A STUDY ON THE ANALYSIS OF TWO-UNIT REDUNDANT REPAIRABLE COMPLEX SYSTEMS

by

SETH THEMBA MOHOTO

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Supervisor: Prof VSS Yadavalli

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To my

PARENTS

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CONTENTS

SUMMARY

Two well-known methods of improving the reliability of a system are

(i) provision of redundant units, and
(ii) repair maintenance.

In a redundant system more units made available for performing the system function when fewer are required actually. There are two major types of redundancy - parallel and standby. In this dissertation we are concerned with both these types.

Some of the typical assumptions made in the analysis of redundant systems are

(i) the repair facility can take up a failed unit for repair at any time, if no other unit is undergoing repair
(ii) the system under consideration is needed all the time

However, we frequently come accross systems where one or more assumptions have to be relaxed. This is the motivation for the detailed study of the models presented in this dissertation.

In this dissertation we present models of redundant systems relaxing one or more of these assumptions simultaneously. More specifically it is a study of stochastic models of redundant systems with 'vacation period' for the repair facility (both standby and parallel systems), and intermittently used systems.

The dissertation contains five chapters. Chapter 1 is introductory in nature and contains a brief description of the mathematical techniques used in the analysis of redundant systems.

In Chapter 2 assumption (i) is relaxed while studying a model of cold standby redundant system with 'vacation period' for the repair facility. In this model the repair facility is not available for a random time immediately after each repair completion. Integral equations for the reliability and availability functions of the system are derived under suitable assumptions.

In Chapter 3, once again assumption (i) is relaxed while studying a model of parallel redundant systems with the same 'vacation period' for the repair facility, explained in the above paragraph.

In Chapter 4, the detailed review of intermittently used systems have been studied.

In Chapter 5, assumption (ii) is relaxed. This chapter is devoted to the study of an intermittently used 2-unit cold standby system with a single repair facility. This study was carried out using the 'correlated alternating renewal process' and the joint forward recurrence times.

All the above models have been studied, when some of the underlying distributions have a non-Markovian nature. They have been analysed using a regeneration point technique.

CHAPTER 1

INTRODUCTION

CHAPTER 1

Introduction

1.1 Introduction

The past two decades have witnessed an upsurge in technological activity due to rapid advancements in the computer and telecommunication industries. As a result there has been a vast increase in the complexity of various systems and in the number of such complicated systems.

The performance of these systems is very important and therefore the functioning of these systems is required to be of a very high standard.

To ensure high system performance, the behaviour of various systems under normal operational conditions had to be examined. Any system analysis in order to be complete must pay due attention to system measures, such as, reliability, availability, mean-time to system failure (MTSF), etc. Advice as to how overall system performance can be enhanced is also required due to the behaviour of such systems. The methods of probability theory and mathematical statistics may be applied to obtain various measures which would reflect on the functioning of the system under study.

Many mathematical models have been proposed in the literature to evaluate measures of system performance and provide methods of improving it. These models describe the operational characteristics of a system taking into account its essential features and can be solved only by stochastic methods. With the recent remarkable advances in electronic engineering and its pervasive applications in modern society, systems have become more sofisticated and their reliability are more crucial. The present work is a stochastic analysis of some mathematical models representing the behaviour of a few complex systems.

1.2 Redundant Systems

One of the principal methods for improving the performance of a system is the introduction of "Redundancy". A redundant system is one in which more system components or units are available for performing the system function when fewer are required actually. Gnedenko, Belyayev and Solovyev (1969) provides an extensive introduction to such systems.

There are several kinds of redundant systems. In this section we describe some important variations of such systems. There are two major types of redundancy: "standby redundancy" and "parallel redundancy".

Standby Redundant Systems

A standby redundant system is one in which one unit is engaged in the actual system function whilst being backed by a number of spares, the unit engaged in the system function is termed the 'online unit' and the spares 'standbys'. Upon failure of the online unit, a standby (if operable) is switched online. The standbys are classified as cold, warm or hot depending on how they are loaded in the standby state. Hot standbys are those in which the standby units are loaded in the same way as the operating unit. Warm standbys have a diminished load and cold standbys are completely unloaded. Thus the probability of failure of a hot standby is the same as that of a unit operating online. Warm standbys may fail but their probability of failure is less than that of a online unit whilst cold standbys never lose their operational ability in the standby state.

Parallel Redundant Systems

An *n*-unit parallel redundant system is one in which all n-units operate simultaneously but the system function requires only one of these units, hence a system failure occurs only when all n-units are in a failed state.

k out of n: F system

The system which consists of *n* units is failed if and only if at least *k* units are failed.

k out of n: G system

The system which consists of n units is good if and only if at least *k* units are good.

1.3 Repairable Systems

In order to improve system performance failed units may be replaced by new ones. The unit which has failed may now be discarded or restored to an operable condition. This restoration process is called repair, and a system which possesses this feature is called a repairable system. The decision to repair a unit is usually taken when the procuring of new units prove to be comparatively expensive or infeasible. Upon failure, the unit is sent to the repair facility for repair. If the repair facility is not available, the unit will wait for repair. The life time of a unit while online, while in standby, and the repair time are taken to be stochastically independent random variables. It is also assumed that the distribution functions of these random variables exist and that they possess probability density functions.

Barlow (1962) considered some repairman problems which bore some feautres of queueing problems. Rau (1964) discussed the problem of finding the optimum value of m in an m out of n: G system for maximizing reliability. Ascher (1968) pointed out some inconsistencies in the modelling of repairable systems by renewal theory. Several authors, notably Barlow and Proschan (1965), Sandler (1963), Shooman (1968), Buzacott (1970) and Doyan and Berssenbrugge (1968) have used continuous time discrete state space Markov renewal process models for describing the behaviour of a repairable system. These methods, however, are not practically feasible for systems with large numbers of states. Gaver (1963), Gnedenko et al (1969), Osaki (1969; 70 a,b) and Srinivasan (1966) have employed the techniques of semi-Markov processes for finding the reliability of a system with exponential failure times. Srinivasan and Subramanian (1977), Subramanian et al (1976), Ravichandran (1979), Natarajan (1980), and Sarma (1982) have used regeneration point technique to analyse repairable systems with arbitrary distributions.

More references in related topics can be found in review papers and books by Subba Rao and Natarajan (1970); Osaki and Nakagawa (1976); Pierskalla and Voelker (1976); Lie, Hwang and Tillman (1977); Kumar and Agharwal (1980); Subramanian and Srinivasan (1980); Osaki (1985); Dhillon and Singh (1980) and Dhillon (1982, 1983).

1.4 Systems with Non-Instantaneous Switchover

In the analysis of standby redundant system it is generally assumed that when the unit operating online fails, a standby unit is automatically switched online with this switchover being instantaneous. This stipulation, however, may be relaxed. Srinivasan (1968), Osaki (1972), Khalil (1977), Subramanian and Ravichandran (1979), Gopalan and Marathe (1978, 1980), Singh et al (1979) and Kalpakam and Shalul Hameed (1980), Sarma (1982), etc. have studied redundant systems incorporating non-negligible switchover times.

1.5 Systems with Imperfect Switch

To transfer a unit from the standby state to the online state, a device known as a "switching device" is required. Generally it has been assumed that this device is perfect in the sense that, it is failure free. However, Gnedenko et al (1969), pointed out that this need not always be the case. Such systems, in which the switching device may fail at a non-negligible rate are called "systems with an imperfect switch". Chow (1971), Osaki (1972), Nakagawa and Osaki (1976a), Nakagawa (1977), Venkatakrishnan (1975), Prakash and Kumar (1970), Srinivasan and Subramanian (1980) and Subramanian and Natarajan (1980) have considered models where the switching device can fail. Subramanian and Sarma (1987) gave a more elaborate state-of-art on such systems.

1.6 Intermittently used systems

A common assumption in most of the literature on redundant systems is that the system is required to perform its intended function continuously. In some practical situations, however, systems are not required continuously but may be in a state of failure during certain time intervals with negligible consequential loss. Systems which possess such characteristics are called "Intermittently used systems". Gaver (1963) pointed out that it is pessimistic to evaluate the performance of such a system solely on the distribution of time to failure, since this failure could have occurred during a time interval when the system was not required, and illustrated his concepts by means of the analysis of a one unit intermittently used system. Later Srinivasan (1966), Nakagawa et al (1976), Srinivasan and Bhaskar (1979a,b,c), Kapur and Kapoor (1978, 1980) extended Gaver's results to the case of two-unit intermittently used systems. Sarma and Natarajan (1982) studied an intermittently used n-unit warm standby system with some restrictions. For the first time, an intermittently used parallel system was studied by Sarma and Hines (1990).

1.7 Measures of System Performance

The previous section briefly described the various types of redundant systems discussed in the existing literature. In this section some of the important measures used to evaluate system performance are discussed [Barlow and Proschan (1975), Gnedenko et al (1969)].

(a) Reliability

Reliability of a system is the probability that the system will adequately perform its desired function for the period of time intended under the operational condition encountered.

Let $\{I(t);\ t \geq 0\}$ be the performance process of the system; then for each fixed t, $I(t)$ is a binary random variable which takes the value 1 if the system operates satisfactorily at a given instant of time t, and takes the value 0, otherwise. Then the reliability function, $R(t)$, is given by

$$R(t) = pr[I(u) = 1;\ \textit{for all}\ \ u\ \ \textit{such that}\ \ 0 \leq u \leq t]$$

The expected value of the random variable denoting the time interval from the point of initial operation of the system to first system failure is termed the "Mean-Time-to-System-ailure" (MTSF). It can be obtained from the reliability function, $R(t)$, from the relation

$$MTSF = \int_0^\infty R(u)du \tag{1.1}$$

provided the integral converges and $R(0) = 1$.

(b) Pointwise or Instantaneous Availability

This is defined to be "the probability that the system is operational within the tolerances at a given instant of time". This may be expressed as, pointwise availability,

$$A(t) = pr[\textit{system is up at } t]$$

$$= pr[I(t) = 1].$$

(c) Asymptotic, Steady-State or Limiting Availability

Asymptotic, Steady-State or Limiting Availabily is defined as

$$A_{\infty} = \lim_{t \to \infty} A(t) \tag{1.2}$$

This may be shown [Barlow and Proschan (1975)] to be the expected fraction of time that the system operates satisfactorily in the long run i.e.

$$A_{\infty} = \frac{\textit{mean uptime}}{\textit{mean uptime} + \textit{mean downtime}} \tag{1.3}$$

(d) Mean Number of Events in (0, t]

Let $N(x, t)$ denote the number of a particular type of event (e.g. system failure) in the time interval $(x, x + t]$

Then the expected number of such events in the time interval $(0, t]$ is given by:

$$E[N(0, t)] = \int_0^t h_1(u)du \tag{1.4}$$

where $h_1(t)$ denotes the first order product density of the events given by

$$h_1(t) = \lim_{\Delta \to 0} E[N(t, \Delta)] / \Delta \tag{1.5}$$

The stationary rate of occurrence of events, N is given by

$$N = \lim_{t \to \infty} [E[N(0, t)] / t] \tag{1.6}$$

1.8 Techniques used in the Analysis of Redundant Systems

This section is a compilation of some of the techniques employed in the analysis of redundant systems.

1.8.1 Renewal Theory

Renewal theory provides a very useful tool for the study of stochastic processes and applied to probability models, and has been used extensively by many authors to analyse various reliability problems. Smith (1958) gave an extensive review and highelighted certain applications of renewal theory, and Feller (1968) made significant contributions to the development of renewal theory. Cox (1962) provides a lucid introduction to renewal theory. A renewal process is a generalization of the Poisson process allowing independent and identically distributed arbitrary distributions of the interarrival times.

Definition 1: A renewal process is a sequence of independent and identically distributed random variables $\{X_n, n = 1, 2, ...,\}$, which are not all zero with probability one.

We assume that these random variables are defined on the same probability space and have finite mean μ. A renewal process is completely determined by means of $f(\cdot)$, the probability density function of X_n. Associated with a renewal process is a counting process

$$\{N(t), t \geq 0\},$$

which represents the number of renewals in the time interval $(0, t]$ [Parzen (1962)].

Definition 2: The expected value of $N(t)$ is called the renewal function and is denoted by $H(t)$. The derivative of $H(t)$, if it exists, is denoted by $h(t)$ and is called the renewal density.

The quantity $h(t)\,\delta t$ is approximately equal to the probability that a renewal occurs in the time interval $(t, t + \delta t)$, for $\delta t > 0$ sufficiently small. The renewal density satisfies the following famous integral equation known as the functional equation of renewal theory

$$h(t) = f(t) + \int_0^t f(u)\,h(t-u)\,du \tag{1.7}$$

The solution of the above equation is given by:

$$h(t) = \sum_{n=1}^{\infty} f^{(n)}(t)$$

where $f^{(n)}(t)$ is the n-fold convolution of $f(t)$ and $f^1(t) = f(t)$.

Renewal theory has been widely used in the analysis of reliability problems. Srinivasan et al (1971) used renewal theory to obtain some operating characteristics of a one-unit system. The integral equation of renewal theory was used by Gnedenko et al (1969) to obtain MTSF of a two-unit standby system. Osaki (1970b), Bhat (1973) applied the integral equation to study several redundant systems. Buzacott (1971) used recent references of theoretic arguments to study some priority redundant systems and many authors have found renewal theory a vital tool in their research.

1.8.2 Semi-Markov and Markov Renewal Processes

In a discrete-time Markov chain the sojourn times are constant, but may be different. In a continuous-time Markov chain, the sojourn times are also random variables. A Markov process can move from one state to another in which the sojourn time is distributed exponentially. On the other hand, a renewal process can revisit a state, in which the state space is only one. However, it can permit an arbitrary distribution of the sojourn time.

Combining a Markov chain and a renewal process, we can get a Markov renewal process or a semi-Markov process.

Consider a stochastic process which makes state transitions in accordance with a Markov chain but, however, with the sojourn times in each state being probabilistic. Denote the state space by the set of non-negative integers $\{0, 1, 2, \ldots\}$ and let the transition probabilities be given by P_{ij}, $i, j = 0, 1, 2, \ldots$. Let $F_{ij}(t)$, $t > 0$ be the conditional distribution function of the sojourn time in state i, given that the next transition will be into state j. Let

$$Q_{ij}(t) = P_{ij} F_{ij}(t), \; i, j = 0, 1, 2, \ldots .$$

Then $Q_{ij}(t)$ denotes the probability that the process makes a transition into state j in an amount of time less than or equal to t given that it just entered state i at $t = 0$.

The functions $Q_{ij}(t)$ satisfy the following conditions:

$$Q_{ij}(0) = 0,$$

$$Q_{ij}(\infty) = P_{ij};$$

$$Q_{ij}(t) = 0; \; i, j = 0, 1, 2, \ldots$$

$$\sum_{j=0}^{\infty} Q_{ij}(t) = 1, \; i = 0, 1, 2, \ldots \tag{1.8}$$

Let J_0 denote the initial state of the process and J_n $(n = 1, 2, \ldots)$ the state of the process of the n-th transition has occurred. Then the process $\{J_n ; \; n = 0, 1, 2, \ldots\}$ is a Markov chain with transition probabilities P_{ij}. This is called an embedded Markov chain. Let $N_i(t)$ denote the number of transitions into state i in $(0, t]$ and define

$$N(t) = \sum_{i=0}^{\infty} N_i(t) \tag{1.9}$$

Now define a stochastic process:
$\{Z(t), \, t \geq 0\}$ where $Z(t) = i$ denotes that the process is in state i at time t. Then $Z(t) = J_{N(t)}$.

Definition 3: The stochastic process $\{Z(t), \, t \geq 0\}$ is called a semi-Markov process (SMP).

Definition 4: The vector stochastic process $\{N_1(t), N_2(t), \ldots, \; t \geq 0\}$ is called a Markov Renewal Process (MRP).

Thus the SMP records the state of the process at each time point, whilst the MRP is a counting process which indicates the number of visits to each state. Let X_i be the random variable denoting the time interval between two successive visits to a particular state i, then it is observed that $\{X_i ; \; i = 0, 1, 2, \ldots\}$ is a renewal process.

Detailed treatment of SMP and MRP can be obtained in Pyke (1961a,b), Cinlar (1975a) and Ross (1970). The survey of Cinlar (1975b) demonstrates the usefulness of the theory of MRP and SMP in applications. Barlow et al (1965) used these processes to obtain the MTSF of a two-unit system. Srinivasan (1968), Cinlar (1975b), Osaki (1970b, 1972, 1980), Arora (1976a,b), Nakagawa and Osaki (1974, 1976) and Nakagawa (1974) have used the theory of SMP to discuss a number of reliability problems.

1.8.3 Stochastic Point Process

The stochastic process considered in the earlier sections may all be treated as special cases of stochastic point processes. Point processes have been studied against different backgrounds resulting in various definitions of such a process being proposed. [See for example Bartlett (1966), Bhaba (1950), Harris (1963), Khinchine (1955)]. A comprehensive definition of point processes is due to Moyal (1962) who deals with such processes in a general space.

Essentially a stochastic point process may be defined as a stochastic process with a continuous time parameter space and a discrete state space.

1.8.4 Product Densities

One of the ways of characterizing a general stochastic point process is by considering product densities [Ramakrishnan (1950, 1954, 1958)], Smith (1955), Srinivasan (1974), Cox and Isham (1980), etc. These densities are analogues of the renewal density in the case of non-renewal processes.

Let $N(x, t)$ be the random variable representing the number of events in the interval $(t, t+x)$. The product density of order n is defined as

$$h_n(x_1, x_2, \ldots, x_n) = \lim_{\Delta_1, \Delta_2, \ldots, \Delta_n \to 0} \frac{E\left[\prod_{i=1}^{n} N(\Delta_i, x_i)\right]}{\Delta_1 \Delta_2 \ldots \Delta_n}$$

$$x_1 \neq x_2 \neq x_3 \neq \ldots \neq x_n \qquad (1.10)$$

a point process is said to be regular if the probability of occurrence of more than one event in an interval of length Δ is $o(\Delta)$.

For regular point process

$$h_n(x_1, \ldots, x_n) = \lim_{\Delta_1, \Delta_2, \ldots, \Delta_n \to 0} \frac{pr[N(\Delta_i, x_i) \geq 1,\ i=1,2,\ldots,n]}{\Delta_1 \Delta_2 \ldots \Delta_n}$$

$$x_1 \neq x_2 \neq x_3 \neq \ldots \neq x_n \qquad (1.11)$$

$h_n(x_1, \ldots, x_n) \cdot \Delta_1\ \Delta_2 \ldots \Delta_n$ represents approximately the joint probability of an event in each of the intervals $(x_1, x_1 + \Delta_1)$, $(x_2, x_2 + \Delta_2)$, ..., $(x_n, x_n + \Delta_n)$.

Even though the functions $h_n\ (\cdot, \cdot, \cdot, .)$ are termed densities, their integration will not give probability densities. The ordinary moments can be obtained by relaxing the condition that all X_i's are distinct.

1.8.5 Regenerative Stochastic Processes

Bellman and Harris (1948) first introduced the idea of a regeneration point while studying population point process. Feller (1949), in the theory of recurrent events, dealt with a special case of regeneration points. Later on, Smith (1955) generalized Feller's results and dealt with more general stochastic point processes possessing such regeneration points familiarly known as regenerative stochastic processes. Kingman (1964) developed a formal theory of such processes.

Consider a stochastic process $Z = \{Z(t);\ t \geq 0\}$ with state space E. Suppose that every time a certain event, R, occurs, the future of the process Z is a probabilistic replica of the future of the process from previous occurrence of R. Such events are called regenerative events and a stochastic process possessing such a characteristic is said to be a regenerative stochastic process. In some special cases of stochastic processes R is the only characteristic, so that the process regenerates itself.

In more general cases, in addition to the occurrence of R, a knowledge of $Z(t)$ is necessary for the prediction of the process. The renewal process discussed above may be considered as a general point process in which each point of occurrence of the event R is a regeneration point. If we further specialize to the case when the interval between successive regenerative events are distributed exponentially, we notice that any point on the time axis is a regenration point. Gnedenko (1964), Srinivasan and Gopalan (1973a,b), Birolini (1974, 1975), Srinivasan and Subramanian (1980), Sarma (1982), etc. have used such regenerative events to study some reliability problems.

CHAPTER 2

A TWO-UNIT COMPLEX SYSTEM WITH VACATION PERIOD FOR THE REPAIR FACILITY

CHAPTER 2

A two-unit complex system with vacation period for the repair facility

2.1 Introduction

One-unit systems have been studied extensively in the past (Osaki, 1985). The availability and reliability of these systems, where the life-time and the repair-time of the unit were arbitrarily distributed have also been studied.

It is generally true that the high reliability and availability can be achieved by redundancy and/or maintenance. We discuss two-unit redundant systems as one of the basic redundant systems since there are many applications of two-unit redundant systems in the real world. For instance we encounter many computing systems comprised of two processors for performing computational demands and achieving high reliability and performance. Examples of such two-unit redundant systems are: a banking system, electronic switching system, a seat reservation system and so on.

We should classify two-unit redundant systems into two categories: one is a two-

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Modified version of this paper is communicated to the *International Journal of Systems Science* unit parallel redundant system and another is a standy redundant system. A two-unit parallel redundant system is a system that both units operate simultaneously if they are available. On the other hand a two-unit standby redundant system is a system, one unit will be operating online and the other will be kept on standby (offline). If the offline is failed in standby state, then it goes for repair.

Two-unit standby redundant systems have attracted the attention of many applied probabilisits and reliability engineers. A bibliography of the work done has been prepared by Osaki and Nakagawa (1976), Kumar and Agarwal (1980), and Lie, Hwang and Tillman (1977). Table 2.1 gives some of the present state-of-art of two-unit standby redundant systems.

From this table it is clear that all the models discussed in the literature so far have the assumption that the repair facility is continuously available to attend the repair of the failed units. But is is reasonable to expect that a 'vacation period' might be needed to get the repair facility ready before the next repair could be taken up. If this vacation is started only when a unit arrives for repair, it is easy to solve the problem, since the vacation period plus the actual repair-time may be taken as the total repair time. But this vacation period starts immediately after each repair completion, so that the facility becomes available at the earliest (Sarma, 1982).

This vacation period of the repair facility is similar to 'dead time' in counter models (Ramakrishnan and Matthews (1953); Takacs (1956; 57)).

In this chapter we consider a two-unit cold standby redundant system in which the repair facility is not available for a random time after each repair completion. This non-availability period of the repair facility is called 'vacation period'.

Following Kendall's notation in queueing theory each model of a standby system will be described by a series of symbols and slashes such as *A/B/C* where *A* indicates the p.d.f. of the life time of a unit while online, *B* that of a unit in standby and *C* the p.d.f. of the repair time. The symbol *G* and *M*, as usual, stands for a general and exponential respectively. E_k stands for *k*-stage Erlangian distribution. The symbol O stands for a cold standby system.

Table 2.1: Two-unit standby redundant systems

System description	Author(s)	Results obtained
1. M/O/E_k	Muth (1966)	MTSF
2. M/M/M	Tin Htun (1966)	A, A_∞
3. M/M/G	Branson and Shah (1971)	MTSF, A_∞
4. M/M/G	Srinivasan and Gopalan (1973a)	R, A
5. G/O/G	Gnedenko (1964)	MTSF
6. G/O/G	Zubova (1964)	R
7. G/O/G	Solovyev (1964)	R and asymptotic behaviour when repair time is small
8. G/O/G	Linton and Bareswell (1973)	MTSF
9. G/O/G	Srinivasan and Gopalan (1973b)	R, A
10. G/O/G	Nakagawa and Osaki (1974)	MTSF, A_∞ Expected number of visits to a state
11. G/M/M	Subramanian (1975)	R, A
12. G/M/G	Birolini (1975)	R, A
13. G/M/G	Subramanian (1977)	R

R: Reliability A: Availability A_∞ : Steady state availability
MTSF: Mean time to system failure

2.2 System Description and Notation

1. The system consists of two identical units. Either unit performs the system function satisfactorily. When one unit is kept online, the other unit is kept as cold standby.
2. There is only one repair facility.
3. Switchover of the unit is instantaneous and switching device is perfect.
4. After each repair completion, the repair facility goes on 'vacation period' for a random period of time.
5. The life time of a unit while operating online is an arbitrarily distributed random variable (r.v.) with probability density function (p.d.f.), $f(\cdot)$.
6. The repair time of a unit is arbitrarily distributed r.v. with p.d.f., $g(\cdot)$.
7. The p.d.f. of the 'vacation period' is exponentially distributed with parameter θ.
8. $\phi(t) = \begin{cases} 1 & \text{if the system is in up state at time } t \\ 0 & \text{if the system is in down state at time } t \end{cases}$
9. E_i : the regenerative event of type i
10. $N_i(t)$: number of E_i events in (0, t].
11. ©: convolution symbol
12. $\bar{F}(t) = 1 - F(t)$
13. $A_i(t) = P[\phi(t) = 1 / E_i \text{ at } t=0]$
14. $R_i(t) = P[\phi(u) = 0, 0 \leq u \leq t / E_i \text{ at } t=0]$
15. $\psi^*(s)$: Laplace transform of $\psi(t)$
16. $\psi^{(n)}(t)$: n-fold convolution of $\psi(t)$

The following regenerative events are used in the reliability and the availability analysis of the system.

E_0 : Event that one unit is just online and the repair for the other unit just commences

E_1 : Event that one unit is just online, and the other unit is waiting for repair when the repair facility is on vacation.

2.3 Auxiliary Functions

The following auxiliary functions are required in the availability and reliability analysis.

$$\alpha_{01}(t) = \lim_{\Delta \to 0} P\{E_1 \ \ in \ \ (t, t+\Delta), \ \ N_0(t) = 0, \ N_1(t) = 1,$$

$$\phi(u) = 0, \ \ 0 \le u \le t \mid E_0\} \ / \ \Delta \tag{2.1}$$

$$\alpha_{10}(t) = \lim_{\Delta \to 0} P\{E_0 \ \ in \ (t, t+\Delta), \ \ N_0(t) = 1, \ \ N_1(t) = 0,$$

$$\phi(u) = 0, \ \ 0 \le u \le t \mid E_0\} \ / \ \Delta \tag{2.2}$$

$$\alpha_{00}(t) = \lim_{\Delta \to 0} P\{E_0 \ \ in \ (t, t+\Delta), \ \ N_0(t) = 1, \ \ N_1(t) = 1,$$

$$\phi(u) = 1, \ \ 0 \le u \le t \mid E_0\} \ / \ \Delta \tag{2.3}$$

$$\alpha_{11}(t) = \lim_{\Delta \to 0} P\{E_1 \ \ in\ (t, t+\Delta),\ N_0(t) = 0,\ N_1(t) = 1,$$

$$\phi(u) = 0,\ 0 \le u \le t,\ |\ E_1\} \ /\ \Delta \qquad (2.4)$$

$$\alpha_{00}(t) = \lim_{\Delta \to 0} P\{E_1 \ \ in\ (t, t+\Delta),\ N_0(t) = 1,$$

$$\phi(u) = 0,\ 0 \le u \le t,\ |\ E_0\} \ /\ \Delta \qquad (2.5)$$

Hence, we obtain,

$$\alpha_{01}(t) = \{\ g(t) \copyright e^{-\theta t}\ \} f(t) \qquad (2.6)$$

$$\alpha_{10}(t) = \{\ \theta e^{-\theta t} \copyright (1 - e^{-\theta t})\ \} f(t) \qquad (2.7)$$

$${}_0\alpha_{11}(t) = \{\theta e^{-\theta t} \copyright e^{-\theta t}\ \} f(t) \qquad (2.8)$$

$${}_1\alpha_{00}(t) = \{g(t) \copyright (1 - e^{-\theta t})\ \} \qquad (2.9)$$

$$\alpha_{00}(t) = {}_1\alpha_{00}(t) + \alpha_{01}(t) \copyright \alpha_{10}(t) + {}_0h_{11}(t) \copyright \alpha_{10}(t) \qquad (2.10)$$

where

$${}_0h_{11}(t) = \sum_{n=1}^{\infty} {}_0\alpha_{11}^{(n)}(t) \qquad (2.11)$$

$$\beta_{00}(t) = \lim_{\Delta \to 0} P\{E_0 \ \ in\ \ (t, t+\Delta),\ N_0(t) = 1\ |\ E_0\} \ /\ \Delta \qquad (2.12)$$

$$\beta_{01}(t) = \lim_{\Delta \to 0} P\{E_1 \ \ in\ \ (t, t+\Delta),\ N_0(t) = 0,\ N_1(t) = 1\ |\ E_0|\} \ /\ \Delta \qquad (2.13)$$

$$\beta_{10}(t) = \lim_{\Delta \to 0} P\{E_0 \;\; in \;\; (t,\, t+\Delta),\;\; N_0(t) = 1,\;\; N_1(t) = 0 \mid E_1\} \,/\, \Delta \qquad (2.14)$$

$$_1\beta_{00}(t) = \lim_{\Delta \to 0} P\{E_0 \;\; in \;\; (t,\, t+\Delta),\;\; N_0(t) = 1,\;\; N_1(t) = 0 \mid E_0\} \,/\, \Delta \qquad (2.15)$$

$$_0\beta_{11}(t) = \lim_{\Delta \to 0} P\{E_1 \;\; in \;\; (t,\, t+\Delta),\;\; N_0(t) = 0,\;\; N_1(t) = 1 \mid E_1\} \,/\, \Delta \qquad (2.16)$$

The above functions (2.12) - (2.16) are obtained as follows:

$$\beta_{01}(t) = \alpha_{01}(t) + g(t)\, F(t) \qquad (2.17)$$

$$\beta_{10}(t) = \alpha_{10}(t) \qquad (2.18)$$

$$_0\beta_{11}(t) = {}_0\alpha_{11}(t) + \{\theta\, e^{-\theta t} \copyright g(t)\}\, F(t) \qquad (2.19)$$

$$_1\beta_{00}(t) = {}_1\alpha_{00}(t) \qquad (2.20)$$

$$\beta_{00}(t) = {}_1\beta_{00}(t) + \beta_{01}(t) \copyright \beta_{10}(t) + \beta_{01}(t) \copyright {}_0k_{11}(t) \copyright \beta_{10}(t) \qquad (2.21)$$

where

$$_0k_{11}(t) = \sum_{n=1}^{\infty} \beta_{11}^{(n)}(t) \qquad (2.22)$$

2.4 Reliability Analysis

Let us define the functions $R_0(t)$, which gives the system performance during the interval $(0, t]$.

$$ {}_0R_0(t) = P[\phi(u) = 0,\ 0 \le u \le t,\ N_0(t) = 0 \mid E_0] \tag{2.23} $$

$$ {}_0R_1(t) = P[\phi(u) = 0,\ 0 \le u \le t,\ N_0(t) = 0 \mid E_1] \tag{2.24} $$

To derive the expression for ${}_0R_0(t)$, the following mutually exclusive and exhaustive cases are considered:

(i) there is E_0 or E_1 event in $(0, t]$

(ii) one or more E_0 avoiding E_1 events occur in $(0, t]$.

$$ {}_0R_0(t) = \bar{F}(t) + {}_0h_{11}(t) \copyright \bar{F}(t) \tag{2.25} $$

$$ {}_0R_1(t) = \bar{F}(t) + \alpha_{01}(t) \copyright {}_0R_1(t) \tag{2.26} $$

Now we obtain the reliability ${}_0R(t)$ using Figure 2.1.

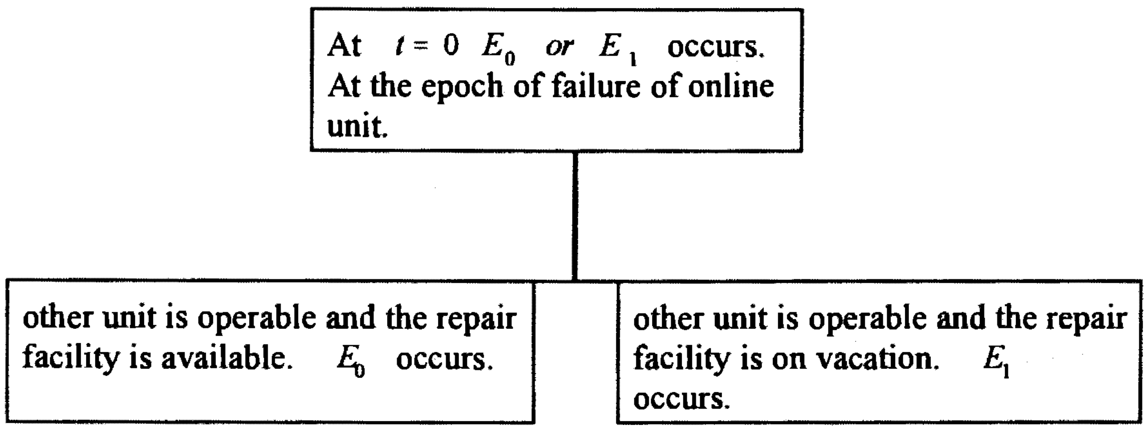
$$ R_0(t) = P[\phi(u) = 0,\ 0 \le u \le t \mid E_0] $$

$$ = {}_0R_0(t) + h_{00}(t) \copyright {}_0R_0(t) \tag{2.27} $$

where

$$ h_{00}(t) = \sum_{n=1}^{\infty} \alpha_{00}^{(n)}(t) \tag{2.28} $$

Figure 2.1


At $t = 0$ E_0 *or* E_1 occurs.
At the epoch of failure of online unit.
other unit is operable and the repair facility is available. E_0 occurs.
other unit is operable and the repair facility is on vacation. E_1 occurs.

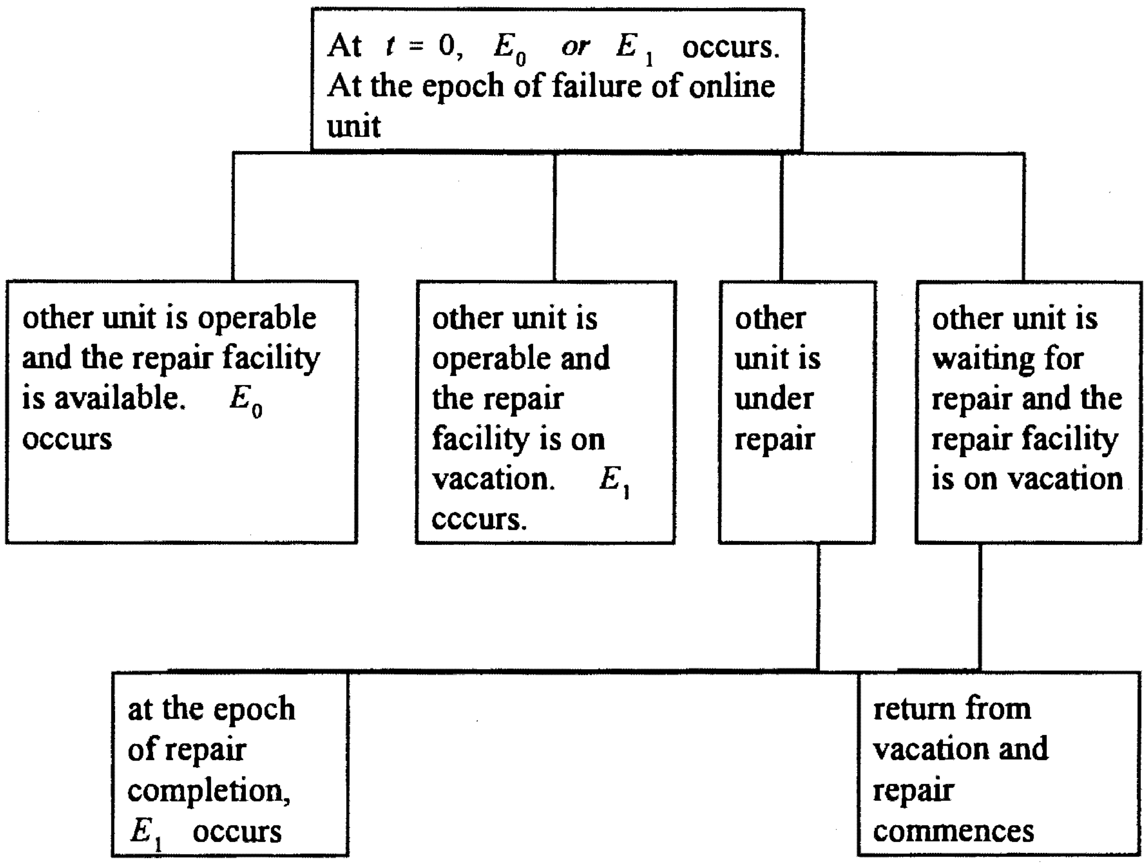
Using $R_0(t)$, we obtain the Mean time Between Failures (MTBF) from the relation

$$MTBF = \lim_{s \to 0} R_0^*(s) \tag{2.29}$$

2.5 Availability Analysis

The availability function $A_0(t)$ can be obtained using the following relation (2.33) and Figure 2.2.

Figure 2.2


At $t = 0$, E_0 or E_1 occurs. At the epoch of failure of online unit
other unit is operable and the repair facility is available. E_0 occurs
other unit is operable and the repair facility is on vacation. E_1 cccurs.
other unit is under repair
other unit is waiting for repair and the repair facility is on vacation
at the epoch of repair completion, E_1 occurs
return from vacation and repair commences

$$A_0(t) = P[\phi(t) = 1 \mid E_0]$$

$$= \bar{F}(t) + \beta_{01}(t) © {}_0A_1(t) \tag{2.32}$$

$${}_0A_1(t) = P[\phi(t) = 1, \ N_0(t) = 0 \mid E_1]$$

$$= \bar{F}(t) + {}_0k_{11}(t) © \bar{F}(t) \tag{2.33}$$

The steady state availability A_∞ can be obtained using the relation

$$A_\infty = \lim_{t \to \infty} A_0(t) = \lim_{s \to 0} s\, A_0^*(s) \tag{2.34}$$

CHAPTER 3

A TWO-UNIT PARALLEL SYSTEM WITH VACATION PERIOD FOR THE REPAIR FACILITY

CHAPTER 3

A two-unit parallel system with vacation period for the repair facility

3.1 Introduction

In the previous chapter two-unit cold standby systems with vacation period has been studied. In this chapter a parallel system will be considered.

Gaver (1963) studied a parallel system with constant state dependent hazard rates and arbitrary repair. He provided the Laplace Transform solution of the reliability of the system and obtained the MTSF and the standby-state unavailability by using supplementary variable technique (Cox and Miller, 1965).

Later Mine and Kawai[2] (1974), Linton and Saw (1974) and Kulshrestha (1968, 70) have studied some parallel redundant systems. Kodama and Deguchi (1974) studied a two-unit parallel redundant system with Erlangian failure and general repair, and obtained MTSF. Linton (1976) extended the results of Kodama et al

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Modified version of this chapter was presented at the *SASA Conference,* 1997.

(1974) for the case when the failure and repair distributions of the one of the units are Erlagian and those of the other are general, by a combination of the phase method (Cohan, 1969) and supplementary variable technique.

Later Subramanian and Ravischandran (1979) extended the results of Linton (1976) to a more general case.

3.2 System description and notation

1. The system consists of two non-identical units connected in parallel.
2. At $t=0$, both units are new and operable, the repair facility (r.f.) is available.
3. There is a single repair facility.
4. With probability (p) unit one will be repaired and with probability $(1-p)$ unit two will be repaired (only when the system is in downstate).
5. The r.f. is not available for a random time and the r.f. needs a 'vacation period'
6. Life-time of the units are random variables distributed exponentially with parameters λ_1, λ_2
7. Repair-Time $\sim$ is a random variable with arbitrary distribution $g_i(\cdot)$, $i = 1, 2$
8. Vacation period is also arbitrarily distributed random variable with p.d.f. $k(\cdot)$.
9. All the random variables are statistically independent.

The following regenrative events are considered for the reliability analysis. (See Table 3.1.)

Table 3.1: Regenerative events

Event	Unit 1	Unit 2	r.f
E_0	0	0	a
E_1	0	r.j.c.	a
E_2	r.j.c.	0	a
E_3	0	0	j.n.a.
E_4	r.j.c.	Q	a
E_5	Q	r.j.c.	a

o = operable

r.j.c. = repair just commences

a = available

Q = queueing for repair

j.n.a. = just not available

3.3 Availability analysis

$A_i(t) = P[\textit{the system is up at } t \,/\, E_i \textit{ at } t=0]$

Now derive the equation for $A_0(t)$, by considering the following mutually exclusive and exhaustive possibilities in the interval $(0, t]$.

(i) neither units fail

(ii) one of the units fail.

$$A_0(t) = e^{-(\lambda_1+\lambda_2)t} + \lambda_2 e^{-(\lambda_1+\lambda_2)t} © A_1 t + \lambda_1 e^{-(\lambda_1+\lambda_2)t} © A_2(t) \qquad (3.1)$$

Similarly we arrive at the other equations. Figures (3.1), (3.2), (3.3) give the various mutually exclusive and exhaustive possibilities in the derivation of $A_1(t)$, $A_2(t)$ and $A_3(t)$ respectively.

$$A_1(t) = e^{-\lambda_1 t}\,\bar{G}_2(t) + [g_2(t)e^{-\lambda_1 t}] © A_3(t) + [g_2(t)(1 - e^{-\lambda_1 t})] © A_2(t) \qquad (3.2)$$

$$A_2(t) = e^{-\lambda_2 t}\,\bar{G}_1(t) + [g_1(t)e^{-\lambda_2 t}] © A_3(t) + [g_1(t)(1 - e^{-\lambda_2 t})] © A_2(t) \qquad (3.3)$$

$$A_3(t) = \bar{K}(t)[1 - (1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})] + [k(t)e^{-(\lambda_1+\lambda_2)t}] © A_0(t)$$

$$+ k(t)[e^{-\lambda_1 t}(1 - e^{-\lambda_2 t})] © A_1 t$$

$$+ k(t)[e^{-\lambda_2 t}(1 - e^{-\lambda_1 t}] © A_2 t$$

$$+k(t)[(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})] © [pA_4(t) + \bar{p}A_3(t)] \qquad (3.4)$$

$$A_4(t) = g_1(t) © A_1(t) \qquad (3.5)$$

$$A_5(t) = g_2 © A_2(t) \qquad (3.6)$$

Figure 3.1

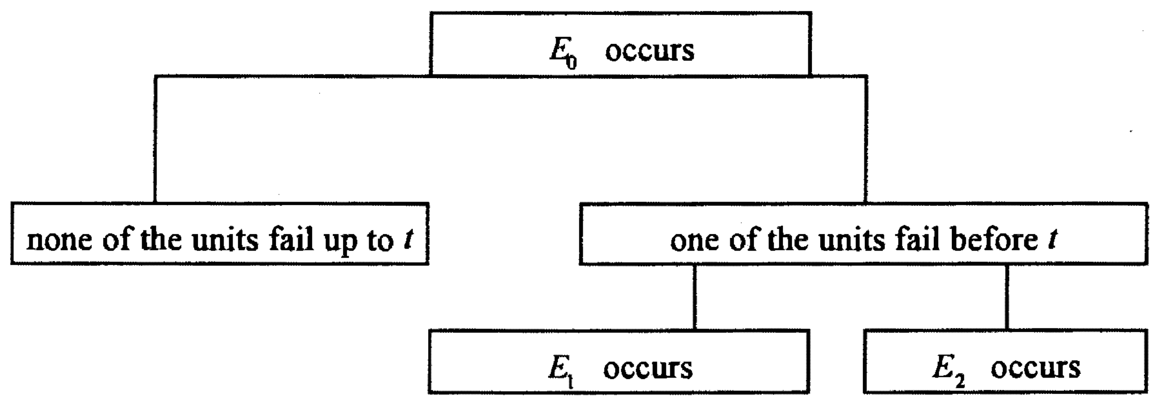
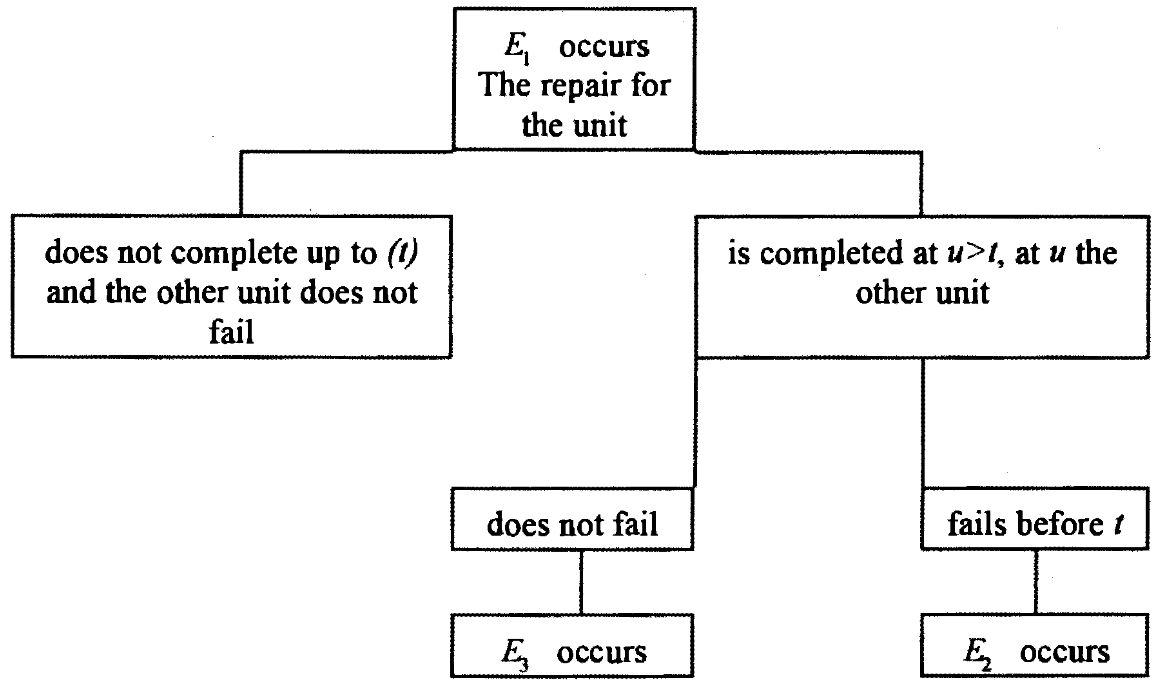

E_0 occurs
none of the units fail up to t
one of the units fail before t
E_1 occurs
E_2 occurs

Figure 3.2


E_1 occurs
The repair for
the unit
does not complete up to *(t)*
and the other unit does not
fail
is completed at *u>t*, at *u* the
other unit
does not fail
fails before *t*
E_3 occurs
E_2 occurs

3.4 Reliability analysis

We now derive the reliability equations for this system. Recall that the system reliability is concerned not only with its state of the system at a particular point in time but with its state over a period of time. These equations are obtained by dropping certain terms in the availability equations.

$$R_i(t) = P[\text{the system is up in } (0, t] \,/\, E_i \text{ at } t=0]$$

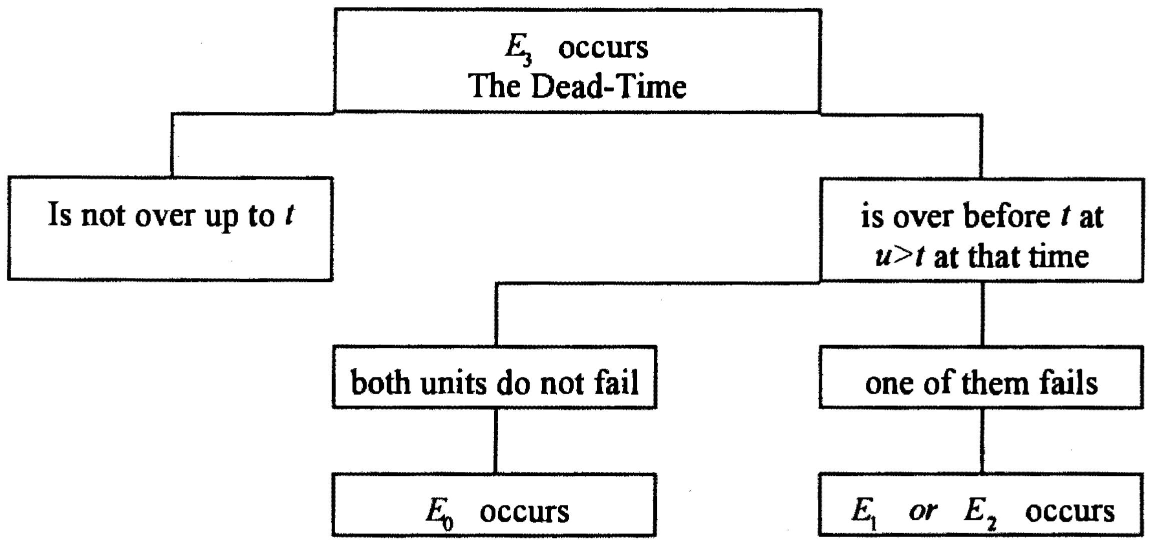
$$R_0(t) = e^{-(\lambda_1+\lambda_2)t} + \lambda_2 e^{-(\lambda_1+\lambda_2)t} © R_1(t) + \lambda_1 e^{-(\lambda_1+\lambda_2)t} © R_2(t) \tag{3.7}$$

$$R_1(t) = e^{-\lambda_1 t}\,\bar{G}_2(t) + [g_2(t)e^{-\lambda_1 t}] © R_3(t) \tag{3.8}$$

$$R_2(t) = \bar{G}_1(t)e^{-\lambda_2 t} + g_1(t)e^{-\lambda_2 t} © R_3(t) \tag{3.9}$$

$$\begin{aligned} R_3(t) = \bar{K}(t)[1(1 - e^{-\lambda_1 t})(1 - e^{-\lambda_2 t})] \\ + [k(t)e^{-(\lambda_1+\lambda_2)t}] © R_0(t) \\ + [k(t)(1 - e^{-\lambda_2 t})e^{-\lambda_1 t}] © R_1(t) \\ + [k(t)e^{-\lambda_2 t}(1 - e^{-\lambda_1 t}) © R_2(t) \end{aligned} \tag{3.10}$$

Figure 3.3


E_3 occurs
The Dead-Time
Is not over up to t
is over before t at
$u>t$ at that time
both units do not fail
one of them fails
E_0 occurs
E_1 or E_2 occurs

Taking the Laplace Transforms from (3.1) to (3.6) and (3.7) to (3.10), we get

$$A_0^*(s) = \frac{1}{\lambda_1+\lambda_2+s} + \frac{\lambda_2}{\lambda_1+\lambda_2+s} A_1^*(s) + \frac{\lambda_1}{\lambda_1+\lambda_2+s} A_2^*(s) \qquad (3.11)$$

$$A_1^*(s) = \frac{1-g_2^*(\lambda_1+s)}{\lambda_1+s} + g_2^*(\lambda_1+s)A_3^*(s) + [g_2^*(s) - g_2^*(\lambda_1+s)]A_2^*(s) \qquad (3.12)$$

$$A_2^*(s) + \frac{1-g_1^*(\lambda_2+s)}{\lambda_2+s} + g_1^*(\lambda_2+s)A_3^*(s) + [g_1^*(s) - g_1^*(\lambda_2+s)]A_2^*(s) \qquad (3.13)$$

$$A_3^*(s) = [\bar{k}^*(\lambda_1+s) + \bar{k}^*(\lambda_2+s) - \bar{k}^*(\lambda_1+\lambda_2+s)] + k^*(\lambda_1+\lambda_2+s)A_0^*(s)$$

$$+ [k^*(\lambda_1+s) - k^*(\lambda_1+\lambda_2+s)]A_1^*+s) + [k^*(\lambda_1+s) - k^*(\lambda_1+\lambda_2+s)]A_2^*(s)$$

$$+ [k^*(\lambda_1+s) + k^*(\lambda_2+s) - k^*(\lambda_1+\lambda_2+s)][pA_4^*(s) + \bar{p}A_3^*(s)] \qquad (3.14)$$

$$A_4^*(s) = g_1^*(s)A_1^*(s) \qquad (3.15)$$

$$A_5^*(s) = g_2^*(s)A_2^*(s) \qquad (3.16)$$

Solving the equations from (3.11) to (3.16), we get $A_0^*(s)$. We obtain the steady state availability A_∞ using the relation

$$A_\infty = \lim_{s \to 0} sA_0^*(s)$$

Similarly taking the Laplace Transforms for the equations (3.7) to (3.10), we get

$$R_0^*(s) = \frac{1}{\lambda_1+\lambda_2+s} + \frac{\lambda_1}{\lambda_1+\lambda_2+s} R_1^*(s) + \frac{\lambda_1}{\lambda_1+\lambda_2+s} R_2^*(s) \tag{3.17}$$

$$R_1^*(s) = \frac{1-g_2^*(\lambda_1+s)}{\lambda_1+s} + g_2^*(\lambda_1+s)\, R_3^*(s) \tag{3.18}$$

$$R_2^*(s) = \frac{1-g_1^*(\lambda_2+s)}{\lambda_2+s} + g_1^*(\lambda_2+s)\, R_3^*(s) \tag{3.19}$$

$$\begin{aligned} R_3^*(s) = {} & [\bar{k}^*(\lambda_1+s) + \bar{k}^*(\lambda_2+s) - \bar{k}^*(\lambda_1+\lambda_2+s)] \\ & + k^*(\lambda_1+\lambda_2+s)\, R_0^*(s) \\ & + [k^*(\lambda_1+s) - k^*(\lambda_1+\lambda_2+s)]\, R_1^*(s) \\ & + [k^*(\lambda_2+s) - k^*(\lambda_1+\lambda_2+s)]\, R_2^*(s) \end{aligned} \tag{3.20}$$

Solving the equations (3.17) to (3.20), we get $R_0^*(s)$. From $R_0^*(s)$, MTSF can be obtained using the relation

$$MTSF = R_0^*(0).$$

SPECIAL CASES

(1) When the units are identical, the results can be easily simplified, i.e. $\lambda_1 = \lambda_2 = \lambda$ and $g_1(t) = g_2(t) = g(t)$. Then the events E_1, E_2 and E_4, E_5 in the table 3.1. The availability equations become

$$A_0(t) = e^{-2\lambda t} + 2\lambda e^{-2\lambda t} © A_1(t) \tag{3.21}$$

$$A_1(t) = e^{-\lambda t}\, \bar{G}(t) + [g(t)e^{-\lambda t}] © A_3(t)$$

$$+ [g(t)\{1 - e^{-\lambda t}\}] © A_0(t) \tag{3.22}$$

$$A_3(t) = \bar{k}(t)[1 - (1 - e^{-\lambda t})^2] + [k(t)e^{-2\lambda t}] © A_0(t)$$

$$+ [2k(t)e^{-\lambda t}(1 - e^{-\lambda t})] © A_1(t)$$

$$+ [k(t)(1 - e^{-\lambda t})^2] © A_3(t) \tag{3.23}$$

Using these equations, the steady state availability A_∞ can be obtained explicitly as

$$A_\infty = \frac{N_A}{D_A}$$

where

$$N_A = \frac{1}{\lambda} + \frac{g^*(\lambda)}{2\lambda}\left[1 - 4k^*(\lambda) + (1 + 4\lambda)k^*(2\lambda)\right]$$

$$D_A = [2k^*(2\lambda)rk^*(\lambda)g^{*\prime}(\lambda) + 2k^{*\prime}(2\lambda)g^*(\lambda) - \frac{k^*(2\lambda)g^*(\lambda)}{2\lambda}$$

$$+ g^*(\lambda)]\int_0^\infty tk(t)dt$$

$$+ [1 - 2g^*(\lambda)k^*(\lambda) + g^*(\lambda) + g^*(\lambda)k^*(2\lambda)]\int_0^\infty tg(t)dt$$

The reliability equations in this special case become

$$R_0(t) = e^{-2\lambda t} + 2\lambda e^{-2\lambda t} © R_1(t) \tag{3.24}$$

$$R_1(t) = e^{-\lambda t}\bar{G}(t) + [g(t)e^{-\lambda t}] © R_3(t) \tag{3.25}$$

$$R_3(t) = \bar{k}(t)[1 - (1 - e^{-\lambda t})^2] + [k(t)\, e^{-2\lambda t}] © R_0(t) \tag{3.26}$$

The MTSF can be obtained from the above equations, and using the relation

$$MTSF = R_0^*(0)$$

$$= \frac{3}{2\lambda} + \frac{g^*(\lambda)}{2\lambda[1-2g^*(\lambda)k^*(\lambda)+g^*(\lambda)k^*(2\lambda)]}$$

(2) In addition to the special case (1), we assume $k(t) = H(t)$, a Heaviside function

$$MTSF = \frac{3}{2\lambda} + \frac{g^*(\lambda)}{2\lambda[1-g^*(\lambda)]}$$

This result is in agreement with Srinivasan and Subramanian (1980).

3.5 Numerical illustration

Table 3.2 presents the steady state availability and the MTSF corresponding to the special case (1), when

$g(t) = \mu^2 te^{-\mu t}$ (a two-stage Erlangian)

and $k(t) = H(t-t_0)$, a Heaviside function. For the parametric values $\frac{1}{\lambda} = 50$ and for different values of $\frac{1}{\mu} = 15,\ 25,\ 35$ A_∞ and MTSF are obtained.

Table 3.2: t_0 versus A_∞ and MTSF

Repair time	t_0	A_∞	MTSF
$\frac{1}{\mu} = 15$	15	0.9918	152.92
	15	0.9809	147.66
	20	0.9011	137.92
	25	0.9490	129.81
	30	0.8999	125.07
	35	0.8598	120.88
	40	0.8307	117.33
	45	0.8080	115.90
	50	0.7899	113.98
$\frac{1}{\mu} = 25$	5	0.8991	122.81
	10	0.8710	117.88
	15	0.8433	114.96
	20	0.8111	112.86
	25	0.7910	110.07
	30	0.7760	108.81
	35	0.7588	106.00
	40	0.7400	105.01
	45	0.7263	104.90
	50	0.7099	103.86
$\frac{1}{\mu} = 35$	5	0.7914	106.78
	10	0.7784	104.14
	15	0.7396	102.98
	20	0.7220	101,11
	25	0.7198	100,70
	30	0.7089	99,91
	35	0.6979	98.01
	40	0.6756	97.66
	45	0.6694	96.78
	50	0.6587	95.01

CHAPTER 4

INTERMITTENTLY USED SYSTEMS - A REVIEW

CHAPTER 4

Intermittently used systems - A Review

4.1 Introduction

The two-unit redundant repairable systems have been studied in the previous chapters with reference to the evaluation of the performance in terms of reliability, pointwise availability and other measures of system performance. There are several facets to the problems of these type. In all the attempts made so far it is assumed that the system under consideration may be needed all the time but there are certain practical situations in which continuous failure free performance may not be necessary. Therefore it may be worthwhile to take into account the fact that the system may not be needed in certain spans of time and the system can be in a downstate during such time intervals without any consequence. In such situations we come across intermittently used systems. While dealing with intermittently used systems the designer does not need to use severe constraints on the components that go to make the system. In modelling, the probability that the system is in the downstate need not be studied. It is enough to study, the probability that the system is not available whenever it is needed. This type of an approach to reliability and its consequences have not been explored by many researchers. We shall give a brief account of the present state of art of intermittently used systems.

Gaver (1964) pointed out that, it is pessimistic to evaluate the performance of an intermittently used system solely on the basis of the distribution of the time to failure. Gaver stressed on the point event called the disappointment characterised by the entry of the system into the downstate when the system is already in the downstate during a need period or the state of the need for the system when the system is already in the failstate he studied the one-unit intermittently used system in connection with a traffic and congestion problem and obtained the system measure, the expected duration of the first disappointment. Later Srinivasan (1966) extended Gaver's analysis to two-unit cold standby systems and obtained the biased involved in the assessment of the system performance when the need pattern is ignored.

Nakagawa, Sawa and Suzuki (1976) made a further study of the one-unit system used intermittently and obtained the system measures

(1) the expected time to the first disappointment
(2) the expected number of disappointments during an interval of time
(3) pointwise unavailability

They also made a preliminary analysis of two-unit cold standby system and obtain Laplace-Stieltjes transform of the three measures given above.

Table 4.1 presents the earlier work done on intermittently used systems. Following Kendal's notation in queueing theory each model is described by a series of symbols and slashes such as *A/B/C* where *A* indicates the probability distribution of the life-time of the online unit, *B* that of a unit in standby and *C* the probability distribution of the repair time.

Table 4.1 Intermittently used System

Number of units	Description of system	Description of need/no-need periods	Authors	Results obtained
2	G/O/G (Dissimilar units)	M/M	Srinivasan (1966)	Expected time to system failure
2	G/M/G	M/M	Nakagawa, Sawa and Suzuki (1976)	Pointwise unavailability
2	G/O/G	M/M	Kapur and Kapoor (1978)	Distribution of time to the first disappontment, probability of a disappointment at time *t*, the expected number of disappointment during (0, *t*].
2	G/O/G	M/M	Srivivasan and Bhaska (1976b)	Mean time to the first disappointment, mean and mean square number of disappontments over any arbitrary interval, duration of the first disappointment.
2	G/M/M	M/M	Srivivasan and Bhaska (1976c)	Mean time to the first disappointment, mean and mean square number of disappontments over any arbitrary interval duration of the first disappointment.
2	G/O/G	M/M	Kapoor and Kapur (1980)	Joint distribution of the first up time and disappointment time.
n	0	M/M	Sambandham (1981)	Time to the first disappointment, distribution of the time between two successive need events, mean and mean square number of disappointments.
n	G/M/M	M/G	Sarma and Natrajan (1982)	Distribution of time to first disappointment and mean time to first disappontment.

The distribution of need and no need period are described as *D/E* respectively in a separate column. As in queueing theory, the symbols G stands for general distribution, M stands for exponential distribution, E_k stands for k-stage Erlangian distribution. Special features like preventive maintenance (a unit which had worked for specified duration of time and due for repair or service) will be indicated in a separate column. If a system is a cold standby system we represent with a symbol O. In case of a system in which the switching device cannot fail no repair is needed for it. The corresponding distribution of the repair time will be represented by the dash (-). Indicating that there is no repair.

The first attempt on n-unit intermittently used standby redundant systems have been made by Subramanian, Sarma and Natarajan (1981), Sarma and Natarajan (1982). They obtained the system measures

(1) expected time to the first disappointment and
(2) expected number of disappointment in an interval.

Various assumptions can be made regarding, whether a need period waits till the system becomes available or not; whether there is continuous monitoring of the system to detect failures or not and so on. A perpetual vigil is set to be kept on the system, if failures are detected as and when they occur irrespective of whether the occurrance is in a need or no need for the system and are taken up for repair immediately if the repair facility is available. If the failures are not detected during a no need period then we say that the system is kept under non-perpetual vigil.

The above two papers of n-unit standby redundant systems studied by Subramanian, Sarma and Natarajan; Sarma and Natarajan have considered the non-perpetual vigil and perpetual vigil respectively.

In the articles cited in the tables the need period of the system is described by a stochastic process

$$\{Z(t),\quad t \geq 0\}$$

with state space *{0, 1}* where the numbers 1 and 0 signifiy the need and no-need states respectively. It has also been assumed that the process $\{Z(t),\ t \geq 0\}$ is Markov, so that the need and no-need period that follow each other constitute a pair of independent random variables which are distributed exponentially.

The first contribution to the intermittently used two-unit parallel system is due to Sarma and Hines (1990). They obtained all the above measures using correlated alternating renewal process, and forward recurrence times.

CHAPTER 5

INTERMITTENTLY USED TWO-UNIT STANDBY SYSTEM

CHAPTER 5

Intermittently used Two-unit Standby System

5.1 Introduction

In this chapter we consider a comparatively simple two-unit system and obtain the system measures like expected time to the first disappointment and the expected number of disappointments in an interval with the help of correlated alternating renewal process (Sarma, 1982).

Baxter (1981) obtained some general measures for the reliability of the repairable one-unit system, by identifying the sequence of periods of operation and repair as an alternating renewal process (Cox, 1962). This type of modelling was possible because of the uptime and the downtime are independent random variables in such a system.

However, consider a two-unit cold standby system, in which the life-time and the repair-time of units are arbitrarily distributed random variables. If at the epoch of the failure of the online unit, the other unit is under repair then the system will enter the downstate. The duration of the system downtime will depend on the elapsed repair-time and therefore the time for which the online unit was operating successfully. Therefore in this example the up-time and the down-time are correlated random variables.

In this case the system remains in the upstate and downstate alternatively. The entire process describing the system behaviour can be thought of as a sequence of circles and each circle representing the uptime and the subsequent downtime. Such a process is called "Correlated Alternating Renewal Process", if it satisfies some more additional conditions. Earlier the joint distribution of uptime and downtime were obtained by Nakagawa and Osaki (1976), for a simple two-unit cold standby system. The joint distribution of uptime and disappointment time for an intermittently used system were studied by Kapoor and Kapur (1980). In this chapter we study the correlated alternating renewal process in detail and apply these results for a two-unit intermittently used cold standby system to obtain the system measures. This is achieved with the help of joint forward recurrance time.

5.2 Correlated Alternating Renewal Process

Consider a system which is alternately in one of two states called for convenience the favourable state and the unfavourable state. Let X_i be the random variable denoting the time spent in the favourable state during the system's i-th visit to the state and Y_i the time spent in the subsequent unfavourable state. We assume that (X_i, Y_i) $i = 1, 2, \ldots$ are statistically independent bivariate random variables. All these random variables are assumed to be defined on the same probability space and have the same distribution as the bivariate random variable $(X\ Y)$ with joint distribution function $F_{X,Y}(x, y)$. The sequence of ordered pairs

$$\{(X_k, Y_k);\ k = 1, 2, \ldots\}$$

will be called correlated alternating renewal process in the following additional assumptions are satisfied:

(i) $X_1, X_2, \ldots$ are independent

(ii) $Y_1, Y_2, \ldots$ are independent

(iii) X_i, Y_j are independent for $i \neq j$

These assumptions imply that the sequence of random variables $\{(X_i + Y_i);\ i = 1, 2, 3, \ldots\}$ are independent and identically distributed and hence form a renewal process.

The following events are identified to study the properties of the correlated alternating renewal process:

E The system enters the favourable state from the unfavourable state.

D The system enters the unfavourable state from the favourable state.

The E and D events occur alternately and give rise to a correlated alternating renewal process. We identify X as the time interval between an E event and the next D event, and Y that between this D event and the next E event, so that $X+Y$ represents the interval between two successive E events. We assume the occurance of an E event at $t=0$, so that the functions $F_{X,Y}(x, y)$ and $f_{X,Y}(x, y)$ are the joint distribution and the corresponding density of X and Y conditioned upon an E event at $t=0$.

5.3 Properties of Correlated Alternating Renewal Process

Using the joint distribution of X and Y, we can obtain the marginal distributions;

$$f_{(X)}(x) = \int_0^\infty f_{X,Y}(x, y)dy \tag{5.1}$$

$$g_Y(y) = \int_0^\infty f_{X,Y}(x, y)dx \tag{5.2}$$

where $f_X(x)$ and $g_Y(y)$ are the marginal distributions of X and Y respectively.

(1) The probability density function of a random variable $X+Y$ which is the interval between two successive E events, is given by

$$f_{X+Y}(t) = \int_0^t f_{X,Y}(u, t-u)du \tag{5.3}$$

(2) Let t be a fixed time point where the system up. Similar to the forward recurrence time for a renewal process we now introduce for the correlated alternating renewal process the joined forward recurrence time as a bivariate random variable (U_t, W_t) corresponding to the time measured from the time instant t to the next D event and the subsequent E event.

For $y > x$, let

$\psi_{D,E}(t, x, y) = \lim_{\Delta_1, \Delta_2 \to 0} \frac{1}{\Delta_1 \Delta_2}$ [*Pr*{the first D event after t occurs in $(t + x, t + x + \Delta_1)$ and the first E event after t occurs in $(t + y, t + y + \Delta_2)$ / the system is up at t, an E event occurred at $t=0$}] for all $x, y \geq 0$.

Then we have

$$\psi_{D,E}(t,x,y) = \begin{cases} f_{X,Y}(t+x, y-x) + \int_0^t h_E(u) f_{X,Y}(t-u+x, y-x)\,du & for \quad y > x \\ 0 & otherwise. \end{cases}$$

(5.4)

where $h_E(\cdot)$ is the renewal density of the E events given by

$$h_E(t) = \sum_{n=1}^{\infty} f_{X+Y}^{(n)}(t) \tag{5.5}$$

where $f_{X+Y}^{(n)}(t)$ is the n-fold convolution of $f_{X+Y}(t)$.

(3) The marginal forward recurrence times of D and E events respectively are given by

$$\psi_D(t_1, x) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr\{\text{The first } D \text{ event after } t \text{ occurs in } (t+x,\ t+x+\Delta) \ /\ \text{the system is up at } t,\ E \text{ at } t=0\}$$

$$= \int_0^{\infty} \psi_{D,E}(t, x, y)\, dy$$

$$= f_X(t + x) + \int_0^{\infty} h_E(u)\, f_X(t-u+x)\,du \tag{5.6}$$

and

$$\psi_E(t, y) = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr\{\text{ the first } E \text{ event after } t \text{ occurs in } t+y,\ t+y+\Delta \ /\text{the system is up at } t,\ E \text{ at } t=0\}$$

$$= \int_0^{\infty} f_{X,Y}(t+u,\ y-u)\,du + \int_0^{y} du \int_0^{t} h_E(w) f_{X,Y}(t-w+v,\ y-u)\,dw \qquad (5.7)$$

(4) The stationary values of the forward recurrence times may be obained as

(a) $\psi_{D,E}(X, Y) = \lim_{t \to \infty} \psi_{D,E}(t, x, y)$

$$= \frac{1}{\mu_1 + \mu_2} \int_x^{\infty} f_{X,Y}(t,\ y-x)\,dt \qquad (5.8)$$

(b) $\psi_D(x) = \lim_{t \to \infty} \psi_D(t, x)$

$$= \frac{1}{\mu_1 + \mu_2} \int_x^{\infty} f_X(t)\,dt \qquad (5.9)$$

(c) $\psi_E(y) = \lim_{t \to \infty} \psi_E(t, y)$

$$= \frac{1}{\mu_1 + \mu_2} \int_y^{\infty} f_{X+Y}(t)\,dt \qquad (5.10)$$

where $\mu_1 = E(X)$ and $\mu_2 = E(Y)$

5.4 An Intermittently used two-unit stand-by system

5.4.1 System description and notation

1. The system consists of two identical units, one operates online and the other is kept as a cold stand-by.
2. There is a single repair facility and the repairs are taken up in first-in-first-out order.
3. Each unit is new after repair.
4. The switchover is instantaneous.
5. Switching device is perfect.
6. Initially the system enters the upstate from the downstate when the system is in the need state.
7. The failure of a unit operating online is detected only when there is a need for the system and the failure remains undetected until the need arises for the system only then remedial measures like replacing the failed online unit with an operable stand-by and attempts to send the failed online unit for repair are considered.
8. If the system enters the downstate when there is a need for the system or if need arises for the system when it is in the downstate, the system remains indefinitely in a state of need until it recovers and then the need period lasts for a span of time governed by the same exponential distribution.
9. The life time of a unit while operating on online is an arbirarily distributed random variable having the p.d.f. $f(\cdot)$.
10. The repair time of a unit is also arbitrarily distributed random variable with p.d.f. $g(\cdot)$
11. The need and the no need period are exponentially distributed random variables with parameters α, β respectively. The need and the no need period occur alternatively.

It is convenient to define the event E_1, in addition to E and D events,

E_1 Event characterized by one unit just online and the other unit just taken for repair, the system is in the need period.

Define:

$Z(t)$ Two valued stochastic process, describing the state of need or noneed for the system at time t, i.e.

$$Z(t) = \begin{cases} 0 & \textit{if the system is in need state} \\ 1 & \textit{if the system is in no-need state} \end{cases}$$

Associated with the process $\{Z(t);\ t \geq 0\}$, we define the following auxiliary functions $\pi_{ij}(t)$, useful to our analysis:

$$\pi_{ij}\ (t) = Pr\{Z(t) = j \mid Z(0) = i\} \quad i, j = 0,, 1, \quad t \geq 0.$$

and can be obtained by renewal theoretic arguments such as:

$$\pi_{00}\ (t) = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\, e^{-(\alpha+\beta)t} \tag{5.11}$$

$$\pi_{01}\ (t) = \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}\, e^{-(\alpha+\beta)t} \tag{5.12}$$

$$\pi_{10}(t) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t} \tag{5.13}$$

$$\pi_{11}(t) = \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta} e^{-(\alpha+\beta)t} \tag{5.14}$$

5.4.2 The Joint Distribution of the Favourable Time and the Unfavourable Time

By identifying X as the time interval between an E event and the next D event and Y as the time interval between this D event and the following E event, the joint density of X and Y is given by figure 5.1.

$$f_{XY}(X, Y) = \pi_{00}(x) f(x) g(x+y) + \int_0^x \pi_{01}(u) f(u) \beta e^{-\beta(x-u)} g(x+y) du$$

$$+ \int_0^x h_{E_1}(u) \pi_{00}(x-u) f(x-u) g(x-u+y) du$$

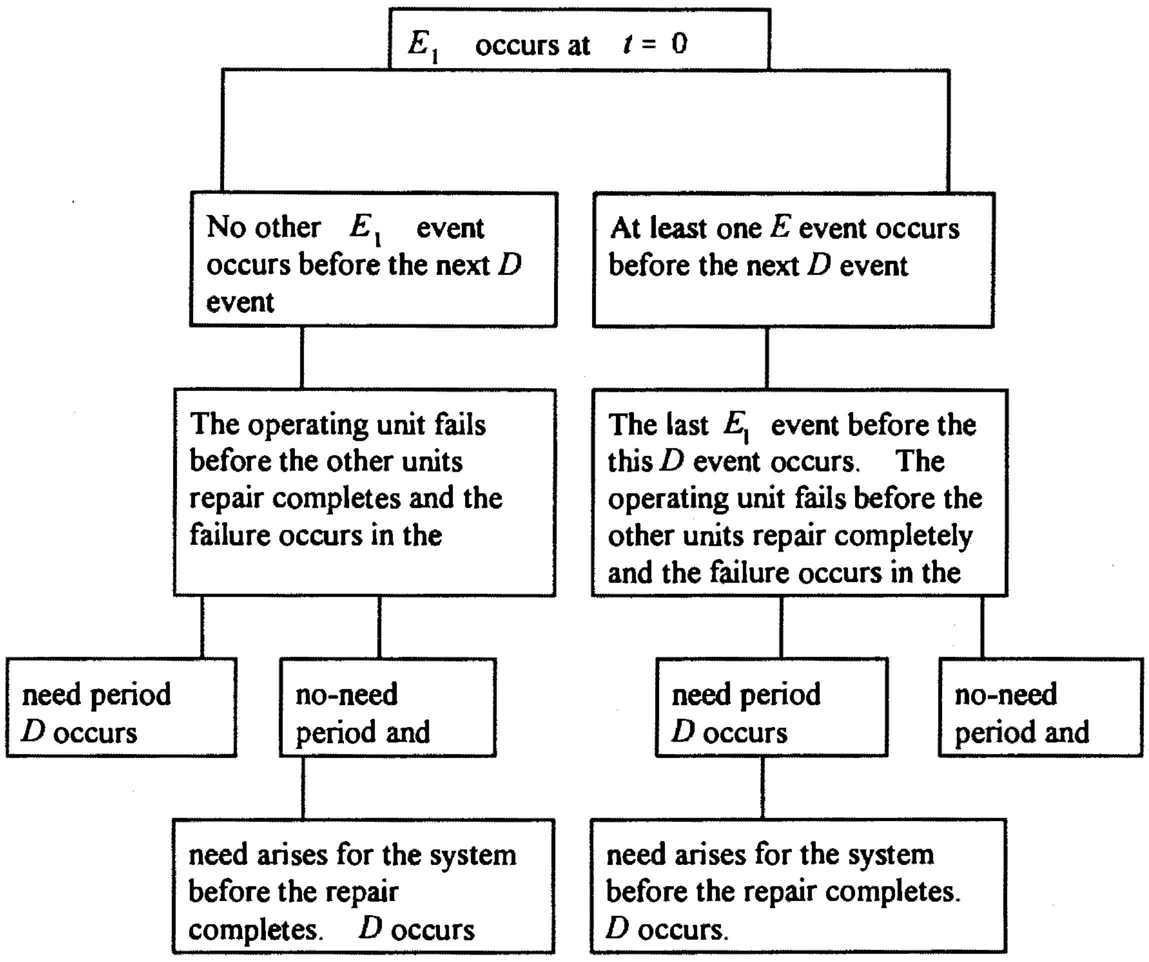
$$+ \int_0^x \int_0^{x-u} h_{E_1}(u) \pi_{01}(v) f(v) \beta e^{-\beta[x-(u+v)]} g(x-u)+y) dv du \tag{5.15}$$

where

$$h_{E_1}(t) = \sum_{n=1}^{\infty} \left[\pi_{00}(t) f(t) G(t) + \int_0^t \pi_{01}(u) f(u) \beta e^{-\beta(t-u)} G(t) du \right]^{(n)}$$

is the renewal density of E_1 events.

Figure 5.1


E_1 occurs at $t = 0$
No other E_1 event occurs before the next D event
At least one E event occurs before the next D event
The operating unit fails before the other units repair completes and the failure occurs in the
The last E_1 event before the this D event occurs. The operating unit fails before the other units repair completely and the failure occurs in the
need period D occurs
no-need period and
need period D occurs
no-need period and
need arises for the system before the repair completes. D occurs
need arises for the system before the repair completes. D occurs.

5.4.3 The Marginal Densities

If the marginal densities of the random variables X and Y are

$f_X(x)$ and $f_Y(y)$

respectively, then

$$
\begin{aligned}
f_X(x) &= \int_0^{\infty} f_{X,Y}(X,\ Y)\,dy \\
&= \pi_{00}(x)\,f(x)\bar{G}(x) + \int_0^{x} \pi_0(u)f(u)\beta e^{-\beta(x-u)}\,g(x+y)du \\
&\quad + \int_0^{x} h_{E_1}(u)\pi_{00}(x-u)\ f(x-u)\bar{G}(x-u)\,du \\
&\quad + \int_0^{x}\int_0^{x-u} h_{E_1}(u)\pi_{01}\ f(v)\beta e^{-\beta[x-(u+v)]}\ \bar{G}(x-u)\,dvdu
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
f_Y(y) &= \int_0^{\infty} f_{X,Y}(x,\ y)\,dx \\
&= \int_0^{\infty} \pi_{00}(x)\,f(x)\ g(x+y)dx + \int_0^{\infty}dx \int_0^{x} \pi_{01}(u)\,f(u)\beta e^{-\beta(x-u)}g(x+y)\,du \\
&\quad + \int_0^{\infty} dx \int_0^{x} h_{E_1}(u)\pi_{00}(x-u)\,f(x-u)\ g(x-u+y)\,du \\
&\quad + \int_0^{\infty}dx \int_0^{x}\int_0^{x-u} h_{E_1}(u)\pi_{01}(v)\,f(v)\beta e^{-\beta[x-(u+v)]}g(x-u+y)\,dvdu
\end{aligned}
\tag{5.17}
$$

The density of the random variable $X+Y$ representing the cycle length is:

$$f_{X+Y}(t) = \int_0^t f_{X,Y}(u,\ t-u)du = g(t)\int_0^t \pi_{00}(u)f(u)du$$

$$+\,g(t)\int_0^t \pi_{01}(u)\ f(u)\ \left(1-e^{-\beta(t-u)}\right)du$$

$$+\int_0^t g(t-u)\int_0^{t-u} h_{E_1}(u)\pi_{01}(v)f(v)\left(1-e^{-\beta(t-(u+v))}\right)dvdu$$

$$+\int_0^t g(t-u)\left[\int_0^{t-u} h_{E_1}(u)\pi_{00}(v)f(v)\ dv\right]du \qquad (5.18)$$

5.5 System measures

Using the joint distribution of the favourable and the unfavourable time and the joint forward recurrence time of the D and E events, expressions for various operating characteristics of an intermittently used system are obtained.

It is noted that (Gaver (1963)), that measures such as reliability and availability will provide a rather pessimistic evaluation of the system performance in this case, since the system may be in the down state during no need periods with negligible consequential loss. Attention will instead be paid to the D event as defined in previous sections.

(1) Time to first D event

Let T_D be the random variable denoting the time to the first D event then T_D has p.d.f. given by

$$f_{T_D}(t) = \psi_D(0, t) = f_X$$

$$\therefore \quad Pr\{T_D > t\} = \int_t^{\infty} f_X(u)du \tag{5.19}$$

The mean value of T_D is given by

$$MTTD = \int_0^{\infty} x\, f_X(x)dx \tag{5.20}$$

(2) The number of D events in the interval $(0, t]$

Taken the point process generated by successive D events to be regular the first order product density for D events is given by

$$h_1(x) = \lim_{\Delta \to 0} \frac{1}{\Delta} E[N(x, \Delta)] = \lim_{\Delta \to 0} \frac{1}{\Delta} Pr\{N(x, \Delta) = 1\}$$

where $N,(\chi, \Delta)$ denotes the number of D events in the time interval $(\chi, \chi+\Delta)$.

Hence

$$h_1(x) = \psi_D(0, \chi) + \int_0^{x} h_{E_1}(u)\psi_D(u, x-u)\, du \tag{5.21}$$

Therefore the expected number of D events in $(0, t]$ is given by

$$E[N(0,t)] = \int_0^t h_1(x)dx = \int_0^t \psi_D(0, \chi)dx + \int_0^t dx \int_0^x h_E(u)\psi_D(u, x-u)\, du$$

$$= \int_0^t f_X(x)dx + \int_0^t dx \int_0^x h_E(u)\, f_X(x-u)\, du \qquad (5.22)$$

(3) The expected duration of a disappointment

The expected duration of a disappointment is given by the expected value of the random variable Y, and

$$E(Y) = \int_0^\infty y\, f_X(y)\, dy \qquad (5.23)$$

5.6 Special case

When $\alpha = 0$ i.e. the system is available continuously. The following results are obtained:

$$f_{X,Y}(x, y) = f(x)g(x+y) + \int_0^x h_E(u) f(x+u)\, g(x-u+y)\, du \qquad (5.24)$$

where

$$h_{E_1}(t) = \sum_{n=1}^{\infty} [f(t)G(t)]^{(n)}$$

and

$$f_X(x) = f(x)\,\bar{G}(x) + \int_0^x h_{E_1}(u)\, f(x-u)\, \bar{G}(x-u)\, du \tag{5.25}$$

$$f_Y(y) = \int_0^\infty f(x)\, g(x+y) du + \int_0^\infty dx \int_0^x h_{E_1}(u) f(x-u)\, g(x-u+y)\, dy \tag{5.26}$$

$$F_{X+Y}(t) = g(t)F(t) + \int_0^t g(t-u) \int_0^{t-u} h_{E_1}(u) f(x)\, dx \tag{5.27}$$

$$= g(x)\, F(t) + \int_0^t g(t-u) h_{E_1}(u) F(t-u)\, Du \tag{5.28}$$

These results are in agreement with Subramanian, Sarma and Natarajan (1983).

Concluding remark and scope of the work

Reliability theory is a very important branch of system engineering and applied probability and deals with methods of evaluating the various measures of performance of a system that may be subject to gradual deterioration. Several methods of redundant systems have been studied in the literature and the following are some of the typical assumptions made in analysing such systems:

(i) the repair facility can take up a failed unit for repair at any time, if no other unit is undergoing repair

(ii) the system under consideration is needed all the time.

This dissertation is a study of redundant repairable systems, relaxing the above two assumptions.

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