colour 1 (eight squares in total). Secondly, queens on all other squares dominate two squares of colour 1, three of colour 2 and three of colour 3. There are $8(2k+1)/3$ edge squares of each colour. To dominate the right number of colours 2 and 3 there must be $8(2k+1)/9$ queens of the second type. This is only possible if $2k+1$ is a multiple of 9. □

Corollary 5.10 *If* Q_{4k+3} *has an edge dominating set and* $2k+1$ *is a multiple of nine, then there are exactly* $(2k+1)/9$ *queens on squares with both their coordinates a multiple of three.* □

We can now give an alternative proof to the one by Weakley [17] for $\gamma(Q_7) = 4$:

Theorem 5.11 $\gamma(Q_7) = 4$.

Proof. Placements establishing $\gamma(Q_7) \leq 4$ can be found in Appendix A. By Theorem 2.1 it thus suffices to show $\gamma(Q_7) \neq 3$. Suppose that there exists a dominating set R of 3 queens on Q_7. The only odd value of j satisfying the inequality in the statement of Theorem 2.2 is $j = 7$. Thus R must be an edge dominating set, but by Theorem 5.9 such a set does not exist, so we have shown $\gamma(Q_7) \neq 3$. □

To study the distribution of queens in (edge) dominating sets we define the following: The *radius* R of a queen is j if the queen lies on the edge of a $j \times j$ sub-board with the centre square on $(0,0)$. Note that the radius of a queen with coordinates (i,j) equals the maximum of $|i|$ and $|j|$. On the simplified board we will refer to $\frac{1}{2}R$, so that we can use the coordinates of the simplified board directly. For a set of queens we can determine the sum of all the radii. We will denote this sum by $\sum R$. Again, we will refer to $\frac{1}{2}\sum R$ on the simplified board. We determined $\frac{1}{2}\sum R$ for all the edge dominating sets of Q_{4k+3} we found with queens in every second row and column (from $k=8$ only symmetric sets) and found for all of them (up to $k=11$) that:

$$\frac{1}{2}\sum R = \frac{4k(k+1)}{3}.$$

We now state the following conjecture:

Conjecture 3: *For an edge dominating set of Q_{4k+3} with queens on every second row and column we have (on the simplified board):*

$$\frac{1}{2}\sum R = \frac{4k(k+1)}{3}.$$

Note that if the above conjecture is true it would also explain why for every third k there is no edge dominating set, because $\sum R$ must be an integer.

We now look at edge dominating sets that also dominate the whole board, and if we assume Conjectures 1 and 2 are true, we can determine for which k it might be possible to find sets that satisfy $\gamma(Q_{4k+3}) = 2k+1$. Henceforth, when we refer to a dominating set D of Q_{4k+3} we assume that, unless stated otherwise, $|D| = 2k+1$ and that there is one queen on each even row and each even column. An (i,i)- or $(i,i+2)$-dominating set D on Q_{4k+3} is the same as a dominating set on Q_{4k+1} with an extra edge that is also dominated. We can therefore use the results in Chapter 4. The following two theorems were first proved in [3] :

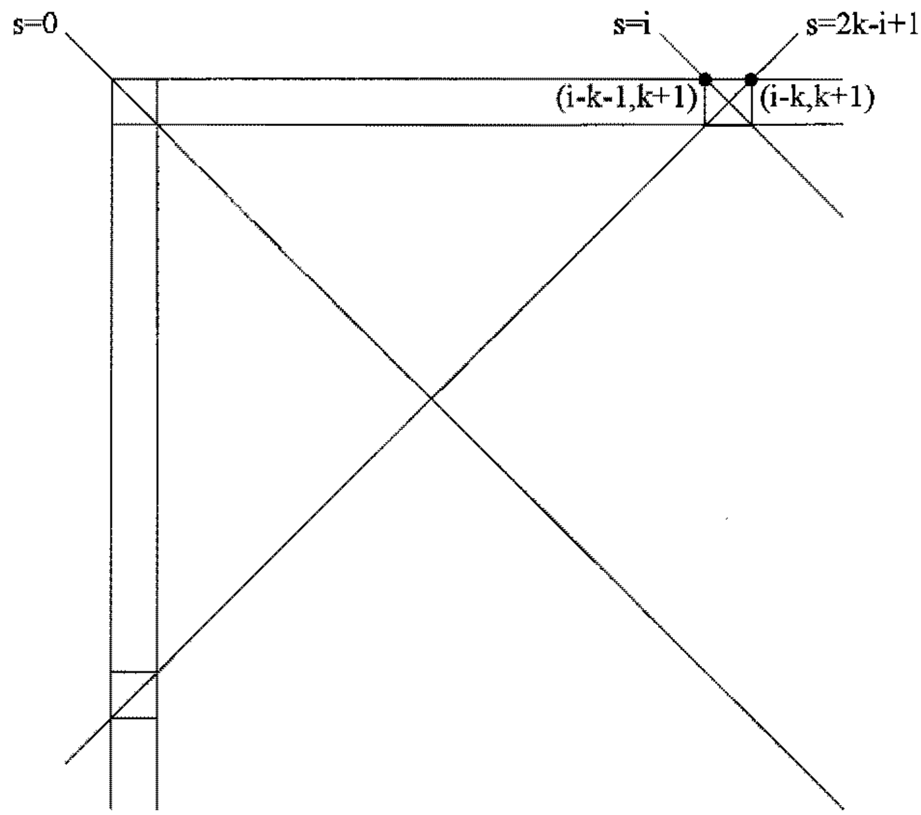

Figure 5.6 There are no $(i, i+2)$-dominating sets on Q_{4k+3}.

Theorem 5.12 *There are no* $(i, i+2)$*-dominating sets on* Q_{4k+3}.

Proof. If D is an $(i, i+2)$-dominating set, then by Theorem 4.2 there are queens on

$$d = 0, \pm 1, \pm 2, \ldots, \pm i, \pm(i+1), \pm(i+3), \pm(i+5), \ldots, \pm(2k-i-1),$$

$$s = 0, \pm 1, \pm 2, \ldots, \pm(i-1), \pm(i+1), \pm(i+3), \ldots, \pm(2k-i-3)$$

with no queens on any other d-diagonal. There are no queens on at least one of $s = i$ and $s = -i$. Without loss of generality say $s = i$ is empty. Thus the squares on $s = i$ on the edge must be dominated by a queen on a d-diagonal. This diagonal must be $d = 2k - i + 1$ (see Figure 5.6). But there is no queen on $d = 2k - i + 1$. Therefore the edge is not dominated. □

Theorem 5.13 *An (i,i)-dominating set on Q_{4k+3} must have exactly one queen on each of*

$$s(d) = 0, \pm 1, \pm 2, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k-i-1), \pm(2k-i+1), \quad (5.1)$$

with no queens on any other diagonal.

Proof. By Theorem 4.2 there must be queens on each of

$$s(d) = 0, \pm 1, \pm 2, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k-i-1). \quad (5.2)$$

It is easy to verify that the only squares on the border that are not dominated by (5.2), are the squares on $s, d = \pm i$. As in the proof of Theorem 5.12 these squares must be dominated by queens on $s, d = \pm(2k-i+1)$. Thus there must be queens on the $2k+1$ diagonals listed in (5.1). Hence there are no queens on any of the other diagonals. □

Note that for a dominating set D on Q_{4k+3} both the core and the body are restricted, and that the number of body diagonals is two more than in the case of Q_{4k+1}, namely $2k - i + 2$. Most of the lemmas and theorems in Chapter 4 can be adapted for dominating sets on Q_{4k+3}. Also, the same techniques can be used to search for dominating sets satisfying $\gamma(Q_{4k+3}) = 2k + 1$. However, no new sets could be found. We now investigate further why there are so few of these sets.

Let c_i be the number of squares on the simplified board dominated diagonally (*i.e.* all the odd-odd squares on the normal board) by the queen on (x_i, y_i), and $\frac{1}{2}R_i$ the radius of the queen on (x_i, y_i).

Lemma 5.14 $c_i = 4(k+1) - 2(\frac{1}{2}R_i)$.

Proof. It is easy to see that queens with the same radius dominate the same number of squares. The centre square dominates $4(k+1)$ queens and if the radius increase by one, the number of squares dominated decreases by two. □

Lemma 5.15 *If D is a dominating set on Q_{4k+3}, then*

$$\sum c_i = 4(k+1)(k+i) - 2i(i-1).$$

Proof. By Theorem 5.13 there is one queen on each of the following diagonals:

$$s/d = \pm 0, 1, 2, ..., i-1, i+1, ..., 2k-i+1$$

Diagonal $s = l(d = l)$ consists of $2k+2-|l|$ squares on the simplified board. Thus we can determine the total number of squares covered by the diagonals (squares that are covered twice are counted twice):

$$\begin{aligned}
\sum c_i &= 2[(2k+2) + 2[(2k+1) + (2k) + ... \\
&\quad +(2k+3-i) + (2k+1-i) + ... + (1+i)]] \\
&= 2[2k+2+2[(i-1)(4k+4-i)/2 + (2k+2)(k-1+1)/2]] \\
&= 2[(2k+2) + (4k+4-i)(i-1) + (2k+2)(k-i+1)] \\
&= 4(k+1)(k+i) - 2i(i-1).
\end{aligned}$$

□

Theorem 5.16 *If D is an (i,i)-dominating set on Q_{4k+3}, then*

$$\frac{1}{2}\sum R = 2(k+1)(k+1-i) + i(i-1).$$

Proof. By Lemma 5.14, $\frac{1}{2}R_i = 2(k+1) - \frac{1}{2}c_i$. Thus we have

$$\begin{aligned}
\frac{1}{2}\sum R &= \gamma \times 2(k+1) - \frac{1}{2}\sum c_i \\
&= 2(2k+1)(k+1) - 2(k+1)(k+i) + i(i-1) \\
&= 2(k+1)(k+1-i) + i(i-1).
\end{aligned}$$

□

If we assume Conjectures 1 and 3 are true, we have calculated $\frac{1}{2}\sum R$ for a dominating set D for Q_{4k+3} in two different ways. We thus have:

$$\frac{4k(k+1)}{3} = 2(k+1)(k+1-i) + i(i-1). \tag{5.3}$$

The first few integer solutions ($k \geq 0$) for this equation are listed in Table 5.3.

k	i
0	1 or 2
2	2 or 5
9	5 or 16
35	16 or 57
132	57 or 210

Table 5.3: Integer solutions for equation (5.3)

The smaller value of i is in each case the valid one. The first two solutions correspond with Q_3 and Q_{11}, for which dominating sets of cardinality $2k+1$ are known. No (5,5)-dominating set D could be found for $k = 9$. We note, however, that for both known dominating sets, k is even.

Chapter 6

Upper bounds for the domination number of Q_n

In this chapter we give an upper bound for $\gamma(Q_n)$ by constructing dominating sets for certain values of n.

As stated in the proof of Theorem 2.1, it is obvious that $\gamma(Q_n)$ is bounded above by $n-2$. As in the case of the lower bounds, no non-trivial upper bounds were known until quite recently. It was proved in [4] that if $n = 108m - 37$, then $\gamma(Q_n) \leq 62m - 23 = \frac{31}{54}n - \frac{95}{54}$. With the lower bound $\gamma(Q_n) \geq \frac{1}{2}n + \mathcal{O}(1)$, this leaves a gap of $\frac{2}{27}n + \mathcal{O}(1)$ between the lower and upper bound. We are now able to narrow this gap by more than half to $\frac{1}{30}$ by showing that if $n = 60m - 11$, then $\gamma(Q_n) \leq 32m - 6 = \frac{8}{15}n - \frac{2}{15}$.

Note that there are restrictions on n, but the set of admissible values for n is an arithmetic progression. For all other values of n, we can create a dominating set by adding queens to a dominating set on a largest admissible board of size less than n. At most one queen is needed for each new row and column. Therefore, the number of added queens is never more than a constant. This will show that $\gamma(Q_n) \leq \frac{8}{15}n + \mathcal{O}(1)$. Note that the pattern formed by the queens to obtain this bound is different from the so-called Z-pattern used in [4] and [12].

We begin with the follwing result which is easily seen to be true (refer to Figure 4.3) and which is therefore stated without proof.

Lemma 6.1 *If an s-diagonal* $s = l \geq 0$ *intersects a row* $y = p$ *or a column* $x = q$, *then any s-diagonal* $s = l'$ *with* $0 \leq l' \leq l$ *intersects all rows* $y = p'$ *with* $p' \geq p$ *and all columns* $x = q'$ *with* $q' \geq q$. *A symmetric statement holds for negative s-diagonals and similar results are true for d-diagonals.*

Theorem 6.2 *For each positive integer m,* $\gamma(Q_{60m-11}) \leq 32m - 6$.

Proof. We will give a set of queens on Q_{4k+1} with at least one queen in every second row and column and at least one queen on each of the diagonals

$$s(d) = 0, \pm1, \pm2, \ldots, \pm(i-1), \pm(i+1), \ldots, \pm(2k-i-1),$$

with $i = 6m - 1$, $k = 15m - 3$ and m any positive integer. According to Theorem 4.2 this will be a dominating set.

The dominating set consists mainly of five groups of queens plus a few more to cover the remaining empty rows, columns and diagonals. See Figures 6.1 and 6.2 for the cases $m = 2$ and $m = 3$.

Again, we use the simplified version of the board. The core queens are the odd queens and their coordinates are given by

$$(1, i-3) + j(1, -3) \text{ and } (-1, -i+3) + j(-1, 3) \text{ for } j = 0, 1, \ldots, 3m-2.$$

These queens are on the diagonals

$$s(d) = -i+2, -i+4, \ldots, -1, 1, 3, \ldots, i-2, \tag{6.1}$$

which cover all the required consecutive odd diagonals.

The body consists of four groups of queens, two of which are exact copies of the core. If we regard (0,0) as the centre of the core queens, then the centres of the exact

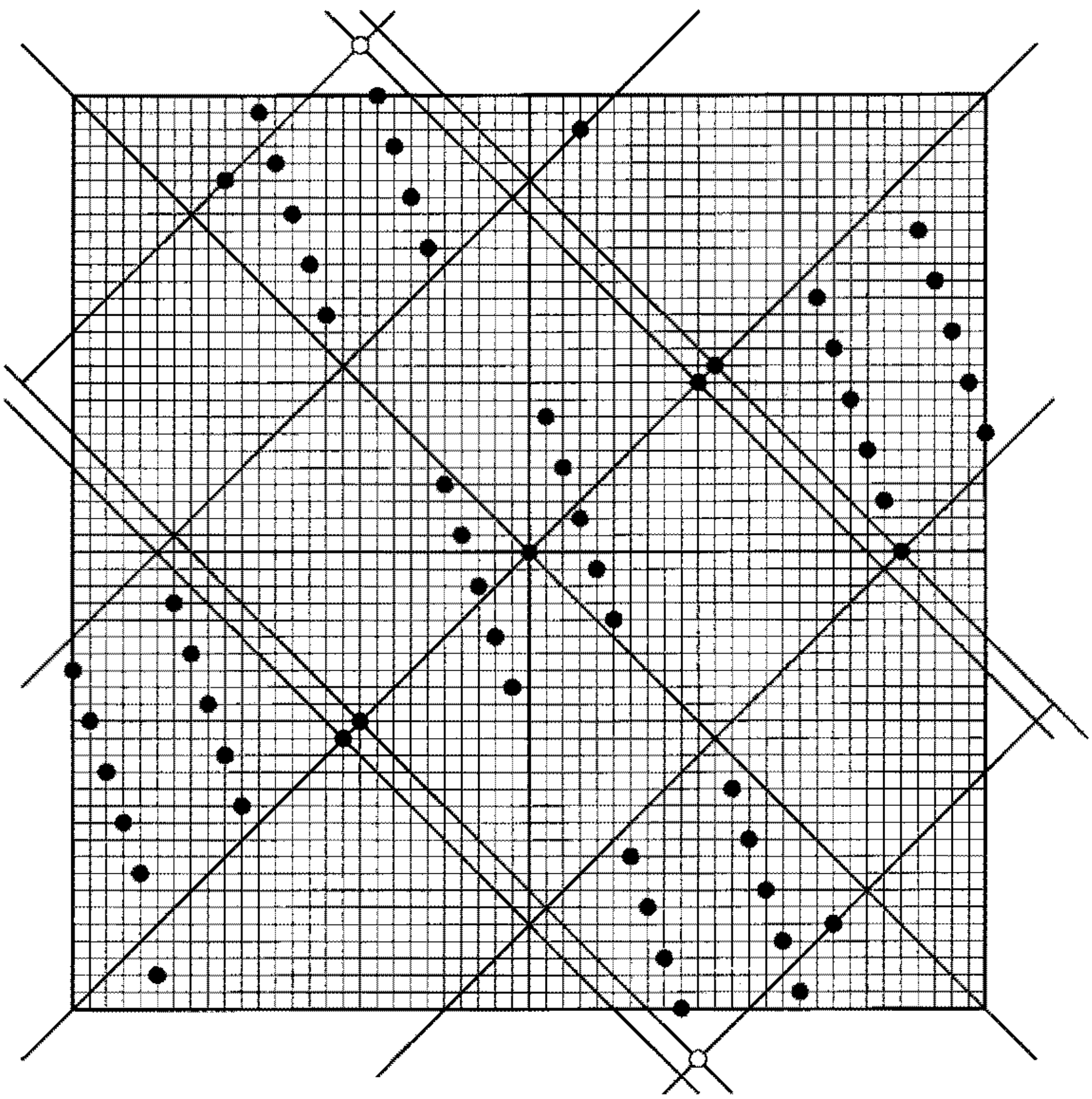

Figure 6.1 $m = 2 \ (\gamma(Q_{109}) \leq 58)$

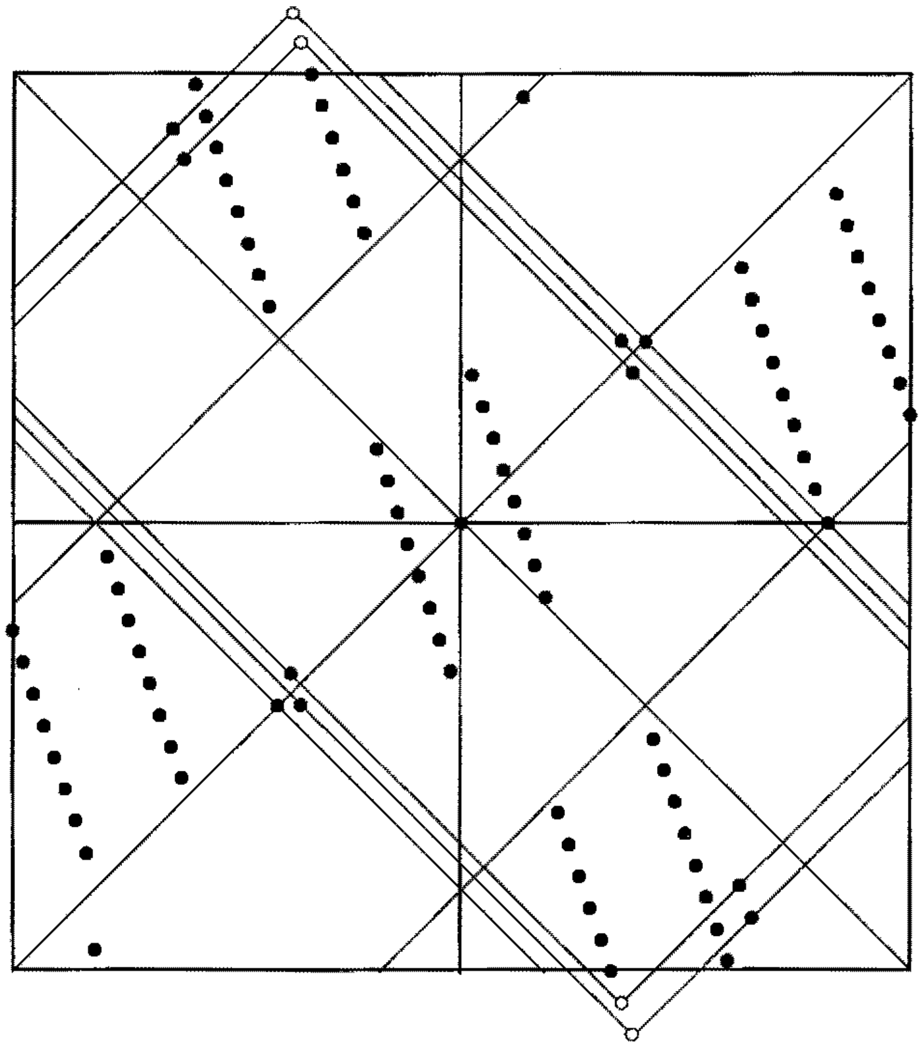

Figure 6.2 $m = 3 \quad (\gamma(Q_{169}) \leq 90)$

copies are at $(2i, i)$ and $(-2i, -i)$. First, consider the copy with centre $(2i, i)$. Because it is a copy of the core, it will also cover consecutive diagonals (of the same parity). We can therefore determine the s- and d-diagonals of the copy by adding $i + 2i$ and $i - 2i$ to (6.1) respectively. This gives:

$$s = 2i+2, 2i+4, \dots, 3i-1, 3i+1, 3i+3, \dots, 4i-2 \tag{6.2}$$

$$d = -2i+2, -2i+4, \dots, -i-1, -i+1, -i+3, \dots, -2. \tag{6.3}$$

Similarly, the copy with centre $(-2i, -i)$ has queens on the diagonals

$$s = -4i+2, -4i+4, \dots, -3i-1, -3i+1, -3i+3, \dots, -2i-2 \tag{6.4}$$

$$d = 2, 4, \dots, i-1, i+1, i+3, \dots, 2i-2. \tag{6.5}$$

The other two groups of queens are also copies of the core with the only difference that if $m \geq 2$, then some of the queens do not fit on the board, and these, of course, cannot be part of the dominating set. The centres of the groups are at $(i, -2i)$ and $(-i, 2i)$. These queens (including those that do not fit on the board) are on the even diagonals

$$s = -2i+2, 2i+4, \dots, -2 \tag{6.6}$$

$$s = 2, 4, \dots, 2i-2 \tag{6.7}$$

$$d = -4i+2, -4i+4, \dots, -2i-2 \tag{6.8}$$

$$d = 2i+2, 2i+4, \dots, 4i-2. \tag{6.9}$$

We see that the diagonals listed from (6.2) to (6.9) are all the even diagonals from $s(d) = -4i+2 = -(2k-i-1)$ to $s(d) = 4i-2 = 2k-i-1$ except $s(d) = 0$ and $s(d) = \pm 2i$.

We have to consider the queens that do not fit on the board, because they are on the extensions of empty diagonals which therefore must be covered in some other way. These queens have y-coordinates bigger than $k = 15m - 3$ or smaller than $-k$. Their coordinates are

$$\begin{aligned} &(-i, 2i) + (1, i-3) + j(1, -3) && \text{for } j = 0, 1, \ldots, m-2 \\ \text{and} \quad &(i, -2i) + (-1, -i+3) + j(-1, +3) && \text{for } j = 0, 1, \ldots, m-2, \end{aligned}$$

which are the same as (remember $i = 6m - 1$)

$$\begin{aligned} &(-6m+2, 18m-6), (-6m+3, 18m-9), \ldots, (-5m, 15m) \\ \text{and} \quad &(6m-2, -18m+6), (6m-3, -18m+9), \ldots, (5m, -15m). \end{aligned}$$

These queens are on the diagonals:

$$\begin{aligned} &s = \pm(2i-2), \pm(2i-4), \ldots \quad m-1 \text{ terms} \\ \text{and} \quad &d = \pm(4i-4), \pm(4i-8), \ldots \quad m-1 \text{ terms} \end{aligned}$$

To summarise, the following lines (corresponding to columns, rows and diagonals) are empty at this stage:

$$\begin{aligned} x &= 0, \pm i, \pm 2i \text{ and } x = \pm(i-1), \pm(i-2), \ldots \quad m-1 \text{ terms} \\ y &= 0, \pm i, \pm 2i \text{ and } y = \pm(k-2), \pm(k-5), \ldots \quad m-1 \text{ terms} \\ d &= 0, \pm 2i \qquad \text{and } d = \pm(4i-4), \pm(4i-8), \ldots \; m-1 \text{ terms} \\ s &= 0, \pm 2i \qquad \text{and } s = \pm(2i-2), \pm(2i-4), \ldots \; m-1 \text{ terms.} \end{aligned}$$

Thus we have $2m + 3$ empty rows and columns each and $2m + 1$ empty s- and d-diagonals each. These lines can be covered by the following procedure:

1. Place queens on (0,0), (i, i) and $(-i, -i)$. This covers the columns x (rows y) $= 0, \pm i$, s-diagonals $s = 0, \pm 2i$ and d-diagonal $d = 0$, leaving $2m$ empty rows, columns and d-diagonals and $2m - 2$ empty s-diagonals.

2. Place a queen on each of $2m - 1$ intersections of an empty row and an empty d-diagonal, leaving one empty row and d-diagonal, say $d = 2i$.

3. Place a queen on each of the $2m-2$ intersections of an empty column and an empty s-diagonal, leaving two empty columns, one of which is $x = -i+1 = -6m+2$, say, and no empty s-diagonals.

4. Place a queen on the intersection of the empty column $x = -6m + 2$ and the remaining empty d-diagonal $d = 2i = 12m - 2$, *i.e.* on the square with coordinates $(-6m+2, 6m)$.

5. Place a queen on the intersection of the remaining empty row and column.

Steps 2 – 5 can be executed in a number of ways. Step 2 is always possible since even the shortest positive (negative, respectively) empty d-diagonal $d = 4i - 4 = 24m - 8$ ($d = -4i + 4 = -24m + 8$, respectively) intersects all the empty rows in the upper (lower) half of the board. This follows from Lemma 6.1 since $d = 24m - 8$ intersects the empty row $y = 12m + 1$ closest to the centre in the upper half of the board in the point $(12m - 9, 12m + 1)$. A symmetric statement holds for the negative d-diagonals. Similarly, Step 3 is always possible by Lemma 6.1 since the shortest empty positive (negative) s-diagonal $s = 12m - 4$ ($s = -12m + 4$) intersects the empty column $x = 5m$ ($x = -5m$) in the upper (lower) half of the board closest to the centre (if $m \geq 2$) in the point $(5m, 7m - 4)$ (respectively $(-5m, -7m + 4)$). The coordinates of the queens in Steps 4 and 5 depend of course on the placements in Steps 2 and 3; the coordinates chosen to execute Step 4 only serve to illustrate that Step 4 is possible, and Step 5 is trivially possible in all cases.

We need $4m + 2$ queens for this procedure. The five copies (minus the few not on the board) require $5(6m - 2) - 2(m - 1)$ queens, giving a total number of $32m - 6$ queens. □

Remark: In some cases it is possible to improve on the above procedure, because if more than two empty lines cross at one point we need fewer queens to cover those lines. For example, if $m = 3$ we can use one fewer queen by placing queens on

$$(0,0), \pm(34,0), \pm(15,17), \pm(16,14), \pm(26,-34), \pm(27,-37), \pm(17,40).$$

The same technique with core patterns that result in more empty lines crossing at the same point would permit us to improve the upper bound even more.

Chapter 7

Domination on hexagonal boards

In this chapter we determine values for γ and i for hexagonal boards, and show that there are only two types of dominating sets for certain boards. We consider a hexagonal board (hive) consisting of hexagonal cells (see Figure 7.1). Note that domination by queens on a square beehive was studied by Theron and Geldenhuys in [16] .

We define a queen on the hexagonal board as a piece that moves along three *lines*, namely along the cells in the same row, up diagonal or down diagonal. A queen dominates a cell if the cell is in the same line as the queen. The problem is to determine the minimum number of queens necessary to dominate all the cells on the board. The *edge* consists of all the cells on the edge of the hive. A cell or a line is *empty* if there is no queen on the cell or line. A cell is *open* if it is not dominated.

Again, this can also be considered as a graph domination problem in the following way: The hexagonal queens graph H_n has the cells of a board with n rows and diagonals as its vertices. Two vertices are adjacent if the two corresponding cells are in the same row or diagonal. A set D of vertices (cells) is a dominating set of H_n if every cell of H_n is either in D or adjacent to a vertex in D. If no two cells of a set I are adjacent then I is an independent set. Let $\gamma(H_n)$ denote the minimum size of a dominating set of H_n, and let $i(H_n)$ denote the minimum size of an independent dominating set of Q_n. Note that for any n, $\gamma(H_n) \leq i(H_n)$.

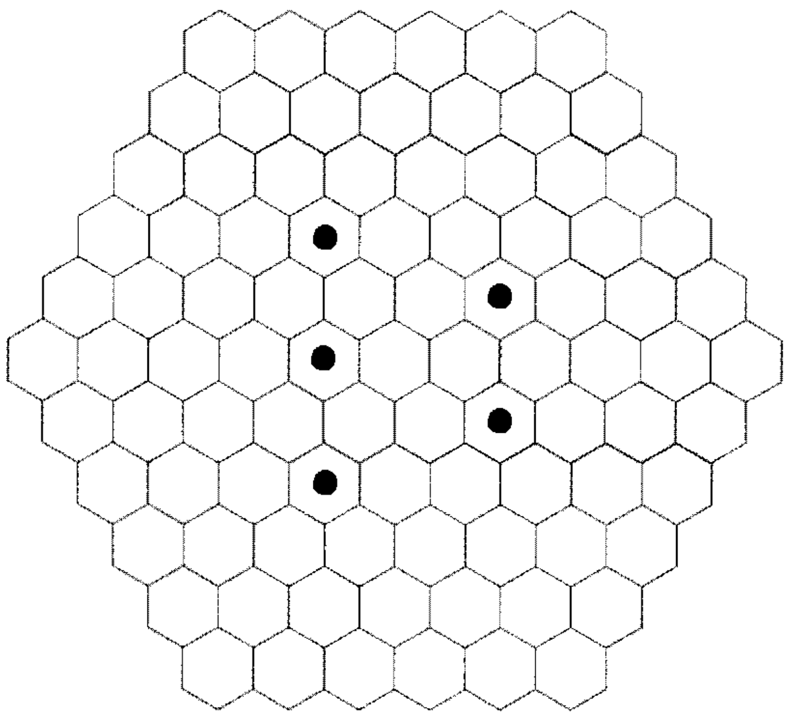

Figure 7.1 A dominating set for H_{11}

We will only consider hives with a centre cell, *i.e.* hives with an odd number of rows and diagonals. Thus we will determine values for $\gamma(H_n)$ and $i(H_n)$ (n odd) and show that there are only two types of dominating sets for H_{4k+3}. The lines are labelled as shown in Figure 7.2.

Each cell has three coordinates, namely row (r), up diagonal (u) and down diagonal (d) which we will denote as (r, u, d). We note the following:

Remark 7.1 *For all cells we have* $r + u + d = 0$

Remark 7.2 *A line with a negative (positive) label intersects an edge line with a positive (negative) label.*

We will now describe a dominating set of queens on H_{4k+3} which was first discovered by Burger and Theron [15] . The placement consists of two columns with $k + 1$

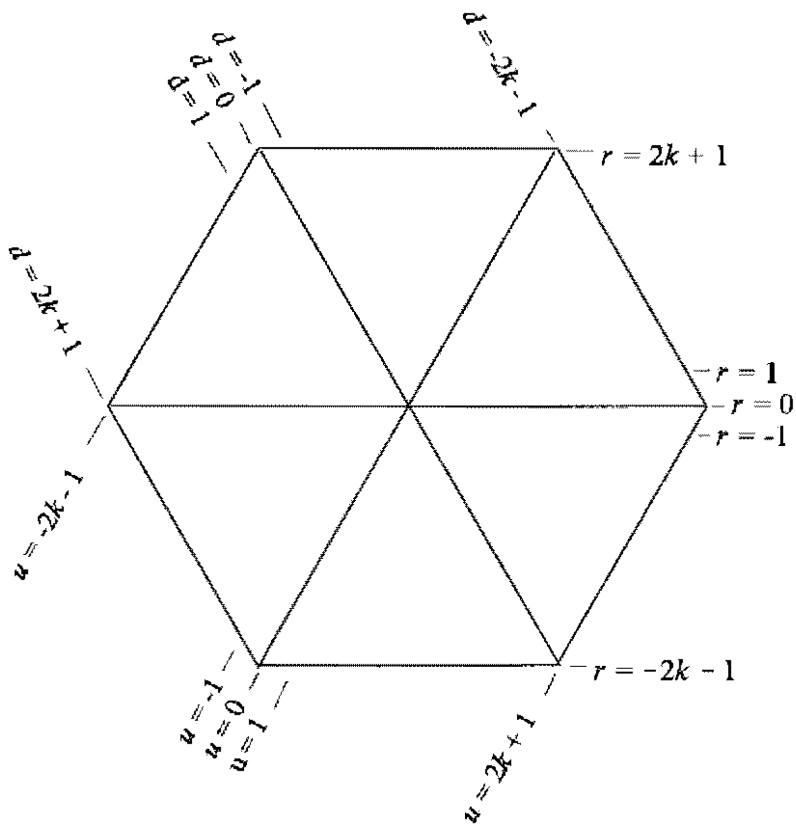

Figure 7.2 Numbering of rows and diagonals

and k queens respectively. Figure 7.1 shows the case $k = 2$. In general, the coordinates for $k \geq 0$ are given below, where the second set of coordinates is undefined (and to be ignored) when $k = 0$ (see Figure 7.3).

$$(2a - k, -a, k - a) \text{ for } a = 0, 1, \ldots, k$$

and

$$(2a + 1 - k, k - a, -1 - a) \text{ for } a = 0, 1, \ldots, k - 1.$$

We will refer to this placement as the *Double Column Placement* (*DCP*). We can see

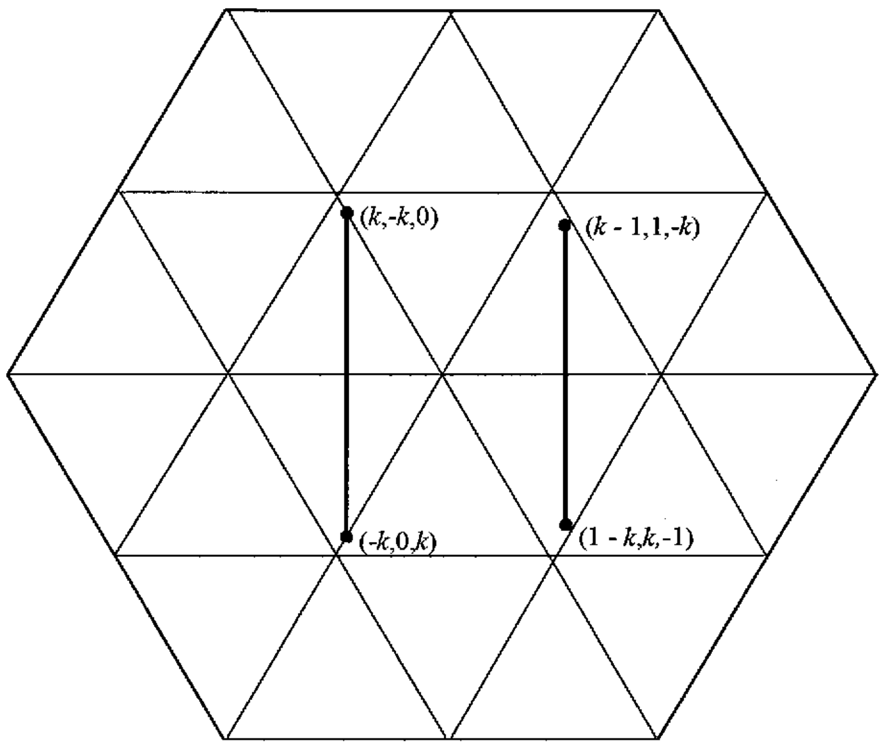

Figure 7.3 Double Column Placement for H_{4k+3}

that each of the rows, up diagonals and down diagonals covered by a DCP has the labels:

$$-k, -k+1, \ldots, -1, 0, 1, \ldots, k-1, k.$$

Thus the $2k+1$ lines closest to the centre are all covered. This is sufficient to dominate the whole hive. Note that the queens form an independent set. Since the case $k = 0$ is trivial, we assume henceforth that $k \geq 1$. We state the following lemma without proof:

Lemma 7.1 *For all* $k \geq 1$, $\gamma(H_{4k+3}) \leq i(H_{4k+3}) \leq 2k+1$.

We define a *ring* as a six-sided convex polygon formed by the union of six lines, where each line consists of at least two cells. The edge is an example of a ring. The edge can

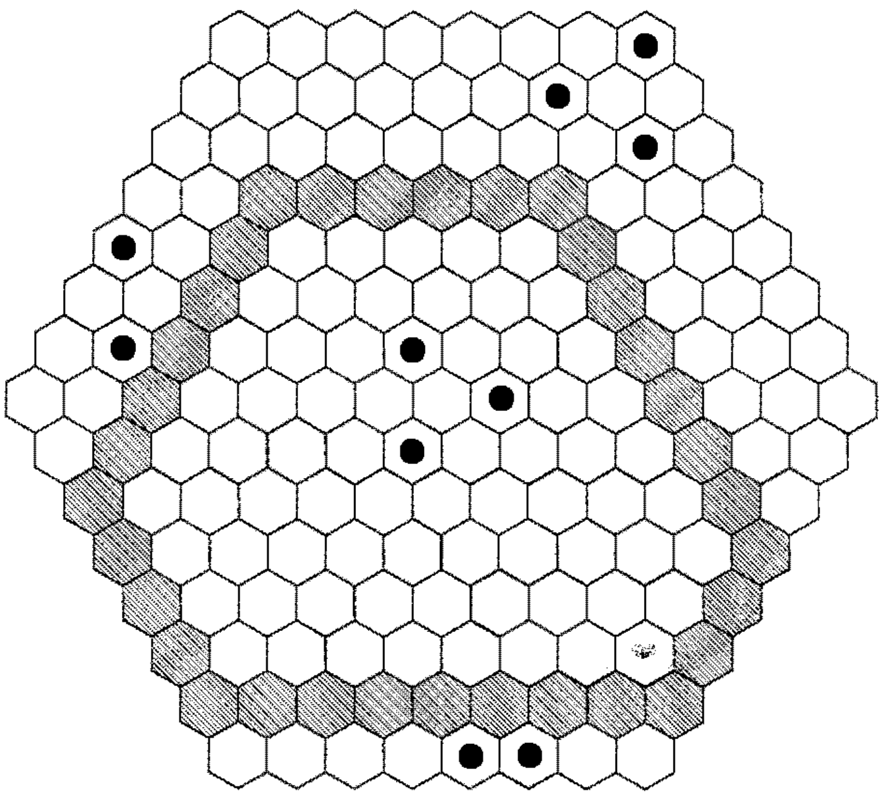

Figure 7.4 Biggest empty ring

be made smaller by replacing one line of the ring with a line closer to the centre, as long as the ring has six sides. For any set of queens on a hive we define the *Biggest Empty Ring* (*BER*), if it exists, as the ring formed by the edge lines, if they are unoccupied, or by replacing each of the occupied edge lines with the empty parallel line closest to the edge line concerned (see Figure 7.4). Let the *distance* a line is replaced be the number of lines outside that side of the BER. Let δ be the sum of all the distances each side is replaced. Note that δ equals the number of cells in the edge minus the number of cells in the BER, because if a line of the ring is replaced by a line just closer to the centre, the number of cells in the ring decreases by one. We now have the following lemma:

Lemma 7.2 *For all $k \geq 1$, if H_{4k+3} has $2k$ or fewer queens, then the BER exists.*

Proof. We only have to verify that the BER always has six sides. If a line of any ring is replaced by a line just closer to the centre, the number of cells in two lines of the ring decreases by one. When constructing the BER, each queen outside the BER caused either one or two such replacements (depending on whether the queen is at a "corner" or not). It is easy to see that each queen outside the BER caused the number of cells in any side to decrease by at most one. There are $2k+2$ cells in each edge line. Thus if there are $2k$ queens, each side of the BER must have at least two cells. □

Let c be the total number of times the BER is dominated by all the queens. Thus if one BER cell is covered m times, it must be counted m times. Let q be the number of queens on the board. We then have the following lemma:

Lemma 7.3 *If the BER exists for a set of q queens on H_{4k+3}, then $c \leq 6q - 2\delta$.*

Proof. There are two types of queens outside the BER (see Figure 7.4):

(1) The queens that lie on the outside of only one line of the BER. Each of them covers four cells of the BER.

(2) Queens that lie on the outside of two lines of the BER. Each of them covers two cells of the BER. Let there be b_1 and b_2 queens of each type respectively. It is easy to see that

$$\delta \leq b_1 + 2b_2,$$

with equality when the queens are independent. Now:

$$\begin{aligned} c &= 6(q - b_1 - b_2) + 4b_1 + 2b_2 \\ &= 6q - 2b_1 - 4b_2 \\ &= 6q - 2(b_1 + 2b_2) \end{aligned}$$

$$\leq \; 6q - 2\delta.$$

Again, equality holds when the queens are independent. □

Lemma 7.4 *If $\gamma(H_{2n+1}) = n$, then*

(a) *the BER is the edge.*

(b) *each edge cell is covered exactly once.*

Proof. From Lemma 7.3 we have $c \leq 6n - 2\delta$. Also $|BER| = 6n - \delta$. For the BER to be dominated we must have $c \geq |BER|$. Thus

$$6n - \delta \leq c \leq 6n - 2\delta$$

or

$$\delta \leq c + 2\delta - 6n \leq 0.$$

Therefore $\delta = 0$. Thus the BER must be the edge. To prove (b) we note that there are $6n$ edge cells and each of the n queens can cover six edge cells. Therefore each edge cell must be dominated exactly once. □

The proofs of the following two results, first proved in [15], now follow easily:

Theorem 7.5 *For all* $k \geq 0$, $i(H_{4k+3}) = \gamma(H_{4k+3}) = 2k + 1$.

Proof. As noted before we need only consider the case $k \geq 1$. We first show that $\gamma(H_{4k+3}) \geq 2k + 1$. Consider any set of $2k$ queens. There are $6(2k + 1) - \delta$ cells in the BER. But by Lemma 7.3, $c \leq 6(2k) - 2\delta$. Thus we have $c \leq 6(2k) - 2\delta < 6(2k + 1) - \delta = |BER|$. Therefore the BER cannot be dominated. The result now follows from Lemma 7.1. □

Theorem 7.6 *For all* $k \geq 0$, $i(H_{4k+1}) = \gamma(H_{4k+1}) = 2k + 1$.

Proof. H_{4k+1} is H_{4k+3} with the edge removed. Therefore the Double Column Placement also dominates H_{4k+1}, which establishes $\gamma(H_{4k+1}) \leq i(H_{4k+1}) \leq 2k + 1$. To show that $\gamma(H_{4k+1}) \geq 2k+1$, $k \geq 1$, we show that $2k$ queens cannot dominate H_{4k+1}. Suppose we have a set of $2k$ queens dominating H_{4k+1}. From Lemma 7.4(b) we see that each cell on the edge is dominated exactly once. The corner cells can only be dominated by a queen on a main diagonal. Consider any main diagonal. There must be one queen on it, and the remaining $2k-1$ queens are on the two sides. A queen in a specific half dominates four cells of the edge in that half and two cells of the edge in the other half. To dominate the same number of edge cells on the two sides, there must be the same number of queens in the two halves. This is a contradiction, because an odd number of queens remains. □

We will now show that there are only two types of minimum dominating sets for H_{4k+3} for all $k \geq 1$. From Lemma 7.4 we see that any dominating set of H_{4k+3} consisting of $2k+1$ queens leaves the edge empty. We use the fact that each edge cell must be dominated exactly once to prove the following lemmas.

Lemma 7.7 *If H_{4k+3} is dominated by $2k+1$ queens, then lines with the same label are either all occupied or all empty.*

Proof. Each of the edge cells is dominated exactly once. Thus if the row $r = a (a > 0)$ is occupied (respectively empty), then $d = 2k+1-a$ and $u = 2k+1-a$ are empty (respectively occupied). But then $u = 2k+1-(2k+1-a) = a$ and $d = a$ are occupied (respectively empty). The arguments for the diagonals are the same. Also, if $a < 0$, the arguments are similar (see Figure 7.5). The lines with label 0 must be occupied, because the corner cells can only be dominated by these lines. □

From Lemma 7.7 we see that we do not have to distinguish between labels of rows and labels of diagonals. Consequently, we will only refer to the set of labels:

$$L = \{-2k-1, -2k, \ldots -2, -1, 0, 1, \ldots, 2k, 2k+1\}$$

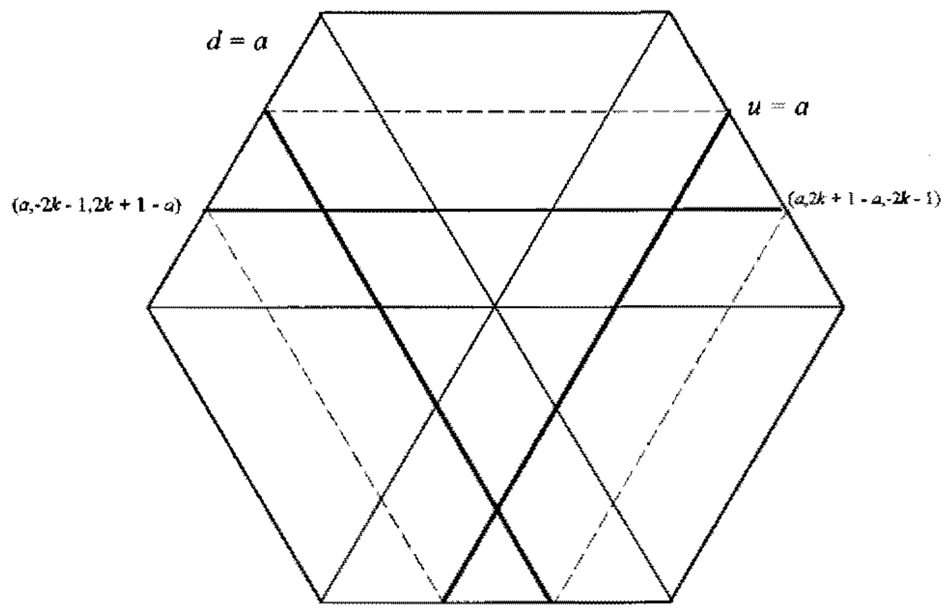

Figure 7.5 Lines with the same label are all occupied (empty)

This set can be partitioned into two disjoint sets: the set representing all the occupied lines (O) and the set representing all the empty lines (E).

Lemma 7.8 *If $a \in O$, then*

$$-a + 2k + 1 \in E \text{ if } a > 0$$

$$-a - 2k - 1 \in E \text{ if } a < 0.$$

Proof. The edge lines are labelled $2k+1$ or $-2k-1$. Suppose $a \in O$. Then since each edge cell is dominated exactly once, it follows from Remarks 7.1 and 7.2 that $-a + 2k + 1 \in E$ if $a > 0$ and $-a - 2k - 1 \in E$ if $a < 0$. □

Lemma 7.9 *If $a, b \in E$ and $|a + b| < 2k + 1$, then*

(a) $-a - b \in O$

(b) $a + b + 2k + 1 \in E$ if $a + b < 0$

(c) $a + b - 2k - 1 \in E$ if $a + b > 0$.

Proof. If lines a and b are empty and they intersect inside the edge, the third line going through the intersection must be occupied. From Remark 7.1 this line must be $-a-b$. Statements (b) and (c) follow from (a) and Lemma 7.8. □

Lemma 7.10 *If* $2a \in E$*, then* $-a \in O$.

Proof. Suppose $2a \in E$ and $-a \in E$. Then from Lemma 7.9(a), $-2a+a=-a \in O$. This is a contradiction. Therefore $-a \in O$. □

Lemma 7.11 *If* $1 \in E$*, then all odd elements of* L *are in* E.

Proof. If $1 \in E$, then from Lemma 7.9(c), $1+1-2k-1=1-2k \in E$. If $1, 1-2k \in E$, then from Lemma 7.9(b), $1+1-2k+2k+1=3 \in E$. If $1, 3 \in E$, then from Lemma 7.9(c), $1+3-2k-1=3-2k \in E$. Continuing in this way, we find the following elements in E:

$$1-2k, 3, 3-2k, 5, 5-2k, ..., 2k-3, -3, 2k-1, -1.$$

These are all the odd elements of L. □

Theorem 7.12 *There are only two types of dominating sets of cardinality* $2k+1$ *for* H_{4k+3}, $k \geq 1$:

(a) $O = \{-2k, -2k+2, ..., -2, 0, 2, ..., 2k-2, 2k\}$

(b) $O = \{-k, -k+1, ..., -1, 0, 1, ..., k-1, k\}$

Proof. Either $1 \in E$ or $1 \in O$. If $1 \in E$, then from Lemma 7.11 we have (a). We must show that (a) is dominating. A cell can only be open if three empty lines intersect in that cell. All empty lines have odd labels. Thus the sum of the coordinates of such a cell would be odd. This is impossible because the sum must be 0.

If $1 \in O$, then by Lemma 7.8 we have $2k \in E$. It follows from Lemma 7.10 that $-k \in O$, and then from Lemma 7.8 that $-k-1 \in E$. If $2k \in E$ and $-k-1 \in E$,

k	$4k+3$	γ	number
1	7	3	1
2	11	5	1
3	15	7	5
4	19	9	56
5	23	11	540
6	27	13	6996

Table 7.1 Number of dominating sets found

then it follows from Lemma 7.9(c) that $-k-2 \in E$. If $2k \in E$ and $-k-2 \in E$, then again from Lemma 7.9(c), $-k-3 \in E$. Continuing in this way, we find:

$$-k-1, -k-2, -k-3, ..., -2k+1, -2k \in E.$$

The whole argument can be repeated with $-2k \in E$ to get:

$$k+1, k+2, ..., 2k-1, 2k \in E.$$

Thus (b) follows, which is also dominating as explained in the case of a DCP. □

We note that in Theorem 7.12 the labels in (a) are double the labels in (b). Thus if we take the coordinates of a dominating set of type (b) and multiply it by two, we have the coordinates of a dominating set of type (a). The reverse can also be done. We therefore have a one-to-one correspondence between all the dominating sets of type (a) and (b). Figure 7.6 shows a few examples.

We can construct minimum dominating sets using the dominating sets of smaller boards. In Figure 7.7 a dominating set of H_{43} is obtained by repeating the pattern of a dominating set of H_{15}. Note that only the central section of the board is shown.

Table 7.1 lists the number of dominating sets found by computer. We see that the number of dominating sets is large for large boards. Dominating sets for H_{4k+1} are even more numerous, and they are not restricted to two types of minimum dominating

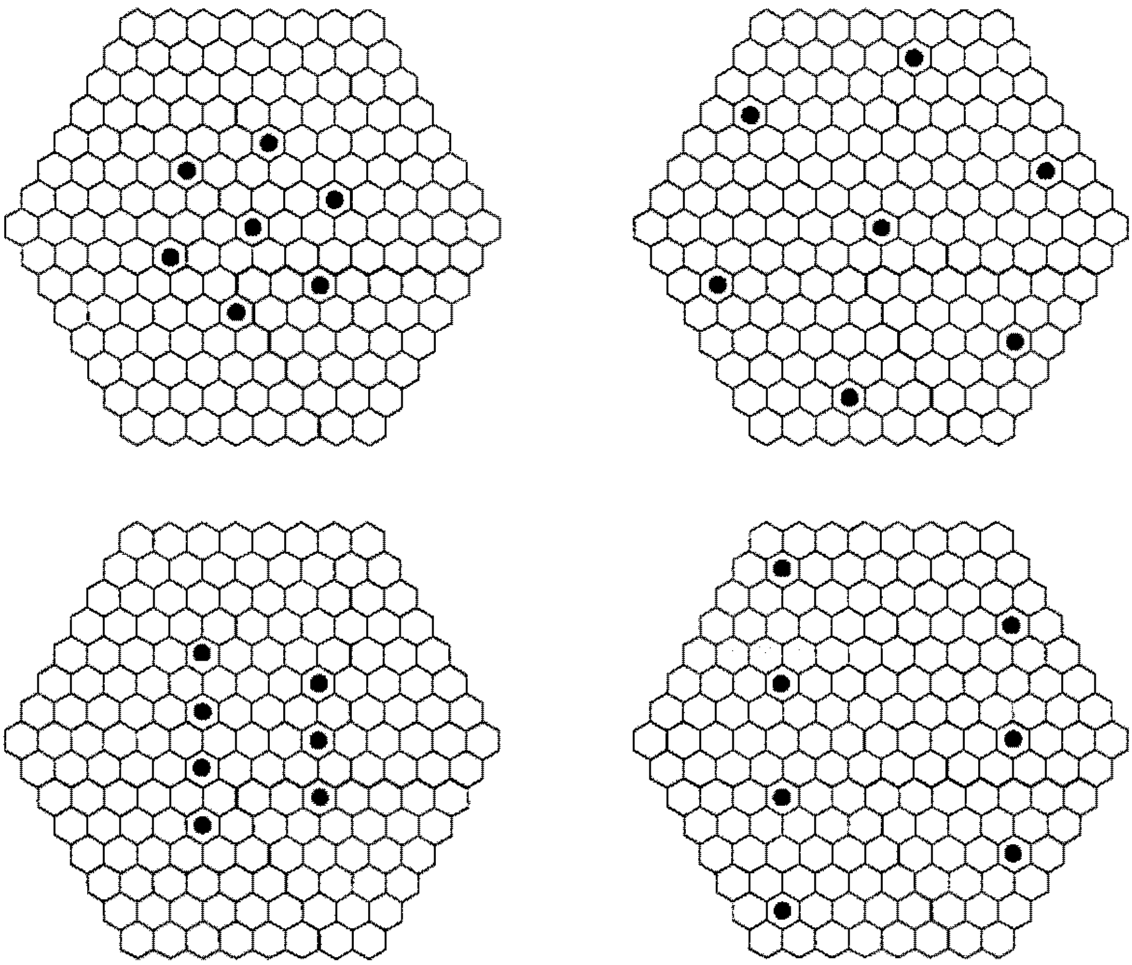

Figure 7.6 Dominating sets of different types for H_{15}

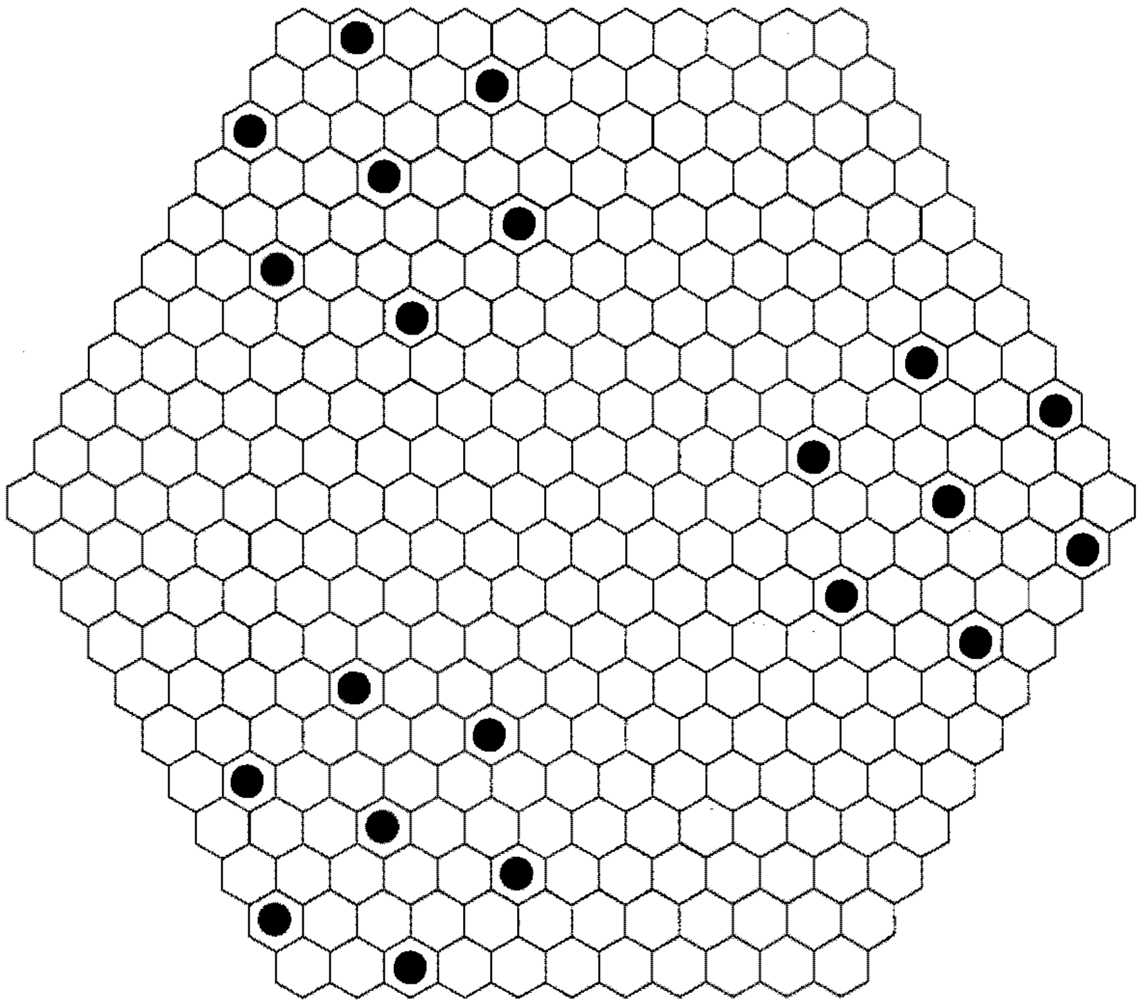

Figure 7.7 The central section of H_{43}

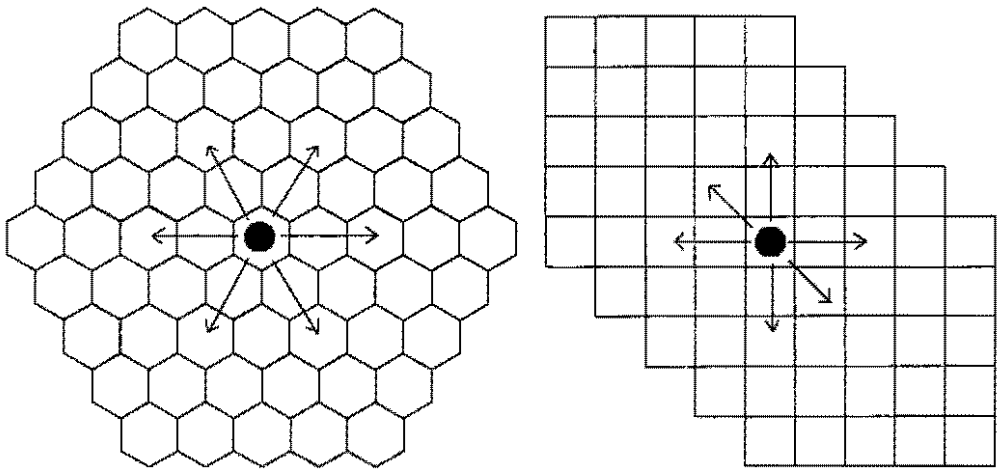

Figure 7.8 Relation between hexagonal boards and chessboards

sets.

In Figure 7.8 we see that the hexagonal domination problem is the same as the queen dominating problem for chessboards with the queens' domination restricted to three lines (row, column and one diagonal) and with two corners of the board cut off. We hope the dominating sets for hexagonal boards can help us to answer some questions for chessboards.

Chapter 8

Irredundance

In this chapter we determine the irredundance number for H_5, H_7, Q_5 and Q_6. We repeat some of the definitions of Chapter 1 here for convenience. The *closed neighbourhood* $N[v]$ of the vertex v in a graph consists of v and the set of vertices adjacent to v. We define the *private neighbourhood* of $v \in S$ as $pn[v, S] = N[v] - N[S - \{v\}]$. If $pn[v, S] \neq \emptyset$ for some vertex v, then every vertex in $pn[v, S]$ is called a *private neighbour* of v. Note that a vertex can be its own private neighbour. We say that a set S of vertices is *irredundant* if for every vertex $v \in S$, v has at least one private neighbour. Note that a minimal dominating set is also irredundant. An irredundant set S is *maximal irredundant* if for every vertex $u \in V - S$, the set $S \cup \{u\}$ is not irredundant, which means that there exists at least one vertex $w \in S \cup \{u\}$ which does not have a private neighbour. The minimum cardinality of a maximal irredundant set in a graph G is called the *irredundance number* and is denoted by $ir(G)$. If a vertex u is added to a set S and it destroys all the private neighbours of some vertex w in S (*i.e.* $pn[w, S] \neq \emptyset$ and $pn[w, S \cup \{u\}] = \emptyset$), we call u a *pn-destroyer*. If u is added to a set S and it has no private neighbours we say u is *pn-less*. For a subset S of vertices in a graph, we say a vertex v (or a cell or a square in the case of hexagonal boards or chessboards) is *open* if it is not dominated by S.

It is well-known that $ir \leq \gamma \leq i$. Up to now no cases of hexagonal or chessboards are known for which $ir < \gamma$. Before we look at hexagonal boards and chessboards separately, we give the following lemmas:

Lemma 8.1 *If S is a maximal irredundant set in $G = (V, E)$, then all open vertices in G must be pn-destroyers.*

Proof. Because S is maximal irredundant any vertex $v \in V - S$ must be a pn-destroyer or pn-less. But an open vertex cannot be pn-less because it is its own pn. Thus open vertices must be pn-destroyers. □

Lemma 8.2 *If S is a maximal irredundant set in a graph G and $|S| < i(G)$, then S is not independent.*

Proof. Suppose S is independent. Then it will be possible to add another independent vertex v. Thus $S \cup \{v\}$ is irredundant because each vertex in $S \cup \{v\}$ is its own private neighbour, meaning S is not maximal irredundant. □

Lemma 8.3 *If S is a maximal irredundant set on a hexagonal board (chessboard) B such that $|S| = \gamma(B) - 1$, then there are at least three open cells (squares).*

Proof. If there are only two open cells (squares), a queen can be added to cover these two cells (squares). This will be a minimal dominating set, which is also irredundant, meaning S is not maximal. □

8.1 Irredundance on hexagonal boards

Lemma 8.4 *If $ir(H_5) = 2$, then:*

(a) *The number of pn-less cells is at most four.*

(b) *Each queen has at least two private neighbours.*

(c) *Each queen's private neighbours can be destroyed from at most six cells.*

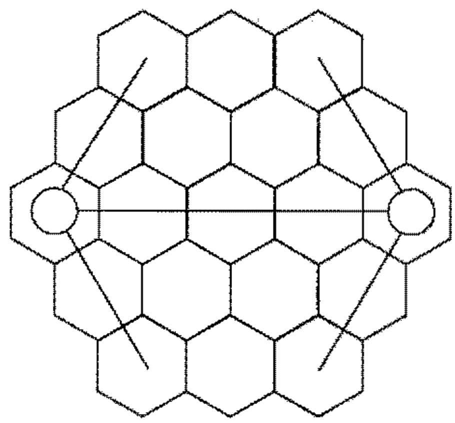

Figure 8.1 If $ir(H_5) = 2$ then there are at most four pn-less cells.

Proof. (a) The pn-less cells are the unoccupied cells not in line with the open cells. There are at least three open cells (Lemma 8.3). Even if there are only two open cells, it is easy to see that there are at most six squares not in line with them (see Figure 8.1). This leaves at most four pn-less cells, because the two queens are also not in line with the open cells.

(b) The two queens are adjacent (Lemma 8.2). Each queen has at least two possible cells per line that can be private neighbours. But the adjacent queen destroys at most one private neighbour per line. Thus there must be at least two private neighbours.

(c) Suppose there are only two private neighbours. If they are in the same line, they can be destroyed from at most four cells in that line, and from at most two cells not on that line. If the private neighbours are not on the same line, the cells which are pn-destroyers are those cells that lie on the intersections of the lines that intersect the two private neighbours. There are at most six such positions (see Figure 8.2). □

Theorem 8.5 $ir(H_5) = 3$.

Proof. We know that $ir(H_5) \leq \gamma(H_5) = 3$. Suppose $ir(H_5) < 3$. It is easy to see that $ir(H_5) > 1$, so consider an irredundant set of H_5 consisting of two queens. There

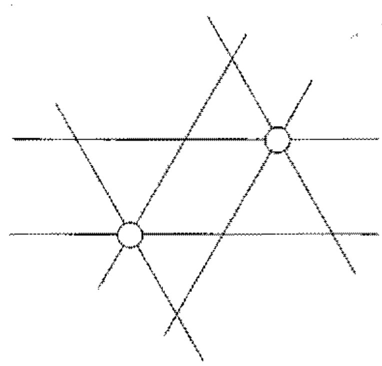

Figure 8.2 If $ir(H_5) = 2$, then there are at most six pn-destroyers

are 17 unoccupied cells. All of them must be either pn-destroyers or pn-less. This is impossible, because each queen's private neighbours can be destroyed from at most 6 cells, and the number of no-pns is at most 4, *i.e.* the total number of pn-destroyers or pn-less cells is $6 + 6 + 4 < 17$. □

Lemma 8.6 *If* $ir\,(H_7) = 2$, *then*:

(a) *There are at least eight open cells.*

(b) *Each queen has at least four private neighbours.*

(c) *There are no pn-destroyers.*

Proof. (a) The two queens are adjacent (Lemma 8.2). Therefore if the BER is the edge, there are at least eight open cells in the BER. If the BER is not the edge, there are even more open cells in the BER (see Lemma 7.3).

(b) Because the queens are adjacent, both queens have two lines that can have private neighbours. It is easy to see that each line has at least two private neighbours.

(c) The private neighbours lie on two lines (at least two on each line). No cell on one of these lines is a pn-destroyer, because then it cannot destroy more than one private neighbour on the other line. A queen not on one of these lines can destroy at most two private neighbours on each line. Thus there must be a queen with only four private

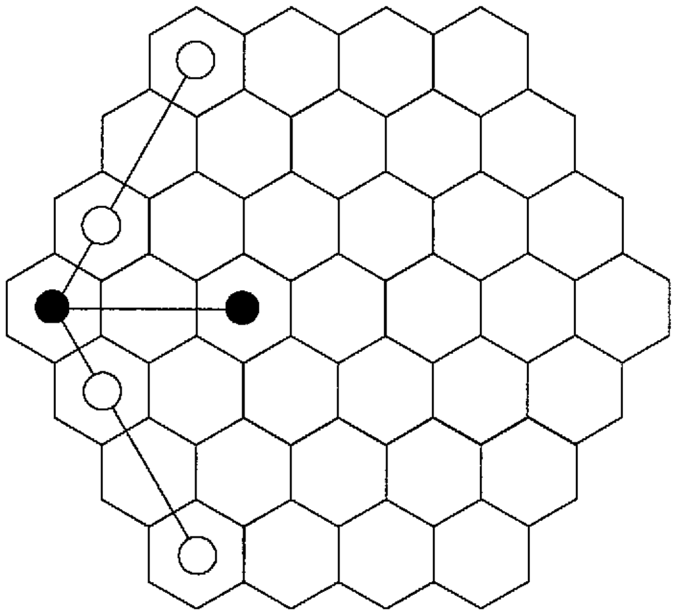

Figure 8.3 If $ir\ (H_7) = 2$, there are no pn-destroyers

neighbours, and such a queen can only be in a corner (see Figure 8.3). It is now easy to see that there are also no pn-destroyers. □

Theorem 8.7 $ir(H_7) = 3$.

Proof. We know $ir \leq \gamma = 3$. Suppose $ir(H_7) = 2$ and consider an irredundant set of H_7 with two queens. By Lemma 8.6 there are no pn-destroyers and there are at least eight open cells. But all open cells must be pn-destroyers (Lemma 8.1), which is a contradiction. Thus the theorem follows. □

8.2 Irredundance in the queens graph

The values for $ir(Q_n)$ for $n = 1, 2, 3$ and 4 are easy to determine by inspection. We now determine $ir\ (Q_5)$.

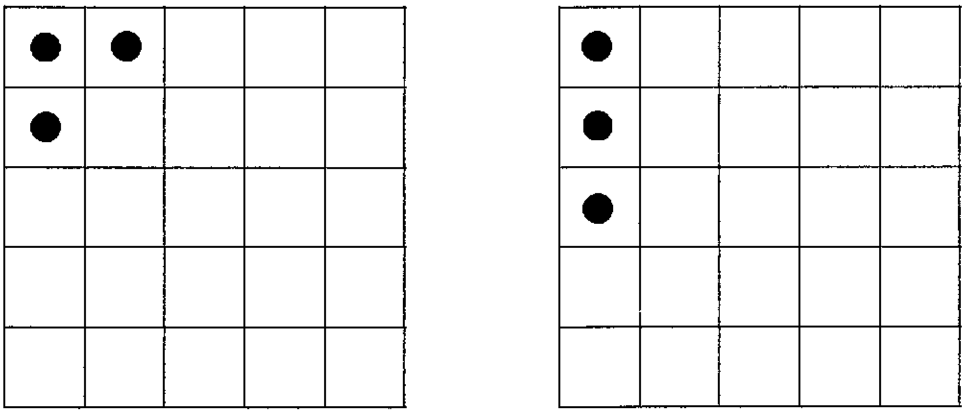

Figure 8.4 If $ir\,(Q_5) = 2$, there are at least seven pn-less squares

Lemma 8.8 *If* $ir\,(Q_5) = 2$, *then*:

(a) *The number of pn-less squares is at most seven.*

(b) *Each queen has at least two private neighbours.*

(c) *Each queen's private neighbours can be destroyed from at most seven cells.*

Proof. (a) Consider an irredundant set of two queens on Q_5. Again, the pn-less squares are the unoccupied squares that are not in line with the open squares. There are at least three open squares (Lemma 8.3). The minimum number of rows and columns three squares can cover, are four. This leaves at most 9 squares, *i.e.* 7 unoccupied squares (see Figure 8.4).

(b) If the two queens are in the same row (column), each queen has at least two private neighbours in its column (row). If the two queens are on the same diagonal, each queen has at least two private neighbours in its row and two in its column.

(c) If a queen has only two private neighbours, the queen must be on the edge (see Figure 8.5). There are then at most four pn-destroyers in line with the private neighbours and at most three pn-destroyers not in line with the private neighbours. It is easy to see that in all other cases there are fewer pn-destroyers. □

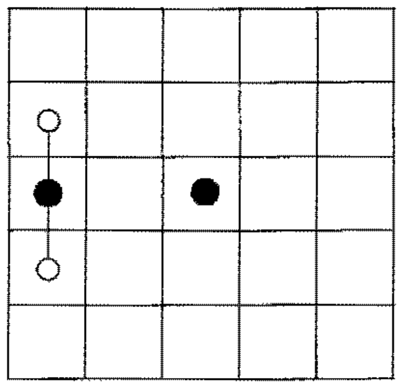

Figure 8.5 If $ir\ (Q_5) = 2$, there are at most seven pn-destroyers

Theorem 8.9 $ir(Q_5) = 3$.

Proof. Suppose $ir\ (Q_5) = 2$ and consider a maximal irredundant set of Q_5 consisting of two queens. There are 23 unoccupied squares. All of them must be either pn-destroyers or pn-less. This is impossible, because each queen's private neighbours can be destroyed from at most seven cells, and the number of no-pns is at most seven, *i.e.* the total number of pn-destroyers or pn-less cells is $7 + 7 + 7 < 23$. □

In the final two results of this thesis we determine $ir\ (Q_6)$.

Lemma 8.10 *If* $ir\ (Q_6) = 2$, *then*:

(a) *The number of pn-less squares is at most* 14.

(b) *Each queen has at least four private neighbours.*

(c) *Each queen's private neighbours can be destroyed from at most five cells.*

Proof. (a) Again, the pn-less squares are the unoccupied squares that are not in line with the open squares. There are at least three open squares (Lemma 8.3). The minimum number of rows and columns three squares can cover, is four. This leaves at most 16 squares, *i.e.* 14 unoccupied squares

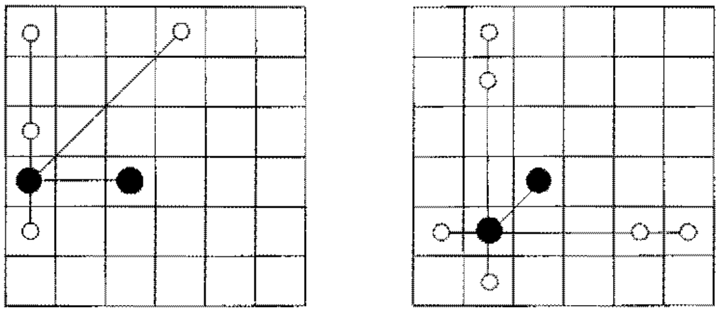

Figure 8.6 If $ir\,(Q_6) = 2$, then each queen has at least four private neighbours

(b) Consider any queen. There are five possible private neighbours in the row (column) of the queen. Depending on the radius of the queen, there are 5, 7 or 9 squares in the same diagonals as the queen which can be private neighbours. If the two queens are in the same row (column), it is clear (see Figure 8.6) that there must be at least three private neighbours per row (column) and one on any of the diagonals that intersect the queen. If the queens are on the same diagonal, there must be at least three private neighbours per row and per column (see Figure 8.6).

(c) Consider any queen. There are at least three private neighbours in the same row (column) as the queen plus at least one other private neighbour. There are at most three squares in the row which will destroy these private neighbours. and at most two other squares. □

Theorem 8.11 $ir(Q_6) = 3$.

Proof. Suppose $ir\,(Q_6) = 2$ and consider a maximal irredundant set of Q_6 consisting of two queens. There are 34 unoccupied cells. All of them must be either pn-destroyers or pn-less. This is impossible, because each queen's private neighbours can be destroyed from at most five cells, and the number of pn-less squares is at most 14, *i.e.* the total number of pn-destroyers or pn-less cells is $14 + 5 + 5 < 34$. □

Using the same method as in the proofs of the above theorems, it can be shown that $ir\ (Q_7) = 4$. However, the proof in this case is much more technical and is therefore omitted.

Appendix A
Dominating sets

For all sets the lower left corner square has coordinates $(1, 1)$.

n = 5

1. 11 13 43	2. 11 15 43	3. 11 22 44	4. 11 23 32	5. 11 23 53
6. 11 24 42	7. 11 25 43	8. 11 25 52	9. 11 33 34	10. 11 33 55
11. 11 34 43	12. 11 24 43	13. 11 45 54	14. 11 35 53	15. 11 34 53
16. 11 34 44	17. 11 24 54	18. 12 14 33	19. 12 14 53	20. 12 21 33
21. 12 21 35	22. 12 21 44	23. 12 23 54	24. 12 33 43	25. 12 33 45
26. 12 33 52	27. 12 34 52	28. 12 35 52	29. 12 42 45	30. 13 23 53
31. 12 33 43	32. 22 23 24	33. 22 24 42	34. 22 24 43	35. 22 33 44
36. 23 32 33	37. 23 33 43			

n = 6

11 35 53

n =7

1. 11 22 46 64	2. 11 27 43 65	3. 11 35 53 77	4. 11 25 54 64
5. 11 35 45 74	6. 11 35 46 63	7. 12 21 37 64	8. 12 26 41 55
9. 31 22 13 56	10. 13 36 41 66	11. 13 36 41 66	11. 14 44 54 64
13. 22 33 44 66			

n = 8 (Please see next page.)

Appendix A

1. 11 12 26 51 65
2. 11 12 46 54 65
3. 11 13 27 52 75
4. 11 13 37 51 75
5. 11 13 37 52 75
6. 11 13 37 53 75
7. 11 13 45 56 82
8. 11 14 17 46 74
9. 11 14 28 52 85
10. 11 14 46 63 78
11. 11 18 24 53 87
12. 11 22 33 57 75
13. 11 22 45 67 83
14. 11 22 46 57 74
15. 11 22 46 64 88
16. 11 23 32 46 85
17. 11 23 32 47 86
18. 11 23 32 48 75
19. 11 23 32 48 84
20. 11 23 32 57 75
21. 11 23 34 42 67
22. 11 23 35 42 54
23. 11 23 35 46 54
24. 11 23 36 62 84
25. 11 23 37 52 66
26. 11 23 37 62 76
27. 11 23 38 42 87
28. 11 23 38 52 86
29. 11 23 46 53 83
30. 11 23 46 54 65
31. 11 23 46 67 74
32. 11 23 47 62 76
33. 11 23 47 63 85
34. 11 23 47 65 82
35. 11 23 48 62 76
36. 11 23 48 63 85
37. 11 24 33 42 67
38. 11 24 35 58 83
39. 11 24 37 53 75
40. 11 24 37 63 76
41. 11 24 38 63 85
42. 11 24 38 63 86
43. 11 24 38 73 85
44. 11 24 42 47 74
45. 11 24 45 56 82
46. 11 24 45 58 82
47. 11 24 47 52 75
48. 11 24 47 52 76
49. 11 24 47 52 77
50. 11 24 47 62 75
51. 11 24 47 62 76
52. 11 24 47 72 75
53. 11 24 47 72 83
54. 11 24 47 72 86
55. 11 24 47 73 82
56. 11 24 48 62 86
57. 11 25 37 64 74
58. 11 25 38 54 64
59. 11 25 43 67 83
60. 11 25 47 64 72
61. 11 26 34 48 83
62. 11 26 34 67 72
63. 11 26 43 58 73
64. 11 26 43 78 84
65. 11 26 47 62 74
66. 11 26 48 62 84
67. 11 26 48 64 82
68. 11 27 33 58 64
69. 11 27 38 43 84
70. 11 27 38 73 82
71. 11 28 43 45 84
72. 11 28 44 66 82
73. 11 28 45 53 54
74. 11 28 45 54 82
75. 11 28 45 54 84
76. 11 28 46 64 82
77. 11 33 44 55 77
78. 11 34 36 64 84
79. 11 34 46 54 65
80. 11 34 46 54 84
81. 11 34 46 57 63
82. 11 34 46 63 76
83. 11 34 46 63 78
84. 11 34 46 63 85
85. 11 34 46 67 73
86. 11 34 47 64 76
87. 11 34 47 73 86
88. 11 34 48 54 84
89. 11 34 48 63 85
90. 11 34 48 63 86
91. 11 34 48 63 87
92. 11 34 48 67 83
93. 11 34 48 73 87
94. 11 35 37 53 73
95. 11 35 37 53 74
96. 11 35 44 67 82
97. 11 36 48 63 84
98. 11 44 45 58 82
99. 11 45 48 54 84
100. 12 13 48 75 81
101. 12 14 21 57 83
102. 12 14 37 61 86
103. 12 15 48 73 81
104. 12 16 38 54 74
105. 12 18 21 55 83
106. 12 21 26 58 73
107. 12 21 33 46 85
108. 12 21 33 47 86
109. 12 21 33 48 75
110. 12 21 33 48 84
111. 12 21 33 57 75
112. 12 21 34 57 83
113. 12 21 35 44 53
114. 12 21 35 67 84
115. 12 21 35 68 74
116. 12 21 35 68 83
117. 12 21 35 74 88

118. 12 21 36 57 84	119. 12 21 36 58 74	120. 12 21 36 63 66
121. 12 21 36 66 83	122. 12 21 37 48 83	123. 12 21 37 64 88
124. 12 21 37 73 88	125. 12 21 38 55 64	126. 12 21 38 55 83
127. 12 21 38 56 74	128. 12 21 38 64 67	129. 12 21 38 66 83
130. 12 21 45 67 83	131. 12 21 45 68 73	132. 12 21 46 55 74
133. 12 21 46 57 83	134. 12 21 46 58 73	135. 12 21 46 64 88
136. 12 21 46 78 83	137. 12 22 26 51 65	138. 12 22 46 51 65
139. 12 22 46 53 83	140. 12 22 48 75 81	141. 12 23 31 44 67
142. 12 23 31 56 88	143. 12 23 35 67 84	144. 12 23 38 66 81
145. 12 23 47 65 81	146. 12 23 48 75 81	147. 12 24 31 43 55
148. 12 24 31 43 88	149. 12 24 31 65 78	150. 12 24 33 41 76
151. 12 24 33 41 85	152. 12 24 35 43 88	153. 12 24 38 63 77
154. 12 24 41 56 77	155. 12 24 47 75 81	156. 12 24 47 75 83
157. 12 24 48 75 81	158. 12 25 31 66 84	159. 12 25 35 74 84
160. 12 25 48 73 81	161. 12 26 31 57 83	162. 12 26 33 51 65
163. 12 26 33 57 63	164. 12 26 42 57 83	165. 12 26 44 51 65
166. 12 27 33 68 74	167. 12 27 41 53 66	168. 12 27 41 68 73
169. 12 27 44 53 65	170. 12 27 44 53 67	171. 12 27 44 61 76
172. 12 27 46 53 83	173. 12 28 33 71 87	174. 12 28 44 66 81
175. 12 28 44 75 81	176. 12 31 46 47 81	177. 12 33 46 72 85
178. 12 33 46 81 85	179. 12 33 48 65 81	180. 12 33 48 75 81
181. 12 34 37 62 85	182. 12 34 46 53 87	183. 12 34 46 55 81
184. 12 34 47 73 86	185. 12 34 48 53 87	186. 12 34 48 73 87
187. 12 34 48 75 81	188. 12 35 42 72 88	189. 12 35 43 54 68
190. 12 35 43 54 88	191. 12 35 44 78 81	192. 12 35 48 72 84
193. 12 35 48 73 81	194. 12 35 48 74 81	195. 12 36 43 58 74
196. 12 36 44 64 84	197. 12 36 44 67 73	198. 12 36 44 68 84
199. 12 36 46 71 83	200. 12 36 46 73 83	201. 12 36 47 64 81
202. 12 37 43 78 84	203. 12 37 44 61 76	204. 12 37 44 61 85
205. 12 37 44 62 85	206. 12 38 43 44 84	207. 12 38 44 51 85
208. 12 38 44 65 84	209. 12 38 44 75 81	210. 12 38 45 71 84
211. 12 42 46 47 81	212. 12 43 44 48 81	213. 12 43 46 78 81
214. 12 44 47 78 81	215. 12 44 48 75 81	216. 12 46 47 53 81
217. 12 46 47 53 83	218. 12 46 47 73 81	219. 13 16 41 64 78
220. 13 21 25 64 78	221. 13 21 27 58 64	222. 13 21 34 57 82
223. 13 21 46 68 74	224. 13 21 46 78 82	225. 13 22 27 52 75
226. 13 22 31 44 67	227. 13 22 31 45 54	228. 13 22 31 45 85
229. 13 22 31 47 74	230. 13 22 31 47 87	231. 13 22 31 48 65
232. 13 22 31 48 76	233. 13 22 31 56 88	234. 13 22 31 57 75

235. 13 22 31 58 85	236. 13 22 31 78 87	237. 13 22 34 41 76
238. 13 22 34 41 85	239. 13 22 37 51 75	240. 13 22 37 76 81
241. 13 22 38 65 81	242. 13 22 38 66 81	243. 13 22 47 71 84
244. 13 22 47 72 84	245. 13 22 47 74 81	246. 13 22 48 61 76
247. 13 24 47 72 88	248. 13 24 48 66 82	249. 13 25 31 64 78
250. 13 25 31 78 84	251. 13 25 32 66 71	252. 13 25 32 78 84
253. 13 25 41 64 78	254. 13 25 44 56 82	255. 13 25 44 58 82
256. 13 25 47 72 84	257. 13 26 41 64 78	258. 13 26 42 64 88
259. 13 27 31 56 84	260. 13 27 31 58 64	261. 13 27 33 52 75
262. 13 27 41 56 84	263. 13 27 41 58 64	264. 13 27 41 64 78
265. 13 27 42 44 67	266. 13 27 42 56 81	267. 13 27 44 52 75
268. 13 27 44 52 76	269. 13 27 44 62 76	270. 13 28 31 54 67
271. 13 28 31 67 74	272. 13 28 32 41 66	273. 13 28 41 44 76
274. 13 28 41 54 66	275. 13 28 44 51 76	276. 13 28 44 61 76
277. 13 28 44 62 86	278. 13 28 45 72 84	279. 13 28 47 72 84
280. 13 31 36 63 88	281. 13 31 37 55 73	282. 13 31 44 55 77
283. 13 31 45 62 78	284. 13 31 47 56 82	285. 13 31 47 58 62
286. 13 31 48 52 67	287. 13 33 37 51 75	288. 13 34 45 71 88
289. 13 34 47 71 88	290. 13 34 48 66 81	291. 13 34 48 73 87
292. 13 35 42 54 68	293. 13 35 44 58 81	294. 13 36 41 64 77
295. 13 36 41 64 78	296. 13 36 41 64 88	297. 13 36 43 58 73
298. 13 36 45 71 82	299. 13 36 47 71 82	300. 13 37 43 78 84
301. 13 37 44 51 75	302. 13 37 44 61 76	303. 13 38 43 66 81
304. 13 38 44 61 85	305. 13 38 44 61 86	306. 13 38 44 71 85
307. 13 38 44 81 85	308. 13 44 47 71 82	309. 13 45 48 71 82
310. 14 16 32 58 72	311. 14 17 41 74 88	312. 14 17 41 77 83
313. 14 18 33 52 76	314. 14 21 36 58 72	315. 14 21 46 58 73
316. 14 21 46 63 78	317. 14 22 35 58 83	318. 14 22 46 65 71
319. 14 22 47 61 75	320. 14 22 48 51 85	321. 14 22 48 62 86
322. 14 23 32 41 66	323. 14 23 32 41 77	324. 14 23 32 57 81
325. 14 23 32 66 81	326. 14 23 33 66 82	327. 14 23 35 72 84
328. 14 23 37 61 86	329. 14 23 37 62 85	330. 14 23 37 62 86
331. 14 23 47 61 85	332. 14 23 48 71 85	333. 14 24 42 77 83
334. 14 25 33 58 82	335. 14 25 41 67 83	336. 14 25 42 77 83
337. 14 26 32 58 72	338. 14 26 32 77 81	339. 14 26 33 67 72
340. 14 26 41 77 83	341. 14 26 42 68 82	342. 14 26 42 77 83
343. 14 26 43 58 71	344. 14 27 43 53 88	345. 14 28 31 61 67
346. 14 28 31 63 67	347. 14 28 32 57 81	348. 14 31 37 58 63
349. 14 31 37 63 78	350. 14 31 46 63 78	351. 14 31 46 67 82

352. 14 31 47 58 62	353. 14 31 48 67 82	354. 14 32 36 58 72
355. 14 32 45 53 68	356. 14 32 45 66 73	357. 14 32 46 58 72
358. 14 33 35 58 81	359. 14 33 37 63 76	360. 14 33 46 63 76
361. 14 33 46 65 72	362. 14 33 46 72 76	363. 14 33 46 72 87
364. 14 33 48 72 76	365. 14 35 41 67 82	366. 14 36 43 52 78
367. 14 36 43 58 72	368. 14 37 41 66 73	369. 14 37 42 58 61
370. 14 37 42 67 81	371. 14 37 44 61 76	372. 14 38 41 66 83
373. 14 38 43 66 81	374. 14 41 44 78 87	375. 14 42 46 68 81
376. 14 43 47 71 88	377. 15 21 34 68 83	378. 15 22 44 58 83
379. 15 22 47 64 71	380. 15 22 47 64 72	381. 15 23 34 67 82
382. 15 23 44 58 81	383. 15 24 42 63 78	384. 15 31 34 67 82
385. 15 32 44 67 82	386. 15 33 44 58 81	387. 15 34 43 57 82
388. 15 34 43 67 82	389. 15 34 43 68 72	390. 15 43 44 58 81
391. 16 21 34 58 72	392. 16 22 44 67 84	393. 16 22 48 62 84
394. 16 23 44 67 71	395. 16 24 32 58 72	396. 16 24 42 67 82
397. 16 24 42 68 82	398. 16 32 34 58 72	399. 16 32 44 58 72
400. 16 32 44 68 84	401. 16 33 43 78 84	402. 16 33 48 61 84
403. 16 33 48 63 81	404. 16 33 48 64 81	405. 16 34 43 55 61
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418. 17 28 33 74 81	419. 17 32 34 57 83	420. 17 32 34 73 88
421. 17 33 43 78 82	422. 17 33 43 78 84	423. 17 33 45 54 64
424. 17 34 42 43 88	425. 17 34 43 57 82	426. 17 34 43 71 88
427. 17 34 43 73 88	428. 17 34 48 54 82	429. 17 35 44 53 71
430. 17 36 44 63 71	431. 17 43 44 78 81	432. 18 21 24 57 82
433. 18 21 24 57 84	434. 18 21 34 57 82	435. 18 22 44 53 67
436. 18 22 44 67 84	437. 18 22 46 62 84	438. 18 22 46 64 81
439. 18 23 42 44 67	440. 18 24 32 53 67	441. 18 24 32 77 83
442. 18 24 42 53 67	443. 18 24 42 66 82	444. 18 24 42 67 73
445. 18 24 42 67 82	446. 18 24 42 77 83	447. 18 27 33 64 74
448. 18 32 34 44 84	449. 18 32 34 56 74	450. 18 34 43 44 84
451. 18 34 43 53 55	452. 18 34 43 53 88	453. 18 34 43 66 81
454. 18 34 44 53 87	455. 18 34 44 54 84	456. 18 34 44 65 84
457. 18 34 44 73 87	458. 18 34 44 75 81	459. 18 34 44 76 84
460. 18 34 44 84 87	461. 18 42 43 44 67	462. 18 43 44 58 64
463. 21 24 26 58 72	464. 21 24 27 56 84	465. 21 24 27 58 84
466. 21 34 45 57 82	467. 21 34 45 57 84	468. 21 34 45 67 83

469. 22 23 26 54 65	470. 22 24 27 43 76	471. 22 24 37 63 76
472. 22 24 38 63 77	473. 22 24 47 63 76	474. 22 24 47 72 86
475. 22 24 48 62 86	476. 22 26 34 57 63	477. 22 26 38 63 84
478. 22 26 43 58 84	479. 22 26 43 68 84	480. 22 26 43 78 84
481. 22 26 44 68 84	482. 22 26 48 54 63	483. 22 26 48 62 84
484. 22 26 48 63 84	485. 22 26 48 64 84	486. 22 27 33 68 74
487. 22 33 44 66 88	488. 22 33 44 68 86	489. 22 34 45 53 65
490. 22 34 45 53 67	491. 22 34 45 53 77	492. 22 34 45 53 78
493. 22 34 45 53 86	494. 22 34 48 53 85	495. 22 34 48 53 87
496. 22 34 48 73 87	497. 22 36 37 63 73	498. 22 44 48 56 83
499. 22 44 48 66 84	500. 23 25 48 71 84	501. 23 27 42 44 67
502. 23 31 48 52 67	503. 23 33 36 62 84	504. 23 33 47 72 86
505. 23 34 45 52 77	506. 23 34 45 67 81	507. 23 34 47 65 81
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511. 23 35 43 73 88	512. 23 35 44 56 81	513. 23 36 42 44 67
514. 23 36 42 77 83	515. 23 36 44 52 65	516. 23 37 43 73 88
517. 23 38 42 51 67	518. 23 38 44 52 85	519. 23 42 46 47 81
520. 23 44 48 56 81	521. 23 44 48 62 86	522. 24 26 34 74 88
523. 24 26 42 62 88	524. 24 26 42 63 88	525. 24 26 42 64 88
526. 24 27 33 52 75	527. 24 27 41 56 84	528. 24 27 42 55 72
529. 24 28 31 61 67	530. 24 28 34 63 77	531. 24 28 42 66 82
532. 24 28 44 62 86	533. 24 33 38 62 85	534. 24 33 45 52 77
535. 24 33 45 62 76	536. 24 33 46 67 72	537. 24 33 47 72 76
538. 24 33 47 72 86	539. 24 33 48 62 86	540. 24 34 35 74 88
541. 24 34 44 54 84	542. 24 34 44 63 85	543. 24 34 44 72 76
544. 24 34 44 72 85	545. 24 34 44 74 88	546. 24 34 44 78 83
547. 24 34 44 78 85	548. 24 34 45 71 88	549. 24 34 47 73 88
550. 24 34 48 71 85	551. 24 35 42 61 78	552. 24 35 43 67 81
553. 24 35 44 72 76	554. 24 36 42 63 77	555. 24 36 42 77 83
556. 24 36 43 52 77	557. 24 36 43 77 82	558. 24 36 44 65 72
559. 24 36 44 67 72	560. 24 36 44 72 76	561. 24 36 44 72 85
562. 24 36 44 72 87	563. 24 36 45 71 82	564. 24 36 47 71 82
565. 24 37 42 56 81	566. 24 37 42 58 61	567. 24 37 44 72 76
568. 24 38 42 51 67	569. 24 38 42 77 83	570. 24 38 43 77 82
571. 24 38 44 62 85	572. 24 38 44 62 86	573. 24 38 44 65 82
574. 24 38 44 66 82	575. 24 38 44 72 76	576. 24 38 44 72 85
577. 24 38 44 72 87	578. 24 44 47 72 86	579. 24 44 48 62 86
580. 25 33 44 58 82	581. 25 33 44 67 81	582. 25 33 44 78 84
583. 25 33 47 64 72	584. 25 34 43 67 81	585. 25 34 43 68 71

586. 25 34 44 46 81	587. 25 34 44 58 83	588. 25 34 46 71 81
589. 25 34 48 71 83	590. 26 33 34 48 83	591. 26 33 43 78 84
592. 26 33 44 67 72	593. 26 33 45 54 62	594. 26 33 45 54 64
595. 26 33 48 62 84	596. 26 33 48 64 82	597. 26 34 43 55 63
598. 26 34 43 57 81	599. 26 34 43 58 71	600. 27 33 42 44 67
601. 27 33 43 78 84	602. 27 35 44 53 72	603. 28 33 34 63 77
604. 28 33 34 74 84	605. 28 33 43 78 84	606. 28 33 44 66 82
607. 28 33 46 64 82	608. 28 34 43 55 82	609. 28 34 48 71 83
610. 28 43 44 45 53	611. 31 34 37 58 61	612. 31 34 37 58 63
613. 31 34 37 63 78	614. 31 34 37 67 81	615. 31 34 37 68 74
616. 31 34 48 67 73	617. 31 34 48 67 82	618. 32 34 37 57 81
619. 33 34 35 46 54	620. 33 34 35 58 81	621. 33 34 37 63 76
622. 33 34 44 78 87	623. 33 34 47 71 88	624. 33 34 48 67 81
625. 33 34 48 73 87	626. 34 35 43 54 88	627. 34 35 44 78 81
628. 34 35 48 72 81	629. 34 37 43 73 88	630. 34 42 46 47 81
631. 34 42 46 67 81	632. 34 43 48 77 84	633. 34 44 46 54 65
634. 34 44 46 54 85	635. 34 44 46 55 81	636. 34 44 48 54 83
637. 34 44 48 73 87	638. 38 44 45 54 83	

n = 9

1. 11 23 37 62 76	2. 11 24 38 63 86	3. 11 24 48 62 86	4. 11 35 58 83 97
5. 11 35 59 73 97	6. 11 45 59 84 98	7. 12 29 55 81 98	8. 13 22 31 58 85
9. 13 28 44 62 86	10. 13 38 44 61 86	11. 13 38 55 61 86	12. 13 39 55 71 97
13. 13 46 54 65 79	14. 14 23 32 41 77	15. 14 49 55 61 96	16. 22 34 48 73 87
17. 22 36 48 67 73	18. 22 44 55 66 88	19. 23 38 55 72 87	20. 24 48 55 62 86
21. 34 47 55 63 76			

n = 10

13 39 55 71 97

n = 11

(2,4) (4,10) (6,6) (8,2) (10,8)

Appendix B

Programmes

Programme 1

```
program domination;        { n=49  (4,6)-dom sets;
assimmetric }
Uses CRT,Printer;
var
 dominating                                  : boolean;
 m1,m2,m3,m4                                 : integer;
 q1,q2,q3,q4,q5,q6,q7,q8,q9,q10,q11,
 q12,q13,q14,q15,q16,q17,q18,q19,q20,q21 : integer;
 cc,m,n,c,k,t,i,a,Qx,Qy,s,d,u,v,w,
 diag,sdiag                                  : integer;
 QueenX,QueenY                    : array[1..25] of integer;
 soekX,soekY                                       : array[-
12..12] of integer;
 SoekD                            : array[1..22] of integer;
 openx,openy,openD                                     : ar-
ray[1..4] of integer;
 OnDiag                                                : ar-
ray[1..4,1..4] of integer;
  f                                            : text;
```

Appendix B

```
Function QueenOK(a,s,Qx : integer) : boolean;
Begin
QueenOK := true;
Qy := s-Qx;
if a=1 then
  begin
    QueenX[1] := Qx;
    QueenY[1] := Qy;
    exit;
  end;
for c := 1 to a-1 do
  if (Qx = QueenX[c])
  or (Qy = QueenY[c])
  or ((Qy-Qx) = (QueenY[c]-QueenX[c])) then
    begin
      QueenOK := false;
      exit;
    end;
QueenX[a] := Qx;
QueenY[a] := Qy;
End;

Procedure DetermineOpenLines;
Begin
```

```
for c := -k to k do
  begin
   SoekX[c] := 0;
   SoekY[c] := 0;
  end;
for c := 1 to 2*k-3 do
  begin
   SoekX[queenX[c]] := 1;
   SoekY[queenY[c]] := 1;
  end;
t := 1;
for c := -k to k do
  if (SoekX[c]=0) then
    begin
     OpenX[t] := c;
     t := t+1;
    end;
t := 1;
for c := -k to k do
  if (SoekY[c]=0) then
    begin
     OpenY[t] := c;
     t := t+1;
    end;
for c := 1 to 2*k-i do SoekD[c] := 0;
for c := 1 to 2*k-i-2 do
```

```
  begin
   diag := QueenY[c] - QueenX[c];
   d := trunc((diag + 2*k-i+1)/2);
   SoekD[d] := 1;
  end;
t := 1;
for c := 1 to 2*k-i do
  if (SoekD[c]=0) then
    begin
     OpenD[t] := c*2-2*k+i-1;
     t := t+1;
    end;
for c := 1 to i+1 do SoekD[c] := 0;
for c := 2*k-i-1 to m-4 do
  begin
   diag := QueenY[c] - QueenX[c];
   d := trunc((diag + i+2)/2);
   SoekD[d] := 1;
  end;
t := 3;
for c := 1 to i+1 do
  if (SoekD[c]=0) then
    begin
     OpenD[t] := c*2-i-2;
     t := t+1;
    end;
```

```
   for u := 1 to 4 do
    for v := 1 to 4 do
      OnDiag[v,u] := 0;
  for u := 1 to 4 do                   {y}
    for v := 1 to 4 do                 {x}
      for w := 1 to 4 do
        if OpenY[u]-OpenX[v] = OpenD[w] then
          begin
           OnDiag[v,u] := w;
           sdiag := OpenY[u] + OpenX[v];
           if abs(sdiag) <= 2*k-i-3 then
             begin
               c := 2*k-i-2 - abs(sdiag);
               if sdiag<0 then c := c+1;
              if (i-abs(sdiag)) mod 2 = 0 then c := c+i-1;
          if (QueenY[c] + QueenX[c]) = (OpenY[u] + OpenX[v]) then
             if Queenx[c] > OpenX[v] then OnDiag[v,u] :=0;
             end;
          end;
End;

BEGIN
{Assign(f,'m\49asop46.pas');}
{Rewrite(f);                          }
clrscr;
cc := 0;
```

Appendix B

```
k := 12;
i := 4;
n := 4*k + 1;            { size of board }
m := 2*k + 1;            { no of Queens   }
writeln(n,'x',n,'    (',i,',',i+2,')');
    for q1 := 5 to 8 do
      if QueenOK(1,17,q1) then
      for q2 := q1-2*k+i+4 to -q1 do          {s=-17}
        if QueenOK(2,-17,q2) then
        for q3 := 3 to 12 do                  {s=+15}
          if QueenOK(3,15,q3) then
          for q4 := -12 to -3 do              {s=-15}
            if QueenOK(4,-15,q4) then
            for q5 := 1 to 12 do              {s=+13}
              if QueenOK(5,13,q5) then
              for q6 := -12 to -1 do          {s=-13}
                if QueenOK(6,-13,q6) then
                for q7 := -1 to 12 do         {s=+11}
                  if QueenOK(7,11,q7) then
                  for q8 := -12 to 1 do       {s=-11}
                    if QueenOK(8,-11,q8) then
                   for q9 := -3 to 12 do            {s=9}
                      if QueenOK(9,9,q9) then
                    for q10 := -12 to 3 do         {s=-9}
                        if QueenOK(10,-9,q10) then
                       for q11 := -5 to 12 do       {s=7}
```

```
                    if QueenOK(11,7,q11) then
 for q12 := -12 to 5 do                                  {s=-7}
   if QueenOK(12,-7,q12) then
   for q13 := -7 to 12 do                                 {s=5}
     if QueenOK(13,5,q13) then
     for q14 := -12 to 7 do                              {s=-5}
       if QueenOK(14,-5,q14) then
       for q15 := -8 to 11 do                             {s=3}
         if QueenOK(15,3,q15) then
         for q16 := -11 to 8 do                          {s=-3}
           if QueenOK(16,-3,q16) then
           for q17 := -9 to 10 do                         {s=1}
             if QueenOK(17,1,q17) then
             for q18 := -10 to 9 do                      {s=-1}
               if QueenOK(18,-1,q18) then
               for q19 := -1 to 3 do                      {s=2}
                 if QueenOK(19,2,q19) then
                 for q20 := -3 to 1 do                   {s=-2}
                   if QueenOK(20,-2,q20) then
                   for q21 := -2 to 2 do                  {s=0}
                     if QueenOK(21,0,q21) then
begin
DetermineOpenLines;
for m1 := 1 to 4 do
 if OnDiag[m1,1]>0 then
 for m2 := 1 to 4 do
```

```
if (OnDiag[m2,2]>0) and (m2<>m1)
and (Ondiag[m1,1]<>OnDiag[m2,2]) then
for m3 := 1 to 4 do
  if (OnDiag[m3,3]>0) and (m3<>m1) and (m3<>m2)
  and (Ondiag[m3,3]<>OnDiag[m1,1])
  and (OnDiag[m3,3]<>OnDiag[m2,2]) then
  for m4 := 1 to 4 do
    if (OnDiag[m4,4]>0) and (m4<>m1)
    and (m4<>m2) and (m4<>m3)
    and (Ondiag[m4,4]<>OnDiag[m1,1])
    and (OnDiag[m4,4]<>OnDiag[m2,2])
    and (Ondiag[m4,4]<>OnDiag[m3,3]) then
      begin
        QueenX[m-3] := OpenX[m1];
        QueenY[m-3] := OpenY[1];
        QueenX[m-2] := OpenX[m2];
        QueenY[m-2] := OpenY[2];
        QueenX[m-1] := OpenX[m3];
        QueenY[m-1] := OpenY[3];
        QueenX[m]   := OpenX[m4];
        QueenY[m]   := OpenY[4];
        cc := cc + 1;
        write(cc,'. ');
    for c := 1 to m do write('(',queenX[c],',',queenY[c],')');
    for c := 1 to m do write(f,queenX[c]:4,queenY[c]:4);
        writeln(f);
```

```
         end;
    end;
    close(f);
    writeln('number=',cc,'  That is all');
    readln;
END.
```

Appendix B

Programme 2

```
Program RyeKolomme31;  {generate all comb of}
                       {rows for 29x29 subbord}
Uses CRT;
Var
y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12,y13,y14,y15,
x,y,q,k,m,n,gamma,Aantopl,rand,d,qns,
c,halfver,halfsom,l,a,ax,bx     : integer;
queenY,queenX                   : array[1..13] of integer;
rypatroon                      : array[-13..13] of integer;
count                                    : longint;
f                                           : text;

Function RyOK(a,bx : integer) : Boolean;
Begin
RyOK := true;
QueenX[a] := bx;
if a=1 then exit;
for c := -l+1 to bx do rypatroon[c] := 0;
for c := 1 to a do rypatroon[QueenX[c]] := 1;
for c := 1 to a-1 do if (((QueenX[c]+bx) mod 2) = 0) then
  begin
    ax := QueenX[c];
    halfsom := trunc((ax+bx)/2);
    Halfver := trunc((bx-ax)/2);
    if (rypatroon[halfsom]=0)          {Theorem 5.5}
```

```
    and (rypatroon[halfver-1]=0) then
      begin
        RyOK := false;
        exit;
      end;
  end;
if a=qns then
  begin
    for c := bx+1 to l-1 do rypatroon[c] := 0;
    for c := 1 to a-1 do
      for d := c+1 to a do
        if (((QueenX[c]+QueenX[d]) mod 2) = 0) then
        begin
          ax := QueenX[c];
          bx := QueenX[d];
          halfsom := trunc((ax+bx)/2);
          halfver := trunc((bx-ax)/2);
         if (rypatroon[halfsom]=0)          {Theorem 5.5}
          and ((rypatroon[halfver-1]=0)
          or (rypatroon[l-halfver]=0)) then
            begin
              RyOK := false;
              exit;
            end;
         if (rypatroon[-halfsom]=0)         {Theorem 5.6}
          and ((rypatroon[halfver-1]=0)
```

```
            or (rypatroon[l-halfver}=0))
          and ((rypatroon[ax]=1) or (rypatroon[bx]=1)) then
              begin
                RyOK := false;
                exit;
              end;
          end;
  end;
End;

BEGIN
clrscr;
k := 7;
gamma := 2*k+1;
count := 0;
l := 14;              {sub-board = 2l+1}
qns := 2*l-2*k-1;      {number of rows with qns cross-
ing subboard}
writeln('board: ',4*k+3);
writeln('total nuber of queens:',gamma);
writeln('sub-board:',2*l+1);
writeln('number of rows with queens crossing subboard:',qns);

for y1 := -13 to -6 do     {most rows in lower half of board}
  if RyOK(1,y1) then
  for y2 := y1+1 to -5 do
```

```
    if RyOK(2,y2) then
    for y3 := y2+1 to -4 do
      if RyOK(3,y3) then
      for y4 := y3+1 to -3 do
        if RyOK(4,y4) then
        for y5 := y4+1 to -2 do
          if RyOK(5,y5) then
          for y6 := y5+1 to -1 do
            if RyOK(6,y6) then
            for y7 := y6+1 to 0 do
              if RyOK(7,y7) then
              for y8 := y7+1 to 8 do
if RyOK(8,y8) then
for y9 := y8+1 to 9 do
  if RyOK(9,y9) then
  for y10 := y9+1 to 10 do
  if RyOK(10,y10) then
    for y11 := y10+1 to 11 do
    if RyOK(11,y11) then
      for y12 := y11+1 to 12 do
        if RyOK(12,y12) then
        for y13 := y12+1 to 13 do
          if RyOK(13,y13) then

begin
count := count+1;
```

```
write(count,'. ');
for c := 1 to 13 do write(QueenX[c]:4);
writeln;
for c := -13 to 13 do write(rypatroon[c]);
writeln;
end;

writeln('Number of patterns = ',count,'.  The End.');
readln;

END.
```

Appendix B

Programme 3

```
Program DominationSq31;    {search edge dominating sets }
                                {for 29x29 subbord (senter)}
Uses CRT;
Var
x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,
y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,
by3,by2,bx1,bx0,
d,s,a,cc,c,k,gamma,Qy,Qx,ry,op,q,j,buiteq,teken,kol,qns,
randY,randX,beginQ,EndQ          : integer;
queenY,queenX                     : array[-3..11] of integer;
Beset                              : array[1..13] of integer;
Uit                                 : array[1..2] of integer;
count                                            : longint;
f                                                   : text;

Function RandOKb(a,Qx,Qy : integer) : boolean;
Begin
randOKb := true;
kol := abs(Qx);
ry := abs(Qy);
if (kol=1) or (kol=2) or (kol=4) or (kol=5)   {empty lines}
or (kol=7) or (kol=8) or (kol=11) then
  begin
    randOKb := false;
    exit;
```

```
  end;
if (ry=1) or (ry=2) or (ry=4) or (ry=5)
or (ry=7) or (ry=8) or (ry=11) then
  begin
    RandOKb := false;
    exit;
  end;
if a=1-buiteq then
         begin
          QueenX[a] := Qx;
          QueenY[a] := Qy;
          exit;
         end;
QueenY[a] := Qy;
QueenX[a] := Qx;
for c := 1-buiteq to a-1 do  {check for queens on same line}
  if (Qx = QueenX[c]) or (Qy = QueenY[c])
  or ((Qy+Qx) = (QueenY[c] + QueenX[c]))
  or ((Qy-Qx) = (QueenY[c] - QueenX[c])) then
    begin
      RandOKb := false;
      exit;
    end;
randY := -j;
for randX := 1-j to j-1 do
  begin
```

```
  cc := 0;
  for c := 1-buiteq to a do
    begin
     if queenX[c] = RandX then cc := cc+1;
    if (queenX[c]+queenY[c]) = (randX+randY) then cc := cc+1;
    if (queenX[c]-queenY[c]) = (randX-randY) then cc := cc+1;
     if cc>1 then
       begin
        randOKb := false;
        exit;
       end;
    end;
  end;
randY := j;
for randX := 1-j to j-1 do
  begin
  cc := 0;
  for c := 1-buiteq to a do
    begin
     if queenX[c] = RandX then cc := cc+1;
    if (queenX[c]+queenY[c]) = (randX+randY) then cc := cc+1;
    if (queenX[c]-queenY[c]) = (randX-randY) then cc := cc+1;
     if cc>1 then
       begin
        randOKb := false;
        exit;
```

```
        end;
     end;
   end;
randX := j;
for randY := 1-j to j-1 do
   begin
   cc := 0;
   for c := 1-buiteq to a do
     begin
      if queenY[c] = RandY then cc := cc+1;
     if (queenX[c]+queenY[c]) = (randX+randY) then cc := cc+1;
     if (queenX[c]-queenY[c]) = (randX-randY) then cc := cc+1;
      if cc>1 then
        begin
         randOKb := false;
         exit;
        end;
     end;
   end;
randX := -j;
for randY := 1-j to j-1 do
   begin
   cc := 0;
   for c := 1-buiteq to a do
     begin
      if queenY[c] = RandY then cc := cc+1;
```

```
    if (queenX[c]+queenY[c]) = (randX+randY) then cc := cc+1;
    if (queenX[c]-queenY[c]) = (randX-randY) then cc := cc+1;
     if cc>1 then
       begin
        randOKb := false;
        exit;
       end;
     end;
  end;
End;

Function RyOK(a,Qy : integer): boolean;
Begin
RyOK := true;
for c := 1-buiteq to 0 do
  if (Qy = QueenY[c])
  or ((QueenY[c] + QueenX[c]) = (Qy-j))
  or ((QueenY[c] - QueenX[c]) = (Qy+j))
  or ((QueenY[c] + QueenX[c]) = (Qy+j))
  or ((QueenY[c] - QueenX[c]) = (Qy-j)) then
    begin
      RyOK := false;
      exit;
    end;
for c := 1 to a-1 do
  if ((QueenY[c]+QueenX[c]) = (Qy-j))
```

```
  or ((QueenY[c]-QueenX[c]) = (Qy-j)) then
    begin
      RyOK := false;
      exit;
    end;
QueenY[a] := Qy;
End;

Function RandOK(a,Qx : integer) : boolean;
Begin
randOK := true;
Qy := QueenY[a];
QueenX[a] := Qx;
kol := abs(Qx);
if (kol=1) or (kol=2) or (kol=4) or (kol=5) or (kol=7)
           or (kol=8) or (kol=11) then
  begin
    RandOK := false;
    exit;
  end;
for c := 1-buiteq to a-1 do {check for queens in the same line}
  if (Qx = QueenX[c])
  or ((Qy+Qx) = (QueenY[c] + QueenX[c]))
  or ((Qy-Qx) = (QueenY[c] - QueenX[c])) then
    begin
      RandOK := false;
```

```
        exit;
      end;
  for c := 1-buiteq to a-1 do      {edge squares in the same col}
      if ((QueenY[c] - QueenX[c]) = (-j-Qx))
      or ((QueenY[c] + QueenX[c]) = (-j+Qx))
      or ((QueenY[c] + QueenX[c]) = (+j+Qx))
      or ((QueenY[c] - QueenX{c}) = (+j-Qx)) then
        begin
          RandOK := false;
          exit;
        end;
  d := abs(Qy - Qx);               {edge squares in the same d-
  diag}
  if (Qy > Qx) then teken := 1 else teken := -1;
  for c := 1-buiteq to a-1 do
    if (abs(QueenY[c] + QueenX[c]) = (2*j - d))
    or (QueenY[c] = (j - d)*teken)
    or (QueenX[c] = (d - j)*teken) then
        begin
          RandOK := false;
          exit;
        end;
  s := abs(Qy + Qx);               {edge squares in the same s-
  diag}
  if (Qy+Qx > 0) then teken := 1 else teken := -1;
  for c := 1-buiteq to a-1 do
```

```
  if (abs(QueenY[c] - QueenX[c]) = (2*j - s))
  or (QueenY[c] = (s - j)*teken)
  or (QueenX[c] = (s - j)*teken) then
      begin
        RandOK := false;
        exit;
      end;
End;

Function Ryeuitgesorteer : boolean;
Begin
Ryeuitgesorteer := true;
cc := 1;
for c := 1 to qns do
  if (Beset[c]=by2) or (Beset[c]=by3) then
    begin
      uit[cc] := c;
      cc := cc+1;
    end;
for c := 1 to uit[1]-1 do QueenY[c] := Beset[c];
for c := uit[1] to uit[2]-1 do QueenY[c] := Beset[c+1];
for c := uit[2] to qns do QueenY[c] := Beset[c+2];
End;

BEGIN
clrscr;
```

```
k := 7;
gamma := 2*k + 1;
j := 14;                    {2j+1 x 2j+1 sub-board}
qns := gamma-4*k+2*j-2;      {occupied rows cutting sub-
board}
buiteq := 8*k-4*j+4;                {qns outside sub-board}
writeln(4*k+3,' x ',4*k+3,' board ');
writeln(2*j+1,' x ',2*j+1,' sub-board');
writeln(buiteq,' queens outside sub-board');
writeln(gamma-buiteq,' queens inside subbord');
writeln(qns,' occupied rows cutting sub-board');
count := 0;
Beset[1] := -13;        Beset[13] := 13;
Beset[2] := -12;        Beset[12] := 12;
Beset[3] := -10;        Beset[11] := 10;
Beset[4] := -9;         Beset[10] := 9;
Beset[5] := -6;         Beset[9] := 6;
Beset[6] := -3;         Beset[8] := 3;
Beset[7] := 0;
for by3 := -11 to 11 do            {queens outside}
  if randOKb(-3,-16,by3) then
  for by2 := -12 to 12 do
    if randOKb(-2,-15,by2) then
    for bx1 := -11 to 11 do
      if randOKb(-1,bx1,-16) then
      for bx0 := -12 to 12 do
```

```
      if randOKb(0,bx0,-15) then if ryeuitgesorteer then
  for x1 := 1-j to j-1 do            {queens inside}
    if RandOK(1,x1) then
      for x2 := 1-j to j-1 do
        if RandOK(2,x2) then
           for x3 := 1-j to j-1 do
            if RandOK(3,x3) then
              for x4 := 1-j to j-1 do
                if RandOK(4,x4) then
                  for x5 := 1-j to j-1 do
                    if RandOK(5,x5) then
              for x6 := 1-j to j-1 do
                if RandOK(6,x6) then
                  for x7 := 1-j to j-1 do
                    if RandOK(7,x7) then
              for x8 := 1-j to j-1 do
                if RandOK(8,x8) then
                  for x9 := 1-j to j-1 do
                    if RandOK(9,x9) then
              for x10 := 1-j to j-1 do
                if RandOK(10,x10) then
                  for x11 := 1-j to j-1 do
                    if RandOK(11,x11) then
begin
   count := count+1;
   write(count,'. ');
```

```
   for c := 1-buiteq to gamma-buiteq do
     write(queenX[c]:4,queenY[c]:3);
   writeln;
end;
writeln('Total= ',count);
readln;
END.
```

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