ON FACTORIZATION STRUCTURES, DENSENESS, SEPARATION AND RELATIVELY COMPACT OBJECTS

by H. J. SIWEYA

MASTER OF SCIENCE

MATHEMATICS

Supervisor: Professor I.W. Alderton

SUMMARY

We define morphism (E, M)-structures in an abstract category, develop their basic properties and present some examples. We also consider the existence of such factorization structures, and find conditions under which they can be extended to factorization structures for certain classes of sources.

There is a Galois correspondence between the collection of all subclasses of X-morphisms and the collection of all subclasses of X-objects. A-epimorphisms diagonalize over A-regular morphisms. Given an (E, M)-factorization structure on a finitely complete category, E-separated objects are those for which diagonal morphisms lie in M. Other characterizations of E-separated objects are given.

We give a bijective correspondence between the class of all (E, M)-factorization structures with M contained in the class of all X-embeddings and the class of all strong limit operators.

We study M-preserving morphisms, M-perfect morphisms and M-compact objects in a morphism (E, M)-hereditary construct, and prove some of their properties which are analogous to the topological ones.

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Signature: 

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HLENGANI JAMES SIWEYA

submitted in part fulfilment for the requirements

for the degree of

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in the subject

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at the

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Supervisor: Professor I.W. Alderton

Date: APRIL 1994

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In Chapter 1 (FACTORIZATION SYSTEMS FOR MORPHISMS, SOURCES AND SINKS), we characterize ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-factorization structures for ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-categories. It is shown that the classes $\underline{\mathbf{E}}$ and $\underline{\mathbf{M}}$ of $\underline{\mathbf{X}}$-morphisms determine each other through the unique diagonalization property or equivalently, that $\underline{\mathbf{E}}$ and $\underline{\mathbf{M}}$ are duals of each other. We also show that $\underline{\mathbf{E}} \subseteq \mathrm{Epi}(\underline{\mathbf{X}})$ if and only if the diagonals lie in $\underline{\mathbf{M}}$. We show that the factorization structures in a category $\underline{\mathbf{X}}$ which can be extended to factorization structures ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}'$) for set-indexed sources are precisely those for which $\underline{\mathbf{X}}$ has products. Swell epimorphisms are discussed and, on a suitable category, are shown to be the extremal epimorphisms. We also discuss (in detail) examples of ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$) -factorization structures on the categories $\underline{\mathbf{Set}}$, $\underline{\mathbf{Grp}}$ and $\underline{\mathbf{Top}}$ (section 1.2).

Chapter 2 (GALOIS CORRESPONDENCE, $\underline{\mathbf{E}}$-SEPARATED OBJECTS AND $\underline{\mathbf{A}}$-EPIMORPHISMS) concerns itself with the *Pumplün–Röhrl* ([PR]) *Galois Correspondence*. It is shown that diagonals belonging to $\underline{\mathbf{M}}$ denote the objects that are $\underline{\mathbf{E}}$-separated in a category $\underline{\mathbf{X}}$ relative to an ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$) -factorization structure on $\underline{\mathbf{X}}$ (Theorem 2.4.1); that $\underline{\mathbf{A}}$-epimorphisms satisfy Bousfield's Characterization Theorem (Theorem 1.4.1) and that on a suitable category $\underline{\mathbf{X}}$, there is a class $\underline{\mathbf{M}}'$ such that ($\underline{\mathbf{A}}$-epi, $\underline{\mathbf{M}}'$) is a factorization structure on $\underline{\mathbf{X}}$ (Proposition 2.5.3); and, finally, that if $\underline{\mathbf{X}}$ has an (epi, mono-source)-factorization for sources, the $\underline{\mathbf{E}}$-separated objects form a (swell epi)-reflective subcategory of $\underline{\mathbf{X}}$ (Proposition 2.5.5), and further, that the class $\underline{\mathbf{E}}$-Sep is $\underline{\mathbf{M}}$-hereditary (Proposition 2.5.7). Examples of $\underline{\mathbf{E}}$-separated objects and $\underline{\mathbf{A}}$-epimorphisms are also provided.

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Chapter 4 ($\underline{\mathbf{M}}$-PERFECT MORPHISMS AND RELATIVELY COMPACT OBJECTS) is devoted to the concepts of $\underline{\mathbf{M}}$-preserving morphisms, $\underline{\mathbf{M}}$-perfect morphisms and $\underline{\mathbf{M}}$-compact objects in a morphism ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-category which is a hereditary construct. We add properties of $\underline{\mathbf{M}}$-preserving morphisms and $\underline{\mathbf{M}}$-perfect morphisms which were not discussed in [HSS] (for example, Remarks 4.1.3, Propositions 4.1.4, 4.1.5 and Lemma 4.1.7). We also show (Example 4.2.2) that for the (dense, closed embedding)-factorization structure on $\underline{\mathbf{Top}}$, $\underline{\mathbf{M}}$-compactness coincides with (topological) compactness. It is shown that $\underline{\mathbf{M}}$-compact objects are both $\underline{\mathbf{M}}$-perfect hereditary and $\underline{\mathbf{M}}$-hereditary (Proposition 4.2.5). We also establish results that are analogous to the topological ones; for instance, that a compact subspace of a $\underline{\mathbf{Top}}_2$-space is closed (Corollary 4.2.8). The rest of the chapter is devoted to the relationship between $\underline{\mathbf{M}}$-compact objects and $\underline{\mathbf{M}}$-perfect morphisms. At the end of this chapter, we indicate other approaches (by some authorities) to categorical compactness.

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In this dissertation, Mor(**X**), Mono(**X**), Epi(**X**) and Iso(**X**) shall denote the classes of morphisms, monomorphisms, epimorphisms and isomorphisms, respectively, in a category **X**. Throughout the dissertation, unless stated, the subcategories are assumed to be *isomorphism-closed*.

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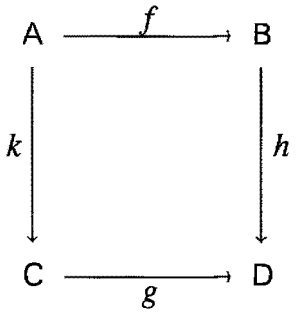
CHAPTER 1

FACTORIZATION SYSTEMS FOR MORPHISMS, SOURCES AND SINKS

1.1 DEFINITIONS

Definition and Notation 1.1.1 ([BO, p. 208])

An $\underline{X}$-morphism $A \xrightarrow{f} B$ is said to have the *unique left lifting property* (the ULLP) for an $\underline{X}$-morphism $C \xrightarrow{g} D$ if, for each commutative square in $\underline{X}$



there exists a unique (diagonal) $\underline{X}$-morphism $B \xrightarrow{d} C$ such that $d \circ f = k$ and $g \circ d = h$. (If f has the ULLP for g, we also say that g has the *unique right lifting property* (the URLP) for f.) We shall denote the fact that f has the ULLP for g by $f \downarrow g$.

Definition 1.1.2

Let $\underline{X}$ be any category. We define $\underline{E}$ and $\underline{M}$ to be classes of $\underline{X}$-morphisms which are closed under composition with isomorphisms in the following sense (provided that these compositions make sense):

(a) If $e \in \underline{E}$ and $h \in \mathrm{Iso}(\underline{X})$, then $h \circ e \in \underline{E}$.

(b) If $m \in \underline{M}$ and $h \in \mathrm{Iso}(\underline{X})$, then $m \circ h \in \underline{M}$.

Definition 1.1.3 (See also [AHS, Definition 14.1])

Given a category $\underline{X}$, the pair $(\underline{E}, \underline{M})$ is called a *factorization structure* on $\underline{X}$ provided that the following two conditions are satisfied:

(1) $\underline{X}$ has $(\underline{E}, \underline{M})$-*factorizations* of morphisms in the sense that each $\underline{X}$-morphism $X \xrightarrow{f} Y$ has a factorization

$$X \xrightarrow{f} Y = X \xrightarrow{e} B \xrightarrow{m} Y,$$

where $e \in \underline{E}$ and $m \in \underline{M}$.

(2) $\underline{X}$ has the *unique* $(\underline{E}, \underline{M})$-*diagonalization property*; in other words, $e \downarrow m$, for each $e \in \underline{E}$ and each $m \in \underline{M}$.

Remark 1.1.4

(a) If $\underline{X}$ has an $(\underline{E}, \underline{M})$-factorization structure , then $\underline{X}$ is called an $(\underline{E}, \underline{M})$-*category*. It must be observed that $\underline{E}$ (respectively, $\underline{M}$) is not necessarily the class Epi($\underline{X}$) (resp., Mono($\underline{X}$)).

(b) If we denote by $\underline{E} \downarrow \underline{M}$ the fact that each $e \in \underline{E}$ diagonalizes over every $m \in \underline{M}$, then, given an $(\underline{E}, \underline{M})$-category $\underline{X}$, the unique $(\underline{E}, \underline{M})$-diagonalization property is equivalent to saying that $\underline{E} \downarrow \underline{M}$.

1.2 EXAMPLES OF $(\underline{E}, \underline{M})$-FACTORIZATION STRUCTURES

Example 1.2.1

For any category $\underline{X}$,

(a) the pair (Iso($\underline{X}$), Mor($\underline{X}$)) is a factorization structure on $\underline{X}$.

(b) the pair (Mor($\underline{X}$), Iso($\underline{X}$)) is a factorization structure on $\underline{X}$.

Proof of (a)

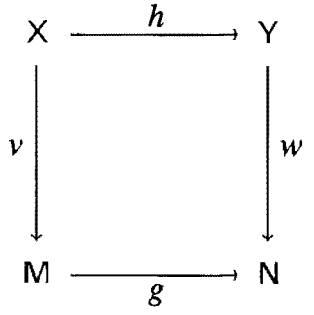
Since Iso($\underline{X}$) is closed under composition, it follows that Iso($\underline{X}$) satisfies Definition 1.1.2(a). Since Iso($\underline{X}$) $\subseteq$ Mor($\underline{X}$), and since Mor($\underline{X}$) is closed under composition, it follows that Mor($\underline{X}$) satisfies Definition 1.1.2(b).

(1) For any $\underline{X}$-morphism $X \xrightarrow{f} Y$, the factorization

$$X \xrightarrow{f} Y \;=\; X \xrightarrow{id_X} X \xrightarrow{f} Y$$

is an (Iso($\underline{X}$), Mor($\underline{X}$))-factorization of f.

(2) Suppose the following square in $\underline{X}$ is commutative with $h \in$ Iso($\underline{X}$) and g any $\underline{X}$-morphism:



Define a diagonal morphism $Y \xrightarrow{d} M$ by $d = v \circ h^{-1}$. Then

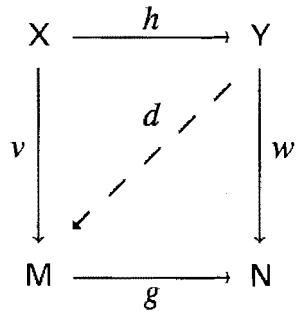
$$d \circ h \;=\; (v \circ h^{-1}) \circ h \;=\; v \circ (h^{-1} \circ h) \;=\; v \circ id_X \;=\; v$$

and

$$g \circ d \;=\; g \circ (v \circ h^{-1}) \;=\; (g \circ v) \circ h^{-1} \;=\; (w \circ h) \circ h^{-1}$$

$$=\; w \circ (h \circ h^{-1}) \;=\; w \circ id_Y \;=\; w,$$

hence the diagram



commutes. To prove uniqueness, suppose $Y \xrightarrow{k} M$ is another **X**-morphism such that $k \circ h = v$ and $g \circ k = w$. Then $d \circ h = v = k \circ h$, so $k = d$, since, being an isomorphism, h is an epimorphism.

(b) That (Mor(**X**), Iso(**X**)) is a factorization structure on **X** follows in a similar way. □

Example 1.2.2(a)

(Epi, Mono) is a factorization structure on **Set**.

Proof

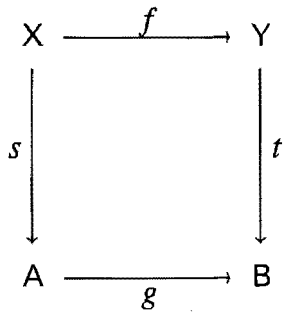
We recall that in **Set** epimorphisms (respectively, monomorphisms or isomorphisms) are precisely surjections (resp., injections or bijections) (see, for example, [HS_1, 6.10(2), 6.3(2) and 5.14(2)]). Since isomorphisms are epimorphisms, and since epimorphisms are closed under composition, it follows that Epi satisfies Definition 1.1.2(a). Since isomorphisms are monomorphisms and monomorphisms are closed under composition, Mono satisfies Definition 1.1.2(b).

(1) If $X \xrightarrow{f} Y$ is a morphism in **Set**, then

$$X \xrightarrow{f} Y = X \xrightarrow{f} f(X) \xrightarrow{i_{f(X)}} Y,$$

where $i_{f(X)}$ is the inclusion of $f(X)$ into Y, is an (Epi, Mono)-factorization of f.

(2) Suppose the square in **Set**



is commutative with $f \in$ Epi(**Set**) and $g \in$ Mono(**Set**). Define $Y \xrightarrow{d} A$ as follows: for each $y \in Y$, $d(y) = s(x)$, whenever $y = f(x)$. We must show that d is well-defined. Suppose $\bar{x} \in X$ also satisfies $f(x) = f(\bar{x})$. Then

$$t(f(x)) = t(f(\bar{x})) \implies g(s(x)) = g(s(\bar{x})) \implies s(x) = s(\bar{x}),$$

since g is injective.

Then, if $x \in X$, we have $(d \circ f)(x) = d(y) = s(x)$, where $y = f(x)$; thus $d \circ f = s$. On the other hand, if $y \in Y$, we find that

$$(g \circ d)(y) = g(d(y)) = g(s(x)) = (g \circ s)(x) = (t \circ f)(x) = t(y),$$

for some x such that $y = f(x)$, thus $g \circ d = t$. And d is unique such that $d \circ f = s$ and $g \circ d = t$. For, if $Y \xrightarrow{d'} A$ is another function in **Set** such that $d' \circ f = s$ and $g \circ d' = t$, then $d \circ f = s = d' \circ f$ and, since f is surjective, $d = d'$. □

Example 1.2.2(b)

(Epi, Mono) is a factorization structure on **Grp**.

Proof

In **Grp**, monomorphisms (respectively, epimorphisms) are precisely the homomorphisms which are injective (resp., surjective) on the underlying sets (see, for example, [HS_1, 6.3(2) and 6.10(2)]). That the pair (Epi, Mono) satisfies Definition 1.1.2 follows by an argument similar to that in the proof of the previous example.

(1) If $(G, \diamond) \xrightarrow{\varphi} (H, \star)$ is a group homomorphism, then

$$(G, \diamond) \xrightarrow{\varphi} (H, \star) = (G, \diamond) \xrightarrow{\varphi'} G/\mathrm{Ker}\,\varphi \xrightarrow{\bar{\varphi}} (H, \star)$$

is an (Epi, Mono)-factorization of φ. Here the homomorphisms φ' and $\bar{\varphi}$ are defined as follows: For each $g \in G$, set $\varphi'(g) = g\mathrm{Ker}\,\varphi$, and, for each $x\mathrm{Ker}\,\varphi \in G/\mathrm{Ker}\,\varphi$, set $\bar{\varphi}(x\mathrm{Ker}\,\varphi) = \varphi(x)$.

(2) Now consider a commutative square in **Grp**

$$\begin{array}{ccc} (G, \diamond) & \xrightarrow{\ e\ } & (H, \star) \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ (K, \theta) & \xrightarrow[\ m\ ]{} & (M, \oplus) \end{array}$$

where $e \in \mathrm{Epi}(\mathbf{Grp})$ and $m \in \mathrm{Mono}(\mathbf{Grp})$. We define $(H, \star) \xrightarrow{d} (K, \theta)$ as follows: For each $x \in H$, set $d(x) = r(y)$ iff $e(y) = x$. (This makes sense because e is an epimorphism.) The map d is well-defined, since if $\bar{y}$ also satisfies $e(\bar{y}) = e(y)$, then

$$s(e(\bar{y})) = s(e(y)) \Rightarrow m(r(\bar{y})) = m(r(y)) \Rightarrow r(\bar{y}) = r(y),$$

since m is a monomorphism.

Given $g \in G$, we find that $(d \circ e)(g) = d(e(g)) = r(g)$, so $d \circ e = r$. On the other hand, for each $y \in H$, we have

$$\begin{aligned}
(m \circ d)(y) &= m(d(y)) \\
&= m(r(x)) \qquad \text{(where } x \text{ is such that } e(x) = y) \\
&= (m \circ r)(x) \\
&= (s \circ e)(x) \qquad \text{(since the square commutes)} \\
&= s(e(x)) \\
&= s(y),
\end{aligned}$$

so that $m \circ d = s$. By [HS_1, 32.9], the map d is a group homomorphism. The homomorphism d is unique such that $d \circ e = r$ and $m \circ d = s$: For, if d' is another morphism satisfying $d' \circ e = r$ and $m \circ d' = s$, then $d \circ e = r = d' \circ e$, and, since e is an epimorphism, we have $d = d'$. □

Example 1.2.3

Apart from the three factorization structures given in Examples 1.2.1 and 1.2.2(a) above, **Set** has the following ($\underline{E}$, $\underline{M}$)-factorization structure:

$$\underline{E} = \{ X \xrightarrow{e} Y \mid X = \emptyset \Rightarrow Y = \emptyset \}$$

and

$$\underline{M} = \{ X \xrightarrow{m} Y \mid m \text{ is a bijection or } X = \emptyset \}.$$

Proof

Given $X \xrightarrow{e} Y \in \underline{E}$ and an isomorphism (a bijection!) $Y \xrightarrow{h} Z$ in **Set**, it must be shown that $h \circ e \in \underline{E}$. If $X = \emptyset$, then $Y = \emptyset$ (since $e \in \underline{E}$). Since h is bijective, we must have $Z = \emptyset$. Hence $h \circ e \in \underline{E}$.

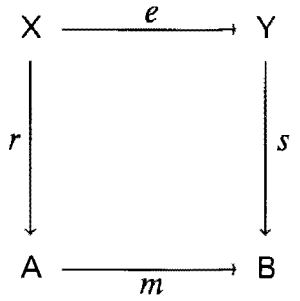
Now let $Z \xrightarrow{h} X \in \mathrm{Iso}(\underline{\mathbf{X}})$ and let $X \xrightarrow{m} Y \in \underline{\mathbf{M}}$. Then h is a bijection. If m is a bijection, then the composition $m \circ h$ is a bijection; hence $m \circ h \in \underline{\mathbf{M}}$. On the other hand, if $X = \emptyset$, then $Z = \emptyset$ (since h is a bijection), hence $m \circ h \in \underline{\mathbf{M}}$. Consequently, the pair $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$ satisfies Definition 1.1.2.

(1) If $X \xrightarrow{f} Y$ is a morphism in **Set** with $X = \emptyset$, then f has the $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$-factorization $\emptyset \xrightarrow{f} Y = \emptyset \xrightarrow{\emptyset} \emptyset \xrightarrow{f} Y$. Otherwise a function $X \xrightarrow{f} Y$ in **Set** has an $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$-factorization

$$X \xrightarrow{f} Y = X \xrightarrow{f} Y \xrightarrow{id_Y} Y$$

satisfying the definitions of $\underline{\mathbf{E}}$ and $\underline{\mathbf{M}}$ above.

(2) Now suppose that the square in **Set**



commutes with $e \in \underline{\mathbf{E}}$ and $m \in \underline{\mathbf{M}}$. If $X = \emptyset$, then $Y = \emptyset$, by definition of $\underline{\mathbf{E}}$ above, so the unique morphism $\emptyset \longrightarrow A$ is the desired diagonal morphism. We shall assume that $X \neq \emptyset$. Then $A \neq \emptyset$, since r is a function. So, by definition of $\underline{\mathbf{M}}$, the function m is a bijection. So, we define $Y \xrightarrow{d} A$ to be the function $m^{-1} \circ s$. Then

$$d \circ e = (m^{-1} \circ s) \circ e = m^{-1} \circ (s \circ e) = m^{-1} \circ (m \circ r)$$

$$= (m^{-1} \circ m) \circ r = id_A \circ r = r$$

and

$$m \circ d = m \circ (m^{-1} \circ s) = (m \circ m^{-1}) \circ s = id_B \circ s = s.$$

And d is unique with this property since m is a bijection. □

Example 1.2.4(a)

(Surjection, embedding) is a factorization structure on **Top**.

Proof

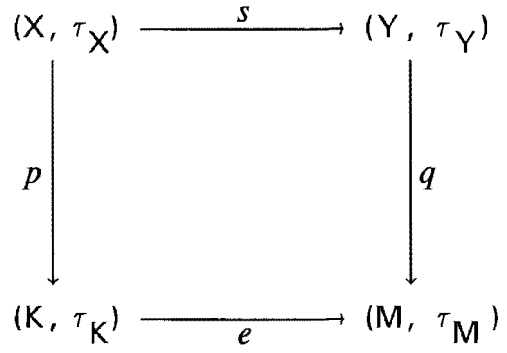
In **Top**, isomorphisms are the homeomorphisms (see e.g. [HS_1, 5.14(4)]). Since homeomorphisms are surjections and surjections are closed under composition, surjections satisfy Definition 1.1.2(a). Also, homeomorphisms are embeddings and embeddings are closed under composition, embeddings satisfy Definition 1.1.2(b).

(1) If $(X, \tau_X) \xrightarrow{f} (Y, \tau_Y)$ is a continuous function between topological spaces, then the factorization

$$(X, \tau_X) \xrightarrow{f} (Y, \tau_Y) = (X, \tau_X) \xrightarrow{f} (f(X), \tau_f) \xrightarrow{i_{f(X)}} (Y, \tau_Y)$$

(where τ_f is the relative topology induced on $f(X)$ by τ_Y and $i_{f(X)}$ is the inclusion of $f(X)$ into Y) is a (surjection, embedding)-factorization of f. The map $(X, \tau_X) \xrightarrow{f} (f(X), \tau_f)$ is continuous since the continuous function $i_{f(X)}$ is initial.

(2) Let the following square in **Top** be commutative with s surjective and e an embedding:



The existence of a unique function $(Y, \tau_Y) \xrightarrow{d} (K, \tau_K)$ completing the above square follows as in Example 1.2.2(a). So, we need only show that $d \in$ Mor(**Top**), that is, d is continuous. Since $e \circ d = q$ and since q is continuous and e is is initial, it follows that d is continuous. □

Example 1.2.4(b)

(Quotient, one-one) is factorization structure on **Top**.

Proof

Given a quotient map $X \xrightarrow{e} Y$, let $Y \xrightarrow{h} Z$ be a homeomorphism (between topological spaces). By [Wl, 9.2], the space Z is a quotient space, so the composition $h \circ e$ is a quotient function; so Definition 1.1.2(a) is satisfied. Since homeomorphisms are one-one and one-one functions are closed under composition, one-one functions satisfy Definition 1.1.2(b).

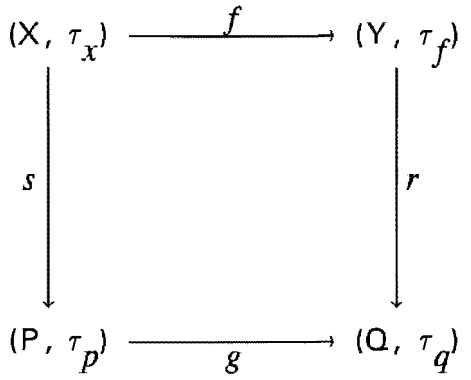
(1) A continuous function $(X, \tau_X) \xrightarrow{f} (Y, \tau_Y)$ in **Top** has the following (quotient, one-one)-factorization:

$$(X, \tau_X) \xrightarrow{f} (Y, \tau_Y) = (X, \tau_X) \xrightarrow{f} (f(X), \sigma) \xrightarrow{i_{f(X)}} (Y, \tau_Y),$$

where σ is the quotient topology induced on $f(X)$ by f and

$i_{f(X)}$ is the inclusion of $f(X)$ into Y. The continuity of the function $i_{f(X)}$ follows from [WI, 9.4].

(2) If the square in **Top**



commutes with f a quotient function (and then, τ_f is the quotient topology induced on Y by f) and g is a one-one function, then f is surjective, being a quotient. Thus we define $(Y, \tau_f) \xrightarrow{d} (P, \tau_p)$ as in Example 1.2.2(a). By a similar argument, the map d is unique such that $d \circ f = s$ and $g \circ d = r$. Continuity of d follows from the fact that $d \circ f = p$ and the fact that p is continuous and f is a quotient. □

Example 1.2.4(c)

(Dense, closed embedding) is a factorization structure on **Top**.

Proof

If $X \xrightarrow{e} Y$ is a dense continuous map and $Y \xrightarrow{h} Z$ is a homeomorphism, then $h \circ e$ is a dense continuous map; for, we have $\overline{(h \circ e)(X)} = \overline{h(e(X))} = h(\overline{e(X)}) = h(Y) = Z$. So dense continuous maps satisfy Definition 1.1.2(a). Since homeomorphisms are closed embeddings, and since closed embeddings are closed under composition, it follows that closed embeddings satisfy Definition 1.1.2(b).

(1) Given a continuous function $(X, \tau) \xrightarrow{f} (Y, \sigma)$ in **Top**, consider $\overline{f(X)}$, the closure $CL_Y(f(X))$ of $f(X)$ in (Y, σ). Then f has the following (dense, closed embedding)-factorization:

$$(X, \tau) \xrightarrow{f} (Y, \sigma) = (X, \tau) \xrightarrow{e} (\overline{f(X)}, \sigma') \xrightarrow{m} (Y, \sigma),$$

where e is the codomain restriction of f, m is the inclusion of $\overline{f(X)}$ into Y and σ' is the relative topology induced on $\overline{f(X)}$ by σ. Since $CL_W(e(X)) = CL_Y(f(X))$ (where $W = CL_Y(f(X))$), the map e is dense. By construction, the map m is an embedding. To see that the inclusion m is closed, let C be closed in $\overline{f(X)}$. Then there exists C′ which is closed in Y such that $C = \overline{f(X)} \cap C'$ (see [WI, 6.3(b)]), so C is closed in Y.

(2) Suppose the following square in **Top** is commutative, with d a dense map and c a closed embedding in **Top**:

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{d} & (Y, \sigma) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ (W, \rho) & \xrightarrow[c]{} & (Z, \lambda) \end{array}$$

Observe that $c(h(X)) \subseteq c(\overline{h(X)})$, so

$$\begin{aligned} \overline{c(h(X))} &\subseteq \overline{c(\overline{h(X)})} \\ &= c(\overline{h(X)}) \qquad (c \text{ is closed}) \\ &\subseteq \overline{c(h(X))} \qquad (c \text{ is continuous, [WI, Theorem 7.2(d)]}). \end{aligned}$$

Hence $c(\overline{h(X)}) = \overline{c(h(X))}$. Choose any $y \in Y$. Then

$$k(y) \in k(Y) = k(\overline{d(X)}) \qquad \text{(since } d \text{ is dense)}$$

$$\subseteq \overline{k(d(X))} \qquad \text{(since } k \text{ is continuous)}$$

$$= \overline{c(h(X))} \qquad \text{(since the square is commutative)}$$

$$= c(\overline{h(X)}).$$

Hence there exists $w \in \overline{h(X)} \subseteq W$ such that $c(w) = k(y)$. We thus define a map $Y \xrightarrow{l} W$ as follows: Given $y \in Y$, let $l(y) = w$, where w is such that $c(w) = k(y)$. The map l is well-defined; for if $\bar{w}$ also satisfies $c(\bar{w}) = k(y)$, then $w = \bar{w}$, since c is injective. Given $x \in X$, we have

$$(c \circ (l \circ d))(x) = (c(l(d(x))) = c(t) \quad (\text{where } t \text{ is such } c(t) = k(d(x)))$$

$$= k(d(x)) = c(h(x)),$$

hence $c \circ (l \circ d) = c \circ h$. But c is an embedding, so $l \circ d = h$. And for each $y \in Y$, we have

$$(c \circ l)(y) = c(l(y)) = c(w) \quad (\text{where } w \text{ is such that } c(w) = k(y))$$

$$= k(y),$$

so that $c \circ l = k$. Therefore, l completes the above square. Uniqueness of l follows from the fact that c is an embedding, and l is continuous by continuity of k and initiality of c. □

Example 1.2.4(d)

(Front dense, front-closed embedding) is a factorization structure on **Top**. (The definitions of front-dense maps and front-closed maps follow.)

Definitions: ([NW, p. 68])

(a) If (X, τ) is a topological space and $A \subseteq X$, the *front-closure* $b(A)$ of A (also called the *b*-closure of A) is defined as follows:

$$b(A) = \{x \in X \mid \text{for each nhood } N \text{ of } x \text{ it holds that } N \cap \overline{\{x\}} \cap A \neq \emptyset\}.$$

Under this definition, a new topology which is called the *front-topology* (or the *b-topology*) is formed on the space X.

(b) (i) A continuous function $f : X \longrightarrow Y$ between topological spaces is said to be *b-dense* provided that $b(f(X)) = Y$.

(ii) An embedding $f : X \longrightarrow Y$ is called *b-closed* if $b(f(X)) = f(X)$.

Some properties (without proof) of the *b*-closure operator are the following (see [SK, 2.1]):

Lemma 1

Given a topological space (X, τ), let $A, B \subseteq X$. Then

(1) $A \subseteq b(A) \subseteq \bar{A}$.

(2) If $A \subseteq B$, then $b(A) \subseteq b(B)$.

(3) $b(A) = b(b(A))$.

(4) $b(A \cup B) = b(A) \cup b(B)$.

Remark 1*:

(a) It is immediate from this lemma that, for any topological space (Y, τ), it holds that $Y = b(Y)$. For,

$$Y \subseteq b(Y) \subseteq \overline{Y} = Y \Longrightarrow Y = b(Y).$$

(b) If $f : X \longrightarrow Y$ is a homeomorphism (between topological spaces), then $b(f(X)) = b(Y) = Y = f(X)$; so a homeomorphism is both a *b-dense* map and a *b-closed* embedding.

Lemma 2

(1) If $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a continuous function, then

$$f(b(V)) \subseteq b(f(V)),$$

for each $V \subseteq X$.

(2) If $m : (X, \tau) \longrightarrow (Y, \sigma)$ is a b-closed embedding, then, for each $V \subseteq X$, it holds that

$$b(m(b(V))) = m(b(V)).$$

Proof

(1) Let $y \in f(b(V))$. It must be shown that $y \in b(f(V))$. Suppose that N is any nhood of y. Then there is some $x \in b(V)$ such that $y = f(x)$. Since f is continuous, there is a nhood $U \subseteq X$ of x such that $f(U) \subset N$. Since $x \in b(V)$, it follows that

$$U \cap \overline{\{x\}} \cap V \neq \emptyset,$$

and so

$$f(U) \cap f(\overline{\{x\}}) \cap f(V) \neq \emptyset.$$

By continuity of f, we have

$$f(U) \cap f(\overline{\{x\}}) \cap f(V) \subseteq N \cap \overline{\{f(x)\}} \cap f(V),$$

which implies that

$$N \cap \overline{\{y\}} \cap f(V) \neq \emptyset.$$

Hence $y \in b(f(V))$.

(2) From (1), we have $m(b(V)) \subseteq b(m(V))$. But

$$V \subseteq b(V) \qquad \text{(Lemma 1(1))}$$

$$\Rightarrow \quad m(V) \subseteq m(b(V))$$

$$\Rightarrow b(m(V)) \subseteq b(m(b(V))), \qquad \text{(Lemma 1(2))}$$

so $m(b(V)) \subseteq b(m(b(V)))$. To prove the reverse inclusion, we note that since m is a b-closed embedding, it follows that

$$\begin{aligned} b(m(b(V))) &\subseteq b(b(m(V))) && \text{(by (1))} \\ &= b(m(V)) && \text{(Lemma 1(3))} \\ &\subseteq b(m(X)) && \text{(since } V \subseteq X\text{)} \\ &= m(X). \end{aligned}$$

Now let $y \in b(m(b(V)))$. It needs to be shown that $y \in m(b(V))$; that is, we must find $x \in b(V)$ such that $m(x) = y$. By the above observation, we have $y \in m(X)$; so there is some $x \in X$ such that $m(x) = y$. Then we need only show that $x \in b(V)$. To this end, let $N \in \tau$ with $x \in N$. Since m is an embedding, it follows that there is some $U \in \sigma$ such that $N = m^{-1}(U)$. Therefore $x \in m^{-1}(U)$, and U is a nhood of y. But $y \in b(m(V))$, so $U \cap \overline{\{y\}} \cap m(V) \neq \emptyset$. Application of m^{-1} to this relation gives:

$$m^{-1}(U) \cap m^{-1}(\overline{\{y\}}) \cap m^{-1}(m(V)) \neq \emptyset.$$

But m is continuous, so

$$N \cap m^{-1}(\overline{\{y\}}) \cap m^{-1}(m(V)) \subseteq N \cap \overline{\{m^{-1}(y)\}} \cap V,$$

and, therefore, $N \cap \overline{\{m^{-1}(y)\}} \cap V \neq \emptyset$. Since m is one-one, it follows that $m^{-1}(y) = x$, thus $\overline{\{m^{-1}(y)\}} = \overline{\{x\}}$. Then $N \cap \overline{\{x\}} \cap V \neq \emptyset$. Hence $x \in b(V)$, that is, $y \in m(b(V))$. Consequently, we have proved that $m(b(V)) = b(m(b(V)))$. □

Lemma 3

If $f : X \longrightarrow Y$ is a homeomorphism between topological spaces, and $V \subseteq X$ is any subset of X, then $b(f(V)) = f(b(V))$.

Proof

Since f is continuous, it follows from Lemma 2 that $f(b(V)) \subseteq b(f(V))$.

To prove the reverse inclusion, let $y \in b(f(V))$. To prove that $y \in f(b(V))$, we must find $x \in b(V)$ so that $f(x) = y$. Since f is bijective, there exists a unique $x \in X$ such that $f(x) = y$. So it needs to be shown that $x \in b(V)$. Let N be a nhood of x. Then $f(N)$ is a nhood of $f(x) = y$. Since $y \in b(f(V))$, it follows that

$$f(N) \cap \overline{\{y\}} \cap f(V) \neq \emptyset.$$

Applying f^{-1} to this relation, we find that

$$f^{-1}(f(N)) \cap f^{-1}(\overline{\{y\}}) \cap f^{-1}(f(V)) \neq \emptyset.$$

Since f is bijective, it follows that $f^{-1}(f(W)) = W$, for all $W \subseteq X$. Since f^{-1} is a continuous bijection, it follows that

$$f^{-1}\overline{\{y\}} = \overline{\{f^{-1}(y)\}} = \overline{\{x\}}.$$

Hence $N \cap \overline{\{x\}} \cap V \neq \emptyset$, for each nhood N of x; that is, $x \in b(V)$. Consequently, $b(f(V)) = f(b(V))$. □

Let us denote by $\mathscr{E}$ the family of all b-dense continuous maps, and by $\mathscr{M}$ the family of all b-closed embeddings.

Lemma 4

Each of $\mathscr{E}$ and $\mathscr{M}$ is closed under composition with isomorphisms, as specified in Definition 1.1.2.

Proof

In **Top**, isomorphisms are the homeomorphic functions (see, for example, [HS_1], Examples 5.4).

Let $X \xrightarrow{e} Y \xrightarrow{h} Z$ be continuous maps in **Top** with $e \in \mathscr{E}$ and h a homeomorphism. It must be shown that $b((h \circ e)(X)) = Z$. We have

$$\begin{aligned} b[(h \circ e)(X)] &= b[h(e(X))] \\ &= h[b(e(X))] && \text{(Lemma 3)} \\ &= h(Y) && (e \text{ is } b\text{-dense}) \\ &= Z, && (h \text{ is surjective}) \end{aligned}$$

so $h \circ e$ is b-dense.

On the other hand, if $A \xrightarrow{h} B \xrightarrow{m} C$ are continuous maps in **Top** with $m \in \mathscr{M}$ and h a homeomorphism, we shall show that $b[(m \circ h)(A)] = (m \circ h)(A)$. We have

$$\begin{aligned} b[(m \circ h)(A)] &= b[m(h(A))] \\ &= b[m(B)] && (h \text{ is surjective}) \\ &= m(B) && (m \text{ is a } b\text{-closed embedding}) \\ &= m[h(A)] \\ &= (m \circ h)(A), \end{aligned}$$

thus $m \circ h \in \mathscr{M}$. □

To complete the example, we proceed as follows:

(1) A function $(X, \tau) \xrightarrow{f} (Y, \sigma)$ in **Top** has the following *(b-dense, b-closed embedding)-factorization*:

$$(X, \tau) \xrightarrow{f} (Y, \sigma) = (X, \tau) \xrightarrow{e} (b_\sigma(f(X)), \sigma') \xrightarrow{m} (Y, \sigma),$$

where m is the inclusion of the b_σ-closure $b_\sigma(f(X))$ into Y, e is the codomain restriction of f (i.e, $e(x) = f(x)$, for each $x \in X$) and σ' is the relative topology induced on $b(f(X))$ by σ. (Here the subscripts on the b's indicate in which topology the b-closure is being taken.) Since $(b_\sigma(f(X)), \sigma')$ is a topological space, we must have

$$b_{\sigma'}(b_\sigma(f(X))) = b_\sigma(f(X)).$$

We also have

$$\begin{aligned} & f(X) \subseteq b_\sigma(f(X)) && \text{(Lemma 1(1))} \\ \Rightarrow\ & b_{\sigma'}(e(X)) = b_{\sigma'}(f(X)) \\ & \subseteq b_{\sigma'}(b_\sigma(f(X))) && \text{(Lemma 1(2))} \\ & = b_\sigma(f(X)). \end{aligned}$$

To prove the reverse inclusion, let $y \in b_\sigma(f(X))$, and let $U \in \sigma'$ with $y \in U$. Then there is some $V \in \sigma$ such that $U = V \cap b_\sigma(f(X))$. Now

$$\begin{aligned} U \cap \mathrm{cl}_{b_\sigma(f(X))}(\{y\}) \cap f(X) &= V \cap b_\sigma(f(X)) \cap \mathrm{cl}_Y(\{y\}) \cap f(X) \\ &= V \cap \mathrm{cl}_Y(\{y\}) \cap f(X) \neq \emptyset, \end{aligned}$$

because $y \in b_\sigma(f(X))$; thus $y \in b_{\sigma'}(f(X))$, and so $b_\sigma(f(X)) \subseteq b_{\sigma'}(f(X))$. Hence e is b-dense. Since m is an inclusion, we have $m(b(f(X)) = b(f(X))$; it is also b-closed because

$$b[m(b(f(X)))] = b[b(f(X))] \qquad \text{(since } m \text{ is an inclusion)}$$

$$= b(f(X)) \qquad \text{(Lemma 1(3))}$$

$$= m[b(f(X)].$$

(2) We consider a commutative square in **Top**, with $e \in \mathscr{E}$ and $m \in \mathscr{M}$.

$$\begin{array}{ccc} (X, \tau) & \xrightarrow{\ e\ } & (Y, \sigma) \\ \Big\downarrow p & & \Big\downarrow q \\ (A, \rho) & \xrightarrow[\ m\ ]{} & (B, \lambda) \end{array}$$

Observe that

$$p(X) \subseteq b(p(X)) \qquad \text{(Lemma 1(1))}$$

$$\Longrightarrow m(p(X)) \subseteq m(b(p(X))),$$

so $b(m(p(X))) \subseteq b(m(b(p(X)))) \qquad$ (Lemma 1(2))

$$= m(b(p(X))) \qquad \text{(Lemma 2(2))}$$

$$\subseteq b(m(p(X))) \qquad \text{(Lemma 2(1))}$$

Hence

$$m(b(p(X))) = b(m(p(X))).$$

Now choose $y \in Y$. Then

$$q(y) \in q(Y) = q(b(e(X)) \qquad (e \text{ is } b\text{-dense})$$

$$\subseteq b(q(e(X))) \qquad \text{(Lemma 2(1))}$$

$$= b(m(p(X))) \qquad \text{(the square is commutative)}$$

$$= m(b(p(X)).$$

Hence there exists $a \in b(p(X)) \subseteq A$ such that $m(a) = q(y)$. Define $d : Y \longrightarrow A$ as follows: for each $y \in Y$, set $d(y) = a$ if and only if $q(y) = m(a)$. Then d is well-defined because m is an injection. That d is unique such that $d \circ e = p$ and $m \circ d = q$ follows exactly as in Example 1.2.4(c). □

1.3 PROPERTIES OF (E, M)-FACTORIZATION STRUCTURES

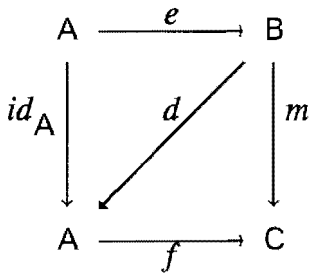
In this section, we show (Proposition 1.3.1) that in an (E, M)-category X, the classes E and M determine each other through the unique diagonalization property; that each of E and M contains Iso(X) and also that M is closed under the formation of all limits. It is shown that the factorization structures (E, M) in X which can be extended to factorization structures (E, M′) for set-indexed sources are precisely those for which X has products (Proposition 1.3.5).

Proposition 1.3.1 (cf. [HSV], 1.2)

Let (E, M) be a factorization structure on X. Then

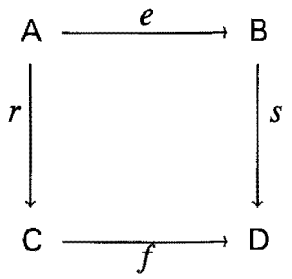
(1) (M, E) is a factorization structure on X^{op}.

(2) If the diagram



commutes, where $e \in E$ and $m \in M$, then $f \in M$. In particular, d is an isomorphism.

(3) An $\underline{\mathbf{X}}$-morphism $f : C \longrightarrow D$ belongs to $\underline{\mathbf{M}}$ if, and only if, for each commutative diagram



with $e \in \underline{E}$, there exists a (not necessarily unique) morphism $B \xrightarrow{d} C$ such that $d \circ e = r$ and $f \circ d = s$.

(4) $\underline{E} \cap \underline{\mathbf{M}} = \mathrm{Iso}(\underline{\mathbf{X}})$.

(5) Each of the classes $\underline{E}$ and $\underline{\mathbf{M}}$ is closed under composition.

(6) Each ($\underline{E}$, $\underline{\mathbf{M}}$)-factorization is unique up to a unique commuting isomorphism.

(7) If $n \circ f \in \underline{\mathbf{M}}$ and $n \in \underline{\mathbf{M}}$ or n is a monomorphism, then $f \in \underline{\mathbf{M}}$.

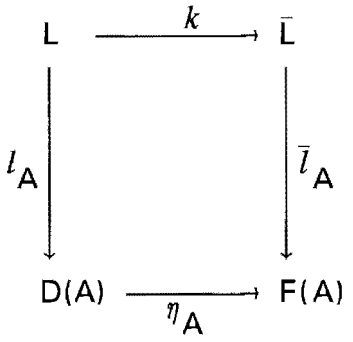
(8) The class $\underline{\mathbf{M}}$ is closed under the formation of products, pullbacks, multiple pullbacks, and limits.

Definitions:

(a) The class $\underline{\mathbf{M}}$ is said to be *closed under the formation of multiple pullbacks* provided that, if a source $(k_i : S \longrightarrow S_i)_I$ is a multiple pullback of a sink $(g_i : S_i \longrightarrow T)_I$ with each g_i in $\underline{\mathbf{M}}$, then $k_i \in \underline{\mathbf{M}}$, for each $i \in I$.

(b) The class $\underline{\mathbf{M}}$ is said to be *closed under the formation of limits* if, whenever $\underline{A}$ is a small category, (L, l_A) and $(\bar{L}, \bar{l}_A)$ are limits of functors $D : \underline{A} \longrightarrow \underline{X}$ and $F : \underline{A} \longrightarrow \underline{X}$, respectively, and $(\eta_A) : D \longrightarrow F$ is a natural transformation with each η_A in $\underline{\mathbf{M}}$, then $k : L \longrightarrow \bar{L}$ belongs to $\underline{\mathbf{M}}$, where k is the unique morphism making the following diagrams (one for each

$A \in Ob(\underline{A})$) commutate:



Proof

For the proofs of (2), (4), (5), (6) and (7), see [AHS], Chapter 14.

(1) (See also [AHS], Proposition 14.3.) Let $f : X \longrightarrow Y$ be an $\underline{X}^{op}$-morphism. Then $f : Y \longrightarrow X$ is an $\underline{X}$-morphism, so it has an ($\underline{E}$, $\underline{M}$)-factorization

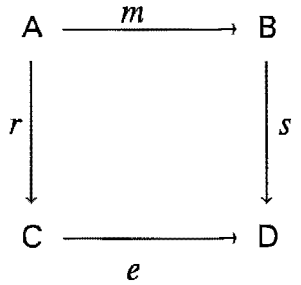
$$Y \xrightarrow{f} X = Y \xrightarrow{e} A \xrightarrow{m} X,$$

where $m \in \underline{M}$ and $e \in \underline{E}$. But then

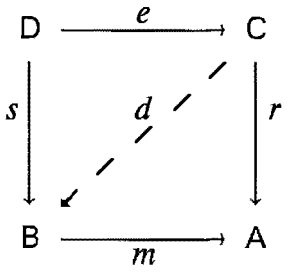
$$X \xrightarrow{f} Y = X \xrightarrow{m} A \xrightarrow{e} Y$$

is an ($\underline{M}$, $\underline{E}$)-factorization of f in $\underline{X}^{op}$.

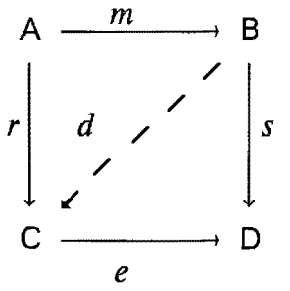
To show that $\underline{X}^{op}$ has the unique ($\underline{M}$, $\underline{E}$)-diagonalization property, consider a commutative square $\underline{X}^{op}$



with $m \in \underline{M}$ and $e \in \underline{E}$. Then there exists a unique $\underline{X}$-morphism $C \xrightarrow{d} B$ such that the following diagram commutes in $\underline{X}$:

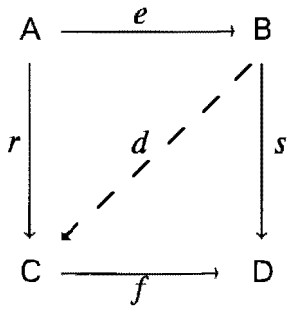


Associated with d is a unique $\underline{\mathbf{X}}^{op}$-morphism $d : B \longrightarrow C$ such that the diagram in $\underline{\mathbf{X}}^{op}$

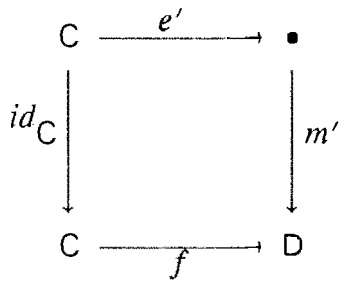


commutes. This follows from the fact that $d \circ e = s$ and $m \circ d = r$.

(3) (See also [AHS], 14.6(3).) Suppose that $C \xrightarrow{f} D \in \underline{\mathbf{M}}$. Since $e \in \underline{\mathbf{E}}$, the unique ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-diagonalization property implies that there exists a $B \xrightarrow{d} C$ such that $r = d \circ e$ and $f \circ d = s$:



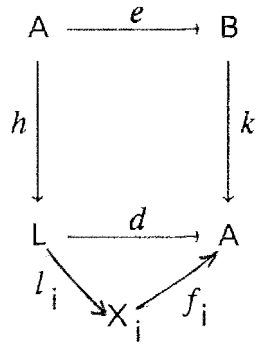
Conversely, suppose the condition is satisfied, and let $f = m' \circ e'$ be an ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-factorization of f. But $f = f \circ id_C$ is also a factorization of f, so the following diagram is commutative:



By hypothesis, there exists an $\underline{X}$-morphism d' such that $d' \circ e' = id_C$ and $f \circ d' = m'$. The result follows from (2) above.

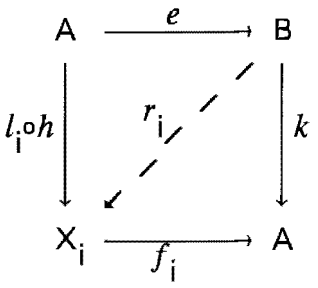
(8)(i) That $\underline{M}$ is closed under the formation of products and pullbacks follows from [AHS, Proposition 14. 15].

(ii) Let the source $(l_i : L \longrightarrow X_i)_I$ be a multiple pullback of a sink $(f_i : X_i \longrightarrow A)_I$. Suppose each $f_i \in \underline{M}$. It must be shown that each $l_i \in \underline{M}$. Then, since $f_i \circ l_i = d$, for some fixed morphism $d : L \longrightarrow A$, we need only show that $d \in \underline{M}$, for then it will follow from (7) that $l_i \in \underline{M}$, for each $i \in I$. We consider the following commutative diagram



(⋆)

where $e \in \underline{E}$. Since each $f_i \in \underline{M}$, it follows that there is a unique morphism $B \xrightarrow{r_i} X_i$ which makes the following diagram commutative, for each $i \in I$:



Thus $f_i \circ r_i = k$ and $r_i \circ e = l_i \circ h$, for each $i \in I$. Since $(L \xrightarrow{l_i} X_i)_I$ is a multiple pullback for the sink $(X_i \xrightarrow{f_i} A)_I$, there is a unique morphism $B \xrightarrow{v} L$ such that $l_i \circ v = r_i$, for each $i \in I$. Now we have

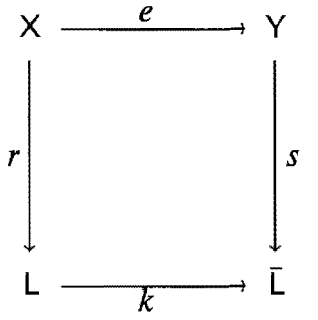
$$l_i \circ (v \circ e) = (l_i \circ v) \circ e = r_i \circ e = l_i \circ h,$$

for each $i \in I$. Since $(L, (l_i)_I, d)$ is a mono-source, it follows that $v \circ e = h$. We also have

$$d \circ v = (f_i \circ l_i) \circ v = f_i \circ (l_i \circ v) = f_i \circ r_i = k,$$

so v makes diagram $(\star)$ commute and, by (7), each $l_i \in \underline{\mathbf{M}}$.

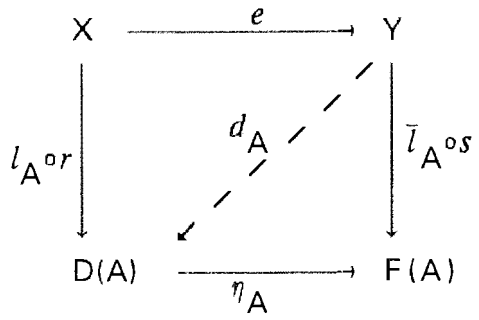
(iii) We shall now show that $\underline{\mathbf{M}}$ is closed under the formation of limits. Suppose each $\eta_A \in \underline{\mathbf{M}}$. It must shown that $k \in \underline{\mathbf{M}}$. Given a commutative square



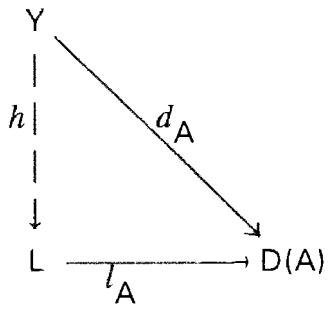
with $e \in \underline{\mathbf{E}}$, we have

$$\eta_A \circ (l_A \circ r) = (\eta_A \circ l_A) \circ r = (\bar{l}_A \circ k) \circ r$$
$$= \bar{l}_A \circ (k \circ r) = \bar{l}_A \circ (s \circ e) = (\bar{l}_A \circ s) \circ e,$$

from which the (**E**, **M**)-diagonalization property implies the existence of a unique **X**-morphism $Y \xrightarrow{d_A} D(A)$ making the following diagram commutative, for each $A \in Ob(\mathbf{A})$:



Since (L, l_A) is a limit of D, there is a unique **X**-morphism $Y \xrightarrow{h} L$ such that $l_A \circ h = d_A$, for each $A \in Ob(\mathbf{A})$:



It remains to be shown that $k \circ h = s$ and $h \circ e = r$. But, for each $A \in Ob(\mathbf{A})$, we have

$$\bar{l}_A \circ (k \circ h) = (\bar{l}_A \circ k) \circ h = (\eta_A \circ l_A) \circ h = \eta_A \circ (l_A \circ h) = \eta_A \circ d_A = \bar{l}_A \circ s.$$

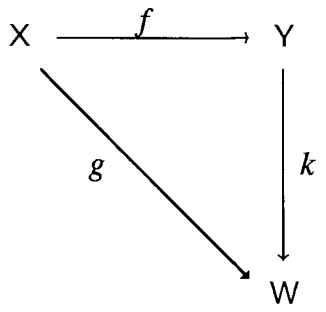
Since limits are mono-sources (see, for example, [HS_1], 20.4), it follows that $k \circ h = s$. In the same way,

$$l_A \circ (h \circ e) = (l_A \circ h) \circ e = d_A \circ e = l_A \circ r$$

implies $h \circ e = r$. So by (3), it follows that $k \in \mathbf{M}$. □

Definition 1.3.2 ([HS_1], 37.8)

Let **A** be a subcategory of a category **X**. An **X**-morphism $X \xrightarrow{f} Y$ is called an **A**-*extendable* morphism if, for each **X**-morphism $X \xrightarrow{g} W$, where $W \in Ob(\mathbf{A})$, there is an **X**-morphism $Y \xrightarrow{k} W$ such that the following triangle is commutative:



Proposition 1.3.3

If **X** has a terminal object T, (**E**, **M**) is a factorization structure on **X** and **B** is the full isomorphism-closed subcategory of **X** which consists of all objects $B \in Ob(\mathbf{X})$ for which the unique morphism $B \longrightarrow T$ belongs to **M**, then

(1) **B** is **E**-reflective in **X**.

(2) If **A** is a subcategory of **X** such that

$$\mathbf{E} = \{ f \in Mor(\mathbf{X}) \mid f \text{ is } \mathbf{A}\text{-extendable} \},$$

then **B** is the **E**-reflective hull of **A** in **X**.

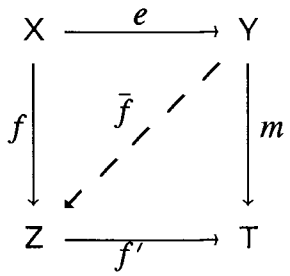
Proof

(1) Let $X \in Ob(\mathbf{X})$. Then there exists a unique **X**-morphism $X \xrightarrow{t} T$. Suppose that

$$X \xrightarrow{t} T = X \xrightarrow{e} Y \xrightarrow{m} T$$

is an (**E**, **M**)-factorization of t. Since $m \in \mathbf{M}$, we must have $Y \in Ob(\mathbf{B})$. If $X \xrightarrow{f} Z \in Mor(\mathbf{X})$ with $Z \in Ob(\mathbf{B})$, there is a unique $Z \xrightarrow{f'} T$ belonging to **M**. Then $f' \circ f = m \circ e$, so

since $\underline{E} \downarrow \underline{M}$, there is a unique morphism $Y \xrightarrow{\bar{f}} Z$ making the following diagram commutative:

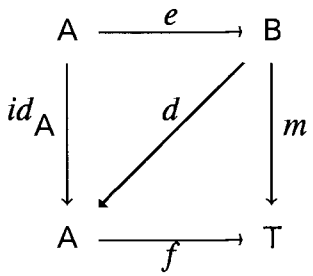


Therefore, (e, Y) is a universal map for X, so the category $\underline{B}$ is reflective in $\underline{X}$. But $e \in \underline{E}$ and the $\underline{X}$-object X was arbitrary, so $\underline{B}$ is $\underline{E}$-reflective in $\underline{X}$.

(2) To show that $\underline{A}$ is a subcategory of $\underline{B}$, choose $A \in Ob(\underline{A})$. It needs to be shown that $A \in Ob(\underline{B})$. To do this it must be shown that the unique $\underline{X}$-morphism $A \xrightarrow{f} T$ belongs to $\underline{M}$. Let

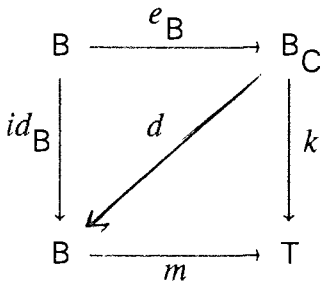
$$A \xrightarrow{f} T = A \xrightarrow{e} B \xrightarrow{m} T$$

be an $(\underline{E}, \underline{M})$-factorization of f. We also have $f \circ id_A = m \circ e$. Since $e \in \underline{E}$, it follows that e is an $\underline{A}$-extendable $\underline{X}$-morphism, so there exists an $\underline{X}$-morphism $B \xrightarrow{d} A$ which makes the following diagram commutative:



(That $m = f \circ d$ follows from the fact that T is a terminal object.) It follows from Proposition 1.3.1(2) that $f \in \underline{M}$, hence $\underline{A} \subseteq \underline{B}$.

To complete the proof, we show that if $\underline{C}$ is any $\underline{E}$-reflective full isomorphism-closed subcategory of $\underline{X}$ which contains $\underline{A}$, then $\underline{B} \subseteq \underline{C}$. Choose any $B \in Ob(\underline{B})$. Then the unique morphism $B \xrightarrow{m} T$ belongs to $\underline{M}$. There also exists a universal morphism $B \xrightarrow{e_B} B_C$ with $B_C \in Ob(\underline{C})$ and $e_B \in \underline{E}$. If k is the unique morphism from B_C to T, then it holds that $k \circ e_B = m$. Since $\underline{E} \downarrow \underline{M}$, there is a morphism $B_C \xrightarrow{d} B$ so that the following diagram commutes:



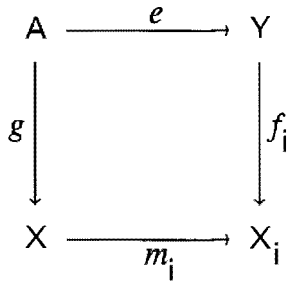
This shows that e_B is a section and hence an isomorphism (see, for example, [HS_1], 36.8). Since $B_C \in Ob(\underline{C})$ and $\underline{C}$ is isomorphism-closed, we get that $B \in Ob(\underline{C})$, as desired. □

Definition 1.3.4 ([HSV, 1(2)])

Let $\underline{E}$ be a class of $\underline{X}$-morphisms and let $\underline{M}$ be a collection of sources in $\underline{X}$ and suppose that both $\underline{E}$ and $\underline{M}$ are closed under composition with $\underline{X}$-isomorphisms. Then $(\underline{E}, \underline{M})$ is a *factorization structure* for sources on $\underline{X}$ if and only if the following conditions hold:

(1) Each source $(X, (f_i))_I$ in $\underline{X}$ has an $(\underline{E}, \underline{M})$-*factorization*; that is, there exist $X \xrightarrow{e} Y \in \underline{E}$ and a source $(Y, (m_i))_I$ in $\underline{M}$ such that $(f_i)_I = (m_i)_I \circ e$.

(2) $\underline{X}$ has the $(\underline{E}, \underline{M})$-*diagonalization property*; that is, $e \downarrow (m_i)_I$, i.e. given commutative squares with $e \in \underline{E}$ and $(m_i) \in \underline{M}$,



there, exists a unique $\underline{X}$-morphism d such that $d \circ e = g$ and $m_i \circ d = f_i$, for each $i \in I$.

Proposition 1.3.5 (See e.g. [HSV, Remark (2)] and [AHS, 15.19(1)])
In any category $\underline{X}$, each of the following statements implies those that follow it. If $\underline{X}$ has an initial object, the three statements are equivalent:

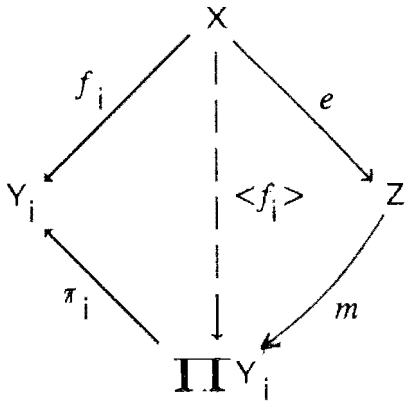
(a) $\underline{X}$ has products (respectively, finite products).

(b) In $\underline{X}$, every morphism factorization structure $(\underline{E}, \underline{M})$ can be extended to a factorization structure $(\underline{E}, \underline{M}')$ for set-indexed sources (resp. finite sources).

(c) The trivial morphism factorization structure $(\mathrm{Mor}(\underline{X}), \mathrm{Iso}(\underline{X}))$ can be extended to a factorization structure for set-indexed sources (resp., finite sources).

Proof

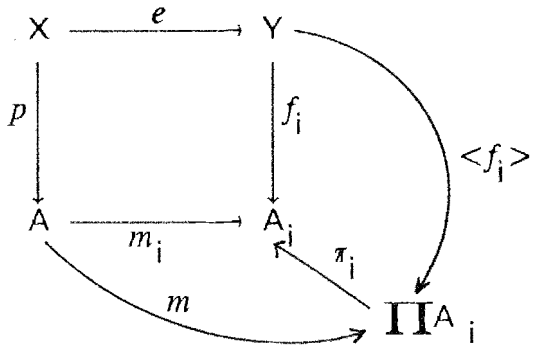
(a) $\Longrightarrow$ (b). Assume that $\underline{X}$ has products and that $(\underline{E}, \underline{M})$ is a morphism factorization structure on $\underline{X}$. Given a set-indexed source $(f_i: X \longrightarrow Y_i)_I$, if $(\prod Y_i, \pi_i)$ is the product of $(Y_i)_I$, then there is a unique $\underline{X}$-morphism $X \xrightarrow{\langle f_i \rangle} \prod Y_i$ such that $\pi_i \circ \langle f_i \rangle = f_i$, for each $i \in I$. Now let $\langle f_i \rangle = m \circ e$ be an $(\underline{E}, \underline{M})$-factorization of $\langle f_i \rangle$:



The factorization $(f_i)_I = ((\pi_i \circ m) \circ e)_I$ is an $(\underline{E}, \underline{M}')$-factorization of $(f_i)_I$, where

$$\underline{M}' = \{(m_i : W \longrightarrow B_i)_I \mid \text{there exists } m : W \longrightarrow \prod B_i \in \underline{M} \text{ such that } m_i = \pi_i \circ m, \text{ for each } i \in I\}.$$

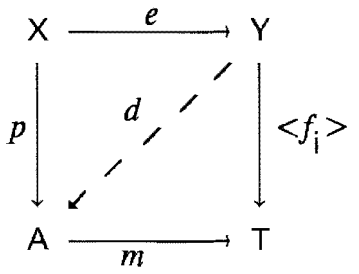
We will show that $(\underline{E}, \underline{M}')$ is a factorization structure for set-indexed sources in $\underline{X}$. Firstly, $\underline{M}'$ is closed under composition with $\underline{X}$-isomorphisms as required by Definition 1.1.2. Consider commutative squares (where X, e, f_i, etc. are not necessarily those in the first part of the proof.)



with $e \in \underline{E}$ and $(m_i)_I \in \underline{M}'$. By definition of $\underline{M}'$, $m_i = \pi_i \circ m$, for some $m \in \underline{M}$, where $(\prod A_i, \pi_i)$ is the product of $(A_i)_I$. There is a unique $\underline{X}$-morphism $\langle f_i \rangle$ such that $\pi_i \circ \langle f_i \rangle = f_i$, for each $i \in I$. We have

$$\pi_i \circ m \circ p = m_i \circ p = f_i \circ e = \pi_i \circ \langle f_i \rangle \circ e,$$

for each $i \in I$, so $m \circ p = \langle f_i \rangle \circ e$. Since $e \downarrow m$, there is a unique $\underline{\mathbf{X}}$-morphism d such that $m \circ d = \langle f_i \rangle$ and $d \circ e = p$:



We need only show that $m_i \circ d = f_i$, for each $i \in I$. But this follows from the equalities

$$m_i \circ d = \pi_i \circ m \circ d = \pi_i \circ \langle f_i \rangle = f_i,$$

for each $i \in I$. Hence $(\underline{\mathbf{E}}, \underline{\mathbf{M}}')$ is a factorization structure for set-indexed sources in $\underline{\mathbf{X}}$. (Similarly, if $\underline{\mathbf{X}}$ has finite products, then $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$ can be extended to a factorization structure for finite sources.)

(b) $\Longrightarrow$ (c). $(\mathrm{Mor}(\underline{\mathbf{X}}), \mathrm{Iso}(\underline{\mathbf{X}}))$ is a particular instance of the factorization structure $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$.

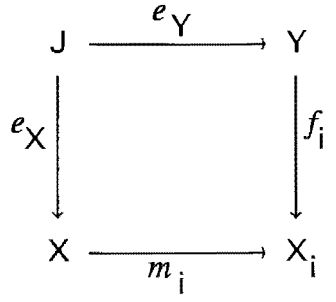
Now we assume that $\underline{\mathbf{X}}$ has an initial object J.

(c) $\Longrightarrow$ (a). Given the trivial factorization structure $(\mathrm{Mor}(\underline{\mathbf{X}}), \mathrm{Iso}(\underline{\mathbf{X}}))$, let $(\mathrm{Mor}(\underline{\mathbf{X}}), \underline{\mathbf{M}})$ be its extension. Suppose that $(X_i)_I$ is a set-indexed class of $\underline{\mathbf{X}}$-objects. For each $i \in I$, there exists a unique $\underline{\mathbf{X}}$-morphism $J \xrightarrow{e_i} X_i$. Let

$$J \xrightarrow{e_i} X_i = J \xrightarrow{e_X} X \xrightarrow{m_i} X_i$$

be a $(\mathrm{Mor}(\underline{\mathbf{X}}), \underline{\mathbf{M}})$-factorization of $(e_i)_I$. It is asserted that

$(X \xrightarrow{m_i} X_i)$ is a product of $(X_i)_I$. For, if $(Y \xrightarrow{f_i} X_i)_I$ is any indexed family of $\underline{X}$-morphisms, then the following squares (one for each $i \in I$) commute,



where e_Y is the unique morphism from J to Y. Since $e_Y \downarrow (m_i)_I$, there is a unique $\underline{X}$-morphism $Y \xrightarrow{d} X$ such that $d \circ e_Y = e_X$ and $m_i \circ d = f_i$, for each $i \in I$. □

Definition 1.3.6 (See e.g. [HS_1], 6.27)

A category $\underline{X}$ is called a *co-(well-powered)* category provided that each $\underline{X}$-object X has a representative class of quotient objects which is a set.

Proposition 1.3.7 (See also [HSV], 3(a))

Let $\underline{X}$ be a co-(well-powered) category and let $(\underline{E}, \underline{M})$ be a factorization structure for set-indexed sources. Then the following statements are equivalent:

(a) There exists an $\underline{M}'$ such that $(\underline{E}, \underline{M}')$ is a factorization structure for arbitrary sources.

(b) $\underline{E} \subseteq \mathrm{Epi}(\underline{X})$.

(c) If $(X, (m)) \in \underline{M}$, then $(X, (m, m)) \in \underline{M}$.

(d) Every $\underline{X}$-section belongs to $\underline{M}$.

Proof. See [AHS], Proposition 15.20.

1.4 EXISTENCE OF FACTORIZATION STRUCTURES

In this section, we shall give an existence theorem (Theorem 1.4.1) for factorization structures. Swell epimorphisms are defined and it is shown that on a category $\underline{X}$ which has (epi, mono-source)-factorizations for 2-sources, swell epimorphisms are the extremal epimorphisms. (Lemma 1.4.5). In Theorem 1.4.7, we show that an (epi, mono-source)-factorizable category is one which is (extremal epi, mono-source)-factorizable, or equivalently, one which is (swell epi, mono-source)-factorizable. Here follows the result due to Bousfield ([BO, 3.1]):

Theorem 1.4.1

Let $\underline{X}$ be a cocomplete category and let $\underline{E}$ be a family of $\underline{X}$-morphisms. Then, for some class $\underline{M}$ of $\underline{X}$-morphisms, $(\underline{E}, \underline{M})$ is a factorization structure on $\underline{X}$ if, and only if, $\underline{E}$ has the following properties:

(a) $\mathrm{Iso}(\underline{X}) \subseteq \underline{E}$.

(b) $\underline{E}$ is closed under composition.

(c) If $e = f \circ \hat{e}$ with $e, \hat{e} \in \underline{E}$, then $f \in \underline{E}$.

(d) $\underline{E}$ is closed under the formation of pushouts.

(e) $\underline{E}$ is closed under the formation of colimits.

(f) **Solution set condition (SSC)**: Each $\underline{X}$-morphism f has a set of factorizations $\{ X \xrightarrow{e_a} B_a \xrightarrow{g_a} Y \}$, with $e_a \in \underline{E}$ for all $a \in \mathcal{A}$ and such that any factorization of f, $X \xrightarrow{e} B \xrightarrow{g} Y$ with $e \in \underline{E}$, can be mapped (in the category $\underline{E}_f$ below) to some member of this set.

Proof

If $(\underline{E}, \underline{M})$ is a factorization structure on $\underline{X}$, then (a) through (e)

follow from Theorem 1.3.1 (dual). So, we need only establish the SSC. Given a $f \in \mathrm{Mor}(\underline{\mathbf{X}})$, the representative set required by the SSC can be taken to be any singleton set consisting of an $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$-factorization of f.

Conversely, suppose that the conditions of the theorem are satisfied. We define $\underline{\mathbf{M}}$ as follows:

$$\underline{\mathbf{M}} = \{ g \in \mathrm{Mor}(\underline{\mathbf{X}}) \mid f \downarrow g, \ \forall f \in \underline{\mathbf{E}} \}.$$

By [AHS, Proposition 14.7], we need only establish the following conditions:

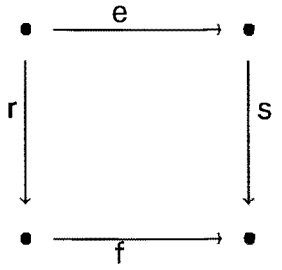
(1) $\mathrm{Iso}(\underline{\mathbf{X}}) \subseteq \underline{\mathbf{E}} \cap \underline{\mathbf{M}}$.

(2) Each of $\underline{\mathbf{E}}$ and $\underline{\mathbf{M}}$ is closed under composition.

(3) $\underline{\mathbf{X}}$ is $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$-factorizable, and each $(\underline{\mathbf{E}}, \underline{\mathbf{M}})$-factorization of an $\underline{\mathbf{X}}$-morphism is unique up to a commuting isomorphism.

For these we proceed as follows:

(1) Consider a commutative diagram

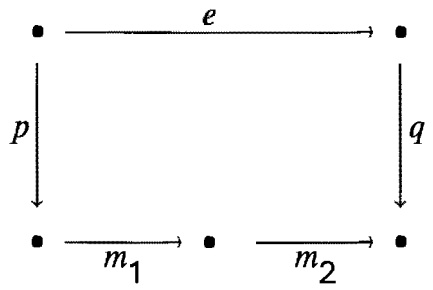


with $e \in \underline{\mathbf{E}}$ and $f \in \mathrm{Iso}(\underline{\mathbf{X}})$. Define $d = f^{-1} \circ s$. Then

$$d \circ e = (f^{-1} \circ s) \circ e = f^{-1} \circ (s \circ e) = f^{-1} \circ (f \circ r) = r$$

and $f \circ d = f \circ (f^{-1} \circ s) = s$. And d is unique such that $d \circ e = r$ and $f \circ d = s$ because f is a monomorphism; so $e \downarrow f$, hence $f \in \underline{\mathbf{M}}$. By (a), $f \in \underline{\mathbf{E}}$, thus $f \in \underline{\mathbf{E}} \cap \underline{\mathbf{M}}$.

(2) Given morphisms m_1, $m_2 \in \underline{\mathbf{M}}$ such that $m_2 \circ m_1$ is defined, consider the following commutative diagram, with $e \in \underline{\mathbf{E}}$:



Then there is a unique morphism d_1 with $m_2 \circ d_1 = q$ and $d_1 \circ e = m_1 \circ p$. Also there is a unique morphism d_2 such that $m_1 \circ d_2 = d_1$ and $d_2 \circ e = p$. Therefore

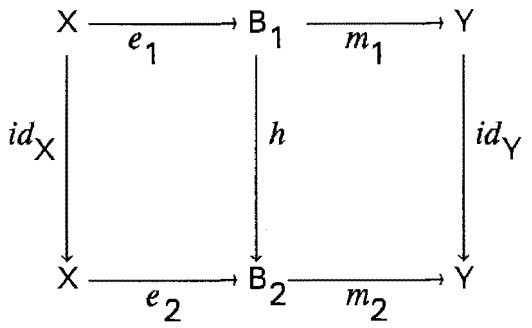
$$(m_2 \circ m_1) \circ d_2 = m_2 \circ (m_1 \circ d_2) = m_2 \circ d_1 = q$$

and $d_2 \circ e = p$. If d were another morphism satisfying the conditions that $(m_2 \circ m_1) \circ d = q$ and $d \circ e = p$, then

$$\begin{aligned} (m_2 \circ m_1) \circ d_2 = (m_2 \circ m_1) \circ d &\Rightarrow m_2 \circ d_1 = m_2 \circ (m_1 \circ d) \\ &\Rightarrow q = m_2 \circ (m_1 \circ d), \end{aligned}$$

and $(m_1 \circ d) \circ e = m_1 \circ p$. But d_1 is unique such that $q = m_2 \circ d_1$ and $d_1 \circ e = m_1 \circ p$, so $m_1 \circ d = d_1$. Since d_2 is unique such that $m_1 \circ d_2 = d_1$ and $d_2 \circ e = p$, it follows that $d_2 = d$. Hence $e \downarrow (m_2 \circ m_1)$, thus $m_2 \circ m_1 \in \underline{\mathbf{M}}$.

(3) Given an $\underline{\mathbf{X}}$-morphism $f : X \longrightarrow Y$, let $\underline{\mathbf{F}}_f$ be the category whose objects are factorizations $X \xrightarrow{e} B \xrightarrow{m} Y$ of f with $e \in \underline{\mathbf{E}}$, and whose morphisms are commutative diagrams:



We will show that $\underline{F}_f$ is cocomplete. Hence let $D : I \longrightarrow \underline{F}_f$ be a functor, where I is a small category. Define functors $D_1 : I \longrightarrow \underline{X}$ and $D_2 : I \longrightarrow \underline{X}$ as follows:

$$D_1(i) = X, \text{ for each } i \in Ob(I)$$

and

$$D_1(m) = id_X, \text{ for each } m \in Mor(I);$$

while

$$D_2(i) = B_i, \text{ whenever } D(i) = X \xrightarrow{e_i} B_i \xrightarrow{m_i} Y,$$

and if $m : i \longrightarrow j \in Mor(I)$, then

$$D_2(m : i \longrightarrow j) = B_i \xrightarrow{h} B_j$$

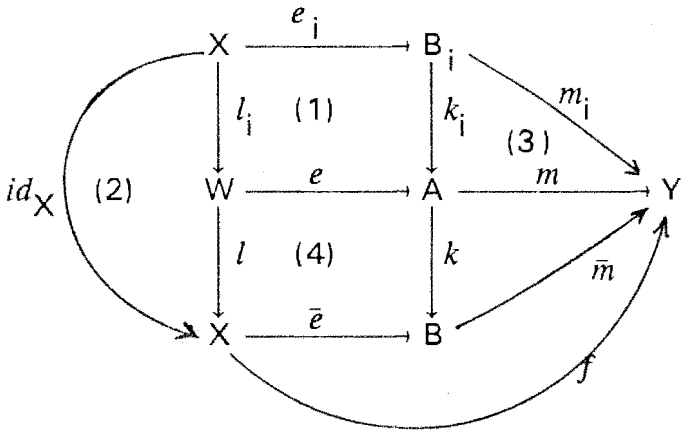
whenever

$$D(m : i \longrightarrow j) = \begin{array}{ccccc} X & \xrightarrow[e_i]{} & B_i & \xrightarrow[m_i]{} & Y \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id_Y} \\ X & \xrightarrow[e_j]{} & B_j & \xrightarrow[m_j]{} & Y \end{array}.$$

Let $\text{Colim } D_1 = (X \xrightarrow{l_i} W)_{i \in Ob(I)}$

and $\text{Colim } D_2 = (B_i \xrightarrow{k_i} A)_{i \in Ob(I)}$.

Consider the following diagram:

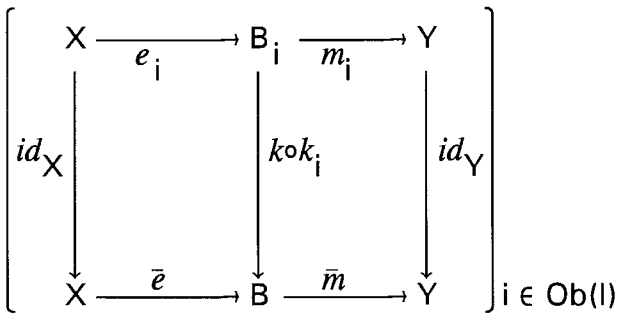


(*)

Since $(X \xrightarrow{l_i} W)_{i \in Ob(I)}$ and $(B_i \xrightarrow{k_i} A)_{i \in Ob(I)}$ are colimits, there is a unique $\underline{X}$-morphism $e : W \longrightarrow A$ such that square (1) commutes. But $(B_i \xrightarrow{m_i} Y)_{i \in Ob(I)}$ and $(X \xrightarrow{id_X} X)_{i \in Ob(I)}$ are natural sinks for D_2 and D_1, respectively; so, by the universal property of colimits, there are unique $\underline{X}$-morphisms $l : W \longrightarrow X$ and $m : A \longrightarrow Y$ such that squares (2) and (3) commute. By condition (e), the morphism e belongs to $\underline{E}$ since each $e_i \in \underline{E}$. Square (4) is a pushout diagram. Since $e \in E$, it follows from condition (d) that $\bar{e} \in E$. Now for each $i \in I$, it holds that

$$m \circ e \circ l_i = m \circ k_i \circ e_i = m_i \circ e_i = f \circ id_X = f \circ l \circ l_i.$$

Since colimits are (extremal epi)-sinks (see, for example, [HS_1, 20.4(dual)]), we have $m \circ e = f \circ l$. But square (4) is a pushout, so there is a unique $\underline{X}$-morphism $\bar{m} : B \longrightarrow Y$ such that $\bar{m} \circ k = m$ and $\bar{m} \circ \bar{e} = f$. It is asserted that the following sink in $\underline{F}_f$ is a colimit of D:



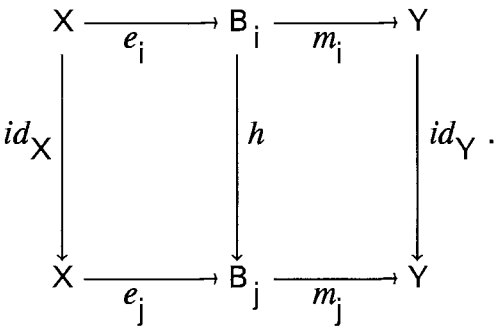
We have, for each $i \in I$,

$$\bar{m} \circ k \circ k_i = m \circ k_i = m_i = id_Y \circ m_i,$$

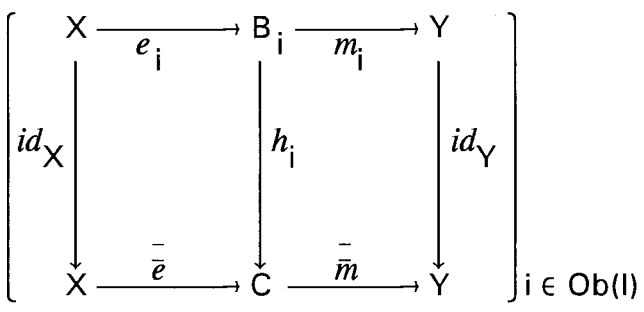
and

$$k \circ k_i \circ e_i = k \circ e \circ l_i = \bar{e} \circ l \circ l_i = \bar{e} \circ id_X,$$

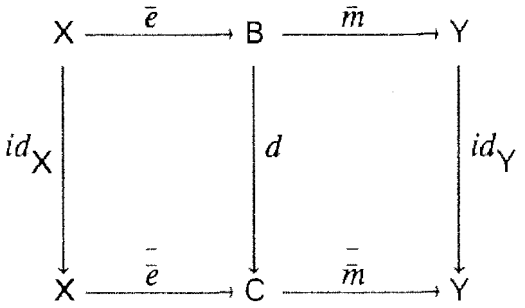
so both squares commute. We first show that this sink is natural for D. Hence let $m : i \longrightarrow j$ be a morphism in I. Its image under D is



It needs to be shown that $k \circ k_j \circ h = k \circ k_i$. But because $(k_i : B_i \longrightarrow A)_{i \in Ob(I)}$ is natural for D_2, we have $k_j \circ h = k_i$, and then $k \circ k_j \circ h = k \circ k_i$. Now suppose that the diagram

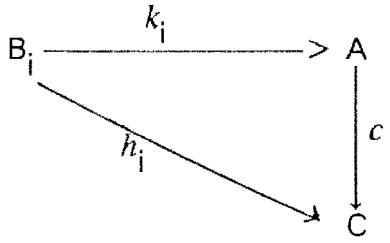


is also a natural sink for D. We seek a unique $\underline{X}$-morphism $d : B \longrightarrow C$ such that the digram

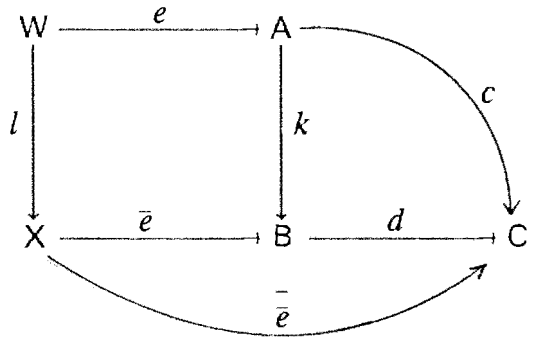


(★★)

commutes, and $d \circ k \circ k_i = h_i$, for each $i \in Ob(I)$. Now $(B_i \xrightarrow{h_i} C)_{i \in Ob(I)}$ is a natural sink for D_2, so we obtain a unique $\underline{X}$-morphism $c : A \longrightarrow C$ such that the triangle



commutes, for each $i \in Ob(I)$.



(★★★)

For each $i \in Ob(I)$, we have that

$$c \circ e \circ l_i = c \circ k_i \circ e_i = h_i \circ e_i = \bar{e} = \bar{e} \circ l \circ l_i.$$

Since colimits are epi-sinks, it follows that $c \circ e = \bar{e} \circ l$. But square (4) is a pushout, so there is a unique $\underline{X}$-morphism

$d : B \longrightarrow C$ such that $d \circ k = c$ and $d \circ \bar{e} = \bar{e}$ (in diagram $(\star\star\star)$ above). For each $i \in Ob(I)$, we also have

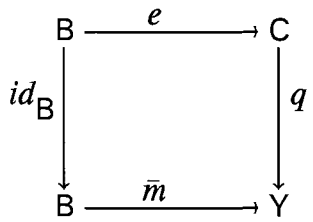
$$\bar{m} \circ d \circ k \circ k_i = \bar{m} \circ c \circ k_i = \bar{m} \circ h_i = id_X \circ m_i = m \circ k_i.$$

But $(k_i : B_i \longrightarrow A)_{i \in Ob(I)}$ is a colimit (and so, an epi-sink), so $\bar{m} \circ d \circ k = m$. Also $\bar{m} \circ d \circ \bar{e} = \bar{m} \circ \bar{e} = f$. By the uniqueness of $\bar{m}$ such that $\bar{m} \circ k = m$ and $\bar{m} \circ \bar{e} = f$ (see diagram $(\star)$), it follows that $\bar{m} \circ d = \bar{m}$. Hence d is such that the squares in diagram $(\star\star)$ commute. We also have that $d \circ k \circ k_i = c \circ k_i = h_i$, for each $i \in Ob(I)$.

Finally, suppose that $\bar{d}$ is also such that $\bar{d} \circ \bar{e} = \bar{e}$, $\bar{m} \circ \bar{d} = \bar{m}$ and $\bar{d} \circ k \circ k_i = h_i$, for each $i \in Ob(I)$. Now, for each $i \in Ob(I)$, we have $\bar{d} \circ k \circ k_i = d \circ k \circ k_i$. Since colimits are epi-sinks, it follows that $\bar{d} \circ k = d \circ k = c$. Since $\bar{d} \circ \bar{e} = \bar{e}$ also, and d is unique such that $d \circ k = c$ and $d \circ \bar{e} = \bar{e}$ (in diagram $(\star\star\star)$), we have $d = \bar{d}$. By the dual of the existence theorem of ([MAC, p. 116]), it follows that $\underline{\mathbf{F}}_f$ has a terminal object, i.e. there exists a factorization $X \xrightarrow{\bar{e}} B \xrightarrow{\bar{m}} Y$ of f with $\bar{e} \in \underline{\mathbf{E}}$, such that if $X \xrightarrow{e} B' \xrightarrow{m} Y$ is any factorization of f with $e \in \underline{\mathbf{E}}$, then there is a unique $\underline{\mathbf{X}}$-morphism $h : B' \longrightarrow B$ so that the following diagram commutes:

$$\begin{array}{ccccc} X & \xrightarrow{e} & B' & \xrightarrow{m} & Y \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id_Y} \\ X & \xrightarrow{\bar{e}} & B & \xrightarrow{\bar{m}} & Y \end{array}$$

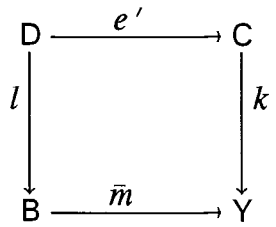
It must now be shown that $\bar{m}$ belongs to **M** as defined above. To do this, it is enough to prove the existence of a unique diagonal morphism in diagrams of the form:



(1)

where $e \in \underline{E}$. This will follow from the equivalence of the following statements:

(i) A unique diagonal morphism exists for commutative diagrams of the form



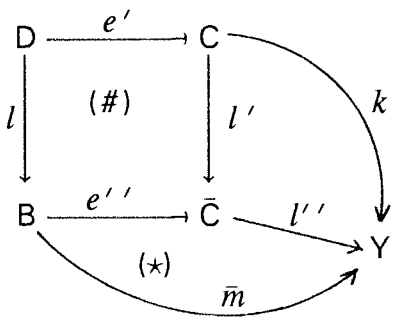
(2)

where $e' \in \underline{E}$.

(ii) A unique diagonal morphism exists for commutative squares of the form (1), where $e \in \underline{E}$.

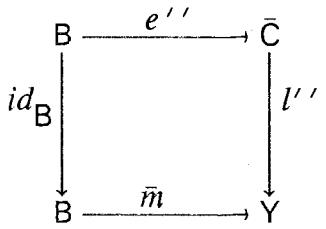
That **(i) ⟹ (ii)** is obvious.

(ii) ⟹ (i). Suppose we have a commutative square, as given in (i). We form a pushout square as indicated in (#) below:



(3)

By condition (d), we have $e'' \in \underline{E}$. By the pushout property, there is a unique $\underline{X}$-morphism $l'' : \bar{C} \longrightarrow Y$ such that the triangles in diagram (3) above commute. From triangle $(\star)$, we have a commutative square of the form given in (ii):



so there is a unique diagonal $\underline{X}$-morphism $d : \bar{C} \longrightarrow B$ such that $\bar{m} \circ d = l''$ and $d \circ e'' = id_B$. Put $\bar{d} = d \circ l'$. Then

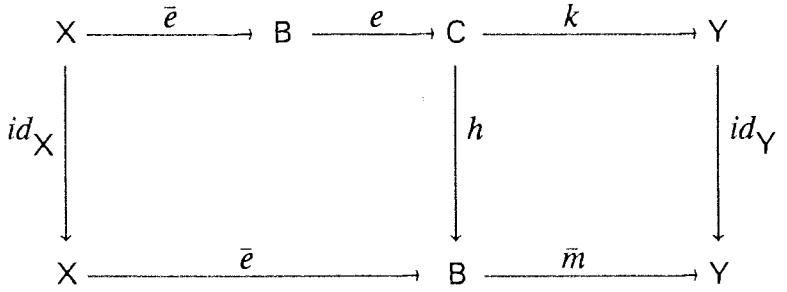
$$\bar{m} \circ \bar{d} = \bar{m} \circ (d \circ l') = (\bar{m} \circ d) \circ l' = l'' \circ l' = k$$

and

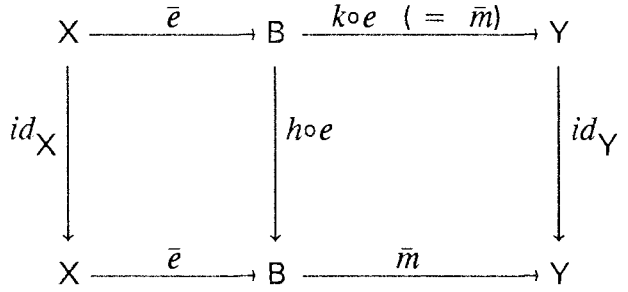
$$\bar{d} \circ e' = (d \circ l') \circ e' = d \circ (l' \circ e') = d \circ (e'' \circ l) = id_X \circ l = l.$$

But pushouts are colimits, and therefore they are epi-sinks (see, for example, [HS_1, 20.4(dual)]); so $\bar{d}$ is the unique diagonal $\underline{X}$-morphism for square (2).

Now it remains to show that there is a unique diagonal $\underline{X}$-morphism for a commutative diagram of the form (1). By condition (b), we have a factorization $X \xrightarrow{e \circ \bar{e}} C \xrightarrow{k} Y$ of the morphism f with $e \circ \bar{e} \in \underline{E}$, which we may assume comes from the representative set:

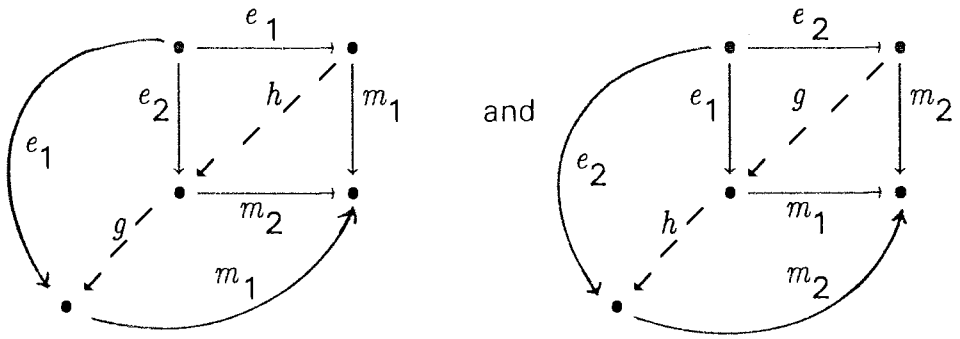


By what we proved initially, there is a unique $\underline{X}$-morphism h which makes the above diagram commutative. It must be shown that h is also that unique diagonal $\underline{X}$-morphism for diagram (1); so it also needs to be shown that $h \circ e = id_B$. But $h \circ e$ makes the following diagram commute:



Since id_B also makes this diagram commutative, it follows from the fact that $X \xrightarrow{\bar{e}} B \xrightarrow{\bar{m}} Y$ is a terminal object that $h \circ e = id_B$.

Now let $m_1 \circ e_1 = f = m_2 \circ e_2$ be any two ($\underline{E}$, $\underline{M}$)-factorizations of f. Then there exist unique $\underline{X}$-morphisms h and g such that each of the following diagrams commutes:



Since $e_i \downarrow m_i$, for each i = 1, 2, we must have $g \circ h = id$ and $h \circ g = id$. Hence h is an isomorphism. Consequently, ($\underline{E}$, $\underline{M}$) is a factorization structure on $\underline{X}$. □

Definition 1.4.2 (See also [HS_1, 17.15(4)])

Let $\underline{M}$ be a class of sources in $\underline{X}$. The category $\underline{X}$ is called an $\underline{M}$-*well-powered* category provided that each $\underline{X}$-object has a representative set of $\underline{M}$-subobjects. (An $\underline{M}$-subobject is a pair (X, f), where $\underline{X}$ is the domain of f and $f \in \underline{M}$.)

Proposition 1.4.3

Let $(\underline{E}_i, \underline{M}_i)_I$ be a family of factorization structures on a cocomplete $\underline{M}$-well-powered category $\underline{X}$ for which ($\underline{E}$, $\underline{M}$) is also a factorization structure with $\underline{E} \subseteq \underline{E}_i$, for each $i \in I$. Then there is some family $\underline{M}'$ of $\underline{X}$-morphisms such that $(\bigcap_I \underline{E}_i, \underline{M}')$ is also a factorization structure on $\underline{X}$.

Proof

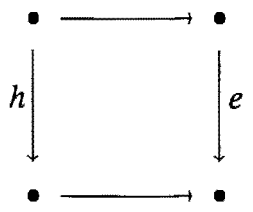
In view of Bousfield's Characterization Theorem, we need only show that $\bigcap_I \underline{E}_i$ satisfies conditions (a) through (f) of Theorem 1.4.1.

(a) Since ($\underline{E}$, $\underline{M}$) is a factorization structure on $\underline{X}$, $\underline{E}$ satisfies the conditions of Theorem 1.4.1. Since $\underline{E} \subseteq \underline{E}_i$, for each $i \in I$, we have $\underline{E} \subseteq \bigcap_I \underline{E}_i$. Since $Iso(\underline{X}) \subseteq \underline{E}$, so $Iso(\underline{X}) \subseteq \bigcap_I \underline{E}_i$.

(b) Suppose $e_1, e_2 \in \bigcap_I \underline{E}_i$. Since each $\underline{E}_i$ is closed under composition, it follows that $e_2 \circ e_1$ (when defined) belongs to each E_i, thus $e_2 \circ e_1 \in \bigcap_I \underline{E}_i$.

(c) Suppose that $e = f \circ e'$, where $e, e' \in \bigcap_I \underline{E}_i$. Then $e, e' \in \underline{E}_i$, for each i, so $f \in \underline{E}_i$, for each i (Theorem 1.4.1(c)), hence $f \in \bigcap_I \underline{E}_i$.

(d) Given a pushout diagram in $\underline{X}$



with $h \in \bigcap_I \underline{E}_i$, we have $h \in \underline{E}_i$, for each i, so that $e \in \underline{E}_i$, for each i (Theorem 1.4.1(d)), hence $e \in \bigcap_I \underline{E}_i$.

(e) Since each $(\underline{E}_i, \underline{M}_i)$ is a factorization structure on $\underline{X}$, by Proposition 1.3.1(8) (dual) each $\underline{E}_i$ is closed under colimits. Consequently, $\bigcap_I \underline{E}_i$ is closed under colimits.

(f) Given an $\underline{X}$-morphism $X \xrightarrow{f} Y$, we consider all factorizations

$$X \xrightarrow{f} Y \;=\; X \xrightarrow{h_j} A_j \xrightarrow{m_j} Y$$

of f, where each m_j belongs to the representative set of $\underline{M}$-subobjects and each $h_j \in \bigcap_I \underline{E}_i$. We assert that this is a solution for f. For suppose,

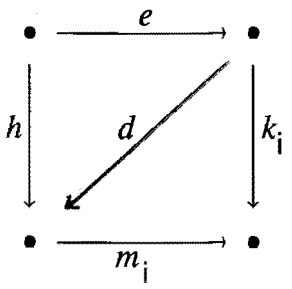
$$X \xrightarrow{f} Y \;=\; X \xrightarrow{\hat{e}} M \xrightarrow{p} Y$$

with $\hat{e} \in \bigcap_I \underline{E}_i$ and $p = m \circ e$ is an $(\underline{E}, \underline{M})$-factorization of p. Then $e \in \underline{E}_i$, for each $i \in I$, so that $e \in \bigcap_I \underline{E}_i$ and (by (b) above)

$e \circ \hat{e} \in \bigcap_I \underline{E}_i$. Hence the factorization $m \circ (e \circ \hat{e})$ of f is isomorphic to one of the factorizations of f in the set above. □

Definition 1.4.4

(a) An $\underline{X}$-morphism e is called a *swell epimorphism* provided that it diagonalizes over mono-sources; that is, whenever $(m_i)_I$ is a mono-source and $k_i \circ e = m_i \circ h$, for each $i \in I$, then $e \downarrow (m_i)_I$; that is, there exists an $\underline{X}$-morphism d such that the following diagram is commutative:



(b) A monomorphism f is called an *extremal monomorphism* if it satisfies the condition that: If $f = h \circ e$ with $e \in \text{Epi}(\underline{X})$, then $e \in \text{Iso}(\underline{X})$.

Dually: An *extremal epimorphism.*

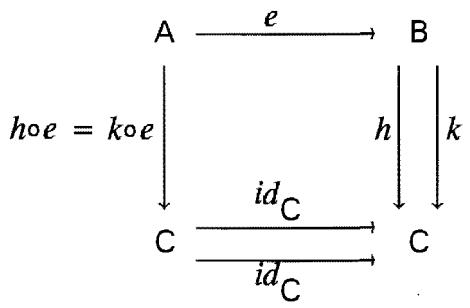
Lemma 1.4.5

Suppose a category $\underline{X}$ has (epi, mono-source)-factorizations for 2-sources. Then in $\underline{X}$ the extremal epimorphisms are the swell epimorphisms.

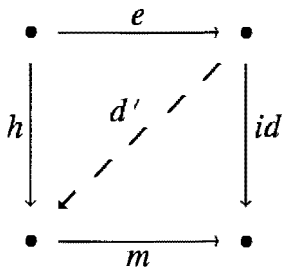
Proof

Given a swell-epimorphism $A \xrightarrow{e} B$, let $B \underset{k}{\overset{h}{\rightrightarrows}} C$ be a pair of $\underline{X}$-morphisms such that $h \circ e = k \circ e$. In the following commutative

diagrams, the pair $(C \xrightarrow{id_C} C,\ C \xrightarrow{id_C} C)$ is a mono-source:

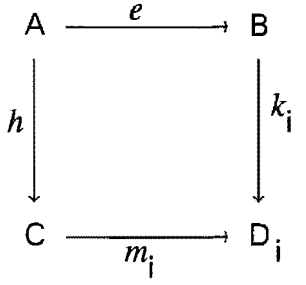


Since e is a swell epimorphism, there is an $\underline{X}$-morphism $B \xrightarrow{d} C$ which completes the above diagram, hence $h = k$. To establish the extremal condition, let $e = m \circ h$ with $m \in \text{Mono}(\underline{X})$. Since m is a mono-source, there is an $\underline{X}$-morphism d' making the following diagram commutative:



Hence m is a retraction, so m is an isomorphism (see e.g $[HS_1]$, Proposition 6.7).

Conversely, suppose that $A \xrightarrow{e} B$ is an extremal epimorphism and suppose that the diagram



commutes, for each $i \in I$, where $(C \xrightarrow{m_i} D_i)_I$ is a mono-source. Let $(e', (m_1, m_2))$ be an (epi, mono-source)-factorization of the two-source

(e, h). We have

$$m_i \circ m_2 \circ e' = m_i \circ h = k_i \circ e = k_i \circ m_1 \circ e',$$

and, since e' is an epimorphism, it follows that $m_i \circ m_2 = k_i \circ m_1$, for each i. To show that m_1 is a monomorphism, we assume that $m_1 \circ l = m_1 \circ \bar{l}$, for some $\underline{X}$-morphisms l and $\bar{l}$. Then, for each $i \in I$,

$$m_i \circ m_2 \circ l = k_i \circ m_1 \circ l = k_i \circ m_1 \circ \bar{l} = m_i \circ m_2 \circ \bar{l}$$

and, since $(m_i)_I$ is a mono-source, $m_2 \circ l = m_2 \circ \bar{l}$. But (m_1, m_2) is a mono-source, so $l = \bar{l}$. Hence m_1 is a monomorphism. Since e is an extremal epimorphism, it follows that m_1 is an isomorphism. Hence the morphism $m_2 \circ m_1^{-1}$ satisfies the relations:

$$(m_2 \circ m_1^{-1}) \circ e = m_2 \circ (m_1^{-1} \circ e) = m_2 \circ (m_1^{-1} \circ m_1) \circ e' = m_2 \circ e' = h$$

and

$$m_i \circ (m_2 \circ m_1^{-1}) \circ e = m_i \circ (m_2 \circ m_1^{-1} \circ m_1 e') = m_i \circ (m_2 \circ e') = m_i \circ h = k_i \circ e,$$

so (since e is an epimorphism) $m_i \circ (m_2 \circ m_1^{-1}) = k_i$, for each $i \in I$. Consequently, $m_2 \circ m_1^{-1}$ is the required diagonal morphism. Thus e is a swell epimorphism, and the lemma is proved. □

Theorem 1.4.6

Every category that has (epi, mono-source)-factorizations is an (extremal epi, mono-source)-category.

Proof. See ([AHS], p. 244). □

Theorem 1.4.7

In any category $\underline{X}$, the following statements are equivalent:

(1) $\underline{X}$ is (epi, mono-source)-factorizable.

(2) $\underline{X}$ is (extremal epi, mono-source)-factorizable.

(3) (Swell epi, mono-source) is a factorization structure for sources in $\underline{X}$.

(4) (Extremal epi, mono-source) is a factorization structure for sources in $\underline{X}$.

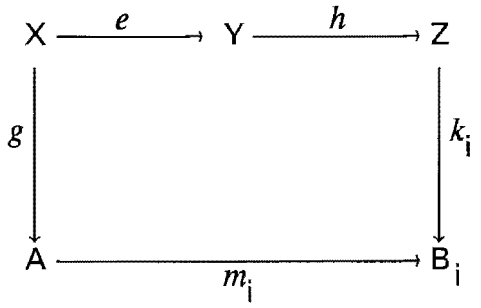
(5) ($\underline{E}$, $\underline{M}$) is a factorization structure for sources in $\underline{X}$, for some class $\underline{E}$ of morphisms and collection $\underline{M}$ of mono-sources.

Proof

(1) $\Rightarrow$ (2). Suppose that $\underline{X}$ is (epi, mono-source)-factorizable. By Theorem 1.4.6, $\underline{X}$ is an (extremal epi, mono-source)-category, hence (extremal epi, mono-source)-factorizable.

(2) $\Rightarrow$ (3). If $\underline{X}$ is (extremal epi, mono-source)-factorizable, then, by Lemma 1.4.5, every source in $\underline{X}$ is (swell epi, mono-source)-factorizable.

It must be shown that mono-sources and swell epimorphisms are closed under composition with $\underline{X}$-isomorphisms as required by Definition 1.3.4. Given a mono-source $(m_i)_I$ and an $\underline{X}$-isomorphism h, let r and s be $\underline{X}$-morphisms such that $(h \circ m_i) \circ r = (h \circ m_i) \circ s$, for each $i \in I$. Then $h \circ (m_i \circ r) = h \circ (m_i \circ s)$, for each $i \in I$. Since h is an $\underline{X}$-monomorphism, it follows that $m_i \circ r = m_i \circ s$, for each $i \in I$. But $(m_i)_I$ is a mono-source, so $r = s$; i.e. $(h \circ m_i)$ is a mono-source. That $(m_i \circ h_i)_I$ (where each h_i is an isomorphism) is also a mono-source is equally easy. Now Let $X \xrightarrow{e} Y$ be a swell epimorphism, let $Y \xrightarrow{h} Z$ be an $\underline{X}$-isomorphism and consider the following commutative squares (one for each $i \in I$):



where $(m_i)_I$ is a mono-source. Then there is an $\underline{X}$-morphism $d : Y \longrightarrow A$ such that $d \circ e = g$ and $m_i \circ d = k_i \circ h$, for each $i \in I$. Define $Z \xrightarrow{k} A$ to be the $\underline{X}$-morphism $k = d \circ h^{-1}$. Then k satisfies

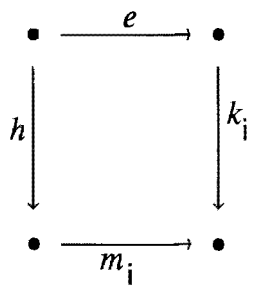
$$k \circ (h \circ e) = (d \circ h^{-1}) \circ h \circ e = d \circ e = g$$

and

$$m_i \circ k = m_i \circ (d \circ h^{-1}) = (m_i \circ d) \circ h^{-1} = (k_i \circ h) \circ h^{-1} = k_i,$$

for each $i \in I$. And k is unique such that $k \circ (h \circ e) = g$ and $m_i \circ k = k_i$, for each $i \in I$, because $(m_i)_I$ is a mono-source. In a similar way, we can show that swell epimorphisms are closed under composition with $\underline{X}$-isomorphisms on the right.

Now let the following square be commutative with e a swell epimorphism and $(m_i)_I$ a mono-source:



By definition, there is a diagonal morphism d which completes the above diagram, for each i. If d' also makes this diagram commute, then $d \circ e = h = d' \circ e$ implies that $d = d'$, since e

(being a swell epimorphism) is an epimorphism. Consequently, (swell epi, mono-source) is a factorization structure for sources in $\underline{X}$.

(3) $\Longrightarrow$ (4). If (swell epi, mono-source) is a factorization structure for sources in $\underline{X}$, then, in particular, we have an (epi, mono-source)-factorization for 2-sources, so the swell epimorphisms and the extremal epimorphisms coincide by Lemma 1.4.5.

(4) $\Longrightarrow$ (5). This is obvious.

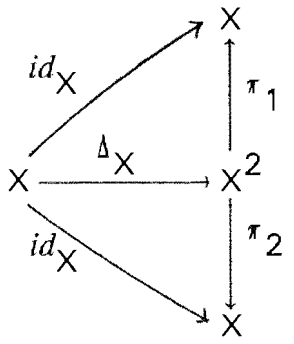
(5) $\Longrightarrow$ (1). By ([AHS], 15.4), $\underline{E}$ is a class of $\underline{X}$-epimorphisms, and so ($\underline{E}$, $\underline{M}$) is an (epi, mono-source)-factorization structure for sources in $\underline{X}$ (also see e.g. [HSV], 1.2(3)). □

1.5 OTHER PROPERTIES OF ($\underline{E}$, $\underline{M}$)-FACTORIZATION STRUCTURES

We start with the following:

Definition 1.5.1

Assume that $\underline{X}$ has finite products. Let $X \in Ob(\underline{X})$. The unique $\underline{X}$-morphism $\Delta_X : X \longrightarrow X^2$ such that the following diagram commutes (π_1 and π_2 are the projections $X^2 \longrightarrow X$) is called the *diagonal morphism*:



We prove that on an (E, M)-category X with products of pairs, the class E consists of epimorphisms if and only if, for each $X \in Ob(\underline{X})$, $\Delta_X \in \underline{M}$ (Theorem 1.5.7).

Definition 1.5.2 ([CA], p. 289)

Let M be a class of X-morphisms. We define $(\underline{M})^{\uparrow}$ and $(\underline{M})^{\downarrow}$ as follows:

(a) $(\underline{M})^{\uparrow} = \{ f \in Mor(\underline{X}) \mid f \downarrow g, \text{ for each } g \in \underline{M} \}$.

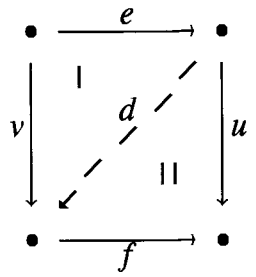
(b) $(\underline{M})^{\downarrow} = \{ g \in Mor(\underline{X}) \mid f \downarrow g, \text{ for each } f \in \underline{M} \}$.

In view of this definition, we have

$$[\, Epi(\underline{X}) \,]^{\downarrow} = \{ g \in Mor(\underline{X}) \mid e \downarrow g, \text{ for each } e \in Epi(\underline{X}) \}.$$

Definition 1.5.3 ([CA], p. 292)

An X-morphism f is called a *strong monomorphism* if it is a monomorphism and belongs to the class $[\, Epi(\underline{X}) \,]^{\downarrow}$; that is, f is a strong monomorphism provided that $f \in Mono(\underline{X}) \cap [\, Epi(\underline{X}) \,]^{\downarrow}$. This means that a monomorphism f is a strong monomorphism if, whenever $f \circ v = u \circ e$ and $e \in Epi(\underline{X})$, there is an X-morphism d which makes the following diagram commutative: (See [KE], p. 129.)



In this case, uniqueness of d follows from the fact that $e \in \text{Epi}(\underline{\mathbf{X}})$. Note also that, if triangle (I) commutes in the above diagram with $e \in \text{Epi}(\underline{\mathbf{X}})$, then $(f \circ d) \circ e = f \circ (d \circ e) = f \circ v = u \circ e$, so that $f \circ d = u$. Thus triangle (II) commutes as well. Dually, triangle (I) commutes whenever (II) does with f a strong monomorphism.

Definition 1.5.4 ([HS_1], 16.13)

Let $X \xrightarrow{f} A$ be an $\underline{\mathbf{X}}$-morphism. The pair (X, f) is called a *regular subobject* of A (and f is then called a *regular monomorphism*) if (X, f) equalizes some pair $A \underset{h}{\overset{g}{\rightrightarrows}} B$ of $\underline{\mathbf{X}}$-morphism.

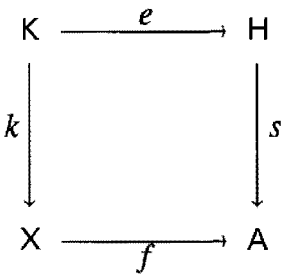
Lemma 1.5.5

(a) Every $\underline{\mathbf{X}}$-regular monomorphism is a strong monomorphism.

(b) Every strong monomorphism is an extremal monomorphism; in particular, a regular monomorphism is an extremal monomorphism.

Proof

(a) Let (X, f) be a regular subobject of $A \in \text{Ob}(\underline{\mathbf{X}})$. Pick $\underline{\mathbf{X}}$-morphisms $A \underset{w}{\overset{t}{\rightrightarrows}} B$ such that $(X, f) \approx \text{Equ}(t, w)$. To begin with, f is an $\underline{\mathbf{X}}$-monomorphisms ([HS_1], 16.15). Let $e \in \text{Epi}(\underline{\mathbf{X}})$. We need only show that $e \downarrow f$; that is, given a commutative square



(⋆)

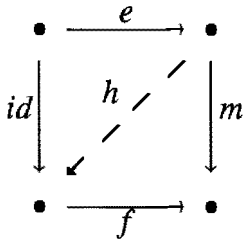
there is a unique $\underline{\mathbf{X}}$-morphism $H \xrightarrow{d} X$ which completes the above square. Now commutativity of the square yields:

$$(t \circ s) \circ e = t \circ (s \circ e) = t \circ (f \circ k) = (t \circ f) \circ k = (w \circ f) \circ k$$

$$= w \circ (f \circ k) = w \circ (s \circ e) = (w \circ s) \circ e$$

and, since e is an epimorphism, $t \circ s = w \circ s$. Since $(X, f) \approx \mathrm{Equ}(t, w)$, there exists a unique $\underline{\mathbf{X}}$-morphism $H \xrightarrow{d} X$ such that $f \circ d = s$. That $d \circ e = k$ follows from the commutativity of triangles (I) and (II) in Definition 1.5.3 above.

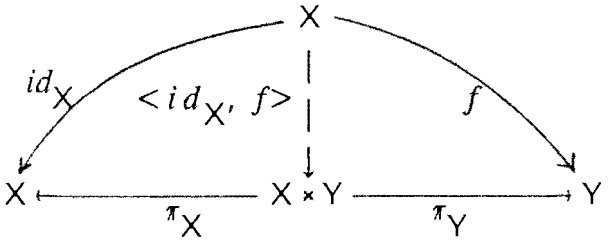
(b) Let f be a strong monomorphism and suppose that $f = m \circ e$, where e is an epimorphism. We need only show that e is an isomorphism. Since e is an epimorphism, there exists a unique $\underline{\mathbf{X}}$-morphism h which renders the following diagram commutative, since $f \in [\mathrm{Epi}(\underline{\mathbf{X}})]^{\downarrow}$:



Thus $h \circ e = id$, so e is a section and, being an epimorphism, it is an isomorphism (see, for example, [HS_1, 6.15]). Hence f is an extremal monomorphism. The second assertion is immediate from the first and part (a). □

Definition 1.5.6

Suppose that the category $\underline{X}$ has products of pairs. If $X \xrightarrow{f} Y$ is an $\underline{X}$-morphism, we define *the graph* of f to be the unique $\underline{X}$-morphism $\langle id_X, f\rangle : X \longrightarrow X \times Y$:



Theorem 1.5.7

Let $(\underline{E}, \underline{M})$ be a morphism factorization structure on a category $\underline{X}$ which has products of pairs. Then the following statements are equivalent:

(1) $\underline{E} \subseteq \mathrm{Epi}(\underline{X})$.

(2) Every $\underline{X}$-extremal monomorphism belongs to $\underline{M}$.

(3) Every $\underline{X}$-strong monomorphism belongs to $\underline{M}$.

(4) Every $\underline{X}$-regular monomorphism belongs to $\underline{M}$.

(5) Every $\underline{X}$-section belongs to $\underline{M}$.

(6) If $X \xrightarrow{f} Z \in \underline{M}$ and $X \xrightarrow{f} Y \in \mathrm{Mor}(\underline{X})$, then $X \xrightarrow{\langle m,\ f\rangle} Z \times Y \in \underline{M}$.

(7) For each $X \xrightarrow{f} Y \in \mathrm{Mor}(\underline{X})$, $\langle id_X, f\rangle \in \underline{M}$.

(8) If $X \xrightarrow{m} Z \in \underline{M}$, then $X \xrightarrow{\langle m,\ m\rangle} Z^2 \in \underline{M}$.

(9) For each $X \in \mathrm{Ob}(\underline{X})$, $X \xrightarrow{\Delta_X} X^2 \in \underline{M}$.

(10) If $g \circ f \in \underline{M}$, then $f \in \underline{M}$.

(11) If $g \circ e \in \underline{M}$ and $e \in \underline{E}$, then $e \in \mathrm{Iso}(\underline{X})$.

(12) $\underline{M} = \{f \in \mathrm{Mor}(\underline{X}) \mid f = g \circ e,\ e \in \underline{E}$ implies $e \in \mathrm{Iso}(\underline{X})\}$.

Proof

The implications

$$(9) \Rightarrow (10) \Rightarrow (11) \Rightarrow (12) \Rightarrow (1)$$

follow from [AHS, Proposition 14.11].

We shall only establish the chain

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (7) \Rightarrow (6) \Rightarrow (8) \Rightarrow (9).$$

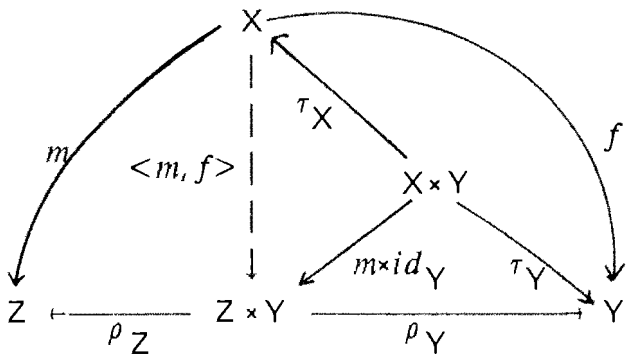
The implications $(2) \Rightarrow (3) \Rightarrow (4)$ follow from Lemma 1.5.5.

(1) $\Rightarrow$ (2). Given an **X**-extremal monomorphism f, let $f = m \circ e$ be its (**E**, **M**)-factorization. By (1), $e \in \mathrm{Epi}(\mathbf{X})$, so, since f is extremal, e is an isomorphism. But **M** is closed under composition on the right with isomorphisms, so $f \in \mathbf{M}$.

(4) $\Rightarrow$ (5). An **X**-section is a regular monomorphism (see, for example, [HS_1], 16.15(1)) and, by (4), it must belong to **M**.

(5) $\Rightarrow$ (7). Given the projection $X \times Y \xrightarrow{\pi_X} X$, we have $\pi_X \circ \langle id_X, f \rangle = id_X$, so $\langle id_X, f \rangle$ is an **X**-section. By (5), the morphism $\langle id_X, f \rangle$ belongs to **M**.

(7) $\Rightarrow$ (6). Let $X \xrightarrow{m} Z \in \mathbf{M}$ and let $X \xrightarrow{f} Y \in \mathrm{Mor}(\mathbf{X})$. Suppose that $(X \times Y, \tau_X, \tau_Y)$ and $(Z \times Y, \rho_Z, \rho_Y)$ are the products of the pairs (X, Y) and (Z, Y), respectively. By commutativity of the following diagram



we have

$$\rho_Z \circ [\, (m \times id_Y) \circ <id_X, f> \,] = [\rho_Z \circ (m \times id_Y)\,] \circ <id_X, f>$$
$$= (m \circ \tau_X) \circ <id_X, f> = m \circ (\tau_X \circ <id_X, f>)$$
$$= m \circ id_X = \rho_Z \circ <m, f>$$

and

$$[\, \rho_Y \circ (m \times id_Y)\,] \circ <id_X, f> = id_Y \circ \tau_Y \circ <id_X, f> = id_Y \circ f$$
$$= id_Y \circ \rho_Y \circ <m, f> = \rho_Y \circ <m, f>.$$

But (ρ_Z, ρ_Y) is a mono-source, so we must have

$$(m \times id_Y) \circ <id_X, f> = <m, f>.$$

Since $id_Y \in \underline{\mathbf{M}}$ (Proposition 1.3.1(4)), we have $m \times id_Y \in \underline{\mathbf{M}}$ (Proposition 1.3.1(8)), so $(m \times id_Y) \circ <id_X, f>$ belongs to $\underline{\mathbf{M}}$ (Proposition 1.3.1(5)), i.e $<m, f> \in \underline{\mathbf{M}}$.

(6) $\Longrightarrow$ (8). Clear.

(8) $\Longrightarrow$ (9). Since $id_X \in \underline{\mathbf{M}}$ (Proposition 1.3.1(4)), we have

$$<id_X, id_X> = \Delta_X \in \underline{\mathbf{M}}.$$

The theorem is now proved. □

CHAPTER 2

GALOIS CORRESPONDENCE, $\underline{E}$-SEPARATED OBJECTS AND $\underline{A}$-EPIMORPHISMS

In this chapter, we give a Galois Correspondence between the collection of all subclasses of $\underline{X}$-morphisms, and the collection of all subclasses of $\underline{X}$-objects (Proposition 2.2.2); we show that the class $\underline{A}$-Epi of $\underline{A}$-epimorphisms are those morphisms that diagonalize over $\underline{A}$-regular morphisms (Proposition 2.1.3); $\underline{A}$-Epi contains the class Epi($\underline{X}$) of $\underline{X}$-epimorphisms (Lemma 2.5.2); it is also shown that there is some class $\underline{M}'$ of $\underline{X}$-morphisms such that ($\underline{A}$-Epi, $\underline{M}'$) is a factorization structure on an $\underline{M}$-well-powered ($\underline{E}$, $\underline{M}$)-category $\underline{X}$ with $\underline{E} \subseteq$ Epi($\underline{X}$) (Theorem 2.5.3).

It is shown that if ($\underline{E}$, $\underline{M}$) is a factorization structure on a finitely complete category $\underline{X}$, then $\underline{E}$-separated objects are precisely those $\underline{X}$-objects Y for which the diagonal $\Delta_Y : Y \longrightarrow Y^2$ belongs to $\underline{M}$ (Theorem 2.4.1); that for a suitable category $\underline{X}$, $\underline{E}$-Sep is a (swell epi)-reflective subcategory (Proposition 2.5.5) and, further, that $\underline{E}$-Sep is $\underline{M}$-hereditary (Proposition 2.5.7) on an ($\underline{E}$, $\underline{M}$)-category $\underline{X}$.

2.1 DEFINITIONS

Definition 2.1.1 ([HS_1], 27Q)

Let $(A, <)$ and $(B, \leq)$ be quasi-ordered classes and let

$$(A, <) \underset{F}{\overset{G}{\rightleftarrows}} (B, \leq)$$

be order-reversing functions. If, for all $a \in A$ and $b \in B$, $a < F(G(a))$

and $b \leq G(F(b))$, then the quadruple $((A, <), (B, \leq), G, F)$ is called a *Galois correspondence* between A and B.

Definition 2.1.2 (cf. [PR], p. 180; [LO_2], 1.7)

(a) For any category $\underline{X}$, we define the relation

$$\sigma \subset Mor(\underline{X}) \times Ob(\underline{X})$$

as follows: $(e, Y) \in \sigma$ if, and only if, for each pair f, g of $\underline{X}$-morphisms with common codomain Y, the relation $f \circ e = g \circ e$ implies that $f = g$.

(b) Given a class $\underline{E}$ of $\underline{X}$-morphisms, the class

$$\underline{E}\text{-Sep} = \{ Y \in Ob(\underline{X}) \mid (e, Y) \in \sigma, \text{ for all } e \in \underline{E} \}$$

of all $\underline{X}$-objects which are *σ-related* to each $\underline{E}$-morphism is called *the class of $\underline{E}$-separated objects* in $\underline{X}$.

(c) Given a class $\underline{A}$ of $\underline{X}$-objects, the class

$$\underline{A}\text{-Epi} = \{ e \in Mor(\underline{X}) \mid (e, Y) \in \sigma, \text{ for all } Y \in \underline{A} \}$$

of all $\underline{X}$-morphisms which are *σ-related* to each $\underline{A}$-object is called *the class of $\underline{A}$-epimorphisms* in $\underline{X}$.

(d) Let $\underline{A}$ be a subcategory of a category X. An $\underline{X}$-morphism e is called an *$\underline{A}$-regular morphism* if there is a pair (f, g) of $\underline{X}$-morphisms whose codomain is in $\underline{A}$ such that $e \approx Equ(f, g)$.

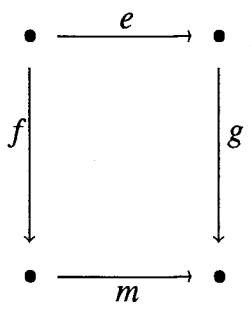
$\underline{A}$-epimorphisms are characterized by the following:

Proposition 2.1.3 ([LO_3, 2.1])

Suppose that a category $\underline{X}$ has equalizers. Given a subcategory $\underline{A}$ of the category $\underline{X}$, a morphism e is an $\underline{A}$-epimorphism if and only if $e \downarrow m$, for all $\underline{A}$-regular morphisms m.

Proof

Suppose that e is an $\underline{A}$-epimorphism, m is an $\underline{A}$-regular morphism, let $m \circ f = g \circ e$ and let $m \approx \mathrm{Equ}(h, k)$, where h and k have a common codomain in $\underline{A}$:

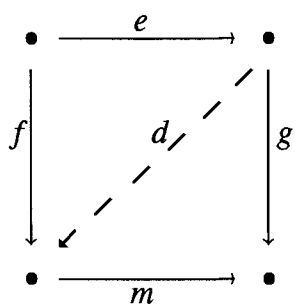


We have

$$(h \circ g) \circ e = h \circ (g \circ e) = h \circ (m \circ f) = (h \circ m) \circ f$$

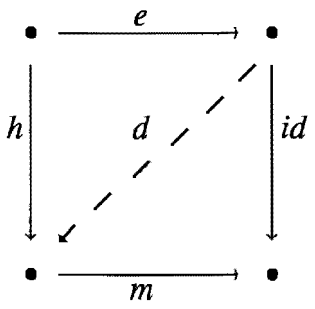
$$= (k \circ m) \circ f = k \circ (m \circ f) = k \circ (g \circ e) = (k \circ g) \circ e.$$

But $e \in \underline{A}$-epi, so $h \circ g = k \circ g$. Since $m \approx \mathrm{Equ}(h, k)$, there exists a morphism d with $m \circ d = g$, and then, $m \circ f = g \circ e = m \circ d \circ e$. But m is a monomorphism (see, for example, [HS_1, 16.4]), so $f = d \circ e$; thus both triangles in the following diagram commute:



Uniqueness of d such that the two triangles commute follows from the fact that m is a monomorphism.

Conversely, suppose that e uniquely diagonalizes over $\underline{A}$-regular morphisms. Let $f \circ e = g \circ e$, where f and g have a common codomain in $\underline{A}$, and let $m \approx \mathrm{Equ}(f, g)$. Then m is an $\underline{A}$-regular morphism, by definition. Since $m \approx \mathrm{Equ}(f, g)$, there is a morphism h such that $m \circ h = e = id \circ e$. By hypothesis e diagonalizes over $\underline{A}$-regular morphisms, so there exists a unique morphism d such that the following diagram commutes:



Thus $(m \circ d) \circ e = m \circ (d \circ e) = m \circ h = id \circ e = e$. Since m is a retraction (as $m \circ d = id$), it is an epimorphism, so, by ([HS_1], 16.7), we have $f = g$. Thus, $e \in \underline{A}$-epi. □

2.2 A GALOIS CORRESPONDENCE

Lemma 2.2.1 (See also [LO_2, 2.9])

For any $\underline{E} \subseteq \mathrm{Mor}(\underline{X})$ and $\underline{A} \subseteq \mathrm{Ob}(\underline{X})$, we have

(i) $\underline{E} \subseteq (\underline{E}\text{-Sep})\text{-Epi}$.

(ii) $\underline{A} \subseteq (\underline{A}\text{-Epi})\text{-Sep}$.

Proof

(i) Given a morphism $e \in \underline{\mathbf{E}}$, assume that $f \circ e = g \circ e$ where f and g have a common codomain belonging to $\underline{\mathbf{E}}$-Sep. Then (by definition) $f = g$, so $e \in (\underline{\mathbf{E}}\text{-Sep})\text{-Epi}$.

(ii) If $e \in \underline{\mathbf{A}}$-Epi, suppose that $f \circ e = g \circ e$ with $A \in \underline{\mathbf{A}}$ a common codomain of f and g. Then $f = g$, so $A \in (\underline{\mathbf{A}}\text{-Epi})\text{-Sep}$. □

The following proposition describes the *Galois* (or *Hausdorff*) *correspondence* between the classes Mor($\underline{\mathbf{X}}$) and Ob($\underline{\mathbf{X}}$) of $\underline{\mathbf{X}}$-morphisms and $\underline{\mathbf{X}}$-objects, respectively. (This correspondence has been called the *Hausdorff Correspondence* by Pumplün and Röhrl ([PR]), and the *Pumplün-Röhrl Galois connection* in [CS].) A similar correspondence was also found by Dikranjan and Giuli in terms of some subclasses of $\underline{\mathbf{M}}$ (defined differently from Definition 1.1.2) and some (closure) operators on $\underline{\mathbf{M}}$. (See [DG_2, Theorem 3.4]). And (in 1992) other Galois Connections were discovered by Castellini et al. See [CKS].

Proposition 2.2.2

Let $\mathscr{E}$ be the collection of all subclasses $\underline{\mathbf{E}}$ of $\underline{\mathbf{X}}$-morphisms and let $\mathscr{A}$ be the collection of all subclasses $\underline{\mathbf{A}}$ of $\underline{\mathbf{X}}$-objects. Suppose these collections, $\mathscr{E}$ and $\mathscr{A}$, are ordered by inclusion. Define two functions $G : \mathscr{E} \longrightarrow \mathscr{A}$ and $F : \mathscr{A} \longrightarrow \mathscr{E}$ as follows:

$$\underline{\mathbf{E}} \longmapsto \underline{\mathbf{E}}\text{-Sep} \quad \text{and} \quad \underline{\mathbf{A}} \longmapsto \underline{\mathbf{A}}\text{-Epi},$$

for each $\underline{\mathbf{E}} \in \mathscr{E}$ and $\underline{\mathbf{A}} \in \mathscr{A}$. Then the quadruple $(\mathscr{E}, \mathscr{A}, G, F)$ is a Galois correspondence.

Proof.

Given $\underline{\mathbf{E}}_1 \subseteq \underline{\mathbf{E}}_2$ in $\mathscr{E}$, we find that

$$\underline{E}_2\text{-Sep} = G(\underline{E}_2) \subseteq G(\underline{E}_1) = \underline{E}_1\text{-Sep}.$$

For, suppose $Y \in \underline{E}_2$-Sep. Then, for each $e \in \underline{E}_2$, $(e, Y) \in \sigma$. Pick $\bar{e}$ in $\underline{E}_1$. Then $\bar{e} \in \underline{E}_2$, so $(\bar{e}, Y) \in \sigma$, hence $Y \in \underline{E}_1$-Sep. Thus G is order-reversing. In a similar way, we can show that F is order-reversing.

That $\underline{E} \subseteq F(G(\underline{E}))$ for each $\underline{E} \in \mathscr{E}$ and $\underline{A} \subseteq G(F(\underline{A}))$ for each $\underline{A} \in \mathscr{A}$ follow from Lemma 2.2.1. Thus $(\mathscr{E}, \mathscr{A}, G, F)$ is a Galois Correspondence as asserted. □

2.3 EXAMPLES OF $\underline{E}$-SEPARATED OBJECTS AND $\underline{A}$-EPIMORPHISMS

Example 1

Let $\underline{X}$ = **Top** and let $\underline{E}$ be the class of all dense continuous maps. Then $\underline{E}$-Sep = **Haus**.

Proof.

To prove that $\underline{E}$-Sep = **Haus**, we make use of a latter result (Theorem 2.4.1). From this result and Theorem 13.7 of [WI], it follows that a topological space $Y \in \underline{E}$-Sep if and only if $\{(y, y) \mid y \in Y\}$ is closed in $Y \times Y$ if and only if $Y \in Ob(\mathbf{Haus})$.

Example 2

Let $\underline{X}$ = **Unif** and let $\underline{E}$ be the class of all dense uniformly continuous maps. Then $\underline{E}$-Sep is the family of all *separated* uniform spaces.

Proof.

Suppose that $(Y, \mathscr{D})$ is not a separated uniform space, where $\mathscr{D}$ is a uniformity on Y. Then there exist $x, y \in Y$ with $x \neq y$ such that $(x, y) \in D$, for each $D \in \mathscr{D}$. We put $X = \{x, y\}$ and give X the uniformity $\mathscr{D}'$ which is initial with respect to the inclusion $X \longrightarrow (X, \mathscr{D})$. Let $X \xrightarrow{e} X$ be the constant map with value y; let $X \xrightarrow{f} Y$ be the inclusion map and let $X \xrightarrow{g} Y$ be the constant function with value y. Then these three functions are uniformly continuous. Observe also that the topology τ' corresponding to $\mathscr{D}$ is $\{X, \emptyset, \{y\}\}$. Moreover, $e(x) = e(y) = y$ so that $\overline{e(X)} = \overline{\{y\}} = X$; thus e is a dense uniformly continuous map. We also have $f \circ e = g \circ e$ but $f \neq g$, so $(Y, \mathscr{D}) \notin \underline{E}\text{-Sep}$.

Conversely, suppose that $(Y, \mathscr{D})$ is a separated uniform space and let $f, g : (X, \xi) \longrightarrow (Y, \mathscr{D})$ be uniformly continuous maps, and let $e : (W, \mathscr{F}) \longrightarrow (X, \xi)$ be a dense uniformly continuous map so that

$$(W, \mathscr{F}) \xrightarrow{e} (X, \xi) \xrightarrow{f} (Y, \mathscr{D}) = (W, \mathscr{F}) \xrightarrow{e} (X, \xi) \xrightarrow{g} (Y, \mathscr{D}).$$

Now if $(Z, \mathscr{C})$ is any uniform space, let $(Z, \tau(\mathscr{C}))$ denote the corresponding topological space. By [WI, Theorem 35.6(b)], the space $(Y, \tau(\mathscr{D}))$ is Hausdorff, and in $\underline{\mathbf{Top}}$ we have the following:

$$(W,\tau(\mathscr{F})) \xrightarrow{e} (X,\tau(\xi)) \xrightarrow{f} (Y,\tau(\mathscr{D})) = (W,\tau(\mathscr{F})) \xrightarrow{e} (X,\tau(\xi)) \xrightarrow{g} (Y,\tau(\mathscr{D})).$$

Since $e : (W, \tau(\mathscr{F})) \longrightarrow (X, \tau(\xi))$ is a dense continuous map, it follows from Example 1 that $f = g$. Hence, $(Y, \mathscr{D}) \in \underline{E}\text{-Sep}$. □

Example 3

Let $\underline{X} = \underline{\mathbf{Top}}$ and let $\underline{E}$ be the family of all front-dense continuous maps. Then $\underline{E}\text{-Sep} = \mathrm{Ob}(\underline{\mathbf{Top}}_0)$.

Proof.

Let $Y \notin \underline{\mathbf{Top}}_0$. Then there are distinct points $y_1, y_2 \in Y$ such that for each open set U it holds that $\{y_1, y_2\} \subseteq U$ or $\{y_1, y_2\} \cap U = \emptyset$. Put $X = \{y_1, y_2\}$ and give it the indiscrete topology. Define a function $X \xrightarrow{e} X$ to be the constant map with value y_2. Then e is continuous; it is also front-dense, since $b(e(X)) = b(\{y_1\}) = X$. Define $f : X \longrightarrow Y$ to be inclusion, and $g : X \longrightarrow Y$ to be the constant map with value y_2. Then f and g are continuous. We also have $f \circ e = g \circ e$, but $f \neq g$. Thus $Y \notin \underline{E}$-Sep.

Conversely, suppose that $Y \in \mathrm{Ob}(\underline{\mathbf{Top}}_0)$. In order to prove that $Y \in \underline{E}$-Sep, we use the characterization given by a later result (Theorem 2.4.1). According to this theorem, we need to show that $b(\Delta) = \Delta$, where Δ is the set $\{(y, y) \mid y \in Y\}$. Since $\Delta \subseteq b(\Delta)$, we just need to show that $b(\Delta) \subseteq \Delta$. So suppose that $(x, y) \in b(\Delta)$, but $x \neq y$. Then there exists an open set U in Y such that $x \in U$, say, but $y \notin U$. Since $(x, y) \in b(\Delta)$, it follows that

$$(U \times X) \cap \overline{\{(x, y)\}} \cap \Delta \neq \emptyset.$$

Hence, there exists $w \in U$ such that, for each nhood W of (w, w) in $Y \times Y$, we have $(x, y) \in W$. But $U \times U$ is a nhood of (w, w), so $(x, y) \in U \times U$, which is impossible. Consequently $x = y$. □

Example 4

Let $\underline{X} = \underline{\mathbf{Top}}$ and let $\underline{E}$ be the family of all *back-dense* continuous maps, where a continuous map $X \xrightarrow{f} Y$ between topological spaces is *back-dense* iff, for each $y \in Y$, there is some $x \in X$ such that $\{y, f(x)\}$ is indiscrete. Then $\underline{E}$-Sep $= \mathrm{Ob}(\underline{\mathbf{Top}}_0)$.

Proof

Suppose $Y \notin Ob(\underline{\mathbf{Top}}_0)$. Then there exist $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that for every open set U it holds that $\{y_1, y_2\} \subseteq U$ or $\{y_1, y_2\} \cap U = \emptyset$. Let $X = \{y_1, y_2\}$ have the indiscrete topology, let $X \xrightarrow{j} Y$ be the inclusion map, and let $X \xrightarrow{g} Y$ be the constant map with value y_1. Then both j and g are continuous. Let $X \xrightarrow{e} X$ be the constant map with value y_1. Then e is continuous and back-dense, and $j \circ e = g \circ e$. Since $j \neq g$, it follows that $Y \notin \underline{E}$-Sep.

Conversely, suppose $Y \in Ob(\underline{\mathbf{Top}}_0)$, $A \xrightarrow{e} B \in \underline{E}$ and let $f, g : B \longrightarrow Y$, be continuous functions such that $f \circ e = g \circ e$. If $f \neq g$, then there exists $b \in B$ such that $f(b) \neq g(b)$. Since Y is a T_0-space, there is some open set U in Y with, say, $f(b) \in U$ but $g(b) \notin U$; so $b \in f^{-1}(U)$. Since e is back-dense, there exists $a \in A$ such that the set $\{b, e(a)\}$ is indiscrete. We note that to say that $\{b, e(a)\}$ is indiscrete is the same as saying that every open set in B which contains one of the points b or $e(a)$, must contain the other point as well. Since each of f and g is continuous, each of the sets $f^{-1}(U)$ and $g^{-1}(U)$ is open. Since $\{b, e(a)\}$ is indiscrete and since $b \in f^{-1}(U)$, we must have $e(a) \in f^{-1}(U)$. But then $f(e(a)) \in U$; and so (by hypothesis) $g(e(a)) \in U$ - which means that $e(a) \in g^{-1}(U)$. But since $\{b, e(a)\}$ is indiscrete and $g^{-1}(U)$ is open, we also have $b \in g^{-1}(U)$, i.e. $g(b) \in U$, which contradicts our choice of U. Hence $f = g$; that is $(Y, \tau) \in \underline{E}$-Sep. □

Example 5:

Let $\underline{X} = \underline{\mathbf{Top}}$ and let $\underline{E}$ be the family of all *c-dense* continuous functions, where a function $X \xrightarrow{f} Y$ is said to be *c-dense* if for each

$y \in Y$, there exists an $x \in X$ such that $f(x) \in \overline{\{y\}}$. Then $\underline{E}$-Sep $= Ob(\underline{Top}_1)$.

Proof

Suppose $Y \notin Ob(\underline{Top}_1)$. Then there are $y_1, y_2 \in Y$ with $y_1 \neq y_2$ such that for each nhood U of y_2, say, it holds that $\{y_1, y_2\} \subseteq U$. Put $X = \{y_1, y_2\}$ and assume X has the subspace topology. Now $\overline{\{y_1\}} = \{y_1, y_2\}$, otherwise $\overline{\{y_1\}} = \{y_1\}$, and then $\{y_2\}$ is open in X. Hence $\{y_2\} = X \cap K$, for some open set K containing y_2, which contradicts the fact that $\{y_1, y_2\} \subseteq K$. Define $X \xrightarrow{e} X$ to be the constant function with value y_2, define $X \xrightarrow{f} Y$ to be the inclusion of X into Y and define $X \xrightarrow{g} Y$ to be the constant function with value y_2. We find that these functions are continuous. And e is c-dense: For $y_1 \in X$, we have $e(y_2) = y_2 \in \overline{\{y_1\}}$, and for y_2, we find that $e(y_2) = y_2 \in \overline{\{y_2\}}$ as indicated above. Then $f \circ e = g \circ e$ but $f \neq g$, so $Y \notin \underline{E}$-Sep. Hence $\underline{E}$-Sep $\subseteq Ob(\underline{Top}_1)$.

Conversely, suppose that $Y \in Ob(\underline{Top}_1)$ and let $A \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be continuous functions such that $f \circ e = g \circ e$, where e is a c-dense function. It must be shown that $f = g$. If not, there is an $x \in X$ such that $f(x) \neq g(x)$. Since Y is a T_1-space, each of $\{f(x)\}$ and $\{g(x)\}$ is closed. Since e is c-dense, there exists $a \in A$ such that $e(a) \in \overline{\{x\}}$, so by continuity of f, we must have that

$$f \circ e(a) \in f\overline{\{x\}} \subseteq \overline{\{f(x)\}} = \{f(x)\};$$

that is, $f \circ e(a) = f(x)$. In a similar way, we can show that $g \circ e(a) = g(x)$. But $f \circ e(a) = g \circ e(a)$, so $f(x) = g(x)$ - a contradiction. Thus $f = g$ and so $Y \in \underline{E}$-Sep. □

Example 6

Let $\underline{\mathbf{X}} = \underline{\mathbf{Top}}$ and let $\underline{\mathbf{E}}$ be the family of all *d-dense* continuous functions, where a function $X \xrightarrow{f} Y$ between topological spaces is said to be *d-dense*, if, for each $y \in Y$, there is some $x \in X$ such that $y \in \overline{\{f(x)\}}$. Then $\underline{\mathbf{E}}$-Sep $= \mathrm{Ob}(\underline{\mathbf{Top}}_1)$.

Proof

We proceed as in Example 5, except that we define $e : X \longrightarrow X$ and $g : X \longrightarrow Y$ to be the constant functions with value y_1. Then the functions e, f and g are continuous. The function e is d-dense: Given $y_1 \in X$, we have $y_1 \in \overline{\{e(y_2)\}} = \overline{\{y_1\}}$ and for $y_2 \in X$, we have $y_2 \in \overline{\{e(y_1)\}} = \overline{\{y_1\}}$. Therefore $f \circ e = g \circ e$ but $f \neq g$. Hence $Y \notin \underline{\mathbf{E}}$-Sep.

Conversely, suppose that $Y \in \mathrm{Ob}(\underline{\mathbf{Top}}_1)$ and assume that $f \circ e = g \circ e$ as in Example 5, where e is d-dense. If $f \neq g$, then there is some $x \in X$ with $f(x) \neq g(x)$. Since e is d-dense, there is some $a \in A$ such that $x \in \overline{\{e(a)\}}$. By continuity of f and since Y is a T_1-space, we have

$$f(x) \in f\overline{\{e(a)\}} \subseteq \overline{\{f(e(a))\}} = \{f(e(a))\},$$

hence $f(x) = f(e(a))$. And similarly, we have $g(x) = g(e(a))$, so that $f(e(a)) \neq g(e(a))$, a contradiction. Hence $f = g$ as desired. □

Example 7

Let $\underline{\mathbf{X}} = \underline{\mathbf{Pos}}$, and let $\underline{\mathbf{E}}$ be the class of all *lower-dense order-preserving* maps. (A function $X \xrightarrow{f} Y$ between partially ordered sets is *lower-dense* iff for each $y \in Y$, there is an $x \in X$ such that $f(x) \leq y$.) Then $\underline{\mathbf{E}}$-Sep is the class of **all partially-ordered sets** whose order is **equality.**

Proof

Suppose that $Y \in \underline{\mathbf{Pos}}$, where the partial order $<_Y$ is *not equality*. Then there exist $y_1 \neq y_2$ in Y such that $y_1 <_Y y_2$. We consider the partially ordered set $X = \{y_1, y_2\}$ with the induced order $<_X$. Define $\{y_1\} \xrightarrow{e} X$ to be the inclusion function, let $X \xrightarrow{f} Y$ be the constant function with value y_1 and let $X \xrightarrow{g} Y$ be the inclusion of X into Y. Then e is lower-dense and order-preserving, f and g are order-preserving and $f \circ e = g \circ e$. But $f \neq g$, so $Y \notin \underline{E}$-Sep.

Conversely, suppose that $(A, \leq_A)$, $(X, \leq_X)$, $(Y, \leq_Y) \in Ob(\underline{\mathbf{Pos}})$, where the partial order $\leq_Y$ is equality. Assume that $f \circ e = g \circ e$, where $A \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ and f and g are *order-preserving maps* with $e \in \underline{E}$. It is asserted that $f = g$. For, suppose $x \in X$. Since e is lower-dense, there exists $a \in A$ such that $e(a) \leq_X x$, i.e. $e(a) = x$, since the order is assumed to be equality. Hence

$$f(x) = f(e(a)) = g(e(a)) = g(x). \text{ Thus } f = g,$$

so $Y \in \underline{E}$-Sep. □

Example 8

Let $\underline{X} = \underline{\mathbf{Top}}$ and suppose that $\underline{A}$ consists of only the two-point discrete space $A = \{0, 1\}$. Then $\underline{A}$-Epi is *the family of all q-dense continuous functions*. (A function $f : X \longrightarrow Y$ between topological spaces is said to be *q-dense* if each clopen neighborhood of each $y \in Y$ meets $f[X]$.)

Proof

Suppose that $X \xrightarrow{e} Y$ is *not q-dense*. Then there is some $y \in Y$ such that $e[X] \cap U = \emptyset$, for some *clopen nhood* U of y, and so

$y \notin e[X]$; thus $e(r) = y$, for no $r \in X$. Define $f : Y \longrightarrow \{0, 1\}$ to be the constant function with value 1 and define $Y \xrightarrow{g} \{1, 0\}$ as follows:

$$g(r) = \begin{cases} 0 & \text{if } r \in U; \\ 1 & \text{if } r \notin U. \end{cases}$$

Since A has the discrete topology, the sets $\emptyset$, A, {0} and {1} are the open sets in A. Since their inverse images $g^{-1}(\emptyset) = \emptyset$, $g^{-1}(A) = Y$, $g^{-1}(\{0\}) = U$ and $g^{-1}(\{1\}) = Y - U$ are open in Y, it follows that g is continuous. The constant function f is (trivially) continuous. Then $f \circ e = g \circ e$ but $f \neq g$. Hence $e \notin \underline{\mathbf{A}}$-Epi.

Conversely, suppose the continuous functions $X \xrightarrow{e} Y \underset{g}{\overset{f}{\rightrightarrows}} \{0, 1\}$ satisfy $f \circ e = g \circ e$, where X and Y are topological spaces and e is *q-dense*. If $f \neq g$, there is $y \in Y$ with $f(y) \neq g(y)$. We assume, without loss of generality, that $f(y) = 0$ and $g(y) = 1$. Since {0, 1} has the discrete topology, each of the subsets {0} and {1} is clopen, and therefore y belongs to the clopen nhood $f^{-1}(\{0\}) \cap g^{-1}(\{1\})$. Since e is q-dense, it follows that

$$e[X] \cap f^{-1}(\{0\}) \cap g^{-1}(\{1\}) \neq \emptyset.$$

Choose $p \in e[X] \cap f^{-1}(\{0\}) \cap g^{-1}(\{1\})$, and find $x \in X$ such that $e(x) = p$. Then

$$0 = f(p) = f \circ e(x) = g \circ e(x) = g(p) = 1,$$

which is impossible. Therefore, $f = g$. □

Example 9

Let $\underline{\mathbf{X}}$ = **Top** and suppose that the two-point indiscrete space A = {0, 1} belongs to $\underline{\mathbf{A}}$. Then $\underline{\mathbf{A}}$-Epi is *the* family of *all surjective continuous maps*.

Proof

If a continuous function $X \xrightarrow{e} Y$ is *not surjective*, there exists some $y \in Y$ such that $e(x) \neq y$, for all $x \in X$. Define $Y \underset{g}{\overset{f}{\rightrightarrows}} A$ as follows:

$$\begin{aligned} f(t) &= g(t) = 1, \text{ whenever } t \neq y; \\ f(y) &= 0; \\ g(y) &= 1. \end{aligned}$$

Then f and g are continuous, and for each $x \in X$, we have $e(x) \neq y$, so $f(e(x)) = g(e(x)) = 1$. Hence $f \circ e = g \circ e$, but $f \neq g$.

The converse follows from the fact that in **Top**, the surjective continuous maps are precisely the epimorphisms (see, for example, [HS_1, 6.10(2)]). □

Example 10

In Example 1, we showed that if **X** = **Top** and e is a dense continuous map, then $e \in$ **Haus**-Epi. Now, we shall show that not every map in **Haus**-Epi is a dense continuous map.

Consider, for instance, the Sierpinski space $X = \{ 0, 1 \}$ with the topology $\{ \emptyset, X, \{ 0 \} \}$. Given any $Y \in Ob(\textbf{Haus})$, then any continuous function $X \xrightarrow{f} Y$ is constant. (Otherwise, Y would no longer be T_2.) The inclusion $\{ 1 \} \xrightarrow{j} X$ is not dense since $\overline{j(\{ 1 \})} = \{ 1 \}$. But $j \in$ **Haus**-Epi, since if $f \circ j = g \circ j$, then $f = g$. □

Example 11

Let **X** = **Top** and let **E** be the family of all c-dense (or d-dense) continuous maps.

By Examples (5) and (6), we know that $\underline{E}$-Sep = Ob($\underline{\mathbf{Top}}_1$) and, consequently, ($\underline{E}$-Sep)-Epi = ($\underline{\mathbf{Top}}_1$)-Epi. By Lemma 2.2.1, we have $\underline{E} \subseteq$ ($\underline{E}$-Sep)-Epi, for each class $\underline{E}$ of $\underline{\mathbf{X}}$-objects. Hence, if e is a *c-dense* (*or d-dense*) continuous map, then $e \in \underline{\mathbf{Top}}_1$-Epi. □

2.4 HAUSDORFF CHARACTERIZATION THEOREM

It is a well-known fact that Hausdorff spaces are precisely those topological spaces X for which the diagonal $\Delta_X : X \longrightarrow X^2$ is closed. In this theorem, we prove that an $\underline{\mathbf{X}}$-object Y is $\underline{E}$-*separated* if and only if the diagonal $\Delta_Y : Y \longrightarrow Y^2$ belongs to $\underline{\mathbf{M}}$, for an ($\underline{E}$, $\underline{\mathbf{M}}$)-factorization structure on a finitely complete category $\underline{\mathbf{X}}$.

Theorem 2.4.1 (cf. [PR, A.2]; [MA, 4.5])

Let ($\underline{E}$, $\underline{\mathbf{M}}$) be a factorization structure on a finitely complete category $\underline{\mathbf{X}}$. Then for any $\underline{\mathbf{X}}$-object Y, the following are equivalent:

(1) $Y \in \underline{E}$-Sep.

(2) For each $X \xrightarrow{f} Y$, the graph of f is in $\underline{\mathbf{M}}$.

(3) For each $X \xrightarrow{f} Y$ and $X \xrightarrow{m} Z \in \underline{\mathbf{M}}$, it holds that $\langle m, f\rangle \in \underline{\mathbf{M}}$.

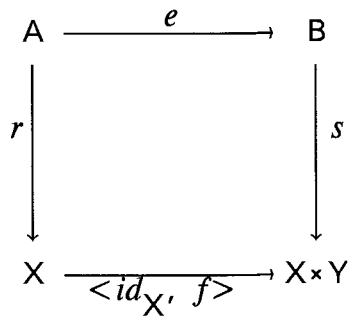
(4) For each $X \xrightarrow{m} Y \in \underline{\mathbf{M}}$, it holds that $\langle m, m\rangle \in \underline{\mathbf{M}}$.

(5) $\Delta_Y : Y \longrightarrow Y^2 \in \underline{\mathbf{M}}$.

(6) If $r, s : X \longrightarrow Y$, then $\mathrm{Equ}(r, s) \in \underline{\mathbf{M}}$.

Proof

(1) $\Rightarrow$ (2). Let $s \circ e = \langle id_X, f\rangle \circ r$, with $e \in \underline{E}$:



By Lemma 2.2.1(i), $\underline{E} \subseteq$ ($\underline{E}$-Sep)-Epi, so $e \in$ ($\underline{E}$-Sep)-Epi. By Proposition 2.1.3, e diagonalizes over ($\underline{E}$-Sep)-regular morphisms, so it is enough to show that $<id_X, f>$ is ($\underline{E}$-Sep)-regular. We have

$$(f \circ \pi_1) \circ <id_X, f> \;=\; f \circ id_X \;=\; f \;=\; \pi_2 \circ <id_X, f>,$$

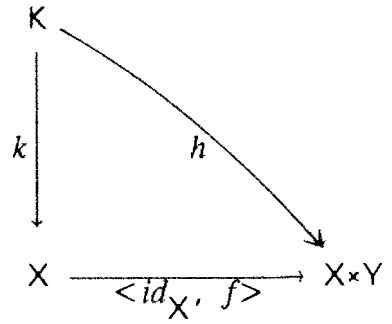
where π_1 and π_2 are the projections $X \times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$, respectively. Given a morphism $K \xrightarrow{h} X \times Y$ with $\pi_2 \circ h = (f \circ \pi_1) \circ h$, define a morphism $K \xrightarrow{k} X$ by $k = \pi_1 \circ h$. Then

$$\pi_2 \circ <id_X, f> \circ k \;=\; f \circ \pi_1 \circ h \;=\; \pi_2 \circ h$$

and

$$\pi_1 \circ <id_X, f> \circ k \;=\; id_X \circ \pi_1 \circ h \;=\; \pi_1 \circ h,$$

so, since (π_1, π_2) is a mono-source, we have $<id_X, f> \circ k = h$. If r also satisfied $<id_X, f> \circ r = h$, then $<id_X, f> \circ r = <id_X, f> \circ k$. Since $<id_X, f>$ is a section (since $\pi_1 \circ <id_X, f> = id_X$), it is a monomorphism, hence $r = k$. Since we proved that k is unique such that the triangle



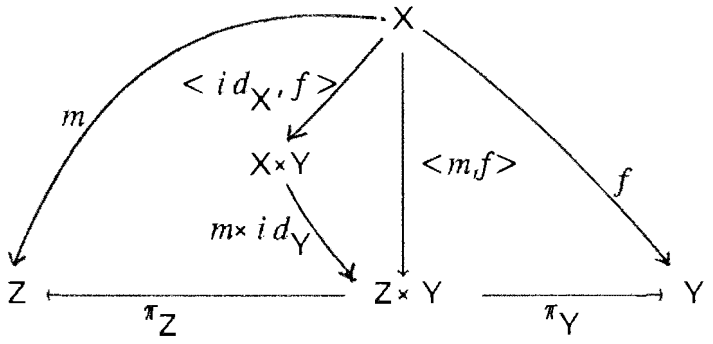
commutes, it follows that $(X, \langle id_X, f\rangle) \approx \mathrm{Equ}(\pi_2, f\circ\pi_1)$, so $e \downarrow \langle id_X, f\rangle$. By Proposition 1.3.1(3), we have $\langle id_X, f\rangle \in \underline{\mathbf{M}}$.

(2) $\Longrightarrow$ (3). Given a morphism $X \xrightarrow{f} Y$, let $X \xrightarrow{m} Z \in \underline{\mathbf{M}}$ and let σ_X and σ_Y be the projections $X\times Y \longrightarrow X$ and $X\times Y \longrightarrow Y$, respectively. Then

$$\pi_Z\circ(m\times id_Y)\circ\langle id_X, f\rangle = m\circ\sigma_X\circ\langle id_X, f\rangle = m\circ(\sigma_X\circ\langle id_X, f\rangle)$$
$$= m\circ id_X = m,$$

and $\pi_Y\circ(m\times id_Y)\circ\langle id_X, f\rangle = \sigma_Y\circ\langle id_X, f\rangle = f,$

where π_Z and π_Y are the projections $Z\times Y \longrightarrow Z$ and $Z\times Y \longrightarrow Y$, respectively. Then the following diagram commutes:



In particular, uniqueness of $\langle m, f\rangle$ such that $\pi_Z\circ\langle m, f\rangle = m$ and $\pi_Y\circ\langle m, f\rangle = f$, ensures that $\langle m, f\rangle = (m\times id_Y)\circ\langle id_X, f\rangle$. By hypothesis, we have $m \in \underline{\mathbf{M}}$, so $m\times id_Y \in \underline{\mathbf{M}}$ (Proposition 1.3.1(4), (8)). We also have $\langle id_X, f\rangle \in \underline{\mathbf{M}}$, so the composition $\langle m, f\rangle = (m\times id_Y)\circ\langle id_X, f\rangle$ belongs to $\underline{\mathbf{M}}$ (Proposition 1.3.1(5)).

(3) $\Rightarrow$ (4). With $Z = Y$ and $f = m$ in (3), we obtain $\langle m, m\rangle \in \underline{\mathbf{M}}$.

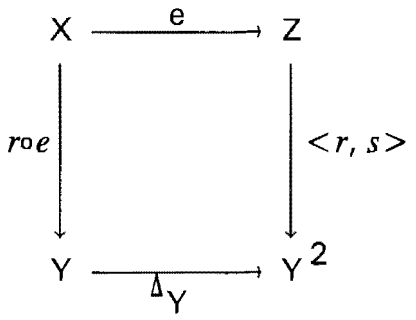
(4) $\Rightarrow$ (5). With $X = Y$ and $m = id_Y$, it will follow from (4) that $\langle id_Y, id_Y\rangle = \Delta_Y \in \underline{\mathbf{M}}$.

(5) $\Rightarrow$ (1). Suppose that $X \xrightarrow{e} Z \in \underline{\mathbf{E}}$ and $Z \underset{s}{\overset{r}{\rightrightarrows}} Y$ are such that $r \circ e = s \circ e$. Then

$$\Delta_Y \circ (r \circ e) = \langle id_Y, id_Y\rangle \circ (r \circ e) = \langle r \circ e, r \circ e\rangle$$

$$= \langle r \circ e, s \circ e\rangle = \langle r, s\rangle \circ e,$$

and so the following diagram commutes:



Put $p = r \circ e = s \circ e$. By (5), $\Delta_Y \in \underline{\mathbf{M}}$, so by the unique diagonalization property there exists a unique $\underline{\mathbf{X}}$-morphism $Z \xrightarrow{d} Y$ such that $d \circ e = p$ and $\Delta_Y \circ d = \langle r, s\rangle$. Since

$$r = p_1 \circ \langle r, s\rangle = p_1 \circ \Delta_Y \circ d = d \quad \text{and}$$

$$s = p_2 \circ \langle r, s\rangle = p_2 \circ \Delta_Y \circ d = d,$$

(where p_1, p_2 are the usual projections $Y^2 \longrightarrow Y$), we must have $r = s$, hence $Y \in \underline{\mathbf{E}}$-Sep.

(1) $\Rightarrow$ (6). Given a pair $X \underset{s}{\overset{r}{\rightrightarrows}} Y$ of $\underline{\mathbf{X}}$-morphisms, let $(C, c) \approx \mathrm{Equ}(r, s)$. By Lemma 2.2.1(i), we have $\underline{\mathbf{E}} \subset (\underline{\mathbf{E}}\text{-Sep})\text{-Epi}$. Since $Y \in \underline{\mathbf{E}}$-Sep, it follows that (C, c) is $(\underline{\mathbf{E}}\text{-Sep})$-regular. By Proposition 2.1.3, the

equalizer (C, c) diagonalizes under all morphisms in $\underline{E}$, hence (C, c) $\in \underline{\mathbf{M}}$.

(6) $\Rightarrow$ (1). Suppose that $f \circ e = g \circ e$, where $e \in \underline{E}$ and Y is the codomain of both f and g. Let $m \approx \text{Equ}(f, g)$ - so that m diagonalizes under $\underline{E}$; in particular, m diagonalizes under the element e of $\underline{E}$. By an argument similar to the second part of the proof of Proposition 2.1.3, it can be shown that m is an isomorphism, so $f = g$. Hence $Y \in \underline{E}$-Sep. □

2.5 PROPERTIES OF $\underline{E}$-SEPARATED OBJECTS AND $\underline{A}$-EPIMORPHISMS

Proposition 2.5.1 (cf. [PR], Lemma A.2)

Let $\underline{E}$ be a class of $\underline{X}$-morphisms. Then, for each family $\underline{A}$ of $\underline{X}$-objects;

(1) $\underline{A}$-Epi satisfies conditions (a) through (e) of Bousfield's Characterization Theorem (Theorem 1.4.1).

(2) $\underline{E}$-Sep is closed under the formation of all mono-sources, and thus under the formation of all limits.

Proof.

(1)(a) Let $f \in \text{Iso}(\underline{X})$ and let $Y \in \underline{A}$ be the codomain of a pair of $\underline{X}$-morphisms h and g such that $h \circ f = g \circ f$. Since f is an epimorphism, we have $h = g$, hence $(f, Y) \in \sigma$. Thus $\text{Iso}(\underline{X}) \subseteq \underline{A}$-Epi.

(b) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be $\underline{A}$-epimorphisms. We want to show that $g \circ f$ is an $\underline{A}$-epimorphism. Suppose a pair of $\underline{X}$-morphisms $C \underset{q}{\overset{r}{\rightrightarrows}} D$ with $D \in \underline{A}$ satisfies $r \circ (g \circ f) = q \circ (g \circ f)$. Then $(r \circ g) \circ f = (q \circ g) \circ f$, so (since $f \in \underline{A}$-Epi) $r \circ g = q \circ g$. But again $g \in \underline{A}$-Epi implies that $r = q$, hence $g \circ f \in \underline{A}$-Epi.

(c) Given $e = f \circ \hat{e}$ with $e, \hat{e} \in \underline{A}$-Epi, let g, h be $\underline{X}$-morphisms with $g \circ f = h \circ f$. Then

$$g \circ e = g \circ (f \circ \hat{e}) = (g \circ f) \circ \hat{e}$$

$$= (h \circ f) \circ \hat{e} = h \circ (f \circ \hat{e}) = h \circ e,$$

and, since $e \in \underline{A}$-Epi, we must have $g = h$. Therefore, $f \in \underline{A}$-Epi.

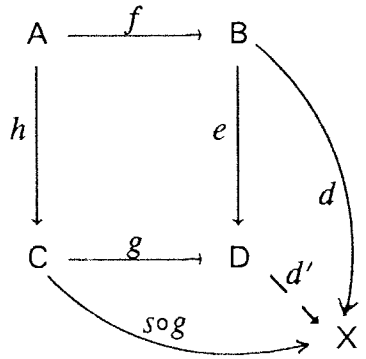
(d) Let

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle e} \\ C & \xrightarrow[\ g\ ]{} & D \end{array}$$

be a pushout diagram with $h \in \underline{A}$-Epi. It must be shown that $e \in \underline{A}$-Epi. Let $D \underset{s}{\overset{r}{\rightrightarrows}} X$ be two $\underline{X}$-morphisms such that $r \circ e = s \circ e$ and $X \in \underline{A}$. Put $r \circ e = s \circ e = d$. Then

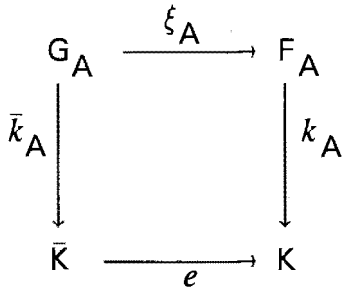
$$
\begin{aligned}
(r \circ g) \circ h &= r \circ (g \circ h) \\
&= r \circ (e \circ f) \\
&= (r \circ e) \circ f \\
&= (s \circ e) \circ f \\
&= s \circ (e \circ f) \\
&= s \circ (g \circ h) \\
&= (s \circ g) \circ h,
\end{aligned}
$$

so $r \circ g = s \circ g$, since $h \in \underline{A}\text{-Epi}$. We also have $(r \circ g) \circ h = (r \circ e) \circ f$. But the given square is a pushout, so there exists a unique $\underline{X}$-morphism $D \xrightarrow{d'} X$ such that the two triangles in the diagram



commute. But r and s are also two such morphisms, so $r = d' = s$, hence $e \in \underline{A}\text{-Epi}$.

(e) Given functors $\underline{Y} \underset{G}{\overset{F}{\rightrightarrows}} \underline{X}$, let (k_A, K) and $(\bar{k}_A, \bar{K})$ be colimits of G and F, respectively, and let $(\xi) : G \longrightarrow F$ be a natural transformation. We must show that if each $\xi_A : G_A \longrightarrow F_A$ belongs to $\underline{A}$-Epi, then so does the unique $\underline{X}$-morphism e which makes the following diagram commutative:



Let $L \in \underline{\mathbf{A}}$ and let $K \underset{g}{\overset{f}{\rightrightarrows}} L$ be $\underline{\mathbf{X}}$-morphisms such that $f \circ e = g \circ e$. Then, for each $A \in \mathrm{Ob}(\underline{\mathbf{Y}})$, it holds that

$$\begin{aligned}(f \circ k_A) \circ \xi_A &= f \circ (k_A \circ \xi_A) = f \circ (e \circ \bar{k}_A) = (f \circ e) \circ \bar{k}_A \\ &= (g \circ e) \circ \bar{k}_A \\ &= g \circ (e \circ \bar{k}_A) \\ &= g \circ (k_A \circ \xi_A) \\ &= (g \circ k_A) \circ \xi_A .\end{aligned}$$

But each $\xi_A \in \underline{\mathbf{A}}$-Epi, so $f \circ k_A = g \circ k_A$. Put $f \circ k_A = r_A = g \circ k_A$. Since (k_A, K) is a colimit of F, there is a unique morphism $K \xrightarrow{h} L$ with $h \circ k_A = r_A$. But each of f and g is also such an h, hence $f = g$.

(2) Suppose that $(f_i : X \longrightarrow X_i)_I$ is a mono-source, where each X_i belongs to $\underline{\mathbf{E}}$-Sep. Suppose $e \in \underline{\mathbf{E}}$ satisfies $r \circ e = s \circ e$, where r and s have a common codomain X. Then $f_i \circ r \circ e = f_i \circ s \circ e$, and so $f_i \circ r = f_i \circ s$, for each $i \in I$ (since each $(e, X_i) \in \sigma$). But $(f_i)_I$ is a mono-source, so $r = s$, hence $X \in \underline{\mathbf{E}}$-Sep.

Finally, limits are mono-sources (see, for example, [HS$_1$, 20.4]), so $\underline{\mathbf{E}}$-Sep must be closed under the formation of limits. □

Lemma 2.5.2 (cf. [PR, Lemma A. 2])

Epi(**X**) $\subseteq$ **A**-Epi, for any category **X** and **A** $\subseteq$ Ob(**X**).

Proof. This is obvious. □

Proposition 2.5.3

If (**E**, **M**) is a factorization structure in an **M**-well-powered cocomplete category **X** and **E** $\subseteq$ Epi(**X**), then, for each family **A** of **X**-objects, there exists a family **M**′ of **X**-morphisms such that (**A**-Epi, **M**′) is a factorization structure on **X**.

Proof.

In view of Theorem 1.4.1 and Theorem 2.5.1 (1), we need only show that **A**-Epi satisfies the *Solution Set Condition*. Since (**E**, **M**) is a factorization structure on **X** and **E** $\subseteq$ Epi(**X**), and since Epi(**X**) $\subseteq$ **A**-Epi (Lemma 2.5.2), it follows that an (**E**, **M**)-factorization of an **X**-morphism is also its (**A**-Epi, **M**)-factorization. So, given a morphism $f : C \longrightarrow D$, we consider all factorizations

$$C \xrightarrow{f} D \;=\; C \xrightarrow{e_i} Y_i \xrightarrow{m_i} D$$

of f, where each m_i belongs to the representative set of **M**-subobjects of D and $e_i \in$ **A**-Epi, for each $i \in I$. We assert that this is a *Solution Set* for f. For, suppose $f = n \circ e$ is a factorization of f with $e \in$ **A**-Epi, and let $n = m \circ \bar{e}$ be an (**E**, **M**)-factorization of n. Since **E** $\subseteq$ Epi(**X**) and Epi(**X**) $\subseteq$ **A**-Epi (Lemma 2.5.2), we have $\bar{e} \circ e \in$ **A**-Epi, by Proposition 2.5.1(1). Therefore, $f = m \circ (\bar{e} \circ e)$ is isomorphic to one of the factorizations of f, and so **A**-Epi satisfies Bousfield's Characterization Theorem 1.4.1. Thus there is a class **M**′ of **X**-morphisms such that (**A**-Epi, **M**′) is a factorization structure on **X**. □

Remark: In [LO_3, Theorem 2.3], it was shown that if $\underline{X}$ has intersections and equalizers, then, for any subcategory $\underline{A}$ of $\underline{X}$, ($\underline{A}$-Epi, $\underline{M}_A$) is a morphism factorization structure on $\underline{X}$, and $\underline{M}_A$ consists of extremal monomorphisms. Here

$$\underline{M}_A = (\underline{A}\text{-Epi})^{\downarrow}.$$

See [LO_3, 2.2].

Definition 2.5.4

If $\underline{X}$ is a category and $\underline{A}$ is a class of $\underline{X}$-objects, then, given an $X \in Ob(\underline{X})$, the source of all $\underline{X}$-morphisms from X to $\underline{A}$-objects is called the *total source* from X to $\underline{A}$.

Proposition 2.5.5

If each source in $\underline{X}$ has an (epi, mono-source)-factorization, then, for each family $\underline{E}$ of $\underline{X}$-morphisms, $\underline{E}$-Sep is a (swell epi)-reflective subcategory.

Proof.

By Theorem 1.4.7, (swell epi, mono-source) is a factorization structure for sources in $\underline{X}$. Given $X \in Ob(\underline{X})$, let $(f_i : X \xrightarrow{f_i} X_i)_I$ be the total source from X to $\underline{E}$-Sep, and consider its (swell epi, mono-source)-factorization:

$$X \xrightarrow{f_i} X_i = X \xrightarrow{e} A \xrightarrow{m_i} X_i.$$

By Proposition 2.5.1(2), $A \in \underline{E}$-Sep. For each $i \in I$, it must be shown that m_i is unique such that $f_i = m_i \circ e$. So, suppose n_i is also such that each $f_i = n_i \circ e$. Then $n_i \circ e = m_i \circ e$. Since a swell epimorphism is an epimorphism (by Lemma 1.4.5), we have $m_i = n_i$. Thus e is a

(swell epi)-reflection morphism corresponding to X, so **E**-Sep is a (swell epi)-reflective subcategory. □

The following definition has been slightly altered to suit the proposition that follows:

Definition 2.5.6 (See, for example, [LO_2, Definition 1.9].)
Let **A** be a subclass (not necessarily a subcategory) of objects in **X**, and let **M** be a class of morphisms in **X**. The class **A** is said to be **M**-*hereditary* in **X** if, for each **X**-morphism $X \xrightarrow{m} A \in \mathbf{M}$ with $A \in \mathbf{A}$, we have that $X \in \mathbf{A}$.

Proposition 2.5.7 (See also [LO_2, Lemma 2.4].)
If (**E**, **M**) is a factorization structure on **X**, then **E**-Sep is **M**-hereditary.

Proof
We want to show that if $X \xrightarrow{m} B \in \mathbf{M}$ and $B \in \mathbf{E}$-Sep, then $X \in \mathbf{E}$-Sep. Hence suppose $Y \underset{g}{\overset{f}{\rightrightarrows}} X$ and $e : A \longrightarrow Y \in \mathbf{E}$ are such that $f \circ e = g \circ e$. Then $(m \circ f) \circ e = (m \circ g) \circ e$. Since $(e, B) \in \sigma$, it follows that $m \circ f = m \circ g$. Put $m \circ g = s$ and $f \circ e = r$. Then the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\ e\ } & Y \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow[\ m\ ]{} & B \end{array}$$

Since $e \in \mathbf{E}$ and $m \in \mathbf{M}$, the unique (**E**, **M**)-diagonalization property implies that there exists a unique diagonal **X**-morphism $d : Y \longrightarrow X$

such that $m \circ d = s$ and $d \circ e = r$. But f and g also satisfy $f \circ e = g \circ e = r$, and $m \circ f = m \circ g = s$, so uniqueness implies that $f = d = g$, hence $X \in \underline{E}$-Sep. □

REMARK

The subcategory $\underline{E}$-Sep is also mono-hereditary. (See [LO_2, 2.5])

Corollary 2.5.8 (See [LO_2, Lemma 2.8])

The factorization structure ($\underline{E}$, $\underline{M}$) on $\underline{X}$ induces a factorization structure ($\underline{E}'$, $\underline{M}'$) on $\underline{E}$-Sep (when $\underline{E}$-Sep is considered a full subcategory of $\underline{X}$).

Proof

Let $f : X \longrightarrow Y$ be a morphism in $\underline{E}$-Sep. (Thus X, Y belong to Ob($\underline{E}$-Sep).) Let

$$X \xrightarrow{f} Y = X \xrightarrow{e} Z \xrightarrow{m} Y$$

be its ($\underline{E}$, $\underline{M}$)-factorization in $\underline{X}$. Since $\underline{E}$-Sep is $\underline{M}$-hereditary, $Z \in \underline{E}$-Sep. We then put

$$\underline{E}' = \{ e : X \longrightarrow Z \mid X, Z \in \underline{E}\text{-Sep}, \ e \in \underline{E} \}$$

and

$$\underline{M}' = \{ m : Z \longrightarrow Y \mid Y \in \underline{E}\text{-Sep}, \ m \in \underline{M} \}. \quad \square$$

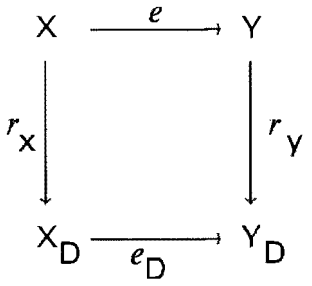
Proposition 2.5.9

Let $\underline{A}$ be a family of $\underline{X}$-objects, let $\underline{B}$ its Pumplün-Röhrl closure; that is, $\underline{B}$ = ($\underline{A}$-Epi)-Sep, let $\underline{C}$ be such that $\underline{A} \subseteq \underline{C} \subseteq \underline{B}$ and let $\underline{D}$ be any reflective subcategory of $\underline{X}$ with $\underline{A} \subseteq \underline{D} \subseteq \underline{B}$. Then, for any $X \xrightarrow{e} Y$ in $\underline{X}$, the following statements are equivalent:

(1) $e \in \underline{\mathbf{A}}$-Epi.

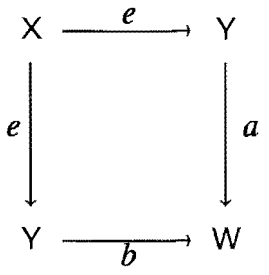
(2) $e \in \underline{\mathbf{C}}$-Epi.

(3) If $X \xrightarrow{r_X} X_D$ and $Y \xrightarrow{r_Y} Y_D$ are $\underline{\mathbf{D}}$-reflections and $X_D \xrightarrow{e_D} Y_D$ is the unique morphism such that the diagram



commutes, then e_D is an epimorphism in $\underline{\mathbf{D}}$.

(4) If the square



is a push-out and $W \xrightarrow{r_W} W_D$ is the $\underline{\mathbf{D}}$-reflection map for W, then $r_W \circ a = r_W \circ b$.

Proof

(1) $\Rightarrow$ (2). Suppose that $e \in \underline{\mathbf{A}}$-Epi. Let $f \circ e = g \circ e$, where $Z \in \underline{\mathbf{C}}$ is the common codomain of both f and g. We assert that $f = g$. For, since $\underline{\mathbf{C}} \subseteq$ ($\underline{\mathbf{A}}$-Epi)-Sep, it is implied by $Z \in \underline{\mathbf{C}}$ that $Z \in$ ($\underline{\mathbf{A}}$-Epi)-Sep. Since $e \in \underline{\mathbf{A}}$-Epi, we must have $(e, Z) \in \sigma$, so that $f = g$. Hence $e \in \underline{\mathbf{C}}$-Epi.

(2) ⇒ (3). Let $f \circ e_D = g \circ e_D$, for some pair $Y_D \underset{g}{\overset{f}{\rightrightarrows}} C$ in Mor(**D**). By hypothesis, **A** ⊆ **C**, so (by Proposition 2.2.2) **C**-Epi ⊆ **A**-Epi. Since $e \in$ **C**-Epi, it follows that $e \in$ **A**-Epi. With $C \in$ Ob(**D**), we have $C \in$ (**A**-Epi)-Sep (since **D** ⊆ **B**), so $(e, C) \in \sigma$. But then

$$f \circ r_Y \circ e = f \circ e_D \circ r_X = g \circ e_D \circ r_X = g \circ r_Y \circ e$$

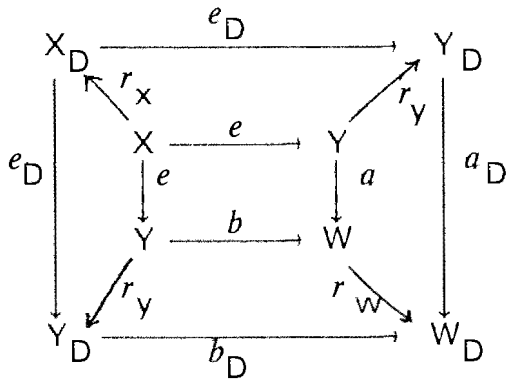
implies that $f \circ r_Y = g \circ r_Y$. Put $f \circ r_Y = p = g \circ r_Y$.

Since $Y \xrightarrow{r_Y} Y_D$ is a **D**-reflection for Y, there exists a unique morphism $Y_D \xrightarrow{h} C$ such that $h \circ r_Y = p$. But each of f and g is such a morphism, so $f = h = g$, hence e_D is an epimorphism in **D**.

(3) ⇒ (4). Let the square

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & Y \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle a} \\ Y & \xrightarrow[\ b\ ]{} & W \end{array}$$

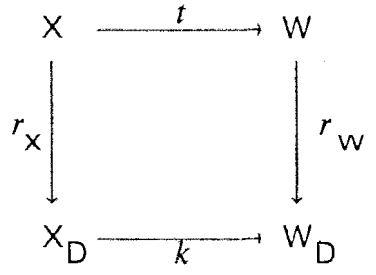
be a pushout square with $W \xrightarrow{r_W} W_D$ the **D**-reflection map for W. Let r_X and r_Y be the D-reflection morphisms for X and Y, respectively. Then the morphisms e_D, a_D and b_D are the unique morphisms such that each of the outside squares in the following diagram commutes:



Put $a \circ e = t = b \circ e$. We have

$$(b_D \circ e_D) \circ r_X = b_D \circ (\circ e_D \circ r_X) = b_D \circ (r_Y \circ e) = (b_D \circ r_Y) \circ e = (r_W \circ b) \circ e$$
$$= r_W \circ (b \circ e) = r_W \circ t$$

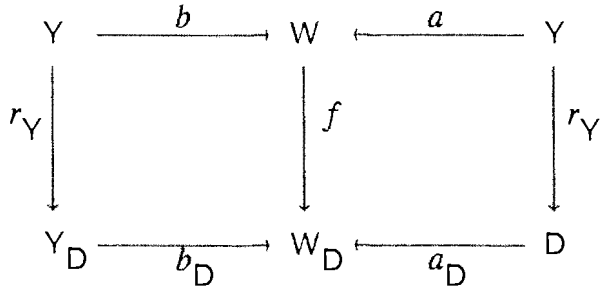
and in a similar way, we find that $(a_D \circ e_D) \circ r_X = r_W \circ t$. Since r_X and r_W are the $\underline{D}$-reflection morphisms for X and W respectively, there is a unique morphism $X_D \xrightarrow{k} W_D$ such that the following rectangle commutes:



Since each of the morphisms $b_D \circ e_D$ and $a_D \circ e_D$ also makes the same rectangle commutative, it follows by uniqueness of k that $b_D \circ e_D = k = a_D \circ e_D$, thus $b_D = a_D$ (since e_D is an epimorphism). We have

$$b_D \circ r_Y \circ e = r_W \circ b \circ e = r_W \circ a \circ e = a_D \circ r_Y \circ e,$$

so there exists a unique morphism $W \xrightarrow{f} W_D$ such that $b_D \circ r_Y = f \circ b$ and $a_D \circ r_Y = f \circ a$.

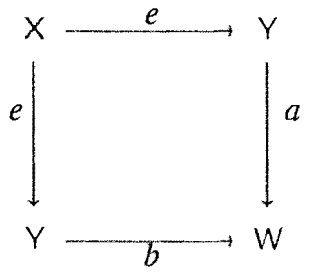


But r_W also satisfies

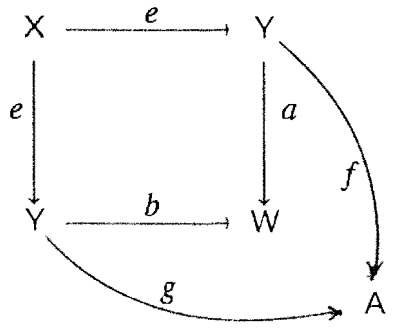
$b_D \circ r_Y = r_W \circ b$ and $a_D \circ r_Y = r_W \circ a$, hence $f = r_W$.

Thus $r_W \circ b = r_W \circ a$, as desired.

(4) $\Rightarrow$ (1). Suppose we have the pushout square



of the diagram $\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle e}\downarrow & & \\ Y & & \end{array}$. Assume that $f \circ e = g \circ e$, where $f, g : Y \longrightarrow A$ with $A \in \underline{\mathbf{A}}$. By definition of a pushout, there is a unique morphism $W \xrightarrow{k} A$ such that the triangles in the following diagram commute:



(*)

Since $W \xrightarrow{r_W} W_D$ is a $\underline{D}$-reflection morphism for W, there is

a unique morphism $W_D \xrightarrow{\bar{k}} A$ such that $\bar{k} \circ r_W = k$. By (4), we have $r_W \circ a = r_W \circ b$, so by the commutativity of (*) we have that

$$f = k \circ a = (\bar{k} \circ r_W) \circ a = \bar{k} \circ (r_W \circ a) = \bar{k} \circ (r_W \circ b) = (\bar{k} \circ r_W) \circ b = k \circ b = g.$$

Hence $e \in \underline{\mathbf{A}}$-Epi. □

CHAPTER 3

STRONG LIMIT OPERATORS

In this chapter, $\underline{X}$ is assumed to be an hereditary construct, that is, a concrete category $U : \underline{X} \longrightarrow \underline{\mathbf{Set}}$ together with the property that the inclusion of each subset Y into the underlying set $U(X)$ of any $\underline{X}$-object X has a unique *initial lift* (defined below). Such initial lifts will be called *embeddings*, and if $Z \longrightarrow X$ is an embedding, then Z will be called a *subobject* of X.

In this chapter, we show, amongst others, that (surjection, embedding) is a factorization structure on $\underline{X}$ (Lemma 3.1.4); that a strong limit operator is both a prelimit operator (Remark 3.1.3(1)) and a closure operator (Propositions 3.2.1 and 3.2.2). We also show that there is a one-to-one correspondence between the class of all strong limit operators and the class of all factorization structures $(\underline{E}, \underline{M})$, where $\underline{M}$ is contained in the class of all $\underline{X}$-embeddings (Theorem 3.2.11). Specifically, that function which assigns to each factorization structure the strong limit operator η (Theorem 3.2.10), and that which assigns to each strong limit operator l (Theorem 3.2.8), the (l-dense, l-closed)-factorization structure are inverses of each other.

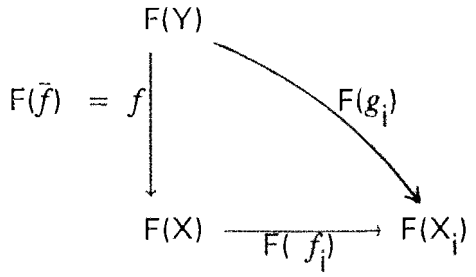
3.1 DEFINITIONS

Definition 3.1.1 ([AHS, 10.57])

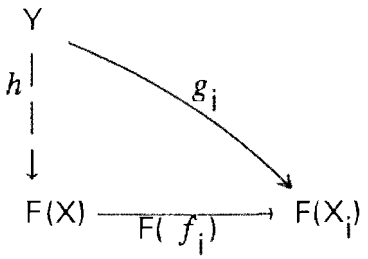
Let $\underline{X} \xrightarrow{F} \underline{A}$ be a functor.

(a) A source $(X \xrightarrow{f_i} X_i)_I$ in $\underline{X}$ is called *initial* provided that for each source $(Y \xrightarrow{g_i} X_i)_I$ in $\underline{X}$ and each $\underline{A}$-morphism

$F(Y) \xrightarrow{f} F(X)$ with $F(f_i) \circ f = F(g_i)$, for each $i \in I$, there exists a unique $\underline{X}$-morphism $Y \xrightarrow{\bar{f}} X$ such that $F(\bar{f}) = f$ and $f_i \circ \bar{f} = g_i$, for each $i \in I$:



(b) A source $(X \xrightarrow{f_i} X_i)_I$ in $\underline{X}$ *lifts* a source $(Y \xrightarrow{g_i} F(X_i))_I$ in $\underline{A}$ provided that there is an $\underline{A}$-isomorphism $Y \xrightarrow{h} F(X)$ such that $F(f_i) \circ h = g_i$, for each $i \in I$:



A concrete category $\underline{X}$ can be regarded simply as a category where the objects are just sets with structure, the morphisms $f : X \longrightarrow Y$ are simply ordinary maps $f : UX \longrightarrow UY$ between the underlying sets which are somehow structure-compatible; and composition of morphisms is simply ordinary composition of maps. Further, no notational distinction will be made between an $\underline{X}$-morphism and its underlying map. No confusion should result.

Hence a source $(X \xrightarrow{f_i} X_i)_I$ in a concrete category $\underline{X}$ is *initial* provided that if $Y \in Ob(\underline{X})$, and $g : U(Y) \longrightarrow U(X)$ is a map such that $f_i \circ g : Y \longrightarrow X_i$ is an $\underline{X}$-morphism for each $i \in I$, then $g : Y \longrightarrow X$ is an $\underline{X}$-morphism. Thus an hereditary construct $\underline{X}$ is characterized by

the following property: If $X \in Ob(\underline{X})$, and Y is a subset of U(X), then there is a unique $\underline{X}$-structure on Y (denote the corresponding $\underline{X}$-object by $\bar{Y}$, say) such that the inclusion map $Y \longrightarrow U(X)$ is an initial $\underline{X}$-morphism $\bar{Y} \longrightarrow X$. (This $\underline{X}$-structure on $\bar{Y}$ is called an *initial structure*.)

Definition 3.1.2

By a *strong limit operator* on $\underline{X}$ we mean a family $l = (l_X)_{X \in Ob(\underline{X})}$, where for any object $X \in Ob(\underline{X})$, l_X assigns to each embedding $Y \subseteq X$ an embedding $l_X(Y) \subseteq X$ such that the following conditions are satisfied:

(1) $Y \subseteq l_X(Y)$.

(2) If $Z \subseteq l_X(Y) = W$, then $l_X(Z) \subseteq l_W(Z)$.

(3) For each $\underline{X}$-morphism $X \xrightarrow{f} Y$ and $Z \subseteq X$, we have $f(l_X(Z)) \subseteq l_Y(f(Z))$. (Note: If Y, $X \in Ob(\underline{X})$, the notation $Y \subseteq X$ means that $U(Y) \subseteq U(X)$, and the inclusion map $U(Y) \longrightarrow U(X)$ is an embedding $Y \longrightarrow X$ in $\underline{X}$. The use of notation such as $f(Z)$ should be clear. This notation just means the set $f(Z)$ endowed with the $\underline{X}$-structure to give the $\underline{X}$-embedding $f(Z) \subseteq Y$. Such (ab)uses of notation will be made frequently, but where they do occur, their meanings should be evident.)

Remarks 3.1.3 (See $[HE_1$, Method 3])

(1) A strong limit operator is a *prelimit operator*; that is, if $X \xrightarrow{f} Y$ and $A \subset Y$ is such that $l_Y(A) = A$, then $l_X(f^{-1}(A)) = f^{-1}(A)$. For, $f^{-1}(A) \subseteq f^{-1}(A) \Rightarrow l_X(f^{-1}(A) \subseteq f^{-1}(A)$ and $f^{-1}(A) \subseteq l_X(f^{-1}(A))$ by Definition 3.1.2(1). Hence $f^{-1}(A) = l_X(f^{-1}(A))$. (See (4) below.)

(2) It is immediate from Definition 3.1.2(1) that $l_X(X) = X$.

(3) If $X \xrightarrow{f} Y$ is an $\underline{X}$-morphism, then the inclusion map $(Uf)(UX) \longrightarrow UY$ is an embedding $f(X) \longrightarrow Y$. By initiality, it follows that $X \longrightarrow f(X)$ is an $\underline{X}$-morphism. It follows from condition (3) that if $Z \subseteq X$, then $f(l_X(Z)) \subseteq l_{f(X)}(f(Z))$.

(4) If A and B are subobjects of $X \in Ob(\underline{X})$, and $A \subseteq B$ and $B \subseteq A$, then $A = B$: From $A \subseteq B$ and $B \subseteq A$ one deduces that $U(A) \subseteq U(B)$ and $U(B) \subseteq U(A)$, so $U(A) = U(B)$. Hence the initial inclusion $A \longrightarrow B$ actually has as underlying map, the identity $id_{U(A)}$. Since the identity $\underline{X}$-morphism $id_B : B \longrightarrow B$ is initial, and initial structures are assumed to be unique, it follows that $A = B$.

Lemma 3.1.4

(Surjection, embedding) is factorization structure on $\underline{X}$.

Proof

Given an $\underline{X}$-morphism $X \xrightarrow{f} Y$, then at **Set**-level, f has a (surjection, injection)-factorization, say:

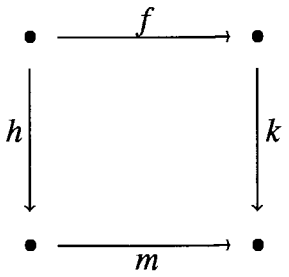
$$UX \xrightarrow{Uf} UY = UX \xrightarrow{Uf} (Uf)(UX) \xrightarrow{m} UY,$$

where m is the inclusion of $(Uf)(UX)$ into UY. Give $(Uf)(UX)$ the $\underline{X}$-structure (denoting the corresponding object by $f[X]$) so that $f[X] \longrightarrow Y$ becomes an embedding in $\underline{X}$. By initiality of this embedding, it follows that $X \longrightarrow f[X]$ is an $\underline{X}$-morphism. It follows that the $\underline{X}$-morphism $f : X \longrightarrow Y$ has the following (surjection, embedding)-factorization:

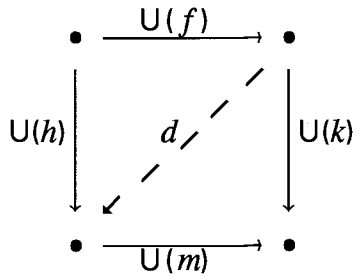
$$X \xrightarrow{f} Y = X \longrightarrow f[X] \longrightarrow Y .$$

(Hereafter, we shall denote by f [X] the middle object of the (surjection, embedding)-factorization of an **X**-morphism $X \xrightarrow{f} Y$.)

Now consider a commutative square in **X**



where f is surjective and m is an embedding. In **Set**, injective maps are monomorphisms and surjections are epimorphisms. Since **Set** is an (epi, mono)-category, there is a diagonal morphism d such that the following diagram commutes in **Set**:



By initiality of m, there exists an **X**-morphism $\bar{d}$ such that $U(\bar{d}) = d$ and $m \circ \bar{d} = k$. We need to show that $\bar{d} \circ f = h$. But $d \circ U(f) = U(h)$ yields $U(\bar{d}) \circ U(f) = U(h)$ or $U(\bar{d} \circ f) = U(h)$, hence $\bar{d} \circ f = h$ by faithfulness of U. And if w also satisfies $w \circ f = h$ and $m \circ w = k$, then $w \circ f = \bar{d} \circ f$, and since f is surjective, $w = \bar{d}$. □

3.2 PROPERTIES OF STRONG LIMIT OPERATORS

In the following proposition we show that a strong limit operator is idempotent and monotone (order-presrving).

Proposition 3.2.1

For every strong limit operator l,

(1) $l_X(l_X(Y)) = l_X(Y)$.

(2) If $Y \subseteq Z \subseteq X$, then $l_X(Y) \subseteq l_X(Z)$.

Proof

(1) We have $l_X(Y) \subseteq l_X(Y)$, so from condition (2) of Definition 3.1.2, it follows that $l_X(l_X(Y)) \subseteq l_{l_X(Y)}(l_X(Y)) = l_X(Y)$. Also, from condition (1) of that definition it follows that $l_X(Y) \subseteq l_X(l_X(Y))$. Hence $l_X(Y) = l_X(l_X(Y))$.

(2) Let $Y \subseteq Z \subseteq X$ be given. Then, $Y \subseteq l_X(Y)$ and $Z \subseteq l_X(Z)$, so we have $Y \subseteq l_X(Z)$. Put $l_X(Z) = T$. By condition (2) of Definition 3.1.2, we have $l_X(Y) \subseteq l_T(Y)$. We need only show that $l_T(Y) \subseteq T$. We find that l_T takes $Y \subseteq l_X(Z) = T$ to $l_T(Y) \subseteq T = l_X(Z)$. The proof is complete. □

We shall need the following observation: Given embeddings $Z \subseteq X$ and $W \subseteq X$, we can obtain the embedding $Z \cup W \subseteq X$, where the domain $Z \cup W$ has underlying set $U(Z) \cup U(W)$.

Proposition 3.2.2 (See e.g [DG_2, 1.2])

If $l = (l_X)_{X \in Ob(\underline{X})}$ is a strong limit operator, the following statements hold, where Y_1 and Y_2 are subobjects of X.

(1) $Y_1 \subseteq l_X(Y_1)$.

(2) $l_X(Y_1) \cup l_X(Y_2) \subseteq l_X(Y_1 \cup Y_2)$.

(3) $l_X[\, l_X(Y_1) \cup l_X(Y_2)\,] = l_X(Y_1 \cup Y_2)$.

Proof

(1) Follows from Definition 3.1.2(1).

(2) We have that $U(Y_1) \subseteq U(Y_1 \cup Y_2) = (UY_1) \cup (UY_2)$. Since $Y_1 \subseteq X$ is an $\underline{\mathbf{X}}$-morphism and $Y_1 \cup Y_2 \subseteq X$ is initial, it follows that the inclusion map $U(Y_1) \longrightarrow U(Y_1 \cup Y_2)$ is an $\underline{\mathbf{X}}$-morphism $Y_1 \longrightarrow Y_1 \cup Y_2$. But the first factor of an embedding is an embedding, so we obtain an embedding $Y_1 \subseteq Y_1 \cup Y_2$. In a similar way, we obtain an embedding $Y_2 \subseteq Y_1 \cup Y_2$. By Proposition 3.2.1(2), it follows that $l_X(Y_1) \subseteq l_X(Y_1 \cup Y_2)$ and $l_X(Y_2) \subseteq l_X(Y_1 \cup Y_2)$. Now we find that

$$U(l_X(Y_1)) \cup U(l_X(Y_2)) \subseteq U(l_X(Y_1 \cup Y_2)).$$

Since $l_X(Y_1) \cup l_X(Y_2) \subseteq X$ is an $\underline{\mathbf{X}}$-morphism and $l_X(Y_1 \cup Y_2) \subseteq X$ is initial, it follows that the inclusion map

$$U(l_X(Y_1)) \cup U(l_X(Y_2)) \longrightarrow U(l_X(Y_1 \cup Y_2))$$

is an $\underline{\mathbf{X}}$-morphism $l_X(Y_1) \cup l_X(Y_2) \longrightarrow l_X(Y_1 \cup Y_2)$.
As a first factor of an embedding, this $\underline{\mathbf{X}}$-morphism is an embedding, so we obtain $l_X(Y_1) \cup l_X(Y_2) \subseteq l_X(Y_1 \cup Y_2)$.

(3) We apply l_X to (2) and use the fact that l_X is idempotent; so we have

$$l_X\{ l_X(Y_1) \cup l_X(Y_2) \} \subseteq l_X\{ l_X(Y_1 \cup Y_2)\} = l_X(Y_1 \cup Y_2).$$

By Definition 3.1.2(1), we have $Y_1 \subseteq l_X(Y_1)$ and $Y_2 \subseteq l_X(Y_2)$. These embeddings give rise to the inclusion maps

$$UY_1 \subseteq U(l_X(Y_1)) \text{ and } UY_2 \subseteq U(l_X(Y_1));$$

so we have an inclusion

$$U(Y_1 \cup Y_2) = (UY_1) \cup (UY_2) \subseteq U(l_X(Y_1)) \cup U(l_X(Y_2)).$$

From the embeddings $l_X(Y_1) \subseteq X$ and $l_X(Y_2) \subseteq X$, we obtain an embedding $l_X(Y_1) \cup l_X(Y_2) \subseteq X$ where the domain $l_X(Y_1) \cup l_X(Y_2)$ has underlying set $(U(l_X(Y_1))) \cup (U(l_X(Y_2)))$. Since $l_X(Y_1 \cup Y_2) \longrightarrow X$ is an $\underline{\mathbf{X}}$-morphism and since $l_X(Y_1) \cup l_X(Y_2) \longrightarrow X$ is initial, it follows that the inclusion map

$$U(Y_1 \cup Y_2) \longrightarrow U(l_X(Y_1) \cup l_X(Y_2))$$

is an $\underline{\mathbf{X}}$-morphism $Y_1 \cup Y_2 \longrightarrow l_X(Y_1) \cup l_X(Y_2)$, and being a first factor of an embedding, this $\underline{\mathbf{X}}$-morphism is an embedding; thus $Y_1 \cup Y_2 \subseteq l_X(Y_1) \cup l_X(Y_2)$. By Proposition 3.2.1(2), it follows that

$$l_X(Y_1 \cup Y_2) \subseteq l_X\{ l_X(Y_1) \cup l_X(Y_2) \}.$$

Hence $l_X\{ l_X(Y_1) \cup l_X(Y_2) \} = l_X(Y_1 \cup Y_2)$. □

Remark 3.2.3:

The above two propositions show that a strong limit operator is a **closure operator**; that is, the operator is expansive (extensive), monotone (order-preserving) and idempotent. (See also [DG_1], [GH], [CM], [LO_1] and [SA].) However, a strong limit operator is not necessarily a **Kuratowski closure operator**. (Because it is not necessarily additive, that is, it need not necessarily hold that

$$l_X(Y_1) \cup l_X(Y_2) = l_X(Y_1 \cup Y_2).)$$

(See [SA, p. 252])

We now give an example of a strong limit operator.

Example 3.2.4

The b-closure operator is a strong limit operator. (See Example 1.2.4(d).)

Proof. Let $A \subseteq (X, \tau)$ where $(X, \tau) \in Ob(\underline{Top})$. (Note that **Top** is an hereditary construct.) Then the b-closure operator transforms A into $b(A) \subseteq (X, \tau)$. When endowed with the relative topology, the inclusion $b(A) \subseteq (X, \tau)$ is an embedding.

(1) That $A \subseteq b(A)$ follows from the properties of the b-closure operator (see Example 1.2.4(d)).

(2) Suppose that $B \subseteq b_X(A) = W$. It needs to be shown that

$$b_X(B) = \{ x \in X \mid \text{for each nhood } N_x \text{ in } X \text{ of } x,\ N_x \cap \overline{\{x\}} \cap B \neq \emptyset \}$$

$$\subseteq \{ y \in b_X(A) \mid \text{for each nhood } U_y \text{ in } b_X(A) \text{ of } y,\ U_y \cap \overline{\{y\}} \cap B \neq \emptyset \}$$

$$= b_W(B).$$

Let $x \in b_X(B)$. Now, for any nhood N_x in $b_X(A)$ of x, we have $N_x = b_X(A) \cap U_x$, where U_x is a nhood in X of x. Since $x \in b_X(B)$, it follows that $U_x \cap \overline{\{x\}} \cap B \neq \emptyset$. Note that $b_X(A) \cap \overline{\{x\}} = \overline{\{x\}}$ in $b_X(A)$, so we have

$$U_x \cap b_X(A) \cap \overline{\{x\}} \cap B \neq \emptyset.$$

That is, $N_x \cap \overline{\{x\}} \cap B \neq \emptyset$, and so $y \in b_W(B)$.

(3) Consider a continuous function $f : X \longrightarrow Y$ between topological spaces with $A \subseteq X$. We will show that $f(b_X A) \subseteq b_Y f(A)$. But this follows from Lemma 2(1) of Example 1.2.4(d).

Hence, the b-closure operator is a strong limit operator. □

Remarks 3.2.5

(a) Nakagawa defined a closure operator cl in terms of an (**E**, **M**)-factorization for an extremal subobject A c X in an (**E**, **M**)-category $\mathscr{C}$. Such a closure operator satisfies the "**continuity**" **property**, it is **idempotent**, **monotone** and **expansive**. (See [NA, pp.146 - 147].) It was further shown that a certain type of factorization structure gives rise to a bireflective subcategory of the category $\underline{\mathbf{Top}}_1$ of T_1-spaces and continuous functions ([NA, Proposition 11]). And conversely ([NA, Proposition 12]), a bireflective subcategory of $\underline{\mathbf{Top}}_1$ which is closed under embeddings gives rise to some factorization structure on $\underline{\mathbf{Top}}_1$.

(b) Given a complete, well-powered concrete category $\mathscr{C}$ with forgetful functor $U : \mathscr{C} \longrightarrow \underline{\mathbf{Set}}$ that preserves monomorphisms, there is a one-one correspondence between the following three families:

(i) The class of hull operators on $\mathscr{C}$.

(ii) The class of hull subobject operators on $\mathscr{C}$.

(iii) The class of strong factorization structures on $\mathscr{C}$.

(For definitions and for the correspondence, see $[LO_1]$.)

(c) In [CM, Proposition 3.6], Cagliari and Mantovani proved that there exists a bijective correspondence between the (**E**, **M**)-factorization structures on a co-well-powered epireflective subcategory $\mathscr{C}$ of $\underline{\mathbf{Top}}$ or of a topological category $U : \underline{\mathbf{A}} \longrightarrow \underline{\mathbf{Set}}$ and the semiclosure operators in $\mathscr{C}$, where **M** is contained in the class of all embeddings. A *semiclosure operator* is defined as follows:

Definition (See [CM, 3.5])

Let $\mathscr{C}$ be a co-well-powered subcategory of the category **Top** of topological spaces and continuous functions. A *semiclosure operator* 1 in $\mathscr{C}$ is an operator which associates, to every subspace Y of X, with X in $\mathscr{C}$, a subspace $1_X(Y)$ of X such that the following properties hold:

(i) $A \subseteq 1_X(A) \subseteq X$;

(ii) the inclusion from Y into $1_X(Y)$ is in $Epi(\mathscr{C})$;

(iii) if $Z \subseteq Y \subseteq X$, then $1_X(Z) \subseteq 1_X(Y)$;

(iv) $1_X(1_X(Y)) = 1_X(Y)$;

(v) if $X \xrightarrow{f} Y \in Mor(\mathscr{C})$, then $f(1_X(A)) \subseteq 1_Y(f(A))$, for each $A \subseteq X$.

As in [CM], we prove in theorem 3.2.11 that if **M** is contained in the class of all embeddings, then there is a bijective correspondence between the class of all strong limit operators and the class of all (**E**, **M**)-factorization structures on a hereditary construct.

Definition 3.2.6

Let l be a strong limit operator on **X**.

(a) An **X**-morphism $X \xrightarrow{f} Y$ is called *l-dense* provided that $l_Y(f[X]) = Y$.

(b) An embedding $X \xrightarrow{m} Y$ in **X** is called *l-closed* provided that $l_Y(m[X]) = m[X]$. (Here "an embedding" means "an initial injection".)

(c) A sink $(f_i : X_i \longrightarrow Y)_I$ is said to be *l-dense* provided that $l_Y(\cup f[X_i]) = Y$.

Lemma 3.2.7

(1) If $X \xrightarrow{h} Y$ is an $\underline{\mathbf{X}}$-morphism and $Z \subseteq W \subseteq X$, then $h(Z) \subseteq h(W)$.

(2) If $X \xrightarrow{h} Y$ is an $\underline{\mathbf{X}}$-isomorphism, then, whenever $Z \subseteq X$, it follows that $h(l_X(Z)) = l_Y(h(Z))$.

Proof

(1) This is easy.

(2) By Definition 3.1.2(3), we have $h(l_X(Z)) \subseteq l_Y(h(Z))$. Since $h^{-1} : Y \longrightarrow X$ is also an $\underline{\mathbf{X}}$-morphism and $h(Z) \subseteq Y$, it follows that

$$h^{-1}(l_Y(h(Z))) \subseteq l_X(h^{-1}(h(Z))) \qquad \text{(Definition 3.1.2(3)}$$
$$= l_X(Z).$$

Applying h to this relation, we find that

$$l_Y(h(Z)) = h(h^{-1}(l_Y(h(Z)))) \subseteq h(l_X(Z)), \quad \text{(By (1))}$$

thus $h(l_X(Z)) = l_Y(h(Z))$. □

Proposition 3.2.8

If l is a strong limit operator on $\underline{\mathbf{X}}$, then (l-dense, l-closed embedding) is a morphism factorization structure on $\underline{\mathbf{X}}$.

Proof

We first show that l-closed embeddings and l-dense morphisms are closed under composition with $\underline{\mathbf{X}}$-isomorphisms, as specified in Definition 1.1.2. By [AHS, 8.14], an $\underline{\mathbf{X}}$-isomorphism is both an initial morphism and an isomorphism in **Set**. But in **Set**, isomorphisms are the bijective maps (see, for example, [HS_1, 5.14(2)]); so $\underline{\mathbf{X}}$-isomorphisms are the initial bijections. Since embeddings are initial by assumption, and since initial morphisms are closed under composition (see, for

example, [AHS, 8.9]), it follows that embeddings are closed under composition with $\underline{\mathbf{X}}$-isomorphisms.

Now let $X \xrightarrow{h} Y \in \mathrm{Iso}(\underline{\mathbf{X}})$ and let $Y \xrightarrow{m} Z$ be an l-closed embedding. It must be shown that $m \circ h$ is l-closed; that is,

$$l_Z((m \circ h)[X]) = (m \circ h)[X].$$

Note that $(m \circ h)[X] \subseteq Z$. So, by Definition 3.1.2(1), we have

$$(m \circ h)[X] \subseteq l_Z((m \circ h)[X]).$$

Put $W = (m \circ h)[X]$. Then Definition 3.1.2(2) ensures that

$$l_Z((m \circ h)[X]) \subseteq l_W((m \circ h)[X]) = l_W(W) = W = (m \circ h)[X].$$

Hence $l_Z((m \circ h)[X]) = (m \circ h)[X]$, thus $m \circ h$ is an l-closed embedding.

On the other hand, if $X \xrightarrow{e} Y$ is an l-dense $\underline{\mathbf{X}}$-morphism and $Y \xrightarrow{h} X$ is an $\underline{\mathbf{X}}$-isomorphism, we have

$$\begin{aligned} l_Z((h \circ e)[X]) &= l_Z(h(e[X])) = h(l_Y(e[X])) && \text{(Lemma 3.2.7)} \\ &= h(Y) && (e \text{ is } l\text{-dense}) \\ &= Z, && (h \text{ is an } \underline{\mathbf{X}}\text{-isomorphism}) \end{aligned}$$

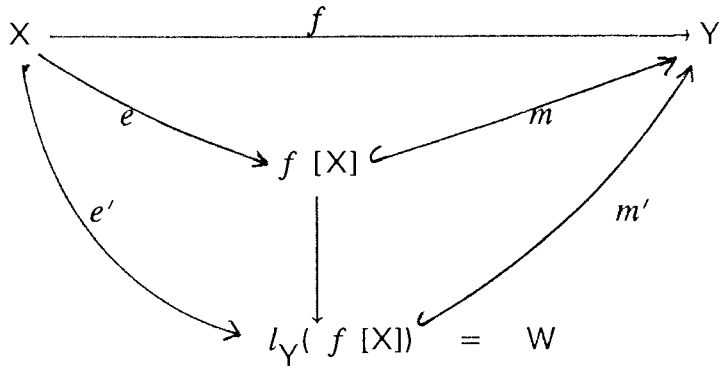
and so $h \circ e$ is l-dense.

Given an $\underline{\mathbf{X}}$-morphism $X \xrightarrow{f} Y$, we form its (surjection, embedding)-factorization, say $m \circ e$:

$$X \xrightarrow{f} Y = X \xrightarrow{e} f[X] \xrightarrow{m} Y.$$

The embedding m is the same as $f[X] \subseteq Y$. Applying l_Y to $f[X]$, we have $l_Y(f[X]) \subseteq Y$, say, an embedding $m' : l_Y(f[X]) \longrightarrow Y$. Let

e' be the composition of e and the embedding $f[X] \subseteq l_Y(f[X])$:



Since m' is initial, it follows that e' is an $\underline{X}$-morphism.

(1) It is asserted that the factorization

$$X \xrightarrow{f} Y = X \xrightarrow{e'} l_Y(f[X]) \xrightarrow{m'} Y$$

is an (l-dense, l-closed embedding)-factorization of f.

(i) $\underline{m' \text{ is } l\text{-closed}}$: (Note that m' is just inclusion.) We have

$$m'(l_Y(f[X])) = l_Y(f[X]).$$

Now it follows that

$$\begin{aligned} l_Y\{m'(l_Y(f[X]))\} &= l_Y\{l_Y(f[X])\} \\ &= l_Y(f[X]) \qquad \text{(Proposition 3.2.1)} \\ &= m'\{l_Y(f[X])\}. \end{aligned}$$

Hence m' is l-closed.

(ii) $\underline{e' \text{ is } l\text{-dense}}$: Put $l_Y(f[X])) = W$. To show that e' is l-dense, it must be shown that

$$l_W(e'[X]) = W = l_Y(f[X]).$$

Since e is the restriction of f, it follows that

$e(X) = f[X]$, so it will be shown that

$$l_W(f[X]) = l_Y(f[X]).$$

Since $f[X] \subseteq l_Y(f[X]) = W$, it follows from Definition 3.1.2(2) that

$$l_Y(f[X]) \subseteq l_W(f[X]).$$

On the other hand, applying l_W to the embedding $f[X] \subseteq l_Y(f[X]) = W$, we obtain

$$\begin{aligned} l_W(f[X]) \subseteq l_W(W) &= W \qquad \text{(Remark 3.1.3(2))} \\ &= l_Y(f[X]). \end{aligned}$$

Hence e' is l-dense.

(2) Now consider a commutative square in $\underline{X}$, where e is an l-dense morphism and m is an l-closed embedding.

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow[\ m\ ]{} & B \end{array}$$

Note that $p[X] \subseteq l_A(p[X]) \implies m(p[X]) \subseteq m(l_A(p[X]))$

$$\begin{aligned} \Rightarrow \quad l_B(m(p[X])) &\subseteq l_B(m(l_A(p[X]))) \qquad \text{(Proposition 3.2.1(2)} \\ &= m(l_A(p[X])) \\ &\subseteq l_B(m(p[X])) \qquad \text{(Definition 3.1.2(3))} \end{aligned}$$

Hence $m(l_A(p[X])) = l_B(m(p[X]))$.

In what follows, no notational distinction is made between an $\underline{X}$-object and its underlying set. Given $y \in Y$, we have

$$
\begin{aligned}
q(y) \in q(Y) &= q(l_Y(e[X])) && (e \text{ is } l\text{-dense}) \\
&\subseteq l_B(q(e[X])) && \text{(Definition 3.1.2(3))} \\
&= l_B(m(p[X])) \\
&= m(l_A(p[X])).
\end{aligned}
$$

Therefore, there is some $t \in l_A(p[X]) \subseteq A$ such that $m(t) = q(y)$. We therefore define $d : Y \longrightarrow A$ as follows: For each $y \in Y$, put $d(y) = t$ provided that $q(y) = m(t)$. The morphism d is well-defined because m is an injection. Uniqueness of d such that $d \circ e = p$ and $m \circ d = q$ follows from the fact that m is an l-closed embedding (and so, a monomorphism). □

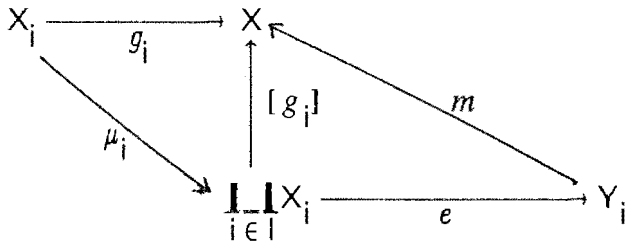
Corollary 3.2.9

If $\underline{X}$ is well-powered and has concrete coproducts, then for each strong limit operator l, $\underline{X}$ is an (l-dense sink, l-closed embedding)-category.

Proof

Since $\underline{X}$ has coproducts, it follows from the dual of Proposition 1.3.5 that the factorization $(\underline{E}, \underline{M})$, with $\underline{E}$ the class of all l-dense morphisms, and $\underline{M}$ the class of all l-closed embeddings, can be extended to a factorization structure $(\underline{E}', \underline{M})$ for set-indexed sinks. Now, since $\underline{X}$ is well-powered and $\underline{M}$ is contained in the class of all embeddings, and hence in $\mathrm{Mono}(\underline{X})$, it follows from the dual of Proposition 1.3.7 that $(\underline{E}', \underline{M})$ can be extended to a factorization structure $(\underline{E}'', \underline{M})$ for arbitrary sinks. It remains to be shown that the class $\underline{E}''$ so obtained is precisely the class of all l-dense sinks. Given a sink $(g_i : X_i \longrightarrow X)_I$ in $\underline{X}$ (where we can assume that I is a set), let $(\mu_i, \coprod_{i \in I} X_i)$ be the coproduct of the family $(X_i)_I$. By the

universal property of coproducts, there is a unique $\underline{X}$-morphism $[g_i] : \coprod_{i\in I} X_i \longrightarrow X$ such that $[g_i]\circ\mu_i = g_i$, for each $i \in I$. Let $[g_i] = m\circ e$ be the $(\underline{E}'', \underline{M})$-factorization of $[g_i]$:



Then $(m\circ(e\circ\mu_i))_I$ is the $(\underline{E}'', \underline{M})$-factorization of the sink $(g_i : X_i \longrightarrow X)_I$. Now

$$l_X([g_i](\coprod_{i\in I} X_i)) = l_X([g_i](\bigcup_{i\in I}\mu_i(X_i))) \quad \text{(since the coproduct is concrete)}$$

$$= l_X(\bigcup_{i\in I}([g_i]\circ\mu_i)(X_i))$$

$$= l_X(\bigcup_{i\in I} g_i(X_i)).$$

Hence $[g_i]$ is an l-dense morphism if and only if $(g_i)_I$ is an l-dense sink. Thus if $(g_i)_I$ is l-dense, then it belongs to E'', and vice versa.

□

Proposition 3.2.10

If $(\underline{E}, \underline{M})$ is a factorization structure on $\underline{X}$, where $\underline{M}$ is contained in the class of all $\underline{X}$-embeddings, and for each embedding $Y \rightarrowtail X$, $\eta_X Y$ is the middle object of its $(\underline{E}, \underline{M})$-factorization (where we can assume that $U(\eta_X Y) \subseteq U(X)$), then $\eta = (\eta_X)$ is a strong limit operator on $\underline{X}$.

Proof

(1) Let $Y \longrightarrow \eta_X Y \longrightarrow X$ be an (**E**, **M**)-factorization of the embedding $Y \subseteq X$. Since the first factor of an embedding is an embedding, it follows that $Y \subseteq \eta_X Y$.

(2) Given an embedding $Y \subseteq X$, form its (**E**, **M**)-factorization $m_1 \circ e_1$ and let $Z \subseteq \eta_X Y$:

$$Y \hookrightarrow X \;=\; Y \xhookrightarrow{e_1} \eta_X Y \xhookrightarrow{m_1} X$$

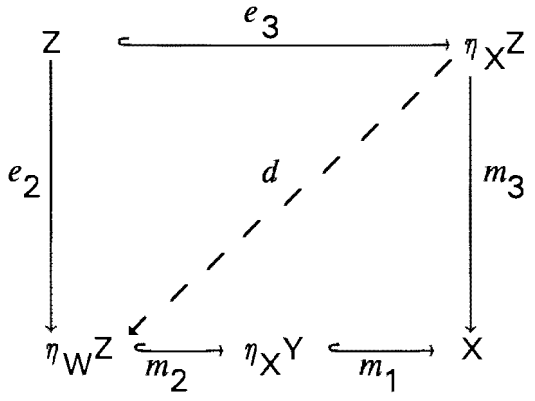
Form an (**E**, **M**)-factorization $m_2 \circ e_2$ of the embedding $m : Z \subseteq \eta_X Y = W$:

$$Z \hookrightarrow \eta_X Y \;=\; Z \xhookrightarrow{e_2} \eta_W Z \xhookrightarrow{m_2} W$$

Now, we form the (**E**, **M**)-factorization $m_3 \circ e_3$ of the embedding $m_1 \circ m : Z \subseteq X$:

$$Z \hookrightarrow X \;=\; Z \xhookrightarrow{e_3} \eta_X Z \xhookrightarrow{m_3} X$$

Note that $m_3 \circ e_3 = m_1 \circ m = m_1 \circ m_2 \circ e_2$. By the (unique) (**E**, **M**)-diagonalization property, there is a unique **X**-morphism $d : \eta_X Z \longrightarrow \eta_W Z$ which makes the following diagram commutative:



Since $m_3 = (m_1 \circ m_2) \circ d$ is an embedding, it follows that its first

factor d is an embedding; so we must have $\eta_X Z \subseteq \eta_W Z$, and condition (2) follows.

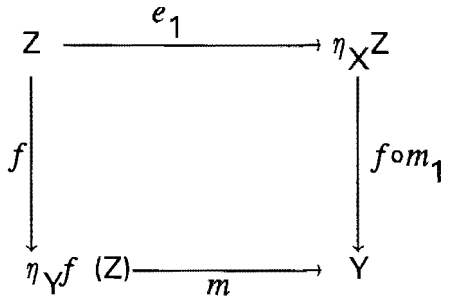
(3) We consider an **X**-morphism $X \xrightarrow{f} Y$ with an embedding $Z \subseteq X$. Let $m_1 \circ e_1$ be the (**E**, **M**)-factorization of $Z \subseteq X$.

$$Z \hookrightarrow X \;=\; Z \xhookrightarrow{e_1} \eta_X Z \xhookrightarrow{m_1} X$$

and let $m \circ e$ be the (**E**, **M**)-factorization of $f(Z) \subseteq Y$:

$$f(Z) \hookrightarrow Y \;=\; f(Z) \xhookrightarrow{e} \eta_Y(f(Z)) \xhookrightarrow{m} Y.$$

Then the following diagram commutes:



If we chase elements, we find the inclusion

$$U(f(\eta_X Z)) \subseteq U(\eta_Y f(Z)).$$

Since $f(\eta_X Z) \hookrightarrow Y$ is an embedding and $\eta_Y f(Z) \subseteq Y$ is initial, it follows that the inclusion $U(f(\eta_X Z)) \subseteq U(\eta_Y f(Z))$ is an **X**-morphism. Being the first factor of an embedding, this inclusion is an embedding, hence $f(\eta_X Z) \subseteq \eta_Y f(Z)$. □

Theorem 3.2.11

There is a bijective correspondence between the class of all (**E**, **M**)-factorization structures on **X** with the class **M** contained in the class of all **X**-embeddings, and the class of all strong limit operators η_X on **X**.

Proof

Given an ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-factorization on $\underline{\mathbf{X}}$, with $\underline{\mathbf{M}}$ contained in the class of all embeddings, we obtain a strong limit operator η as in Proposition 3.2.10 on $\underline{\mathbf{X}}$. From this strong limit operator we now obtain a factorization structure ($\underline{\mathbf{E}}'$, $\underline{\mathbf{M}}'$) = (η-dense, η-closed embedding) as in Proposition 3.2.8. We shall show that ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$) = ($\underline{\mathbf{E}}'$, $\underline{\mathbf{M}}'$). Suppose $m : X \longrightarrow Y \in \underline{\mathbf{M}}'$ and factorize m as an isomorphism followed by an initial inclusion:

$$X \xrightarrow{m} Y = X \xrightarrow{m'} m(X) \xrightarrow{j} Y,$$

where m' denotes the codomain restriction of m, and j denotes the inclusion map. By Theorem 1.3.1(4), (5), it follows that $j \in \underline{\mathbf{M}}'$. Consider the ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-factorization of j:

$$m(X) \xrightarrow{j} Y = m(X) \xrightarrow{e} Z \xrightarrow{n} Y,$$

where e and n are again initial inclusions. Now

$$\begin{aligned} Z &= \eta_Y(m(X)) && \text{(by definition of } \eta_Y) \\ &= m(X) && \text{(since } j \text{ is } \eta\text{-closed).} \end{aligned}$$

Hence e is the identity morphism, and so $j = n \in \underline{\mathbf{M}}$. It follows now that $m \in \underline{\mathbf{M}}$.

On the other hand, suppose $m : X \longrightarrow Y \in \underline{\mathbf{M}}$. To form the ($\eta$-dense, η-closed embedding)-factorization, we first form the (surjection, embedding)-factorization of m, namely:

$$X \xrightarrow{m} Y = X \xrightarrow{m'} m(X) \xrightarrow{j} Y,$$

where m' is the codomain restriction of m, and j is the inclusion map. Observe that $j \in \underline{\mathbf{M}}$. We then apply η_Y to $m(X)$ to get the (η-dense, η-closed embedding)-factorization. Thus:

$$X \xrightarrow{m} Y \;=\; X \xrightarrow{m'} m(X) \hookrightarrow \eta_Y(m(X)) \hookrightarrow Y.$$

But $\eta_Y(m(X))$ is the middle object in the ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$)-factorization of $j : m(X) \hookrightarrow Y$ which itself belongs to $\underline{\mathbf{M}}$. Hence $\eta_Y(m(X)) = m(X)$. Thus $m \in \underline{\mathbf{M}}'$. Consequently, $\underline{\mathbf{M}} = \underline{\mathbf{M}}'$, and so ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$) = ($\underline{\mathbf{E}}'$, $\underline{\mathbf{M}}'$), by the dual of Theorem 1.3.1(3).

On the other hand, if l is a strong limit on $\underline{\mathbf{X}}$, we form a factorization structure ($\underline{\mathbf{E}}$, $\underline{\mathbf{M}}$) = (l-dense, l-closed embedding) (Proposition 3.2.8). From this factorization structure, we obtain a strong limit operator η on $\underline{\mathbf{X}}$ (Proposition 3.2.10). It must be shown that $l = \eta$. Given an embedding $j : Y \subseteq X$, we obtain $\eta_X Y$ as the middle object in the (l-dense, l-closed embedding)-factorization of j as follows: First construct the (surjection, embedding)-factorization of j, which is just

$$Y \xhookrightarrow{j} X \;=\; Y \xrightarrow{id_Y} Y \xhookrightarrow{j} X.$$

Then applying l to this, we obtain the required factorization:

$$Y \longrightarrow l_X Y \hookrightarrow X.$$

The middle object is $l_X Y$, so $\eta_X Y = l_X Y$, hence $l = \eta$. □

CHAPTER 4

$\underline{M}$-PERFECT MORPHISMS AND RELATIVELY COMPACT OBJECTS

We shall assume that the category $\underline{X}$ is a morphism ($\underline{E}$, $\underline{M}$)-hereditary construct with finite products. That is, $\underline{X}$ is a morphism ($\underline{E}$, $\underline{M}$)-category and is a hereditary construct (as defined in Chapter 3).

In this chapter, we define $\underline{M}$-compactness and $\underline{M}$-perfectness in the category $\underline{X}$. Section 4.1 deals with the properties of $\underline{M}$-preserving and $\underline{M}$-perfect morphisms; both classes of morphisms are closed under composition (Proposition 4.1.6), and they are also closed under composition with isomorphisms (Lemma 4.1.2 and Proposition 4.1.4). In Proposition 4.1.8, we shall show that the class of $\underline{M}$-perfect $\underline{X}$-morphisms is finitely productive. In Section 4.2, properties of $\underline{M}$-compact $\underline{X}$-objects are presented, and it is shown, amongst others, that the class **Comp($\underline{X}$)** of $\underline{M}$-compact $\underline{X}$-objects is closed under the formation of finite (nonempty) products (Proposition 4.2.3), and that **Comp($\underline{X}$)** is both $\underline{M}$-perfect hereditary and $\underline{M}$-hereditary. See Proposition 4.2.5.

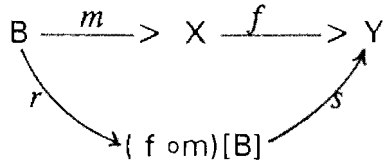
4.1 PROPERTIES OF $\underline{M}$-PERFECT MORPHISMS

We recall that (surjection, embedding) is a factorization structure on every hereditary construct (Lemma 3.1.4).

DEFINITION 4.1.1 (cf. [BR, 2.1])

Let $f : X \longrightarrow Y$ be an $\underline{X}$-morphism.

(1) f is called an $\underline{M}$-*preserving morphism* provided that, for each $m : B \longrightarrow X$ in $\underline{M}$, if



is the (surjection, embedding)-factorization of $f \circ m$, then $s \in \underline{M}$.

(2) f is called an $\underline{M}$-*perfect morphism* provided that, for each $\underline{X}$-object Z, the product morphism $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is $\underline{M}$-preserving.

Lemma 4.1.2

If $f : X \longrightarrow Y$ is $\underline{M}$-preserving and $B \xrightarrow{m} X \in \underline{M}$, then $f \circ m$ is $\underline{M}$-preserving.

Proof

Suppose that $m_1 : C \longrightarrow B \in \underline{M}$, and let $s \circ r = (f \circ m) \circ m_1$ be the (surjection, embedding)-factorization of $(f \circ m) \circ m_1$. Since $\underline{M}$ is closed under composition (Proposition 1.3.1(5)), we have $m \circ m_1 \in \underline{M}$. Since f is $\underline{M}$-preserving, we must have $s \in \underline{M}$. Hence $f \circ m$ is $\underline{M}$-preserving. □

Remarks 4.1.3

(1) Since $\underline{X}$-isomorphisms belong to $\underline{M}$ (Proposition 1.3.1(4)), it follows that $\underline{M}$-preserving morphisms are closed under composition with $\underline{X}$-isomorphisms on the the right.

(2) $\underline{M}$-preserving morphisms are also closed under composition with $\underline{X}$-isomorphisms on the left.

Proof

Let $X \xrightarrow{f} Y$ be an **M**-preserving and let $Y \xrightarrow{g} Z \in \mathrm{Iso}(\mathbf{X})$. Given a morphism $B \xrightarrow{m} X$ in **M**, we consider the (surjection, embedding)-factorization $(g \circ f) \circ m = s \circ r$ of $(g \circ f) \circ m$. Then $f \circ m = (g^{-1} \circ s) \circ r$. Since g^{-1} is an embedding and embeddings are closed under composition, it follows that $(g^{-1} \circ s) \circ r$ is the (surjection, embedding)-factorization of $f \circ m$. Since f is **M**-preserving, we have $g^{-1} \circ s \in \mathbf{M}$. But $g^{-1} \in \mathbf{M}$ (Proposition 1.3.1(4)), so $s \in \mathbf{M}$ (Proposition 1.3.1(7)). Hence $g \circ f$ is **M**-preserving. □

Proposition 4.1.4

If $X \xrightarrow{f} Y$ is **M**-perfect and $B \xrightarrow{m} X \in \mathbf{M}$, then $f \circ m$ is **M**-perfect.

Proof

For any **X**-object Z, we have $(f \circ m) \times id_Z = (f \times id_Z) \circ (m \times id_Z)$. Since f is **M**-perfect, it follows that $f \times id_Z$ is **M**-preserving. Since $id_Z \in \mathbf{M}$ (Proposition 1.3.1(4)), it follows that $m \times id_Z \in \mathbf{M}$ (Proposition 1.3.1(8)). By Lemma 4.1.2, we find that $(f \times id_Z) \circ (m \times id_Z)$ is **M**-preserving, hence $f \circ m$ is **M**-perfect. □

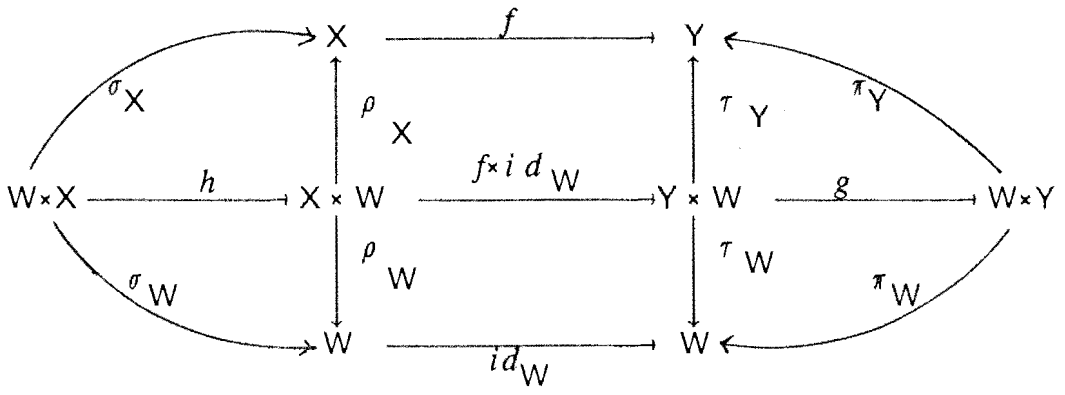
Proposition 4.1.5

If $X \xrightarrow{f} Y$ is **M**-perfect, then the product morphism $id_W \times f$ is **M**-preserving, for any $W \in \mathrm{Ob}(\mathbf{X})$.

Proof

Let $(W \times X, \sigma_W, \sigma_X)$ and $(X \times W, \rho_X, \rho_W)$ be the products of (W, X) and (X, W), respectively; and let $(Y \times W, \tau_Y, \tau_W)$ and $(W \times Y, \pi_W, \pi_Y)$ be the products of (Y, W) and (W, Y), respectively. Then there are

(unique) morphisms $h : W\times X \longrightarrow X\times W$ and $Y\times W \xrightarrow{g} W\times Y$ which make the following diagram commutative:



Also there is a (unique) morphism $r : X\times W \longrightarrow W\times X$ such that $\sigma_W \circ r = \rho_W$ and $\sigma_X \circ r = \rho_X$. Then

$$\rho_W \circ h \circ r = \sigma_W \circ r = \rho_W \circ id_{X\times W} \quad \text{and} \quad \rho_X \circ h \circ r = \sigma_X \circ r = \rho_X \circ id_{X\times W}.$$

But (ρ_W, ρ_X) is a mono-source, so $h \circ r = id_{X\times W}$. And in a similar way, we find that $r \circ h = id_{W\times X}$. Consequently, the morphisms h and g are isomorphisms. By uniqueness of the product morphism $id_W \times f$ such that

$$\pi_Y \circ (id_W \times f) = f \circ \sigma_X \quad \text{and} \quad \pi_W \circ (id_W \times f) = id_W \circ \sigma_W,$$

it follows from the above diagram that

$$id_W \times f = g \circ (f \times id_W) \circ h.$$

Since f is $\underline{\mathbf{M}}$-perfect, the product morphism $f\times id_W$ is $\underline{\mathbf{M}}$-preserving. By Lemma 4.1.2, the morphism $(f\times id_W)\circ h$ is $\underline{\mathbf{M}}$-preserving (since $h \in \underline{\mathbf{M}}$). By Remark 4.1.3, the composition $g\circ(f\times id_W)\circ h = id_W \times f$ is $\underline{\mathbf{M}}$-preserving as was to be shown. □

Proposition 4.1.6

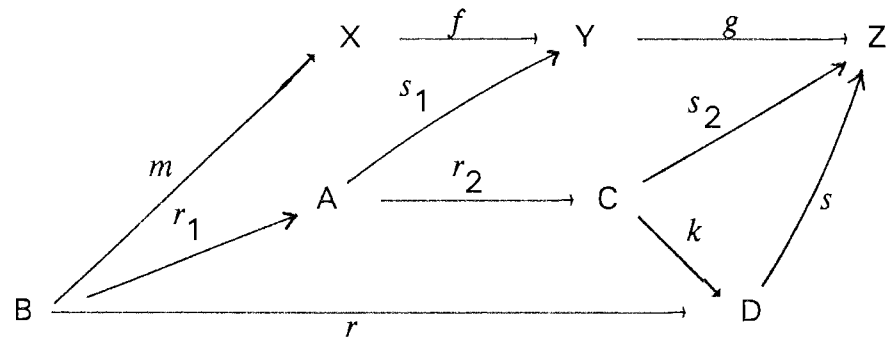
The classes of $\underline{M}$-preserving morphisms and $\underline{M}$-perfect morphisms are closed under composition.

Proof

(1) Let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be $\underline{M}$-preserving $\underline{X}$-morphisms. Given $B \xrightarrow{m} X \in \underline{M}$, let

$$B \xrightarrow{g \circ f \circ m} Z \;=\; B \xrightarrow{r} \bullet \xrightarrow{s} Z$$

be the (surjection, embedding)-factorization of $(g \circ f) \circ m$, let $s_1 \circ r_1$ be the (surjection, embedding)-factorization of $f \circ m$, and let $s_2 \circ r_2$ be the (surjection, embedding)-factorization of $g \circ s_1$. Since f is $\underline{M}$-preserving, we have $s_1 \in \underline{M}$. Since g is $\underline{M}$-preserving, we have $s_2 \in \underline{M}$. But $s_2 \circ (r_2 \circ r_1)$ and $s \circ r$ are both (surjection, embedding)-factorizations of $g \circ f \circ m$. By Proposition 1.3.1(6), there is an isomorphism k with $s \circ k = s_2$ and $k \circ r_2 \circ r_1 = r$:



By Proposition 1.3.1(4), we have $k^{-1} \in \underline{M}$, hence $s = s_2 \circ k^{-1}$ belongs to $\underline{M}$ (Proposition 1.3.1(5)).

(2) Given $\underline{M}$-perfect $\underline{X}$-morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, it must be shown that $(g \circ f) \times id_W : X \times W \longrightarrow Z \times W$ is $\underline{M}$-preserving.

Since

$$(g \circ f) \times id_W = (g \times id_W) \circ (f \times id_W)$$

and since each of f and g is **M**-perfect, it follows by definition that both $f \times id_W$ and $g \times id_W$ are **M**-preserving, so by (1) their composition $(g \times id_W) \circ (f \times id_W)$ is **M**-preserving. □

In the next proposition we shall need the following:

Lemma 4.1.7

If $X \xrightarrow{f} Y$ is **M**-perfect, so is the product morphism $f \times id_Z$, for every **X**-object Z.

Proof

Given $W \in Ob(\mathbf{X})$, we have

$$(f \times id_Z) \times id_W = f \times id_{Z \times W},$$

which is **M**-preserving by **M**-perfectness of f. Thus $f \times id_Z$ is **M**-perfect. □

Proposition 4.1.8 (cf. [BR, 2.3])

The class of **M**-perfect morphisms is finitely productive; that is, given a finite set I with $X_i \xrightarrow{f_i} Y_i$ an **M**-perfect morphism, for each $i \in I$, then the product morphism $\prod f_i : \prod X_i \longrightarrow \prod Y_i$ is **M**-perfect.

Proof

It suffices, by induction, to establish the result for $I = \{1, 2\}$. Let $f_1 : X_1 \longrightarrow Y_1$ and $f_2 : X_2 \longrightarrow Y_2$ be **M**-perfect morphisms and let Z be an **X**-object. We need to show that the morphism

$f_1 \times f_2 \times id_Z$ is **M**-preserving. But we have

$$f_1 \times f_2 \times id_Z = (f_1 \times id_{Y_2} \times id_Z) \circ (id_{X_1} \times f_2 \times id_Z).$$

Since each f_i is **M**-perfect, it follows that $f_1 \times id_{Y_2} \times id_Z$ is **M**-preserving (Lemma 4.1.7) and $f_2 \times id_Z$ is **M**-perfect (Lemma 4.1.7). By Proposition 4.1.5, the product morphism $id_{X_1} \times f_2 \times id_Z$ is **M**-preserving. By Proposition 4.1.6, the composition

$$(f_1 \times id_{Y_2} \times id_Z) \circ (id_{X_1} \times f_2 \times id_Z)$$

is **M**-preserving. □

Proposition 4.1.9

If every surjective morphism is in **E**, then

(1) **M** is the family of **M**-preserving embeddings.

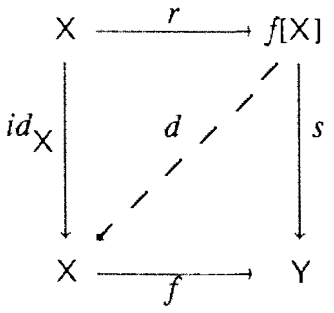
(2) **M** is contained in the family of **M**-perfect morphisms.

Proof

(1) Let $X \xrightarrow{f} Y \in \mathbf{M}$ and let

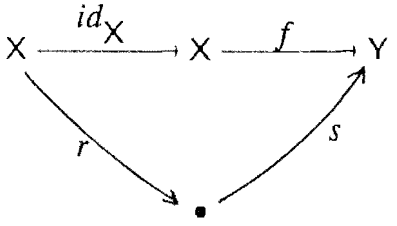
$$X \xrightarrow{f} Y = X \xrightarrow{r} f[X] \xrightarrow{s} Y$$

be its (surjection, embedding)-factorization. It needs to be shown that $s \in \mathbf{M}$ and f is an embedding. By hypothesis, we have $r \in \mathbf{E}$, so it follows from the unique diagonalization property that there is a morphism $d : f[X] \longrightarrow X$ such that $d \circ r = id_X$ and $f \circ d = s$; that is, the following diagram commutes:



From Proposition 1.3.1(2), it follows that f is an embedding, and d is an isomorphism. Consequently, the morphism $r = d^{-1}$ is an isomorphism. By Proposition 1.3.1(4) and (5), the morphism $s = f \circ r^{-1}$ belongs to **M**.

Conversely, suppose that $X \xrightarrow{f} Y$ is an $\underline{\mathbf{M}}$-preserving embedding, and let $s \circ r$ be its (surjection, embedding)-factorization. Since $id_X \in \underline{\mathbf{M}}$, it follows from the definition of an $\underline{\mathbf{M}}$-preserving morphism that $s \in \underline{\mathbf{M}}$:



Now r is a surjection and an embedding (being the first factor of an embedding), so by Proposition 1.3.1(4), r is an isomorphism. But then this also means that $r \in \underline{\mathbf{M}}$. Consequently, the morphism f belongs to $\underline{\mathbf{M}}$, by Proposition 1.3.1(5).

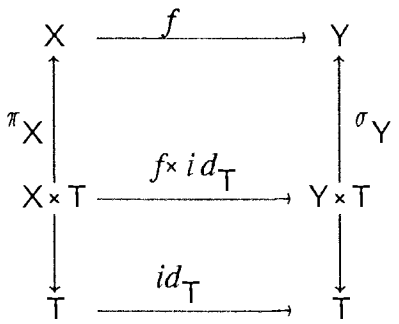
(2) Let $X \xrightarrow{f} Y \in \underline{\mathbf{M}}$. It will be shown that, for any $Z \in \mathrm{Ob}(\underline{\mathbf{X}})$, the product morphism $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is $\underline{\mathbf{M}}$-preserving. Since $id_Z \in \underline{\mathbf{M}}$, we have $f \times id_Z \in \underline{\mathbf{M}}$ (Proposition 1.3.1(8)). By (1), $\underline{\mathbf{M}}$ is the family of $\underline{\mathbf{M}}$-preserving embeddings, so $f \times id_Z$ is $\underline{\mathbf{M}}$-preserving. Hence f is $\underline{\mathbf{M}}$-perfect. □

Proposition 4.1.10

If **X** has a terminal object, then in **X** every **M**-perfect morphism is **M**-preserving.

Proof

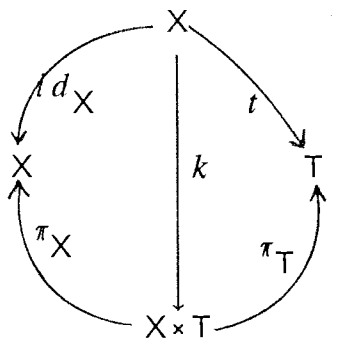
Let $X \xrightarrow{f} Y$ be an **M**-perfect morphism and let T be a terminal object of **X**. In the following commutative diagram, defining the product morphism $f \times id_T$;



the projections π_X and σ_Y are isomorphisms: for, the diagram

$$(X \xleftarrow{id_X} X \xrightarrow{t} T)$$

is a product of X and T, so there is a morphism k making the following diagram commutative:



We have

$$\pi_X \circ k \circ \pi_X = \pi_X \circ id_{X \times T} \quad \text{and} \quad \pi_T \circ k \circ \pi_X = t \circ \pi_X = \pi_T \circ id_{X \times T}.$$

Since (π_X, π_T) is a mono-source, it follows that $k \circ \pi_X = id_{X \times T}$. Since

it also holds that $\pi_X \circ k = id_X$, it follows that π_X is an isomorphism. And in a similar way, we can show that σ_Y is an isomorphism. Since f is **M**-perfect, it follows that $f \times id_T$ is **M**-preserving. By Remark 4.1.3, we must have that the morphism

$$f = \sigma_Y \circ (f \times id_T) \circ \pi_X^{-1}$$

is **M**-preserving. □

Corollary 4.1.11

If **X** has a terminal object and **E** contains all surjective morphisms, then, for any **X**-morphism f, each condition below implies those that follow it; in particular, for **X**-embeddings f, all three conditions are equivalent:

(1) $f \in \mathbf{M}$.

(2) f is **M**-perfect.

(3) f is **M**-preserving.

Proof

(1) $\Rightarrow$ (2). Follows from Proposition 4.1.9(2).

(2) $\Rightarrow$ (3). Follows from Proposition 4.1.10.

If f is an embedding, then **(3) $\Rightarrow$ (1)** follows from Proposition 4.1.9(1).

□

4.2 PROPERTIES OF M-COMPACT OBJECTS

In this section, we use the following theorem (due to Mrówka) to obtain a (relative) categorical notion of compact objects in a morphism (E, M)-hereditary construct X.

Mrówka's Theorem: (cf. [MR, p. 20] and [MA, 1.1])
Let (X, τ) be a topological space. Then (X, τ) is a compact space if and only if, for every topological space (Y, δ), the projection

$$\pi_1 : (Y, \delta)\times(X, \tau) \longrightarrow (Y, \delta)$$

is a closed mapping.

Proof

Suppose that $S \subseteq Y\times X$ is closed and let $\pi_1 : S \longrightarrow Y$ be the projection of S to Y. Let y_0 be any point of $\overline{\pi_1(S)}$. Then there exists a net $\{y_r \mid r \in \mathbb{R}\}$ of points of $\pi_1(S)$ which converges to y_0. Now, for each $r \in \mathbb{R}$ we can find $x_r \in X$ such that $\langle y_r, x_r\rangle \in S$. Since X is compact, it follows that the net $\{x_r \mid r \in \mathbb{R}\}$ contains a convergent subnet $\{x_{r_q} \mid q \in \mathbb{Q}\}$. Then the net $\{\langle y_{r_q}, x_{r_q}\rangle \mid q \in \mathbb{Q}\}$ is also convergent; so let us denote its limit by p_0. Then $p_0 \in S$ and y_0 is the projection of p_0, so $y_0 \in \pi_1(S)$.

Conversely, suppose that the projection $\pi_1 : (Y, \delta)\times(X, \tau) \longrightarrow (Y, \tau)$ is a closed mapping and let βX be the set of ultrafilters on the set X. For each $A \subset X$, set

$$A^* = \{\, \mathcal{U} \in \beta X \mid A \in \mathcal{U}\}.$$

Then $\{A^* \mid A \subset X\}$ is a base for a topology σ on βX. Let $\xi \subset \beta X \times X$ be the convergence relation of (X, τ) defined by

$$\xi = \{ (\mathcal{U}, x) \mid \mathcal{U} \text{ converges to } x \text{ in } \tau \}$$

Then, for any topology τ, ξ is closed (see [MA, Lemma 1.2]). Now, let C be the set of all ultrafilters which converge in τ to at least one point. C is dense in βX; for, if A^* is nonempty, there exists $x \in A$ and the principal ultrafilter

$$\text{princ}(x) = \{ A \subset X \mid x \in A \} \in A^* \cap C.$$

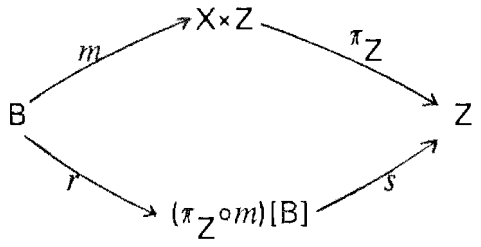
But C is the image of ξ under the projection

$$\pi_1 : (\beta X, \sigma) \times (X, \tau) \longrightarrow (\beta X, \sigma).$$

Therefore, if π_1 is closed, then all ultrafilters converge, thus (X, τ) is compact. And this completes the proof. □

Definition 4.2.1

An $\underline{X}$-object X is called an $\underline{M}$-*compact object* provided that, for each $\underline{X}$-object Z, the projection map $\pi_Z : X \times Z \longrightarrow Z$ is $\underline{M}$-preserving; that is, if $B \xrightarrow{m} X \times Z$ belongs to $\underline{M}$ and $\pi_Z \circ m$ has the following (surjection, embedding)-factorization, then the embedding $s \in \underline{M}$:



We shall denote by $\underline{\text{Comp}(X)}$ the class of all $\underline{M}$-compact $\underline{X}$-objects.

EXAMPLE 4.2.2

Let $\underline{X} = \underline{Top}$ and let $(\underline{E}, \underline{M})$ be the (dense, closed embedding)-factorization structure on $\underline{Top}$. Then the $\underline{M}$-compact objects are the compact spaces; so $\underline{M}$-compact $\cap$ $\underline{E}$-Sep is the family of all compact Hausdorff spaces.

Proof.

Observe that in $\underline{Top}$, the closed continuous functions are precisely the $\underline{M}$-preserving continuous functions, where $\underline{M}$ is the family of all closed embeddings in $\underline{Top}$. Given a compact space X, let $Y \in Ob(\underline{Top})$. Then the projection $\pi_Y : X \times Y \longrightarrow Y$ is closed ([MA, Theorem 1.1]). Now let

$$B \xrightarrow{m} X \times Y \xrightarrow{\pi_Y} Y = X \xrightarrow{r} Z \xrightarrow{s} Y$$

be the (surjection, embedding)-factorization of $\pi_Y \circ m$, where m is a closed embedding. Then $\pi_Y \circ m$ is closed, so the composition $s \circ r$ is closed, and since r is continuous and surjective, it follows that s is closed, hence s is a closed embedding. Consequently, X is $\underline{M}$-compact.

The converse is just a reformulation of Mrówka's theorem.

Since $\underline{M}$-compact spaces are the compact spaces and $\underline{E}$-separated spaces are the Hausdorff spaces (Examples 2.3(1)), it follows that $\underline{Comp(X)} \cap \underline{E}$-Sep is the class of all compact Hausdorff spaces. □

Proposition 4.2.3 (cf. [LO_1, 4.3])

$\underline{Comp(X)}$ is closed under the formation of finite (nonempty) products.

Proof

Suppose $A, B \in \underline{Comp(X)}$. It must be shown that $A \times B \in \underline{Comp(X)}$.

Given $Z \in Ob(\underline{X})$, we have $(A\times B)\times Z \cong A\times(B\times Z)$, say $h : (A\times B)\times Z \longrightarrow A\times(B\times Z)$ is an isomorphism such that, if σ_Z, $\pi_{B\times Z}$ and ρ_Z are the projection morphisms $(A\times B)\times Z \longrightarrow Z$, $A\times(B\times Z) \longrightarrow B\times Z$ and $B\times Z \longrightarrow Z$, respectively, then $\sigma_Z = (\rho_Z \circ \pi_{B\times Z}) \circ h$:

$$\begin{array}{ccc} (A\times B)\times Z & \xrightarrow{\ \sigma_Z\ } & Z \\ \Big\downarrow{\scriptstyle h} & & \Big\uparrow{\scriptstyle \rho_Z} \\ A\times(B\times Z) & \xrightarrow[\ \pi_{B\times Z}\ ]{} & B\times Z \end{array}$$

Since A is $\underline{\mathbf{M}}$-compact, the projection morphism $\pi_{B\times Z}$ is $\underline{\mathbf{M}}$-preserving. But B is $\underline{\mathbf{M}}$-compact, so the projection ρ_Z is $\underline{\mathbf{M}}$-preserving, hence the composition $\rho_Z \circ \pi_{B\times Z}$ is $\underline{\mathbf{M}}$-preserving (Proposition 4.1.6). Since $h \in \underline{\mathbf{M}}$ (Proposition 1.3.1(4)), it follows that $(\rho_Z \circ \pi_{B\times Z}) \circ h = \sigma_Z$ is $\underline{\mathbf{M}}$-preserving (Lemma 4.1.2). Hence $A\times B$ is $\underline{\mathbf{M}}$-compact as was to be shown. □

Remark:

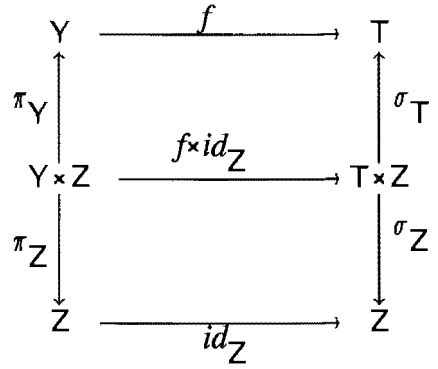
In [HSS, Proposition 4.11], it is assumed that $\underline{X}$ has a terminal object and that all surjective morphisms belong to the class $\underline{\mathbf{E}}$ of $\underline{X}$-morphisms. In the following proposition, we shall prove a similar result without the assumption on the surjective morphisms but the existence of a terminal object.

Proposition 4.2.4 (cf. [LO_1, 4.2])

If $\underline{X}$ has a terminal object T, then an $\underline{X}$-object Y is $\underline{\mathbf{M}}$-compact if, and only if, $Y \xrightarrow{f} T$ is $\underline{\mathbf{M}}$-perfect.

Proof

Suppose that Y is **M**-compact. By an argument similar to that used in the proof of Proposition 4.1.10, we can show that the projection $\sigma_Z : T\times Z \longrightarrow Z$ in the following commutative diagram - defining the product morphism $f\times id_Z$ - is an isomorphism:



Thus $f\times id_Z = \sigma_Z^{-1}\circ\pi_Z$. Since Y is **M**-compact, it follows that the projection π_Z is **M**-preserving, for each $Z \in Ob(\underline{X})$. By Remark 4.1.3(2), the composition $\sigma_Z^{-1}\circ\pi_Z = f\times id_Z$ is **M**-preserving, for each $Z \in Ob(\underline{X})$; that is, f is **M**-perfect.

Conversely, suppose that $Y \xrightarrow{f} T$ is **M**-perfect. Then the product morphism $Y\times Z \xrightarrow{f\times id_Z} T\times Z$ is **M**-preserving, for each $Z \in Ob(\underline{X})$. Since σ_Z is an isomorphism, it follows from Remark 4.1.3(2) that π_Z is **M**-preserving. Hence Y is **M**-compact. □

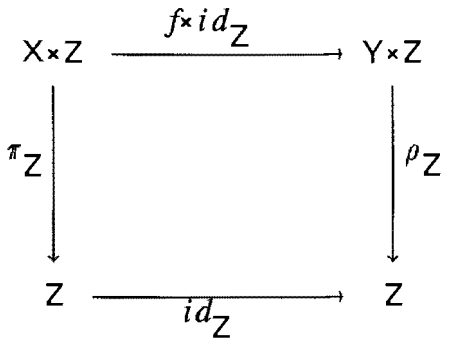
Proposition 4.2.5 (cf. [MA, Theorem 4.4 (1)])

The class **Comp(X)** of **M**-compact objects is **M**-perfect hereditary and **M**-hereditary. (A class **A** of **X**-objects is said to be ***M**-hereditary* (respectively, ***M**-perfect hereditary)* if, whenever $f : X \longrightarrow A$ belongs to **M** (resp. *f is **M**-perfect*) and $A \in \underline{A}$, then X is an object in **A**.)

Proof

Let $X \xrightarrow{f} Y$ be **M**-perfect with Y an **M**-compact **X**-object. We

wish to show that X is also **M**-compact. Let Z be any **X**-object. Since f is **M**-perfect, $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is **M**-preserving. Since Y is **M**-compact, the projection $\rho_Z : Y \times Z \longrightarrow Z$ is **M**-preserving.

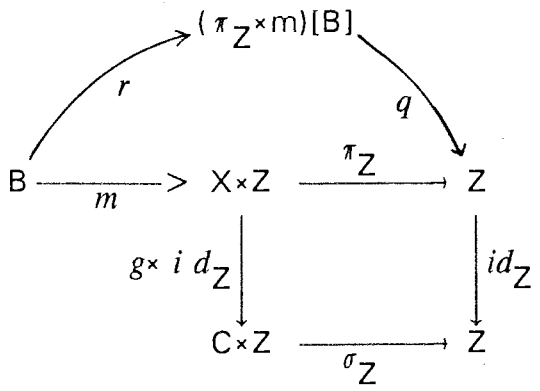


Therefore, the projection

$$\pi_Z = id_Z \circ \pi_Z = \rho_Z \circ (f \times id_Z)$$

is **M**-preserving, by Proposition 4.1.6. So X is **M**-compact, by definition.

Now suppose $X \xrightarrow{g} C$ belongs to **M** with C an **M**-compact **X**-object. For any **X**-object Z with $B \xrightarrow{m} X \times Z \in \mathbf{M}$, let $q \circ r = \pi_Z \circ m$ be the (surjection, embedding)-factorization of $\pi_Z \circ m$, where π_Z is the projection $X \times Z \longrightarrow Z$. We assert that $q \in \mathbf{M}$. By Proposition 1.3.1(8), the product morphism $g \times id_Z : X \times Z \longrightarrow C \times Z$ belongs to **M**, so $(g \times id_Z) \circ m$ belongs to **M** as well (Proposition 1.3.1(5)). Hence $q \circ r = \sigma_Z \circ ((g \times id_Z) \circ m)$ is the (surjection, embedding)-factorization of $\sigma_Z \circ (g \times id_Z) \circ m$, where σ_Z is the projection $C \times Z \longrightarrow Z$:



But C is $\underline{M}$-compact, hence $q \in \underline{M}$. Thus X is $\underline{M}$-compact. □

Corollary 4.2.6

If Y is a compact topological space and $X \xrightarrow{f} Y$ is a closed embedding or a perfect continuous map, then X is compact. (A function $f : X \longrightarrow Y$ between topological spaces is *perfect* if, for each topological space Z, the product function

$$f \times id_Z : X \times Z \longrightarrow Y \times Z$$

is a closed function (see [MA, p. 346])).

Proof

Let $\underline{X}$ = $\underline{Top}$ with the factorization structure ($\underline{E}$, $\underline{M}$) = (dense, closed embedding). We have that Y is a compact space. By Example 4.2.2, it follows that Y is an $\underline{M}$-compact space. By Proposition 4.2.5, since $X \xrightarrow{f} Y$ is a closed embedding, it follows that X is $\underline{M}$-compact, and accordingly, X is a compact space.

On the other hand, f is a perfect continuous function if and only if $f \times id_Z : X \times Z \longrightarrow Y \times Z$ is a closed continuous function; i.e. $f \times id_Z$ is $\underline{M}$-preserving, for each topological space Z. Hence the perfect continuous functions are precisely the $\underline{M}$-perfect ones, with $\underline{M}$ the

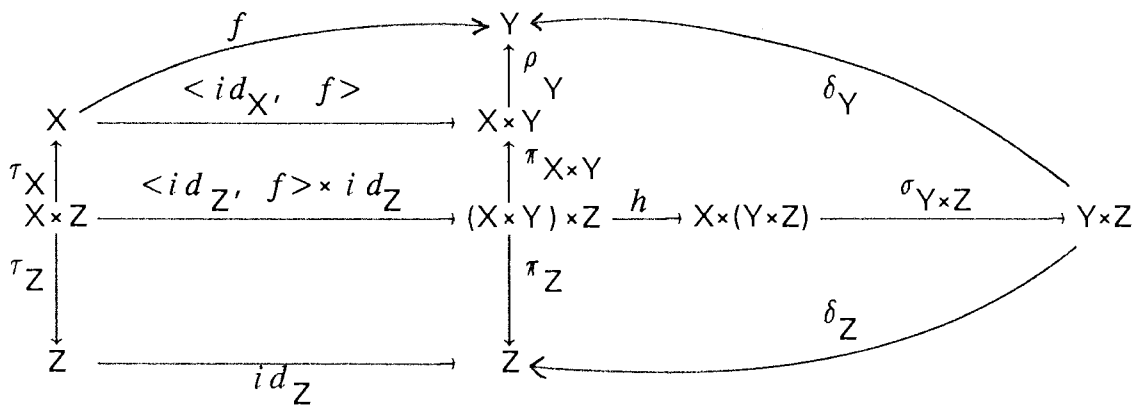
class of all closed embeddings. If Y is compact (i.e. **M**-compact), then it follows from Proposition 4.2.5 that X is compact. □

Proposition 4.2.7 (cf. [MA], 4.6.)

Suppose **X** is finitely complete. Given an **X**-morphism $X \xrightarrow{f} Y$, where X is **M**-compact and $Y \in$ **E**-Sep, then f must be **M**-perfect.

Proof

Let $Z \in Ob(\mathbf{X})$. Since X is **M**-compact, it follows that the projection $\sigma_{Y\times Z} : X\times(Y\times Z) \longrightarrow Y\times Z$ is **M**-preserving. Since Y is in **E**-Sep, we must have $\langle id_X, f\rangle : X \longrightarrow X\times Y \in \mathbf{M}$, by the Hausdorff Characterization Theorem 2.4.1. Since $id_Z \in \mathbf{M}$, the product morphism $\langle id_X, f\rangle \times id_Z \in \mathbf{M}$ (Proposition 1.3.1(8)). Let $(X\times Z, \tau_X, \tau_Z)$ be the product of $\{X, Z\}$, let $((X\times Y)\times Z, \pi_{X\times Y}, \pi_Z)$ be the product of $\{X\times Y, Z\}$ and let $(Y\times Z, \delta_Y, \delta_Z)$ be the product of $\{Y, Z\}$. If h is an isomorphism from $(X\times Y)\times Z$ to $X\times(Y\times Z)$ such that the upper and the lower right-hand triangles in the following diagram commute (where ρ_Y is the projection $X\times Y \longrightarrow Y$), then $h\circ(\langle id_X, f\rangle \times id_Z) \in \mathbf{M}$, by Proposition 1.3.1(5):



Thus, the composition $\sigma_{Y\times Z}\circ h\circ(\langle id_X, f\rangle \times id_Z)$ is **M**-preserving (Proposition 4.1.2). By uniqueness of the product morphism $f\times id_Z$, it

follows that $f \times id_Z = \sigma_{Y \times Z} \circ h \circ (\langle id_X, f \rangle \times id_Z)$. It has been proved that $f \times id_Z$ is **M**-preserving. Hence f is **M**-perfect. □

Corollary 4.2.8

Every compact subspace of a Hausdorff topological space is closed.

Proof

Let **X** = **Top** with the factorization structure (**E**, **M**) = (dense, closed embedding). Suppose Y is a Hausdorff space and let X be a compact subspace of Y. By Example 4.2.2, the space X is **M**-compact, and, by Example 2.3(1), Y ∈ **E**-Sep. By the proposition, the inclusion function $X \xrightarrow{j} Y$ is an **M**-perfect embedding. The category **Top** has a terminal object, namely, a singleton $\{\star\}$. By Proposition 4.1.10, the inclusion j is **M**-preserving and, thus closed. In particular, the space X is closed in Y. □

Proposition 4.2.9 (cf. [BR, 2.2])

Suppose **X** has concrete pullbacks and let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be any **X**-morphisms.

(1) If $g \circ f$ is **M**-preserving and g is an embedding, then f is **M**-preserving.

(2) If $g \circ f$ is **M**-perfect and g is an embedding, then f is **M**-perfect.

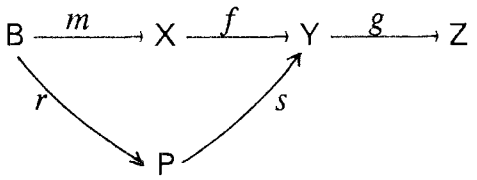
(3) If $g \circ f$ is **M**-perfect, Y ∈ **E**-Sep and every surjective morphism is in **E**, then f is **M**-perfect.

Proof

(1) If $g \circ f : X \longrightarrow Z$ is **M**-preserving, g is an embedding and $B \xrightarrow{m} X \in \mathbf{M}$, let $s \circ r = f \circ m$ be the (surjection, embedding)-factorization of $f \circ m$:

$$B \xrightarrow{m} X \xrightarrow{f} Y \;=\; B \xrightarrow{r} P \xrightarrow{s} Y$$

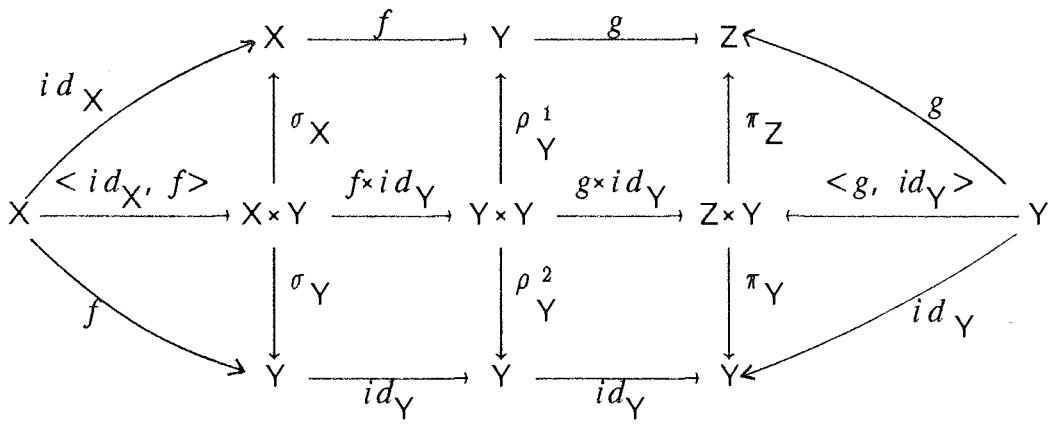
Then $(g\circ s)\circ r = (g\circ f)\circ m$, and $g\circ s$ is an embedding (since it is a composition of embeddings), so $(g\circ s)\circ r$ is a (surjection, embedding)-factorization of $(g\circ f)\circ m$:



But $g\circ f$ is $\underline{\mathbf{M}}$-preserving, so $g\circ s \in \underline{\mathbf{M}}$. Since g is an embedding, it follows from Proposition 1.3.1(7) that $s \in \underline{\mathbf{M}}$.

(2) Given that $g\circ f$ is $\underline{\mathbf{M}}$-perfect and g is an embedding, we need to show that $f\times id$ is $\underline{\mathbf{M}}$-preserving. Since $g\circ f$ is $\underline{\mathbf{M}}$-perfect, we have $(g\circ f)\times id$ is $\underline{\mathbf{M}}$-preserving. But $(g\circ f)\times id = (g\times id)\circ(f\times id)$. Since $g\times id$ is an embedding (Proposition 1.3.1(8)), it follows by (1) that $f\times id$ is $\underline{\mathbf{M}}$-preserving. Hence f is $\underline{\mathbf{M}}$-perfect.

(3)) We shall refer to the following commutative diagrams, where $(X\times Y, \sigma_X, \sigma_Y)$ is the product of $\{X, Y\}$, $(Z\times Y, \pi_Z, \pi_Y)$ is the product of $\{Z, Y\}$, and ρ^1_Y, ρ^2_Y are the usual projections $Y^2 \longrightarrow Y$:



We have

$$\pi_Z \circ [(g \circ f) \times id_Y] \circ \langle id_X, f \rangle = \pi_Z \circ (g \times id_Y) \circ (f \times id_Y) \circ \langle id_X, f \rangle$$

$$= g \circ \rho^1_Y \circ (f \times id_Y) \circ \langle id_X, f \rangle = g \circ f \circ \pi_X \circ \langle id_X, f \rangle = g \circ f,$$

and $\pi_Z \circ \langle g, id_Y \rangle \circ f = g \circ f$ is immediate. Hence

$$\pi_Z \circ [(g \circ f) \times id_Y] \circ \langle id_X, f \rangle = \pi_Z \circ \langle g, id_Y \rangle \circ f.$$

We also have

$$\pi_Y \circ [(g \circ f) \times id_Y] \circ \langle id_X, f \rangle = \pi_Y \circ (g \times id_Y) \circ (f \times id_Y) \circ \langle id_X, f \rangle$$

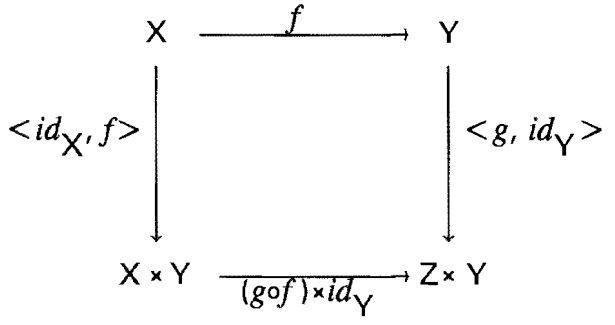
$$= id_Y \circ \rho^2_Y \circ (f \times id_Y) \circ \langle id_X, f \rangle = id_Y \circ \pi_Y \circ \langle id_X, f \rangle = id_Y \circ f$$

$$= \pi_Y \circ \langle g, id_Y \rangle \circ f$$

But (π_Z, π_Y) is a mono-source, so

$$[(g \circ f) \times id_Y] \circ \langle id_X, f \rangle = \langle g, id_Y \rangle \circ f,$$

that is, the following diagram commutes:



Since $\underline{\mathbf{X}}$ has pullbacks, it follows that $\underline{\mathbf{X}}$ is finitely complete; so Hausdorff Characterization Theorem 2.4.1 applies. Now since $Y \in \underline{\mathbf{E}}$-Sep, it follows that $\langle id_X, f \rangle \in \underline{\mathbf{M}}$. Since every surjective morphism is in $\underline{\mathbf{E}}$, it holds that $\langle id_X, f \rangle$ is $\underline{\mathbf{M}}$-perfect (Proposition 4.1.9(2)). By hypothesis, $g \circ f$ is $\underline{\mathbf{M}}$-perfect, so

$(g \circ f) \times id_Y$ is $\underline{M}$-perfect (Lemma 4.1.7). Thus the composition

$$[(g \circ f) \times id_Y] \circ \langle id_X, f \rangle$$

is $\underline{M}$-perfect (Proposition 4.1.6), hence $\langle g, id_Y \rangle \circ f$ is $\underline{M}$-perfect. Since $\langle g, id_Y \rangle$ is an embedding (being a section), it follows from (2) that f is $\underline{M}$-perfect. □

4.3 OTHER NOTIONS OF COMPACTNESS

(a) Compactness in categories can be approached via closure operators; see for instance, Castellini, G:

Castellini, G: *Compact objects, surjectivity of epimorphisms and compactifications*, Cahiers Top. Géom. Diff. Cat. 31 (1990), 53 - 65.

Castellini, G: *Regular closure operators and compactness*, Cahiers Top. Géom. Diff. Cat. 31 (1992), 21 - 31.

(b) Categorically compactness in an algebraic setting was extensively studied by Fay and Walls:

T. H. Fay: *Compact modules*, Comm. Alg. 16 (1988), 1209 - 1219.

T. H. Fay and G. L. Walls: *Compact nilpotent groups*, Comm. Alg. 17(9) (1989), 2255 - 2268.

T. H. Fay and G. L. Walls: *Categorically compact locally nilpotent groups*, Comm. Alg. 18 (1990), 3423 - 3435.

T. H. Fay: *Remarks on the Mal'cev completion of torsion-free locally nilpotent groups*, to appear.

(c) Another type of categorical compactness is that introduced by Ánh and Wiegandt ([AW]) in terms of a functor:

Ánh, P. N and Wiegandt, R: *Compactness in categories and interpretations*, Proc. EECS (1990), to appear.

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